J. Spectr. Theory 14 (2024), 1475–1512
DOI 10.4171/JST/529

On the closure of the Airault–Mckean–Moser locus for elliptic KdV potentials via Darboux transformations

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Abstract. We study the general elliptic KdV potentials, which can be expressed (up to adding a constant) as

$$q_{\mathbf{p}}(z) := \sum_{j=1}^{n} m_j(m_j+1)\wp(z-p_j), \quad m_j \in \mathbb{N}.$$

We give an elementary proof of the theorem that the singularity

$$\mathbf{p} = (\overbrace{p_1,\dots,p_1}^{m_1(m_1+1)/2},\dots,\overbrace{p_n,\dots,p_n}^{m_n(m_n+1)/2})$$

is contained in the closure of the elliptic Airault–Mckean–Moser locus, which was proved previously by Treibich and Verdier in the late 1980s using purely algebro-geometric methods. Our proof is based on Darboux transformations and does not use algebraic geometry. This solves an open problem posed by Gesztesy, Unterkofler, and Weikard [Trans. Amer. Math. Soc. 358 (2006), 603–656]. Some applications are also given.

1. Introduction

Let $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ and $E_\tau := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ be a flat torus. Let $\wp(z)$ be the Weierstrass $\wp$-function with periods $\omega_1 = 1$ and $\omega_2 = \tau$, $\zeta(z) := -\int^z \wp(\xi)d\xi$ be the Weierstrass zeta function with two quasi-periods η_k:

$$\eta_k := 2\zeta(\frac{\omega_k}{2}) = \zeta(z+\omega_k) - \zeta(z), \quad k = 1, 2, \tag{1.1}$$

and $\sigma(z) := \exp\int^z \zeta(\xi)d\xi$ be the Weierstrass sigma function. Notice that $\zeta(z)$ is an odd meromorphic function with simple poles at $\Lambda_\tau := \mathbb{Z}+\mathbb{Z}\tau$, while $\sigma(z)$ is an odd entire function with simple zeros at Λ_τ and satisfies the following transformation law

$$\sigma(z+\omega_k) = -e^{\eta_k(z+\frac{1}{2}\omega_k)}\sigma(z), \quad k = 1, 2. \tag{1.2}$$

Mathematics Subject Classification 2020: 37K10 (primary); 14H70 (secondary).
Keywords: elliptic KdV potentials, Darboux transformations.

In this paper, we give new proofs of several theorems (see Theorems 1.1, 1.3, and 1.4 below) related to the elliptic KdV potentials, or equivalently, Picard potentials. These theorems and our new proofs will play important roles in our future study of the partial differential equation

$$\Delta u + e^u = 8\pi \sum_{j=1}^{n} m_j \delta_{p_j}, \quad \text{on } E_\tau,$$

where δ_{p_j} denotes the Dirac measure at p_j. Although these theorems were not stated precisely the same as the current form appearing in this paper, they can be deduced from Treibich and Verdier's papers [26, 27, 29, 31]. Treibich and Verdier's proofs are purely algebro-geometric. Our new proofs are quite elementary in the sense that we mainly use the Darboux transformation instead of using algebraic geometry. In particular, this solves an open question raised by Gesztesy, Unterkofler, and Weikard [12].

According to Gesztesy and Weikard [15], an elliptic function $q(z)$ is called *a Picard potential* if, for any $E \in \mathbb{C}$, all solutions of the linear ODE (denoted by $\mathrm{L}(q; E)$)

$$y''(z) = (q(z) - E)y(z), \quad z \in \mathbb{C} \tag{1.3}$$

are meromorphic in $\mathbb{C}$; in other words, the local monodromy matrix of (1.3) at any singularity is always the identity matrix I_2, so the monodromy representation of (1.3) would be reduced to a group homomorphism $\rho\colon \pi_1(E_\tau) \to \mathrm{SL}(2, \mathbb{C})$. This class of linear ODEs might be simplest from the monodromy aspect. It was proved in [12, Theorem 1.1] that $q(z)$ is a Picard potential if and only if up to adding a constant, $q(z)$ is expressed as

$$q(z) = \sum_{j=1}^{n} m_j(m_j + 1)\wp(z - p_j), \tag{1.4}$$

where $m_j \in \mathbb{N}$, $p_j \in E_\tau$ satisfies $p_i \neq p_j$ for $i \neq j$ and the following conditions

$$\sum_{j=1, \neq i}^{n} m_j(m_j + 1)\wp^{(2k-1)}(p_i - p_j) = 0 \quad \text{for } 1 \le k \le m_i, 1 \le i \le n. \tag{1.5}$$

It is interesting to note that Picard potentials are closely related to the theory of the stationary KdV hierarchy [14, 15]. As in [11, 12], an elliptic function $q(z)$ is called an *elliptic KdV potential* if $q(z)$ is an algebro-geometric solution of the stationary KdV hierarchy; see Section 2 for a brief overview of this theory. The stationary KdV hierarchy has been widely studied in the literature; see e.g., [1, 7, 8, 10, 12, 15–17, 22, 24–32] and references therein. In the seminal work [15], Gesztesy and Weikard proved the following remarkable result.

Theorem A ([15]). *An elliptic function $q(z)$ is an elliptic KdV potential if and only if $q(z)$ is a Picard potential.*

Theorem A implies that an elliptic function $q(z)$ is an elliptic KdV potential if and only if up to adding a constant, $q(z)$ is of the form (1.4) satisfying (1.5). When $m_i = 1$ for all i, the condition (1.5) becomes

$$\sum_{j=1,\neq i}^{n} \wp'(p_i - p_j) = 0 \quad \text{for } 1 \leq i \leq n,$$

which was first studied by Airault, Mckean, and Moser [1].

Now, we fix $N \in \mathbb{N}$. Given any $\mathbf{p} = (p_1, \ldots, p_N) \in \mathrm{Sym}^N E_\tau := E_\tau^N / S_N$ (i.e., the symmetric space of the N-th copy of E_τ), we define

$$q_{\mathbf{p}}(z) := 2 \sum_{j=1}^{N} \wp(z - p_j).$$

As in [1, 12], we define the elliptic Airault–Mckean–Moser (AMM) locus of poles $\mathcal{L}_N \subset \mathrm{Sym}^N E_\tau$ by

$$\begin{aligned} \mathcal{L}_N &:= \left\{ \mathbf{p} = (p_1, \ldots, p_N) \in \mathrm{Sym}^N E_\tau \;\middle|\; \begin{array}{l} p_i \neq p_j \text{ for } i \neq j, \\ q_{\mathbf{p}}(z) \text{ is a KdV potential} \end{array} \right\} \\ &= \left\{ \mathbf{p} = (p_1, \ldots, p_N) \in \mathrm{Sym}^N E_\tau \;\middle|\; \begin{array}{l} p_i \neq p_j \text{ for } i \neq j, \\ \sum_{j \neq i}^{N} \wp'(p_i - p_j) = 0 \text{ for all } i \end{array} \right\} \end{aligned} \tag{1.6}$$

in the collisionless case. In the presence of collisions, it is convenient to introduce the following locus:

$$\mathcal{A}_N := \{\mathbf{p} = (p_1, \ldots, p_N) \in \mathrm{Sym}^N E_\tau \mid q_{\mathbf{p}}(z) \text{ is a KdV potential}\}.$$

That is, in contrast to the elliptic AMM locus $\mathcal{L}_N$, several numbers of p_i's might collapse together for $\mathbf{p} \in \mathcal{A}_N$, i.e.,

$$\mathbf{p} = (\overbrace{p_1, \ldots, p_1}^{m_1(m_1+1)/2}, \ldots, \overbrace{p_n, \ldots, p_n}^{m_n(m_n+1)/2}), \tag{1.7}$$

where $p_i \neq p_j$ for any $i \neq j$,

$$N = \frac{1}{2} \sum_{i=1}^{n} m_i(m_i + 1),$$

and

$$q_{\mathbf{p}}(z) = 2 \sum_{i=1}^{N} \wp(z - p_i) = \sum_{i=1}^{n} m_i(m_i + 1) \wp(z - p_i). \tag{1.8}$$

Clearly, $\mathcal{L}_N \subset \mathcal{A}_N$. Let $\overline{\mathcal{L}_N}$ and $\overline{\mathcal{A}_N}$ be the closure of $\mathcal{L}_N$ and $\mathcal{A}_N$ in $\mathrm{Sym}^N E_\tau$, respectively. The characterization of the closure of the elliptic AMM locus is given by the following interesting result.

Theorem 1.1 ([29, 31]). *There holds* $\overline{\mathcal{L}_N} = \overline{\mathcal{A}_N} = \mathcal{A}_N$.

Remark 1.2. Theorem 1.1 shows that any $\mathbf{p} \in \mathcal{A}_N$ belongs to the closure of the elliptic AMM locus. Theorem 1.1 was first proved by Treibich and Verdier [29, 31].

On the other hand, Gesztesy, Unterkofler, and Weikard [12] proved the same statement as Theorem 1.1 for rational KdV potentials and simply-periodic KdV potentials. Their proofs apply the Darboux transformation and do not need to use algebraic geometry. In [12, Remark 3.18], they proposed the open question of whether this result for elliptic KdV potentials (i.e., Theorem 1.1) can be proved via the Darboux transformation either. The main purpose of this paper is to give a positive answer to this open question, namely we will give a new proof of Theorem 1.1 via the Darboux transformation.

It is interesting to note that the Darboux transformation can also be applied to give new proofs of some other statements concerning the covering map, which was first introduced by Krichever [17]. There are two ways to define the covering map. In the seminal work [17], Krichever considered the elliptic KdV potential $q_{\mathbf{p}}(z)$ with $\mathbf{p} \in \mathcal{L}_N$:

$$q_{\mathbf{p}}(z) = 2\sum_{j=1}^{N} \wp(z - p_j), \quad p_i \neq p_j \text{ in } E_\tau \text{ for } i \neq j.$$

Recall (1.6) that $\mathbf{p} \in \mathcal{L}_N$ if and only if

$$\sum_{j=1, \neq i}^{N} \wp'(p_i - p_j) = 0 \quad \text{for } 1 \leq i \leq N. \tag{1.9}$$

In [17], Krichever proved that, except for finitely many E's, the ODE

$$y''(z) = (q_{\mathbf{p}}(z) - E)y(z) = \Big[2\sum_{j=1}^{N} \wp(z - p_j) - E\Big]y(z)$$

has a solution of the following form that is now known as *Krichever's ansatz* in the literature:

$$\varphi(z; \kappa, \alpha) := e^{-\kappa z} \cdot \sum_{i=1}^{N} d_i \Psi(z - p_i; \alpha), \tag{1.10}$$

where $\alpha \in E_\tau \setminus \{0\}$, $\kappa \in \mathbb{C}$, $d_i \in \mathbb{C}$ for all i and

$$\Psi(z; \alpha) := -\frac{\sigma(z - \alpha)}{\sigma(z)\sigma(\alpha)} e^{\zeta(\alpha) z}.$$

To find such a solution $\varphi(z)$, Krichever introduced the following two $N \times N$ matrices $L(\alpha)$ and $T(\alpha)$:

$$L(\alpha) = (L_{ij})_{N\times N} \quad \text{with } L_{ij} = (1-\delta_{ij})\Psi_{ij}(\alpha) := \begin{cases} 0 & \text{if } i = j, \\ \Psi_{ij}(\alpha) & \text{if } i \neq j, \end{cases}$$

and

$$T(\alpha) = (T_{ij})_{N\times N} \quad \text{with } T_{ij} = \delta_{ij} \sum_{k\neq i} \wp_{ik} + (1-\delta_{ij})\Psi'_{ij}(\alpha),$$

where we use the notations $\Psi_{ij}(\alpha) := \Psi(p_i - p_j; \alpha)$, $\Psi'_{ij}(\alpha) := \frac{d}{dz}\Psi(p_i - p_j; \alpha)$, and $\wp_{ij} := \wp(p_i - p_j)$ for $i \neq j$. Among other things, Krichever obtained the following results.

(i) $L(\alpha)$ and $T(\alpha)$ commute if and only if (1.9) holds, i.e., $\mathbf{p} \in \mathcal{L}_N$.

(ii) The vector $\vec{d} = (d_1, \dots, d_N)^t$, which consists of the coefficients in (1.10), is a common eigenvector of $L(\alpha)$ and $T(\alpha)$ with κ being the corresponding eigenvalue of $L(\alpha)$.

(iii) The pair (κ, α) satisfies $R(\kappa, \alpha) = 0$, where $R(\kappa, \alpha)$ is the characteristic polynomial of $L(\alpha)$, i.e.,

$$R(\kappa, \alpha) = R_{\mathbf{p}}(\kappa, \alpha) := \det(\kappa I_N - L(\alpha)).$$

This $R(\kappa, \alpha)$ will be called *Krichever polynomial* in this paper.

(iv) The algebraic curve defined by

$$\Gamma_N := \{(\kappa, \alpha) \mid R(\kappa, \alpha) = 0\}$$

is an N-sheeted covering over E_τ, where the covering map $\pi\colon \Gamma_N \to E_\tau$ is defined by $\pi(\kappa, \alpha) := \alpha$. This covering map π will be called *Krichever covering map* in this paper.

We will briefly review this theory in Section 4. Note that this covering map π was originally defined only for $\mathbf{p} \in \mathcal{L}_N$.

On the other hand, there is another way to introduce the covering map for a general elliptic KdV potential $q_{\mathbf{p}}(z)$ of the form (1.8). This alternative way is based on the Baker–Akhiezer function of the associated ODE $\mathrm{L}(q_{\mathbf{p}}; E)$,

$$y''(z) = (q_{\mathbf{p}}(z) - E)y(z) = \Big[\sum_{i=1}^{n} m_i(m_i+1)\wp(z-p_i) - E\Big]y(z). \tag{1.11}$$

It is well known (cf. [11]) that, for the elliptic KdV potential $q_{\mathbf{p}}(z)$, there is an associated *spectral polynomial* $Q_{q_{\mathbf{p}}}(E)$ of odd degree as well as the corresponding *spectral* *curve*

$$\Gamma_{q_{\mathbf{p}}} := \{P := (E, \mathcal{C}) \in \mathbb{C}^2 \mid \mathcal{C}^2 - Q_{q_{\mathbf{p}}}(E) = 0\}.$$

The one-point compactification of this hyperelliptic curve $\Gamma_{q_{\mathbf{p}}}$, i.e., by joining the point at infinity (denoted by P_∞), is denoted by $\overline{\Gamma_{q_{\mathbf{p}}}}$. We also define the involution $*$ on $\overline{\Gamma_{q_{\mathbf{p}}}}$ by

$$*: \overline{\Gamma_{q_{\mathbf{p}}}} \to \overline{\Gamma_{q_{\mathbf{p}}}}, \quad P = (E, \mathcal{C}) \mapsto P^* = (E, -\mathcal{C}),\ P_\infty^* = P_\infty.$$

It is well known (cf. [11]) that, given $P = (E, \mathcal{C}) \in \Gamma_{q_{\mathbf{p}}}$, there is an associated Baker–Akhiezer function $\psi(P; z)$, which solves (1.11) and is unique up to multiplying a constant. Furthermore, we have the following.

(a) $\psi(P; z)$ is elliptic of the second kind as a meromorphic function of $z \in \mathbb{C}$; namely, there are $\lambda_i = \lambda_i(P) \in \mathbb{C} \setminus \{0\}$'s such that

$$\psi(P; z + \omega_i) = \lambda_i \psi(P; z), \quad i = 1, 2,$$

and $\lambda_i(P^*) = 1/\lambda_i(P)$ for $i = 1, 2$.

(b) When $\mathcal{C}^2 - Q_{q_{\mathbf{p}}}(E) \neq 0$, i.e., $P \neq P^*$, then $\psi(P; z)$ and $\psi(P^*; z)$ are linearly independent solutions of (1.11). In this case, the monodromy matrices of (1.11) with respect to $(\psi(P; z), \psi(P^*; z))$ is given by

$$\begin{pmatrix} \psi(P; z + \omega_i) \\ \psi(P^*; z + \omega_i) \end{pmatrix} = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix} \begin{pmatrix} \psi(P; z) \\ \psi(P^*; z) \end{pmatrix}, \quad i = 1, 2. \tag{1.12}$$

See Section 2 for a brief overview of this theory. Denote

$$\mathfrak{m} := \sum_{i=1}^{n} m_i \quad \text{for } q_{\mathbf{p}} \text{ given by (1.8)}.$$

Since $\psi(P; z)$ is a solution of (1.11), $\psi(P; z)$ has at most poles at p_i with order m_i. Note from (1.12) that the zeros of $\psi(P; z)$ are well defined on E_τ, and as an elliptic function of the second kind, the number of zeros of $\psi(P; z)$ on E_τ equals the number of poles by counting multiplicity. Then the classical theory of elliptic functions says that, up to multiplying a nonzero constant, $\psi(P; z)$ can be expressed as

$$\psi(P; z) = e^{c(P)z} \frac{\prod_{i=1}^{\mathfrak{m}} \sigma(z - a_i(P))}{\prod_{i=1}^{n} \sigma(z - p_i)^{m_i}},$$

for some $a_i(P) \in E_\tau$ and some constant $c(P) \in \mathbb{C}$. Therefore, the spectral curve $\Gamma_{q_{\mathbf{p}}}$ can be embedded into $\mathrm{Sym}^{\mathfrak{m}} E_\tau$ by mapping P to the zero set $\{a_1(P), \ldots, a_{\mathfrak{m}}(P)\} \in \mathrm{Sym}^{\mathfrak{m}} E_\tau$ of $\psi(P; z)$. Note that $\mathrm{Sym}^{\mathfrak{m}} E_\tau$ has a natural addition map to E_τ:

$$\{a_1, \ldots, a_{\mathfrak{m}}\} \mapsto \sum_{i=1}^{\mathfrak{m}} a_i - \sum_{j=1}^{n} m_j p_j,$$

so the composition will give rise to a covering map $\sigma_{q_{\mathbf{p}}}\colon \overline{\Gamma_{q_{\mathbf{p}}}} = \Gamma_{q_{\mathbf{p}}} \cup \{P_\infty\} \to E_\tau$, which is defined by

$$\sigma_{q_{\mathbf{p}}}(P) := \sum_{i=1}^{\mathfrak{m}} a_i(P) - \sum_{j=1}^{n} m_j p_j \quad \text{for all } P \in \Gamma_{q_{\mathbf{p}}}.$$

The degree of this covering map $\sigma_{q_{\mathbf{p}}}$ is defined as $\deg \sigma_{q_{\mathbf{p}}} = \#\sigma_{q_{\mathbf{p}}}^{-1}(z)$ counted with multiplicity. The basic question is how to compute it.

The interesting thing is that, when $\mathbf{p} \in \mathcal{L}_N$, we will see in Theorem 4.6 that there is a birational map $\vartheta\colon \Gamma_{q_{\mathbf{p}}} \to \Gamma_N$ such that $\sigma_{q_{\mathbf{p}}} = \pi \circ \vartheta$, where $\pi\colon \Gamma_N \to E_\tau$ is precisely the aforementioned Krichever covering map. For this reason, this $\sigma_{q_{\mathbf{p}}}$ for general $\mathbf{p} \in \mathcal{A}_N$ will also be called *Krichever covering map* in this paper.

The next result gives the formula of $\deg \sigma_{q_{\mathbf{p}}}$ for all $\mathbf{p} \in \mathcal{A}_N = \overline{\mathcal{L}_N}$.

Theorem 1.3. [26, 27] *For any $\mathbf{p} \in \mathcal{A}_N = \overline{\mathcal{L}_N}$, the Krichever covering map*

$$\sigma_{q_{\mathbf{p}}}\colon \overline{\Gamma_{q_{\mathbf{p}}}} \to E_\tau$$

always has degree N.

Theorem 1.3 was first proved by Treibich [26, 27]. Here we give a new proof of Theorem 1.3 via the Darboux transformation. More precisely, we will prove Theorem 1.3 first for $\mathbf{p} \in \mathcal{L}_N$ via Krichever's theory [17], and then for general $\mathbf{p} \in \mathcal{A}_N$ via Ehlers–Knörrer's theorem [9, 10] concerning the Darboux transformation for the KdV hierarchy. We will briefly review Ehlers–Knörrer's theorem in Section 2.

Let $K(E_\tau)$ and $K(\overline{\Gamma_{q_{\mathbf{p}}}})$ be the field of rational functions on E_τ and $\overline{\Gamma_{q_{\mathbf{p}}}}$, respectively. Then Theorem 1.3 indicates that $K(\overline{\Gamma_{q_{\mathbf{p}}}})$ is a finite field extension over $K(E_\tau)$ with

$$[K(\overline{\Gamma_{q_{\mathbf{p}}}})\colon K(E_\tau)] = \deg \sigma_{q_{\mathbf{p}}} = N.$$

A basic question is *how to find a primitive generator of this field extension*.

The results of [3, 4, 19, 20] for Lamé and Darboux–Treibich–Verdier potentials indicate that this primitive generator is related to the monodromy (1.12); see Section 5 for details. This motivates us to define a rational function $\mathbf{z}_{\mathbf{p}}\colon \overline{\Gamma_{q_{\mathbf{p}}}} \to \mathbb{C} \cup \{\infty\}$ by

$$\begin{aligned}\mathbf{z}_{\mathbf{p}}(P) := {}& \zeta\Big(\sum_{k=1}^{\mathfrak{m}} a_k(P) - \sum_{j=1}^{n} m_j p_j\Big) - \sum_{k=1}^{\mathfrak{m}} \zeta(a_k(P) - p_n) \\ &+ \sum_{j=1}^{n-1} m_j \zeta(p_j - p_n) \quad \text{for all } P \in \Gamma_{q_{\mathbf{p}}}.\end{aligned}$$

Remark that we can replace p_n with any fixed p_i in the definition of $\mathbf{z}_{\mathbf{p}}$; see Section 5. The next result shows that this $\mathbf{z}_{\mathbf{p}}$ can be taken as a primitive generator.

Theorem 1.4 ([26,27]). *$\mathbf{z_p}$ is a primitive generator of the field extension $K(\overline{\Gamma_{q_{\mathbf{p}}}})$ over $K(E_\tau)$.*

Again, Theorem 1.4 can be deduced from Treibich [26, 27]. Here our new proof of Theorem 1.4 is based on the Darboux transformation and Krichever's theory [17]. In particular, we will see in Section 5 that, for $\mathbf{p} \in \mathcal{L}_N$, the minimal polynomial of $\mathbf{z_p}$ coincides with the Krichever polynomial $R_{\mathbf{p}}(\kappa, \alpha)$.

The rest of this paper is organized as follows. In Section 2, we give a brief overview of the theory of the stationary KdV hierarchy and the Darboux transformation. In Section 3, we introduce our new observation concerning the Darboux transformation, which allows us to connect $\mathbf{p} \in \mathcal{A}_N \setminus \mathcal{L}_N$ with $\mathbf{p} \in \mathcal{L}_N$ and plays a key role in this paper. This idea is also the main novelty of our approach in this paper. As an application, we can prove Theorem 1.1. In Section 4, we first briefly review Krichever's famous theory [17] for elliptic KdV potentials $q_{\mathbf{p}}(z)$ with $\mathbf{p} \in \mathcal{L}_N$. We will see that Theorem 1.3 for $\mathbf{p} \in \mathcal{L}_N$ easily follows from Krichever's theory. Then we can prove Theorem 1.3 for $\mathbf{p} \in \mathcal{A}_N \setminus \mathcal{L}_N$ via the Darboux transformation. Similarly we can prove Theorem 1.4 in Section 5.

2. Preliminaries

2.1. Stationary KdV hierarchy

In this section, we review basic facts on the stationary KdV hierarchy following Gesztesy and Holden [11, Chapter 1]. Given a meromorphic function $q(z)$ in $\mathbb{C}$, we define $\{f_\ell(z)\}_{\ell \in \mathbb{N} \cup \{0\}}$ recursively by

$$f_0 = 1, \quad f_\ell' = -\frac{1}{4} f_{\ell-1}^{(3)} + q f_{\ell-1}' + \frac{1}{2} q' f_{\ell-1}, \quad \ell \in \mathbb{N}. \tag{2.1}$$

Explicitly, one finds

$$f_0 = 1, \quad f_1 = \frac{1}{2} q + c_1, \quad f_2 = -\frac{1}{8}(q'' - 3q^2) + c_1 \frac{1}{2} q + c_2, \quad \ldots.$$

Here $\{c_\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{C}$ denote integration constants that naturally arise when solving (2.1). It is known (cf. [11, Theorem D.1]) that $f_j(z)$ are *differential polynomials* of $q(z)$ for all j. In particular, $f_j(z)$ is meromorphic in z.

Consider a second-order differential operator of Schrödinger type

$$L = -\frac{d^2}{dz^2} + q,$$

and a $2g+1$-order differential operator

$$P_{2g+1} = \sum_{j=0}^{g}\Big(f_j \frac{d}{dz} - \frac{1}{2} f_j'\Big) L^{g-j}, \quad g \in \mathbb{N} \cup \{0\}.$$

By the recursive relation (2.1), a direct computation leads to (note that $[\cdot,\cdot]$ denotes the commutator symbol)

$$[L, P_{2g+1}] = -2 f_{g+1}', \quad g \in \mathbb{N} \cup \{0\}.$$

In particular, (L, P_{2g+1}) represents the celebrated *Lax pair* [18] of the KdV hierarchy. Varying $g \in \mathbb{N} \cup \{0\}$, the stationary KdV hierarchy is then defined in terms of the vanishing of the commutator of L and P_{2g+1} by

$$\text{s-KdV}_g(q) = [L, P_{2g+1}] = -2 f_{g+1}' = 0, \quad g \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

For example,

$$\text{s-KdV}_0(q) = -q' = 0, \quad \text{s-KdV}_1(q) = \frac{1}{4} q^{(3)} - \frac{3}{2} q q' + c_1(-q') = 0, \quad \ldots$$

represent the first two equations of the stationary KdV hierarchy. By definition, the set of solutions of (2.2), with g ranging in $\mathbb{N} \cup \{0\}$ and c_ℓ in $\mathbb{C}$, represents the class of *algebro-geometric KdV solutions*. As in [11,12], it will be convenient to abbreviate algebro-geometric KdV solutions q simply as *KdV potentials*.

Next, we introduce a polynomial $\Phi_q = \Phi_{q,g}$ of degree g with respect to the spectral parameter $E \in \mathbb{C}$ by

$$\Phi_q(z; E) = \Phi_{q,g}(z; E) := \sum_{\ell=0}^{g} f_{g-\ell}(z) E^{\ell}, \quad g \in \mathbb{N} \cup \{0\}. \tag{2.3}$$

The recursive relation (2.1) and (2.2) together imply that this meromorphic function $\Phi_q(z; E)$ solves

$$\Phi'''(z) - 4(q(z) - E)\Phi'(z) - 2q'(z)\Phi(z) = 0.$$

Consequently,

$$Q_q(E) = Q_{q,2g+1}(E) := \frac{1}{2}\Phi_q \Phi_q'' - \frac{1}{4}\Phi_q'^2 - (q - E)\Phi_q^2 \tag{2.4}$$

is a monic polynomial in E of degree $2g+1$ that is independent of z. Since the equality $[L, P_{2g+1}] = 0$ implies $P_{2g+1}^2 \phi = -Q_q(E)\phi$ for any $L\phi = E\phi$, we conclude that

$$P_{2g+1}^2 + Q_q(L) = 0, \tag{2.5}$$

a celebrated theorem by Burchnall and Chaundy [2]. Equation (2.5) leads naturally to the hyperelliptic curve Γ_q of (arithmetic) genus g:

$$\Gamma_q := \{(E, \mathcal{C}) \in \mathbb{C}^2 \mid \mathcal{C}^2 - Q_q(E) = 0\}.$$

Remark 2.1. As mentioned in [11, Remark 1.5], if $q(z)$ satisfies one stationary KdV equation s-KdV$_g(q) = 0$ for some g, then it also satisfies s-KdV$_k(q) = 0$ for any $k > g$. In this paper, we say that q is *a genus g KdV potential* if g is the smallest integer such that s-KdV$_g(q) = 0$, i.e., s-KdV$_{g-1}(q) \neq 0$ for any choices of integration constants c_k's. For a genus g KdV potential $q(z)$, the corresponding $Q_q(E) = Q_{q,2g+1}(E)$ is called its *spectral polynomial* and Γ_q in (2.5) is called its *spectral curve*.

2.2. Baker–Akhiezer functions and Darboux transformations

Let $q(z)$ be a KdV potential of genus g. First we recall the following result from [23, 32].

Theorem 2.2 ([23] or [32, Theorem 1]). *Let $q(z)$ be a KdV potential. Then the following statements hold.*

(1) *Any pole z_0 of $q(z)$ is a regular singular point of*

$$y''(z) = [q(z) - E]y(z). \tag{2.6}$$

Furthermore, the Laurent expansion of $q(z)$ near z_0 is given by $\frac{k(k+1)}{(z-z_0)^2} + O(1)$ for some $k \in \mathbb{N}$ (i.e., the residue of $q(z)$ at z_0 is 0).

(2) *For any $E \in \mathbb{C}$, all solutions of (2.6) are meromorphic in $\mathbb{C}$.*

Remark that in the case when the spectral curve Γ_q associated with $q(z)$ is non-singular, Theorem 2.2 also follows from Its and Matveev [16].

Now, we recall the notion of Baker–Akhiezer functions. Given any $P = (E, \mathcal{C}) \in \Gamma_q$ and recalling the meromorphic function $\Phi_q(z; E)$ in (2.3), we define the fundamental meromorphic function $\phi(P; z)$ by

$$\phi(P; z) := \frac{i\mathcal{C}(P) + \frac{1}{2}\Phi_q'(z; E)}{\Phi_q(z; E)}, \quad z \in \mathbb{C}, \tag{2.7}$$

where $i = \sqrt{-1}$ and $\mathcal{C}(P)$ denotes the second component of $P = (E, \mathcal{C})$. Then, by (2.4) and $\mathcal{C}(P)^2 = Q_q(E)$, a direct computation implies that $\phi(P; z)$ solves the following Ricatti equation (cf. [10, Lemma 2.1]):

$$\phi'(P; z) = q(z) - E - \phi(P; z)^2. \tag{2.8}$$

Proposition 2.3. *The meromorphic function $\phi(P;z)$ has only simple poles with integer residues.*

Proof. Let z_0 be a pole of $\phi(P;z)$. If z_0 is not a pole of $q(z)$, it follows from (2.8) that $\phi(P;z) = (z-z_0)^{-1} + O(1)$. If z_0 is a pole of $q(z)$, i.e., $q(z) = \frac{k(k+1)}{(z-z_0)^2} + O(1)$ for some $k \in \mathbb{N}$ by Theorem 2.2, again we conclude from (2.8) that $\phi(P;z) = \frac{c}{z-z_0} + O(1)$ with $c \in \{-k, k+1\}$. ■

Fixing any $z_0 \in \mathbb{C} \setminus \{\text{poles of } q(z)\}$, the Baker–Akhiezer function $\psi(P;z,z_0)$ on $\Gamma_q = \overline{\Gamma_q} \setminus \{P_\infty\}$ is defined by

$$\psi(P;z,z_0) := \exp\Bigg(\int_{z_0}^{z} \phi(P;\xi)d\xi\Bigg), \qquad P = (E,\mathcal{C}) \in \Gamma_q,\ z \in \mathbb{C}, \tag{2.9}$$

where the integral path is chosen a smooth non-selfintersecting path from z_0 to z which avoids singularities of $\phi(P;z)$. Clearly, Proposition 2.3 implies that $\psi(P;z,z_0)$ is independent of the choice of the integral path, and so $\psi(P;z,z_0)$ is single-valued in $\mathbb{C}$. Since

$$\phi(P;z) = \frac{\psi'(P;z,z_0)}{\psi(P;z,z_0)},$$

and $\phi(P;z)$ solves the Ricatti equation (2.8), a direct computation shows that both $\psi(P;z,z_0)$ and $\psi(P^*;z,z_0)$ solve

$$y''(z) = [q(z) - E]y(z), \tag{2.10}$$

and (cf. [10, Lemma 2.1])

$$\psi(P;z,z_0)\psi(P^*;z,z_0) = \frac{\Phi_q(z;E)}{\Phi_q(z_0;E)}, \tag{2.11}$$

$$W(\psi(P;z,z_0), \psi(P^*;z,z_0)) = \frac{2i\mathcal{C}(P)}{\Phi_q(z_0;E)}, \tag{2.12}$$

where $' = \frac{d}{dz}$ and $W(f,g) = f'g - fg'$ denotes the Wronskian of f, g. In particular, it follows from Theorem 2.2 that the Baker–Akhiezer functions $\psi(P;\cdot,z_0)$ and $\psi(P^*;\cdot,z_0)$ are meromorphic in $\mathbb{C}$. Since different choices of z_0 give the same solution of the linear ODE (2.10) up to multiplying a constant, we omit the notation z_0 and write

$$\psi(P;z,z_0) = \psi(P;z), \quad \psi(P^*;z,z_0) = \psi(P^*;z)$$

just for convenience. Clearly, (2.12) implies that

$$\psi(P;z), \psi(P^*;z) \textit{ are linearly independent solutions of } (2.10)$$
$$\textit{if and only if } \mathcal{C}(P) \neq 0, \textit{ i.e., } Q_q(E) \neq 0. \tag{2.13}$$

Now, we recall the following Ehlers–Knörrer's theorem [9, 10] concerning the well-known Darboux transformation for the stationary KdV hierarchy.

Theorem 2.4 ([9, 10]). *Suppose $q(z)$ is a genus g KdV potential with the associated spectral polynomial $Q_{q,2g+1}(E)$. For any $P_0 = (E_0, \mathcal{C}_0) \in \Gamma_q$, we let $y(z)$ be any solution of*

$$y''(z) = [q(z) - E_0]y(z), \tag{2.14}$$

and define a new potential $\tilde{q}(z)$ via the Darboux transformation

$$\tilde{q}(z) := q(z) - 2(\ln y(z))''. \tag{2.15}$$

Then $\tilde{q}(z)$ is also a KdV potential. More precisely, the following statements hold.

(1) *If $y(z)$ is not the Baker–Akhiezer function of (2.14), then $\tilde{q}(z)$ is a genus $g+1$ KdV potential with the associated spectral polynomial*

$$Q_{\tilde{q},2g+3}(E) = (E - E_0)^2 Q_{q,2g+1}(E).$$

(2) *If $y(z) = \psi(P_0; z)$ is the Baker–Akhiezer function and E_0 is not a multiple zero of $Q_{q,2g+1}(E)$ (i.e., either $Q_{q,2g+1}(E_0) \neq 0$ or E_0 is a simple zero of $Q_{q,2g+1}(E)$), then $\tilde{q}(z)$ is a genus g KdV potential isospectral to $q(z)$.*

(3) *If $y(z) = \psi(P_0; z)$ is the Baker–Akhiezer function and E_0 is a multiple zero of $Q_{q,2g+1}(E)$, then $\tilde{q}(z)$ is a genus $g-1$ KdV potential with the associated spectral polynomial*

$$Q_{\tilde{q},2g-1}(E) = (E - E_0)^{-2} Q_{q,2g+1}(E).$$

2.3. Elliptic KdV potentials and monodromy

In this paper, we are only interested in *elliptic KdV potentials,* i.e., those solutions of the stationary KdV hierarchy that are elliptic functions. As explained in Section 1, they must be of the form (1.8) up to adding a constant.

Fix $N \in \mathbb{N}$. Let $\mathbf{p} \in \mathcal{A}_N$ and $q(z) = q_{\mathbf{p}}(z)$ be the corresponding elliptic KdV potential given by (1.8). Assume that $q_{\mathbf{p}}(z)$ is of genus g, then the corresponding $\Phi_{q_{\mathbf{p}}}(z; E)$ in (2.3) is an *elliptic* function because $f_j(z)$'s are differential polynomials of $q_{\mathbf{p}}(z)$. From here and (2.7)–(2.9), we conclude that $\phi(P; z) = \frac{\psi'(P;z)}{\psi(P;z)}$ is also elliptic and so the Baker–Akhiezer function $\psi(P; z)$ is *elliptic of the second kind*, i.e.,

$$\psi(P; z + \omega_j) = \lambda_j(P)\psi(P; z), \quad j = 1, 2, \tag{2.16}$$

for some constants $\lambda_j(P) \in \mathbb{C} \setminus \{0\}$. Then, as explained in Section 1, up to multiplying a nonzero constant, $\psi(P; z)$ can be expressed as

$$\psi(P; z) = e^{c(P)z} \frac{\prod_{k=1}^{\mathfrak{m}} \sigma(z - a_k(P))}{\prod_{k=1}^{n} \sigma(z - p_k)^{m_k}}, \tag{2.17}$$

for some $a_k(P) \in E_\tau$ and some constant $c(P) \in \mathbb{C}$. For later usage, we define the couple $(r(P), s(P)) \in \mathbb{C}^2$ by

$$\begin{cases} -2\pi i s(P) := c(P) - \eta_1\Big(\sum_{k=1}^{\mathfrak{m}} a_k(P) - \sum_{k=1}^{n} m_k p_k\Big) \\ 2\pi i r(P) := c(P)\tau - \eta_2\Big(\sum_{k=1}^{\mathfrak{m}} a_k(P) - \sum_{k=1}^{n} m_k p_k\Big), \end{cases} \tag{2.18}$$

then we see from (2.17) and the transformation law (1.2) that

$$\begin{aligned} \psi(P; z + \omega_j) &= e^{c(P)\omega_j - \eta_j(\sum_{k=1}^{\mathfrak{m}} a_k(P) - \sum_{k=1}^{n} m_k p_k)}\psi(P; z) \\ &= \begin{cases} e^{-2\pi i s(P)}\psi(P; z) & \text{for } j = 1, \\ e^{2\pi i r(P)}\psi(P; z) & \text{for } j = 2. \end{cases} \end{aligned} \tag{2.19}$$

Note that by the Legendre relation $\tau\eta_1 - \eta_2 = 2\pi i$, (2.18) is equivalent to

$$\begin{cases} r(P) + s(P)\tau = \sum_{k=1}^{\mathfrak{m}} a_k(P) - \sum_{k=1}^{n} m_k p_k, \\ r(P)\eta_1 + s(P)\eta_2 = c(P). \end{cases} \tag{2.20}$$

Remark that since $\psi(P; z)$ is a solution of the ODE $\mathrm{L}(q_{\mathbf{p}}; E)$,

$$y''(z) = (q_{\mathbf{p}}(z) - E)y(z) = \Big[\sum_{j=1}^{n} m_j(m_j + 1)\wp(z - p_j) - E\Big]y(z), \tag{2.21}$$

both the constant $c(P)$ (and so $(r(P), s(P))$) and the spectral parameter E can be expressed in terms of the zeros $a_j(P)$'s; see Lemma 4.1 for example.

On the other hand, since (2.11) says that $\psi(P; z)\psi(P^*; z)$ is an elliptic function, we conclude from (2.19) that

$$(r(P^*), s(P^*)) \equiv -(r(P), s(P)) \mod \mathbb{Z}^2. \tag{2.22}$$

Let us turn to the monodromy of (2.21). As explained in Section 1, since all solutions of (2.21) are meromorphic in $\mathbb{C}$, the monodromy representation of (2.21) reduces to a group homomorphism $\rho: \pi_1(E_\tau) \to SL(2, \mathbb{C})$. Consequently, the monodromy group is abelian and has two generators $M_1(E), M_2(E) \in SL(2, \mathbb{C})$, where $M_j(E)$ denotes the monodromy matrix under the basic loop $z \to z + \omega_j$ of $\pi_1(E_\tau)$. Although $M_j(E)$'s depend on the choice of solutions, they are unique up to a common conjugation. In particular,

$$\Delta_j(E) := \operatorname{tr} M_j(E), \quad j = 1, 2,$$

are independent of the choice of solutions. This $\Delta_j(E)$ is called *Hill's discriminant* in the Floquet theory and is known as a non-constant entire function of E (cf. [13]). Define

$$\mathcal{E}_{\mathbf{p}} := \{E \in \mathbb{C} \mid \Delta_1(E)^2 = \Delta_2(E)^2 = 4\},$$

Then it was proved in [15, Proposition 5.6] that $\mathcal{E}_{\mathbf{p}}$ *is a finite set*. This fact plays a crucial role in Gesztesy–Weikard's proof of Theorem A.

According to the monodromy of $\mathrm{L}(q_{\mathbf{p}}; E)$, there are two cases in general.

Definition 2.5. (1) $\mathrm{L}(q_{\mathbf{p}}; E)$ is *completely reducible* if the monodromy matrices $M_1(E)$ and $M_2(E)$ can be diagonalized simultaneously.

(2) $\mathrm{L}(q_{\mathbf{p}}; E)$ is *not completely reducible* if $M_1(E)$ and $M_2(E)$ cannot be diagonalized simultaneously.

Recalling (2.13), we have the following two cases.

Case (a): $P = (E, \mathcal{C}) \in \Gamma_{q_{\mathbf{p}}}$ such that $\mathcal{C}^2 = Q_{q_{\mathbf{p}}}(E) \neq 0$. Then the two Baker–Akhiezer functions $\psi(P; z), \psi(P^*; z)$ are linearly independent solutions of (2.21), and with respect to $(\psi(P; z), \psi(P^*; z))$, the monodromy matrices are given by (recall (2.19) and (2.22))

$$M_1(E) = \begin{pmatrix} e^{-2\pi i s(P)} & 0 \\ 0 & e^{2\pi i s(P)} \end{pmatrix}, \quad M_2(E) = \begin{pmatrix} e^{2\pi i r(P)} & 0 \\ 0 & e^{-2\pi i r(P)} \end{pmatrix}, \tag{2.23}$$

namely $\mathrm{L}(q_{\mathbf{p}}; E)$ is completely reducible. Due to (2.23), this $(r(P), s(P))$ is called the *monodromy data for* $\mathrm{L}(q_{\mathbf{p}}; E)$. Furthermore, we have the following subcases.

Case (a)-1: $E \notin \mathcal{E}_{\mathbf{p}}$. We have either $e^{2\pi i s(P)} \neq \pm 1$ or $e^{2\pi i r(P)} \neq \pm 1$. This is the generic case since $\mathcal{E}_{\mathbf{p}}$ is finite. Then if $y(z)$ is a nontrivial solution of $\mathrm{L}(q_{\mathbf{p}}; E)$ that is elliptic of the second kind, then $y(z)$ must be one of the two Baker–Akhiezer functions (up to multiplying a constant).

Case (a)-2: $E \in \mathcal{E}_{\mathbf{p}}$. We have $e^{2\pi i s(P)} = \pm 1$ and $e^{2\pi i r(P)} = \pm 1$, so

$$M_1(E) = \pm I_2, \quad M_2(E) = \pm I_2.$$

Then any solution of $\mathrm{L}(q_{\mathbf{p}}; E)$ is elliptic of the second kind.

Case (b): $P = (E, \mathcal{C}) \in \Gamma_{q_{\mathbf{p}}}$ such that $\mathcal{C}^2 = Q_{q_{\mathbf{p}}}(E) = 0$. Then $P = P^* = (E, 0)$, i.e., $\psi(P; z) = \psi(P^*; z)$, and it follows from (2.22) that $(r(P), s(P)) \in \frac{1}{2}\mathbb{Z}^2$, which implies

$$\lambda_j(P) = \lambda_j^{-1}(P) = \pm 1, \quad j = 1, 2,$$

or equivalently,

$$\Delta_j(E) = \pm 2 \quad \text{for } j = 1, 2,$$

i.e., $E \in \mathcal{E}_{\mathbf{p}}$.

Case (b)-1: $\mathrm{L}(q_{\mathbf{p}}; E)$ is completely reducible. Then

$$M_1(E) = \pm I_2, \quad M_2(E) = \pm I_2,$$

and any solution of $\mathrm{L}(q_{\mathbf{p}}; E)$ is elliptic of the second kind.

Case (b)-2: $\mathrm{L}(q_{\mathbf{p}}; E)$ is not completely reducible. Then it is not difficult to prove (see e.g., [6, Theorem 2.4] for $q_{\mathbf{p}}(z)$ being the Darboux–Treibich–Verdier potentials) that, up to a common conjugation, $M_1(E)$ and $M_2(E)$ can be normalized as

$$M_1(E) = \pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M_2(E) = \pm\begin{pmatrix} 1 & 0 \\ \mathcal{D} & 1 \end{pmatrix}, \quad \mathcal{D} \in \mathbb{C} \cup \{\infty\}. \tag{2.24}$$

In this case, the constant $\mathcal{D}$ is called the *monodromic data* for $\mathrm{L}(q_{\mathbf{p}}; E)$. Remark that if $\mathcal{D} = \infty$, then (2.24) should be understood as

$$M_1(E) = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_2(E) = \pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

3. The closure of the elliptic AMM locus $\mathcal{L}_N$

This section is denoted to the proof of Theorem 1.1. Fix $N \in \mathbb{N}$. Let $\mathbf{p} \in \mathcal{A}_N$ be given by (1.7) and the corresponding elliptic KdV potential $q(z) = q_{\mathbf{p}}(z)$ be given by (1.8). Suppose its genus is $g \geq 1$. Then the corresponding $\Phi_{q_{\mathbf{p}},g}(z; E)$ in (2.3) is an elliptic function. Here we recall the following result from [12] for later usage.

Theorem 3.1 ([12, Theorem 2.15]). $\Phi_{q_{\mathbf{p}},g}(z; E)$ *admits the following expression:*

$$\Phi_{q_{\mathbf{p}},g}(z; E) = A_0(E) + \sum_{i=1}^{n}\sum_{j=1}^{m_i} A_{i,j}(E)\wp(z - p_i)^j$$

where $A_0(E)$, $A_{i,j}(E) \in \mathbb{C}[E]$ *are all polynomials of* E *with* $\deg_E A_0(E) = g$ *and* $\deg_E A_{i,j}(E) < g$ *for any* i, j.

Since the Baker–Akhiezer function $\psi(P; z)$ is expressed as (2.16), we have

$$\phi(P; z) = \frac{\psi'(P; z)}{\psi(P; z)} = c(P) + \sum_{j=1}^{\mathfrak{m}} \zeta(z - a_j(P)) - \sum_{i=1}^{n} m_i \zeta(z - p_i), \tag{3.1}$$

$$\phi'(P; z) = \sum_{i=1}^{n} m_i \wp(z - p_i) - \sum_{j=1}^{\mathfrak{m}} \wp(z - a_j(P)).$$

Consequently, the new KdV potential given by the Darboux transformation

$$
\begin{aligned}
q_P(z) &:= q_{\mathbf{p}}(z) - 2(\ln \psi(P;z))'' = q_{\mathbf{p}}(z) - 2\phi'(P;z) \\
&= \sum_{i=1}^{n} (m_i - 1)m_i \wp(z - p_i) + 2\sum_{j=1}^{\mathfrak{m}} \wp(z - a_j(P))
\end{aligned}
\tag{3.2}
$$

is also elliptic, and its genus is either g or $g-1$ by Theorem 2.4. Furthermore,

$$
\frac{1}{2}\sum_{i=1}^{n} m_i(m_i - 1) + \mathfrak{m} = \frac{1}{2}\sum_{i=1}^{n} m_i(m_i + 1) = N,
$$

i.e., the corresponding

$$
\mathbf{p}_P := (\overbrace{p_1, \ldots, p_1}^{m_1(m_1-1)/2}, \ldots, \overbrace{p_n, \ldots, p_n}^{m_n(m_n-1)/2}, a_1(P), \ldots, a_{\mathfrak{m}}(P))
\tag{3.3}
$$

of the new elliptic KdV potential $q_P(z)$ *belongs to the same locus* $\mathcal{A}_N$ as $\mathbf{p}$ of the original potential $q(z) = q_{\mathbf{p}}$. This is an interesting property for elliptic KdV potentials. Remark that $\mathcal{A}_N$ might be not connected in general. Our key observation is the following.

Lemma 3.2. *Under the above notations,* $\mathbf{p}_P \to \mathbf{p}$ *in* $\mathrm{Sym}^N E_\tau$ *as* $P \to P_\infty$ *in the spectral curve* $\overline{\Gamma_{q_{\mathbf{p}}}}$. *In particular,* $\mathbf{p}_P$ *belongs to the same connected component of* $\mathcal{A}_N$ *as* $\mathbf{p}$ *for any* $P \in \Gamma_{q_{\mathbf{p}}} = \overline{\Gamma_{q_{\mathbf{p}}}} \setminus \{P_\infty\}$.

Proof. It suffices to prove that up to a subsequence, $\mathbf{p}_P \to \mathbf{p}$ as $P = (E, \mathcal{C}) \to P_\infty$, i.e., as $E \to \infty$.

Recall $\mathbf{a}(P) = (a_1(P), \ldots, a_{\mathfrak{m}}(P)) \in \mathrm{Sym}^{\mathfrak{m}} E_\tau$, where $\mathfrak{m} = \sum_{i=1}^{n} m_i$. Since $\mathrm{Sym}^{\mathfrak{m}} E_\tau$ is compact, up to a subsequence, we may assume

$$
\mathbf{a}(P) \to \mathbf{a}(\infty) = (a_1(\infty), \ldots, a_{\mathfrak{m}}(\infty)) \in \mathrm{Sym}^{\mathfrak{m}} E_\tau \quad \text{as } E \to \infty.
\tag{3.4}
$$

Recalling (3.1) and (2.8), we have

$$
\begin{aligned}
q_{\mathbf{p}}(z) - E &= \phi'(P;z) + \phi(P;z)^2 \\
&= c(P)^2 + 2c(P)D(z;\mathbf{a}(P)) + D(z;\mathbf{a}(P))^2 + F(z;\mathbf{a}(P)),
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
D(z;\mathbf{a}(P)) &:= \sum_{j=1}^{\mathfrak{m}} \zeta(z - a_j(P)) - \sum_{i=1}^{n} m_i \zeta(z - p_i), \\
F(z;\mathbf{a}(P)) &:= \sum_{i=1}^{n} m_i \wp(z - p_i) - \sum_{j=1}^{\mathfrak{m}} \wp(z - a_j(P)) = D'(z;\mathbf{a}(P)).
\end{aligned}
\tag{3.6}
$$

By (3.4), we see that both $D(z;\mathbf{a}(P))$ and $F(z;\mathbf{a}(P))$ converge uniformly for z outside the singularities $\{a_j(\infty)\}_{j=1}^{\mathfrak{m}} \cup \{p_i\}_{i=1}^{n}$. From here and (3.5), we see that

$$c(P) \to \infty \iff E \to \infty. \tag{3.7}$$

Differentiating both sides of (3.5), we have

$$\sum_{i=1}^{n} m_i(m_i+1)\wp'(z-p_i) \\ = 2c(P)F(z;\mathbf{a}(P)) + 2D(z;\mathbf{a}(P))F(z;\mathbf{a}(P)) + F'(z;\mathbf{a}(P)). \tag{3.8}$$

Then the right-hand side of (3.8) converges uniformly outside the singularities $\{p_i\}_{i=1}^{n}$ as $E \to \infty$. Since $c(P) \to \infty$, we obtain

$$F(z;\mathbf{a}(P)) \to 0 \quad \text{as } E \to \infty,$$

i.e.

$$\mathbf{a}(\infty) = (\overbrace{p_1,\ldots,p_1}^{m_1}, \overbrace{p_2,\ldots,p_2}^{m_2}, \ldots, \overbrace{p_n,\ldots,p_n}^{m_n}). \tag{3.9}$$

This, together with (3.3), proves $\mathbf{p}_P \to \mathbf{p}$ as $E \to \infty$. ■

Lemma 3.2 has the following consequence, which indicates that two elliptic KdV potentials, with their corresponding $\mathbf{p}$'s belonging to the same connected component of $\mathcal{A}_N$, might have *different genera*.

Corollary 3.3. *Let $\mathbf{p} \in \mathcal{A}_N$ be given by (1.7) and the corresponding elliptic KdV potential $q_{\mathbf{p}}$ be given by (1.8) such that its genus $g \geq 2$. Suppose E_0 is a multiple root of the spectral polynomial $Q_{q_{\mathbf{p}},2g+1}(E)$. Then the elliptic KdV potential $q_{P_0}(z)$ (defined in (3.2) with $P_0 = (E_0, 0)$) is of genus $g-1$ and its spectral polynomial $Q_{q_{P_0},2g-1}(E) = (E-E_0)^{-2}Q_{q_{\mathbf{p}},2g+1}(E)$.*

Proof. This assertion follows directly from Theorem 2.4. ■

For any $\mathbf{p} \in \mathcal{A}_N$ given by (1.7), we define

$$\sharp(\mathbf{p}) := \max_{0\leq i\leq n} m_i \geq 1.$$

Then $\sharp(\mathbf{p})(\sharp(\mathbf{p})+1) \leq 2N$ and

$$\mathbf{p} \in \mathcal{L}_N \iff \sharp(\mathbf{p}) = 1.$$

The following lemma is also crucial in the proof of Theorem 1.1.

Lemma 3.4. *Except finitely many P's in $\Gamma_{q_{\mathbf{p}}}$, the zeros $a_j(P)$'s of $\psi(P;z)$ satisfy*

$$\{a_1(P), \ldots, a_{\mathfrak{m}}(P)\} \cap \{p_1, \ldots, p_n\} = \emptyset \quad \text{in } E_\tau, \tag{3.10}$$

and

$$a_i(P) \neq a_j(P) \text{ in } E_\tau \quad \text{for all } i \neq j. \tag{3.11}$$

In particular, $\sharp(\mathbf{p}_P) = \sharp(\mathbf{p}) - 1$ *if* $\sharp(\mathbf{p}) \geq 2$.

Proof. Recalling Theorem 3.1, we define

$$\sum := \{P = (E, \mathcal{C}) \in \Gamma_{q_{\mathbf{p}}} \mid A_{i,m_i}(E) = 0 \text{ for some } 1 \leq i \leq n\},$$

which is clearly a finite set. Fix any $P = (E, \mathcal{C}) \in \Gamma_{q_{\mathbf{p}}} \setminus \sum$. Then for any $i \in [1, n]$, p_i is a pole of $\Phi_{q_{\mathbf{p}},g}(z; E)$ with order $2m_i$ (since the local exponent of $\Phi_{q_{\mathbf{p}},g}(z; E)$ at p_i must be one of $\{-2m_i, 1, 2m_i + 2\}$). Since the local exponents of $\psi(P; z)$ and $\psi(P^*; z)$ at p_i are either $-m_i$ or $m_i + 1$, and (see (2.11))

$$\psi(P; z)\psi(P^*; z) = \frac{\Phi_{q_{\mathbf{p}},g}(z; E)}{\Phi_{q_{\mathbf{p}},g}(z_0; E)},$$

we conclude that p_i is a pole of $\psi(P; z)$ with order m_i for any i. Together with the expression (2.16) of $\psi(P; z)$, we have $a_j(P) \neq p_i$ in E_τ for any i, j, namely (3.10) holds. Consequently, $a_j(P)$ is a simple zero of $\psi(P; z)$ for any j, so (3.11) holds. In particular, if $\sharp(\mathbf{p}) = \max_{1 \leq i \leq n} m_i \geq 2$, then we have $\sharp(\mathbf{p}_P) = \max\{1, m_i - 1\}_{i=1}^n = \sharp(\mathbf{p}) - 1$. ■

The next result shows that $\mathcal{A}_N$ is closed in $\mathrm{Sym}^N E_\tau$, i.e., the limit of elliptic KdV potentials is also an elliptic KdV potential. This natural result might be well known to experts in this field, but we cannot find an explicit reference, so we would like to give a proof for completeness.

Lemma 3.5. $\overline{\mathcal{L}_N} \subset \overline{\mathcal{A}_N} = \mathcal{A}_N$.

Proof. Let $\{\mathbf{p}_k\}$ be a sequence of points in $\mathcal{A}_N$ such that $\mathbf{p}_k \to \mathbf{p}_\infty$. Then

$$\begin{aligned} q_k(z) &:= \sum_{i=1}^{n_k} m_{k,i}(m_{k,i} + 1)\wp(z - p_{k,i}) \\ \to q_\infty(z) &:= \sum_{i=1}^{n_\infty} m_{\infty,i}(m_{\infty,i} + 1)\wp(z - p_{\infty,i}), \end{aligned}$$

where $p_{k,i} \neq p_{k,j}$ for any $i \neq j$ and $k \in \mathbb{N} \cup \{\infty\}$, and

$$\sum_{i=1}^{n_k} m_{k,i}(m_{k,i} + 1) = \sum_{i=1}^{n_\infty} m_{\infty,i}(m_{\infty,i} + 1) = 2N.$$

Our goal is to prove that $q_\infty(z)$ is an elliptic KdV potential, i.e., $\mathbf{p}_\infty \in \mathcal{A}_N$.

Take $z_0 \in E_\tau$ such that none of $z_0 + \mathbb{R}$, $z_0 + \mathbb{R}\tau$ cross any singularities of $\{q_k(z)\}_{k=1}^{\infty}$. Consider the following Hill's equation:

$$y''(x) = [q_k(z_0 + x) - E]y(x), \quad x \in \mathbb{R},\ k \in \mathbb{N} \cup \{\infty\}, \tag{3.12}$$

where $q_k(z_0 + \cdot)$ is a complex-valued smooth non-constant periodic function of period $\omega_1 = 1$ on $\mathbb{R}$. Let $y_{k,1}(x)$ and $y_{k,2}(x)$ be any two linearly independent solutions of (3.12). Then so are $y_{k,1}(x + \omega_1)$ and $y_{k,2}(x + \omega_1)$ and hence there exists a monodromy matrix $M_k(E) \in \mathrm{SL}(2, \mathbb{C})$ such that

$$\begin{pmatrix} y_{k,1}(x + \omega_1) \\ y_{k,2}(x + \omega_1) \end{pmatrix} = M_k(E) \begin{pmatrix} y_{k,1}(x) \\ y_{k,2}(x) \end{pmatrix}.$$

Define the *Hill's discriminant* $\Delta_k(E)$ by

$$\Delta_k(E) := \operatorname{tr} M_k(E),$$

which is clearly an invariant of (3.12), i.e., does not depend on the choice of linearly independent solutions. This $\Delta_k(E)$ is a non-constant entire function and plays a fundamental role since it encodes all the spectrum information of the associated operator; see e.g., [13, 21] and references therein.

Let $c_k(x; E)$ and $s_k(x; E)$ be the special fundamental system of solutions of (3.12) satisfying the initial values

$$c_k(0; E) = s_k'(0; E) = 1, \quad c_k'(0; E) = s_k(0; E) = 0.$$

Then we have

$$\Delta_k(E) = c_k(\omega_1; E) + s_k'(\omega_1; E).$$

From here we easily see that

$$\lim_{k \to \infty} \Delta_k(E) = \Delta_\infty(E).$$

Now, we fix any $E \in \mathbb{C}$ such that $\Delta_\infty(E) \neq \pm 2$. Then $\Delta_k(E) \neq \pm 2$ for large k. Consider

$$y''(z) = [q_k(z) - E]y(z), \quad z \in \mathbb{C}. \tag{3.13}$$

For large $k \neq \infty$ such that $\Delta_k(E) \neq \pm 2$, since $q_k(z)$ is an elliptic KdV potential, it follows from Section 2.3 that the two Baker–Akhiezer functions (denoted by $\tilde{y}_{k,1}(z), \tilde{y}_{k,2}(z)$) of (3.13) are *linearly independent* and satisfy

$$\begin{pmatrix} \tilde{y}_{k,1}(z + \omega_j) \\ \tilde{y}_{k,2}(z + \omega_j) \end{pmatrix} = \begin{pmatrix} \lambda_{k,j} & 0 \\ 0 & \lambda_{k,j}^{-1} \end{pmatrix} \begin{pmatrix} \tilde{y}_{k,1}(z) \\ \tilde{y}_{k,2}(z) \end{pmatrix}, \quad j = 1, 2, \tag{3.14}$$

with

$$\lambda_{1,j} + \lambda_{1,j}^{-1} = \Delta_k(E) \to \Delta_\infty(E) \neq \pm 2.$$

So

$$\lambda_{\infty,1} := \lim_{k\to\infty} \lambda_{k,1} \neq \pm 1. \tag{3.15}$$

Furthermore, $\tilde{y}_{k,j}(z)$ can be expressed as

$$\tilde{y}_{k,j}(z) = e^{c_{k,j}z} \frac{\prod_{j=1}^{\sum_i m_{k,i}} \sigma(z-a_{k,j})}{\prod_{i=1}^{n_k} \sigma(z-p_{k,i})^{n_{k,i}}}, \quad j=1,2.$$

By using the same argument as (3.4)–(3.7), we have $c_{k,j} \not\to \infty$ as $k \to \infty$. So, up to a subsequence, $\tilde{y}_{k,j}(z)$ converges to some function

$$\tilde{y}_{\infty,j}(z) = e^{c_{\infty,j}z} \frac{\prod_{j=1}^{\sum_i m_{\infty,i}} \sigma(z-a_{\infty,j})}{\prod_{i=1}^{n_\infty} \sigma(z-p_{\infty,i})^{n_{\infty,i}}}, \quad j=1,2,$$

uniformly outside the singularities $\{p_{\infty,i}\}_i$, and $\tilde{y}_{\infty,j}(z)$ solves

$$y''(z) = [q_\infty(z) - E]y(z), \quad z \in \mathbb{C}. \tag{3.16}$$

By (3.14)–(3.15), we have

$$\begin{pmatrix} \tilde{y}_{\infty,1}(z+\omega_1) \\ \tilde{y}_{\infty,2}(z+\omega_1) \end{pmatrix} = \begin{pmatrix} \lambda_{\infty,1} & 0 \\ 0 & \lambda_{\infty,1}^{-1} \end{pmatrix} \begin{pmatrix} \tilde{y}_{\infty,1}(z) \\ \tilde{y}_{\infty,2}(z) \end{pmatrix},$$

with $\lambda_{\infty,1} \neq \lambda_{\infty,1}^{-1}$. Therefore, $\tilde{y}_{\infty,1}(z)$, $\tilde{y}_{\infty,2}(z)$ are linearly independent solutions of (3.16). This proves that all solutions of (3.16) are meromorphic as long as $\Delta_\infty(E) \neq \pm 2$, i.e., for all $E \in \mathbb{C}$ except for countably many E's because $\Delta_\infty(E)$ is a non-constant entire function. However, by the standard Frobenius' method, it is easy to see (cf. [32, Lemma 6]) that the set

$$\{E \in \mathbb{C} \mid \text{all solutions of (3.16) are meromorphic}\}$$

is either a finite set or equal to $\mathbb{C}$. Therefore, we conclude that all solutions of (3.16) are meromorphic for all $E \in \mathbb{C}$, i.e., $q_\infty(z)$ is a Picard potential and hence an elliptic KdV potential by Theorem A. This proves that $\mathbf{p}_\infty \in \mathcal{A}_N$, so $\overline{\mathcal{A}_N} = \mathcal{A}_N$ and $\overline{\mathcal{L}_N} \subset \mathcal{A}_N$. ∎

Now, we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.5, it suffices to prove

$$\mathcal{A}_N \subset \overline{\mathcal{L}_N}. \tag{3.17}$$

For any $1 \le k \le k_0$, where k_0 is the largest integer satisfying $k_0(k_0+1) \le 2N$, we define

$$\mathcal{A}_{N,k} := \{\mathbf{p} \in \mathcal{A}_N \mid \sharp(\mathbf{p}) = k\}.$$

Then

$$\mathcal{A}_N = \bigcup_{k=1}^{k_0} \mathcal{A}_{N,k}, \quad \mathcal{A}_{N,1} = \mathcal{L}_N.$$

If $k_0 = 1$ we are done. So we consider the general case $k_0 \ge 2$. Since Lemmas 3.2 and 3.4 together imply

$$\mathcal{A}_{N,k} \subset \overline{\mathcal{A}_{N,k-1}} \quad \text{for any } 2 \le k \le k_0,$$

we immediately obtain

$$\overline{\mathcal{A}_{N,k_0}} \subset \overline{\mathcal{A}_{N,k_0-1}} \subset \cdots \subset \overline{\mathcal{A}_{N,1}} = \overline{\mathcal{L}_N}$$

and so (3.17) holds. This completes the proof. ■

4. The degree of the Krichever covering map

4.1. Definition of the Krichever covering map

Let $\mathbf{p} \in \mathcal{A}_N$ be given by (1.7) and the corresponding elliptic KdV potential $q(z) = q_{\mathbf{p}}(z)$ be in (1.8). Suppose it is of genus g and recall its associated spectral curve $\Gamma_{q_{\mathbf{p}}}$. As explained in Section 1, $\Gamma_{q_{\mathbf{p}}}$ can be mapped into $\mathrm{Sym}^{\mathfrak{m}} E_\tau$ through $i_{q_{\mathbf{p}}}\colon \Gamma_{q_{\mathbf{p}}} \to \mathrm{Sym}^{\mathfrak{m}} E_\tau$, which is defined by

$$i_{q_{\mathbf{p}}}(P) := (a_1(P), \ldots, a_{\mathfrak{m}}(P)), \tag{4.1}$$

where $(a_1(P), \ldots, a_{\mathfrak{m}}(P)) \in \mathrm{Sym}^{\mathfrak{m}} E_\tau$ is the zero set of the Baker–Akhiezer function $\psi(P; z)$ in (2.16). Remark that since the local exponent of $\psi(P; z)$ at each p_i is either $-m_i$ or $m_i + 1$, and $\psi(P; z)$ must have poles at some p_i, so the local exponents of $\psi(P; z)$ at such poles p_i's are $-m_i$, which implies $p_i \notin \{a_1(P), \ldots, a_{\mathfrak{m}}(P)\}$ as long as p_i is a pole of $\psi(P; z)$. Therefore, for any $P \in \Gamma_{q_{\mathbf{p}}}$ there are i's (depending on P) such that

$$p_i \notin \{a_1(P), \ldots, a_{\mathfrak{m}}(P)\}. \tag{4.2}$$

Lemma 4.1. *For any $i \in [1, n]$ satisfying (4.2), the zeros $a_j(P)$'s of $\psi(P; z)$ satisfy*

$$E = (1 - 2m_i) \sum_{k=1}^{\mathfrak{m}} \wp(p_i - a_k(P)) + \sum_{j=1, j \ne i}^{n} m_j(m_j + 2m_i)\wp(p_i - p_j), \tag{4.3}$$

$$c(P) = \sum_{j=1, j \ne i}^{n} m_j \zeta(p_i - p_j) - \sum_{k=1}^{\mathfrak{m}} \zeta(p_i - a_k(P)). \tag{4.4}$$

Proof. Recall (3.5)–(3.6) that

$$\sum_{j=1}^{n} m_j(m_j+1)\wp(z-p_j) - E = \Big[c(P) + \sum_{j=1}^{\mathfrak{m}} \zeta(z-a_j(P)) - \sum_{i=1}^{n} m_i \zeta(z-p_i)\Big]^2 + \sum_{i=1}^{n} m_i \wp(z-p_i) - \sum_{j=1}^{\mathfrak{m}} \wp(z-a_j(P)).$$

Fixing any $i \in [1,n]$ such that (4.2) holds, we compute the local expansion of the above formula at $z = p_i$, then it is easy to see that the coefficient of the term $(z-p_i)^{-1}$ gives (4.4) and the coefficient of the term $(z-p_i)^0$ gives (4.3). We omit the details. ∎

Proposition 4.2. *The map $i_{q_{\mathbf{p}}}: \Gamma_{q_{\mathbf{p}}} \to \mathrm{Sym}^{\mathfrak{m}} E_\tau$ defined by (4.1) is one-to-one and hence $i_{q_{\mathbf{p}}}$ is an embedding.*

Proof. Suppose $i_{q_{\mathbf{p}}}(P_1) = i_{q_{\mathbf{p}}}(P_2)$ for some $P_1 = (E_1, \mathcal{C}_1)$ and $P_2 = (E_2, \mathcal{C}_2)$ in $\Gamma_{q_{\mathbf{p}}}$. Then (4.3) implies that $E_1 = E_2$ and so either $\mathcal{C}_1 = \mathcal{C}_2$ or $\mathcal{C}_1 = -\mathcal{C}_2 \neq 0$. If $\mathcal{C}_1 = -\mathcal{C}_2 \neq 0$, we have $P_2 = P_1^* \neq P_1$, then the Baker–Akhiezer functions $\psi(P_1; z)$, $\psi(P_1^*; z)$ are linearly independent and so have no common zeros, clearly a contradiction with $i_{q_{\mathbf{p}}}(P_1) = i_{q_{\mathbf{p}}}(P_2) = i_{q_{\mathbf{p}}}(P_1^*)$. Thus, $\mathcal{C}_1 = \mathcal{C}_2$, or equivalently, $P_1 = P_2$. ∎

As explained in Section 1, the above embedding $i_{q_{\mathbf{p}}}$ induces the Krichever covering map $\sigma_{q_{\mathbf{p}}}: \overline{\Gamma_{q_{\mathbf{p}}}} = \Gamma_{q_{\mathbf{p}}} \cup \{P_\infty\} \to E_\tau$, which is defined by

$$\sigma_{q_{\mathbf{p}}}(P) := \sum_{i=1}^{\mathfrak{m}} a_i(P) - \sum_{j=1}^{n} m_j p_j \quad \text{for all } P \in \Gamma_{q_{\mathbf{p}}}, \tag{4.5a}$$

$$\sigma_{q_{\mathbf{p}}}(P_\infty) := \lim_{P \to P_\infty} \sigma_{q_{\mathbf{p}}}(P) = \lim_{P \to P_\infty} \sum_{i=1}^{\mathfrak{m}} a_i(P) - \sum_{j=1}^{n} m_j p_j = 0 \text{ in } E_\tau, \tag{4.5b}$$

where the last equality follows from (3.4) and (3.9) in Lemma 3.2. Recalling (2.20) and (2.22), we have

$$\begin{aligned} \sigma_{q_{\mathbf{p}}}(P^*) &= r(P^*) + s(P^*)\tau = -(r(P) + s(P)\tau) \quad \text{in } E_\tau \\ &= -\sigma_{q_{\mathbf{p}}}(P) \quad \text{for all } P \in \Gamma_{q_{\mathbf{p}}}. \end{aligned} \tag{4.6}$$

Theorem 4.3. *The map $\sigma_{q_{\mathbf{p}}}: \overline{\Gamma_{q_{\mathbf{p}}}} \to E_\tau$ defined by (4.5) is a finite morphism.*

Proof. Consider $\wp(\sigma_{q_{\mathbf{p}}}(P)) = \wp(\sigma_{q_{\mathbf{p}}}(E, \mathcal{C}))$ with $P = (E, \mathcal{C}) \in \Gamma_{q_{\mathbf{p}}}$, i.e., $\mathcal{C}^2 = Q_{q_{\mathbf{p}}}(E)$. Since (4.6) implies $\wp(\sigma_{q_{\mathbf{p}}}(E, \mathcal{C})) = \wp(\sigma_{q_{\mathbf{p}}}(E, -\mathcal{C}))$, $\wp(\sigma_{q_{\mathbf{p}}}(E, \mathcal{C}))$ only depends on E and is a meromorphic function of $E \in \mathbb{C}$. From here and

$$\lim_{E \to \infty} \wp(\sigma_{q_{\mathbf{p}}}(E, \mathcal{C})) = \lim_{P \to P_\infty} \wp(\sigma_{q_{\mathbf{p}}}(P)) = \infty,$$

we conclude that $\wp(\sigma_{q_{\mathbf{p}}}(E,\mathcal{C}))$ is a rational function of E, i.e.,

$$\wp(\sigma_{q_{\mathbf{p}}}(P)) = \wp(\sigma_{q_{\mathbf{p}}}(E,\mathcal{C})) = \frac{P_1(E)}{P_2(E)} \tag{4.7}$$

for some coprime polynomials $P_1(E), P_2(E) \in \mathbb{C}[E]$. This implies that $\sigma_{q_{\mathbf{p}}}\colon \overline{\Gamma_{q_{\mathbf{p}}}} \to E_\tau$ is a finite morphism.

A similar argument applies to $\mathcal{C}\wp'(\sigma_{q_{\mathbf{p}}}(P))$ and implies

$$\frac{\wp'(\sigma_{q_{\mathbf{p}}}(P))}{\mathcal{C}} = \frac{\mathcal{C}\wp'(\sigma_{q_{\mathbf{p}}}(E,\mathcal{C}))}{\mathcal{C}^2} = \frac{P_3(E)}{P_4(E)} \tag{4.8}$$

for some coprime polynomials $P_3(E), P_4(E) \in \mathbb{C}[E]$. ■

Theorem 4.3 implies that the degree $\deg \sigma_{q_{\mathbf{p}}}$ is well defined. A basic question is how to compute it. Our strategy contains two main steps. Step 1 is to compute the degree for $\mathbf{p} \in \mathcal{L}_N$ via Krichever's ansatz [17]. Step 2 is to apply the Darboux transformation to relate the degree for $\mathbf{p} \in \mathcal{A}_N$ with the degree for $\hat{\mathbf{p}} \in \mathcal{L}_N$. For this purpose, we need to review Krichever's theory in the following section.

4.2. Computing $\deg \sigma_{q_{\mathbf{p}}}$ for $\mathbf{p} \in \mathcal{L}_N$ via Krichever's theory

In this section, we briefly review Krichever's construction [17] of the algebraic curve Γ_N for an elliptic KdV potential $q_{\mathbf{p}}(z)$ with $\mathbf{p} = (p_1, \ldots, p_N) \in \mathcal{L}_N$. As an application, we can easily prove $\deg \sigma_{q_{\mathbf{p}}} = N$. Since Krichever's original theory [17] is for the more general KP equation, we prefer to provide all the necessary details of Krichever's theory in the KdV case to make the paper self-contained.

As mentioned in Section 1, for the ODE $\mathrm{L}(q_{\mathbf{p}}; E)$,

$$y''(z) = (q_{\mathbf{p}}(z) - E)y(z) = \Big[2\sum_{j=1}^{N} \wp(z-p_j) - E\Big]y(z), \tag{4.9}$$

Krichever [17] studied the following Krichever's ansatz:

$$\varphi(z;\kappa,\alpha) := e^{-\kappa z} \cdot \sum_{i=1}^{N} d_i \Psi(z-p_i;\alpha), \tag{4.10}$$

where $\alpha \in E_\tau \setminus \{0\}$, $\kappa \in \mathbb{C}$, $d_i \in \mathbb{C}$ for all i and

$$\Psi(z;\alpha) := -\frac{\sigma(z-\alpha)}{\sigma(z)\sigma(\alpha)} e^{\zeta(\alpha)z}. \tag{4.11}$$

Notice that $\Psi(z;\alpha)$ is doubly periodic in α, which implies that $\Psi(z;\alpha)$ is well defined for $\alpha \in E_\tau$. It follows from (1.2) that

$$e^{-\kappa(z+\omega_k)}\Psi(z+\omega_k;\alpha) = e^{(\zeta(\alpha)-\kappa)\omega_k-\eta_k\alpha}e^{-\kappa z}\Psi(z;\alpha), \quad k=1,2, \tag{4.12}$$

so Krichever's ansatz $\varphi(z;\kappa,\alpha)$ is elliptic of the second kind.

In [17, (15), p. 284], Krichever introduced the following two $N\times N$ matrices $L(\alpha)$ and $T(\alpha)$:

$$L(\alpha) = (L_{ij})_{N\times N} \quad \text{with } L_{ij} = (1-\delta_{ij})\Psi_{ij}(\alpha) := \begin{cases} 0 & \text{if } i=j, \\ \Psi_{ij}(\alpha) & \text{if } i\neq j, \end{cases}$$

and

$$T(\alpha) = (T_{ij})_{N\times N} \quad \text{with } T_{ij} = \delta_{ij}\sum_{k\neq i}\wp_{ik} + (1-\delta_{ij})\Psi'_{ij}(\alpha),$$

where we use the notations $\Psi_{ij}(\alpha) := \Psi(p_i-p_j;\alpha)$, $\Psi'_{ij}(\alpha) := \frac{d}{dz}\Psi(p_i-p_j;\alpha)$ and $\wp_{ij} := \wp(p_i-p_j)$ for $i\neq j$.

Let $\vec{d} = (d_1,\ldots,d_N)^t \neq 0$. By substituting $\varphi(z;\kappa,\alpha)$ defined by (4.10) into (4.9) and equating the coefficients of $(z-p_j)^{-2}$ and $(z-p_j)^{-1}$ to zero for all $j\in[1,N]$, it was proved in [17, p. 284] that κ and $\vec{d}$ satisfy the following two equations:

$$(L(\alpha)-\kappa I_N)\vec{d} = 0 \quad \text{and} \quad \Big(T(\alpha) - \frac{1}{2}(E-\wp(\alpha)-\kappa^2)I_N\Big)\vec{d} = 0, \tag{4.13}$$

where I_N is the $N\times N$ identity matrix. So the vector $\vec{d}$ must be a common eigenvector of $L(\alpha)$ and $T(\alpha)$, and the value κ must be an eigenvalue of the matrix $L(\alpha)$. Clearly, the two equations in (4.13) are compatible if and only if the two matrices commute, i.e.,

$$[L(\alpha), T(\alpha)] = L(\alpha)T(\alpha) - T(\alpha)L(\alpha) = 0. \tag{4.14}$$

It was proved in [17, Lemma 1] that one has the equality $[L(\alpha),T(\alpha)]=0$ if and only if $\sum_{j\neq i}^N \wp'(p_i-p_j) = 0$ for all i, or equivalently, $\mathbf{p}\in\mathcal{L}_N$.

Consider the characteristic polynomial of $L(\alpha)$:

$$R(\kappa,\alpha) = R_{\mathbf{p}}(\kappa,\alpha) := \det(\kappa I_N - L(\alpha)), \tag{4.15}$$

which will be called the *Krichever polynomial* in this paper. Although the matrix $L(\alpha)$ has essential singularities at $\alpha=0$, by introducing a diagonal matrix

$$G(\alpha) = (G_{ij})_{N\times N}, \quad \text{where } G_{ij} = \delta_{ij}e^{\zeta(\alpha)p_i},$$

we see that $L(\alpha) = G(\alpha)\widetilde{L}(\alpha)G^{-1}(\alpha)$, where the matrix $\widetilde{L}(\alpha)$ does not have any essential singularities. Consequently, $R(\kappa, \alpha)$ can be written as a polynomial of κ as follows:

$$R(\kappa, \alpha) = \sum_{i=0}^{N} r_i(\alpha)\kappa^i = \kappa^N + \cdots,$$

where $r_i(\alpha)$ are all elliptic functions with poles only at $\alpha = 0$. In particular, $R(\kappa, \alpha) \in \mathbb{C}[\wp(\alpha), \wp'(\alpha)][\kappa]$.

Let Γ_N be the algebraic curve defined by

$$\Gamma_N := \{(\kappa, \alpha) \mid R(\kappa, \alpha) = 0\},$$

then the algebraic curve Γ_N is an N-sheeted covering over E_τ via the Krichever covering map

$$\pi\colon \Gamma_N \to E_\tau, \quad \pi(\kappa, \alpha) := \alpha,$$

and the branch points coincide with the zeros of $\frac{\partial R}{\partial \kappa}(\kappa, \alpha)$ on Γ_N. As a meromorphic function on Γ_N, it was proved by Krichever [17, Lemma 2] that $\frac{\partial R}{\partial \kappa}(\kappa, \alpha)$ has only $2(N-1)$ poles on Γ_N. For any meromorphic function, the number of zeros is equal to the number of poles by counting multiplicities. Consequently, $\frac{\partial R}{\partial \kappa}(\kappa, \alpha)$ has only $2(N-1)$ zeros on Γ_N, which implies that except finitely many α's in E_τ, the κ-polynomial $R(\kappa, \alpha)$ has no multiple roots. See more detailed arguments of this fact in [17, Lemma 2]. Therefore, the set

$$\mathcal{B}_{\mathbf{p}} := \{\alpha \in E_\tau \mid \text{the } \kappa\text{-polynomial } R_{\mathbf{p}}(\kappa, \alpha) \text{ has multiple roots}\}$$

is a *finite* set.

Denote $\omega_0 = 0$, $\omega_3 = \omega_1 + \omega_2$, and define $E_\tau[2] := \left\{\frac{\omega_k}{2} \,\middle|\, 0 \le k \le 3\right\}$ to be the set of 2-torsion points in E_τ. The following result, which will be a consequence of Krichever's theory, is crucial for us to compute the degree $\deg \sigma_{q_{\mathbf{p}}}$ for $\mathbf{p} \in \mathcal{L}_N$.

Proposition 4.4. *Let* $\mathbf{p} \in \mathcal{L}_N$ *and* $\alpha \notin \mathcal{B}_{\mathbf{p}} \cup E_\tau[2]$. *Then for each* $(\kappa_i(\alpha), \alpha)$ *satisfying* $R_{\mathbf{p}}(\kappa, \alpha) = 0$, *there exists a unique* $P_i = (E_i, \mathcal{C}_i) \in \Gamma_{q_{\mathbf{p}}}$ *such that*

$$\sigma_{q_{\mathbf{p}}}(P_i) = \alpha, \quad \kappa_i(\alpha) = \zeta(\alpha) - r(P_i)\eta_1 - s(P_i)\eta_2. \tag{4.16}$$

Moreover,

$$E_i \neq E_j \quad \textit{for } i \neq j. \tag{4.17}$$

Proof. Under our assumption $\alpha \in E_\tau \setminus (\mathcal{B}_{\mathbf{p}} \cup E_\tau[2])$, the κ-polynomial $R_{\mathbf{p}}(\kappa, \alpha)$ has N distinct roots, namely the matrix $L(\alpha)$ has N distinct eigenvalues $\kappa_1(\alpha), \ldots, \kappa_N(\alpha)$ and $\kappa_i(\alpha) \neq \kappa_j(\alpha)$ for $i \neq j$. For each eigenvalue $\kappa_i(\alpha)$, up to multiplying a constant, there is a unique nonzero eigenvector

$$\vec{d}_i = (d_{i,1}, \ldots, d_{i,N})^t$$

satisfying $L(\alpha)\vec{d}_i = \kappa_i(\alpha)\vec{d}_i$. Since $\mathbf{p} \in \mathscr{L}_N$ implies (4.14), we have

$$L(\alpha)T(\alpha)\vec{d}_i = T(\alpha)L(\alpha)\vec{d}_i = \kappa_i(\alpha)T(\alpha)\vec{d}_i,$$

so $T(\alpha)\vec{d}_i$ is also an eigenvector of $L(\alpha)$ with respect to the eigenvalue $\kappa_i(\alpha)$. Because $\vec{d}_1, \dots, \vec{d}_N$ are linearly independent, we have

$$T(\alpha)\vec{d}_i = \mu_i(\alpha)\vec{d}_i$$

for some $\mu_i(\alpha) \in \mathbb{C}$ that is uniquely determined by $(\kappa_i(\alpha), \alpha)$, i.e., $\vec{d}_i$ is also an eigenvector of $T(\alpha)$ with respect to the eigenvalue $\mu_i(\alpha)$. Define E_i by the following equation:

$$\frac{1}{2}(E_i - \wp(\alpha) - \kappa_i^2(\alpha)) = \mu_i(\alpha), \tag{4.18}$$

then $\vec{d}_i$ solves the equation (4.13) with $\kappa = \kappa_i(\alpha)$ and $E = E_i$, i.e.,

$$(L(\alpha) - \kappa_i(\alpha)I_N)\vec{d}_i = 0, \tag{4.19a}$$

$$\Big(T(\alpha) - \frac{1}{2}(E_i - \wp(\alpha) - \kappa_i^2(\alpha))I_N\Big)\vec{d}_i = 0. \tag{4.19b}$$

Consequently, if we let

$$\varphi_i(z) = \varphi(z; \kappa_i, \alpha, E_i) := e^{-\kappa_i(\alpha)z} \cdot \sum_{k=1}^{N} d_{i,k}\Psi(z - p_k; \alpha),$$

then (4.19) implies that $\varphi_i''(z) - (q_{\mathbf{p}}(z) - E_i)\varphi_i(z)$ has *no poles* and satisfies the same transformation law (4.12) as $e^{-\kappa_i(\alpha)z}\Psi(z;\alpha)$. Consequently,

$$\frac{\varphi_i''(z) - (q_{\mathbf{p}}(z) - E_i)\varphi_i(z)}{e^{-\kappa_i(\alpha)z}\Psi(z;\alpha)}$$

is an elliptic function with only one simple pole at $z = \alpha$ and so is a constant. Since it has a zero at $z = 0$, by using (4.11), this constant must be 0. Therefore, $\varphi_i(z)$ is a solution of

$$y''(z) = (q_{\mathbf{p}}(z) - E_i)y(z). \tag{4.20}$$

Define r_i, s_i by

$$\begin{cases} -2\pi i s_i = (\zeta(\alpha) - \kappa_i(\alpha)) - \eta_1\alpha, \\ 2\pi i r_i = (\zeta(\alpha) - \kappa_i(\alpha))\tau - \eta_2\alpha, \end{cases}$$

then we see from (4.12) that

$$\varphi_i(z + \omega_k) = e^{(\zeta(\alpha) - \kappa_i(\alpha))\omega_k - \eta_k\alpha}\varphi_i(z) = \begin{cases} e^{-2\pi i s_i}\varphi_i(z) & \text{if } k = 1, \\ e^{2\pi i r_i}\varphi_i(z) & \text{if } k = 2, \end{cases}$$

namely $e^{-2\pi i s_i}$ (resp. $e^{2\pi i r_i}$) is an eigenvalue of the monodromy matrix $M_1(E_i)$ (resp. $M_2(E_i)$) of (4.20).

On the other hand, the Legendre relation $\tau\eta_1 - \eta_2 = 2\pi i$ implies

$$\begin{cases} \alpha = r_i + s_i\tau, \\ \kappa_i(\alpha) = \zeta(\alpha) - r_i\eta_1 - s_i\eta_2. \end{cases} \tag{4.21}$$

By $\alpha \notin E_\tau[2]$, we obtain $(r_i, s_i) \notin \frac{1}{2}\mathbb{Z}^2$, say $s_i \notin \frac{1}{2}\mathbb{Z}$ for example. Then $e^{-2\pi i s_i} \neq \pm 1$ as an eigenvalue of the monodromy matrix $M_1(E_i)$ of (4.20), so it follows from Case (a)-1 in Section 2.3 that $\varphi_i(z)$ is one of the two Baker–Akhiezer functions of (4.20), namely we may assume $\varphi_i(z) = \varphi(P_i; z)$ for $P_i = (E_i, \mathscr{C}_i) \in \Gamma_{q_{\mathbf{p}}}$. Then it follows from Section 2.3 that $(r_i, s_i) = (r(P_i), s(P_i))$, which implies (4.16) by applying (4.21) and (4.6).

Finally, suppose $E_i = E_j$ for some $i \neq j$, then $P_i = P_j$ or $P_i = P_j^*$. If $P_i = P_j$, it follows from (4.16) that $\kappa_i(\alpha) = \kappa_j(\alpha)$, a contradiction. So $P_i = P_j^*$, which together with (4.6) imply

$$\alpha = \sigma_{q_{\mathbf{p}}}(P_i) = \sigma_{q_{\mathbf{p}}}(P_j^*) = -\sigma_{q_{\mathbf{p}}}(P_j) = -\alpha, \tag{4.22}$$

a contradiction to $\alpha \notin E_\tau[2]$. This proves (4.17). ∎

Theorem 4.5. *For any* $\mathbf{p} \in \mathscr{L}_N$, *there holds* $\deg \sigma_{q_{\mathbf{p}}} = N$.

Proof. Denote $d := \deg \sigma_{q_{\mathbf{p}}}$. We want to prove $d = N$. Choose $\alpha \notin \mathscr{B} \cup E_\tau[2]$ such that the κ-polynomial $R_{\mathbf{p}}(\kappa, \alpha)$ has N distinct roots, denoted by $\kappa_1(\alpha), \ldots, \kappa_N(\alpha)$. By Proposition 4.4, for each $\kappa_i(\alpha)$, there associates a unique point $P_i = (E_i, \mathscr{C}_i) \in \Gamma_{q_{\mathbf{p}}}$ such that $\sigma_{q_{\mathbf{p}}}(P_i) = \alpha$ and $E_i \neq E_j$ for $i \neq j$. This implies $\#\sigma_{q_{\mathbf{p}}}^{-1}(\alpha) \geq N$ and so $d \geq N$.

Conversely, to prove $d \leq N$, we can also choose $\alpha \notin \mathscr{B} \cup E_\tau[2]$ such that

$$\sigma_{q_{\mathbf{p}}}^{-1}(\alpha) = \{P_i = (E_i, \mathscr{C}_i) \mid i = 1, \ldots, d\} \quad \text{with } P_i \neq P_j \text{ for } i \neq j.$$

For each P_i, we define

$$\kappa(P_i) := \zeta(\alpha) - r(P_i)\eta_1 - s(P_i)\eta_2, \tag{4.23}$$

then we claim that there are $d_j \in \mathbb{C}$'s such that up to multiplying a nonzero constant, the Baker–Akhiezer function $\psi(P_i; z)$ can be expressed as

$$\psi(P_i; z) = e^{-\kappa(P_i)z} \cdot \sum_{j=1}^{N} d_j \Psi(z - p_j; \alpha). \tag{4.24}$$

Indeed, we can choose $d_j \in \mathbb{C}$'s such that

$$\psi(P_i; z) - e^{-\kappa(P_i)z} \cdot \sum_{j=1}^{N} d_j \Psi(z - p_j; \alpha)$$

has no poles. Furthermore, by $\alpha = \sigma_{q_{\mathbf{p}}}(P_i) = r(P_i) + s(P_i)\tau$ and (4.23), we see that

$$e^{-\kappa(P_i)(z+1)}\Psi(z+1;\alpha) = e^{-2\pi i s(P_i)}e^{-\kappa(P_i)z}\Psi(z;\alpha),$$
$$e^{-\kappa(P_i)(z+\tau)}\Psi(z+\tau;\alpha) = e^{2\pi i r(P_i)}e^{-\kappa(P_i)z}\Psi(z;\alpha),$$

namely $e^{-\kappa(P_i)z}\Psi(z;\alpha)$ satisfies the same transformation law as $\psi(P_i; z)$, so

$$\frac{\psi(P_i; z) - e^{-\kappa(P_i)z} \cdot \sum_{j=1}^{N} d_j \Psi(z - p_j; \alpha)}{e^{-\kappa(P_i)z}\Psi(z;\alpha)}$$

is an elliptic function, which has only one simple pole at $z = \alpha$ and also a zero at $z = 0$, so as before we conclude that it is identically 0. This proves (4.24). Consequently, $\kappa = \kappa(P_i)$, $E = E_i$, and $\vec{d} = (d_1, \ldots, d_N)^t$ solve (4.13), so

$$R_{\mathbf{p}}(\kappa(P_i), \alpha) = 0, \quad i = 1, \ldots, d.$$

Recall that $P_i \neq P_j$ for $i \neq j$. Suppose $\kappa(P_i) = \kappa(P_j)$ for some $i \neq j$. Then it is seen from (4.18) that $E_i = E_j$ and so $P_i = P_j^*$. Again, this leads to (4.22), which contradicts with $\alpha \notin E_\tau[2]$. Thus,

$$\kappa(P_i) \neq \kappa(P_j) \quad \text{for } 1 \leq i \neq j \leq d,$$

which finally implies $d \leq N = \deg_\kappa R_{\mathbf{p}}(\kappa, \alpha)$ and so $d = N$. The proof is complete. ■

Proposition 4.4 and Theorem 4.5 say that for any $\alpha \notin \mathcal{B} \cup E_\tau[2]$, the fiber $\sigma_{q_{\mathbf{p}}}^{-1}(\alpha)$ is one-to-one and onto the fiber $\pi^{-1}(\alpha)$, which induces a birational map ϑ such that $\sigma_{q_{\mathbf{p}}} = \pi \circ \vartheta$ as follows. Define

$$\vartheta\colon \Gamma_{q_{\mathbf{p}}} \to \Gamma_N, \quad \vartheta(P) := (\kappa, \wp(\alpha), \wp'(\alpha)), \tag{4.25}$$

where (note $\mathfrak{m} = N$ for $\mathbf{p} \in \mathcal{L}_N$)

$$\alpha := \sigma_{q_{\mathbf{p}}}(P) = \sum_{j=1}^{\mathfrak{m}} a_j(P) - \sum_{j=1}^{N} p_j = r(P) + s(P)\tau,$$

$$\begin{aligned} \kappa = \kappa(P) &:= \zeta(\sigma_{q_{\mathbf{p}}}(P)) - r(P)\eta_1 - s(P)\eta_2 \\ &= \zeta(r(P) + s(P)\tau) - r(P)\eta_1 - s(P)\eta_2. \end{aligned} \tag{4.26}$$

Recall (4.7)–(4.8) that $\wp(\alpha) = \wp(\sigma_{q_{\mathbf{p}}}(P))$ and $\wp'(\alpha)/\mathcal{C} = \wp'(\sigma_{q_{\mathbf{p}}}(P))/\mathcal{C}$ are rational functions of E. Since (1.1), (2.22), and (4.26) together imply

$$\kappa(P^*) = -\kappa(P),$$

$\mathcal{C}\kappa(P)$ is a meromorphic function of E in $\mathbb{C}$. Since $\lim_{P \to P_\infty} \mathcal{C}\kappa(P) = \infty$ by (4.26), we conclude that $\kappa(P)/\mathcal{C}$ is also a rational function of E.

Note from (4.26) that the poles of κ are precisely the fiber $\sigma_{q_{\mathbf{p}}}^{-1}(0)$, i.e.,

$$\sigma_{q_{\mathbf{p}}}^{-1}(0) = \kappa^{-1}(\infty),$$

so the map ϑ can be extended from $\overline{\Gamma_{q_{\mathbf{p}}}}$ to $\overline{\Gamma_N}$. In conclusion, we have the following result.

Theorem 4.6. *The map* $\vartheta\colon \overline{\Gamma_{q_{\mathbf{p}}}} \to \overline{\Gamma_N}$ *defined by* (4.25)–(4.26) *is a birational map as follows:*

$$\wp(\alpha) = \frac{P_1(E)}{P_2(E)}, \quad \wp'(\alpha) = \mathcal{C}\frac{P_3(E)}{P_4(E)}, \quad \kappa = \mathcal{C}\frac{P_5(E)}{P_6(E)},$$

where $P_j(E) \in \mathbb{C}[E]$. *In particular,* $\sigma_{q_{\mathbf{p}}} = \pi \circ \vartheta$.

4.3. Proof of Theorem 1.3 for $\mathbf{p} \in \mathcal{A}_N$

Now, we consider

$$\mathbf{p}(\overbrace{p_1, \ldots, p_1}^{m_1(m_1+1)/2}, \ldots, \overbrace{p_n, \ldots, p_n}^{m_n(m_n+1)/2}) \in \mathcal{A}_N \quad \text{with } \sharp(\mathbf{p}) = \max_{0 \le i \le n} m_i \ge 2.$$

Let $q(z) = q_{\mathbf{p}}(z)$ be the corresponding elliptic KdV potential in (1.8) and suppose its genus is g. To prove Theorem 1.3 for such $\mathbf{p}$, we need to relate this elliptic KdV potential $q_{\mathbf{p}}(z)$ with another elliptic KdV potential $q_{\hat{\mathbf{p}}}(z)$ with $\hat{\mathbf{p}} \in \mathcal{L}_N$ via the Darboux transformation, and show that the degrees of their Krichever covering maps are the same.

By Lemma 3.4, we can take $P_0 = (E_0, \mathcal{C}_0) \in \Gamma_{q_{\mathbf{p}}}$ with $\mathcal{C}_0 \neq 0$ such that

$$\{a_1(P_0), \ldots, a_{\mathfrak{m}}(P_0)\} \cap \{p_1, \ldots, p_n\} = \emptyset.$$

Recalling (3.1)–(3.3), we obtain the following new elliptic KdV potential via the Darboux transformation:

$$\begin{aligned} q_{P_0}(z) = q_{\mathbf{p}_{P_0}}(z) &:= q(z) - 2(\ln \psi(P_0; z))'' \\ &= \sum_{i=1}^{n} (m_i - 1) m_i \wp(z - p_i) + 2 \sum_{j=1}^{\mathfrak{m}} \wp(z - a_j(P_0)) \end{aligned} \tag{4.27}$$

with the corresponding

$$\mathbf{p}_{P_0} = (\overbrace{p_1, \ldots, p_1}^{m_1(m_1-1)/2}, \ldots, \overbrace{p_n, \ldots, p_n}^{m_n(m_n-1)/2}, a_1(P_0), \ldots, a_{\mathfrak{m}}(P_0)) \in \mathcal{A}_N$$

satisfying $\sharp(\mathbf{p}_{P_0}) = \sharp(\mathbf{p}) - 1$. Furthermore, it follows from Theorem 2.4 and $\mathcal{C}_0 \neq 0$ that $q_{P_0}(z)$ is also of genus g and has the same spectral polynomial as $q_{\mathbf{p}}$, namely their spectral curve are the same: $\Gamma_{q_{P_0}} = \Gamma_{q_{\mathbf{p}}}$.

For each $E \in \mathbb{C}$, we consider the two linear equations:

$$y''(z) = (q_{\mathbf{p}}(z) - E)y(z), \tag{4.28}$$

$$y''(z) = (q_{P_0}(z) - E)y(z). \tag{4.29}$$

We denote the two Baker–Akhiezer functions of (4.28) by $\psi(P; z)$, $\psi(P^*; z)$ as before, while the two Baker–Akhiezer functions of (4.29) by $\hat{\psi}(P; z)$, $\hat{\psi}(P^*; z)$. Recalling (2.19), we denote the corresponding monodromy data of $\psi(P; z)$ and $\hat{\psi}(P; z)$ by $(r(P), s(P))$ and $(\hat{r}(P), \hat{s}(P))$, respectively.

Proposition 4.7. *Under the above notations,*

$$\{(r(P), s(P)), (r(P^*), s(P^*))\} = \{(\hat{r}(P), \hat{s}(P)), (\hat{r}(P^*), \hat{s}(P^*))\} \tag{4.30}$$

holds for any $P = (E, \mathcal{C}) \in \Gamma_{q_{\mathbf{p}}} = \Gamma_{q_{P_0}}$.

Proof. Define a linear differential operator A_{P_0} with its potential being an elliptic function as follows:

$$A_{P_0} := \frac{d}{dz} - \frac{\psi'(P_0; z)}{\psi(P_0; z)}.$$

Then it is classical (cf. [10]) that $A_{P_0} y(z)$ solves (4.29) as long as $y(z)$ solves (4.28). Consider any $P = (E, \mathcal{C}) \in \Gamma_{q_{\mathbf{p}}}$, there are two cases.

Case 1: $E \neq E_0$. Recalling the Baker–Akhiezer functions $\psi(P; z)$, $\psi(P^*; z)$ of (4.28), we see that $A_{P_0}\psi(P; z)$, $A_{P_0}\psi(P^*; z)$ are nontrivial solutions of (4.29) and satisfy the *same* transformation law (2.19) as $\psi(P; z)$ and $\psi(P^*; z)$, respectively! Indeed, it was proved in [10] that $A_{P_0}\psi(P; z)$, $A_{P_0}\psi(P^*; z)$ are precisely the two Baker–Akhiezer functions $\hat{\psi}(P; z)$, $\hat{\psi}(P^*; z)$ of (4.29). Thus, (4.30) holds.

Case 2: $E = E_0$. Since $\mathcal{C}_0 \neq 0$ implies that $\psi(P_0; z)$ and $\psi(P_0^*; z)$ are linearly independent, there is a constant $c \neq 0$ such that $A_{P_0}\psi(P_0^*; z) = \frac{c}{\psi(P_0^*; z)}$, so $A_{P_0}\psi(P_0^*; z)$ is again a nontrivial solution of (4.29) with $E = E_0$ and satisfies the *same* transformation law (2.19) as $\psi(P_0; z)$ (because (2.22) implies that $\psi(P_0^*; z)^{-1}$ satisfies the same transformation law (2.19) as $\psi(P_0; z)$). Again, it was proved in [10] that this $A_{P_0}\psi(P^*;z)$ is one of the two Baker–Akhiezer functions $\hat{\psi}(P;z)$, $\hat{\psi}(P^*;z)$ of (4.29), so

$$(r(P), s(P)) \in \{(\hat{r}(P), \hat{s}(P)), (\hat{r}(P^*), \hat{s}(P^*))\}.$$

Clearly, (4.30) follows from here and (2.22). ■

With the help of Proposition 4.7, we can compare the degree of the covering maps of $q_{\mathbf{p}}(z)$ with that of $q_{P_0}(z)$ in (4.27).

Proposition 4.8. *There holds*

$$\deg \sigma_{q_{\mathbf{p}}} = \deg \sigma_{q_{P_0}},$$

where $\sigma_{q_{\mathbf{p}}}$ and $\sigma_{q_{P_0}}$ are the Krichever covering maps of $q_{\mathbf{p}}(z)$ and $q_{P_0}(z)$ respectively.

Proof. Denote $d = \deg \sigma_{q_{\mathbf{p}}}$. We choose $\alpha \in E_\tau \setminus E_\tau[2]$ such that

$$\sigma_{q_{\mathbf{p}}}^{-1}(\alpha) = \{P_i = (E_i, \mathscr{C}_i) \mid i = 1, \ldots, d\} \quad \text{with } P_i \neq P_j \text{ for } i \neq j.$$

Then it follows from (4.6) that

$$\alpha = r(P_j) + s(P_j)\tau \quad \text{for all } j.$$

Since $\alpha \notin E_\tau[2]$, the same argument as the proof of Theorem 4.5 implies $E_i \neq E_j$ for any $i \neq j$.

For each such P_j, it follows from Proposition 4.7 that there is $\hat{P}_j \in \{P_j, P_j^*\}$ such that $(\hat{r}(\hat{P}_j), \hat{s}(\hat{P}_j)) = (r(P_j), s(P_j))$ and so

$$\sigma_{q_{P_0}}(\hat{P}_j) = \hat{r}(\hat{P}_j) + \hat{s}(\hat{P}_j^*)\tau = r(P_j) + s(P_j)\tau = \alpha.$$

Since $E_i \neq E_j$ implies $\hat{P}_i \neq \hat{P}_j$ for any $i \neq j$, we obtain $\deg \sigma_{q_{\mathbf{p}}} \leq \deg \sigma_{q_{P_0}}$.

On the other hand, the Darboux transformation is invertible, namely $q_{\mathbf{p}}(z)$ can be obtained from $q_{P_0}(z)$ via the Darboux transformation by the Baker–Akhiezer function of (4.29) with $E = E_0$, Therefore, the same argument also shows that $\deg \sigma_{q_{P_0}} \leq \deg \sigma_{q_{\mathbf{p}}}$. This completes the proof. ■

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\mathbf{p} \in \mathscr{A}_N$. The case $\sharp(\mathbf{p}) = 1$, i.e., $\mathbf{p} \in \mathscr{L}_N$ was proved in Theorem 4.5. Thus, we may assume $\sharp(\mathbf{p}) \geq 2$. Then the above argument implies the existence of $\mathbf{p}' = \mathbf{p}_{P_0} \in \mathscr{A}_N$ with $\sharp(\mathbf{p}') = \sharp(\mathbf{p}) - 1$ such that $\deg \sigma_{q_{\mathbf{p}}} = \deg \sigma_{q_{\mathbf{p}'}}$. Clearly, we can repeat this argument finite times and finally obtain $\hat{\mathbf{p}} \in \mathscr{A}_N$ with $\sharp(\hat{\mathbf{p}}) = 1$, i.e., $\hat{\mathbf{p}} \in \mathscr{L}_N$, such that $\deg \sigma_{q_{\mathbf{p}}} = \deg \sigma_{q_{\hat{\mathbf{p}}}}$. Since Theorem 4.5 proves $\deg \sigma_{q_{\hat{\mathbf{p}}}} = N$, we conclude $\deg \sigma_{q_{\mathbf{p}}} = N$. This completes the proof.

Remark that since Lemma 3.2 says that $\mathbf{p}' = \mathbf{p}_{P_0}$ belongs to the same connected component of $\mathcal{A}_N$ as $\mathbf{p}$, the $\hat{\mathbf{p}} \in \mathcal{L}_n$ can be chosen to belong to the same connected component of $\mathcal{A}_N$ as $\mathbf{p}$. ∎

5. Proof of Theorem 1.4

The purpose of this section is to prove Theorem 1.4. Let $\mathbf{p} \in \mathcal{A}_N$ be given by (1.7) and the corresponding elliptic KdV potential $q(z) = q_{\mathbf{p}}(z)$ be in (1.8). Suppose it is of genus g and recall its associated spectral curve $\Gamma_{q_{\mathbf{p}}}$. Let $K(E_\tau)$ and $K(\overline{\Gamma_{q_{\mathbf{p}}}})$ be the field of rational functions on E_τ and $\overline{\Gamma_{q_{\mathbf{p}}}}$, respectively. Then Theorem 1.3 indicates that $K(\overline{\Gamma_{q_{\mathbf{p}}}})$ is a finite field extension over $K(E_\tau)$ with

$$[K(\overline{\Gamma_{q_{\mathbf{p}}}}) : K(E_\tau)] = \deg \sigma_{q_{\mathbf{p}}} = N. \tag{5.1}$$

In order to find a primitive generator of this field extension, as mentioned in Section 1, we define $\mathbf{z}_{\mathbf{p}} \colon \overline{\Gamma_{q_{\mathbf{p}}}} \to \mathbb{C} \cup \{\infty\}$ by

$$\begin{aligned}\mathbf{z}_{\mathbf{p}}(P) := \zeta\Big(\sum_{k=1}^{\mathfrak{m}} a_k(P) - \sum_{j=1}^{n} m_j p_j\Big) - \sum_{k=1}^{\mathfrak{m}} \zeta(a_k(P) - p_n) \\ + \sum_{j=1}^{n-1} m_j \zeta(p_j - p_n) \quad \text{for all } P \in \Gamma_{q_{\mathbf{p}}}.\end{aligned}$$

The reason of defining this function is clear from the following results.

Lemma 5.1. *For any $P = (E, \mathcal{C}) \in \Gamma_{q_{\mathbf{p}}}$, we have*

$$\mathbf{z}_{\mathbf{p}}(P) = \zeta(\sigma_{q_{\mathbf{p}}}(P)) - r(P)\eta_1 - s(P)\eta_2. \tag{5.2}$$

Consequently, the poles of $\mathbf{z}_{\mathbf{p}}$ on $\overline{\Gamma_{q_{\mathbf{p}}}}$ are precisely the fiber $\sigma_{q_{\mathbf{p}}}^{-1}(0)$.

Proof. Define

$$\mathcal{B}_0 := \{P \in \Gamma_{q_{\mathbf{p}}} \mid \{a_1(P), \ldots, a_{\mathfrak{m}}(P)\} \cap \{p_1, \ldots, p_n\} \neq \emptyset \text{ in } E_\tau\}.$$

Then Lemma 3.4 says that $\mathcal{B}_0$ is finite. For any $P \in \Gamma_{q_{\mathbf{p}}} \setminus \mathcal{B}_0$, we can apply (4.4) with $i = n$ and (2.20) to obtain (5.2). Then (5.2) also holds for $P \in \mathcal{B}_0$ by continuity. Consequently, the last statement follows from (5.2). ∎

Remark 5.2. By (4.4) and the proof of Lemma 5.1, clearly we can replace p_n with any fixed p_i in the definition of $\mathbf{z}_{\mathbf{p}}$.

If $\mathbf{p} \in \mathcal{L}_N$, it follows from (5.2) that this $\mathbf{z}_{\mathbf{p}}$ is precisely the rational function $\kappa(P) \in K(\overline{\Gamma_{q_{\mathbf{p}}}})$ given in (4.26) and Theorem 4.6.

For general $\mathbf{p} \in \mathcal{A}_N$, thanks to (5.2), the same argument as Theorem 4.6 implies that

$$\mathbf{z_p}(P) = \mathcal{C}\frac{P_7(E)}{P_8(E)}$$

for some coprime polynomials $P_7(E), P_8(E) \in \mathbb{C}[E]$.

Thus, $\mathbf{z_p} \in K(\overline{\Gamma_{q_\mathbf{p}}})$, then the minimal polynomial $W_\mathbf{p}(\mathbf{z}) \in K(E_\tau)[\mathbf{z}]$ of $\mathbf{z_p}$ exists with degree

$$d_\mathbf{p} := \deg W_\mathbf{p}(\mathbf{z}).$$

Lemma 5.1 shows that $\mathbf{z}_{q_\mathbf{p}}$ has no poles over $E_\tau^\times := E_\tau \setminus \{0\}$, which implies that the minimal polynomial $W_\mathbf{p}(\mathbf{z})$ with $\sigma_{q_\mathbf{p}} = \alpha$ is a monic polynomial in $\mathbb{C}[\wp(\alpha), \wp'(\alpha)][\mathbf{z}]$, and we write $W_\mathbf{p}(\mathbf{z}) = W_\mathbf{p}(\mathbf{z}, \alpha)$ to emphasize this fact.

Since $\deg_\mathbf{z} W_\mathbf{p}(\mathbf{z}, \alpha) | \deg \sigma_{q_\mathbf{p}}$ and $\deg \sigma_{q_\mathbf{p}} = N$, we have $d_\mathbf{p} | N$. Clearly, to prove that $\mathbf{z_p}$ is a primitive generator is equivalent to prove $d_\mathbf{p} = N$. The following result proves this result for $\mathbf{p} \in \mathcal{L}_N$. Recall (4.15) that the Krichever polynomial $R_\mathbf{p}(\kappa, \alpha)$ is well defined for $\mathbf{p} \in \mathcal{L}_N$.

Lemma 5.3. *Let* $\mathbf{p} \in \mathcal{L}_N$. *Then for any* $\alpha \in E_\tau$,

$$W_\mathbf{p}(\mathbf{z}, \alpha) = R_\mathbf{p}(\mathbf{z}, \alpha), \tag{5.3}$$

Here $R_\mathbf{p}(\kappa, \alpha)$ *is the Krichever polynomial defined in* (4.15). *In particular,* $\mathbf{z_p}(P) = \kappa(P)$ *is a primitive generator of the finite field extension* (5.1).

Proof. Since $p \in \mathcal{L}_N$, Remark 5.2 says $\mathbf{z_p}(P) = \kappa(P)$.

It suffices to prove (5.3) for any $\alpha \notin \mathcal{B}_\mathbf{p} \cup E_\tau[2]$. For such α, the κ-polynomial $R_\mathbf{p}(\kappa, \alpha)$ has N distinct roots, denoted by $\kappa_1(\alpha), \ldots, \kappa_N(\alpha)$. For each $\kappa_i(\alpha)$, we see from Proposition 4.4 that there is a unique point $P_i = (E_i, \mathcal{C}_i) \in \Gamma_{q_\mathbf{p}}$ such that $\sigma_{q_\mathbf{p}}(P_i) = \alpha$,

$$\kappa_i(\alpha) = \zeta(\alpha) - r(P_i)\eta_1 - s(P_i)\eta_2 = \mathbf{z_p}(P_i),$$

and $E_i \neq E_j$ for $i \neq j$. This implies

$$W_\mathbf{p}(\kappa_i(\alpha), \alpha) = 0 \quad \text{for all } 1 \leq i \leq N. \tag{5.4}$$

Since the $\mathbf{z}$-polynomial $R_\mathbf{p}(\mathbf{z}, \alpha)$ has no multiple roots for such α, we conclude from (5.4) that $R_\mathbf{p}(\mathbf{z}, \alpha) | W_\mathbf{p}(\mathbf{z}, \alpha)$. This implies

$$N = \deg_\mathbf{z} R_\mathbf{p}(\mathbf{z}, \alpha) \leq \deg_\mathbf{z} W_\mathbf{p}(\mathbf{z}, \alpha) \leq N,$$

so $\deg_\mathbf{z} W_\mathbf{p}(\mathbf{z}, \alpha) = N$ and (5.3) holds. ∎

Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\mathbf{p} \in \mathcal{A}_N$. The case $\sharp(\mathbf{p}) = 1$, i.e., $\mathbf{p} \in \mathcal{L}_N$, was proved in Lemma 5.3. Thus, we may assume $\sharp(\mathbf{p}) \geq 2$. Recalling the argument in Section 4.3, we conclude the existence of $\hat{\mathbf{p}} \in \mathcal{L}_N$ which belongs to the same component of $\mathcal{A}_N$ as $\mathbf{p}$ such that the following statement hold:

(1) $\deg \sigma_{q_{\mathbf{p}}} = \deg \sigma_{q_{\hat{\mathbf{p}}}} = N$;

(2) the spectral curves are the same, i.e., $\Gamma_{q_{\mathbf{p}}} = \Gamma_{q_{\hat{\mathbf{p}}}}$;

(3) the conclusion of Proposition 4.7 still holds after replacing $\mathbf{p}_{P_0}$ with $\hat{\mathbf{p}}$.

Again, let $\alpha \notin \mathcal{B}_{\hat{\mathbf{p}}} \cup E_\tau[2]$ such that

$$\sigma_{q_{\hat{\mathbf{p}}}}^{-1}(\alpha) = \{P_i = (E_i, \mathcal{C}_i) \mid i = 1, \ldots, N\} \quad \text{with } P_i \neq P_j \text{ for } i \neq j,$$

and $E_i \neq E_j$ for all $i \neq j$. Then Lemma 5.3 implies that $\mathbf{z}_{\hat{\mathbf{p}}}(P_i)$'s are N distinct roots of its minimal polynomial $W_{\hat{\mathbf{p}}}(\mathbf{z}, \alpha)$.

By Proposition 4.7, there exists $\widetilde{P}_i \in \{P_i, P_i^*\}$ such that

$$(r(\widetilde{P}_i), s(\widetilde{P}_i)) = (\hat{r}(P_i), \hat{s}(P_i)). \tag{5.5}$$

where $(r(P), s(P))$, $(\hat{r}(P), \hat{s}(P))$ denote the corresponding monodromy data for the associated Baker–Akhiezer functions of $q_{\mathbf{p}}$ and $q_{\hat{\mathbf{p}}}$, respectively. Then

$$\sigma_{q_{\mathbf{p}}}^{-1}(\alpha) = \{\widetilde{P}_i \mid 1 \leq i \leq N\},$$

and it follows from (5.2) and (5.5) that

$$\begin{aligned} \mathbf{z}_{\hat{\mathbf{p}}}(P_i) &= \zeta(\alpha) - \hat{r}(P_i)\eta_1 - \hat{s}(P_i)\eta_2 \\ &= \zeta(\alpha) - r(\widetilde{P}_i)\eta_1 - s(\widetilde{P}_i)\eta_2 = \mathbf{z}_{\mathbf{p}}(\widetilde{P}_i), \end{aligned}$$

so $\mathbf{z}_{\hat{\mathbf{p}}}(P_i)$'s are also the N distinct roots of $W_{\mathbf{p}}(\mathbf{z}, \alpha)$. Since $\deg_{\mathbf{z}} W_{\mathbf{p}}(\mathbf{z},\alpha) \leq N$, we conclude that

$$\deg W_{\mathbf{p}}(\mathbf{z},\alpha) = N = \deg W_{\hat{\mathbf{p}}}(\mathbf{z},\alpha)$$

and

$$W_{\mathbf{p}}(\mathbf{z},\alpha) = W_{\hat{\mathbf{p}}}(\mathbf{z},\alpha).$$

In particular, $\mathbf{z}_{\mathbf{p}}$ is a primitive generator of the field extention (5.1). ■

For any $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, we define

$$Z_{r,s}(\tau) := \zeta(r + s\tau) - r\eta_1 - s\eta_2.$$

Here we note that $Z_{r,s}(\tau) \equiv \infty$ if $(r, s) \in \mathbb{Z}^2$ and $Z_{r,s}(\tau) \equiv 0$ if $(r, s) \in \frac{1}{2}\mathbb{Z}^2 \setminus \mathbb{Z}^2$. Define $Z_{\mathbf{p}}(r, s)$ by

$$Z_{\mathbf{p}}(r, s) := W_{\mathbf{p}}(Z_{r,s}(\tau), r + s\tau).$$

Given $(r,s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and $\alpha = r + s\tau$, we see from Lemma 5.1 that for $P \in \sigma_{q_{\mathbf{p}}}^{-1}(\alpha)$,

$$\mathbf{z}_{\mathbf{p}}(P) = Z_{r,s}(\tau), \tag{5.6}$$

so $Z_{\mathbf{p}}(r,s) = 0$.

Conversely, suppose $(r,s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ satisfies $Z_{\mathbf{p}}(r,s) = 0$. Then $Z_{r,s}(\tau)$ is a root of $W_{\mathbf{p}}(\mathbf{z},\alpha)$ with $\alpha = r + s\tau$, so there is $P \in \sigma_{q_{\mathbf{p}}}^{-1}(\alpha)$ such that (5.6) holds, which implies $(r,s) = (r(P), s(P))$. Therefore, the following result holds.

Theorem 5.4. *Let $\mathbf{p} \in \mathcal{A}_N$ and $(r,s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. Then $(r,s) = (r(P), s(P))$ for some $P \in \Gamma_{q_{\mathbf{p}}}$ if and only if $Z_{\mathbf{p}}(r,s) = 0$.*

We conclude this section with the following remark.

Remark 5.5. (1) Note that the original Krichever polynomial $R_{\mathbf{p}}(\kappa,\alpha)$ is well defined only for $\mathbf{p} \in \mathcal{L}_N$, while the minimal polynomial $W_{\mathbf{p}}(\mathbf{z}_{\mathbf{p}},\alpha)$ and the corresponding quantity $Z_{\mathbf{p}}(r,s)$ are well defined for all $\mathbf{p} \in \mathcal{A}_N$. Lemma 5.3 shows $W_{\mathbf{p}}(\mathbf{z},\alpha) = R_{\mathbf{p}}(\mathbf{z},\alpha)$ for $\mathbf{p} \in \mathcal{L}_N$.

(2) When $q_{\mathbf{p}}(z)$ is the Lamé potential or the general Darboux–Treibich–Verdier potential $\sum_{i=0}^{3} m_i(m_i+1)\wp\big(z - \frac{\omega_i}{2}\big)$, the quantity $Z_{\mathbf{p}}(r,s)$ was already studied in [5, 20], where it was proved that $Z_{\mathbf{p}}(r,s)$ is a modular form as a holomorphic function of τ for $(r,s) \in \mathbb{Q}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, which has important applications to PDE problems.

(3) The proof of Theorem 1.4 implies $W_{\mathbf{p}}(\mathbf{z},\alpha) = W_{\hat{\mathbf{p}}}(\mathbf{z},\alpha)$, where $\hat{\mathbf{p}} \in \mathcal{L}_N$ belongs to the same component of $\mathcal{A}_N$ as $\mathbf{p}$. This leads to a natural question.

> *Is the minimal polynomial $W_{\mathbf{p}}(\mathbf{z},\alpha)$ invariant for all $\mathbf{p}$ in a connected component $\mathcal{A}_N^0$ of $\mathcal{A}_N$? Is the monodromy representation of the ODE $L(q_{\mathbf{p}}; E)$ invariant for all $\mathbf{p} \in \mathcal{A}_N^0$?*

It seems that this question has not been settled in the literature. We would like to study this question elsewhere.

Acknowledgements. The authors thank the referee for careful reading and valuable comments.

Funding. Z. Chen was supported by National Science and Technology Major Project (No. 2022ZD0117000) and NSFC (No. 12222109).

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Received 23 January 2024; revised 6 August 2024.

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