J. Spectr. Theory 14 (2024), 1383–1407
DOI 10.4171/JST/517

Real diffusion with complex spectral gap

Jean-François Bony and Laurent Michel

Abstract. The low-lying eigenvalues of the generator of a Langevin process are known to satisfy the Eyring–Kramers law in the low temperature regime under suitable assumptions. These eigenvalues are generically real. We construct generators whose spectral gap is given by non-real eigenvalues or by a real eigenvalue having a Jordan block.

1. Introduction

The generator of a diffusion process is generally a differential operator of order two with real coefficients. In the last decades, the asymptotic of its low-lying eigenvalues has been obtained [3, 4, 9, 10, 13, 14, 16, 17] in the low temperature regime (Eyring–Kramers law), see [2] for a general presentation. These results provide sharp informations on metastability or on return to equilibrium. For reversible processes, the generator is a self-adjoint operator on an appropriate Hilbert space and then its spectrum is always real. For irreversible processes, the generator is no longer self-adjoint on the natural Hilbert space and one can hope to observe non-real eigenvalues or Jordan's blocks. But, as recalled at the end of this part, there are strong constrains on the low-lying spectrum of generators which make such phenomena unlikely and explain why non-real spectra have not been obtained up to now. The goal of this paper is to construct generators with pathologic spectral gap.

We first discuss spectral properties of generators in the general setting of [3] and send the reader to this paper for precise statements and to the references of the previous paragraph for slightly different settings. In [3], we consider the operator on $L^2(\mathbb{R}^d)$

$$P = -h \operatorname{div} \circ A \circ h\nabla + \frac{1}{2}(b \cdot h\nabla + h \operatorname{div} \circ b) + c, \tag{1.1}$$

where the symmetric matrix $A = (a_{j,k}(x,h))_{j,k}$, the vector field $b = (b_j(x,h))_j$ and the function $c(x,h)$ are smooth and real-valued. Moreover, these functions are

---

Mathematics Subject Classification 2020: 35P20 (primary); 35J10, 47A10, 60J35, 81Q20 (secondary).

Keywords: scattering theory, semiclassical asymptotics, resonances, microlocal analysis, Schrödinger operators.

symbols and have an asymptotic expansion in power of the parameter h, which is proportional to the temperature. We assume that P has an invariant distribution which has a Gibbs form. More precisely, there exists a confining smooth Morse function f such that

$$P(e^{-f/h}) = P^*(e^{-f/h}) = 0.$$

Let $1 \leq n_0 < +\infty$ denote the number of minima of f. Hypoelliptic and hypocoercive assumptions are also made. Under these assumptions, P is maximal accretive and has domain

$$\mathcal{D}(P) = \{u \in L^2(\mathbb{R}^d);\ Pu \in L^2(\mathbb{R}^d)\},$$

as proved in [13, Section 3]. The evolution equation naturally associated to P is the heat (or Fokker–Planck) equation

$$\begin{cases} h\partial_t u(t,x) = -Pu(t,x), \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}$$

where $u_0(x) \in L^2(\mathbb{R}^2)$ is the initial data. The low-lying spectrum of P is given by the following result (see [3, Theorem 3]).

Theorem 1.1 (Eyring–Kramers law). *There exists $\lambda_* > 0$ such that, for h small enough, P has exactly n_0 eigenvalues counted with their algebraic multiplicity $\lambda_1(h), \dots, \lambda_{n_0}(h)$ in $\{z \in \mathbb{C}; \operatorname{Re} z \leq \lambda_* h\}$. Moreover, $\lambda_1(h) = 0$ is simple with* $\operatorname{Ker} P = e^{-f/h}\mathbb{C}$. *For $n = 2, \dots, n_0$, the eigenvalue $\lambda_n(h)$ satisfies the asymptotic*

$$\lambda_n(h) = a_n(h) h e^{-2S_n/h} \quad \text{with } a_n(h) \simeq \sum_{j \geq 0} a_n^j h^j,$$

$S_n = f(s_n) - f(m_n) > 0$ for some particular saddle point s_n and minimum m_n, $a_n^0 \neq 0$ explicitly known and $a_n^j \in \mathbb{R}$ for all $j \neq 0$.

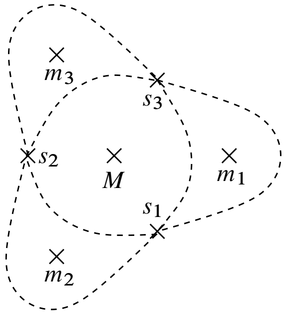
Note that the first eigenvalue $\lambda_1 = 0$ is always real. Since all the coefficients a_n^j are real, it is not possible to use the Eyring–Kramers law to construct an operator with non-real small eigenvalues. Moreover, the imaginary part of λ_n is always extremely small. More precisely, the following statement holds true.

Remark 1.2. For all $n = 1, \dots, n_0$, we have

$$|\operatorname{Im} \lambda_n| = \mathcal{O}(h^\infty) \operatorname{Re} \lambda_n. \tag{1.3}$$

On the other hand, the particular form (1.1) of the generator P induces symmetries on its spectrum, as remarked on [17, p. 15]. More precisely, since the coefficients of P are real-valued and the domain of P is stable by complex conjugation, we get

$$\overline{(P - \lambda)u} = (P - \bar{\lambda})\bar{u}, \tag{1.4}$$



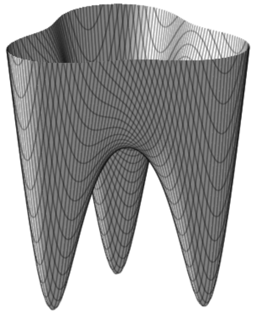


Figure 1. The structure of the critical points of f and an example of such a Morse function.

for all $\lambda \in \mathbb{C}$ and $u \in \mathcal{D}(P)$. This implies the following property which is also satisfied for $\mathcal{P}\mathcal{T}$-symmetric operators (see for instance [1] for the bifurcation of eigenvalues from the real axis to the complex plane).

Remark 1.3. The spectrum of P is invariant by complex conjugation.

In particular, when f has exactly two minima, P has two small eigenvalues $\lambda_1 = 0$ and λ_2 by Theorem 1.1. Since $\lambda_2 = \overline{\lambda_2}$ by Remark 1.3, these two small eigenvalues are always real and simple for h small enough (see [17, Remark 1.10]).

More generally, if the asymptotic expansion of λ_n given by the Eyring–Kramers law is different from that of the other eigenvalues, then λ_n is real and simple for h small enough. As an example, if the (Arrhenius) exponential factors S_n are all different, then all the small eigenvalues λ_n are real and simple for h small enough. This shows that the exponentially small eigenvalues of the generator of a diffusion as in (1.1) are generically real.

We now construct operators of the form (1.1) with non-real small eigenvalues or Jordan blocks. From the two previous paragraphs, the associated Morse function f must have at least 3 minima and some of exponential factors $S_n = f(s_n) - f(m_n)$ must coincide.

2. Statement of the results

On $\mathbb{R}^2$, we consider a smooth Morse function f with $f(x) = x^2$ outside a compact set and which is invariant under R, the rotation of angle $2\pi/3$ around 0. Moreover, we assume that the set of critical points of f consists of 3 (global) minima m_1, m_2, m_3, 3 saddle points s_1, s_2, s_3 and 1 (local) maximum M as in Figure 1. Let P_0 be the

Witten Laplacian associated to the function f, that is

$$P_0 = d_f^* \circ d_f \quad \text{with } d_f = e^{-f/h} \circ h\nabla \circ e^{f/h} = \begin{pmatrix} h\partial_{x_1} + \partial_{x_1} f \\ h\partial_{x_2} + \partial_{x_2} f \end{pmatrix}. \tag{2.1}$$

A classical computation shows that this operator has the form

$$P_0 = -h^2\Delta + |\nabla f|^2 - h\Delta f.$$

Since f is a compactly supported perturbation of x^2, P_0 is self-adjoint on the domain of the harmonic oscillator $\mathcal{D}(P_0) = H^2(\mathbb{R}^2) \cap \langle x\rangle^{-2} L^2(\mathbb{R}^2)$, has a compact resolvent, $P_0 \geq 0$ and

$$\operatorname{Ker} P_0 = e^{-f/h}\mathbb{C}.$$

The spectrum of Witten Laplacians in such a geometric configuration has been studied in [14, Section 7.4], [19, Section 7C3] and [18, Section 9.3]. We send the reader to [12] or to the second edition of the book [8] for details on Witten Laplacians.

Throughout the paper, we set $S = f(s) - f(m) > 0$ and $\mu(s) < 0$ denotes the unique negative eigenvalue of Hess $f(s)$. Since f is invariant by rotation, these quantities do not depend on the minimum m and the saddle point s where they are computed. The bottom of the spectrum of P_0 is given by the following result.

Proposition 2.1 (Low eigenvalues of P_0). *There exists $\lambda_* > 0$ such that, for h small enough, P_0 has exactly three eigenvalues counted with multiplicity $\lambda_1(h), \lambda_2(h), \lambda_3(h)$ in $]-\infty, \lambda_* h]$. Moreover,*

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 \quad \textit{and} \quad \lambda_2 \sim \frac{3|\mu(s)||\det \operatorname{Hess} f(m)|^{1/2}}{\pi|\det \operatorname{Hess} f(s)|^{1/2}} h e^{-2S/h}.$$

This proposition is mainly a consequence of previous results (see [14, 19]). The unique novelty is that λ_2 has multiplicity two. This point and other spectral properties of P_0 are proved in Section 3. Since f is invariant under R, so are P_0 and all its eigenspaces.

We now construct an operator having a non-real spectral gap. For that, we perturb the operator P_0 by an anti-adjoint differential operator of order one. More precisely, we consider the operator

$$P_{\mathrm{com}} = P_0 + B \quad \text{with } B = \frac{1}{2}(b \cdot h\nabla + h \operatorname{div} \circ b). \tag{2.2}$$

We require that the vector-valued function $b(x,h) \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^2)$ is a compactly supported real symbol of class $S(h^\infty)$, where

$$S(r) = \{b(x,h) \in C^\infty(\mathbb{R}^2);\ \forall \alpha \in \mathbb{N}^2,\ \exists C_\alpha > 0,$$
$$\forall x \in \mathbb{R}^2,\ \forall h \in]0,1],\ |\partial_x^\alpha b(x,h)| \leq C_\alpha r(h)\},$$

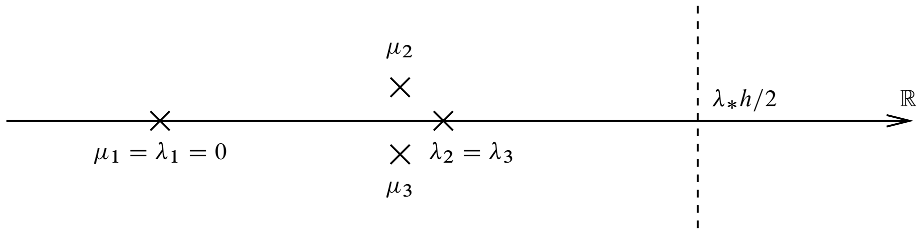


Figure 2. The low-lying eigenvalues of P_0 and P_{com}.

and $b \in S(h^\infty)$ means that $b \in S(h^j)$ for all $j \in \mathbb{N}$. In particular, P_{com} is closed on the domain $\mathcal{D}(P_0)$. We also assume that

$$B(e^{-f/h}) = 0. \tag{2.3}$$

Then, the operator P_{com} enters into the setting of (1.1).

Theorem 2.2 (Non-real eigenvalues). *Let $r(h) = \mathcal{O}(h^\infty)$ be a positive function. There exists a function $b(x,h) \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^2) \cap S(r)$ with (2.3) such that the spectrum of P_{com} satisfies*

$$\sigma(P_{\mathrm{com}}) \cap \{z \in \mathbb{C};\, \operatorname{Re} z < \lambda_* h/2\} = \{\mu_1(h), \mu_2(h), \mu_3(h)\},$$

for h small enough, with $\mu_1 = 0$, $\mu_2 = \lambda_2 + \mathcal{O}(r)$, $\mu_3 = \overline{\mu_2}$ and

$$\operatorname{Im} \mu_2 \neq 0.$$

Here and in the sequel, $\sigma(T)$ denotes the spectrum of the operator T and the eigenvalues $\mu_\bullet(h)$ are simple for h small enough. The setting of Theorem 2.2 is illustrated in Figure 2. The symbol $b(x,h)$ is only partially explicit (see Lemma 4.1, (4.6) and (5.1)). In particular, its size may be way more smaller than r. Then, the imaginary part of μ_2 and μ_3 is very small. But, as explained in (1.3), it is always the case in the general setting.

Theorem 2.2 is proved using the perturbation theory at fixed h small enough. In particular, its proof shows that operators as in (2.2) with a small enough anti-adjoint part B have a non-real spectral gap as soon as the leading term coming from the perturbation theory does not vanish (see Lemma 4.4). In this sense, the situation of Theorem 2.2 is generic.

For h small enough, let Π_{μ_j} denote the spectral projection of P_{com} associated to the eigenvalue μ_j. Using the Cauchy formula, it can be written

$$\Pi_{\mu_j} = \frac{1}{2i\pi} \oint_\gamma (z - P_{\mathrm{com}})^{-1}\, dz,$$

where γ is a sufficiently small loop around μ_j positively oriented. Relations (1.1) and (2.3) give

$$\Pi_{\mu_1} = \frac{e^{-f(x)/h}}{\|e^{-f/h}\|^2}\langle e^{-f/h}, \cdot\rangle \quad \text{and} \quad \overline{\Pi_{\mu_2} u} = \Pi_{\mu_3}\bar{u}. \tag{2.4}$$

Let $u(x,h)$ be an eigenvector of P_{com} associated to the eigenvalue μ_2. From (2.4), $\bar{u}$ is an eigenvector associated to the eigenvalue μ_3. Then, $(\operatorname{Re} u, \operatorname{Im} u)$ is a basis of $\operatorname{Im}\Pi_{\mu_2} \oplus \operatorname{Im}\Pi_{\mu_3} = \operatorname{Ker}(P_{\mathrm{com}} - \mu_2) \oplus \operatorname{Ker}(P_{\mathrm{com}} - \mu_3)$. In particular, u cannot be a real (or purely imaginary) function. From [3, Corollary 1.6] and Theorem 2.2, the solution of the evolution equation (1.2) associated to P_{com} satisfies the following metastable behavior.

Corollary 2.3. *Consider P_{com} as in Theorem 2.2 with h small enough. For all $u_0 \in L^2(\mathbb{R}^2)$, the solution $u = e^{-tP_{\mathrm{com}}/h}u_0$ of (1.2) can be written*

$$\begin{aligned} e^{-tP_{\mathrm{com}}/h}u_0 &= u_1 + e^{-t\mu_2/h}u_2 + e^{-t\mu_3/h}u_3 + \varepsilon(t) \\ &= u_1 + e^{-t\operatorname{Re}\mu_2/h}\Big(\cos\Big(t\operatorname{Im}\frac{\mu_2}{h}\Big)u_{\mathrm{c}} + \sin\Big(t\operatorname{Im}\frac{\mu_2}{h}\Big)u_{\mathrm{s}}\Big) + \varepsilon(t), \end{aligned} \tag{2.5}$$

with $u_j = \Pi_{\mu_j}u_0$ for $j = 1,2,3$, $u_{\mathrm{c}} = u_2 + u_3$, $u_{\mathrm{s}} = iu_3 - iu_2$ and

$$\|\varepsilon(t)\|_{L^2(\mathbb{R}^2)} \leq Ce^{-t/C}\|u_0\|_{L^2(\mathbb{R}^2)},$$

for some constant $C > 0$ independent of t, h, u_0.

If the function u_0 is real-valued, (2.4) implies that $\overline{u_2} = u_3$ and then u_{c} and u_{s} are also real-valued. If in addition u_2, u_3, u_{c} or u_{s} does not vanish identically, the discussion below (2.4) shows that $(u_{\mathrm{c}}, u_{\mathrm{s}})$ is a basis of $\operatorname{Im}\Pi_{\mu_2} \oplus \operatorname{Im}\Pi_{\mu_3}$. In that case,

$$t \mapsto \cos\Big(t\operatorname{Im}\frac{\mu_2}{h}\Big)u_{\mathrm{c}} + \sin\Big(t\operatorname{Im}\frac{\mu_2}{h}\Big)u_{\mathrm{s}},$$

is a non-vanishing periodic function of period $2\pi h|\operatorname{Im}\mu_2|^{-1}$ which reaches all the directions of $\operatorname{Im}\Pi_{\mu_2} \oplus \operatorname{Im}\Pi_{\mu_3}$. Then, the subprincipal term in (2.5), which measures the return to equilibrium, is oscillating. Nevertheless, this phenomenon may be difficult to see in the applications since (1.3) implies that this subprincipal term decays more quickly than it oscillates.

We now construct an operator having a spectral gap with a Jordan block. For that, we consider perturbations of P_0 of the form

$$P_{\mathrm{Jor}} = d_f^* \circ (1 + \chi(x,h))\operatorname{Id} \circ d_f + B \quad \text{where } B = \frac{1}{2}(b \cdot h\nabla + h\operatorname{div} \circ b),$$

Id denotes the 2×2 identity matrix, $\chi \in C_0^\infty(\mathbb{R}^2;\mathbb{R}) \cap S(h^\infty)$ and $b \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^2) \cap S(h^\infty)$. For h small enough, such an operator falls within the general framework of (1.1).

Theorem 2.4 (Jordan block). *Let $r(h) = \mathcal{O}(h^\infty)$ be a positive function. There exist functions $\chi(x,h) \in C_0^\infty(\mathbb{R}^2;\mathbb{R}) \cap S(r)$ and $b(x,h) \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^2) \cap S(r)$ with (2.3) such that, for h small enough,*

$$\sigma(P_{\mathrm{Jor}}) \cap \{z \in \mathbb{C};\ \operatorname{Re} z < \lambda_* h/2\} = \{\lambda_1, \lambda_2\} \text{ of multiplicity } 1 \text{ and } 2 \text{ respectively,}$$

and P_{Jor} has a non-trivial Jordan block associated with the eigenvalue λ_2.

Let Π_{λ_1} and Π_{λ_2} be the spectral projectors of P_{Jor} associated to λ_1 and λ_2 respectively. From Theorem 2.4 and (1.4), there exists an orthonormal basis of real-valued functions, denoted (e_1, e_2), of $\operatorname{Im} \Pi_{\lambda_2}$ such that $\Pi_{\lambda_2} P_{\mathrm{Jor}} \Pi_{\lambda_2}$ expressed in the basis (e_1, e_2) writes

$$\begin{pmatrix} \lambda_2 & \rho \\ 0 & \lambda_2 \end{pmatrix},$$

for some $\rho(h) \in \mathbb{R} \setminus \{0\}$ (see (6.25)). Note that e_1 and e_2 are unique modulo multiplication by ± 1. By construction, the constant ρ is very small. More precisely,

$$|\rho(h)| = \mathcal{O}(h^\infty \lambda_2) = \mathcal{O}(h^\infty e^{-2S/h}). \tag{2.6}$$

But, as for the imaginary part of the eigenvalues (1.3), this is a general fact: any Jordan block associated with a small eigenvalue of an operator of the form (1.1) satisfies an estimate similar to (2.6). Indeed, all the terms in the asymptotic expansion of the interaction matrices are self-adjoint (see [3, Section 6]).

It is difficult to construct by perturbation theory an operator of the form (2.2) satisfying Theorem 2.4. Indeed, Lemma 4.4 shows that such operators enter into the setting of Theorem 2.2 as soon as the leading term in the perturbation theory does not vanish. This is why we consider here more general perturbations which allow to "generate all the possible" leading terms (see [20, Section 4] for similar ideas in resonances theory).

Contrary to Theorem 2.2, the spectral situation of Theorem 2.4 is unstable. Generically, a small perturbation (in the setting of (1.1)) splits the double eigenvalue λ_2 into two non-real conjugate eigenvalues. This is general fact concerning the Jordan blocks. Moreover, the second eigenvalue of P_0 and P_{Jor} is the same. The proof of Theorem 2.4 allows to change slightly the second eigenvalue of P_{Jor}, but the actual statement simplifies the result.

Combining with [3, Corollary 1.6], the time evolution equation associated to P_{Jor} satisfies the following property.

Corollary 2.5. *Consider P_{Jor} as in Theorem 2.4 with h small enough. For all $u_0 \in L^2(\mathbb{R}^2)$, the solution $u = e^{-tP_{\mathrm{Jor}}/h} u_0$ of (1.2) can be written*

$$e^{-tP_{\mathrm{Jor}}/h} u_0 = u_1 + te^{-t\lambda_2/h} u_2 + e^{-t\lambda_2/h} u_3 + \varepsilon(t),$$

with $u_1 = \Pi_{\lambda_1} u_0$, $u_2 = -\rho\langle e_2, \Pi_{\lambda_2} u_0\rangle e_1$, $u_3 = \Pi_{\lambda_2} u_0$ and

$$\|\varepsilon(t)\|_{L^2(\mathbb{R}^2)} \leq Ce^{-t/C} \|u_0\|_{L^2(\mathbb{R}^2)},$$

for some constant $C > 0$ independent of t, h, u_0.

In particular, we have the sharp return to equilibrium result

$$\|e^{-tP_{\mathrm{Jor}}/h} - \Pi_{\lambda_1}\| \sim \alpha te^{-t\lambda_2/h},$$

in the limit $t \to +\infty$ for h small enough and some positive constant $\alpha(h) > 0$. This estimate shows that the return to equilibrium is not purely exponentially decreasing in general and that some powers of t may appear.

Until now, we have only considered the spectral gap given by exponentially small eigenvalues, corresponding to several minima. But, if we study higher eigenvalues, it is more simple to have non-real spectrum. For $\varepsilon \in \mathbb{R}$, consider the operator

$$\mathscr{P} = -h^2\Delta + x^2 - 2h + \varepsilon(x_1 h\partial_{x_2} - x_2 h\partial_{x_1}).$$

It enters in the setting of (1.1) with the Morse function $f(x) = x^2/2$ which has a unique minimum at $x = 0$. The bottom of its spectrum is given as follows.

Proposition 2.6. *For $h > 0$ and $\varepsilon \neq 0$, we have*

$$\sigma(\mathscr{P}) \cap \{z \in \mathbb{C};\ \operatorname{Re} z < 4h\} = \{0, 2h + i\varepsilon h, 2h - i\varepsilon h\},$$

and these eigenvalues are simple.

Then, this operator has a non-real spectral gap. Nevertheless, it is not given by exponentially small eigenvalues responsible of metastable dynamics. In this simple well situation, the Eyring–Kramers law only provides the asymptotic of 0, the first eigenvalue of $\mathscr{P}$. Note also that, for $\varepsilon \neq 0$ fixed, these eigenvalues do no longer satisfy (1.3).

The rest of the paper is organized as follows. In the next section, we collect some properties of the reference operator P_0 used in the sequel. Section 4 is devoted to the construction of the anti-adjoint perturbation B based on properties of nodal sets. This construction allows to prove Theorem 2.2 (resp. Theorem 2.4) in Section 5 (resp. Section 6) combining perturbation theory and previous results of [3]. Lastly, Proposition 2.6 is obtained in Section 7 by direct computations.

3. Spectral properties of P_0

This part is devoted to the proof of Proposition 2.1 and to other technical results on P_0. From Theorem 1.1, there exists $\lambda_* > 0$ such that, for h small enough, P_0 has exactly three eigenvalues counted with multiplicity $0 = \lambda_1(h) < \lambda_2(h) \leq \lambda_3(h)$ in $]-\infty, \lambda_* h]$. Moreover, λ_2 and λ_3 are exponentially small. Eventually, the asymptotic

$$\lambda_2, \lambda_3 \sim \frac{3|\mu(s)||\det \operatorname{Hess} f(m)|^{1/2}}{\pi |\det \operatorname{Hess} f(s)|^{1/2}} h e^{-2S/h},$$

is a direct consequence of [19, Section 7C3] (see also [3, 14]). We denote

$$\Pi = \mathbb{1}_{[\lambda_1, \lambda_3]}(P_0), \quad \Pi_1 = \mathbb{1}_{\{\lambda_1\}}(P_0), \quad \Pi_{23} = \mathbb{1}_{[\lambda_2, \lambda_3]}(P_0),$$

the spectral projectors of P_0. They satisfy $\Pi = \Pi_1 + \Pi_{23}$,

$$\overline{\Pi_\bullet u} = \Pi_\bullet \bar{u} \quad \text{and} \quad R\Pi_\bullet = \Pi_\bullet R, \tag{3.1}$$

since P_0 commutes with R and the complex conjugation. Here, R is viewed as the rotation acting on functions (i.e., $R(f) = f \circ R$ for $f \in L^2(\mathbb{R}^d)$).

Let $\chi \in C_0^\infty(\mathbb{R}^2; [0,1])$ be supported near m_1 with $\chi = 1$ near m_1. We set

$$\psi_1 = \frac{\chi(x) e^{-f(x)/h}}{\|\chi e^{-f/h}\|}, \quad \psi_2 = \psi_1 \circ R, \quad \psi_3 = \psi_1 \circ R^2, \tag{3.2}$$

with the estimates

$$\|\chi e^{-f/h}\| \sim \sqrt{\pi h} (\det \operatorname{Hess} f(m))^{-1/4} \tag{3.3a}$$

and

$$\|e^{-f/h}\| \sim \sqrt{3\pi h} (\det \operatorname{Hess} f(m))^{-1/4}. \tag{3.3b}$$

Since f is invariant by rotation, these quantities do not depend on the minimum m where they are computed. The function ψ_j is localized near m_j from Figure 1, and the family $(\psi_j)_j$ is orthonormal. We then set

$$\varphi_j = \Pi \psi_j.$$

We have $\psi_j \in C_0^\infty(\mathbb{R}^2)$ and $\varphi_j \in \mathcal{S}(\mathbb{R}^2)$ since $\varphi_j \in \mathcal{D}(P_0^N)$ for all $N \in \mathbb{N}$. A classical result (see the proof of [11, Proposition 2.5]) yields

$$\varphi_j = \psi_j + \mathcal{O}(e^{-\delta/h}), \tag{3.4}$$

showing that the family $(\varphi_j)_j$ is an almost orthonormal basis of $\operatorname{Im} \Pi$. Furthermore, (3.1) implies that

$$\text{the function } \varphi_j \text{ is real for all } j \text{ and } \varphi_j = \varphi_k R^{j-k} \text{ for all } j, k. \tag{3.5}$$

Eventually, (3.2)–(3.4) give

$$\frac{\sqrt{3}}{\|e^{-f/h}\|}e^{-f/h} = \varphi_1 + \varphi_2 + \varphi_3 + \mathcal{O}(e^{-\delta/h}). \tag{3.6}$$

We can now show that $\lambda_2 = \lambda_3$.

Lemma 3.1. *For h small enough, the second eigenvalue λ_2 of P_0 has multiplicity* 2.

Proof. We prove this result by contradiction. Assume that λ_2 has multiplicity one and let u be a normalized eigenvector. From (1.4), we can always choose u real-valued. In the basis $(\varphi_j)_j$ of $\operatorname{Im}\Pi$, this function can be written

$$u = u_1\varphi_1 + u_2\varphi_2 + u_3\varphi_3, \tag{3.7}$$

for some $u_j \in \mathbb{R}$. Since $(\varphi_j)_j$ is almost orthonormal,

$$1 = \|u\|^2 = u_1^2 + u_2^2 + u_3^2 + \mathcal{O}(e^{-\delta/h}). \tag{3.8}$$

Applying the rotation R, (3.5) and (3.7) gives

$$u \circ R = u_1\varphi_2 + u_2\varphi_3 + u_3\varphi_1.$$

On the other hand, we have $P(u \circ R) = (Pu) \circ R = \lambda_2 u \circ R$. Since λ_2 is simple, there exists $\alpha \in \mathbb{C}$ such that $u \circ R = \alpha u$, that is

$$u_1 = \alpha u_2, \quad u_2 = \alpha u_3, \quad u_3 = \alpha u_1. \tag{3.9}$$

Since u and $u \circ R$ are real valued, we necessarily have $\alpha \in \mathbb{R}$. Relation (3.9) implies $u_j = \alpha^3 u_j$ for $j = 1, 2, 3$. Since at least one of the u_j is non-zero from (3.8), we get $\alpha^3 = 1$ and then $\alpha = 1$. Thus, $u_1 = u_2 = u_3$ and $|u_1| = 3^{-1/2} + \mathcal{O}(e^{-\delta/h})$. On the other hand, u and $e^{-f/h}$ are orthogonal since they belong to two different eigenspaces of the self-adjoint operator P_0. Combining the previous properties with (3.6), it comes

$$\begin{aligned} 0 &= |\langle \sqrt{3}\|e^{-f/h}\|^{-1}e^{-f/h}, u\rangle| \\ &= |u_1||\langle \varphi_1 + \varphi_2 + \varphi_3, \varphi_1 + \varphi_2 + \varphi_3\rangle| + \mathcal{O}(e^{-\delta/h}) \\ &= \sqrt{3} + \mathcal{O}(e^{-\delta/h}), \end{aligned}$$

which provides a contradiction for h small enough. We have just proved that λ_2 has multiplicity at least two. Since this multiplicity ccannot be larger than two, we get the lemma. ∎

4. Construction of the anti-adjoint perturbation B

The anti-adjoint part of P is chosen of the form $B = \varepsilon \mathcal{B}$ with

$$\mathcal{B} = d_f^* \circ G \circ d_f \quad \text{with } G = \begin{pmatrix} 0 & g \\ -g & 0 \end{pmatrix}, \tag{4.1}$$

with d_f defined in equation (2.1), for some constant $\varepsilon(h) \in]0, +\infty[$ and some function $g(x,h) \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$ fixed in the sequel.

Lemma 4.1. *The operator $\mathcal{B}$ is formally anti-adjoint, $\mathcal{B}e^{-f/h} = 0$ and*

$$\mathcal{B} = \frac{1}{2}(b \cdot h\nabla + h \operatorname{div} \circ b) \quad \textit{with } b(x,h) = \begin{pmatrix} h\partial_{x_2} g - 2g\partial_{x_2} f \\ -h\partial_{x_1} g + 2g\partial_{x_1} f \end{pmatrix} \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^2).$$

Proof. The definition of $\mathcal{B}$ given in (4.1) immediately implies that $\mathcal{B}$ is formally anti-adjoint and that $\mathcal{B}e^{-f/h} = 0$. Moreover, a direct computation gives

$$\begin{aligned}
\mathcal{B} &= (-h\partial_{x_1} + \partial_{x_1} f, -h\partial_{x_2} + \partial_{x_2} f)\begin{pmatrix} 0 & g \\ -g & 0 \end{pmatrix}\begin{pmatrix} h\partial_{x_1} + \partial_{x_1} f \\ h\partial_{x_2} + \partial_{x_2} f \end{pmatrix} \\
&= (-h\partial_{x_1} + \partial_{x_1} f)g(h\partial_{x_2} + \partial_{x_2} f) - (-h\partial_{x_2} + \partial_{x_2} f)g(h\partial_{x_1} + \partial_{x_1} f) \\
&= (\partial_{x_1} f)g(h\partial_{x_2}) + (h\partial_{x_2})g(\partial_{x_1} f) - (\partial_{x_2} f)g(h\partial_{x_1}) - (h\partial_{x_1})g(\partial_{x_2} f) \\
&\quad - (h\partial_{x_1})g(h\partial_{x_2}) + (h\partial_{x_2})g(h\partial_{x_1}) \\
&= \frac{1}{2}(b \cdot h\nabla + h \operatorname{div} b),
\end{aligned}$$

and the lemma follows. ∎

Since we see B as a perturbation of P_0 and want to use the Kato's theory, we seek the function $g \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$ such that $\Pi_{23}\mathcal{B}\Pi_{23} \neq 0$. For that, let (u,v) be a real-valued orthonormal basis of $\operatorname{Im}\Pi_{23}$. From (4.1), we have

$$\begin{aligned}
\langle \mathcal{B}u, v \rangle &= \int_{\mathbb{R}^2} G e^{-f/h} h\nabla e^{f/h} u \cdot e^{-f/h} h\nabla e^{f/h} v \, dx \\
&= \int_{\mathbb{R}^2} \tilde{g}(\partial_{x_2}\tilde{u}\partial_{x_1}\tilde{v} - \partial_{x_1}\tilde{u}\partial_{x_2}\tilde{v}) \, dx,
\end{aligned} \tag{4.2}$$

with

$$\tilde{u} = e^{f/h} u, \quad \tilde{v} = e^{f/h} v, \quad \tilde{g} = h^2 e^{-2f/h} g. \tag{4.3}$$

Thus, if $\partial_{x_2}\tilde{u}\partial_{x_1}\tilde{v} - \partial_{x_1}\tilde{u}\partial_{x_2}\tilde{v}$ does not vanish identically, it is possible to find g such that $\langle \mathcal{B}u, v \rangle \neq 0$. This justify the next intermediate result.

Lemma 4.2. *For h small enough, we have $\partial_{x_2}\tilde{u}\partial_{x_1}\tilde{v} - \partial_{x_1}\tilde{u}\partial_{x_2}\tilde{v} \not\equiv 0$.*

Proof. We prove this lemma by contradiction. If it does not hold true, we have

$$\partial_{x_2}\tilde{u}\partial_{x_1}\tilde{v} - \partial_{x_1}\tilde{u}\partial_{x_2}\tilde{v} \equiv 0, \tag{4.4}$$

for a sequence of positive h which goes to 0. Roughly speaking, this equation means that the level sets of $\tilde{u}$ and $\tilde{v}$ are the same. This leads to consider the nodal sets of u and v whose we recall now the general properties.

Proposition 4.3. *Let w be a real-valued eigenvector of P_0 associated to the eigenvalue λ_2 (in particular, $w \in \mathcal{S}(\mathbb{R}^2;\mathbb{R})$ and $w \not\equiv 0$). Then,*

(1) *the open set $\mathbb{R}^2 \setminus w^{-1}(0)$ has precisely two connected components $\Omega_\pm^w$ on which $\pm w > 0$,*

(2) *the nodal set $w^{-1}(0)$ is a (unique) smooth curve without crossing on which $\nabla w \neq 0$,*

(3) *if w_1 and w_2 are two of such eigenvectors, then $w_1^{-1}(0) \cap w_2^{-1}(0) \neq \emptyset$.*

Proof of Proposition 4.3. This result collects classical properties of nodal sets and we send the reader to the corresponding papers for the proofs. First, [7, Section VI.6] (see also [6]) shows that $\mathbb{R}^2 \setminus w^{-1}(0)$ has at most two connected components. This result, originally stated in domains, extends to our setting since the potential $|\nabla f|^2 - h\Delta f$ is confining because $f(x) = x^2$ outside a compact set. Moreover, if $\mathbb{R}^2 \setminus w^{-1}(0)$ has only one connected component, this function has a constant sign and cannot be orthogonal to the positive function $e^{-f/h}$, an eigenvector of P_0 associated to its first eigenvalue λ_1. Summing up, $\mathbb{R}^2 \setminus w^{-1}(0)$ has precisely two connected components.

The structure of the nodal set $w^{-1}(0)$ is described in [5, Theorem 2.5] in the present two-dimensional case (see also [5, Theorem 2.2] in the general case). Outside of isolated critical points, $w^{-1}(0)$ is the reunion of smooth curves without crossing on which $\nabla w \neq 0$. At the critical points, a finite number of nodal curves cross and form an equiangular system. If such a critical point exists, then there will be more than two connected components in $\mathbb{R}^2 \setminus w^{-1}(0)$. Thus, there is no critical point and $w^{-1}(0)$ is the reunion of smooth curves without crossing. Consider such a curve $t \mapsto x(t)$ and assume that $x(t)$ does not go to infinity as $t \to +\infty$. Then, it comes back in a bounded set for a sequence of arbitrarily large times. By compactness, there exist a sequence $(t_k)_{k\in\mathbb{N}}$ with $t_k \to +\infty$ as $k \to +\infty$ and $x_\infty \in \mathbb{R}^2$ such that $x(t_k) \to x_\infty$ as $k \to \infty$. Therefore, $x_\infty \in w^{-1}(0)$ and $w^{-1}(0)$ is a piece of curve $\mathcal{C}$ near x_∞. Eventually, $x(t_k) \in \mathcal{C}$ for k large enough and $t \mapsto x(t)$ is periodical. We have just proved that a curve in $w^{-1}(0)$ is either periodical or goes to infinity. In particular, each curve in $w^{-1}(0)$ generates a connected component in $\mathbb{R}^2 \setminus w^{-1}(0)$. Since this set has precisely two connected components, $w^{-1}(0)$ must be composed of a unique curve on which $\nabla w \neq 0$. Since w changes sign across $w^{-1}(0)$, the connected components of

$\mathbb{R}^2 \setminus w^{-1}(0)$ can be labeled $\Omega_\pm^w$ in a such way that $\pm w > 0$ on $\Omega_\pm^w$. This proves (1) and (2).

It remains to show (3). For that, we follow the proof of [5, Lemma 4.2]. Assume that $w_1^{-1}(0) \cap w_2^{-1}(0) = \emptyset$. Since $w_1^{-1}(0)$ is a single curve, we have $w_1^{-1}(0) \subset \Omega_-^{w_2}$ or $w_1^{-1}(0) \subset \Omega_+^{w_2}$. We can suppose that $w_1^{-1}(0) \subset \Omega_-^{w_2}$. Then, $\Omega_-^{w_1} \subsetneq \Omega_-^{w_2}$ or $\Omega_+^{w_1} \subsetneq \Omega_-^{w_2}$. We can suppose that $\Omega_-^{w_1} \subsetneq \Omega_-^{w_2}$. Eventually, by Courant's minimum principle, the first eigenvalue of the operator P_0 restricted to $\Omega_-^{w_1}$ with Dirichlet boundary condition is greater than the first eigenvalue of the operator P_0 restricted to $\Omega_-^{w_2}$ with Dirichlet boundary condition, whereas these two quantity are equal to λ_2. This is a contradiction and (3) follows. ∎

We now come back to the proof of Lemma 4.2. From (4.3), the zeros of $\tilde{u}$ (resp. $\tilde{v}$) are those of u (resp. v). Moreover, Proposition 4.3 (2) shows that

$$\nabla \tilde{u} = e^{f/h} \nabla u + u \nabla e^{f/h} = e^{f/h} \nabla u \neq 0, \tag{4.5}$$

on $\tilde{u}^{-1}(0)$. Let x_0 be a point of $\tilde{u}^{-1}(0) \cap \tilde{v}^{-1}(0)$ which is not empty from Proposition 4.3 (3), and consider the curve $x(t) \in \mathbb{R}^2$ solution of

$$\begin{cases} \partial_t x(t) = \begin{pmatrix} \partial_{x_2} \tilde{u}(x(t)) \\ -\partial_{x_1} \tilde{u}(x(t)) \end{pmatrix}, \\ x(0) = x_0. \end{cases}$$

The definition of $x(t)$ gives $\partial_t \tilde{u}(x(t)) = (\partial_{x_1} \tilde{u} \partial_{x_2} \tilde{u} - \partial_{x_2} \tilde{u} \partial_{x_1} \tilde{u})(x(t)) = 0$, showing that $\tilde{u}(x(t)) = 0$ for all $t \in \mathbb{R}$. Combined with Proposition 4.3 (2) and (4.5), it implies that $x(t)$ is a parametrization of $\tilde{u}^{-1}(0)$. On the other hand, (4.4) yields

$$\partial_t \tilde{v}(x(t)) = (\partial_{x_1} \tilde{v} \partial_{x_2} \tilde{u} - \partial_{x_2} \tilde{v} \partial_{x_1} \tilde{u})(x(t)) = 0,$$

showing as before that $\tilde{v}(x(t)) = \tilde{v}(x_0) = 0$ for all $t \in \mathbb{R}$. This proves $u^{-1}(0) = v^{-1}(0)$ from Proposition 4.3 (2). Using Proposition 4.3 (1), we deduce $\Omega_\pm^u = \Omega_\pm^v$ or $\Omega_\pm^u = \Omega_\mp^v$. It implies $\langle u, v \rangle > 0$ or $\langle u, v \rangle < 0$ respectively. On the other hand, we have $\langle u, v \rangle = 0$ since (u, v) is orthogonal. This contradiction finishes the proof of Lemma 4.2. ∎

Let $\chi \in C_0^\infty(\mathbb{R}^2; [0, 1])$ with $\operatorname{supp} \chi \subset B(0, 1)$ and $\chi = 1$ on $B(0, 1/2)$. From Lemma 4.2, there exists $x_0 = x_0(h) \in \mathbb{R}^2$ for h small enough such that

$$(\partial_{x_2} \tilde{u} \partial_{x_1} \tilde{v} - \partial_{x_1} \tilde{u} \partial_{x_2} \tilde{v})(x_0) \neq 0.$$

By continuity ($\tilde{u}, \tilde{v} \in C^\infty(\mathbb{R}^2)$), there exists $\nu = \nu(h) \in]0, 1]$ such that $\partial_{x_2} \tilde{u} \partial_{x_1} \tilde{v} - \partial_{x_1} \tilde{u} \partial_{x_2} \tilde{v}$ does not change it sign in $B(x_0, \nu)$. We then set

$$g(x, h) = \langle x_0 \rangle^{-1} e^{-1/\nu} \chi\Big(\frac{x - x_0}{\nu}\Big) \in C_0^\infty(\mathbb{R}^2; \mathbb{R}), \tag{4.6}$$

which satisfies, for h small enough and $\alpha \in \mathbb{N}^2$,

$$\forall x \in \mathbb{R}^2 \quad |\partial_x^\alpha g(x,h)| = \langle x_0 \rangle^{-1} \nu^{-|\alpha|} e^{-1/\nu} \Big| \chi^{(\alpha)}\Big(\frac{x - x_0}{\nu}\Big)\Big| \leq M_\alpha \langle x_0 \rangle^{-1} \tag{4.7}$$

for some constant $M_\alpha > 0$. Combining with Lemma 4.1 and $f = x^2$ outside a compact set, it shows that $b(x,h)$ is a symbol of class $S(1)$. Moreover, using $\tilde{g} = h^2 e^{-2f/h} g$ and (4.2), this construction yields

$$\langle \mathcal{B} u, v \rangle = \beta, \tag{4.8}$$

for h small enough and some constant $\beta(h) \neq 0$.

Lemma 4.4. *In any real-valued orthonormal basis of* $\operatorname{Im} \Pi_{23}$*, the operator* $\Pi_{23} \mathcal{B} \Pi_{23}$ *writes*

$$\Pi_{23} \mathcal{B} \Pi_{23} = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix},$$

for h small enough and some constant $\gamma(h) \in \mathbb{R} \setminus \{0\}$.

Proof. In a real-valued orthonormal basis (e_1, e_2) of $\operatorname{Im} \Pi_{23}$, we have

$$\Pi_{23} \mathcal{B} \Pi_{23} = \begin{pmatrix} \langle e_1, \mathcal{B} e_1 \rangle & \langle e_1, \mathcal{B} e_2 \rangle \\ \langle e_2, \mathcal{B} e_1 \rangle & \langle e_2, \mathcal{B} e_2 \rangle \end{pmatrix}.$$

Since e_1, e_2 are real-valued, (4.1) gives $\langle e_1, \mathcal{B} e_1 \rangle = \langle e_2, \mathcal{B} e_2 \rangle = 0$ and $\langle e_2, \mathcal{B} e_1 \rangle = -\langle e_1, \mathcal{B} e_2 \rangle$. Let us assume that $\langle e_1, \mathcal{B} e_2 \rangle = 0$ for a sequence of positive h which goes to 0. In that case, the previous relations imply $\langle e_j, \mathcal{B} e_k \rangle = 0$ for all $j, k \in \{1, 2\}$. Since (e_1, e_2) is a basis of $\operatorname{Im} \Pi_{23}$, it yields $\langle \mathcal{B} u, v \rangle = 0$ in contradiction with (4.8). Summing up, $\gamma(h) := \langle e_1, \mathcal{B} e_2 \rangle \neq 0$ for h small enough. ∎

5. Proof of Theorem 2.2

We now apply the perturbation theory for all h fixed small enough. Let $P_{\mathrm{com}} = P_0 + B$ with

$$B = \varepsilon \mathcal{B},$$

where $\mathcal{B}$ has been constructed in Section 4.

Proposition 5.1. *The operator P_{com} is closed on the domain of P_0. Moreover, for h small enough, there exist $\varepsilon_0(h) > 0$ and three analytic functions*

$$\varepsilon \mapsto \lambda_1(\varepsilon, h), \lambda_2(\varepsilon, h), \lambda_3(\varepsilon, h)$$

defined for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ *with* $\lambda_1(\varepsilon, h) = 0$*,*

$$\begin{cases} \lambda_2(\varepsilon, h) = \lambda_2(h) + i\gamma(h)\varepsilon + \mathcal{O}_h(\varepsilon^2), \\ \lambda_3(\varepsilon, h) = \lambda_2(h) - i\gamma(h)\varepsilon + \mathcal{O}_h(\varepsilon^2), \end{cases}$$

such that

$$\sigma(P_{\mathrm{com}}) \cap \{z \in \mathbb{C};\, \operatorname{Re} z < h\lambda_*/2\} = \{\lambda_1, \lambda_2, \lambda_3\},$$

for h *small enough and* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

In this statement, the notation $\mathcal{O}_h(1)$ designs a function bounded by a constant which may depend on h, and the eigenvalues are counted with multiplicity. The constant $\gamma(h) \in \mathbb{R} \setminus \{0\}$ is whose of Lemma 4.4.

Proof of Proposition 5.1. Since B is a relatively compact perturbation of P_0 from Lemma 4.1, the operator P_{com} is well defined and closed on the domain of P_0 (see [15, Theorem IV.1.11]). Moreover, $\varepsilon \mapsto P_0 + \varepsilon\mathcal{B}$ is a holomorphic family of unbounded operators in the sense of [15, Section VII]. Recall that $\lambda_1 = 0$ is a simple eigenvalue of P_0. From Lemma 3.1, λ_2 is a double eigenvalue which is semisimple since P_0 is self-adjoint. On the other hand, Lemma 4.4 shows that the eigenvalues of $\Pi_{23}\mathcal{B}\Pi_{23}$ are $\pm i\gamma$. Since $\gamma \neq 0$, these eigenvalues are different. Then, by the perturbation theory of spectrum, more precisely the perturbation theory of finite systems of eigenvalues (see [15, Section VII.1.3]) and the reduction process for semisimple eigenvalues (see [15, Section II.2.3]), there exist analytic functions $\varepsilon \mapsto \lambda_1(\varepsilon, h), \lambda_2(\varepsilon, h), \lambda_3(\varepsilon, h)$ defined for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ with $\varepsilon_0(h) > 0$ such that

$$\begin{cases} \lambda_1(0, h) = 0, \\ \lambda_2(\varepsilon, h) = \lambda_2(h) + i\gamma(h)\varepsilon + \mathcal{O}_h(\varepsilon^2), \\ \lambda_3(\varepsilon, h) = \lambda_2(h) - i\gamma(h)\varepsilon + \mathcal{O}_h(\varepsilon^2), \end{cases}$$

and $\sigma(P_{\mathrm{com}}) \cap \{z \in \mathbb{C};\, \operatorname{Re} z < h\lambda_*/2\} = \{\lambda_1, \lambda_2, \lambda_3\}$. Since 0 is always an eigenvalue of P_{com} by Lemma 4.1, we have $\lambda_1(\varepsilon, h) = 0$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ after a possible shrinking of ε_0. ∎

The asymptotic expansions in Proposition 5.1 and $\gamma(h) \in \mathbb{R} \setminus \{0\}$ yield that

$$\operatorname{Im} \lambda_2(\varepsilon, h) \neq 0 \quad \text{and} \quad \operatorname{Im} \lambda_3(\varepsilon, h) \neq 0,$$

for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1] \setminus \{0\}$ with $\varepsilon_1(h) > 0$ small enough. We eventually choose

$$\varepsilon(h) = \min(\varepsilon_0(h), \varepsilon_1(h), r(h)). \tag{5.1}$$

Thus, $b(x, h)$ is a symbol of order at most $r(h)$ from Lemma 4.1 and (4.7). In the domain $\{z \in \mathbb{C};\ \operatorname{Re} z < \lambda_* h/2\}$, P_{com} has three eigenvalues $\mu_1(h) = 0$, $\mu_2(h) =$

$\lambda_2(\varepsilon(h), h)$, and $\mu_3(h) = \lambda_3(\varepsilon(h), h)$ with $\operatorname{Im} \mu_2 \neq 0$ and $\operatorname{Im} \mu_3 \neq 0$. From (1.4), we automatically have $\mu_3 = \overline{\mu_2}$. Finally, we can write

$$P_{\mathrm{com}} - z = (B(P_0 - z)^{-1} - 1)(P_0 - z),$$

for $z \in B(0,1) \setminus \sigma(P_0)$ with

$$B(P_0 - z)^{-1} = B(P_0 + i)^{-1}(1 + (z + i)(P_0 - z)^{-1}) = \mathcal{O}(r \operatorname{dist}(z, \sigma(P_0))^{-1}),$$

from Lemma 4.1. If $\operatorname{dist}(z, \sigma(P_0)) \geq Mr$ with $M > 1$ large enough, then one has that $\| B(P_0 - z)^{-1} \| \leq 1/2$ and $P_{\mathrm{com}} - z$ is invertible. It implies that $\mu_2 = \lambda_2 + \mathcal{O}(r)$ and finishes the proof of Theorem 2.2.

6. Proof of Theorem 2.4

To find a setting with a Jordan block, we consider operators of the form

$$P_\tau = P_0 + \tau_1 P_1 + \tau_2 P_2 + \tau_3 P_3 + \tau_4 \mathcal{B},$$

where $\tau = (\tau_1, \tau_2, \tau_3, \tau_4) \in \mathbb{R}^4$, $\mathcal{B}$ has been constructed in Section 4 and P_ν is as follows. For $\nu = 1, 2, 3$, let $\chi_\nu \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$ be supported near s_ν and equal to 1 in a neighborhood of s_ν. We also assume that $\chi_2 = \chi_1 \circ R$ and $\chi_3 = \chi_1 \circ R^2$ (see Figure 1). Then, P_ν is defined by

$$P_\nu = d_f^* \circ \chi_\nu \circ d_f. \tag{6.1}$$

Summing up, P_τ can be written

$$P_\tau = d_f^* \circ \chi \circ d_f + \tau_4 \mathcal{B},$$

with $\chi = 1 + \tau_1 \chi_1 + \tau_2 \chi_2 + \tau_3 \chi_3$. Thus, for τ small enough, P_τ enters into the setting of (1.1).

For $j = 1, 2, 3$, let $\phi_j(x)$ denote the global quasimode of P_0 supported near the connected component of $\{f < f(s)\}$ containing m_j constructed in [3, Section 4] (see also [17]). More precisely, this real-valued function can be written

$$\phi_j(x) = \theta_j(x)(v_j(x) + 1)e^{-f(x)/h} = \tilde{v}_j(x)e^{-f(x)/h}, \tag{6.2}$$

where $\theta_j \in C_0^\infty(\mathbb{R}^2)$ is a plateau function near the connected component of $\{f < f(s)\}$ containing m_j and $v_j \in C^\infty(\mathbb{R}^2)$ is given near the support of θ_j by

$$v_j(x) = \begin{cases} C_0^{-1} \displaystyle\int_0^{\ell_s^j(x,h)} \zeta(r) e^{-r^2/2h} \, dr & \text{near } s, \text{ one of the two saddle points close to } m_j. \\ 1 & \text{outside,} \end{cases} \tag{6.3}$$

Here, we say that a saddle point s is "close to a minimum m" if s is in the closure of the connected component of $\{f < f(s)\}$ containing m (see Figure 1). The function $\zeta \in C_0^\infty(\mathbb{R}; [0,1])$ is even and satisfies $\zeta(r) = 1$ for r near 0,

$$C_0 = \int_0^{+\infty} \zeta(r) e^{-r^2/2h} \, dr = \sqrt{\frac{\pi h}{2}} (1 + \mathcal{O}(e^{-\delta/h})).$$

The function $\ell_s^j(x,h) \simeq \ell_{s,0}^j(x) + \ell_{s,1}^j(x)h + \cdots$ is smooth with $\ell_{s,0}^j(s) = 0$ and $\nabla \ell_{s,0}^j(s) \neq 0$. As in (3.2), we can make these constructions so that

$$\phi_2 = \phi_1 \circ R \quad \text{and} \quad \phi_3 = \phi_1 \circ R^2. \tag{6.4}$$

We choose χ_ν in (6.1) such that $\chi_\nu = 1$ near the support of $\theta_j \nabla v_j$ if s_ν is close to m_j. By comparison with Section 3, we have $\phi_j = 2\psi_j + \mathcal{O}(e^{-\delta/h})$ for some $\delta > 0$, but ϕ_j is a better quasimode than ψ_j (see Lemma 6.1 below).

We define the geometric quantities $S = f(s) - f(m)$ and

$$C_1 = \frac{2|\mu(s)|}{|\det \operatorname{Hess} f(s)|^{1/2}},$$

where $\mu(s)$ is given above Proposition 2.1. The quasimodes ϕ_j's satisfy the following property.

Lemma 6.1. *For all $\nu, j, k \in \{1,2,3\}$, we have*

$$\langle P_\nu \phi_j, \phi_k \rangle \sim \begin{cases} C_1 h^2 e^{-2S/h} & \text{if } s_\nu \text{ is close to } m_j = m_k, \\ -C_1 h^2 e^{-2S/h} & \text{if } s_\nu \text{ is close to } m_j \neq m_k, \\ 0 & \text{otherwise,} \end{cases} \tag{6.5}$$

$$\|P_0 \phi_j\|^2 = \mathcal{O}(h^\infty) e^{-2S/h} \quad \text{and} \quad \|P_\nu \phi_j\|^2 = \mathcal{O}(h^\infty) e^{-2S/h}. \tag{6.6}$$

Here, the notation "s_ν is close to $m_j \neq m_k$" means that $j \neq k$ and that s_ν is close to m_j and m_k. Roughly speaking, it means that s_ν is between m_j and m_k (see Figure 1).

Proof. This result is similar to [3, Proposition 5.1] (see also [17, Section 4B]). We only explain here the ideas of the proof and the necessary changes, and we send the reader to [3] for the details.

Combining (6.1) and (6.2) leads to

$$\langle P_\nu \phi_j, \phi_k \rangle = \langle \chi_\nu d_f \phi_j, d_f \phi_k \rangle = \langle \chi_\nu e^{-f/h} h \nabla \tilde{v}_j, e^{-f/h} h \nabla \tilde{v}_k \rangle. \tag{6.7}$$

Using $\nabla\tilde{v}_j = (v_j+1)\nabla\theta_j + \theta_j\nabla v_j$ and $e^{-f/h} = \mathcal{O}(e^{-(S+\delta)/h})$ on the support of $(v_j+1)\nabla\theta_j$, the previous equation becomes

$$\langle P_\nu\phi_j, \phi_k\rangle = h^2\int \chi_\nu\theta_j\theta_k\nabla v_j\cdot\nabla v_k e^{-2f/h}dx + \mathcal{O}(e^{-2(S+\delta)/h}). \tag{6.8}$$

From (6.3), we have on the support of θ_j

$$\nabla v_j = \sum_{s \text{ close to } m_j} C_0^{-1}\zeta(\ell_s^j)e^{-(\ell_s^j)^2/2h}\nabla\ell_s^j.$$

If s_ν is close to $m_j = m_k$, (6.8) writes

$$\langle P_\nu\phi_j, \phi_k\rangle = h^2C_0^{-2}\int \theta_j^2\zeta(\ell_{s_\nu}^j)^2|\nabla\ell_{s_\nu}^j|^2e^{-2(f+\frac{(\ell_{s_\nu}^j)^2}{2})/h}dx + \mathcal{O}(e^{-2(S+\delta)/h}).$$

The asymptotic of such an integral has been obtained in [3, equation (5.5)] using the Laplace method. This computation gives

$$\langle P_\nu\phi_j, \phi_k\rangle \sim C_1h^2e^{-2S/h}, \tag{6.9}$$

when s_ν is close to $m_j = m_k$. Assume now that s_ν is close to m_j and m_k with $m_j \neq m_k$. In this case, we have $\ell_{s_\nu}^j = -\ell_{s_\nu}^k$ (see [3, discussion below equation (4.6)]). Then, (6.8) and the parity of ζ give

$$\langle P_\nu\phi_j, \phi_k\rangle = -h^2C_0^{-2}\int \theta_j\theta_k\zeta(\ell_{s_\nu}^j)^2|\nabla\ell_{s_\nu}^j|^2e^{-2(f+\frac{(\ell_{s_\nu}^j)^2}{2})/h}dx + \mathcal{O}(e^{-2(S+\delta)/h}).$$

As before, the Laplace method implies

$$\langle P_\nu\phi_j, \phi_k\rangle \sim -C_1h^2e^{-2S/h}, \tag{6.10}$$

when s_ν is close to $m_j \neq m_k$. Finally, if s_ν is not close to m_j or m_k, we directly get from (6.7) and the support properties of χ_ν, θ_j and θ_k that

$$\langle P_\nu\phi_j, \phi_k\rangle = 0, \tag{6.11}$$

in that case. Summing up, (6.5) follows from (6.9), (6.10), and (6.11).

It remains to show (6.6). The first estimate is a direct consequence of [3, Proposition 5.1 (ii) and (iii)]. On the other hand, using (6.2) and $P_\nu e^{-f/h} = 0$, we deduce

$$P_\nu\phi_j = [P_\nu, \theta_j](v_j+1)e^{-f/h} + \theta_jP_\nu(v_je^{-f/h}).$$

Since $e^{-f/h} = \mathcal{O}(e^{-(S+\delta)/h})$ on the support of $(v_j+1)\nabla\theta_j$, the first term is $\mathcal{O}(e^{-(S+\delta)/h})$ in L^2 norm. Concerning the second term, we remark that χ_ν is constant (equal to 0 or 1) near each connected component of the support of $\theta_j\nabla v_j$ if the support of θ_j has been chosen sufficiently close to the connected component of $\{f < f(s)\}$ containing m_j. Then, (6.1) and (6.3) lead to

$$\begin{aligned}\theta_j P_\nu(v_j e^{-f/h}) &= \theta_j d_f^* \chi_\nu d_f v_j e^{-f/h} = \theta_j d_f^* \chi_\nu e^{-f/h} h\nabla v_j \\ &= \chi_\nu \theta_j d_f^* e^{-f/h} h\nabla v_j = \chi_\nu \theta_j P_0(v_j e^{-f/h}).\end{aligned}$$

It is proved in [3, below equation (5.6)] that $\theta_j P_0(v_j e^{-f/h}) = \mathcal{O}(h^\infty)e^{-S/h}$. Summing up,

$$P_\nu \phi_j = \mathcal{O}(h^\infty)e^{-S/h},$$

and (6.6) follows. ∎

We construct a basis of the 2-dimensional spectral space of P_τ associated to the eigenvalues close to λ_2. We set

$$\begin{cases} \tilde{e}_1(x) = \dfrac{1}{\sqrt{8}\|e^{-f/h}\|}(2\phi_1(x) - \phi_2(x) - \phi_3(x)), \\ \tilde{e}_2(x) = \dfrac{\sqrt{3}}{\sqrt{8}\|e^{-f/h}\|}(\phi_2(x) - \phi_3(x)), \end{cases} \tag{6.12}$$

with $\|e^{-f/h}\|$ estimated in (3.3b). The idea behind this choice of functions is that

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\-1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}, \tag{6.13}$$

form an orthonormal basis of $\mathbb{R}^3$ (the first vector corresponding to $\|e^{-f/h}\|^{-1}e^{-f/h}$). For $\nu \in \{1,2,3\}$, let $\widetilde{\mathcal{P}}_\nu \in M_{2\times 2}(\mathbb{R})$ be the matrix of coefficients

$$(\widetilde{\mathcal{P}}_\nu)_{j,k} = C_2^{-1}h^{-1}e^{2S/h}\langle P_\nu \tilde{e}_j, \tilde{e}_k\rangle \quad \text{with } C_2 = \frac{|\mu(s)||\det \operatorname{Hess} f(m)|^{1/2}}{4\pi|\det \operatorname{Hess} f(s)|^{1/2}}.$$

The asymptotic of these matrices are provided by the next result.

Lemma 6.2. *For all $j,k \in \{1,2\}$, we have*

$$\langle \tilde{e}_j, \tilde{e}_k\rangle = \delta_{j,k} + \mathcal{O}(e^{-\delta/h}). \tag{6.14}$$

Moreover, the matrices $\widetilde{\mathcal{P}}_\nu$ satisfy modulo $o(1)$ terms

$$\widetilde{\mathcal{P}}_1 = \begin{pmatrix}3 & -\sqrt{3}\\ -\sqrt{3} & 1\end{pmatrix}, \quad \widetilde{\mathcal{P}}_2 = \begin{pmatrix}0 & 0\\ 0 & 4\end{pmatrix}, \quad \widetilde{\mathcal{P}}_3 = \begin{pmatrix}3 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}. \tag{6.15}$$

Proof. From (6.2), we have $\phi_\nu = 2e^{-f/h}$ near m_ν and $\phi_\nu = \mathcal{O}(e^{-\delta/h})$ outside. It implies

$$\|e^{-f/h}\|^{-2}\langle\phi_j, \phi_k\rangle = \frac{4}{3}\delta_{j,k} + \mathcal{O}(e^{-\delta/h}),$$

thanks to (3.3). Combining this relation with (6.12), we get (6.14).

To show (6.15), it is enough to combine (6.5) and (6.12). For instance,

$$\begin{aligned}\langle P_3\tilde{e}_1,\tilde{e}_2\rangle &= \frac{\sqrt{3}}{8\|e^{-f/h}\|^2}\langle P_3(2\phi_1-\phi_2-\phi_3),(\phi_2-\phi_3)\rangle\\ &= \frac{\sqrt{3}}{8\|e^{-f/h}\|^2}\big(2\langle P_3\phi_1,\phi_2\rangle - 2\langle P_3\phi_1,\phi_3\rangle - \langle P_3\phi_2,\phi_2\rangle\\ &\qquad + \langle P_3\phi_2,\phi_3\rangle - \langle P_3\phi_3,\phi_2\rangle + \langle P_3\phi_3,\phi_3\rangle\big)\\ &= \frac{\sqrt{3}C_1h^2e^{-2S/h}}{8\|e^{-f/h}\|^2}(0+2+0+0+0+1)+o(he^{-2S/h})\\ &= \sqrt{3}C_2he^{-2S/h}+o(he^{-2S/h}),\end{aligned}$$

thanks to (3.3) and

$$C_2 = \frac{C_1|\det \operatorname{Hess} f(m)|^{1/2}}{8\pi}.$$

This provides the desired asymptotic of $(\widetilde{\mathscr{P}}_3)_{1,2}$. The other coefficients can be computed the same way. ∎

We now apply perturbation theory for h fixed small enough. For τ small enough (depending on h), P_τ has 3 small eigenvalues counted with multiplicity: $\lambda_1^\tau = 0$ associated to the eigenspace $e^{f/h}\mathbb{C}$ and $\lambda_2^\tau, \lambda_3^\tau$ such that $\lambda_j^\tau \to \lambda_2$ as $\tau \to 0$ for $j = 2, 3$. Mimicking Section 3, we introduce the spectral projectors

$$\Pi^\tau = \mathbb{1}_{\{\lambda_1,\lambda_2,\lambda_3\}}(P_\tau), \quad \Pi_1^\tau = \mathbb{1}_{\{\lambda_1\}}(P_\tau), \quad \Pi_{23}^\tau = \mathbb{1}_{\{\lambda_2,\lambda_3\}}(P_\tau),$$

which satisfy $\Pi^\tau = \Pi_1^\tau + \Pi_{23}^\tau$, $\Pi_1^\tau = \Pi_1$ and $\overline{\Pi_\bullet^\tau u} = \Pi_\bullet^\tau \bar{u}$ as in (3.1). Moreover, $\tau \to \Pi_\bullet^\tau$ is analytic in a real neighborhood of 0 which may depend on h. We define

$$\hat{e}_1^\tau = \Pi_{23}^\tau\tilde{e}_1 \quad \text{and} \quad \hat{e}_2^\tau = \Pi_{23}^\tau\tilde{e}_2.$$

for $j = 1, 2$. These real-valued functions satisfy the following property.

Lemma 6.3. *For $j = 1, 2$ and τ small enough (depending on h), we have*

$$\hat{e}_j^\tau = \tilde{e}_j + \mathscr{O}(h^\infty)e^{-S/h},$$

in $H^2(\mathbb{R}^2)$.

Proof. Using the Cauchy formula, we can write

$$\begin{aligned}\hat{e}_j^\tau &= \Pi_{23}\tilde{e}_j - \frac{1}{2i\pi}\oint_\gamma (P_\tau - z)^{-1}\,dz\,\tilde{e}_j + \frac{1}{2i\pi}\oint_\gamma (P_0-z)^{-1}\,dz\,\tilde{e}_j\\ &= \Pi_{23}\tilde{e}_j + \frac{1}{2i\pi}\oint_\gamma (P_\tau-z)^{-1}(P_\tau-P_0)(P_0-z)^{-1}\,dz\,\tilde{e}_j,\end{aligned}$$

where γ is a simple loop around λ_2, oriented counterclockwise, which depends on h but not on τ. This implies

$$\hat{e}_j^\tau = \Pi_{23}\tilde{e}_j + \mathcal{O}_h(\tau) = \Pi_{23}\tilde{e}_j + \mathcal{O}(h^\infty)e^{-S/h}, \tag{6.16}$$

in $H^2(\mathbb{R}^2)$ for τ small enough.

Since $\Pi_{23} = \Pi - \Pi_1$, (6.16) gives

$$\hat{e}_j^\tau = \Pi\tilde{e}_j - \Pi_1\tilde{e}_j + \mathcal{O}(h^\infty)e^{-S/h}.$$

Using (6.4) and that f is invariant under R, we get

$$\begin{aligned}
\Pi_1\tilde{e}_1 &= e^{-f/h}\frac{1}{\sqrt{8}\|e^{-f/h}\|^3}\langle e^{-f/h}, 2\phi_1 - \phi_2 - \phi_3\rangle \\
&= e^{-f/h}\frac{1}{\sqrt{8}\|e^{-f/h}\|^3}\langle e^{-f/h}, 2\phi_1 - \phi_1 R - \phi_1 R^2\rangle \\
&= e^{-f/h}\frac{1}{\sqrt{8}\|e^{-f/h}\|^3}\langle e^{-f/h}, 2\phi_1 - \phi_1 - \phi_1\rangle = 0.
\end{aligned}$$

The same way, $\Pi_1\tilde{e}_2 = 0$. These relations follow in fact from (6.13). Summing up,

$$\hat{e}_j^\tau = \Pi\tilde{e}_j + \mathcal{O}(h^\infty)e^{-S/h}, \tag{6.17}$$

in $H^2(\mathbb{R}^2)$ for τ small enough.

Finally, we work as in [17, Lemma 4.9] to remove the projector Π (see also [3, equation (5.13)]). The Cauchy formula gives

$$\begin{aligned}
\Pi\tilde{e}_j &= \tilde{e}_j - \frac{1}{2i\pi}\oint_{\partial B(0,\lambda_* h/2)} (P_0 - z)^{-1}\, dz\, \tilde{e}_j - \frac{1}{2i\pi}\oint_{\partial B(0,\lambda_* h/2)} z^{-1}\, dz\, \tilde{e}_j \\
&= \tilde{e}_j - \frac{1}{2i\pi}\oint_{\partial B(0,\lambda_* h/2)} z(P_0 - z)^{-1}\, dz\, P_0\tilde{e}_j.
\end{aligned}$$

Combining with (6.6), (6.12), and (6.17), the lemma follows. ■

From (6.12), (6.13), and Lemma 6.3, $(\hat{e}_1^\tau, \hat{e}_2^\tau)$ is almost orthonormal and then is a basis of $\operatorname{Im}\Pi_{23}^\tau$. We orthonormalize $(\hat{e}_1^\tau, \hat{e}_2^\tau)$ into (e_1^τ, e_2^τ) by the Gram–Schmidt process. It means

$$e_1^\tau = \|\hat{e}_1^\tau\|^{-1}\hat{e}_1^\tau \quad \text{and} \quad e_2^\tau = \|\hat{e}_2^\tau - \langle e_1^\tau, \hat{e}_2^\tau\rangle e_1^\tau\|^{-1}(\hat{e}_2^\tau - \langle e_1^\tau, \hat{e}_2^\tau\rangle e_1^\tau). \tag{6.18}$$

In particular, (e_1^τ, e_2^τ) is a orthonormal basis of $\operatorname{Im}\Pi_{23}^\tau$, e_j^τ is real-valued and $\tau \to e_j^\tau$ is analytic for $j = 1, 2$ and τ near 0. We now define the interaction matrix $Q(\tau)$. More precisely,

let $Q(\tau)$ be the matrix of the operator $\Pi_{23}^\tau P_\tau \Pi_{23}^\tau$ expressed in the basis (e_1^τ, e_2^τ).

More prosaically, it means that $Q_{j,k}(\tau) = \langle e_j^\tau, P_\tau e_k^\tau \rangle$. From the previous discussion, $Q(\tau)$ is well defined for τ small,

$$Q \in C^\infty(\mathbb{R}^4; M_{2\times 2}(\mathbb{R})), \tag{6.19}$$

and $Q(0) = \lambda_2 \operatorname{Id}$. Moreover, its partial derivatives satisfy the following property.

Lemma 6.4. *We have*

$$\partial_{\tau_\nu} Q(0) = C_2 h e^{-2S/h} (\widetilde{\mathcal{P}}_\nu + o(1)), \tag{6.20}$$

for $\nu = 1, 2, 3$ *and*

$$\partial_{\tau_4} Q(0) = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \tag{6.21}$$

where the $\widetilde{\mathcal{P}}_\nu$*'s are defined in Lemma 6.2 and* $\gamma(h) \in \mathbb{R} \setminus \{0\}$ *is the constant given by Lemma 4.4 and associated to the basis* (e_1^0, e_2^0) *of* $\operatorname{Im} \Pi_{23}$.

Proof. We compute these derivatives using the classical trick in the reduction process (see [15, Section II.2.3]). We can write

$$\begin{aligned} \partial_{\tau_\nu} Q_{j,k}(0) &= \partial_{\tau_\nu} (Q_{j,k} - \delta_{j,k}\lambda_2)(0) = \partial_{\tau_\nu} \langle e_j^\tau, (P_\tau - \lambda_2) e_k^\tau \rangle (0) \\ &= \langle (\partial_{\tau_\nu} e_j^0), (P_0 - \lambda_2) e_k^0 \rangle + \langle e_j^0, (\partial_{\tau_\nu} P_\tau)(0) e_k^0 \rangle \\ &\quad + \langle e_j^0, (P_0 - \lambda_2)(\partial_{\tau_\nu} e_k^0) \rangle \\ &= \langle e_j^0, (\partial_{\tau_\nu} P_\tau)(0) e_k^0 \rangle, \end{aligned} \tag{6.22}$$

since $(P_0 - \lambda_2) e_j^0 = (P_0 - \lambda_2) e_k^0 = 0$. For $\nu = 1, 2, 3$, (6.22) gives

$$\partial_{\tau_\nu} Q_{j,k}(0) = \langle e_j^0, P_\nu e_k^0 \rangle, \tag{6.23}$$

with P_ν defined in (6.1).

From (6.14) and Lemma 6.3, we have

$$\| \hat{e}_1^\tau \| = 1 + \mathcal{O}(e^{-\delta/h}) \quad \text{and} \quad \| \hat{e}_2^\tau - \langle e_1^\tau, \hat{e}_2^\tau \rangle e_1^\tau \| = 1 + \mathcal{O}(e^{-\delta/h}).$$

Using again Lemma 6.3, (6.18) becomes

$$\begin{cases} e_1^\tau = \tilde{e}_1 + \mathcal{O}(e^{-\delta/h}) \tilde{e}_1 + \mathcal{O}(h^\infty) e^{-S/h}, \\ e_2^\tau = \tilde{e}_2 + \mathcal{O}(e^{-\delta/h}) \tilde{e}_1 + \mathcal{O}(e^{-\delta/h}) \tilde{e}_2 + \mathcal{O}(h^\infty) e^{-S/h}, \end{cases} \tag{6.24}$$

where the $\mathcal{O}(e^{-\delta/h})$'s are constants. Then, equation (6.20) follows from (6.15), (6.23), and (6.24).

On the other hand, (6.23) gives $\partial_{\tau_4} Q_{j,k}(0) = \langle e_j^0, \mathcal{B} e_k^0 \rangle$. In other words, $\partial_{\tau_4} Q(0)$ is the operator $\Pi_{23} \mathcal{B} \Pi_{23}$ expressed in the basis (e_1^0, e_2^0) of $\operatorname{Im} \Pi_{23}$. Then, (6.21) follows directly from Lemma 4.4. ∎

Proof of Theorem 2.4. From Lemma 6.4 and (6.15), $(\partial_{\tau_\nu} Q(0))_{\nu=1,2,3,4}$ is a basis of $M_{2\times 2}(\mathbb{R})$. Thus,

$$d_0 Q\colon \mathbb{R}^4 \simeq T_0\mathbb{R}^4 \to T_{\lambda_2 \operatorname{Id}} M_{2\times 2}(\mathbb{R}) \simeq M_{2\times 2}(\mathbb{R}),$$

is an isomorphism. By the inverse function theorem, $\tau \mapsto Q(\tau)$ is a local diffeomorphism from a neighborhood of 0 to a neighborhood of $\lambda_2 \operatorname{Id}$, for h small enough. Note that the neighborhoods may depend on h. Then, there exists $\tau(h) \in \mathbb{R}^4$ with $|\tau| < r$ such that

$$Q(\tau) = \begin{pmatrix} \lambda_2 & \rho \\ 0 & \lambda_2 \end{pmatrix}, \tag{6.25}$$

for some $\rho(h) \neq 0$. Since Q is the operator P_τ restricted to its stable eigenspace $\operatorname{Im} \Pi_{23}^\tau$ in the basis (e_1^τ, e_2^τ), (6.25) shows that $P_{\mathrm{Jor}} := P_\tau$ has a non-trivial Jordan block associated with the eigenvalue λ_2 and Theorem 2.4 follows. ∎

7. Proof of Proposition 2.6

We write $\mathcal{P} = \mathcal{P}_0 + \varepsilon\mathcal{B}$ with

$$\mathcal{P}_0 = -h^2\Delta + x^2 - 2 \quad \text{and} \quad \mathcal{B} = x_1 h\partial_{x_2} - x_2 h\partial_{x_1}.$$

The operator $\mathcal{P}_0$ is the harmonic oscillator

$$\mathcal{P}_0 = d_f^* \circ d_f = a_1^* \circ a_1 + a_2^* \circ a_2 - 2,$$

with the creation operators

$$a_j^* = -h\partial_{x_j} + x_j$$

and annihilation operators

$$a_j = h\partial_{x_j} + x_j.$$

The spectrum of $\mathcal{P}_0$ is $2h\mathbb{N}$ and the eigenspace associated to $2nh$ is the $(n+1)$-dimensional space

$$E_n = \operatorname{Vect}\{a_1^{*k} a_2^{*n-k} e^{-f/h};\ k = 0, \dots, n\}.$$

A direct computation gives

$$[\mathcal{B}, a_1^*] = -h a_2^* \quad \text{and} \quad [\mathcal{B}, a_2^*] = h a_1^*,$$

showing that E_n is stable by $\mathcal{B}$. Let $\mathcal{B}_n$ denote the restriction of $\mathcal{B}$ to E_n. Summing up the previous arguments, we deduce

$$\sigma(\mathcal{P}) = \bigcup_{n=0}^{+\infty} 2nh + \varepsilon\sigma(\mathcal{B}_n), \tag{7.1}$$

where $\sigma(\mathcal{B}_n) \subset i\mathbb{R}$ since $\mathcal{B}_n$ is anti-adjoint as $\mathcal{B}$.

To get the spectral gap of $\mathcal{P}$, it remains to compute $\sigma(\mathcal{B}_1)$. In the basis $(x_1 e^{-f/h}, x_2 e^{-f/h})$ of E_1, the matrix of $\mathcal{B}_1$ takes the form

$$\mathcal{B}_1 = \begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix}.$$

Thus, $\sigma(\mathcal{B}_1) = \{ih, -ih\}$ and Proposition 2.6 follows from (7.1).

References

[1] A. Benbernou, N. Boussekkine, N. Mecherout, T. Ramond, and J. Sjöstrand, Non-real eigenvalues for $\mathcal{PT}$-symmetric double wells. *Lett. Math. Phys.* **106** (2016), no. 12, 1817–1835 Zbl 1372.35201 MR 3569646

[2] N. Berglund, Kramers' law: validity, derivations and generalisations. *Markov Process. Related Fields* **19** (2013), no. 3, 459–490 Zbl 1321.58035 MR 3156961

[3] J.-F. Bony, D. L. Peutrec, and L. Michel, Eyring–Kramers law for Fokker–Planck type differential operators. *J. Eur. Math. Soc. (JEMS)* (2024), DOI 10.4171/jems/1461

[4] A. Bovier, V. Gayrard, and M. Klein, Metastability in reversible diffusion processes. II. Precise asymptotics for small eigenvalues. *J. Eur. Math. Soc. (JEMS)* **7** (2005), no. 1, 69–99 Zbl 1105.82025 MR 2120991

[5] S. Y. Cheng, Eigenfunctions and nodal sets. *Comment. Math. Helv.* **51** (1976), no. 1, 43–55 Zbl 0334.35022 MR 0397805

[6] R. Courant, *Ein allgemeiner Satz zur Theorie der Eigenfunktionen selbstadjungierter Differentialausdrücke*, Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. **1923** (1923), 81–84. JFM 49.0342.01

[7] R. Courant and D. Hilbert, *Methods of mathematical physics.* Vol. I. Interscience Publishers, Inc., New York, 1953 Zbl 0053.02805 MR 0065391

[8] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*. Study edn., Texts Monogr. Phys., Springer, Berlin, 1987 Zbl 0619.47005 MR 0883643

[9] G. Di Gesù, T. Lelièvre, D. Le Peutrec, and B. Nectoux, Sharp asymptotics of the first exit point density. *Ann. PDE* **5** (2019), no. 1, article no. 5 Zbl 1431.60087 MR 3975562

[10] B. Helffer, M. Klein, and F. Nier, Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach. *Mat. Contemp.* **26** (2004), 41–85 Zbl 1079.58025 MR 2111815

[11] B. Helffer and J. Sjöstrand, Multiple wells in the semiclassical limit. I. *Comm. Partial Differential Equations* **9** (1984), no. 4, 337–408 Zbl 0546.35053 MR 0740094

[12] B. Helffer and J. Sjöstrand, Puits multiples en mécanique semi-classique. IV. étude du complexe de Witten. *Comm. Partial Differential Equations* **10** (1985), no. 3, 245–340 Zbl 0597.35024 MR 0780068

[13] F. Hérau, M. Hitrik, and J. Sjöstrand, Tunnel effect for Kramers-Fokker-Planck type operators. *Ann. Henri Poincaré* **9** (2008), no. 2, 209–274 Zbl 1141.82011 MR 2399189

[14] F. Hérau, M. Hitrik, and J. Sjöstrand, Tunnel effect and symmetries for Kramers–Fokker–Planck type operators. *J. Inst. Math. Jussieu* **10** (2011), no. 3, 567–634 Zbl 1223.35246 MR 2806463

[15] T. Kato, *Perturbation theory for linear operators*. Second edn., Grund. Math. Wiss. 132, Springer, Berlin etc., 1976 Zbl 0342.47009 MR 0407617

[16] C. Landim, M. Mariani, and I. Seo, Dirichlet's and Thomson's principles for non-selfadjoint elliptic operators with application to non-reversible metastable diffusion processes. *Arch. Ration. Mech. Anal.* **231** (2019), no. 2, 887–938 Zbl 1412.35093 MR 3900816

[17] D. Le Peutrec and L. Michel, Sharp spectral asymptotics for nonreversible metastable diffusion processes. *Probab. Math. Phys.* **1** (2020), no. 1, 3–53 Zbl 1484.60086 MR 4408002

[18] D. Le Peutrec, F. Nier, and C. Viterbo, Bar codes of persistent cohomology and arrhenius law for p-forms. 2002, arXiv:2002.06949v1

[19] L. Michel, About small eigenvalues of the Witten Laplacian. *Pure Appl. Anal.* **1** (2019), no. 2, 149–206 Zbl 1447.35236 MR 3949372

[20] J. Sjöstrand, Semiclassical resonances generated by nondegenerate critical points. In *Pseudodifferential operators (Oberwolfach, 1986)*, pp. 402–429, Lecture Notes in Math. 1256, Springer, Berlin, 1987 MR 0897789

Received 10 October 2023; revised 5 April 2024.

Jean-François Bony
Institut de Mathématiques de Bordeaux, CNRS (UMR 5251), Université de Bordeaux, 351, cours de la Libération, 33405 Talence, France; bony@math.u-bordeaux.fr

Laurent Michel
Institut de Mathématiques de Bordeaux, CNRS (UMR 5251), Université de Bordeaux, 351, cours de la Libération, 33405 Talence, France; lamichel@math.u-bordeaux.fr