J. Spectr. Theory 14 (2024), 1647–1660
DOI 10.4171/JST/527

Weak Hölder continuity of Lyapunov exponent for Gevrey quasi-periodic Schrödinger cocycles

Licheng Fang and Fengpeng Wang

Abstract. We prove the large deviation theorem (LDT) for quasi-periodic dynamically defined Gevrey Schrödinger cocycles with weak Liouville frequency. We show that the associated Lyapunov exponent is log-Hölder continuous, while the frequency satisfies $\beta(\omega) = 0$.

1. Introduction and the main results

Since the discovery of quasi-crystals by Nobel laureate Schechtman [23], the spectral problems of the one-dimensional discrete Schrödinger operators with quasi-periodic potentials have been extensively studied, see [2, 10, 11, 16, 22] and references therein. A *one-dimensional discrete Schrödinger operator* is the following self-adjoint lattice operator acting on $\ell^2(\mathbb{Z})$:

$$(H\phi)(n) := \phi(n+1) + \phi(n-1) + v(x+n\omega)\phi(n), \quad n \in \mathbb{Z},$$

where $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, $\omega \in \mathbb{R}\backslash\mathbb{Q}$, and $v\colon \mathbb{T} \to \mathbb{R}$ are called *phase, frequency,* and *potential,* respectively. For an energy $E \in \mathbb{R}$, we consider the associated eigen-equation

$$\phi(n+1) + \phi(n-1) + v(x+n\omega)\phi(n) = E\phi(n). \tag{1.1}$$

It is well known that the properties of the solution ϕ of equation (1.1) are intimately connected to the spectral properties of the Schrödinger operator with potential v, see [9] for instance. To be specific, the solution to (1.1) can be expressed via transfer matrices,

$$\begin{pmatrix} \phi(n+1) \\ \phi(n) \end{pmatrix} = \begin{pmatrix} E - v(x+n\omega) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi(n) \\ \phi(n-1) \end{pmatrix},$$

Mathematics Subject Classification 2020: 37D25 (primary); 47B36 (secondary).
Keywords: log-Hölder continuity, Lyapunov exponents, Gevrey potential, quasi-periodic Schrödinger cocycles.

which induces an $\mathrm{SL}(2,\mathbb{R})$-cocycle, called the *Schrödinger cocycle*, over $\mathbb{T}$ as follows:

$$(\omega, A^E)\colon \mathbb{T}\times\mathbb{R}^2 \to \mathbb{T}\times\mathbb{R}^2, \quad (x,w)\mapsto (x+\omega, A^E(x)\cdot w),$$

where the map $A^E\colon \mathbb{T}\to \mathrm{SL}(2,\mathbb{R})$ is given by

$$A^E(x) := \begin{pmatrix} E-v(x) & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.2}$$

We denote the iterates of (ω, A^E) by $(\omega, A^E)^n = (n\omega, A_n^E(\cdot))$, where, for $n\ge 1$,

$$\begin{aligned} A_n^E(x) &= A^E(x+(n-1)\omega)\cdots A^E(x), \\ A_{-n}^E(x) &= [A^E(x-n\omega)]^{-1}\cdots[A^E(x-\omega)]^{-1}, \end{aligned}$$

and $A_0^E(x)$ is identity matrix. We are interested in the asymptotic behavior of the norm of $A_n^E(x)$ as $n\to\infty$. The subadditive ergodic theorem ensures the existence and almost-sure equivalency of $L(\omega,E)$, called *Lyapunov exponent*,

$$\begin{aligned} L(\omega,E) :&= \lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}}\log\|A_n^E(x)\|dx \\ &= \lim_{n\to\infty}\frac{1}{n}\log\|A_n^E(\omega,x)\|, \text{ a.e. } x\in\mathbb{T}. \end{aligned} \tag{1.3}$$

As one of the most basic concepts in mathematics, the Lyapunov exponent is decisive for the long-term evolution of the system (1.1).

In this paper, we focus on the continuity of the Lyapunov exponent with respect to the energy, which plays an important role in mathematics. It is well known that the continuity of the Lyapunov exponent $L(\omega,E)$ depends sensitively on the base dynamics. It depends on the smoothness of $A^E(x)$ (see (1.2)) if the base is elliptic. To be specific, when the base dynamics is quasi-periodic, the Lyapunov exponent is always continuous for the analytic case, while it could be discontinuous for the smooth case, even for bounded type frequencies. On the other hand, the arithmetic properties of ω will also influence the dynamics of the Schrödinger cocycles. For a fixed irrational $\omega\in\mathbb{T}$, consider its fraction expansion $\omega=[a_1,a_2,\dots]$ with convergents $\frac{p_n}{q_n}$ for $n\in\mathbb{N}$. Define

$$\beta=\beta(\omega):=\limsup_n \frac{\log q_{n+1}}{q_n}\in[0,\infty]. \tag{1.4}$$

Recall that ω satisfies the *Diophantine condition* (D.C.) if $q_{n+1}=O(q_n^a)$ for some a, and the *strong Diophantine condition* (S.D.C) if $q_{n+1}=O(q_n(\log q_n)^a)$. It is then clear from (1.4) that S.D.C. $\subsetneq$ D.C. $\subsetneq \{\omega:\beta(\omega)=0\}$. Those values ω with $\beta(\omega)>0$ are usually called *Liouville numbers*, while *super Liouvillean* means $\beta(\omega)=\infty$ and the weak Liouville number satisfies $\beta(\omega)<\infty$. It is also clear that $\beta(\omega)=0$ if ω is a Brjuno number.

In the study of Schrödinger operators, the Schrödinger cocycle turned out to be a very important tool, which allows us to study Schrödinger operators by using dynamical systems. The Schrödinger cocycle (ω, A^E) is *uniformly hyperbolic* if there are $C > 0$ and $\lambda > 1$ such that $\|A_n^E(x)\| \geq C\lambda^{|n|}$ for every $x \in \mathbb{T}$ and $n \in \mathbb{Z}$. We write $\mathcal{UH} = \{E \in \mathbb{R} : (\omega, A^E)$ is uniformly hyperbolic$\}$. Clearly, $L(\omega, E) > 0$ for every $E \in \mathcal{UH}$. The converse is in general not true, and hence we denote the set of energies at which (ω, A^E) is *non-uniformly hyperbolic* by $\mathcal{NUH} = \{E \in \mathbb{R} : L(\omega, E) > 0$ and $E \notin \mathcal{UH}\}$. Therefore, if we set $\mathcal{Z} = \{E \in \mathbb{R} : L(\omega, E) = 0\}$, we have $\mathbb{R} = \mathcal{Z} \cup \mathcal{NUH} \cup \mathcal{UH}$.

Since the uniform hyperbolicity of the Schrödinger cocycle is equivalent to E belonging to the resolvent set as shown by Lenz [21] (see also Johnson [17]), and the potential v has a bounded norm, i.e., $\|v\|_{\mathbb{T}} := \sup_{x\in\mathbb{T}} |v(x)| < \infty$, then for $E \in (-\infty, -2 + \|v\|_{\mathbb{T}}] \cup [2 + \|v\|_{\mathbb{T}}, +\infty)$, the Schrödinger cocycle is uniformly hyperbolic, thus the continuity of the Lyapunov exponent is not an issue. So, without loss of generality, we assume that $|E| \leq 2 + \|v\|_{\mathbb{T}}$. On the other hand, due to the fact that the Lyapunov exponent is upper semi-continuous, it is continuous in $\mathcal{Z}$. Therefore, we only need to focus on the continuity of the Lyapunov exponent on the non-uniformly hyperbolic ($\mathcal{NUH}$) set.

About the continuity of the Lyapunv exponent, fruitful results have been obtained recently. For analytic Schrödinger cocycles, i.e., $A^E \in \mathcal{C}^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$, the Lyapunov exponent is always continuous. Bourgain and Jitomirskaya [5] proved that $L(\omega, E)$ is continuous in E for any ω and jointly continuous in (ω, E) at every (ω_0, E) with irrational ω_0, which plays an important role in solving the Ten Martini Problem, see [1]. For frequency ω satisfying the strong Diophantine condition, Goldstein and Schlag [14] developed a powerful tool, the *avalanche principle,* and proved Hölder continuity in E of $L(\omega, E)$ in the regime of $L > 0$. For all Diophantine frequencies and some weak Liouville frequencies ω, You and Zhang [25] and Han and Zhang [15] proved optimal Hölder regularity of the Lyapunov exponent $L(\omega, E)$ in the regime of $L > 0$. For irrational rotation, Bochi [3] and Furman [12] showed that the Lyapunov exponent of the Schrödinger cocycle is never continuous whenever $A^E \in \mathcal{C}^0(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$. Therefore, if one wants to obtain some regularity of the Lyapunov exponent, finer smoothness than $\mathcal{C}^0$ is necessary. As for the smooth Schrödinger cocycles, i.e., $A^E \in \mathcal{C}^k(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ with $k = 1, 2, \ldots, \infty$, what is interesting is whether the Lyapunov exponent is still always continuous just like $A^E \in \mathcal{C}^\omega(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$. The answer is no, since Wang and You [24] constructed examples of discontinuity of the Lyapunov expoent in smooth quasi-periodic $\mathrm{SL}(2,\mathbb{R})$ cocycles (including the Schrödinger cocycles) even when ω is of bounded type. However, if additional conditions are added, the continuity of the Lyapunov exponent is still valid. For example, Cai, Chavaudret, You, and Zhou [6] showed that if $A^E \in \mathcal{C}^k(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$, the Lyapunov exponent is continuous, even Hölder continuous, provided that the base frequency ω satisfies the Diophantine condition and the cocycle is $\mathscr{C}^{k,k'}$ almost reducible.

In this paper, we consider the Schrödinger cocycles belonging to the Gevrey-class, i.e., $v(x) \in G^s(\mathbb{T})$, $s > 1$. The general definition of the Gevrey function is, there are some constants $M, K > 0$ such that

$$\sup_{x\in\mathbb{T}} |\partial^m v(x)| \leq MK^m (m!)^s \quad \text{for all } m \geqslant 0,$$

which is equivalent to the following exponential-type decay of the Fourier coefficients of v, i.e.

$$|\hat{v}(k)| \leq M \exp(-\rho|k|^{1/s}) \quad \text{for all } k \in \mathbb{Z} \tag{1.5}$$

for some constants M; $\rho > 0$, see [18, Chapter V.2] for more details. It is clear from (1.5) that Gevrey-class functions have weaker smoothness than $\mathscr{C}^\omega$ (if $s = 1$, $G^s(\mathbb{T}) = \mathscr{C}^\omega(\mathbb{T})$) and stronger smoothness than $\mathscr{C}^\infty$. There are some continuity results of the Lyapunov exponent in this class, studied by Klein [19] for the first time.

For example, Klein [19] extended the work by Bourgain [14] and Goldstein and Schlag [4] from the analytic class to the Gevrey class, proving the continuity of the associated Lyapunov exponent for any $s > 1$ in energy E, in the regime of positive Lyapunov exponent, assuming the frequencies ω satisfying a strong Diophantine condition. Afterwards, the method developed in [19] has been applied extensively in dealing with spectral theory of Gevrey Schrödinger operators. Cheng, Ge, You, and Zhou [7] proved the continuity of the Lyapunov exponent whenever $s < 2$ and the frequency is Diophantine. Recently, Ge, Wang, You, and Zhao [13] showed that the Lyapunov exponent is discontinuous when $s > 2$ and ω is of bounded type, which shows that $G^2(\mathbb{T})$ is the transition space for the continuity of the Lyapunov exponent.

In order to prove the continuity of the Lyapunov exponent, the main technical tool is the so-called *large deviation theorem* (LDT) for the logarithmic average of transfer matrices. According to (1.3), by Kingman's subadditive ergodic theorem (see [8] for instance), one has for a.e. $x \in \mathbb{T}$,

$$\frac{1}{n} \log \|A_n^E(\omega, x)\| \to L(\omega, E) \quad \text{as } n \to \infty.$$

The LDT provides a quantitative version of this convergence:

$$\operatorname{mes}\left\{x \in \mathbb{T} : \left|\frac{1}{n} \log \|A_n^E(\omega, x)\| - L_n(\omega, E)\right| > \varepsilon\right\} \leq \sigma(n, \varepsilon),$$

where $\varepsilon = o(1)$, $\sigma(n, \varepsilon) \to 0$ as $n \to \infty$, and $L_n(\omega, E) = \frac{1}{n}\int_{\mathbb{T}} \log \|A_n^E(\omega, x)\| dx$.

For the sake of our convenience, we set $u_n(\omega, E, x) = \frac{1}{n} \log \|A_n^E(\omega, x)\|$. Then the main result of this paper is as follows.

Theorem 1.1. *For any $\varepsilon > 0$, $b > 1$, $s > 1$, we denote $\delta = b(s-1)$, which means $s < 1 + \delta$. Let C_v be a constant depending on v. There exist absolute constants c_0, c_1 such that if $n > N(\varepsilon, C_v)$ and $\beta = \beta(\omega) < c_0 n^{-\delta}\varepsilon/C_v$, then*

$$\begin{aligned}\operatorname{mes}(\Omega_n(\varepsilon)) &:= \operatorname{mes}\{x \in \mathbb{T} : |u_n(\omega, E, x) - L_n(\omega, E)| > \varepsilon\} \\ &< \exp\Big\{-\frac{c_1}{C_v^3}\varepsilon^3 n^{1-2\delta}\Big\},\end{aligned} \tag{1.6}$$

where $1 - 2\delta > 0$, that is, $\delta < \frac{1}{2}$.

Remark 1. Due to the positivity of $1 - 2\delta$, this LDT (1.6) holds for certain Gevrey potentials $G^s(\mathbb{T})$, i.e., $1 < s < 1 + \delta < 3/2$, which is a class of potentials that is close to be analytic.

Remark 2. We should mention that in Theorem 1.1 the frequency ω satisfies $\beta(\omega) < c_0 n^{-\delta}\varepsilon/C_v := C(v, \varepsilon, \delta, n)$, which is a weak Liouville number, and it also depends on the scale n, in contrast with $\beta(\omega) < C(v, \varepsilon)$ in [25, Theorem 1].

Once we have the LDT (1.6), we are able to prove the continuity of the Lyapunov exponent. In order to do that, we need to assume that the Lyapunov exponent is positive, that is,

$$L(\omega, E) > \gamma > 0 \quad \text{for all } E \in [E_1, E_2]. \tag{1.7}$$

Theorem 1.2. *Assume that $\beta(\omega) = 0$ and suppose (1.7) holds, then*

$$|L(\omega, E) - L(\omega, E')| \le C \exp[-c(\log|E - E'|^{-1})^{1-2\delta}], \tag{1.8}$$

where c and C are some constants depend on v and γ, and δ is the constant in Theorem 1.1.

Remark 3. Due to the appearance of the constant δ, the continuity of the Lyapunov exponent is the log-Hölder continuity instead of the sharper τ-Hölder continuity in [25, Theorem 2], where they treat the analytic case, i.e., $s = 1$.

Remark 4. Inequality (1.8) shows that we improved the results of [7, 19, 25] in the sense that the Lyapunov exponent is log-Hölder continuous under the Gevrey potential $G^s(\mathbb{T})$ with $s < 3/2$ and the frequency satisfies $\beta(\omega) = 0$, which is a larger set than D.C.

This paper is organized as follows. In Section 2 we give the basic definitions and we show rigorously the approximation (truncation) argument, following some ideas and techniques in [19, Section 2] or [20] along the way. Section 3 is devoted to the proof of the large deviation theorem, and once the LDT is proved, we are able to combine it with the avalanche principle ([14, Proposition 2.2]) to prove the log-Hölder continuity of the Lyapunov exponent.

2. Description of the approximation process

In this section, we introduce the Fourier truncation and its application to the Gevrey-class functions. This idea of dealing with Gevrey-class potetials has been originally developed by Klein in [19].

For a Gevrey-class potential $v(x) \in G^s(\mathbb{T})$, where $s > 1$, we consider the truncation of $v(x)$, i.e.,

$$v_n(x) := \sum_{|k| \le \tilde{n}} \hat{v}(k) e^{ikx}, \tag{2.1}$$

where $\tilde{n} = \deg v_n$ will be determined later. Since $v_n(x)$ is a 2π-periodic, real analytic function on $\mathbb{R}$, it can be extended to a 2π-periodic holomorphic function on $\mathbb{C}$ by

$$v_n(z) = \sum_{|k| \le \tilde{n}} \hat{v}(k) e^{ikz}.$$

In order to get the uniform boundedness in n of $v_n(z)$, we have to restrict $v_n(z)$ to the strip $[|\Im z| < \rho_n]$, where $\rho_n := \frac{\rho}{2}\tilde{n}^{1/s-1}$. Indeed, if $z = x + iy$, $|y| < \rho_n$, then

$$\begin{aligned} |v_n(z)| &= \Big|\sum_{|k| \le \tilde{n}} \hat{v}(k) e^{ikz}\Big| \le \sum_{|k| \le \tilde{n}} |\hat{v}(k)| e^{-ky} \le M \sum_{|k| \le \tilde{n}} e^{-\rho|k|^{1/s}} e^{|k||y|} \\ &\le 2M \sum_{k=0}^{\tilde{n}} e^{-\rho|k|^{1/s} + |k||y|} \le 2M \sum_{k=0}^{\tilde{n}} e^{-\frac{\rho}{2}|k|^{1/s}} \\ &\le 2M \sum_{k=0}^{\infty} e^{-\frac{\rho}{2}|k|^{1/s}} := B < \infty \end{aligned}$$

where B is a constant which depends on ρ, s, M, and we used $|y| < \rho_n = \frac{\rho}{2}\tilde{n}^{1/s-1} \le \frac{\rho}{2}|k|^{1/s-1}$ for $|k| \le \tilde{n}$ since $s > 1$.

In view of the truncation (2.1), we have $|v(x) - v_n(x)| \le Ce^{-c\tilde{n}^{1/s}}$ for all $x \in \mathbb{R}$, where $C, c > 0$ depend on ρ and s. We will need a super-exponentially small error in how $v_n(x)$ approximates $v(x)$. Hence, $\tilde{n}$ should be chosen such that $e^{-c\tilde{n}^{1/s}} \le e^{-n^b}$ for some $b > 1$. So if $\tilde{n} := n^{bs}$, then the width of the holomorphic extension $v_n(x)$ will be $\rho_n = \frac{\rho}{2} n^{bs(-1+1/s)} = \frac{\rho}{2} n^{-b(s-1)} := \frac{\rho}{2} n^{-\delta}$, where $\delta := b(s-1) > 0$, and, to ensure that $b = \frac{\delta}{s-1} > 1$, $s < \delta + 1$.

From the description above, we conclude that, for every integer $n \ge 1$, we have a function $v_n(x)$ on $\mathbb{T}$ such that

$$|v(x) - v_n(x)| < e^{-cn^b},$$

and $v_n(x)$ has a holomorphic extension $v_n(z)$ to the strip $[|\Im z| < \rho_n]$, where $\rho_n = \frac{\rho}{2} n^{-\delta}$, for which

$$|v_n(z)| \le B \quad \text{for all } z \in [|\Im z| < \rho_n].$$

We now substitute $v_n(x)$ for $v(x)$ in the definition of the transfer matrix $A^E(x)$ (see (1.2)) and get

$$\widetilde{A}_n^E(\omega, x) := \prod_{j=n-1}^{0} \widetilde{A}^E(x + j\omega),$$

where

$$\widetilde{A}^E(x) := \begin{pmatrix} E - v_n(x) & -1 \\ 1 & 0 \end{pmatrix}.$$

We denote by

$$\widetilde{L}_n(\omega, E) := \frac{1}{n} \int_{\mathbb{T}} \log \|\widetilde{A}_n^E(\omega, x)\| dx.$$

The norm $\|\cdot\|$ of the transfer matrix and the substituted transfer matrix are

$$\|A^E(x)\| = \left\| \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix} \right\| \le B + |E| + 2 \le 4 + 2B := e^{C_v},$$

$$\|\widetilde{A}^E(x)\| = \left\| \begin{pmatrix} E - v_n(x) & -1 \\ 1 & 0 \end{pmatrix} \right\| \le B + |E| + 2 \le 4 + 2B := e^{C_v},$$

for any $x \in \mathbb{T}$, where we used $|v(x)| \le \sup_{x\in\mathbb{T}} |v(x)| \le B$, $|v_n(x)| \le B$ and $|E| \le 2 + B$. Thus,

$$\frac{1}{n} \log \|\widetilde{A}_n^E(\omega, x)\| \le C_v.$$

Since

$$\|A^E(x) - \widetilde{A}^E(x)\| \le \sup_{x\in\mathbb{T}} |v(x) - v_n(x)| \le e^{-cn^b},$$

by Trotter's formula,

$$\begin{aligned} A_n^E(\omega, x) - \widetilde{A}_n^E(\omega, x) = \sum_{j=0}^{n-1} A^E(x + (n-1)\omega) \cdots A^E(x + (j+1)\omega) \\ \cdot (A^E(x + j\omega) - \widetilde{A}^E(x + j\omega)) \\ \cdot \widetilde{A}^E(x + (j-1)\omega) \cdots \widetilde{A}^E(x); \end{aligned}$$

we then have

$$\begin{aligned} \|A_n^E(\omega, x) - \widetilde{A}_n^E(\omega, x)\| &\le \sum_{j=0}^{n-1} e^{C_v} \cdots e^{C_v} e^{-cn^b} e^{C_v} \cdots e^{C_v} \\ &\le n e^{nC_v} e^{-cn^b}. \end{aligned}$$

Since

$$\|\widetilde{A}_n^E(\omega, x)\| \ge |\det \widetilde{A}_n^E(\omega, x)| = 1,$$

and $b > 1$, for large n such that $n \gtrsim C_v{}^{\frac{1}{b-1}}$,

$$\begin{aligned}&\left|\frac{1}{n}\log\|A_n^E(\omega,x)\| - \frac{1}{n}\log\|\widetilde{A}_n^E(\omega,x)\|\right|\\&\quad= \left\|\frac{1}{n}\log\Big(1+\frac{\|A_n^E(\omega,x)\|-\|\widetilde{A}_n^E(\omega,x)\|}{\|\widetilde{A}_n^E(\omega,x)\|}\Big)\right\|\\&\le \frac{1}{n}\frac{\|A_n^E(\omega,x)-\widetilde{A}_n^E(\omega,x)\|}{\|\widetilde{A}_n^E(\omega,x)\|} \le \frac{1}{n}\|A_n^E(\omega,x)-\widetilde{A}_n^E(\omega,x)\| \le e^{-cn^b},\end{aligned}$$

where c may stand for different constants simultaneously and by averaging

$$|L_n(\omega,E) - \widetilde{L}_n(\omega,E)| \le e^{-cn^b}.$$

We now summarize all of the above into the following.

Lemma 2.1. *For fixed parameters ω, E, we have a 2π-periodic function*

$$\tilde{u}_n(\omega,E,x) := \frac{1}{n}\log\|\widetilde{A}_n^E(\omega,x)\|,$$

which extends to the strip $[|\Im z| < \rho_n]$, $\rho_n \approx n^{-\delta}$, $\delta = b(s-1)$ to a subharmonic function $\tilde{u}_n(\omega,E,z)$ so that[1]

$$|\tilde{u}_n(\omega,E,z)| \le C_v \quad \text{for all } z \in [|\Im z| < \rho_n].$$

Moreover, for fixed $b > 1$, i.e., $s < 1+\delta$, if $n \gtrsim C_v{}^{\frac{1}{b-1}}$, and recalling that one has $u_n(\omega,E,x) = \frac{1}{n}\log\|A_n^E(\omega,x)\|$ and $L_n(\omega,E) = \frac{1}{n}\int_{\mathbb{T}}\log\|A_n^E(\omega,x)\|dx$, they are well approximated by their substitutes

$$|u_n(\omega,E,x) - \tilde{u}_n(\omega,E,x)| < e^{-cn^b}, \tag{2.2}$$

$$|L_n(\omega,E) - \widetilde{L}_n(\omega,E)| < e^{-cn^b}, \tag{2.3}$$

where c is a constant, and $\widetilde{L}_n(\omega,E) = \int_{\mathbb{T}}\tilde{u}_n(\omega,E,x)dx$.

3. Proof of the large deviation theorem and the continuity of the Lyapunov exponent

Recall that $\tilde{u}_n(\omega,E,x) = \frac{1}{n}\log\|\widetilde{A}_n^E(\omega,x)\|$. We define

$$w_n := n^{-\delta}\tilde{u}_n(\omega,E,x),$$

[1] The bound here is uniform in n.

where δ is the constant in Lemma 2.1. With our variables, we note that w_n depends on ω, E, and x. We will write $w_n(x)$ or $w_n(\omega, E, x)$ whenever we want to make this dependence explicit.

Expand $w_n(x)$ into its Fourier series and denote the Fourier coefficient by $\widehat{w}(k)$, that is,

$$w_n(x) = \sum_{k\in\mathbb{Z}} \widehat{w}(k)e^{ikx}, \quad \widehat{w}(k) = \int_{x\in\mathbb{T}} w_n(x)e^{-ikx}dx. \tag{3.1}$$

Consider the Féjèr average $w_R(x)$ of $w_n(x)$ along the orbit and denote the Féjèr kernel as $F_R(k)$

$$w_R(x) = \sum_{|j|<R} \frac{R-|j|}{R^2} w_n(x+j\omega), \quad F_R(k) = \sum_{|j|<R} \frac{R-|j|}{R^2} e^{ikj\omega}.$$

Combining with (3.1), this leads to

$$w_R(x) = \sum_{|j|<R} \frac{R-|j|}{R^2} \sum_{k\in\mathbb{Z}} \widehat{w}(k)e^{ik(x+j\omega)} = \sum_{k\in\mathbb{Z}} \widehat{w}(k)F_R(k)e^{ikx}.$$

We truncate $w_R(x)$ into two parts:

$$w_{\mathrm{I}}(x) = \sum_{0<|k|<K} \widehat{w}(k)F_R(k)e^{ikx}, \quad w_{\mathrm{II}}(x) = \sum_{|k|\geq K} \widehat{w}(k)F_R(k)e^{ikx};$$

this leads to

$$\begin{aligned} w_R(x) &= \langle w_n(x)\rangle + w_{\mathrm{I}}(x) + w_{\mathrm{II}}(x) \\ &= n^{-\delta}\widetilde{L}_n + w_{\mathrm{I}}(x) + w_{\mathrm{II}}(x). \end{aligned} \tag{3.2}$$

Before turning to the proof of the LDT, we need the following lemmas. We should mention that some of these estimates for Gevrey class have been first obtained by [19]; e.g., (3.4) is [19, (1.14)].

Lemma 3.1. *For any n, $R \in \mathbb{N}$, $k \in \mathbb{Z}$,*

$$|w_n(x) - w_R(x)| \leq 2C_v \frac{R}{n^{1+\delta}}, \tag{3.3}$$

$$|\widehat{w}(k)| \lesssim \frac{n^{-\delta}\sup_{|\Im z|<\rho_n} |\tilde{u}_n(z)|}{\rho_n |k|} = \frac{C_v}{|k|}, \tag{3.4}$$

$$|F_R(k)| \leq \frac{6}{1+R^2\|k\omega\|^2}. \tag{3.5}$$

Proof. These results were obtained in [25, Lemma 2.2] for Schrödinger cocycles with analytic potential. We follow the same approach here. Let us point out a small change in (3.4): since the width of the holomorphic extension $\tilde{u}_n(x)$ is $\rho_n \approx n^{-\delta}$, the Fourier coefficients of $\tilde{u}_n(x)$ are closely related to ρ_n, we refer to [4, Corollary 4.7] for more details. This is the reason why we define $w_n(x)$ to be $n^{-\delta}\tilde{u}_n(x)$. ■

Lemma 3.2. *For any $\kappa > 0$, sufficiently large n, and small $\beta = \beta(\omega)$, a specific choice of $R(n,\kappa), K(n,\kappa) \in \mathbb{N}$ implies*

$$|w_n(x) - w_R(x)| < \frac{\kappa}{3}, \tag{3.6}$$

$$|w_I| < \frac{\kappa}{3}, \tag{3.7}$$

$$\|w_{\mathrm{II}}\|_2^2 \leq (6C_v)^2 \frac{2}{K}. \tag{3.8}$$

Proof. We take

$$R = \left[\frac{\kappa}{6C_v} n^{1+\delta}\right], \quad K = \left[\exp\left\{\frac{\kappa^2}{200{C_v}^2} R\right\}\right], \quad \beta < \frac{\kappa}{40C_v}.$$

Since $q_n \to +\infty$ monotonically, there is an $m \in \mathbb{N}$ suth that $(\kappa/10C_v)R \in [q_m, q_{m+1})$.

We begin with (3.6). Inequality (3.3) leads to

$$|w_n(x) - w_R(x)| \leq 2C_v \frac{(\frac{\kappa}{6}C_v)n^{1+\delta}}{n^{1+\delta}} < \kappa/3.$$

As for the proof of (3.7), we divide w_I into three parts, that is,

$$w_{\mathrm{I}} = \sum_{1 \leq |k| < \frac{q_m}{4}} \hat{w}(k) F_R(k) e^{ikx} + \sum_{\frac{q_m}{4} \leq |k| < \frac{q_{m+1}}{4}} \hat{w}(k) F_R(k) e^{ikx} + \sum_{\frac{q_{m+1}}{4} \leq |k| < K} \hat{w}(k) F_R(k) e^{ikx}$$

and we obtain that (see the proof of [25, Lemma 2.1] for more details)

$$\begin{aligned} |w_{\mathrm{I}}| &\lesssim C_v\left(\frac{q_m}{R} + \frac{\log q_{m+1}}{q_m} + \frac{\log q_{m+1}}{R} + \frac{\log K}{q_{m+1}} + \frac{\log K}{R}\right) \\ &\lesssim C_v\left(\frac{\kappa}{10C_v} + 2\frac{\log q_{m+1}}{q_m} + \frac{\log K}{\kappa R/10C_v} + \frac{\log K}{R}\right) \\ &\lesssim C_v\left(\frac{\kappa}{10C_v} + 4\beta + \frac{20C_v}{\kappa}\frac{\log K}{R}\right) \lesssim \frac{\kappa}{10} + \frac{\kappa}{10} + \frac{\kappa}{10} \lesssim \frac{\kappa}{3}, \end{aligned}$$

where we used $\frac{\log q_{m+1}}{q_m} < 2\beta$ when β is positive and $\frac{\log q_{m+1}}{q_m}$ could be arbitrarily small if $\beta = 0$, provided q_m is large enough.

As for the proof of (3.8), combing with (3.4) and (3.5), we have

$$\begin{aligned}\|w_{\mathrm{II}}\|_2^2 &= \Big\| \sum_{|k|\geq K} \widehat{w}(k) F_R(k) e^{ikx} \Big\|_2^2 \leq \sum_{|k|\geq K} |\widehat{w}(k) F_R(k)|^2 \\ &\leq \sum_{|k|\geq K} \Big|\frac{C_v}{|k|}\cdot 6\Big|^2 \leq (6C_v)^2 \frac{2}{K}.\end{aligned}$$

We now come to the proof of Theorem 1.1.

Proof. In view of (3.6)–(3.8), and taking (3.2) into account, we have

$$\begin{aligned}&\operatorname{mes}\{x\in\mathbb{T} : |w_n(x) - \langle w_n(x)\rangle| > \kappa\} \\ &\quad\leq \operatorname{mes}\Big\{x\in\mathbb{T} : |w_n(x) - w_R(x)| > \frac{\kappa}{3}\Big\} \\ &\qquad + \operatorname{mes}\Big\{x\in\mathbb{T} : |w_I| > \frac{\kappa}{3}\Big\} + \operatorname{mes}\Big\{x\in\mathbb{T} : |w_{\mathrm{II}}| > \frac{\kappa}{3}\Big\} \\ &\quad\leq \frac{1}{(\frac{\kappa}{3})^2}\|w_{\mathrm{II}}\|_2^2 \leq \frac{9}{\kappa^2}(6C_v)^2\frac{2}{K} \\ &\quad\leq \frac{18}{\kappa^2}(6C_v)^2\cdot\exp\Big(-\frac{\kappa^3}{3000C_v^3}n^{1+\delta}\Big) \leq \exp\Big(-\frac{\kappa^3}{6000C_v^3}n^{1+\delta}\Big),\end{aligned}$$

where we used $K > \exp\{(\kappa^3/3000C_v^3)n^{1+\delta}\}$ in the last line but one. We recall that $w_n(\omega, E, x) = n^{-\delta}\tilde{u}_n(\omega, E, x)$, which leads to

$$\operatorname{mes}\{x\in\mathbb{T} : |\tilde{u}_n(\omega, E, x) - \widetilde{L}_n(\omega, E)| > n^\delta\kappa\} \leq \exp\Big\{-\frac{\kappa^3}{6000C_v^3}n^{1+\delta}\Big\}.$$

Now, taking $\kappa = \frac{n^{-\delta}\varepsilon}{3}$, then for$\beta < \frac{\kappa}{40C_v} = \frac{n^{-\delta}\varepsilon}{120C_v}$, we have

$$\operatorname{mes}\Big\{x\in\mathbb{T} : |\tilde{u}_n(\omega, E, x) - \widetilde{L}_n(\omega, E)| > \frac{\varepsilon}{3}\Big\} \leq \exp\Big\{-\frac{\varepsilon^3}{2\times 10^5 C_v^3}n^{1-2\delta}\Big\}. \tag{3.9}$$

In view of (2.2) and (2.3), and we choose large n such that $e^{-cn^b} < \varepsilon/3$, combing with (3.9) leads to

$$\begin{aligned}\operatorname{mes}(\Omega_n(\varepsilon)) &:= \operatorname{mes}\{x\in\mathbb{T} : |u_n(\omega, E, x) - L_n(\omega, E)| > \varepsilon\} \\ &\leq \operatorname{mes}\Big\{x\in\mathbb{T} : |u_n(\omega, E, x) - \tilde{u}_n(\omega, E, x)| > \frac{\varepsilon}{3}\Big\} \\ &\quad + \operatorname{mes}\Big\{x\in\mathbb{T} : |\tilde{u}_n(\omega, E, x) - \widetilde{L}_n(\omega, E)| > \frac{\varepsilon}{3}\Big\} \\ &\quad + \operatorname{mes}\Big\{x\in\mathbb{T} : |L_n(\omega, E) - \widetilde{L}_n(\omega, E)| > \frac{\varepsilon}{3}\Big\} \\ &\leq \operatorname{mes}\Big\{x\in\mathbb{T} : |\tilde{u}_n(\omega, E, x) - \widetilde{L}_n(\omega, E)| > \frac{\varepsilon}{3}\Big\} \\ &\leq \exp\Big\{-\frac{\varepsilon^3}{2\times 10^5 C_v^3}n^{1-2\delta}\Big\}.\end{aligned}$$

■

As for the proof of Theorem 1.2. Firstly, we show the following lemma.

Lemma 3.3. *Assume that* $\beta(\omega) = 0$ *and suppose that* $L(\omega, E) > \gamma > 0$ *for* $E \in [E_1, E_2]$. *Then for all* $n \in \mathbb{N}$ *large enough*

$$|L(\omega, E) + L_n(\omega, E) - 2L_{2n}(\omega, E)| < \exp(-cn^{1-2\delta}), \tag{3.10}$$

where δ *is the constant in Theorem* 1.1 *and* $c = c(\gamma) > 0$.

Proof. The proof of the result (3.10) uses the LDT we established in Theorem 1.1 and the avalanche principle (see [14, Proposition 2.2]), and follows the same pattern (the standard iteration approach) as the proof of the corresponding result for the analytic case, where at each iteration process, the $\beta(\omega)$ has something to do with iteration scale since $\beta(\omega) < c_0 n^{-\delta}\varepsilon/C_v$. We obtain this result (3.10) as the iteration continues, while $\beta(\omega)$ converges to zero. For more details, the reader is referred to [25, Proposition A.3] or [4, Chapter VII]. ∎

Secondly, it is obvious that for any $n \in \mathbb{N}$,

$$\|\partial_E A_n^E\| \leq n(e^{C_v})^{n-1},$$

and then

$$|L_n(\omega, E) - L_n(\omega, E')| \leq e^{nC_v}|E - E'|. \tag{3.11}$$

In view of (3.10) and (3.11), for any $|E - E'| \ll 1$, $E, E' \in [E_1, E_2]$, we finally have

$$\begin{aligned} |L(\omega, E) - L(\omega, E')| &\leq |L(\omega, E) + L_n(\omega, E) - 2L_{2n}(\omega, E)| \\ &\quad + |L(\omega, E') + L_n(\omega, E') - 2L_{2n}(\omega, E')| \\ &\quad + |L_n(\omega, E) - L_n(\omega, E')| \\ &\quad + 2|L_{2n}(\omega, E) - L_{2n}(\omega, E')| \\ &\leq 2\exp(-cn^{1-2\delta}) + 3e^{2nC_v}|E - E'| \\ &\leq C\exp[-c(\log|E - E'|^{-1})^{1-2\delta}], \end{aligned}$$

where the last step follows from $n \sim \log \frac{1}{|E-E'|}$. This proves Theorem 1.2. ∎

Funding. L. Fang was partially supported by NSFC (12301235) and the Natural Science Foundation of Shandong Province (ZR2022QA011). F. Wang was partially supported by NSFC (12001551).

References

[1] A. Avila and S. Jitomirskaya, The Ten Martini Problem. *Ann. of Math. (2)* **170** (2009), no. 1, 303–342 Zbl 1166.47031 MR 2521117

[2] J. Bellissard, B. Iochum, E. Scoppola, and D. Testard, Spectral properties of one-dimensional quasi-crystals. *Comm. Math. Phys.* **125** (1989), no. 3, 527–543 Zbl 0825.58010 MR 1022526

[3] J. Bochi, Discontinuity of the Lyapunov exponent for non-hyperbolic cocycles. Unpublished, 1999

[4] J. Bourgain, *Green's function estimates for lattice Schrödinger operators and applications*. Ann. of Math. Stud. 158, Princeton University Press, Princeton, NJ, 2005 Zbl 1137.35001 MR 2100420

[5] J. Bourgain and S. Jitomirskaya, Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. *J. Stat. Phys.* **108** (2002), 1203–1218 Zbl 1039.81019 MR 1933451

[6] A. Cai, C. Chavaudret, J. You, and Q. Zhou, Sharp Hölder continuity of the Lyapunov exponent of finitely differentiable quasi-periodic cocycles. *Math. Z.* **291** (2019), no. 3-4, 931–958 Zbl 1482.37031 MR 3936094

[7] H. Cheng, L. Ge, J. You, and Q. Zhou, Global rigidity for ultra-differentiable quasiperiodic cocycles and its spectral applications. *Adv. Math.* **409** (2022), article no. 108679 Zbl 1506.37033 MR 4477429

[8] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*. Study edn., Texts Monogr. Phys., Springer, Berlin, 1987 Zbl 0619.47005 MR 0883643

[9] D. Damanik, Schrödinger operators with dynamically defined potentials. *Ergodic Theory Dynam. Systems* **37** (2017), no. 6, 1681–1764 Zbl 06823048 MR 3681983

[10] D. Damanik, A. Gorodetski, and W. Yessen, The Fibonacci Hamiltonian. *Invent. Math.* **206** (2016), no. 3, 629–692 Zbl 1359.81108 MR 3573970

[11] D. Damanik and M. Landrigan, Log-dimensional spectral properties of one-dimensional quasicrystals. *Proc. Amer. Math. Soc.* **131** (2003), no. 7, 2209–2216 Zbl 1075.81024 MR 1963769

[12] A. Furman, On the multiplicative ergodic theorem for uniquely ergodic systems. *Ann. Inst. H. Poincaré Probab. Statist.* **33** (1997), no. 6, 797–815 Zbl 0892.60011 MR 1484541

[13] L. Ge, Y. Wang, J. You, and X. Zhao, Transition space for the continuity of the Lyapunov exponent of quasiperiodic Schrödinger cocycles. 2021, arXiv:2102.05175v1

[14] M. Goldstein and W. Schlag, Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions. *Ann. of Math. (2)* **154** (2001), no. 1, 155–203 Zbl 0990.39014 MR 1847592

[15] R. Han and S. Zhang, Large deviation estimates and Hölder regularity of the Lyapunov exponents for quasi-periodic Schrödinger cocycles. *Int. Math. Res. Not. IMRN* (2022), no. 3, 1666–1713 Zbl 07471362 MR 4373222

[16] A. Hof, O. Knill, and B. Simon, Singular continuous spectrum for palindromic Schrödinger operators. *Comm. Math. Phys.* **174** (1995), no. 1, 149–159 Zbl 0839.11009 MR 1372804

[17] R. A. Johnson, Exponential dichotomy, rotation number, and linear differential operators with bounded coefficients. *J. Differential Equations* **61** (1986), no. 1, 54–78 Zbl 0608.34056 MR 0818861

[18] Y. Katznelson, *An introduction to harmonic analysis*. John Wiley & Sons, New York etc., 1968 Zbl 0169.17902 MR 0248482

[19] S. Klein, Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey-class function. *J. Funct. Anal.* **218** (2005), no. 2, 255–292 Zbl 1068.81023 MR 2108112

[20] S. Klein, Localization for quasiperiodic Schrödinger operators with multivariable Gevrey potential functions. *J. Spectr. Theory* **4** (2014), no. 3, 431–484 Zbl 1454.81084 MR 3291922

[21] D. Lenz, Singular spectrum of Lebesgue measure zero for one-dimensional quasicrystals. *Comm. Math. Phys.* **227** (2002), no. 1, 119–130 Zbl 1065.47035 MR 1903841

[22] J. Liang, Y. Wang, and J. You, Hölder continuity of Lyapunov exponent for a family of smooth Schrödinger cocycles. *Ann. Henri Poincaré* **25** (2024), no. 2, 1399–1444 MR 4703421

[23] D. Schechtman, I. Blech, D. Gratias, and J. Cahn, Metallic phase with long-range orientational order and no translational symmetry. *Phys. Rev. Lett.* **53** (1984), 1951–1954

[24] Y. Wang and J. You, Examples of discontinuity of Lyapunov exponent in smooth quasiperiodic cocycles. *Duke Math. J.* **162** (2013), no. 13, 2363–2412 Zbl 1405.37032 MR 3127804

[25] J. You and S. Zhang, Hölder continuity of the Lyapunov exponent for analytic quasiperiodic Schrödinger cocycle with weak Liouville frequency. *Ergodic Theory Dynam. Systems* **34** (2014), no. 4, 1395–1408 Zbl 1315.39004 MR 3227161

Received 11 April 2024; revised 5 August 2024.

Licheng Fang
School of Mathematics and Information Science, Yantai University, Qingquan Road 30, 264005 Yantai, P. R. China; licheng@ytu.edu.cn

Fengpeng Wang
School of Mathematics (Zhuhai), Sun Yat-Sen University, Daxue Road 2, 519082 Zhuhai, P. R. China; wangfp6@mail.sysu.edu.cn