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Isoperimetric inequalities for inner parallel curves

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Abstract. We prove weighted isoperimetric inequalities for smooth, bounded, and simply-connected domains. More precisely, we show that the moment of inertia of inner parallel curves for domains with fixed perimeter attains its maximum for a disk. This inequality, which was previously only known for convex domains, allows us to extend an isoperimetric inequality for the magnetic Robin Laplacian to non-convex centrally symmetric domains. Furthermore, we extend our isoperimetric inequality for moments of inertia, which are second moments, to p-th moments for all p smaller than or equal to two. We also show that the disk is a strict local maximiser in the nearly circular, centrally symmetric case for all p strictly less than three, and that the inequality fails for all p strictly bigger than three.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply-connected domain. For any $t \geq 0$, we define the corresponding inner parallel curve S_t by

$$S_t := S_t(\Omega) = \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) = t\}. \tag{1.1}$$

The systematic study of the geometric structure and regularity of inner parallel curves was initiated in [4, 11, 16], see also [27, 28] and references therein.

By [28, Theorem 4.4.1] and [27, Proposition A.1], the inner parallel curve S_t is a finite union of piecewise smooth simple curves for almost every $t \geq 0$. Hartman [16, Corollary 6.1] showed that

$$|S_t| \leq |\partial\Omega| - 2\pi t \quad \text{for almost every } t \geq 0 \text{ with } S_t \neq \varnothing, \tag{1.2}$$

where $|S_t|$ and $|\partial\Omega|$ denote the length of S_t and $\partial\Omega$, respectively.

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The moment of inertia of the inner parallel curve S_t computed with respect to its centroid

$$c(t) := \frac{1}{|S_t|} \int_{S_t} x \, \mathrm{d}\mathcal{H}^1(x) \tag{1.3}$$

is given by

$$\int_{S_t} |x - c(t)|^2 \, \mathrm{d}\mathcal{H}^1(x), \tag{1.4}$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. In this paper, we address the following question.

Question. *Fixing the perimeter of Ω, what shape of Ω maximises the moment of inertia of the inner parallel curve S_t as defined in (1.4) for given $t \geq 0$?*

Our main result states that the optimal shape is attained for a disk.

Theorem 1.1 (An isoperimetric inequality for moments of inertia). *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply-connected domain. Then, for almost every $t \geq 0$,*

$$\int_{S_t(\Omega)} |x - c(t)|^2 \, \mathrm{d}\mathcal{H}^1(x) \leq \int_{S_t(\mathcal{B})} |x|^2 \, \mathrm{d}\mathcal{H}^1(x), \tag{1.5}$$

where $\mathcal{B}$ is the disk centred at the origin and with the same perimeter as Ω. Here $S_t(\cdot)$ and $c(t)$ are defined in (1.1) and (1.3), respectively. When $t \in \left[0, \frac{|\partial\Omega|}{2\pi}\right)$, the equality in (1.5) is attained if and only if Ω is a disk.

Note that $c(t) = 0$ if Ω is a disk centred at the origin. In the setting where S_t is a closed curve, the statement of Theorem 1.1 can be deduced from a result due to Hurwitz [18, pp. 396–397] combined with (1.2). For instance, this argument applies for convex domains Ω, see [19, p. 12] for further details. In general, S_t can consist of several connected components, see for example Figure 1.1, and Theorem 1.1 is novel in this case. The classical result by Hurwitz itself provides an isoperimetric inequality for the moment of inertia of the boundary of a planar domain under fixed perimeter constraint and essentially coincides with the statement of Theorem 1.1 in the special case $t = 0$. The recent contribution [23] proves a quantitative version of the inequality by Hurwitz and addresses the higher-dimensional setting.

The smoothness assumption for the domain is not optimal. It is used in the proof for Proposition 2.1 below, see [16], see also [28, Theorem 4.4.1] and [27, Proposition A.1]. For piecewise C^2-domains satisfying a similar result to Proposition 2.1 below, we can also obtain (1.5) using the same rest of the proof. In general, Theorem 1.1 fails for non-simply-connected domains; see Remark 4.1 below.

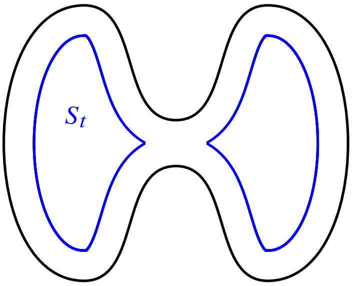

Figure 1.1. A schematic representation of the inner parallel curve S_t in a dumbbell-like domain. Note that S_t can be disconnected.

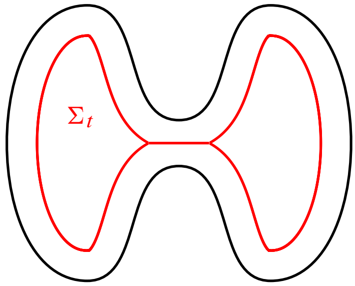

Figure 1.2. A schematic illustration of the connected curve Σ_t in the case of a dumbbell-like domain. The proof of Theorem 1.2 will show that the segment connecting the two connected components of S_t is doubly covered.

Our proof of Theorem 1.1 relies on an explicit construction of a *closed* curve Σ_t. An illustration for Σ_t is shown in Figure 1.2 (where Σ_t inherits the symmetry of Ω).

Theorem 1.2 (Covering inner parallel curves with a closed curve). *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply-connected domain. Then, for almost every $t \geq 0$ with $S_t \neq \varnothing$, there exists a closed and piecewise smooth curve Σ_t with*

$$S_t \subset \Sigma_t \quad \textit{and} \quad |\Sigma_t| \leq |\partial\Omega| - 2\pi t,$$

where S_t was defined in (1.1).

The technical result Theorem 1.2 is of independent interest as we obtain an improved version of (1.2) taking the distance between different connected components of S_t into account, see Corollary 7.1 below.

More generally, we can consider p-th moments and ask for which $p \in (0, \infty)$ we have

$$\int_{S_t(\Omega)} |x - c(t)|^p \, \mathrm{d}\mathcal{H}^1(x) \leq \int_{S_t(\mathcal{B})} |x|^p \, \mathrm{d}\mathcal{H}^1(x) \quad \text{for almost every } t \geq 0, \tag{1.6}$$

where $c(t)$ is the centroid of $S_t(\Omega)$ and $\mathcal{B}$ is a disk centred at the origin with $|\partial\Omega| = |\partial\mathcal{B}|$. For $t \geq 0$ small enough, we have $|S_t(\Omega)| = |S_t(\mathcal{B})|$ and $S_t(\Omega)$ is a closed curve, see Lemma 2.3 (ii) below, so (1.6) reduces to

$$\frac{(2\pi)^p}{|S_t(\Omega)|^{p+1}} \int_{S_t(\Omega)} |x - c(t)|^p \, \mathrm{d}\mathcal{H}^1(x) \leq 1.$$

So we may ask if we have

$$\frac{(2\pi)^p}{|\Gamma|^{p+1}} \int_{\Gamma} |x|^p \, \mathrm{d}\,\mathcal{H}^1(x) \leq 1 \tag{1.7}$$

for all closed Lipschitz curves Γ with the origin as its centroid.

Note that the centroid $c(t)$ is independent of t for all centrally symmetric domains Ω, or for example for domains with two not necessarily orthogonal axes of symmetry. To keep things simple, we focus on the centrally symmetric case.

Theorem 1.3 (An isoperimetric inequality for p-th moments). *The followings statements hold.*

(i) *The statement of Theorem 1.1 extends to (1.6) in the case $p \in (0, 2]$.*

(ii) *For $p < 3$, the boundary of a disk is a strict local maximiser among nearly circular, centrally symmetric closed Lipschitz curves Γ of the left-hand side in the inequality (1.7).*

(iii) *For $p > 3$, (1.7) does not hold, not even locally near boundary of the disk. More precisely, there exists a sequence of nearly circular, centrally symmetric closed Lipschitz curves $(\Gamma_n)_{n\in\mathbb{N}}$ converging uniformly to the boundary of the disk for which*

$$\frac{(2\pi)^p}{|\Gamma_n|^{p+1}} \int_{\Gamma_n} |x|^p \, \mathrm{d}\,\mathcal{H}^1(x) > 1.$$

This naturally leads to the following conjecture.

Conjecture 1.4. (1.7) *holds for all $p \leq 3$ and all closed Lipschitz curves Γ with the origin as its centroid.*

In the case $p \in (0, 2]$, Theorem 1.3 follows from Theorem 1.1 and Theorem 1.2 using Jensen's inequality. For the local optimality for $p < 3$ in Theorem 1.3 (ii), we follow a Fuglede-type argument [12]. From these computations, we also obtain Theorem 1.3 (iii), where symmetry breaking occurs for $p > 3$.

Theorem 1.1 and Theorem 1.3 are of general interest as (weighted) isoperimetric inequalities have recently received great attention [1, 3, 7, 10], see also [5, 9, 13, 24, 25] on quantitative isoperimetric inequalities. In the present paper, we consider the moment of inertia of the inner parallel curves S_t and compare it with the corresponding quantity for a disk of the same perimeter. This is a relatively unusual setting as our constraints do not involve the area of the domain Ω, but only its perimeter. Note that under fixed area constraint the p-th moment of the boundary is minimised by the disk for all $p \geq 1$ (cf. [3, Theorem 2.1]), which is in contrast to our result under fixed perimeter.

In the case of centrally symmetric domains, or more generally for domains Ω for which the centroid $c(t)$ defined in (1.3) is independent of t, we can deduce from Theorem 1.1 a result going back to Hadwiger [14], see (7.1) by integrating over t.

As an application of Theorem 1.1, we obtain an isoperimetric inequality for the magnetic Robin Laplacian. More precisely, considering the magnetic Robin Laplacian with a negative boundary parameter β and a sufficiently small constant magnetic field b, the ground state energy is expressed as follows:

$$\lambda_1^{\beta,b}(\Omega) = \inf_{\substack{u\in H^1(\Omega)\\ \|u\|_{L^2(\Omega)}=1}} \left(\int_\Omega |(-\mathrm{i}\nabla - b\mathbf{A})u|^2 + \beta \int_{\partial\Omega} |u|^2 \,\mathrm{d}\,\mathcal{H}^1(x) \right),$$

where $\mathbf{A}$ is a vector field in Ω with $\operatorname{curl} \mathbf{A} = 1$. It was shown in [19, Theorem 4.8] that the corresponding ground state energies for convex and centrally symmetric domains Ω and a disk $\mathcal{B}$ of the same perimeter satisfy $\lambda_1^{\beta,b}(\Omega) \le \lambda_1^{\beta,b}(\mathcal{B})$. Using Theorem 1.1, we can remove the convexity assumption on Ω.

Theorem 1.5 (An isoperimetric inequality for the magnetic Robin Laplacian). *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply-connected domain. Assume that Ω is centrally symmetric or, more generally, that the centroid of $S_t(\Omega)$ is independent of t for all $t \ge 0$ with $S_t(\Omega) \neq \varnothing$. Let $\beta < 0$ be the negative Robin parameter, and let $0 < b < b_0(|\partial\Omega|, \beta)$, where $b_0(|\partial\Omega|, \beta)$ depends on $|\partial\Omega|$ and β. Then the lowest eigenvalue of the magnetic Robin Laplacian on Ω with constant magnetic field of strength b and Robin boundary conditions with parameter β satisfies*

$$\lambda_1^{\beta,b}(\Omega) \le \lambda_1^{\beta,b}(\mathcal{B}), \tag{1.8}$$

where $\mathcal{B} \subset \mathbb{R}^2$ is the disk having the same perimeter as Ω. Equality in (1.8) occurs if and only if Ω is a disk.

If $|\partial\Omega| = 2\pi$, then we have the explicit expression $b_0(|\partial\Omega|, \beta) = \min(1, 4\sqrt{-\beta})$.

Structure of the paper. The rest of the paper is organised as follows. In Section 2, we introduce some notation and auxiliary results on inner parallel curves. In Section 3, we prove Theorem 1.2. We use this in Section 4 to prove Theorem 1.1. In Section 5, we show Theorem 1.3. More precisely, the proof and the precise statement of Theorem 1.3 (i) can be found in Corollary 5.1 and for Theorem 1.3 (ii), (iii) we refer to Proposition 5.6. Some background material on the magnetic Robin Laplacian and the proof of Theorem 1.5 are given in Section 6. In Section 7, we present two simple applications of Theorem 1.1 and Theorem 1.2, namely Corollary 7.1 on an improved version of (1.2), and Section 7.2 on moments of inertia of domains.

2. Preliminaries

2.1. Notation

We introduce for a piecewise-C^1 mapping $\gamma\colon [0,L] \to \mathbb{R}^2$ the length of the closed and not necessarily simple curve parametrised by γ,

$$\ell(\gamma) := \int_0^L |\gamma'(s)|\,\mathrm{d}s.$$

We also use the notation $\gamma([a,b]) = \{\gamma(s)\colon s \in [a,b]\}$ for $a, b \in [0,L]$, $a < b$. We say that $\gamma_1, \gamma_2\colon [0,L] \to \mathbb{R}^2$ *parametrise the same curve* if there exists a continuous bijection $\psi\colon [0,L] \to [0,L]$ such that $\gamma_1 = \gamma_2 \circ \psi$. A subset $U \subset \mathbb{R}^2$ is said to be *centrally symmetric* if it coincides with its reflection $\{-x\colon x \in U\}$ with respect to the origin.

2.2. Inner parallel curves

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply-connected smooth domain. In this subsection we recall some properties of the inner parallel curves of Ω.

Parametrisation of the boundary. Let us denote by $L = |\partial\Omega|$ the perimeter of Ω. Consider the arc-length parametrisation of $\partial\Omega$ oriented in the counter-clockwise direction,

$$s \in \mathbb{R}/(L\mathbb{Z}) \mapsto \gamma(s) = (\gamma_1(s), \gamma_2(s))^\top \in \mathbb{R}^2,$$

which identifies $\partial\Omega$ with $\mathbb{R}/(L\mathbb{Z}) \simeq [0,L)$; the function γ is smooth which matches with the smoothness hypothesis we imposed on $\partial\Omega$.

The vector $\gamma'(s) = (\gamma_1'(s), \gamma_2'(s))^\top$ is the unit tangent vector to $\partial\Omega$ at $\gamma(s)$ and points in the counter-clockwise direction. The unit normal vector at $\gamma(s)$ pointing inwards the domain Ω is given by

$$\mathbf{n}(s) = (-\gamma_2'(s), \gamma_1'(s))^\top.$$

We introduce the curvature

$$\kappa(s) := \gamma_2''(s)\gamma_1'(s) - \gamma_1''(s)\gamma_2'(s) \tag{2.1}$$

of $\partial\Omega$ at the point $\gamma(s)$. In particular, the Frenet formula

$$\gamma''(s) = \kappa(s)\mathbf{n}(s), \tag{2.2}$$

holds. Recall that, since $\partial\Omega$ is a smooth closed simple curve, the total curvature identity [20, Corollary 2.2.2] yields

$$\int_0^L \kappa(s)\,\mathrm{d}s = 2\pi. \tag{2.3}$$

We remark that within the chosen sign convention the curvature of a convex domain is non-negative.

Properties of inner parallel curves. We define the *in-radius* of Ω by

$$r_{\mathrm{i}}(\Omega) := \max_{x\in\Omega} \rho(x),$$

where ρ is the distance function given by

$$\rho\colon \Omega \to \mathbb{R}_+, \quad \rho(x) := \inf_{y\in\partial\Omega} |x-y|. \tag{2.4}$$

Recall that the inner parallel curve for Ω is the level set of the distance function

$$S_t = \{x \in \overline{\Omega}\colon \rho(x) = t\}, \quad t \in [0, r_{\mathrm{i}}(\Omega)).$$

For almost every $t \in (0, r_i(\Omega))$, the inner parallel curve S_t is a finite union of disjoint piecewise smooth simple closed curves, and the curve S_t admits a parametrisation as in Proposition 2.1 below, which was proved in [16], see also [28, Theorem 4.4.1] and [27, Proposition A.1] for more modern presentations and further refinements.

Proposition 2.1. *There exists a subset $\mathcal{L} \subset [0, r_{\mathrm{i}}(\Omega))$, whose complement is of Lebesgue measure zero, such that for any $t \in \mathcal{L}$, there exist $m \in \mathbb{N}$ and*

$$0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m \le L,$$

such that the inner parallel curve S_t consists of the union of finitely many smooth curves parametrised by

$$[a_k, b_k] \ni s \mapsto \gamma(s) + t\mathbf{n}(s), \quad k \in \{1, 2, \ldots, m\},$$

which forms a union of finitely many piecewise-smooth simple closed curves.

Consider the mapping

$$(s,t) \in \mathbb{R}/(L\mathbb{Z}) \times \big(0, r_{\mathrm{i}}(\Omega)\big) \mapsto \Phi(s,t) := \gamma(s) + t\mathbf{n}(s) \in \mathbb{R}^2 \tag{2.5}$$

According to [21, Theorem 5.25], there exists $t_\star \in (0, r_{\mathrm{i}}(\Omega))$ such that the restriction of the mapping Φ in (2.5) to the set $\mathbb{R}/(L\mathbb{Z}) \times (0, t_\star)$ is a smooth diffeomorphism onto its range. The range of this restriction is then given by a tubular neighbourhood of $\partial\Omega$. It is not difficult to verify that $S_t = \Phi(\mathbb{R}/(L\mathbb{Z}), t)$ for all $t \in (0, t_\star)$. However, for $t \geq t_\star$ the same property, in general, does not hold. It can also be easily checked that for all $t \in (0, t_\star)$, the inner parallel curve S_t is connected and $|S_t| = L - 2\pi t$.

Lemma 2.2. *Let $t \in \mathcal{L}$ and the associated numbers $m \in \mathbb{N}$, $\{a_k\}_{k=1}^m$, $\{b_k\}_{k=1}^m$, be as in Proposition 2.1. Then, for any $k \in \{1, 2, \ldots, m\}$ and any $s_0 \in [a_k, b_k]$, it holds that $\kappa(s_0) \leq \frac{1}{t}$.*

Proof. Let us introduce the notation

$$\mathsf{m}(t) := \gamma(s_0) + t\mathbf{n}(s_0).$$

By the Frenet formula (2.2), one has $\gamma''(s_0) = \kappa(s_0)\mathbf{n}(s_0)$ and by Taylor's formula near s_0 we get

$$\gamma(s) = \gamma(s_0) + (s - s_0)\gamma'(s_0) + \frac{1}{2}(s - s_0)^2\kappa(s_0)\mathbf{n}(s_0) + \mathcal{O}(|s - s_0|^3), \quad s \to s_0.$$

Consequently, using orthogonality of $\gamma'(s_0)$ and $\mathbf{n}(s_0)$, we get

$$\operatorname{dist}(\gamma(s), \mathsf{m}(t))^2 = t^2 + (1 - t\kappa(s_0))(s - s_0)^2 + \mathcal{O}(|s - s_0|^3), \quad s \to s_0.$$

Since $\mathsf{m}(t) \in S_t$, then $\operatorname{dist}(\gamma(s), \mathsf{m}(t)) \geq t$ for s in a neighbourhood of s_0, which is possible only when $1 - t\kappa(s_0) \geq 0$. ■

In the next lemma we provide a simple construction of a closed but not necessarily simple curve which contains S_t. The geometric bound as in Theorem 1.2 on its length will only hold for t not larger than the inverse of the maximum of the curvature for the curve γ:

$$\kappa_{\max}(\Omega) := \max_{s \in \mathbb{R}/(L\mathbb{Z})} \kappa(s)$$

Lemma 2.3. *For $t \in (0, r_{\mathrm{i}}(\Omega))$, the mapping*

$$s \in \mathbb{R}/(L\mathbb{Z}) \mapsto \alpha_t(s) := \gamma(s) + t\mathbf{n}(s) \in \mathbb{R}^2$$

parametrises a smooth closed, not necessarily simple curve such that

(i) $S_t \subset \alpha_t([0, L])$ *for all* $t \in \mathcal{L}$;

(ii) $\ell(\alpha_t) = L - 2\pi t$ *for all* $t \leq \frac{1}{\kappa_{\max}(\Omega)}$;

(iii) $\ell(\alpha_t) > L - 2\pi t$ *for all* $\frac{1}{\kappa_{\max}(\Omega)} < t < r_{\mathrm{i}}(\Omega)$.

Remark 2.4. According to [26], the domain Ω contains a disk of radius $\frac{1}{\kappa_{\max}(\Omega)}$. In other words, it holds that

$$r_{\mathrm{i}}(\Omega) \geq \frac{1}{\kappa_{\max}(\Omega)}. \tag{2.6}$$

The equality occurs for some special types of domains such as a disk or (going beyond smooth domains) for a convex hull of two disjoint disks of equal radius. The original work [26] is hardly available and the complete proof can be found in [17, Proposition 2.1]. Inequality (2.6) shows that, in general, a more sophisticated method than in Lemma 2.3 is needed to construct for any $t \in (0, r_{\mathrm{i}}(\Omega))$ a closed curve of length not larger than $L - 2\pi t$, which contains the inner parallel curve S_t.

Proof of Lemma 2.3. Notice that smoothness of γ on $\mathbb{R}/(L\mathbb{Z})$ ensures that α_t is smooth on $\mathbb{R}/(L\mathbb{Z})$ as well. It is also clear from Proposition 2.1 that $S_t \subset \alpha_t([0, L])$ for all $t \in \mathcal{L}$.

Using the identity (2.3), we get for any $t \leq \frac{1}{\kappa_{\max}(\Omega)}$

$$\ell(\alpha_t) = \int_0^L |\dot{\alpha}(s)| \,\mathrm{d}s = \int_0^L |1 - t\kappa(s)| \,\mathrm{d}s = \int_0^L (1 - t\kappa(s)) \,\mathrm{d}s = L - 2\pi t.$$

Analogously, we get for any $t \in \left(\frac{1}{\kappa_{\max}(\Omega)}, r_{\mathrm{i}}(\Omega)\right)$

$$\ell(\alpha_t) = \int_0^L |\dot{\alpha}(s)| \,\mathrm{d}s = \int_0^L |1 - t\kappa(s)| \,\mathrm{d}s > \int_0^L (1 - t\kappa(s)) \,\mathrm{d}s = L - 2\pi t. \qquad \blacksquare$$

In the remainder of this subsection we will discuss the properties of S_t for a centrally symmetric domain Ω. The central symmetry of Ω is inherited by $\partial\Omega$. Consequently, if $y = \gamma(s) \in \partial\Omega$, we know that $-y \in \partial\Omega$ too; moreover the centroid of $\partial\Omega$ is the origin, so

$$\int_0^L \gamma(s) \,\mathrm{d}s = 0.$$

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply-connected, centrally symmetric smooth domain. Then, for all $t \in (0, r_{\mathrm{i}}(\Omega))$, the inner parallel curve $S_t \subset \Omega$ is centrally symmetric.*

Proof. Let $x \in S_t$ be fixed. Then $\rho(x) = t$ and there exists a point $y \in \partial\Omega$ such that $|x - y| = t$. Observe now that $\rho(-x) \leq t$, because $-y \in \partial\Omega$. In the case that $\rho(-x) < t$, there would exist a point $z \in \partial\Omega$ such that $|z + x| < t$. Since $-z \in \partial\Omega$, we would get that $\rho(x) < t$, leading to a contradiction. Thus, we infer that $\rho(-x) = t$ and hence $-x \in S_t$. $\blacksquare$

2.3. An auxiliary geometric inequality

The aim of this subsection is to provide a geometric inequality, which will be used in the proof of Theorem 1.2.

Hypothesis 2.6. Let $c_1, c_2 \in \mathbb{R}^2$ and $t > 0$ be fixed. Let a smooth simple non-closed curve $\Gamma \subset \mathbb{R}^2$ be parametrised by the arc-length via the mapping $\gamma\colon [s_1, s_2] \to \mathbb{R}^2$, $s_1 < s_2$. Assume that the following properties hold:

- (i) $p_j := \gamma(s_j) \in \partial\mathcal{B}_t(c_j)$ for $j = 1, 2$;
- (ii) $\gamma'(s_j)$ is tangent to $\partial\mathcal{B}_t(c_j)$ in the counterclockwise direction for $j = 1, 2$.y;
- (iii) Γ can be extended up to a closed simple curve so that $\mathcal{B}_t(c_1) \cup \mathcal{B}_t(c_2)$ is surrounded by this extension.

Proposition 2.7. *Under Hypothesis 2.6 the following geometric inequality holds:*

$$|\Gamma| \geq |c_1 - c_2| + t \int_{s_1}^{s_2} \kappa(s)\,\mathrm{d}s, \tag{2.7}$$

where κ is the curvature of Γ defined as in (2.1).

Proof. The proof relies on an abstract result due to Chillingworth [8, Theorem 3.3], which states that two closed homotopic curves are regularly homotopic if the curves are direct, that is, the corresponding curves in the covering space are simple.

The curve Γ is homotopic to its projection Σ on the convex hull of the two disks. After modifying Σ suitably so we avoid nullhomotopic loops, we can extend Γ and Σ to closed direct curves $\widetilde{\Gamma}$, $\widetilde{\Sigma}$ using the same extension. By [8, Theorem 3.3], $\widetilde{\Gamma}$ and $\widetilde{\Sigma}$ are regularly homotopic, so the integral over their curvatures agree: $\int_{\widetilde{\Gamma}} \kappa = \int_{\widetilde{\Sigma}} \kappa$, see for example [30]. Since Γ and Σ were extended using the same extension, we also get $\int_\Gamma \kappa = \int_\Sigma \kappa$. Finally, one can check that Σ satisfies (2.7) and by $|\Gamma| \geq |\Sigma|$, we obtain (2.7) for Γ. ∎

3. Proof of Theorem 1.2 – Covering inner parallel curves

The aim of this section is to prove the following theorem, which yields Theorem 1.2.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply-connected domain and let S_t for $t \geq 0$ be the corresponding inner parallel curves defined in (1.1). Then there exists a subset $\mathcal{L} \subset [0, r_\mathrm{i}(\Omega))$ such that $(0, r_\mathrm{i}(\Omega)) \setminus \mathcal{L}$ is of Lebesgue measure zero, and for any $t \in \mathcal{L}$, there exists a piecewise smooth continuous mapping $\sigma_t\colon \mathbb{R}/(L\mathbb{Z}) \to \mathbb{R}^2$ such that*

(i) $S_t \subset \sigma_t([0, L])$;

(ii) $\ell(\sigma_t) \le L - 2\pi t$;

(iii) *for centrally symmetric domains* Ω*, the curve* $\sigma_t([0, L])$ *is centrally symmetric too.*

In the following, let the set $\mathcal{L}$ be as in Proposition 2.1. Let $t \in \mathcal{L}$ be fixed. Then, there exist $m \in \mathbb{N}$ and

$$0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m \le L$$

such that the inner parallel curve S_t is given by

$$S_t = \bigcup_{k=1}^{m} \{\gamma(s) + t\mathbf{n}(s) : s \in [a_k, b_k]\}.$$

Without loss of generality, we can always reparametrise the boundary of Ω so that $a_1 = 0$. In the following we assume that such a re-parametrisation is performed and that $a_1 = 0$. Moreover, for the sake of convenience we also set $a_{m+1} := L$ and $b_{m+1} := b_1$.

The inner parallel curve S_t, $t \in \mathcal{L}$, is not necessarily connected and, in general, it consists of finitely many piecewise-smooth, simple, closed curves. Note that, while S_t consists of finitely many piecewise smooth simple closed curves, the pieces $C_k := \{\gamma(s) + t\mathbf{n}(s) : s \in [a_k, b_k]\}$ are not necessarily closed curves. Our aim is to construct a piecewise smooth, closed, not necessarily simple curve, which contains S_t and whose length is not larger than $L - 2\pi t$. The idea is to connect the terminal point of $\{\gamma(s) + t\mathbf{n}(s) : s \in [a_k, b_k]\}$ with the starting point of $\{\gamma(s) + t\mathbf{n}(s) : s \in [a_{k+1}, b_{k+1}]\}$ for all $k \in \{1, 2, \dots, m\}$.

Using the computation in the proof of Lemma 2.3, and that $\kappa(s) \le \frac{1}{t}$ for all $s \in [a_k, b_k]$ by Lemma 2.2, we get the following expression for the length of S_t:

$$|S_t| = \sum_{k=1}^{m} \int_{a_k}^{b_k} |1 - t\kappa(s)| \, \mathrm{d}s = \sum_{k=1}^{m} \int_{a_k}^{b_k} (1 - t\kappa(s)) \, \mathrm{d}s. \tag{3.1}$$

Let us define the points

$$\mathfrak{p}_k = \gamma(a_k) + t\mathbf{n}(a_k), \quad \mathfrak{q}_k := \gamma(b_k) + t\mathbf{n}(b_k), \quad \text{for } k \in \{1, 2, \dots, m+1\},$$

and the line segments connecting them

$$\mathcal{I}_k := \{(1 - s)\mathfrak{q}_k + s\mathfrak{p}_{k+1} : s \in [0, 1]\}, \quad k \in \{1, 2, \dots, m\}.$$

In the case that $b_m = L$, the line segment $\mathcal{I}_m$ reduces to a single point. Note that, even when $b_k < a_{k+1}$, the line segment $\mathcal{I}_k$ could reduce to a single point. The piecewise smooth continuous mapping $\sigma_t \colon \mathbb{R}/(L\mathbb{Z}) \to \mathbb{R}^2$ defined by

$$\sigma_t(s) = \begin{cases} \gamma(s) + t\mathbf{n}(s), & s \in [a_k, b_k] \quad \text{for } k \in \{1, \ldots, m\}, \\ \dfrac{a_{k+1} - s}{a_{k+1} - b_k}\mathfrak{q}_k + \dfrac{s - b_k}{a_{k+1} - b_k}\mathfrak{p}_{k+1}, & s \in [b_k, a_{k+1}] \quad \text{for } k \in \{1, \ldots, m\}, \end{cases}$$

parametrises a closed, not necessarily simple curve in $\mathbb{R}^2$. The property (i) in the formulation of Theorem 3.1 follows from Proposition 2.1 and the construction of the mapping σ_t.

Note that the curve $\gamma([b_k, a_{k+1}])$, $k \in \{1, \ldots, m\}$, satisfies Hypothesis 2.6 with $c_1 = \mathfrak{q}_k$ and $c_2 = \mathfrak{p}_{k+1}$. Thus, it follows from Proposition 2.7 and since γ is parametrised by arc-length that for any $k \in \{1, 2, \ldots, m\}$

$$|\mathcal{I}_k| \le |\gamma([b_k, a_{k+1}])| - t\int_{b_k}^{a_{k+1}} \kappa(s)\,\mathrm{d}s = a_{k+1} - b_k - t\int_{b_k}^{a_{k+1}} \kappa(s)\,\mathrm{d}s.$$

Combining formula (3.1) for the length of S_t with the upper bounds on the lengths of line segments $\{\mathcal{I}_k\}_{k=1}^m$ we get

$$\begin{aligned} \ell(\sigma_t) = |S_t| + \sum_{k=1}^m |\mathcal{I}_k| &\le \sum_{k=1}^m \int_{a_k}^{b_k} (1 - t\kappa(s))\,\mathrm{d}s + \sum_{k=1}^m \int_{b_k}^{a_{k+1}} (1 - t\kappa(s))\,\mathrm{d}s \\ &\le L - t\int_0^L \kappa(s)\,\mathrm{d}s = L - 2\pi t, \end{aligned}$$

where we used the total curvature identity (2.3) in the last step. Hence, we get the property (ii) in the formulation of Theorem 3.1.

Finally, if Ω is centrally symmetric, then by Lemma 2.5 so is S_t. Consequently, to every piece C_k of S_t joining $\mathfrak{p}_k$ to $\mathfrak{q}_k$, there corresponds a curve C_{k^*} which is the symmetric of C_k about the origin. This forces the number m of the curves C_k to be even, unless it is equal to one, and therefore we get that the corresponding joining segments $(\mathcal{I}_k)_{1 \le k \le m}$ constitute a centrally symmetric set. This proves that the image of σ_t is centrally symmetric, thereby establishing (iii) in the formulation of Theorem 3.1. This proves Theorem 1.2.

4. Proof of Theorem 1.1 – An isoperimetric inequality for moments of inertia

Let $t \in \mathcal{L}$ and the mapping σ_t be as constructed in the proof of Theorem 3.1, which defines a closed curve Σ_t. Since the moment of inertia of a curve about a point p is minimal when p is the centroid of the curve, it suffices to prove (1.5) with $c(t)$ the centroid of Σ_t. Let us introduce the notation $L_t := \ell(\sigma_t)$ and re-parametrise the curve Σ_t by the arc-length via the mapping $\widetilde{\sigma}_t : \mathbb{R}/(L_t\mathbb{Z}) \to \mathbb{R}^2$. Clearly, we have $\widetilde{\sigma}_t \in H^1(\mathbb{R}/(L_t\mathbb{Z}))$ thanks to the regularity of σ_t. Furthermore, by centring the coordinates at the centroid $c(t)$ of Σ_t, we can assume that $c(t) = 0$, and consequently

$$\int_0^{L_t} \widetilde{\sigma}_t(s)\,\mathrm{d}s = 0. \tag{4.1}$$

Using the inclusion $S_t \subset \widetilde{\Sigma}_t$ and applying the Wirtinger inequality [15, Section 7.7],

$$\int_0^{L_t} |\widetilde{\sigma}_t(s)|^2\,\mathrm{d}s \le \frac{|L_t|^2}{4\pi^2} \int_0^{L_t} |\widetilde{\sigma}_t'(s)|^2\,\mathrm{d}s, \tag{4.2}$$

we get

$$\int_{S_t} |x|^2\,\mathrm{d}\mathcal{H}^1(x) \le \int_0^{L_t} |\widetilde{\sigma}_t(s)|^2\,\mathrm{d}s \le \frac{|L_t|^3}{4\pi^2} \le \frac{(L-2\pi t)^3}{4\pi^2}, \tag{4.3}$$

where we employed that $L_t \le L - 2\pi t$ in the last step. Therefore, (1.5) is proved.

Assuming that there is equality in (1.5), then we get from (4.3) that $L_t = L - 2\pi t$ and there is equality is (4.2). Under the conditions (4.1) and $|\tilde{\sigma}_t'(s)| = 1$, equality happens in (4.2) if and only if $\tilde{\sigma}_t(s) = \frac{L_t}{2\pi}\mathrm{e}^{\pm \mathrm{i}2\pi(s-s_0)/L_t}$ for some $s_0 \in \mathbb{R}$ and Σ_t is a circle (here we identify $\mathbb{C}$ and $\mathbb{R}^2$). Moreover, knowing that $L_t = L - 2\pi t$ and $S_t = \Sigma_t$, we get that S_t is the circle of centre 0 and radius $\frac{L}{2\pi} - t$, and consequently, the domain Ω with perimeter L contains the disk $\mathcal{B}$ of radius $\frac{L}{2\pi}$, hence $\Omega = \mathcal{B}$ is a disk, thanks to the geometric isoperimetric inequality.

Finally, if Ω is a disk, then S_t is a circle of radius $\frac{L}{2\pi} - t$ and equality in (1.5) occurs.

Remark 4.1. The hypothesis of simple connectivity is necessary in Theorem 1.1 as we demonstrate in the following example. Let $a \in \left(0, \frac{1}{4}\right)$ be a parameter, which we will send to zero later. Consider the annulus $\Omega_a := B_{1-a}(0) \setminus \overline{B_a(0)}$, where we denote by $B_r(x)$ the open disk centred at $x \in \mathbb{R}^2$ of radius $r > 0$. Then the corresponding $\mathcal{B}$ with the same perimeter as Ω_a is $\mathcal{B} := B_1(0)$. Furthermore, for all $t < \frac{1}{4}$, we have $S_t(\Omega_a) = \partial B_{a+t}(0) \cup \partial B_{1-(a+t)}(0)$, and $S_t(\mathcal{B}) = \partial B_{1-t}(0)$. Also note that the centroid of each $S_t(\Omega_a)$ is the origin. Then

$$\frac{1}{2\pi} \int_{S_t(\Omega_a)} |x|^2 \, \mathrm{d}\,\mathcal{H}^1(x) = (a+t)^3 + (1-(a+t))^3$$

and

$$\frac{1}{2\pi} \int_{S_t(\mathcal{B})} |x|^2 \, \mathrm{d}\,\mathcal{H}^1(x) = (1-t)^3.$$

Letting $a \to 0$, we find that

$$\frac{1}{2\pi} \int_{S_t(\Omega_a)} |x|^2 \, \mathrm{d}\,\mathcal{H}^1(x) \to t^3 + (1-t)^3 > \frac{1}{2\pi} \int_{S_t(\mathcal{B})} |x|^2 \, \mathrm{d}\,\mathcal{H}^1(x),$$

where the convergence is uniform for $t \in [t_1, t_2]$ for any $0 < t_1 < t_2 < \frac{1}{4}$. This shows that Theorem 1.1 cannot hold in this case.

5. Proof of Theorem 1.3 – An isoperimetric inequality for p-th moments

In this section, we study p-th moments of inner parallel curves and prove Theorem 1.3. We show Theorem 1.3 (i) (extension of Theorem 1.1 to p-th moments, for $0 \leq p \leq 2$) in Corollary 5.1 with the help of Jensen's inequality. Proposition 5.6 yields Theorem 1.3 (ii) and (iii).

Corollary 5.1. *Let $p \in [0,2]$ and suppose that $\Omega \subset \mathbb{R}^2$ is a smooth, bounded and simply-connected domain. Then, for almost every $t \geq 0$,*

$$\int_{S_t(\Omega)} |x - c(t)|^p \, \mathrm{d}\,\mathcal{H}^1(x) \leq \int_{S_t(\mathcal{B})} |x|^p \, \mathrm{d}\,\mathcal{H}^1(x), \tag{5.1}$$

where $c(t) \in \mathbb{R}^2$ is the centroid of $S_t(\Omega)$ and where $\mathcal{B}$ is the disk centred at the origin and with the same perimeter as Ω. Here $S_t(\cdot)$ is defined in (1.1), and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. For $p \neq 0$ and $t \in \left[0, \frac{|\partial\Omega|}{2\pi}\right)$, the equality in (5.1) is attained if and only if Ω is a disk

Proof. For $p = 0$, (5.1) reduces to the well-known bound $|S_t| \leq L - 2\pi t$, see (1.2).

Let us take $p \in (0,2]$. Since $\frac{2}{p} \geq 1$, we write by Jensen's inequality,

$$\left(\int_{S_t(\Omega)} |x - c(t)|^p \, \mathrm{d}\,\mathcal{H}^1(x) \right)^{2/p} \leq |S_t(\Omega)|^{\frac{2}{p}-1} \int_{S_t(\Omega)} |x - c(t)|^2 \, \mathrm{d}\,\mathcal{H}^1(x).$$

To finish the proof, we use that $|S_t| \leq L - 2\pi t$, apply Theorem 1.1, and note that

$$\int_{S_t(\mathcal{B})} |x|^p \,\mathrm{d}\mathcal{H}^1(x) = 2\pi\Big(\frac{L}{2\pi} - t\Big)^{1+p}.$$

Since the inequality in Theorem 1.1 is strict for all $t \in \big[0, \frac{|\partial\Omega|}{2\pi}\big)$ when Ω is not a disk, this also holds for (5.1). ∎

Secondly, we formulate the corresponding variational problem to Theorem 1.3.

Definition 5.2. Given $p > 0$, we define

$$C_p := \sup_{\Gamma} \frac{\int_{\Gamma} |x|^p \,\mathrm{d}\mathcal{H}^1(x)}{\int_{\partial\mathcal{B}} |x|^p \,\mathrm{d}\mathcal{H}^1(x)} = \sup_{\Gamma} \frac{(2\pi)^p}{|\Gamma|^{p+1}} \int_{\Gamma} |x|^p \,\mathrm{d}\mathcal{H}^1(x), \tag{5.2}$$

where the supremum is taken over all centrally symmetric, closed Lipschitz curves Γ and $\mathcal{B}$ is a disk centred at the origin with $|\partial\mathcal{B}| = |\Gamma|$.

Remark 5.3. By scaling, we find that $C_p \in (0, \infty)$. By testing with $\Gamma = \partial\mathcal{B}$, we find $C_p \geq 1$ for all p. We have already shown that $C_p = 1$ for all $p \in (0, 2]$, see the end of the proof of Theorem 1.1 (or alternatively the result by Hurwitz [18]) combined with Jensen's inequality as in the proof of Corollary 5.1. In fact, using Jensen's inequality as in the proof of Corollary 5.1, one can show that C_p is non-decreasing in p.

Remark 5.4 (Existence of an optimising curve in (5.2)). We only sketch a way of proving the existence of an optimising curve in (5.2) here. First note that by scaling, we can restrict ourselves to smooth centrally symmetric curves satisfying $|\Gamma| = 1$. We can approximate these curves by piecewise linear centrally symmetric curves with $|\Gamma| = 1$, so we can take the supremum over such curves instead. Using a finite-step mirroring argument, the supremum stays the same if we only consider convex piecewise linear centrally symmetric curves with $|\Gamma| = 1$. More precisely, in this context we say that a piecewise linear curve is convex if it is the boundary of a convex polygon.

Now, consider a sequence of such curves $(\Gamma_n)_{n\in\mathbb{N}}$ such that the corresponding expression in the supremum in (5.2) converges to C_p. Assume that the Γ_n are parametrised by arc-length by $\gamma_n\colon [0,1] \to \mathbb{R}^2$. By $|\Gamma_n| = 1$ and the central symmetry, we have $\Gamma_n \subset B_1(0)$. Due to the arc-length parametrisation of the γ_n, we obtain that up to a subsequence, they converge uniformly to a Lipschitz continuous function $\gamma\colon [0,1] \to \mathbb{R}^2$ with Lipschitz constant at most one. To see this, one can for instance use the Arzelà–Ascoli theorem. Since the Γ_n are convex, the corresponding curve Γ is convex, too.

Furthermore, we have $|\Gamma| = 1$. Proving this is the key step and it uses the convexity. It can be seen by approximating Γ by piecewise linear convex centrally symmetric curves with the same length as Γ, and then showing that the lengths of these piecewise linear convex curves have to be close to $|\Gamma_n| = 1$. Finally, using that γ is Lipschitz continuous with Lipschitz constant at most one and $|\Gamma| = 1$, we get $|\gamma'(s)| = 1$ almost everywhere, so

$$\lim_{n\to\infty} \int_{\Gamma_n} |x|^p \, \mathrm{d}\mathcal{H}^1(x) = \lim_{n\to\infty} \int_0^1 |\gamma_n(s)|^p \, \mathrm{d}s = \int_0^1 |\gamma(s)|^p \, \mathrm{d}s = \int_{\Gamma} |x|^p \, \mathrm{d}\mathcal{H}^1(x).$$

Together with $|\Gamma| = 1$, this proves the optimality of the convex curve Γ. Also note that any optimal curve needs to be convex.

The next proposition shows that C_p is not constant.

Proposition 5.5. *We have $C_p > 1$ for all p large enough.*

Proof. Consider the curve Γ parametrised by $\gamma\colon [0,1] \to \mathbb{R}^2$, $\gamma(s) = (\gamma_1(s), \gamma_2(s))^\top$ with $\gamma_2(s) = 0$ for all $s \in [0,1]$ and

$$\gamma_1(s) = \begin{cases} -\frac{1}{4} + s & \text{for } s \in \left[0, \frac{1}{2}\right], \\ \frac{3}{4} - s & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Γ is a closed curve that is parametrised by arc-length, centrally symmetric, its shape is the doubly covered interval $[-\frac{1}{4}, \frac{1}{4}] \times \{0\}$, and its length is $|\Gamma| = 1$. We have

$$\lim_{p\to\infty} \left(\int_{\Gamma} |x|^p \, \mathrm{d}\mathcal{H}^1(x) \right)^{1/p} = \lim_{p\to\infty} \left(\int_0^1 |\gamma_1(s)|^p \, \mathrm{d}s \right)^{1/p} = \sup_{s\in[0,1]} |\gamma_1(s)| = \frac{1}{4},$$

and therefore, by $|\Gamma| = 1$,

$$\lim_{p\to\infty} \left(\frac{(2\pi)^p}{|\Gamma|^{p+1}} \int_{\Gamma} |x|^p \, \mathrm{d}\mathcal{H}^1(x) \right)^{1/p} = \frac{2\pi}{4} > 1.$$

This proves $C_p > 1$ for all p large enough. ∎

This leads to the question of determining the critical value

$$p_* := \sup\{p : p > 0 \text{ and } C_p = 1\}.$$

The following proposition shows that $p_* \leq 3$, see (ii), which we conjecture to be optimal (see Conjecture 1.4) since the disk is a local optimiser for $p < 3$, see (i) below.

Proposition 5.6. *The following statements hold.*

(i) *Let $p < 3$. Then the disk is a local optimiser among centrally symmetric curves in the following sense. If $r\colon \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ is continuous with $r(\theta) = r(\theta + \pi)$ for all θ, and the curve Γ_ε is parametrised by $\gamma_\varepsilon\colon \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2$ with*

$$\gamma_\varepsilon(\theta) = \begin{pmatrix} R_\varepsilon(\theta)\cos(\theta) \\ R_\varepsilon(\theta)\sin(\theta) \end{pmatrix}, \quad \text{where } R_\varepsilon(\theta) = 1 + \varepsilon r(\theta), \tag{5.3}$$

then

$$\frac{(2\pi)^p}{|\Gamma_\varepsilon|^{p+1}} \int_{\Gamma_\varepsilon} |x|^p \,\mathrm{d}\mathcal{H}^1(x) \leq 1$$

for all $\varepsilon > 0$ small enough; furthermore, the inequality is strict if r is non-constant.

(ii) *Let $p > 3$. Then the disk is not optimal, not even locally: There exists a sequence of nearly circular, centrally symmetric closed Lipschitz curves $(\Gamma_n)_{n\in\mathbb{N}}$ converging uniformly to the boundary of the disk for which*

$$\frac{(2\pi)^p}{|\Gamma_n|^{p+1}} \int_{\Gamma_n} |x|^p \,\mathrm{d}\mathcal{H}^1(x) > 1.$$

Proof. Consider the curve Γ_ε defined by the parametrisation of Γ_ε in (5.3). Since we assume that $r(\theta) = r(\theta + \pi)$, the curve Γ_ε is centrally symmetric.

It is straightforward to check that with γ_ε defined as in (5.3),

$$\begin{aligned} |\gamma_\varepsilon(\theta)|^p &= 1 + \varepsilon p r(\theta) + \frac{\varepsilon^2}{2} p(p-1)|r(\theta)|^2 + \mathcal{O}(\varepsilon^3), \\ |\gamma_\varepsilon'(\theta)| &= 1 + \varepsilon r(\theta) + \frac{\varepsilon^2}{2}|r'(\theta)|^2 + \mathcal{O}(\varepsilon^3), \\ |\Gamma_\varepsilon| &= 2\pi + \varepsilon \int_0^{2\pi} r(\theta)\,\mathrm{d}\theta + \frac{\varepsilon^2}{2}\int_0^{2\pi} |r'(\theta)|^2\,\mathrm{d}\theta + \mathcal{O}(\varepsilon^3). \end{aligned}$$

With the above formulas in hand, we get

$$\frac{|\Gamma_\varepsilon|^{p+1}}{(2\pi)^p} = 2\pi + \varepsilon(p+1)\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta + \varepsilon^2(p+1)\Big(\frac{1}{2}\int_0^{2\pi} |r'(\theta)|^2\,\mathrm{d}\theta + \frac{p}{4\pi}\Big(\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta\Big)^2\Big) + \mathcal{O}(\varepsilon^3),$$

and

$$\int_{\Gamma_\varepsilon} |x|^p\,\mathrm{d}\mathcal{H}^1(x) - \frac{|\Gamma_\varepsilon|^{p+1}}{(2\pi)^p} = \frac{\varepsilon^2 p}{2}(\mathcal{F}(r) + \mathcal{O}(\varepsilon)), \tag{5.4}$$

where

$$\begin{aligned}\mathcal{F}(r) &= (p+1)\int_0^{2\pi} |r(\theta)|^2\,\mathrm{d}\theta - \frac{p+1}{2\pi}\Big(\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta\Big)^2 - \int_0^{2\pi} |r'(\theta)|^2\,\mathrm{d}\theta\\ &= \int_0^{2\pi}\Big[(p+1)\Big(r(\theta) - \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta\Big)^2 - |r'(\theta)|^2\Big]\,\mathrm{d}\theta.\end{aligned}$$

We expand r as a Fourier series and notice that the coefficients of the odd indices will vanish, thanks to the symmetry condition on r. More precisely, we have

$$\begin{aligned} r(\theta) - \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta &= \sum_{n\geq 2} a_n\cos(n\theta) + \sum_{n\geq 2} b_n\sin(n\theta),\\ r'(\theta) &= -\sum_{n\geq 2} n a_n\sin(n\theta) + \sum_{n\geq 2} n b_n\cos(n\theta).\end{aligned}$$

By Parseval's identity, we write

$$\begin{aligned}\int_0^{2\pi}\Big(r(\theta) - \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta\Big)^2\,\mathrm{d}\theta &= \pi\sum_{n\geq 2}\big(|a_n|^2 + |b_n|^2\big),\\ \int_0^{2\pi} |r'(\theta)|^2\,\mathrm{d}\theta &= \pi\sum_{n\geq 2} n^2\big(|a_n|^2 + |b_n|^2\big).\end{aligned}$$

Hence,

$$\int_0^{2\pi} |r'(\theta)|^2\,\mathrm{d}\theta \geq 4\int_0^{2\pi}\Big(r(\theta) - \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta\Big)^2\,\mathrm{d}\theta, \tag{5.5}$$

and consequently, for $p < 3$

$$\mathcal{F}(r) \leq (p-3)\int_0^{2\pi}\left(r(\theta) - \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta\right)^2 \mathrm{d}\theta \leq 0.$$

For $p < 3$, $\mathcal{F}(r)$ vanishes if and only if the function r is constant. To conclude the proof of (i), we take $\varepsilon \to 0$ and note that the disk is a strict local maximiser if and only if we have for all non-constant r and ε small enough

$$\int_{\Gamma_\varepsilon} |x|^p \,\mathrm{d}\mathcal{H}^1(x) - \frac{|\Gamma_\varepsilon|^{p+1}}{(2\pi)^p} < 0.$$

Taking $\varepsilon \to 0$ and using (5.4), we obtain the desired result.

For (ii), note that choosing $r(\theta) := \sin(2\theta)$, we have equality in (5.5), which leads to

$$\mathcal{F}(r) = (p-3)\int_0^{2\pi}\left(r(\theta) - \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\,\mathrm{d}\theta\right)^2 \mathrm{d}\theta.$$

And for $p > 3$, we get $\mathcal{F}(r) > 0$, which yields the claim by (5.4) for ε small enough. ■

Remark 5.7. Proposition 5.6 does not address the case $p = 3$ since it is a degenerate case: the quantity $\mathcal{F}(r)$ is zero for the optimal choice $r(\theta) := \sin(2\theta)$.

Note that, by Remark 5.3, C_p is non-decreasing in p. Furthermore, we deduce that C_p is left-continuous by the continuity in p of the expression inside the supremum in (5.2) for every fixed Γ. Hence,

$$\{p \in [0,\infty) : C_p = 1\} = [0, p_*].$$

So in order to prove Conjecture 1.4, namely that $p_* = 3$, it suffices to show $C_p = 1$ for all $p < 3$.

Remark 5.8 (Optimal curve for large p). The curve Γ considered in the proof of Proposition 5.5 is the optimal curve in the case $p = \infty$ when replacing the L^p norms by the corresponding supremum norms. For values of $p \in (3,\infty)$, we conjecture the optimal curve to be a deformed circle that degenerates into the curve from the proof of Proposition 5.5 as p increases, compare also with the proof of Proposition 5.6 (ii) above.

6. Proof of Theorem 1.5 – Applications to the magnetic Robin Laplacian

In this section, we show that Theorem 1.1 can be used to relax the assumptions on the domain in the isoperimetric inequality for the lowest eigenvalue of the magnetic Robin Laplacian on a bounded domain with a negative boundary parameter, recently obtained in [19] by the second and the third authors of the present paper.

The operator we study involves the vector potential (magnetic potential)

$$\mathbf{A}(x) := \frac{1}{2}(-x_2, x_1)^\top \quad (x = (x_1, x_2)).$$

and two parameters, $b \geq 0$ standing for the intensity of the magnetic field and $\beta \leq 0$, the Robin parameter, appearing in the boundary condition. Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected smooth domain. Our magnetic Robin Laplacian, $\mathsf{H}_\Omega^{\beta,b}$, is the self-adjoint operator defined by the closed, symmetric, densely defined and lower semi-bounded quadratic form

$$\mathfrak{q}_\Omega^{\beta,b}[u] := \|(\nabla - \mathrm{i}b\mathbf{A})u\|^2_{L^2(\Omega;\mathbb{C}^2)} + \beta\|u\|^2_{L^2(\partial\Omega)}, \quad \operatorname{dom}\mathfrak{q}_\Omega^{\beta,b} := H^1(\Omega),$$

and it is characterised by

$$\begin{aligned}\operatorname{dom}\mathsf{H}_\Omega^{\beta,b} = &\{u \in H^1(\Omega) : \text{there exists } w \in L^2(\Omega) \text{ such that} \\ &\qquad \mathfrak{q}_\Omega^{\beta,b}[u,v] = (w,v)_{L^2(\Omega)} \text{ for all } v \in \operatorname{dom}\mathfrak{q}_\Omega^{\beta,b}\},\\ \mathsf{H}_\Omega^{\beta,b}u :=& -(\nabla - \mathrm{i}b\mathbf{A})^2 u = w.\end{aligned}$$

Denoting by ν the unit inward normal vector on $\partial\Omega$, we observe that functions in $\operatorname{dom}\mathsf{H}_\Omega^{\beta,b}$ satisfy the (magnetic) Robin boundary condition

$$\nu\cdot(\nabla - \mathrm{i}b\mathbf{A})u = \beta u \quad \text{on } \partial\Omega.$$

The isoperimetric inequality obtained in [19] concerns the lowest eigenvalue of $\mathsf{H}_\Omega^{\beta,b}$, which we express in the variational form as follows:

$$\lambda_1^{\beta,b}(\Omega) := \inf_{u\in H^1(\Omega)\setminus\{0\}} \frac{\mathfrak{q}_\Omega^{\beta,b}[u]}{\|u\|^2_{L^2(\Omega)}}.$$

Denoting by $\mathcal{B}$ the disk in $\mathbb{R}^2$ centred at the origin, with radius R and having the same perimeter $2\pi R = |\partial\Omega|$ as the domain Ω, it is known that the following inequality holds (see [19, Theorem 4.8, Corollary 4.9]):

$$\lambda_1^{\beta,b}(\Omega) \leq \lambda_1^{\beta,b}(\mathcal{B}),$$

provided that

(i) $\beta < 0$ and $0 < b < \min(R^{-2}, 4\sqrt{-\beta}\, R^{-3/2})$ (i.e., the magnetic field's intensity b is of moderate strength); and

(ii) the *inner parallel curves* of Ω obey the condition

$$\int_{S_t} |x - x_0|^2 \, \mathrm{d}\,\mathcal{H}^1(x) \leq \frac{(L - 2\pi t)^3}{4\pi^2} \tag{6.1}$$

for some fixed point $x_0 \in \mathbb{R}^2$ and almost all $t \in (0, r_{\mathrm{i}}(\Omega))$. This condition holds for instance, when $\Omega \subset \mathcal{B}$ or when Ω is *convex* and *centrally symmetric* (see [19, Proposition 4.4])

Proof of Theorem 1.5. In view of Theorem 1.1, the condition in (6.1) holds with $x_0 = 0$ for all bounded centrally symmetric simply-connected smooth domains or, more generally, with x_0 being the centroid of all $S_t(\Omega)$ for all simply-connected smooth domains Ω such that the centroid of the inner parallel curve $S_t(\Omega)$ is independent of t. Thus, we relaxed the convexity assumption on the domain Ω. We obtain Theorem 1.5 with the choice $b_0(|\partial\Omega|, \beta) = \min\{R^{-2}, 4\sqrt{-\beta} R^{-3/2}\}$, where $R := \frac{|\partial\Omega|}{2\pi}$. ■

7. Some direct consequences of Theorems 1.1 and 1.2

7.1. A refined bound on the length of the disconnected inner parallel curve

In this subsection we use Theorem 3.1 to get a refined upper bound on the length of the inner parallel curve S_t in the situation when S_t consists of several connected components.

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected smooth domain with perimeter $L > 0$. Let the inner parallel curve $S_t \subset \Omega$ be as in (1.1). For any $t \in \mathcal{L}$, we have by Proposition 2.1 for some $N \in \mathbb{N}$

$$S_t = \bigcup_{n=1}^{N} \Gamma_n, \quad |S_t| \leq L - 2\pi t,$$

where $\{\Gamma_n\}_{n=1}^N$ are piecewise-smooth closed simple curves that are pairwise disjoint. In the case that S_t is connected, one has $N = 1$. However, in general, N can be an arbitrarily large integer number. In the case that $N = 2$, we immediately get as a consequence of Theorem 3.1

$$|S_t| + 2\,\mathrm{dist}(\Gamma_1, \Gamma_2) \leq L - 2\pi t.$$

This observation can be generalised to the case of arbitrary $N \in \mathbb{N}$ to improve Hartman's bound (1.2), see [16], on the length of the inner parallel curve S_t.

Corollary 7.1. *For all $t \in \mathcal{L}$, it holds that*

$$|S_t| + \sum_{n=1}^{N} \operatorname{dist}(\Gamma_n, S_t \setminus \Gamma_n) \leq L - 2\pi t.$$

Proof. The length of the closed piecewise-smooth curve parametrised by the mapping σ_t constructed in the proof of Theorem 3.1 is given by

$$\ell(\sigma_t) = |S_t| + \sum_{k=1}^{m} |\mathcal{I}_k|.$$

Every $\mathcal{I}_k$ connects some $\Gamma_{n(k)}$ with some $\Gamma_{n(k+1)}$, and for each $n \in \{1, \dots, N\}$ there is at least one $k \in \{1, \dots, m\}$ such that $n = n(k)$. Hence, we get that

$$\sum_{k=1}^{m} |\mathcal{I}_k| \geq \sum_{k=1}^{m} \operatorname{dist}(\Gamma_{n(k)}, \Gamma_{n(k+1)}) \geq \sum_{n=1}^{N} \operatorname{dist}(\Gamma_n, S_t \setminus \Gamma_n).$$

Thus, we conclude that

$$|S_t| + \sum_{n=1}^{N} \operatorname{dist}(\Gamma_n, S_t \setminus \Gamma_n) \leq \ell(\sigma_t) \leq L - 2\pi t. \qquad \blacksquare$$

7.2. Moments of inertia of domains

In this subsection, we apply Theorem 1.1 to recover an isoperimetric upper bound on the moment of inertia for the domain Ω itself leading to an alternative proof of a result due to Hadwiger [14].

Assume that $\Omega \subset \mathbb{R}^2$ is a bounded, simply-connected, centrally symmetric domain. Then

$$\int_{\Omega} |x|^2 \, \mathrm{d}x \leq \int_{\mathcal{B}} |x|^2 \, \mathrm{d}x, \tag{7.1}$$

where $\mathcal{B} \subset \mathbb{R}^2$ is a disk centred at the origin with the same perimeter as Ω.

Remark 7.2. The isoperimetric inequality (7.1) can be derived from the inequality by Hadwiger [14], where only convex domains were considered. Let $\mathcal{K} \subset \mathbb{R}^2$ be a bounded convex domain with a Lipschitz boundary. We can translate the domain $\mathcal{K}$ so that the origin becomes the centroid of $\mathcal{K}$ in the sense that

$$\int_{\mathcal{K}} x \, \mathrm{d}x = 0.$$

Let $\mathcal{B}' \subset \mathbb{R}^2$ be the disk centred at the origin of the same perimeter as $\mathcal{K}$. It is proved in [14] that

$$\int_{\mathcal{K}} |x|^2 \, \mathrm{d}x \le \int_{\mathcal{B}'} |x|^2 \, \mathrm{d}x. \tag{7.2}$$

Let us define the domain $\mathcal{K}$ as the convex hull of the bounded simply-connected centrally symmetric smooth $\Omega \subset \mathbb{R}^2$. Then, the perimeter of $\mathcal{K}$ does not exceed the perimeter of Ω. This is a well-known fact, whose proof can be found, e.g., in [29]. Moreover, the convex domain $\mathcal{K}$ is centrally symmetric as well. Therefore, the origin is the centroid of $\mathcal{K}$. Hence, we get from (7.2)

$$\int_{\Omega} |x|^2 \, \mathrm{d}x \le \int_{\mathcal{K}} |x|^2 \, \mathrm{d}x \le \int_{\mathcal{B}'} |x|^2 \, \mathrm{d}x \le \int_{\mathcal{B}} |x|^2 \, \mathrm{d}x,$$

where we used that the perimeter of Ω is larger than or equal to the perimeter of $\mathcal{K}$, so the radius of $\mathcal{B}'$ does not exceed the radius of $\mathcal{B}$.

Remark 7.3. If we furthermore assume that Ω is smooth, we have the following proof of (7.1) using Theorem 1.1. Recall that $\Omega \subset \mathbb{R}^2$ is a bounded simply-connected smooth domain with the perimeter $L > 0$ and the origin being the centroid of S_t for almost every $t \in (0, r_{\mathrm{i}}(\Omega))$, and that $\mathcal{B} \subset \mathbb{R}^2$ is the disk of radius $R = \frac{L}{2\pi}$, having thus the same perimeter as Ω. By the geometric isoperimetric inequality, we have $|\Omega| \le |\mathcal{B}|$ and therefore it holds that $R \ge r_{\mathrm{i}}(\Omega)$.

Recall the co-area formula in two dimensions (see [2, Theorem 4.20] and [22]). If $\mathcal{A} \subset \mathbb{R}^2$ is an open set, $f\colon \mathcal{A} \to \mathbb{R}$ is a Lipschitz continuous real-valued function, and $g\colon \mathcal{A} \to \mathbb{R}$ is an integrable function, then we have

$$\int_{\mathcal{A}} g(x) |\nabla f(x)| \, \mathrm{d}x = \int_{\mathbb{R}} \int_{f^{-1}(t)} g(x) \, \mathrm{d}\mathcal{H}^1(x) \, \mathrm{d}t. \tag{7.3}$$

Applying the co-area formula (7.3) with $\mathcal{A} = \Omega$, $g(x) = |x|^2$ and $f(x) = \rho(x)$ (the distance function to the boundary of Ω defined in (2.4)) we get using the inequality in Theorem 1.1,

$$\begin{aligned}\int_{\Omega} |x|^2 \, \mathrm{d}x &= \int_0^{r_{\mathrm{i}}(\Omega)} \int_{S_t} |x|^2 \, \mathrm{d}\mathcal{H}^1(x) \le \int_0^{r_{\mathrm{i}}(\Omega)} \frac{(L - 2\pi t)^3}{4\pi^2} \, \mathrm{d}t \\ &\le \int_0^R \frac{(L - 2\pi t)^3}{4\pi^2} \, \mathrm{d}t = 2\pi \int_0^R (R - t)^3 \, \mathrm{d}t = \frac{\pi R^4}{2} = \int_{\mathcal{B}} |x|^2 \, \mathrm{d}x.\end{aligned}$$

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