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Intersecting the Twin Dragon with rational lines

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Abstract. The Knuth Twin Dragon is a compact subset of the plane with fractal boundary of Hausdorff dimension $s = (\log\lambda)/(\log\sqrt{2})$, $\lambda^3 = \lambda^2 + 2$. Although the intersection with a generic line has Hausdorff dimension $s - 1$, we prove that this does not occur for lines with rational parameters. We further describe the intersection of the Twin Dragon with the two diagonals as well as with various axis parallel lines.

Dedicated to Professor Jörg Thuswaldner on the occasion of his 50th birthday

1. Introduction

We investigate the intersections of the Knuth Twin Dragon with rational lines. Let $\alpha = -1 + i$, then

$$\mathcal{K} = \left\{ \sum_{k=1}^{\infty} \frac{d_k}{\alpha^k} \,:\, d_k \in \{0, 1\} \right\}$$

is the Knuth Twin Dragon. The Hausdorff dimension of its boundary ∂K is $\mathscr{s} = \frac{\log\lambda}{\log\sqrt{2}} \approx 1.5236$, where λ is the real number satisfying $\lambda^3 = \lambda^2 + 2$. For lines

$$\Delta_{p,q,r} = \{x + iy \in \mathbb{C} \,:\, px + qy = r\} \tag{1.1}$$

with $p, q, r \in \mathbb{Z}$, we show that the α-expansions of $\mathcal{K} \cap \Delta_{p,q,r}$ are recognized by a finite automaton.

By a result of John Marstrand [5], the intersection of $\partial\mathcal{K}$ with Lebesgue almost all lines going through $\mathcal{K}$ has Hausdorff dimension $\mathscr{s} - 1$, meaning that in the set of all parameter triples $(p, q, r) \in \mathbb{R}^3$ for which $\Delta_{p,q,r} \cap \mathcal{K} \neq \emptyset$, the exceptional cases form a Lebesgue null set. We obtain here that the Hausdorff dimension of the intersection of the boundary of the Twin Dragon with rational lines is never equal to $\mathscr{s} - 1$.

Further, we revisit results by Akiyama and Scheicher [1] and add uncountably many examples of horizontal, vertical, and diagonal lines.

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We mention that similar results were obtained in [4] for lines intersecting the Sierpinski carpet F. The set F has Hausdorff dimension $\frac{\log 8}{\log 3}$. Manning and Simon showed that, given a slope $\alpha \in \mathbb{Q}$, the intersection of F with the line $y = \alpha x + \beta$ is strictly less than $\frac{\log 8}{\log 3} - 1$ for Lebesgue almost every β.

2. Main statement and proof

We first recall the notions of a canonical number system and its fundamental domain. Let β be an algebraic integer and $\mathcal{N} = \{0, 1, \dots, |N(\beta)| - 1\}$, where $N(x)$ denotes the norm of x over $\mathbb{Q}(\beta)/\mathbb{Q}$. The pair $(\beta, \mathcal{N})$ is called a *canonical number system* (CNS) if each $\gamma \in \mathbb{Z}[\beta]$ admits a representation of the form

$$\gamma = \sum_{k=0}^{n} d_k \beta^k, \quad d_k \in \mathcal{N}. \tag{2.1}$$

We call β the *radix* or *base* and $\mathcal{N}$ the set of *digits*. The representation (2.1) is unique up to leading zeros.

The Knuth Twin Dragon $\mathcal{K}$ appears as the *fundamental domain* of the CNS $(\alpha, \mathcal{N})$, where $\alpha = -1 + i$ is the root of the polynomial $x^2 - 2x - 2$ and $\mathcal{N} = \{0, 1\}$. The fundamental domain of a CNS is the set of all numbers that can be expressed with purely negative exponents. Since $\alpha^4 = -4$, it is often useful to consider groups of four digits:

$$\sum_{k=1}^{\infty} \frac{d_k}{\alpha^k} = \sum_{k=1}^{\infty} \frac{\sum_{j=0}^{3} d_{4k-j}\alpha^j}{\alpha^{4k}} = \sum_{k=1}^{\infty} \frac{b_k}{(-4)^k},$$

with the possibilities for $b_k = \sum_{j=0}^{3} d_{4k-j}\alpha^j$ being

$$\begin{array}{llll}
[0000]_\alpha = 0, & [0001]_\alpha = 1, & [0010]_\alpha = -1+i, & [0011]_\alpha = i, \\
[0100]_\alpha = -2i, & [0101]_\alpha = 1-2i, & [0110]_\alpha = -1-i, & [0111]_\alpha = -i, \\
[1000]_\alpha = 2+2i, & [1001]_\alpha = 3+2i, & [1010]_\alpha = 1+3i, & [1011]_\alpha = 2+3i, \\
[1100]_\alpha = 2, & [1101]_\alpha = 3, & [1110]_\alpha = 1+i, & [1111]_\alpha = 2+i.
\end{array}$$

In other words, we have

$$\mathcal{K} = \left\{ \sum_{k=1}^{\infty} \frac{b_k}{(-4)^k} \ : \ b_k \in \mathcal{D} \right\},$$

with

$$\mathcal{D} = \{-1-i, -1+i, -2i, -i, 0, i, 1-2i, 1, 1+i, 1+3i, 2, 2+i, 2+2i, 2+3i, 3, 3+2i\}.$$

Points in the intersection of $\mathcal{K}$ with lines $\Delta_{p,q,r} = \{x + iy \ : \ px + qy = r\}$ can now be characterized by their digit expansion in the following way.

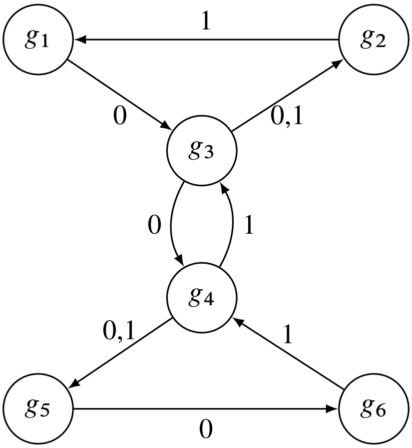

Figure 1. An automaton characterizing $\partial\mathcal{K}$ (in base α), where all states are initial and terminal.

Lemma 2.1. *We have* $z \in \mathcal{K} \cap \Delta_{p,q,r}$ *if and only if there is a digit sequence* $b_1 b_2 \cdots \in \mathcal{D}^{\mathbb{N}}$ *with*

$$z = \sum_{k=1}^{\infty} \frac{b_k}{(-4)^k} \quad \textit{and} \quad r = \sum_{k=1}^{\infty} \frac{p\,\Re(b_k) + q\,\Im(b_k)}{(-4)^k}.$$

Here, $\Re(b)$ denotes the real part and $\Im(b)$ denotes the imaginary part of $b \in \mathbb{C}$.

We show that we can characterize the digit expansion of the points in the intersection $\Delta_{p,q,r} \cap \mathcal{K}$ via a Büchi automaton, that is a finite automaton that accepts infinite paths. Using this representation, we are able to calculate the Hausdorff dimension of the intersection $\mathcal{K} \cap \Delta_{p,q,r}$ as well as the Hausdorff dimension of $\partial K \cap \Delta_{p,q,r}$.

Definition 2.2. A *Büchi automaton* is a 5-tuple (Q, A, E, I, T), where the set $Q = \{q_1, \dots, q_N\}$ is a finite set of *states*, A is a finite *alphabet*, $E \subset Q \times A \times Q$ is a set of *edges* and $I, T \subset Q$ the set of *initial* and *terminal states*. Let A^* denote the set of all (finite) words and A^ω denote the set of all (right) infinite words. A word $w \in A^*$, $w = w_1 \cdots w_n$, is *accepted* by the automaton if and only if there are states $q_{i_0}, \dots, q_{i_n}$ such that $q_{i_0} \in I$, $q_{i_n} \in T$ and $(q_{i_{k-1}}, w_k, q_{i_k}) \in E$ for all k. We call such a finite path *successful*, and we call an infinite path successful if and only if infinitely many subpaths are successful. An infinite word $w \in A^\omega$ is accepted by the automaton if there exists an infinite successful path with *label* w. The set of all $w \in A^\omega$ that are accepted by the automaton is called its *ω-language*.

Büchi automata are really helpful to describe self-similar sets. The automaton in Figure 1 characterizes all infinite sequences of digits 0, 1 in base α that give rise to boundary points in $\partial\mathcal{K}$; see [3,7].

Let L_1, L_2 be two ω-languages on the same alphabet that are accepted by $\mathcal{A}$, $\mathcal{B}$, respectively. It can be necessary to create automata accepting the union of the languages or their intersection. The union is not difficult: one just uses the union of states and edges, as well as the union of terminal and initial states. The intersection generally requires heavy computations, especially in the non-deterministic case, where a larger framework than Büchi automata needs to be used. But it becomes easy in some cases. We prove one particular case that is useful to prove our main statements.

Lemma 2.3. *Let L_1, L_2 be two ω-languages on the same alphabet A accepted by Büchi automata. If one of the automata has only terminal states, then there is a Büchi automaton accepting $L_1 \cap L_2$.*

Proof. Define $\mathcal{A} \times \mathcal{B} = (Q_{\mathcal{A}} \times Q_{\mathcal{B}}, A, E, I_{\mathcal{A}} \times I_{\mathcal{B}}, T_{\mathcal{A}} \times T_{\mathcal{B}})$, where E consists of the edges $(a,b) \xrightarrow{d} (a',b')$ with $a \xrightarrow{d} a'$ and $b \xrightarrow{d} b'$. Let $w \in A^{\omega}$ be a word that is accepted by $\mathcal{A} \times \mathcal{B}$. Then there exists an infinite path in the automaton. Projecting to the first coordinate gives an infinite path through $\mathcal{A}$. Therefore, we have $w \in L_1$ and with the same reasoning $w \in L_2$. Now let $w \in L_1 \cap L_2$. There exists a path $a_0 a_1 \cdots$ through $\mathcal{A}$ and a path $b_0 b_1 \cdots$ through $\mathcal{B}$. Then $(a_0, b_0)(a_1, b_1) \cdots$ is a path in the product automaton. Assume w.l.o.g. that all states of $\mathcal{A}$ are terminal. Then, for every finite subpath $b_0 b_1 \cdots b_k$ accepted by $\mathcal{B}$, the corresponding path $a_0 a_1 \cdots a_k$ in $\mathcal{A}$ is also accepted, hence $(a_0, b_0)(a_1, b_1) \cdots$ is successful. ■

In general, if $\Delta_{p,q,r} \cap \mathcal{K}$ is described by a Büchi automaton $\mathcal{A}$ and the boundary $\partial\mathcal{K}$ by a Büchi automaton $\mathcal{G}$, then $\partial\mathcal{K} \cap \Delta_{p,q,r}$ is described by the product automaton $\mathcal{A} \times \mathcal{G}$. Interpreting this Büchi automaton as a graph directed construction for the set $\partial\mathcal{K} \cap \Delta_{p,q,r}$, we have a way to compute the Hausdorff dimension of this set via results of Mauldin and Williams [6]. Let us state and prove our main statements.

Theorem 2.4. *Let $p, q, r \in \mathbb{Z}$, $\Delta_{p,q,r}$ as in (1.1) and $\mathcal{K}$ the Knuth Twin Dragon. Then the intersection $\mathcal{K} \cap \Delta_{p,q,r}$ can be described by a Büchi automaton.*

Proof. For $s, s' \in \mathbb{Z}$, we define an edge relation by

$$s \xrightarrow{b} s' \iff s' = p\,\Re(b) + q\,\Im(b) - 4s. \tag{2.2}$$

Now consider a path $-r = s_0 \xrightarrow{b_1} s_1 \xrightarrow{b_2} \cdots \xrightarrow{b_n} s_n$. Then

$$s_n = (-4)^n(-r) + \sum_{k=1}^{n} (-4)^{n-k}\big(p\,\Re(b_k) + q\,\Im(b_k)\big),$$

i.e.,

$$\frac{s_n}{(-4)^n} = -r + \sum_{k=1}^{n} \frac{p\,\Re(b_k) + q\,\Im(b_k)}{(-4)^k}.$$

Using Lemma 2.1, we immediately get that

$$(x,y)=[0.b_1b_2b_3\cdots]_{-4}\in\mathcal{K}\cap\Delta_{p,q,r}\quad\text{if and only if}\quad\lim_{n\to\infty}\frac{s_n}{(-4)^n}=0.$$

We now show that the elements s_n lying on paths starting with $s_0=-r$ and satisfying $\lim_{n\to\infty}\frac{s_n}{(-4)^n}=0$ are bounded by a constant $c(p,q)$. Indeed, we have

$$\frac{s_n}{(-4)^n}=-r+\sum_{k=1}^{n}\frac{p\,\Re(b_k)+q\,\Im(b_k)}{(-4)^k}=-\sum_{k=n+1}^{\infty}\frac{p\,\Re(b_k)+q\,\Im(b_k)}{(-4)^k},$$

and therefore

$$|s_n|=4^n\left|\sum_{k=n+1}^{\infty}\frac{p\,\Re(b_k)+q\,\Im(b_k)}{(-4)^k}\right|\le\frac{\max\{|p\,\Re(b)+q\,\Im(b)|:b\in\mathcal{D}\}}{3}=c(p,q).$$

Defining the set of states $Q=\{s\in\mathbb{Z}:|s|\le c(p,q)\}\cup\{-r\}$, $I=\{-r\}$, $T=Q$ and edges as in 2.2, gives us the desired Büchi automaton. ■

Theorem 2.5. *Let $p,q,r\in\mathbb{Z}$, $\Delta_{p,q,r}$ as in (1.1) and $\mathcal{K}$ the Knuth Twin Dragon. Then, the Hausdorff dimension of the intersection $\partial\mathcal{K}\cap\Delta_{p,q,r}$ is never $\mathfrak{s}-1$, where $\mathfrak{s}$ is the Hausdorff dimension of $\partial\mathcal{K}$.*

Proof. The Büchi automaton of Theorem 2.4 gives rise to a description of the intersection $\mathcal{K}\cap\Delta_{p,q,r}$ as one of the attractors of a graph directed construction (*GIFS*) with attractors $(K_s)_{s\in Q}$:

$$K_{-r}=\mathcal{K}\cap\Delta_{p,q,r},\quad\text{with}\quad K_s=\bigcup_{s\xrightarrow{b}s'\in\mathcal{A}}\frac{K_{s'}+b}{-4}\qquad(s\in Q).$$

As mentioned above, $\partial\mathcal{K}$ is also the attractor of a GIFS:

$$\partial\mathcal{K}=\bigcup_{g\in Q'}K_g,\quad\text{with}\quad K_g=\bigcup_{g\xrightarrow{b}g'\in\mathcal{G}}\frac{K_{g'}+b}{-4}\qquad(g\in Q'),$$

where $\mathcal{G}$ is the automaton characterizing $\partial\mathcal{K}$ in base -4. The automaton $\mathcal{G}$ can be obtained from the automaton $\mathcal{G}'$ of Figure 1 as follows.

- The set of states Q' is the same as for $\mathcal{G}'$; all states are initial and terminal.
- There is an edge from g to g' in $\mathcal{G}$ whenever there is a path of length 4 from g to g' in $\mathcal{G}'$. The label of this edge in $\mathcal{G}$ is the digit vector $[d_1d_2d_3d_4]_\alpha$ corresponding to the labels d_1,d_2,d_3,d_4 in $\mathcal{G}'$ along the path of length 4.

In that way, $\mathcal{A}$ and $\mathcal{G}$ are built on the same alphabet. By Lemma 2.3, the intersection $\mathcal{A} \times \mathcal{G}$ is a Büchi automaton describing the intersection $\Delta_{p,q,r} \cap \partial\mathcal{K}$. By Mauldin and Williams [6], the Hausdorff dimension of a GIFS attractor can be computed from the spectral radius β of the incidence matrix of a strongly connected component of the associated automaton; see further details in Remark 2.6. In particular, in our case,

$$\dim_H(\partial\mathcal{K} \cap \Delta_{p,q,r}) = \frac{\log \beta}{\log 4},$$

where the involved number β is an algebraic integer.

Now, the dimension of the boundary of the Twin Dragon is $\mathfrak{s} = \frac{\log \lambda}{\log \sqrt{2}}$, with $\lambda^3 = \lambda^2 + 2$. To have $\frac{\log \beta}{\log 4} = \mathfrak{s} - 1$, we need $\beta = \frac{\lambda^4}{4}$. However, the minimal polynomial of $\frac{\lambda^4}{4}$ is $4x^3 - 9x^2 + 2x - 1$, thus $\frac{\lambda^4}{4}$ is not an algebraic integer. ■

Remark 2.6. We shortly explain why the results of Mauldin and Williams [6] indeed apply to our setting. All the similarities in our graphs are contractions of the form $T(x) = \frac{x+b}{-4}$, with the same ratio $-\frac{1}{4}$. Therefore, if G denotes any of our graphs, we only need to check the existence of nonoverlapping compact sets $J_1, \ldots, J_n$ (one for each node $1, \ldots, n$ of G) with the property

$$\forall\, i \in \{1, \ldots, n\},\ J_i \supset \bigcup_{i \xrightarrow{T} j \in G} T(J_j),$$

each union being nonoverlapping.

For the graph $G = \mathcal{G}$ of our paper (with states $g \in Q'$), the intersections of $\mathcal{K}$ with its six neighboring tiles in the plane tiling generated by $\mathcal{K}$ are compact sets playing the role of the J_i's, that is, satisfying the above nonoverlapping conditions; see for example [2]. These intersections are exactly the sets K_g defined in the proof of Theorem 2.5.

Now, the graph $G = \mathcal{A} \times \mathcal{G}$ of our paper can be interpreted as a *subgraph* of $\mathcal{G}$: taking the product of $\mathcal{A}$ and $\mathcal{G}$ means to select paths of $\mathcal{G}$. The states of $\mathcal{A} \times \mathcal{G}$ are of the form (r, g), for some integers r and $g \in Q'$. Defining

$$K_{r,g} := \Delta_{p,q,-r} \cap K_g,$$

we obtain compact sets fulfilling the nonoverlapping requirements mentioned above.

3. Further results on intersections of the Twin Dragon with rational lines

In this section, we want to extend the work of [1], where the intersections with the x-and the y-axis are calculated. The intersections of these lines with ∂K are signi-

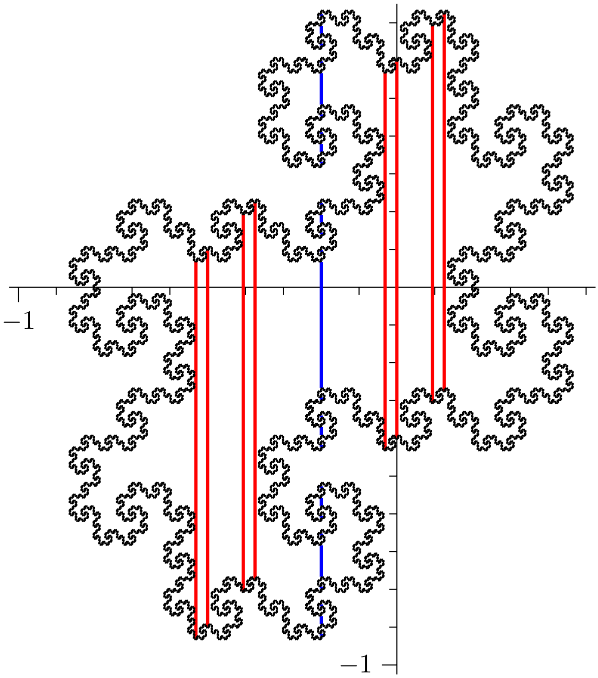

Figure 2. The Knuth Twin Dragon $\mathcal{K}$ and its intersection with $\Delta_{1,0,r}$ for some r as in Theorem 3.1 (red) and with $\Delta_{1,0,-1/5}$ (blue).

ficatively different from the expected result for intersections of fractals and lines, as they consist only of two points. First, we show that their result extends to uncountably many axis-parallel lines (where we do not have finite automata), and using the self-similar structure, to diagonal lines. Then, we give one example of a more complicated intersection.

Theorem 3.1. *Let $a_1a_2\cdots$ be a sequence in $\{0,1\}^\omega$ not ending in $(01)^\omega$, and*

$$r = \sum_{k=1}^{\infty} \frac{2a_k}{(-4)^k}.$$

Then

$$\partial\mathcal{K} \cap \Delta_{1,0,r} = \left\{r + \left(r - \tfrac{2}{5}\right) i,\ r + \left(r + \tfrac{3}{5}\right) i\right\},$$

and $\mathcal{K} \cap \Delta_{1,0,r}$ is the closed line segment $r + \left[r - \frac{2}{5}, r + \frac{3}{5}\right] i$.

Proof. We first use Lemma 2.1 to describe $\mathcal{K} \cap \Delta_{1,0,r}$, that is, we determine the sequences $b_1b_2\cdots \in \mathcal{D}$ such that $\Re\left(\sum_{k=1}^{\infty} b_k(-4)^{-k}\right) = r$, i.e.,

$$\sum_{k=1}^{\infty} \frac{2a_k - \Re(b_k)}{(-4)^k} = 0.$$

Since $\Re(b_k) \in \{-1, 0, 1, 2, 3\}$, we have $2a_k - \Re(b_k) \in \{-3, -2, \ldots, 2, 3\}$ and thus

$$\left| \sum_{k=n+1}^{\infty} \frac{2a_k - \Re(b_k)}{(-4)^k} \right| \leq \frac{1}{4^n} \quad \text{for all } n \geq 0.$$

Moreover, equality holds if and only if $2a_k - \Re(b_k)$ is alternately 3 and -3, which implies that a_k is alternately 1 and 0, which we have excluded. This gives that

$$\left| \sum_{k=n+1}^{\infty} \frac{2a_k - \Re(b_k)}{(-4)^k} \right| < \frac{1}{4^n} \quad \text{and} \quad \sum_{k=n+1}^{\infty} \frac{2a_k - \Re(b_k)}{(-4)^k} = \sum_{k=1}^{n} \frac{\Re(b_k) - 2a_k}{(-4)^k} \in \frac{\mathbb{Z}}{4^n}$$

for all $n \geq 1$, hence $\Re(b_k) = 2a_k$ for all $k \geq 1$. For the corresponding sequences $d_1 d_2 \cdots$ (with $\sum_{j=0}^{3} d_{4k-j} \alpha^j = b_k$) this implies that

$$d_{4k-3} d_{4k-2} d_{4k-1} d_{4k} \in \{a_k 000, a_k 011, a_k 100, a_k 111\} \quad \text{for all } k \geq 1. \tag{3.1}$$

Now consider sequences $d_1 d_2 \cdots$ of the form (3.1) in the boundary automaton $\mathcal{G}$ given in Figure 1. The only paths labeled by $abcc$, $a, b, c \in \{0, 1\}$, starting from g_1, g_2, g_5 and g_6, respectively, are

$$g_1 \xrightarrow{0000} g_6,\ g_1 \xrightarrow{0011} g_2,\ g_2 \xrightarrow{1000} g_5,\ g_2 \xrightarrow{1011} g_1,$$
$$g_5 \xrightarrow{0100} g_6,\ g_5 \xrightarrow{0111} g_2,\ g_6 \xrightarrow{1100} g_5,\ g_6 \xrightarrow{1111} g_1.$$

Therefore, for an infinite successful path of the form (3.1) starting from g_1, g_2, g_5 or g_6, the sequence $a_1 a_2 \cdots$ is alternately 0 and 1, which we have excluded. Hence, it suffices to consider paths that are in g_3 and g_4 after $4k$ steps for all $k \geq 0$. From

$$g_3 \xrightarrow{a100} g_4 \quad \text{and} \quad g_4 \xrightarrow{a011} g_3 \qquad (a \in \{0, 1\}),$$

we see that the only points in $\partial\mathcal{K} \cap \Delta_{1,0,r}$ are

$$\sum_{k=1}^{\infty} \frac{a_k \alpha^3}{(-4)^k} + \sum_{k=1}^{\infty} \frac{\alpha^6 + \alpha + 1}{16^k} = r\,(1+i) + \frac{3i}{5},$$
$$\sum_{k=1}^{\infty} \frac{a_k \alpha^3}{(-4)^k} + \sum_{k=1}^{\infty} \frac{\alpha^5 + \alpha^4 + \alpha^2}{16^k} = r\,(1+i) - \frac{2i}{5}.$$

Since $r\,(1+i) \in \mathcal{K}$, $\mathcal{K} \cap \Delta_{1,0,r}$ is the line segment between these points. ■

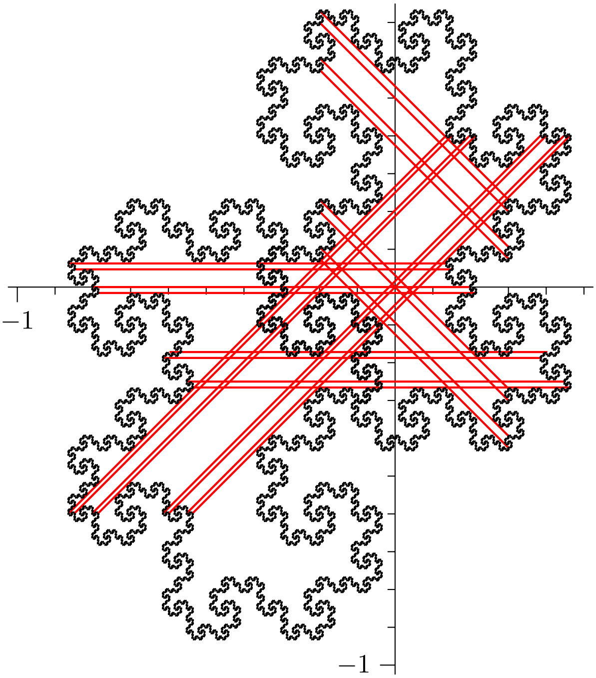

Figure 3. The intersection of $\mathcal{K} = \alpha^{-1}\big(\mathcal{K} \cup (\mathcal{K}+1)\big)$ with lines $\Delta_{0,1,r/2}$, $\Delta_{1,1,-r}$, and $\Delta_{1,-1,r/2}$ for some r as in Theorem 3.1.

Theorem 3.2. *For* $-\frac{8}{15} < r < \frac{2}{15}$, *we have*

$$\begin{aligned} -2i\,(\mathcal{K} \cap \Delta_{0,1,r/2}) &= (\mathcal{K} \cap \Delta_{1,0,r}) + \{0, i\}, \\ (-1+i)\,(\mathcal{K} \cap \Delta_{1,1,-r}) &= \mathcal{K} \cap \Delta_{1,0,r}\,, \\ (-1+i)\,(\mathcal{K} \cap \Delta_{1,-1,r/2}) &= (\mathcal{K} \cap \Delta_{0,1,r/2}) + \{0, 1\}, \\ 2\,(1+i)\,(\mathcal{K} \cap \Delta_{1,-1,r/2}) &= (\mathcal{K} \cap \Delta_{1,0,r}) + \{-2i, -i, 0, i\}. \end{aligned}$$

In particular, for r *as in Theorem 3.1, the intersections* $\mathcal{K} \cap \Delta_{0,1,r/2}$, $\mathcal{K} \cap \Delta_{1,1,-r}$ *and* $\mathcal{K} \cap \Delta_{1,-1,r/2}$ *are closed line segments with endpoints*

$$\begin{aligned} \partial\mathcal{K} \cap \Delta_{0,1,r/2} &= \partial(\mathcal{K} \cap \Delta_{0,1,r/2}) = \left\{-\tfrac{4}{5} - \tfrac{r}{2} + \tfrac{r}{2}\,i, \tfrac{1}{5} - \tfrac{r}{2} + \tfrac{r}{2}\,i\right\}, \\ \partial\mathcal{K} \cap \Delta_{1,1,-r} &= \partial(\mathcal{K} \cap \Delta_{1,1,-r}) = \left\{-\tfrac{1}{5} + \left(\tfrac{1}{5} - r\right) i, \tfrac{3}{10} - \left(\tfrac{3}{10} + r\right) i\right\}, \\ \partial\mathcal{K} \cap \Delta_{1,-1,r/2} &= \partial(\mathcal{K} \cap \Delta_{1,-1,r/2}) = \left\{-\tfrac{3}{5} + \tfrac{r}{2} - \tfrac{3}{5}\,i, \tfrac{2}{5} + \tfrac{r}{2} + \tfrac{2}{5}\,i\right\}. \end{aligned}$$

Proof. Note that $\alpha\,\mathcal{K} = \mathcal{K} \cup (\mathcal{K}+1)$ and

$$\alpha\,\Delta_{1,1,-r} = \Delta_{1,0,-r}, \quad \alpha\,\Delta_{0,1,r/2} = \Delta_{1,1,-r}, \quad \alpha\,\Delta_{1,-1,r/2} = \Delta_{0,1,r/2}.$$

Moreover, we have

$$(\mathcal{K}+1)\cap\Delta_{1,0,r} = \emptyset = (\mathcal{K}-1)\cap\Delta_{1,0,r} = (\mathcal{K}+\alpha)\cap\Delta_{1,0,r}$$

since $-\frac{8}{15} < r < \frac{2}{15}$ and

$$\begin{aligned}\min\{x : x+iy\in\mathcal{K}\} &= \sum_{k=1}^{\infty}\left(\frac{3}{(-4)^{2k-1}}+\frac{-1}{(-4)^{2k}}\right) = -\sum_{k=1}^{\infty}\frac{13}{16^k} = -\frac{13}{15},\\ \max\{x : x+iy\in\mathcal{K}\} &= \sum_{k=1}^{\infty}\left(\frac{-1}{(-4)^{2k-1}}+\frac{3}{(-4)^{2k}}\right) = \sum_{k=1}^{\infty}\frac{7}{16^k} = \frac{7}{15}.\end{aligned}$$

Using these geometric properties, we obtain that

$$\begin{aligned}\alpha\,(\mathcal{K}\cap\Delta_{1,1,-r}) &= \big(\mathcal{K}\cup(\mathcal{K}+1)\big)\cap\Delta_{1,0,r} = \mathcal{K}\cap\Delta_{1,0,r}\,,\\ \alpha^2\,(\mathcal{K}\cap\Delta_{0,1,r/2}) &= \big(\mathcal{K}\cup(\mathcal{K}+1)\cup(\mathcal{K}+\alpha)\cup(\mathcal{K}+\alpha+1)\big)\cap\Delta_{1,0,r}\\ &= (\mathcal{K}\cap\Delta_{1,0,r})\cup\big((\mathcal{K}+i)\cap\Delta_{1,0,r}\big)\\ &= (\mathcal{K}\cap\Delta_{1,0,r})+\{0,i\},\\ \alpha\,(\mathcal{K}\cap\Delta_{1,-1,r/2}) &= \big(\mathcal{K}\cup(\mathcal{K}+1)\big)\cap\Delta_{0,1,r/2} = (\mathcal{K}\cap\Delta_{0,1,r/2})+\{0,1\},\\ \alpha^3\,(\mathcal{K}\cap\Delta_{1,-1,r/2}) &= \alpha^2\,(\mathcal{K}\cap\Delta_{0,1,r/2})-\{0,2i\}\\ &= (\mathcal{K}\cap\Delta_{1,0,r})+\{-2i,-i,0,i\}.\end{aligned}$$

For r as in Theorem 3.1, we have $-\frac{8}{15} < r < \frac{2}{15}$ since

$$\begin{aligned}\min\left\{\sum_{k=1}^{\infty}\frac{2a_k}{(-4)^k} : a_1a_2\cdots\in\{0,1\}^{\omega}\right\} &= -\sum_{k=1}^{\infty}\frac{8}{16^k} = -\frac{8}{15},\\ \max\left\{\sum_{k=1}^{\infty}\frac{2a_k}{(-4)^k} : a_1a_2\cdots\in\{0,1\}^{\omega}\right\} &= \sum_{k=1}^{\infty}\frac{2}{16^k} = \frac{2}{15},\end{aligned}$$

and the minimum and maximum are attained only for the sequences $(10)^{\omega}$ and $(01)^{\omega}$, which we have excluded. Therefore, Theorem 3.1 and the formulae above give that

$$\begin{aligned}\mathcal{K}\cap\Delta_{1,1,-r} &= -\tfrac{1+i}{2}\left(r\,(1+i)+\left[-\tfrac{2}{5},\tfrac{3}{5}\right]i\right) = -r\,i+\left[-\tfrac{1}{5},\tfrac{3}{10}\right](1-i),\\ \mathcal{K}\cap\Delta_{0,1,r/2} &= \tfrac{i}{2}\left(r\,(1+i)+\left[-\tfrac{2}{5},\tfrac{8}{5}\right]i\right) = r\,\tfrac{-1+i}{2}+\left[-\tfrac{4}{5},\tfrac{1}{5}\right],\\ \mathcal{K}\cap\Delta_{1,-1,r/2} &= \tfrac{1-i}{4}\left(r\,(1+i)+\left[-\tfrac{12}{5},\tfrac{8}{5}\right]i\right) = \tfrac{r}{2}+\left[-\tfrac{3}{5},\tfrac{2}{5}\right](1+i),\end{aligned}$$

which proves the statements for the intersection of $\mathcal{K}$ with lines. For the intersections of $\partial\mathcal{K}$ with lines, it only remains to check that the points in

$$\alpha^{-2}\big((\mathcal{K}\cap\Delta_{1,0,r})\cap((\mathcal{K}\cap\Delta_{1,0,r})+i)\big) = \left\{\tfrac{1}{\alpha^2}\left(r\,(1+i)+\tfrac{3i}{5}\right)\right\}$$

and

$$\alpha^{-1}\big((\mathcal{K} \cap \Delta_{0,1,r/2}) \cap ((\mathcal{K} \cap \Delta_{0,1,r/2}) + 1)\big) = \left\{\tfrac{1}{\alpha^3}\left(r\,(1+i) - \tfrac{2}{5}\,i\right)\right\}$$

are not in ∂K. By the proof of Theorem 3.1, the digit expansion

$$[.a_1 100 a_2 011 a_3 100 a_4 011 \cdots]_\alpha = r\,(1+i) + \tfrac{3}{5}\,i$$

is given by a path starting only from g_3 in the boundary automaton $\mathcal{G}$. Dividing by α^2 adds 00 in front of the expansion, but g_3 cannot be reached by 00, hence $\frac{1}{\alpha^2}\left(r\,(1+i) + \frac{3i}{5}\right)$ is not on the boundary of K. Similarly, the digit expansion

$$[.a_1 011 a_2 100 a_3 011 a_4 100 \cdots]_\alpha = r\,(1+i) - \tfrac{2}{5}\,i$$

is given by a path starting from g_4 in the boundary automaton $\mathcal{G}$, and g_4 cannot be reached by 000, thus $\frac{1}{\alpha^3}\left(r\,(1+i) - \frac{2}{5}\,i\right)$ is not on the boundary of K. This proves that all intersections of $\mathcal{K}$ with the given lines are line segments. ■

We can use this method to find a vertical line with a more interesting intersection. For example, if we look at $\Delta_{1,0,-1/4}$, we see that the only expansion $\sum_{k=1}^{\infty} \frac{b_k}{(-4)^k}$ with $b_k \in \mathcal{D}$ having real part $-1/4$ is $b_1 b_2 \cdots = 100\cdots$. In base α, we must therefore have $d_1 d_2 d_3 d_4 \in \{0001, 0101, 1010, 1110\}$, which correspond to the digits 1, $1-2i$, $1+3i$, $1+i \in \mathcal{D}$. The remaining digit sequences $d_5 d_6 \cdots$ give points in the set $\frac{1}{\alpha^4}(\mathcal{K} \cap \Delta_{1,0,0})$, thus

$$\mathcal{K} \cap \Delta_{1,0,-1/4} = -\tfrac{1}{4} + \left(\left[-\tfrac{9}{10}, -\tfrac{13}{20}\right] \cup \left[-\tfrac{2}{5}, \tfrac{1}{10}\right] \cup \left[\tfrac{7}{20}, \tfrac{3}{5}\right]\right) i.$$

We go on with $\Delta_{1,0,-1/4+1/16}$ and see that points in the intersection have imaginary part with an expansion in base -4 starting with two digits in $\{-2, 0, 1, 3\}$ and ending with digits in $\{-1, 0, 1, 2\}$. For the limit $\Delta_{1,0,-1/5}$ of lines of this form, we obtain the following intersection with $\mathcal{K}$, see Figure 2.

Theorem 3.3. *We have*

$$\mathcal{K} \cap \Delta_{1,0,-1/5} = \left\{ -\frac{1}{5} + \sum_{k=1}^{\infty} \frac{d_k}{(-4)^k}\, i \;:\; d_k \in \{-2, 0, 1, 3\} \text{ for all } k \geq 1 \right\},$$

and a point is in $\partial\mathcal{K} \cap \Delta_{1,0,-1/5}$ if and only if it is of the form $-\frac{1}{5} + \sum_{k=1}^{\infty} d_k (-4)^{-k} i$, where $d_1 d_2 \cdots$ is a path in the automaton in Figure 4.

Proof. Since $-\frac{1}{5} = \sum_{k=1}^{\infty} (-4)^{-k}$, we obtain in the same way as in the proof of Theorem 3.1 that $\Re\big(\sum_{k=1}^{\infty} b_k (-4)^{-k}\big) = -\frac{1}{5}$ with $b_k \in \mathcal{D}$ if and only if $\Re(b_k) = 1$ for all $k \geq 1$, i.e., $b_k \in \{1-2i, 1, 1+i, 1+3i\}$. The corresponding 4-digit blocks in base α are 0101, 0001, 1110, and 1010. This proves the characterization of $\mathcal{K} \cap \Delta_{1,0,-1/5}$.

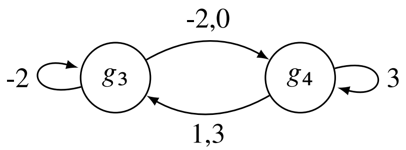

Figure 4. Automaton recognizing the imaginary parts of points in $\partial\mathcal{K} \cap \Delta_{1,0,-1/5}$ in base -4.

In the boundary automaton, the digit blocks 0101, 0001, 1110, and 1010 are accepted only from g_3 and g_4, and we have the transitions

$$g_3 \xrightarrow{0101} g_3,\ g_3 \xrightarrow{0001} g_4,\ g_3 \xrightarrow{0101} g_4,\ g_4 \xrightarrow{1010} g_4,\ g_4 \xrightarrow{1010} g_3,\ g_4 \xrightarrow{1110} g_3.$$

Taking imaginary parts of the corresponding numbers in $\mathcal{D}$ gives the automaton in Figure 4. ■

Theorem 3.4. *The Hausdorff dimension of* $\mathcal{K} \cap \Delta_{1,0,-1/5}$ *is* 1 *and*

$$\dim_H(\partial\mathcal{K} \cap \Delta_{1,0,-1/5}) = \tfrac{\log 3}{\log 4} \approx 0.7925 > \mathfrak{s} - 1.$$

Proof. We can interpret the intersection with $\Delta_{1,0,-1/5}$ as the self-similar digit tile in $\mathbb{R}$ with $A = -4$ and $D = \{-2, 0, 1, 3\}$. Since D is a complete residue system modulo 4, this tile has non-empty interior and therefore is of dimension 1.

For the boundary, we have $\partial\mathcal{K} \cap \Delta_{1,0,-1/5} = K_3 \cup K_4$, with

$$-4K_3 = (K_3 - 2) \cup (K_4 - 2) \cup K_4, \quad -4K_4 = (K_3 + 1) \cup (K_3 + 3) \cup (K_4 + 3).$$

Therefore, by [6], the Hausdorff dimension of $\partial\mathcal{K} \cap \Delta_{1,0,-1/5}$ is $\log\beta/\log 4$, where β is the Perron–Frobenius eigenvalue of the matrix $\left(\begin{smallmatrix}1 & 2\\ 2 & 1\end{smallmatrix}\right)$, i.e., $\beta = 3$. ■

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