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Projections of totally disconnected thin fractals with very thick shadows on $\mathbb{R}^d$

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Abstract. We study an extreme scenario of the Marstrand projection theorem for which a fractal has the property that its orthogonal projection is the same as the orthogonal projection of its convex hull. We extend results in current literature and establish checkable criteria for self-affine sets to have this property. Using this, we show that every convex polytope on $\mathbb{R}^d$ contains a totally disconnected compact set, which is a union of self-affine sets, of dimension as close to 1 as possible, as well as a rectifiable 1-set, such that the fractal projects to an interval in every 1-dimensional subspace and its convex hull is the given polytope. Other convex sets and projections onto higher dimensional subspaces are also discussed.

1. Introduction

The Marstrand projection theorem [14], which describes the relationship between the dimensions of Borel sets and their projections through a measure-theoretic statement, is now the landmark result of the projection theory of fractals.

Theorem 1.1 (Marstrand, 1954). *Let $E \subset \mathbb{R}^2$ be a Borel set, and let $\alpha = \dim_H E$. If $\alpha \le 1$, then*

$$\dim_H(\pi_\theta(E)) = \alpha \qquad \text{for almost every } \theta \in [0, \pi).$$

If $\alpha > 1$, then

$$\mathcal{L}^1(\pi_\theta(E)) > 0 \qquad \text{for almost every } \theta \in [0, \pi),$$

where $\mathcal{L}^1$ is the Lebesgue measure.

It is known that this result can be generalized to higher dimensions with projections on k-dimensional subspaces, see [15] for more details. There has been a huge amount of research studying different ways of improving this theorem under various

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conditions. In one direction, one can obtain a more precise dimension estimate for the exceptional set of directions. This starts from the result of R. Kaufman [12] who proved that if $s < \dim_H(E) < 1$, then

$$\dim_H\{\theta : \dim_H(\pi_\theta(E)) < s\} \le s.$$

There have been substantial improvements and generalization on this topic. In the other direction, people are interested in determining the classes of sets for which "almost all" can be replaced by "all" in the statement of the Marstrand's theorem. For example, Peres and Shmerkin [17] showed that self-similar sets with dense rotation group satisfy $\dim_H(\pi_\theta(K)) = \min\{\dim_H(K), 1\}$ for *all* $\theta \in [0, \pi)$. We refer the reader to [6] by Falconer, Fraser, and Jin for some of these recent developments.

This paper considers the Marstrand projection theorem in its most extreme scenario. In particular, we examine how small the Hausdorff dimension of a compact totally disconnected set on $\mathbb{R}^d$ can be while still ensuring its projections are exactly an interval in *all* one-dimensional subspaces. We aim to present a systematic study of such sets and give different interesting examples. Let us set up the following two terminologies.

Definition 1.2. Let K be a Borel set on $\mathbb{R}^d$ with convex hull C and let $1 \le k < d$. We say that K *projects a very thick shadow on k-dimensional subspace W* where $W \in G(d,k)$ if we have $\pi_W(K) = \pi_W(C)$. We say that K satisfies the (k-dimensional) *everywhere very thick shadow condition* if K projects very thick shadows on every $W \in G(d,k)$.

Here, $G(d,k)$ is the set of all k-dimensional subspaces in $\mathbb{R}^d$. We also have a dual concept for convex sets.

Definition 1.3. Let C be a closed convex set in $\mathbb{R}^d$. We say that C is (*k-dimensional*) *fractal-decomposable* with a set K if

(1) $\text{conv}(K) = C$, and

(2) $\pi_W(K) = \pi_W(C)$ for all $W \in G(d,k)$.

Clearly, if K is connected, the projection of K is connected and hence, the closed convex hull of K is 1-dimensional fractal-decomposable. Our main interest is asking how "disconnected" K can be if it satisfies the above two definitions. Moreover, we would hope that K possesses a nice topology and geometry, so compactness of K is desirable, as well as that K is generated by self-similar or self-affine iterated function systems.

From Marstrand's theorem, we know already that the set must have Hausdorff dimension at least one. Another landmark projection theorem by Besicovitch also showed that purely unrectifiable 1-sets cannot be fractal-decomposable either (see

Section 6 for the precise definitions about rectifiability). Our main result demonstrates that other than these two constraints, all convex polytopes are fractal-decomposable using compact sets of dimension close to 1 or even rectifiable 1-sets.

Theorem 1.4. *Let C be a convex polytope on $\mathbb{R}^d$. Then*

(1) *For all $\epsilon > 0$, there exists a totally disconnected compact set K, such that $1 \le \dim_H(K) \le 1 + \epsilon$ and where K is a finite union of self-affine sets, such that C is 1-dimensional fractal-decomposable with the set K.*

(2) *There exists a totally disconnected compact rectifiable 1-set K such that C is 1-dimensional fractal-decomposable with the set K.*

The very first totally disconnected compact set which projects very thick shadows on every line was discovered by Mendivil and Taylor [16]. Without noticing Mendivil–Taylor's work, Falconer and Fraser [5] demonstrated some examples of self-similar sets that project very thick shadows while studying the visibility conjecture of fractals. They mentioned a checking criterion for self-similar sets having very thick shadows everywhere. Farkas [8] studied interval projections of self-similar sets and gave an example of a totally disconnected self-similar fractal of dimension arbitrarily close to 1 existing inside the unit square. In the example, rotations were required and there was no indication that the example could be generalized to other polygons or to higher dimensions. We give an example based only on fractal squares and rigorously prove the criterion used by previous researchers on $\mathbb{R}^d$ for checking when a self-affine set has a very thick shadow everywhere. Moreover, as another interesting contribution of the paper, for the projection onto all 1-dimensional subspaces, we provide another natural criterion using the connected components of the first iteration (see Theorems 3.1 and 3.2). We can prove Theorem 1.4 (1) conveniently with this new criterion.

The everywhere very thick shadow condition has been a useful sufficient condition to study the visibility conjecture: *If $F \subset \mathbb{R}^2$ is Borel and $\dim_H(F) \ge 1$, then the Hausdorff dimension of the visible part of F in the direction θ is equal to 1 for almost all angles θ.* Falconer and Fraser [5] showed that the conjecture is true for self-similar sets under the convex open set condition on $\mathbb{R}^2$ and the everywhere very thick shadow condition. Other more general cases for the visibility conjecture of self-affine or self-similar sets were studied in [11, 19]. Therefore, the construction given in this paper (e.g., in Figure 3) produces examples for the visibility conjecture.

Theorem 1.4 gives a complete answer to convex polytopes. One may be interested in other convex sets as well as projections onto other subspaces of dimension greater than 1. Unfortunately, with certain simple observations, total disconnectedness is indeed an impossible requirement for fractal decomposability if we go beyond the condition for Theorem 1.4. A thorough discussion is given in the last section of the paper.

We organize our paper as follows: in Section 2, we set up our terminologies used in this paper. In Section 3, we prove our classification theorem for self-affine sets. In Section 4, we discuss fractal decomposablility for unit cubes. Examples existing in current literature are also exhausted in this section. In Section 5, we discuss the fractal decomposablility for general polytopes using self-affine sets. In Section 6, we discuss the fractal decomposablility for polytopes using rectifiable sets. Theorem 1.4 is proved in Sections 5 and 6. Finally, we close our paper with remarks on general cases.

2. Preliminaries

The goal of this section is to set up the basic terminologies throughout this paper. Let $\Phi = \{\phi_1, \dots, \phi_N\}$ be a collection of contractive maps from $\mathbb{R}^d \to \mathbb{R}^d$. Φ generates an *iterated function system (IFS)* with a unique nonempty compact attractor $K = K_\Phi$. A *self-affine* IFS consists of maps $\phi_i(x) = T_i x + t_i$ where T_i are invertible matrices on $\mathbb{R}^d$ with the operator norm $\|T_i\| < 1$ and $t_i \in \mathbb{R}^d$. A *self-similar* IFS is a special case of a self-affine IFS in which all T_i are of the form $r_i O_i$, where $0 < r_i < 1$ and O_i is an orthogonal transformation.

We use the standard multi-index notation to describe our IFS. Namely, we let $\Sigma = \{1, \dots, N\}$ and $\Sigma^k = \Sigma \times \cdots \times \Sigma$ (k times). For each $\sigma = (\sigma_1, \dots, \sigma_k) \in \Sigma^k$,

$$\phi_\sigma(x) = \phi_{\sigma_1} \circ \cdots \circ \phi_{\sigma_k}(x).$$

It is well known that the Hausdorff dimension of the attractor of a self-similar IFS under the open set condition is the unique s such that

$$\sum_{i=1}^{N} r_i^s = 1.$$

The Hausdorff dimension of a self-affine IFS is, however, much more difficult to compute. Let $\Phi = \{\phi_1, \dots, \phi_N\}$ be a self-affine IFS. Write each $\phi_i(x) = T_i x + b_i$, where T_i are linear transformations. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping that is contracting and nonsingular. The *singular values* $1 > \alpha_1 \geq \cdots \geq \alpha_N > 0$ of T are the positive square roots of the eigenvalues of T^*T, where T^* is the adjoint of T. For $0 \leq s \leq n$, we define the *singular value function* as

$$\varphi^s(T) = \alpha_1 \alpha_2 \cdots \alpha_{r-1} \alpha_r^{s-r+1},$$

where r is the smallest integer greater than or equal to s. We define the affinity dimension as

$$\dim_a(\Phi) = \inf\left\{s : \sum_{k=1}^{\infty}\sum_{\sigma\in\Sigma^k}\varphi^s(T_\sigma) < \infty\right\}. \tag{2.1}$$

Falconer proved that

$$\dim_H(K_\Phi) \le \dim_a(\Phi)$$

and that equality holds for almost all translations as long as $\|T_i\| < 1/3$ [3]. Solomyak later relaxed the condition to $\|T_i\| < 1/2$ [20]. One cannot expect equality to hold generally for the well-known Bedford–McMullen carpet. In a recent deep result by Bárány, Hochman and Rapaport [1], however, equality does hold for affine IFS under certain natural separation and irreducibility conditions.

2.1. Total disconnectedness

A topological space X is totally disconnected if the only connected components of X are the singletons. Many definitions of total disconnectedness appear in the literature. For the convenience of the reader, we collect all alternate equivalent definitions below that are useful for our discussions. This theorem follows from [2, Propositions 3.1.8 and 3.1.11], as well as [21, Theorem 29.7]. [1]

Theorem 2.1. *If X is a locally compact separable metric space, then the following are equivalent:*

(1) *All connected components are singletons.*

(2) *For all $x \neq y$, there exist disjoint open sets U, V such that $x \in U$ and $y \in V$ where $U \cup V = X$.*

(3) *For every pair of disjoint closed sets E, F, there exist disjoint open sets U, V such that $E \subset U$ and $F \subset V$ where $U \cup V = X$.*

(4) *There exists a basis of X such that every basis element is a clopen set in X.*

In particular, the following proposition holds true [2, p.91].

Proposition 2.2. *A finite union of totally disconnected closed sets is totally disconnected.*

We note that closedness cannot be removed as the rationals and irrationals are both totally disconnected, while their union is not. We now give the following sufficient condition for total disconnectedness in fractals generated by an IFS.

[1] In [21, Theorem 29.7], rim-compact means that every point has an open set with compact boundary, which is true for a locally compact space, see p.288 of the same book.

Proposition 2.3. *Let $\Phi = \{\phi_1, \dots, \phi_N\}$ be an IFS. Let C be a closed and connected set such that*

$$C \supset \bigcup_{i=1}^{N} \phi_i(C).$$

For any level $k \geq 1$, we let

$$\bigcup_{\sigma \in \Sigma^k} \phi_\sigma(C) = R_{1,k} \cup \cdots \cup R_{m_k,k}, \tag{2.2}$$

where $R_{j,k}$ denotes the jth connected component of the kth iteration. Then, K is totally disconnected if

$$\lim_{k\to\infty} \max_{1 \leq j \leq m_k} \operatorname{diam}(R_{j,k}) = 0.$$

Proof. As C is connected, all $\phi_\sigma(C)$ are connected. Hence, there can only be finitely many connected components at each level k, so $m_k < \infty$ in (2.2). Let $x, y \in K$ be distinct. Then $|x - y| > 0$. Choose the iteration k where $\operatorname{diam}(R_{j,k}) < |x - y|$ for all $1 \leq j \leq m_k$. Then x and y belong to different connected components of the kth level. So the connected components in the kth stage provides the separation. Thus, K is totally disconnected. ∎

We remark that another useful criterion for checking total disconnectedness specifically for fractal squares can be found in [18] (see also [13]).

2.2. Notions about convex geometry

We use some notions from convex geometry. We can see [9] for details. A hyperplane E is the level set of a linear functional, i.e., $E = \{x : f(x) = \alpha\}$ for some linear functional $f : \mathbb{R}^d \to \mathbb{R}$. Let $A \subset \mathbb{R}^d$ be closed and convex. Then E is called a *supporting hyperplane* of A if $A \cap E \neq \varnothing$ and A is contained in exactly one of the two closed half-spaces $\{f \leq \alpha\}$ or $\{f \geq \alpha\}$.

A *convex polytope* C is a closed convex set that is a convex hull of finitely many points. It is also known that a convex polytope also admits a half-space representation, i.e.,

$$C = \bigcap_{i=1}^{M} \mathcal{H}_i$$

where $\mathcal{H}_i$ is a closed half-space for some linear functional f. Recall that a point x is called an *extreme point* of a convex set C if $x = \lambda y + (1 - \lambda)z$ for some $\lambda \in [0, 1]$ and $y, z \in C$ implies that $y = z$. A point x is called an *exposed point* of a convex set C if there exists a supporting hyperplane H such that $H \cap C = \{x\}$. We would like to note that all exposed points are extreme points, but the converse is not necessarily true, see the track and field example [9, p.75].

3. Classification theorems

In this section, we provide a classification for which self-affine fractals in $\mathbb{R}^d$ project very thick shadows in every k-dimensional subspace, where $1 \le k < d$. Suppose K is a self-affine fractal whose IFS consists of the generating contractions

$$\{\phi_1, \ldots, \phi_N\}.$$

Let $C = C_K$ represent the convex hull of K. Let $\mathcal{L}_k(K)$ be the set of all $d-k$ dimensional affine subspaces that intersect C. When we say *an affine subspace* W of dimension $d-k$, we mean that W is a translated $d-k$ dimensional subspace, i.e., $W = V + x$ for some $V \in G(d, d-k)$ and $x \in \mathbb{R}^d$.

The following theorem provides an equivalence by only checking the first level iteration of self-affine sets. It was first used by Mendivil and Taylor in [16] in their special case. It has also been mentioned for the case of $\mathbb{R}^2$ without proof by [5] and [8]. There appears to be no explicit formal proof written down. Although the proof is not too difficult to experts in fractal geometry, we provide the proof here for the sake of completeness.

Theorem 3.1. *Using the above notations, the following are equivalent:*

(1) *For all $W \in \mathcal{L}_k(K)$, there is some $i \in \{1, \ldots, N\}$ such that $W \cap \phi_i(C) \neq \varnothing$.*

(2) *K projects very thick shadows in every k-dimensional subspace.*

Proof. ($\Longrightarrow$): It suffices to claim that for all $W \in \mathcal{L}_k(K)$ and $n \in \mathbb{N}$, there exists some $\sigma_n \in \Sigma^n$ such that $\phi_{\sigma_n}(C) \cap W \neq \varnothing$. Indeed, we can take $x_n \in \phi_{\sigma_n}(K) \subset \phi_{\sigma_n}(C)$. By passing into subsequence if necessary, we can assume $x_n \to x \in K$ and we have

$$\operatorname{dist}(x_n, W) \le \operatorname{diam}(\phi_{\sigma_n}(K)) \to 0$$

as $n \to \infty$. Hence, $x \in W$. As all $d-k$ dimensional affine subspaces hit K, the projection of K onto any k-dimensional subspace must be equal to that of C. This proves (2).

We now justify our claim by induction. Fix $W \in \mathcal{L}_k(K)$. By our assumption (1), $W \cap \phi_i(C) \neq \varnothing$ for some $i \in \{1, 2, \ldots, N\}$. Taking $\sigma_1 = i$, we proved the case $n = 1$. For an inductive hypothesis, assume that at the nth iteration, there exists some map ϕ_{σ_n} such that $W \cap \phi_{\sigma_n}(C) \neq \varnothing$. Because the IFS is self-affine, the nth level map, ϕ_{σ_n}, from the hypothesis must have an inverse, $\phi_{\sigma_n}^{-1}$. From our inductive hypothesis, we have $\phi_{\sigma_n}(C) \cap W \neq \varnothing$. It implies that $C \cap \phi_{\sigma_n}^{-1}(W) \neq \varnothing$. Notice that $\phi_{\sigma_n}^{-1}(W)$ is still a $(d-k)$ dimensional affine subspace, so the above implies that $\phi_{\sigma_n}^{-1}(W) \in \mathcal{L}_k(K)$. So, by our assumption, there exists some $i \in \{1, \ldots, N\}$ such that

$$\phi_i(C) \cap \phi_{\sigma_n}^{-1}(W) \neq \varnothing.$$

Applying the forward map ϕ_{σ_n} to the above, we have

$$\phi_{\sigma_n}(\phi_i(C)) \cap W \neq \varnothing.$$

Hence, for $\phi_{\sigma_{n+1}} = \phi_{\sigma_n} \circ \phi_i$, we have $\phi_{\sigma_{n+1}}(C) \cap W \neq \varnothing$. So we have shown that (1) $\implies$ (2) holds.

($\Longleftarrow$): Suppose that the self-affine fractal K has a very thick shadow in every k-dimensional subspace. Then we know that $\pi_V(K) = \pi_V(C)$ for all $V \in G(d,k)$. Fix $W \in \mathcal{L}_k(K)$. Take V to be the k-dimensional subspace orthogonal to W. As W passes through C, we can find $z \in W \cap C$. Let y be the orthogonal projection of z onto V, then we have that $y \in V \cap \pi_V(C)$. Moreover, $\pi_V^{-1}(\{y\}) = W$. By our assumption that $\pi_V(C) = \pi_V(K)$, there exists $x \in K$ such that $\pi_V(x) = y$. Hence, $x \in \pi_V^{-1}(\{y\}) = W$. Recall that

$$K \subset \bigcup_{i=1}^{N} \phi_i(C).$$

Therefore, for some $i \in \{1, \dots, N\}$, we have $x \in \phi_i(C)$. In particular, this shows that $\phi_i(C) \cap W \neq \varnothing$. ∎

Checking condition (1) in Theorem 3.1 may still be a tough computational task. The following theorem gives a third equivalent condition when $k = 1$ using the convex hulls of subsets of the connected components of the first iteration of the generating IFS. This third condition is a new contribution, to the best of our knowledge. Moreover, as we will see, this provides a much more effective checking criterion for self-affine fractals.

To state the theorem, we now decompose the image of the first iteration of the convex hull into the disjoint union of r connected components, denoted by R_j where $j \in \{1, \dots, r\}$. That is,

$$\bigcup_{i=1}^{N} \phi_i(C) = R_1 \cup \cdots \cup R_r.$$

We are now ready to state our main theorem.

Theorem 3.2. *Using the prior notations, the following are equivalent:*

(1) *K has a very thick shadow in every 1-dimensional subspace.*

(2) *For all proper indexing sets $I \subset \{1, \dots, r\}$,*

$$\mathrm{conv}\Big(\bigcup_{i\in I} R_i\Big) \cap \mathrm{conv}\Big(\bigcup_{i\in I^c} R_i\Big) \neq \varnothing.$$

Proof. ($\Longrightarrow$): We assume that K has a very thick shadow in every 1-dimensional subspace. Then for all $W \in G(d, 1)$, we have $\pi_W(K) = \pi_W(C)$ is an interval, where $\text{conv}(K) = C$. Suppose for a contradiction, there exists some proper indexing set $I \subset \{1, \ldots, r\}$ such that

$$\text{conv}\Big(\bigcup_{i \in I} R_i\Big) \cap \text{conv}\Big(\bigcup_{i \in I^c} R_i\Big) = \varnothing.$$

For ease, we let A and B denote the above sets, respectively. By the hyperplane separation theorem, the convex nonempty sets A and B can be properly separated by some hyperplane $H = \{x : \langle n, x\rangle = c\}$. This implies that $A \subset H^- = \{x : \langle n, x\rangle < c\}$ and $B \subset H^+ = \{x : \langle n, x\rangle > c\}$. Let W be the 1-dimensional subspace spanned by n. If we project A and B onto W, then $\pi_W(A) \cap \pi_W(B) = \varnothing$. Thus, $\pi_W(A) \cup \pi_W(B)$ is a union of two disjoint closed intervals. But K is a subset of $\bigcup_{i=1}^r R_i$, implying that

$$\pi_W(K) \subset \pi_W(A) \cup \pi_W(B).$$

As I is proper, K intersects both A and B. Hence, $\pi_W(K)$ cannot be an interval, contradicting our assumption (1).

($\Longleftarrow$) Assume that for any indexing set $I \subset \{1, 2, \ldots, r\}$, we have that

$$\text{conv}\Big(\bigcup_{i \in I} R_i\Big) \cap \text{conv}\Big(\bigcup_{i \in I^c} R_i\Big) \neq \varnothing.$$

For a contradiction, suppose that the condition of very thick shadows fails. By the contrapositive of Theorem 3.1, there exists some hyperplane $H \in \mathcal{L}_1(K)$ such that for all $i \in \{1, \ldots, N\}$, we have $H \cap \phi_i(C) = \varnothing$. Note that each $\phi_i(C)$ is connected. The definition of connected components implies that

$$R_j = \bigcup_{\{i : \phi_i(C) \subset R_j\}} \phi_i(C).$$

We see that H cannot intersect any connected component $R_1, \ldots, R_r$. So, there must exist some indexing set $I \subset \{1, \ldots, r\}$ such that

$$\bigcup_{i \in I} R_i \subset H^+ \quad \text{and} \quad \bigcup_{i \in I^c} R_i \subset H^-$$

where H^+, H^- are the upper and lower half plane determined by H. Moreover, $I \neq \varnothing$ because if all $R_i \in H^+$ or H^-, then H would not intersect C. Since the convex hull of a set is the smallest convex set containing the original set and $H^\pm$ are convex, we see that

$$\text{conv}\Big(\bigcup_{i \in I} R_i\Big) \subset H^+ \text{ and } \text{conv}\Big(\bigcup_{i \in I^c} R_i\Big) \subset H^-.$$

Therefore, there does exist some proper indexing set I such that

$$\operatorname{conv}\Big(\bigcup_{i\in I} R_i\Big) \cap \operatorname{conv}\Big(\bigcup_{i\in I^c} R_i\Big) = \varnothing.$$

This contradicts our original assumption. The proof is complete. ∎

4. Fractal-decomposability for cubes

Let us first revisit some known examples in literature and provide a full characterization, for those fractals, when they have very thick shadows on all 1-dimensional subspaces. All these examples were on the unit square. In the last example, we provide a new construction which works in higher dimensions that shows the fractal decomposability for the unit cubes in $\mathbb{R}^d$.

1. Mendivil–Taylor self-affine fractals. Mendivil and Taylor first discovered a self-affine fractal whose convex hull is the unit square such that it has very thick shadows in every projection (see Figure 1). Let $0 < t < \frac{1}{2} < s < 1$ and $s + t < 1$. The contractive maps are defined by

$$\phi_1(x, y) = (tx, sy),\ \phi_2(x, y) = (sx, ty + (1 - t)),$$

$$\phi_3(x, y) = (sx + (1 - s), ty),\ \phi_4(x, y) = (tx + (1 - x), sy + (1 - s)).$$

They obtained a sufficient condition on t, s such that the corresponding invariant set projects very thick shadows in every direction. We now let $R_i = \phi_i([0, 1]^2)$; that is, R_i is the rectangle that is the image of ϕ_i applied on the unit square. In the following argument, we prove that the condition given by Mendivil and Taylor is a complete description of t, s for the everywhere very thick shadow condition.

Example 4.1. Let $0 < t < \frac{1}{2} < s < 1$ and $s + t < 1$. Then, the Mendivil–Taylor self-affine set has very thick shadows in all directions if and only if

$$\frac{1 - \sqrt{2s - 1}}{2} \le t.$$

Proof. We use the characterization about the convex hull in Theorem 3.2. By the rotational symmetry of the rectangles inside the square, condition (2) holds if and only if

(i) The rectangle R_1 intersects the convex hull of the other three rectangles.

(ii) The convex hull of $R_1 \cup R_3$ intersects the convex hull of $R_2 \cup R_4$.

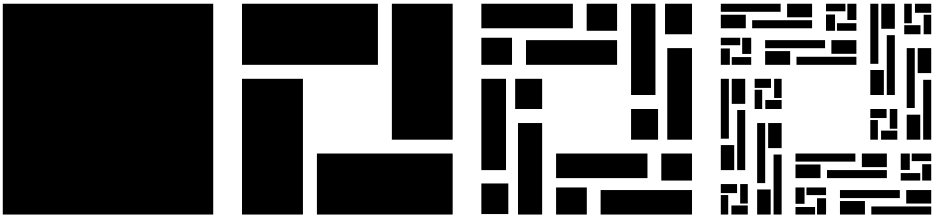

Figure 1. Mendivil–Taylor fractal

Denote the straight line joining $(1-s,0)$ and $(0,1-t)$ by

$$\ell_1 : f(x,y) = (1-t)x + (1-s)y = (1-t)(1-s).$$

Similarly, denote the straight line joining $(t,0)$ and $(s,1-t)$ by

$$\ell_2 : g(x,y) = (1-t)x - (s-t)y = (1-t)t.$$

Condition (i) is achieved if and only if the right upper corner point (t,s) of R_1 lies above ℓ_1, meaning that $f(t,s) \geq (1-t)(1-s)$. Plugging in and rearranging, we have

$$(1-t)(1-s-t) \leq s(1-s).$$

Similarly, (ii) is achieved if and only if $g(1-s,t) \leq t(1-t)$, which means that

$$(1-t)(1-s-t) \leq t(s-t).$$

Indeed, for $s+t \leq 1$, we have $t(s-t) \leq s(1-s)$, so (ii) implies (i). Thus, the Mendivil–Taylor fractal has a very thick shadow in every direction if and only if $(1-t)(1-s-t) \leq t(s-t)$. Solving for t, we obtain the desired characterization. ■

2. A rotated square in the middle. This example was considered by Falconer and Fraser in [5] and Farkas in [8].

Example 4.2. Let $0 < r < 1/2$. In Figure 2, we consider the self-similar IFS on the unit square generated by

$$\phi_1(x) = rx, \quad \phi_2(x) = rx + (1-r,0), \quad \phi_3(x) = rx + (0,1-r),$$
$$\phi_4(x) = rx + (1-r,1-r), \quad \phi_5(x) = rR_{\pi/4}x + \left(\frac{1}{2}, \frac{1}{2} - \frac{\sqrt{2}}{2}r\right).$$

Then (1) the attractor projects very thick shadows in every 1-dimensional subspace if and only if $r \geq 1/3$, and (2) the attractor is totally disconnected if $r < (2 + \frac{1}{\sqrt{2}})^{-1}$.

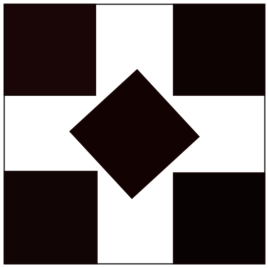

Figure 2. IFS for Example 4.2

Proof. (1) We apply (2) from Theorem 3.2. Notice that the attractor projects very thick shadows if and only if the corner square always intersects the convex hull of the remaining four squares. A direct check shows that the rotated square is always inside the convex hull of the three corner squares. Hence, by symmetry, the condition is equivalent to the right-hand corner of the square $[0,r]^2$, i.e., (r,r) lying above the line $x+y=1-r$, which is the line joining $(0,1-r)$ and $(1-r,0)$. Hence, $r+r \geq 1-r$, meaning $r \geq 1/3$.

(2) By ensuring that the corner (r,r) is not intersecting the rotated square, which is exactly when $r < \frac{1}{2+\frac{1}{\sqrt{2}}}$, all squares do not intersect each other. Hence, the fractal is totally disconnected. ■

Consequently, a totally disconnected self-similar set projecting very thick shadows in every 1-dimensional subspace exists if we take $1/3 \leq r < \frac{1}{2+1/\sqrt{2}}$. Farkas indicated how to use this pattern to create self-similar totally disconnected sets whose Hausdorff dimension is arbitrarily close to 1. However, there were no explicit mappings written down and it is unclear how one can generalize to higher dimensions. In the following example, we provide an explicit solution based on fractal squares/cubes.

3. Cross and corner fractal cubes.

Example 4.3 (Cross and Corner Fractal Cubes). We introduce a fractal cube construction that we call the *cross and corner fractal*. To begin, fix some odd positive integer n. Divide the unit cube into $n \times \cdots \times n$ smaller cubes on $\mathbb{R}^d$. We denote by $\mathcal{P}_d$ the set of all $d \times d$ permutation matrices that map the coordinate hyperplane $x_d = 0$ to another coordinate plane $x_j = 0$ for some $j = 1, \ldots, d$.

Take the cross in the center of our grid, whose digits are

$$\texttt{Cross} = \bigcup_{\sigma \in \mathcal{P}_d} \sigma \left\{ \left(j, \frac{n-1}{2}, \ldots, \frac{n-1}{2} \right) : j = 1, \ldots, n-2 \right\}.$$

We say the *diagonals* are all the unit cubes such that each of them is connected only by a corner, from $(\epsilon_1, \dots, \epsilon_d)$ to $(1-\epsilon_1, \dots, 1-\epsilon_d)$, for $\epsilon_1, \dots, \epsilon_d \in \{0, 1\}$.

For each corner of the unit cube, we choose cubes on the diagonal consecutively until we hit the convex hull of the cross. For example, at the origin, we choose cubes on the diagonal until we hit the hyperplane $x_1 + \cdots + x_d = \frac{(n-1)(d-1)}{2} + 1$, which is the convex hull of the points $(1, \frac{n-1}{2}, \dots, \frac{n-1}{2})$, $(\frac{n-1}{2}, 1, \frac{n-1}{2}, \dots, \frac{n-1}{2})$,..., $(\frac{n-1}{2}, \dots, \frac{n-1}{2}, 1)$. As these diagonal cubes are of the form $(j, j, \dots, j)$ where j takes the values $1, \dots, n-1$, when we plug them into the hyperplane equation, we find that

$$N_d = \left\lfloor \frac{(n-1)(d-1)}{2d} + \frac{1}{d} \right\rfloor \tag{4.1}$$

cubes are needed. By symmetry, we choose from every corner of the cubes, N_d consecutive diagonal cubes from the corner. Collect all those digits as the set `Corner`. Then our digit set is $\mathcal{D} = \texttt{Cross} \cup \texttt{Corner}$, and our IFS is $\{\frac{1}{n}(x+d) : d \in \mathcal{D}\}$. In Figure 3, the first two iterations of the 2-dimensional cross and corner fractal squares are shown. In Figure 4, the cross and one side of the corners of the fractal construction are shown.

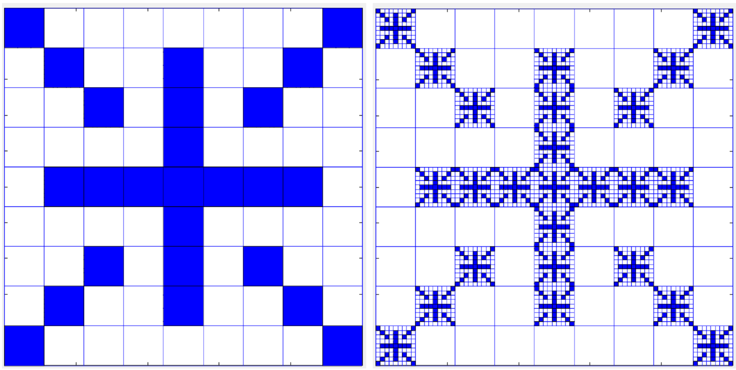

Figure 3. A two-dimensional picture for the cross and corner fractal

We call K_n to be the self-similar fractal cubes that this IFS generates. Then,
(1) K_n is totally disconnected.
(2) K_n projects very thick shadows in every 1-dimensional subspace.
(3) $\dim_H(K_n) \to 1$ as $n \to \infty$.

Proof. (1) To see that K_n is totally disconnected, we apply Proposition 2.3. At the kth stage of the iteration, the connected components are either a cross inside a cube of length $1/k$ or a union of corner cubes which lies in 2^d cubes meeting at a point. As this union of corner cubes does not intersect the cross, this component has diameter at

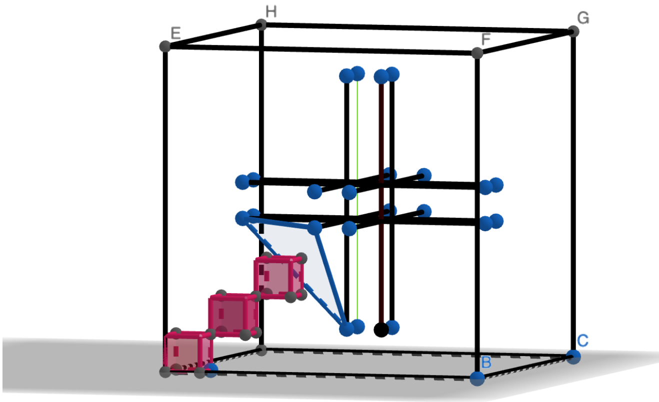

Figure 4. A part of the three-dimensional picture for the cross and corner fractal. The whole construction is to include the diagonal cubes starting from all corners

most $\sqrt{2d}/k$. Therefore, the maximum of the diameters of the connected components at the kth iteration is bounded above by $\sqrt{2d}/k$, which tends to 0 as $k \to \infty$. This shows K_n is totally disconnected.

(2) We now show that this fractal projects to intervals for all lines on $\mathbb{R}^d$. Note that by our construction, the convex hull of the cross must intersect the convex hull of the connected components at each corner. Therefore, it is not possible to separate any connected components by hyperplanes. By Theorem 3.2, our proof is complete.

(3) For a fixed odd integer $n > 2$, there are $d(n-2) + 2^d N_d$ number of cubes chosen for this pattern where N_d is given by (4.1). Hence

$$\dim_H(K_n) = \frac{\log(d(n-2) + 2^d N_d)}{\log n}.$$

As $d(n-2) + 2^d N_d = O(n)$, the above expression goes to 1 as $n \to \infty$. ∎

5. Fractal-decomposability for other convex sets

In the previous section, we demonstrate that cubes on $\mathbb{R}^d$ are fractal decomposable with self-similar sets of dimension arbitrarily close to one. We now extend our study to other convex polytopes in this section. Let $\overrightarrow{AB}$ denote the line segment joining A and B with direction pointing from point A to point B. A simplex on $\mathbb{R}^d$ is the convex hull of $d+1$ points $\{A_0, \ldots, A_d\}$ where the vectors $\{\overrightarrow{A_0A_1}, \ldots, \overrightarrow{A_0A_d}\}$ are linearly independent. We denote it by $\mathcal{S}(A_0, \ldots, A_d)$.

Given a simplex $\mathcal{S} = \mathcal{S}(A_0, \ldots, A_d)$, we consider another similar unrotated image of $\mathcal{S}$, denoted by $\mathcal{S}(A'_0, \ldots, A'_d) = r\mathcal{S}(A_0, \ldots, A_d) + t$, where $0 < r < 1$ and t is a translation vector such that $\mathcal{S}(A'_0, \ldots, A'_d)$ lies in the interior of $\mathcal{S}(A_0, \ldots, A_d)$. Fix $\lambda > 0$. Notice that for all $i = 0, 1, \ldots, d$, the set $\{\overrightarrow{A_i A_j} : j \neq i, j \in \{0, \ldots, d\}\}$ forms a basis for $\mathbb{R}^d$. We now define $d + 1$ affine maps inside $\mathcal{S}$ by the following relations. For $i = 0, 1, \ldots, d$,

$$\begin{aligned} T_i(\overrightarrow{A_i A_j}) &= \lambda \overrightarrow{A_i A_j}, \quad j \neq i - 1, \text{ and} \\ T_i(\overrightarrow{A_i A_{i-1}}) &= \overrightarrow{A_i A'_{i+1}}, \quad j = i - 1, \end{aligned} \tag{5.1}$$

where $j = 0, 1, \ldots, d$. Here $\lambda \overrightarrow{A_i A_j}$ denotes the scaled vector of $\overrightarrow{A_i A_j}$ with A_i fixed. We identify the addition as the addition on the cyclic group of $d + 1$ elements, i.e., if $i = 0$, we identify $-1 = d$. $\Phi_\lambda = \{T_0, \ldots, T_d\}$ defines a self-affine IFS whose attractor has convex hull exactly equal to $\mathcal{S}(A_0, \ldots, A_d)$. Moreover, A_i is the fixed point of T_i for each i. In this IFS,

$$T_i(\mathcal{S}(A_0, \ldots, A_d)) = \mathcal{S}(A_i, \{A_i + \lambda \overrightarrow{A_i A_j} : j \neq i - 1, i\}, A'_{i+1}).$$

Figures 5 and 6 illustrate the IFS on $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

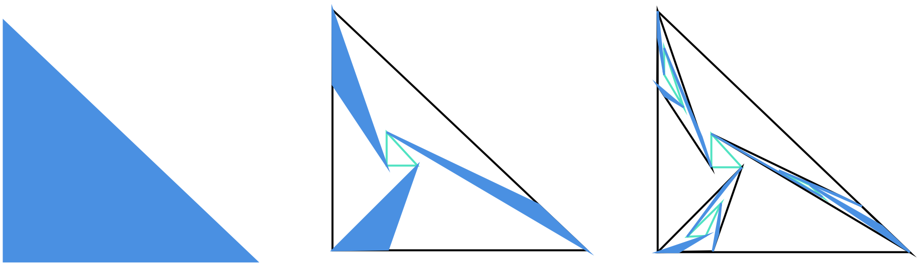

Figure 5. Illustration of the triangular IFS on $\mathbb{R}^2$.

Theorem 5.1. *The attractor K_λ of the IFS Φ_λ defined above is totally disconnected, projects very thick shadows in every 1-dimensional subspace, and has* $\dim_H(K_\lambda) \to 1$ *as* $\lambda \to 0$.

Consequently, all simplices on $\mathbb{R}^d$ are fractal-decomposable with self-affine sets whose Hausdorff dimension is arbitrarily close to 1.

As $T_i(\mathcal{S}(A_0, \ldots, A_d))$ are mutually disjoint, it follows that the self-affine set must be totally disconnected. In Proposition 5.2, we verify condition (2) from Theorem 3.2. Hence, the invariant set projects very thick shadows in every 1-dimensional subspace. Finally, we show that the Hausdorff dimension can be made arbitrarily close to 1 in Proposition 5.3.

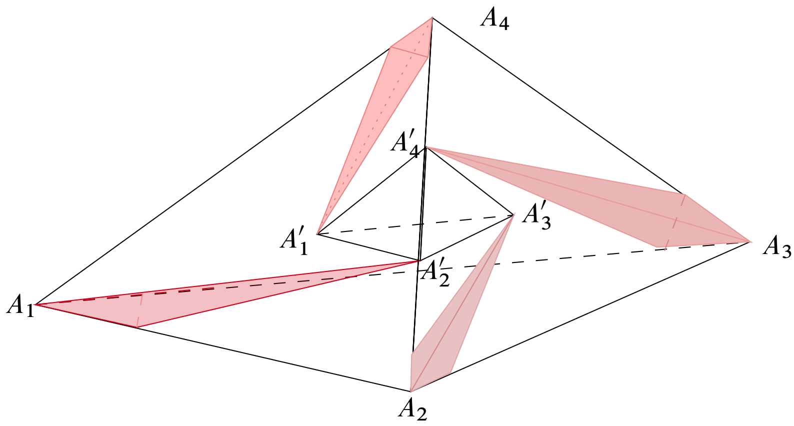

Figure 6. An illustration of the IFS on the simplex on $\mathbb{R}^3$

Proposition 5.2. *Let $\mathcal{J}_i = T_i(\mathcal{S}(A_0, \ldots, A_d))$. Then for all $I \subsetneq \{0, 1, \ldots, d\}$,*

$$\operatorname{conv}\Big(\bigcup_{i\in I} \mathcal{J}_i\Big) \cap \operatorname{conv}\Big(\bigcup_{i\in I^c} \mathcal{J}_i\Big) \neq \varnothing.$$

Proof. Let $I \subsetneq \{0, 1, \ldots, d\}$ and let $i \in I$. We first notice that if $i \in I$, then the line segment $\overrightarrow{A_i A'_{i+1}} \in \operatorname{conv}\big(\bigcup_{i\in I} \mathcal{J}_i\big)$ by (5.1). The proof is based on a simple geometrical fact.

Fact. Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be two parallel line segments in $\mathbb{R}^2$ such that the vectors point in the same direction. Then the line segment $\overrightarrow{AD}$ and $\overrightarrow{CB}$ must intersect.

The proof of this fact is elementary (e.g., we can put $\overrightarrow{AB}$ and $\overrightarrow{CD}$ on $y = 0$ and $y = 1$, respectively, and solve for the intersection), so we omit it. We now have two cases:

(1) Suppose that $i - 1, i + 1 \in I^c$. We note that the line segment $\overrightarrow{A_i A'_{i+1}} \in \operatorname{conv}\big(\bigcup_{i\in I} \mathcal{J}_i\big)$. On the other hand, the points A'_i and A_{i+1} are in $\operatorname{conv}\big(\bigcup_{i\in I^c} \mathcal{J}_i\big)$. Hence, the line segment $\overrightarrow{A'_i A_{i+1}} \in \operatorname{conv}\big(\bigcup_{i\in I^c} \mathcal{J}_i\big)$ as well. From our construction of the smaller simplex, $\overrightarrow{A_i A_{i+1}}$ and $\overrightarrow{A'_i A'_{i+1}}$ are two parallel line segments pointing in the same direction. Thus, the points $A_i, A'_i, A_{i+1}, A'_{i+1}$ lie in the same two-dimensional plane. In particular, $\overrightarrow{A_i A'_{i+1}}$ and $\overrightarrow{A'_i A_{i+1}}$ must intersect by the *Fact*. This means that the intersection of the convex hulls in the statement cannot be empty.

Because of (1), if I consists of only one element, then the conclusion of the proposition must hold. The same is true if I^c consists of only one element as we can switch the role of I and I^c in the proof of case (1). Hence, we can assume that both I and I^c contain at least two elements for case (2).

(2) If (1) does not hold and I, I^c has at least two elements, then $i-1$ or $i+1 \in I$. In this case, we let

$$j = \min\{i' \geq i : i' \in I, i'+1 \in I^c\},\ k = \max\{i' < i : i'+1 \in I, i' \in I^c\}.$$

Since I, I^c has at least two elements and (1) does not hold, we see that $k+1 \neq j$ and $j+1 \neq k$. Moreover, we have $j \in I$, $j+1 \in I^c$, $k \in I^c$ and $k+1 \in I$. This means that the points $A'_{j+1}, A_{k+1} \in \operatorname{conv}(\bigcup_{i \in I} \mathcal{J}_i)$ and $A_{j+1}, A'_{k+1} \in \operatorname{conv}(\bigcup_{i \in I^c} \mathcal{J}_i)$, as well as their respective line segments. We now consider the plane generated by $A_{j+1}, A'_{j+1}, A_{k+1}, A'_{k+1}$. The *Fact* implies that the line segment $\overrightarrow{A_{j+1}A'_{k+1}}$ and $\overrightarrow{A'_{j+1}A_{k+1}}$ must intersect, so the intersection of the convex hulls cannot be empty. This completes the whole proof. ∎

Proposition 5.3. *With respect to the above notations, we have*

$$\lim_{\lambda \to 0} \dim_H(K_\lambda) = 1.$$

Proof. We give an estimate of the affinity dimension (2.1) of the affine IFS. To do so, we need to give an estimate of the singular values of the linear part of T_i. We let $\alpha_{i,1} \geq \alpha_{i,2} \geq \ldots \geq \alpha_{i,d} > 0$ be the singular values of T_i. Also, let

$$\alpha_1 = \max\{\alpha_{i,1} : i = 0, \ldots, d\},\ \alpha_2 = \max\{\alpha_{i,2} : i = 0, \ldots, d\}.$$

As A_i is the fixed point of T_i, by putting A_i as the origin of the coordinates, we may assume T_i is a linear transformation. Moreover, the subspace H_i containing the face of the simplex generated by the points $\{A_j : j \in \{0, 1, \ldots, d\} \setminus \{i-1\}\}$ is the eigenspace of T_i with eigenvalue λ. Let $\mathcal{B}_i$ be an orthonormal basis for H_i; we extend $\mathcal{B}_i$ to an orthonormal basis for $\mathbb{R}^d$ by adding one more vector. Then T_i admits a matrix representation of the form $\begin{bmatrix} \lambda I_{d-1} & \mathbf{u} \\ \mathbf{0} & u_d \end{bmatrix}$ where I_{d-1} is the $(d-1)\times(d-1)$ identity matrix and $\mathbf{u} = (u_1, \ldots, u_{d-1})^\top$. Hence,

$$T_i^* T_i = \begin{bmatrix} \lambda^2 I_{d-1} & \lambda \mathbf{u} \\ \lambda \mathbf{u} & \sum_{i=1}^d u_i^2 \end{bmatrix}.$$

By the Cauchy interlacing theorem applied to the principal minor $\lambda^2 I_{d-1}$, we have $\alpha_2 \leq \lambda$ (they are indeed equal if $d \geq 3$). Recall that the singular value function is sub-multiplicative. Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{\sigma\in\Sigma^k}\varphi^s(T_\sigma) &\le \sum_{k=1}^{\infty}\sum_{\sigma\in\Sigma^k}(\alpha_{1,\sigma_1}\alpha_{2,\sigma_1}^{s-1})\cdots(\alpha_{1,\sigma_k}\alpha_{2,\sigma_k}^{s-1})\\ &\le \sum_{k=1}^{\infty}\sum_{\sigma\in\Sigma^k}\alpha_1^k(\alpha_2^{s-1})^k\\ &= \sum_{k=1}^{\infty}((d+1)\alpha_1\alpha_2^{s-1})^k.\end{aligned}$$

The above summation is finite if and only if $(d+1)\alpha_1\alpha_2^{s-1} < 1$. So we must have

$$\dim_a(K) \le 1 + \frac{\log((d+1)\alpha_1)}{\log\alpha_2^{-1}}.$$

Since $\alpha_2 \to 0$ as $\lambda \to 0$, we must have $\dim_H(K_\lambda) \le \dim_a(K) \to 1$. Proposition 5.2 implies that K_λ projects very thick shadows, so $\dim_H(K_\lambda) \ge 1$. Hence, our proposition follows. ∎

Proof of Theorem 1.4 (1). We note that if C is a convex polytope, then C is a finite union of simplices. Denote all these simplices by $\{\Delta_1, \Delta_2, \dots, \Delta_N\}$. Next, by Theorem 5.1, there exists an affine IFS Φ_j whose attractor K_j projects very thick shadows in every 1-dimensional subspace and has Hausdorff dimension arbitrarily close to 1 for each Δ_j, where $1 \le j \le N$. By Proposition 2.2, the union

$$K = \bigcup_{j=1}^{N} K_j$$

is totally disconnected. Moreover, K is perfect since each K_j is perfect and the finite union of perfect sets is perfect. K also has Hausdorff dimension arbitrarily close to 1 by the countable stability of Hausdorff dimension. It remains to demonstrate that $\pi_W(C) = \pi_W(K)$ for all $W \in G(d,1)$. Consider the following:

$$\pi_W(C)=\pi_W\left(\bigcup_{j=1}^{N}\Delta_j\right)=\bigcup_{j=1}^{N}\pi_W(\Delta_j)=\bigcup_{j=1}^{N}\pi_W(K_j)=\pi_W\left(\bigcup_{j=1}^{N}K_j\right)=\pi_W(K).$$

So every convex polytope is fractal-decomposable with a finite union of self-affine sets whose Hausdorff dimension is arbitrarily close to 1, completing the proof of Theorem 1.4 (1). ∎

Remark. It is possible to obtain a very thin self-similar set inside any triangle on $\mathbb{R}^2$ projecting very thick shadows in every direction. To see this, let Δ be a triangle. We notice a simple fact that every triangle is self-similar with 4 maps of contraction ratio 1/2. For $\lambda > 0$ given in the affine IFS construction (see Figure 5), we now partition the triangle into 4^n similar triangles where $n = n_\lambda$ is the integer such that

$$2^{-n} \leq \frac{\lambda}{100} < 2^{-n+1}.$$

We now take all those small triangles that intersect $\Delta_i = T_i(\Delta)$, for $i = 1, 2, 3$. These triangles form the first iteration of the self-similar IFS. Call this IFS Φ. Because the self-similar triangulation covers each Δ_i, the convex hull condition of Theorem 3.2 still holds, ensuring we have very thick shadows in every direction. Additionally, because there is a small triangle separating Δ_i, the diameter of each connected component goes to 0, and we have that the attractor K of Φ is totally disconnected by Proposition 2.3.

Finally, we calculate the Hausdorff dimension of K. We know that the contraction ratio is $\frac{1}{2^n}$. By standard volume counting, we estimate the number of maps covering each Δ_i is $O(2^{2n}\lambda)$. Therefore, for some universal constant $C > 0$,

$$\dim_H(K) \leq \frac{\log(C2^{2n}\lambda)}{\log 2^n} = 1 + \frac{\log(C2^n\lambda)}{\log 2^n} < 1 + \frac{\log 200C}{\log 2^n}.$$

Since $n \to \infty$ as $\lambda \to 0$, we see that $\dim_H(K) \to 1$ as $\lambda \to 0$.

6. A compact fractal-decomposable rectifiable set

In this section, our objective is to construct a rectifiable compact set that is fractal-decomposable. Let us recall some terminologies. A 1-*set* is a set E with finite and positive one-dimensional Hausdorff measure. A 1-set E is called *rectifiable* if there exists a countable collection of sets A_i and Lipschitz functions f_i such that

$$\mathcal{H}^1\left(E \setminus \bigcup_{i=1}^{\infty} f_i(A_i)\right) = 0.$$

E is called *purely unrectifiable* if $\mathcal{H}^1(E \cap F) = 0$ for all rectifiable sets F. The well-known Besicovitch projection theorem states the following.

Theorem 6.1 (Besicovitch projection theorem, see [15]). *For a purely unrectifiable 1-set on $\mathbb{R}^2$, the projection of E must have measure zero for almost all directions.*

Because of this theorem, purely unrectifiable sets do not project very thick shadows in almost all directions.

We now prove Theorem 1.4 (2), which predicts that every convex set is fractal-decomposable with compact totally disconnected rectifiable 1-sets. This question was first brought to Alan Chang by the first-named author. He later on discussed with Tuomas Orponen, who provided to us a workable idea of the construction. We would like to thank Alan Chang and Tuomas Orponen for supplying the main idea of the proof in the key lemma below, which utilizes the venetian blind construction. Since the lemma is on dimension 2, we parametrize the projection by π_θ, $\theta \in [0, \pi)$, where π_θ is the orthogonal projection onto the line $y = x(\tan\theta)$.

Lemma 6.2. *Let $E_0 = [0, 1] \times \{0\}$. Then there exists a totally disconnected rectifiable compact 1-set $E \subset [0, 1]^2$ such that $\pi_\theta(E) \supset \pi_\theta(E_0)$ for $\theta \in [0, \pi)$.*

Proof. It suffices to construct E such that the conclusion holds for $\theta \in [0, \frac{\pi}{2}]$. To extend the conclusion to $\theta \in [\frac{\pi}{2}, \pi)$, we need only reflect E about the line $x = \frac{1}{2}$. Unioning E and its reflection gives us our desired result.

Now, we construct E for $\theta \in [0, \frac{\pi}{2}]$ using the venetian blind construction[2]. To begin, $L(\mathbf{a}, \mathbf{b})$ represents the line segment connecting $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. Let $\{\varepsilon_n\}$ be a rapidly decreasing sequence, which is specified later. Let

$$E_1 = L\Big((0, 0),\ \Big(\frac{1}{2}, \varepsilon_1\Big)\Big) \cup L\Big(\Big(\frac{1}{2}, 0\Big),\ (1, \varepsilon_1)\Big).$$

Suppose E_i is constructed. Let $\mathbf{e_2} = (0, 1)$. Then, we define

$$E_{i+1} = \bigcup_{L(\mathbf{a},\mathbf{b}) \subset E_i} \Big(L\Big(\mathbf{a},\ \frac{\mathbf{a}+\mathbf{b}}{2} + \varepsilon_{i+1}\mathbf{e_2}\Big) \cup L\Big(\frac{\mathbf{a}+\mathbf{b}}{2}, \mathbf{b} + \varepsilon_{i+1}\mathbf{e_2}\Big)\Big).$$

There are 2^j line segments in E_j. For simplicity, we write $E_j = \bigcup_{k=1}^{2^j} L_{k,j}$, where $L_{k,j}$ is the line segment ending at the vertical line $x = \frac{k}{2^j}$. We can think of $L_{k,j}$ as the "blind". This construction of E_i is shown in Figure 7. Recall the Hausdorff metric between two compact sets is defined as

$$d_H(E, F) = \inf\{\delta : E \subset (F)_\delta \text{ and } F \subset (E)_\delta\},$$

where $(E)_\delta$ is the δ-neighborhood of the set E. It is well known that the set of all compact sets forms a complete metric space under d_H.

Our lemma follows by establishing these four claims:

(1) $E = \lim_{n\to\infty} E_n$ under the Hausdorff metric.

(2) E is totally disconnected.

[2] Another example of this construction can be found on page 104 of [4].

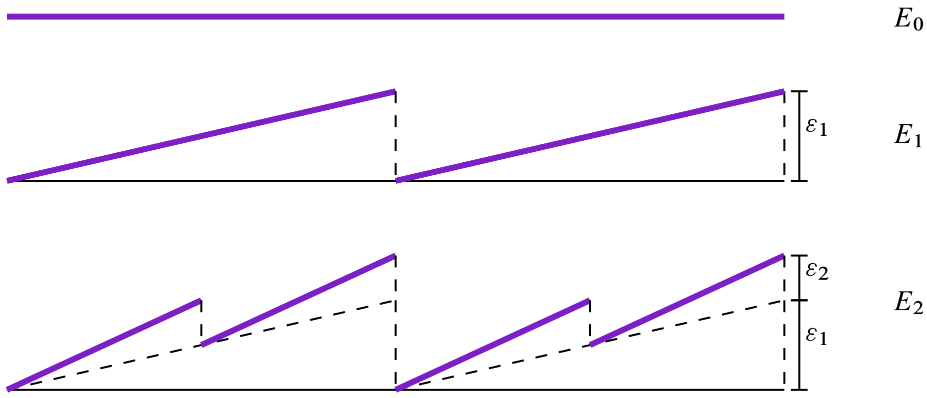

Figure 7. Venetian blind construction

(3) $\pi_\theta(E) \supset \pi_\theta(E_0)$ for all $\theta \in [0, \frac{\pi}{2}]$.

(4) E is a rectifiable 1-set.

(1) To prove the first claim, notice that $E_{i+1} \subset (E_i)_{\varepsilon_{i+1}}$ and $E_i \subset (E_{i+1})_{\varepsilon_{i+1}}$. So $d_H(E_i, E_{i+1}) < \varepsilon_{i+1}$. Then, $d_H(E_m, E_n) < \sum_{i=n}^{m} \varepsilon_{i+1}$ by the triangle inequality. If we choose $\{\varepsilon_i\}$ to be a summable sequence, we see that E_n is a Cauchy sequence in the Hausdorff metric. By completeness, E_n converges to some compact set E.

(2) We now show the second claim. The vertical segment from $(\frac{1}{2}, 0)$ to $(\frac{1}{2}, t)$ where $t \in [0, \varepsilon_1]$ is not in E. So there exist bounded disjoint open sets U_0 and U_1 such that

$$E \cap \left(\left\{(x, y) : 0 < x < \frac{1}{2}\right\} \cup \left\{\mathbf{b}_0, \mathbf{u}_{1/2}\right\}\right) \subseteq U_0 \quad \text{and}$$

$$E \cap \left(\left\{(x, y) : \frac{1}{2} < x < 1\right\} \cup \left\{\mathbf{b}_{1/2}, \mathbf{u}_1\right\}\right) \subseteq U_1$$

where we denote by $\mathbf{u}_{i2^{-n}}$ and $\mathbf{b}_{i2^{-n}}$ the uppermost point and bottom most points of E on the line $x = i2^{-n}$. For induction, suppose U_σ, $\sigma \in \{0, 1\}^n$, is an open set such that

$$E \cap \left(\left\{(x, y) : \frac{i}{2^n} < x < \frac{i+1}{2^n}\right\} \cup \{\mathbf{b}_{i2^{-n}}, \mathbf{u}_{(i+1)2^{-n}}\}\right) \subseteq U_\sigma.$$

Further, by the venetian blind construction, there exist vertical jumps whenever the x-coordinate is a dyadic point, giving us that

$$E \cap \left(\left\{(x, y) : \frac{2i}{2^{n+1}} < x < \frac{2i+1}{2^{n+1}}\right\} \cup \{\mathbf{b}_{i2^{-n}}, \mathbf{u}_{(2i+1)2^{-(n+1)}}\}\right) \subseteq U_{\sigma 0} \quad \text{and}$$

$$E \cap \left(\left\{(x, y) : \frac{2i+1}{2^{n+1}} < x \le \frac{2i+2}{2^{n+1}}\right\} \cup \{\mathbf{b}_{(2i+1)2^{-(n+1)}}, \mathbf{u}_{(i+1)2^{-n}}\}\right) \subseteq U_{\sigma 1},$$

where $U_{\sigma 0}$ and $U_{\sigma 1}$ are disjoint open sets inside U_σ. Thus,

$$E \subset \bigcap_{n=1}^{\infty} \bigcup_{\sigma\in\Sigma^n} U_\sigma, \quad \text{where } \Sigma = \{0,1\}.$$

All U_σ are disjoint from one another by construction, and one can ensure the maximum of the diameters of U_σ among $\sigma \in \Sigma^n$ tends to zero as $n \to \infty$ since $\{\varepsilon_n\}$ is summable. Hence, E is totally disconnected.

(3) The third claim follows from showing that any line with slope $-\tan\theta$ for $\theta \in [0, \frac{\pi}{2}]$ passing through E_0 must also pass through E. Let L be such a line through E_0. By the venetian blind construction, L must also pass through E_n for all n, i.e., $L \cap E_n \neq \varnothing$. As $E_n \to E$ in the Hausdorff metric, we see L must also pass through E. Hence $\pi_\theta(E) \supset \pi_\theta(E_0)$.

(4) For the fourth claim, we show that E is indeed rectifiable by constructing a curve with finite $\mathcal{H}^1$ measure containing E. Define the curve Γ_1 as

$$\Gamma_1 = E_1 \cup \mathcal{V}_1$$

where

$$\mathcal{V}_1 = L\Big(\Big(\frac{1}{2}, 0\Big),\ \Big(\frac{1}{2}, \varepsilon_1\Big)\Big) \cup L\left((1,0)\ (1,\varepsilon_1)\right) := v_{1,1} \cup v_{2,1}.$$

That is, Γ_1 is the union of E_1 with the collection $\mathcal{V}_1$ of vertical line segments that connect the disjoint line segments of E_1. Suppose Γ_j is constructed, such that

$$\Gamma_j = E_j \cup \mathcal{V}_j.$$

We construct Γ_{j+1} as follows. Recall that $E_j = \bigcup_{k=1}^{2^j} L_{k,j}$. Additionally, for each $r = 1, 2, \dots, 2^{j+1}$, we consider $L_{k_r,j}$ to be the unique line segment with the largest y-coordinate that intersects $x = \frac{r}{2^{j+1}}$. Define $v'_{r,j+1}$ to be the vertical line segment at $x = \frac{r}{2^{j+1}}$ of length ε_{j+1}, beginning at $L_{k_r,j}$, and moving vertically ε_{j+1} units. Then, we have $\mathcal{V}_{j+1}$ is the collection of vertical line segments $v_{r,j+1}$ defined as

$$v_{r,j+1} = \begin{cases} v'_{r,j+1} & \text{if } r \text{ is odd} \\ v_{r/2,j} \cup v'_{r,j+1} & \text{if } r \text{ is even.} \end{cases}$$

Then, we can define Γ_{j+1} as

$$\Gamma_{j+1} = E_{j+1} \cup \mathcal{V}_{j+1}.$$

Γ_3 is represented by the bolded line in Figure 8.

In the proof of the first claim, we showed that $E_n \to E$ under d_H. Since the maximal distance between Γ_n and Γ_{n+1} is ε_{n+1} under d_H, we can apply the same strategy to show that Γ_n converges to some compact set Γ.

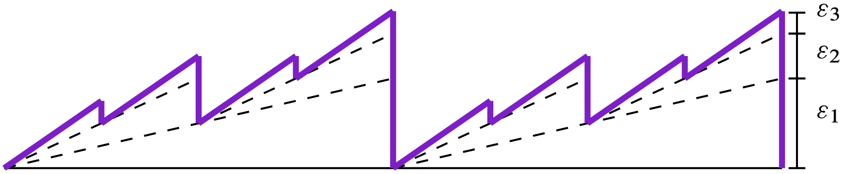

Figure 8. Γ_3 curve

We claim that $E \subset \Gamma$. First, recall that a point $x \in E = \lim_{n\to\infty} E_n$ in Hausdorff metric if and only if there is a sequence of points $x_n \in E_n$ such that $\lim_{n\to\infty} x_n = x$ (see, e.g., [2, p.72]). As we know that $E_n \subset \Gamma_n$ and Γ is the limit of Γ_n under the Hausdorff metric, the claim follows.

Recall that a continuum is a compact connected metric space. To complete our proof, we need the following theorem taken from [7].

Theorem 6.3 ([7, Theorem 3.18, p.39]). *Let $\{\Gamma_n\}$ be a sequence of continua in $\mathbb{R}^d$ convergent in the Hausdorff metric to a compact set Γ. Then Γ is a continuum and*

$$\mathcal{H}^1(\Gamma) \le \liminf_{n\to\infty} \mathcal{H}^1(\Gamma_n).$$

Moreover, a continuum with finite $\mathcal{H}^1$ measure is rectifiable (cf. [7, Theorem 3.14, p.36]*).*

Γ_n is compact and connected. Therefore, $\{\Gamma_n\}$ is indeed a sequence of continua that converges to Γ under d_H. We can then apply Theorem 6.3 to see that Γ itself is a continuum – and hence connected – as well as that

$$\mathcal{H}^1(\Gamma) \le \liminf_{n\to\infty} \mathcal{H}^1(\Gamma_n).$$

It remains to compute $\mathcal{H}^1(\Gamma_n)$ which is exactly the total length of the line segments in $E_n \cup \mathcal{V}_n$. Note that

$$\mathcal{H}^1(\mathcal{V}_n) = \sum_{j=1}^{n} 2^j \varepsilon_j$$

and by elementary geometry and the Pythagorean theorem, the length of one of the "blinds" in E_n is equal to

$$\sqrt{\left(\frac{1}{2^n}\right)^2 + \left(\frac{\varepsilon_1}{2^{n-1}} + \frac{\varepsilon_2}{2^{n-2}} + \cdots + \varepsilon_n\right)^2} = \frac{1}{2^n} \cdot \sqrt{1 + \left(\sum_{j=1}^{n} 2^j \varepsilon_j\right)^2}.$$

Therefore, summing all 2^n "blinds" which are of the same length, we have

$$\mathcal{H}^1(\Gamma_n) = \sum_{j=1}^{n} 2^j \varepsilon_j + \sqrt{1 + \left(\sum_{j=1}^{n} 2^j \varepsilon_j\right)^2}.$$

If we choose our rapidly decreasing positive sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ such that it satisfies $\sum_{j=1}^{\infty} 2^j \varepsilon_j < \infty$, then we see that

$$\mathcal{H}^1(\Gamma) \le \liminf_{n\to\infty} \mathcal{H}^1(\Gamma_n) < \infty.$$

Additionally, since Γ contains vertical line segments at each dyadic point, we know that $\mathcal{H}^1(\Gamma) > 0$. Therefore, Γ is rectifiable.

Finally, by the monotonicity of measure, $\mathcal{H}^1(E) \le \mathcal{H}^1(\Gamma) < \infty$. We now notice that as $\pi_0(E) \supset [0, 1]$, and the projection does not increase the Hausdorff measure, so we have $\mathcal{H}^1(E) \ge \mathcal{H}^1(\pi_0(E)) \ge 1 > 0$. Thus, E is a 1-set. As $E \subset \Gamma$ and Γ is rectifiable, E is also rectifiable, justifying our last claim. ∎

We use Lemma 6.2 to prove Theorem 1.4 (2). The proof in high dimensions requires an induction, so we first settle it for a polygon on $\mathbb{R}^2$.

Proposition 6.4. *Let C be a convex polygon on $\mathbb{R}^2$. Then there exists a compact totally disconnected rectifiable 1-set $K \subset C$ such that $\pi_\theta(K) = \pi_\theta(C)$ for all $\theta \in [0, \pi)$. Hence, polygons are fractal-decomposable by a compact totally disconnected rectifiable 1-set.*

Proof. Let C be an n-sided polygon, and let $L_1, L_2, L_3, \dots, L_n$ be the sides of the polygon. We claim that for each $j = 1, \dots, n$, there exists a compact totally disconnected rectifiable set $E_j \subset C$ such that $\pi_\theta(E_j) \supset \pi_\theta(L_j)$ for all $\theta \in [0, \pi)$. Then we take E to be the union of E_j, which is still a compact totally disconnected rectifiable set. The union is a subset of C, and since $\pi_\theta(C) = \bigcup_{j=1}^{n} \pi_\theta(L_j)$, this union gives us

$$\pi_\theta(C) \subset \bigcup_{j=1}^{n} \pi_\theta(E_j) = \pi_\theta(E).$$

But $E \subset C$, and we have the other inclusion for the projection as well. Thus, E is our desired set.

To justify the claim, we take a side $L = L_j$ for some j. Suppose that L connects with the other two sides, called L' and L'', at some vertices. We now cut $L = \ell' \cup \ell''$ into two lines where L' connects with ℓ' and similarly for the others. Consider $L' \cup \ell'$ and $L'' \cup \ell''$. We have two cases depending on the angle between L' and ℓ' (and, respectively, L'' and ℓ'').

(i) Suppose that L' makes an obtuse angle with ℓ' (i.e., the angle lies between $[\pi/2, \pi)$). We simply take E in Lemma 6.2 for ℓ'. For ε_j sufficiently small, E must be inside the polygon and $\pi_\theta(E) = \pi_\theta(\ell')$ for all $\theta \in [0, \pi)$.

(ii) Suppose that L' makes an acute angle with ℓ'. Without loss of generality, assume that $\ell' = [0, 1] \times \{0\}$, and L' starts at $(1, 0)$, making an acute angle with ℓ'. Due to the acute angle, the E constructed in Lemma 6.2 cannot lie in C. However, we can further decompose ℓ' into countable interior disjoint line segments towards the vertex:

$$\ell' = \bigcup_{j=0}^{\infty} \ell_j, \ \ell_j = [1 - 2^{-j}, 1 - 2^{-j-1}] \times \{0\}.$$

We apply Lemma 6.2 for each ℓ_j to construct $E_j \subset C$ (by taking sufficiently small ε_j). Define $E = \bigcup_{j=1}^{\infty} E_j \cup \{(1, 0)\}$. Then $\pi_\theta(E) \supset \pi_\theta(\ell')$ for all $\theta \in [0, \pi)$. It remains to check if E is closed and totally disconnected. The set E is closed because if $x = (x_1, x_2) \in \mathbb{R}^2 \setminus E$, then for $x_1 < 1$ and for all sufficiently small $\delta > 0$, the ball $B(x, \delta) \subset (1 - 2^{-j}, 1 - 2^{-j-1}) \times \mathbb{R}$. But in this strip, $E \cap [1 - 2^{-j}, 1 - 2^{-j-1}] \times \mathbb{R} = E_j$ and E_j is closed. There exists some δ_1 such that the ball $B(x, \delta_1) \subset \mathbb{R}^2 \setminus E$. Hence, $\mathbb{R}^2 \setminus E$ is an open set, showing that E is closed. Finally, it is a routine check that E is totally disconnected using Theorem 2.1 (2), so we omit this detail. The proof of this proposition is now complete. ■

Proof of Theorem 1.4 (2). We prove the theorem by induction on dimension d. By Proposition 6.4, the theorem has been proven for $d = 2$. Suppose that for all $(d-1)$-dimensional convex polytopes C, there exists a rectifiable 1-set K such that $K \subset C$ and K projects very thick shadows in every one-dimensional subspace. We now prove that it is also true for dimension d.

To prove the statement, let C be a convex polytope of dimension d. Suppose first that C is lying on a $(d-1)$-dimensional hyperplane H_0. By the induction hypothesis, we construct the rectifiable 1-set K inside $C \cap H_0$. Here, for any $(d-1)$-dimensional hyperplanes H such that H intersects C, $H \cap H_0$ is a $(d-2)$-dimensional hyperplane which intersects C. As K projects very thick shadows in every 1-dimensional subspace, every $d-2$-dimensional subspace must intersect K. So $H \cap H_0 \cap K \neq \varnothing$, and hence $H \cap K \neq \varnothing$.

Suppose now that C is not lying in any $(d-1)$-dimensional hyperplane. Then C admits a half-space representation:

$$C = \bigcap_{i=1}^{M} \mathcal{H}_i,$$

where $\mathcal{H}_i$ are closed half-spaces. Moreover, each $\mathcal{H}_i \cap C$ is a $(d-1)$-dimensional convex polytope. We now construct rectifiable 1-sets K_i for each $\mathcal{H}_i \cap C$ by the induction hypothesis, and define

$$K = \bigcup_{i=1}^{M} K_i.$$

Then each $(d-1)$-dimensional hyperplane must intersect one of the faces $\mathcal{H}_i \cap C$. By the induction hypothesis, the hyperplane H_i must intersect K_i. As a consequence, every $(d-1)$-dimensional hyperplane must intersect K. As K is a finite union of compact totally disconnected rectifiable 1-sets, it must be a compact totally disconnected rectifiable 1-set as well. The proof is complete. ■

7. Remarks and open questions

This paper provides a detailed study regarding the projections of very thick shadows onto 1-dimensional subspaces for polytopes on $\mathbb{R}^d$. We conclude this paper with this section by discussing more general cases.

7.1. Other convex sets

To begin with, we establish the following proposition showing that 1-dimensional fractal-decomposability is not possible for general convex sets. The definition of exposed points can be found in Section 2.

Proposition 7.1. *Let C be a closed convex set in $\mathbb{R}^d$. Suppose that C is fractal-decomposable with the totally disconnected Borel set K. Then all exposed points of C are also in K, and the set of exposed points is totally disconnected.*

Proof. Let x be an exposed point of C. There exists some hyperplane H supporting x such that $H \cap (C \setminus \{x\}) = \varnothing$. We now project C and K onto the orthogonal complement of H, denoted by $H^\perp$, which is a 1-dimensional subspace. Since C is fractal-decomposable, we have that $\pi_{H^\perp}(C) = \pi_{H^\perp}(K)$. Therefore, there must exist some $y \in K$ such that

$$\pi_{H^\perp}(x) = \pi_{H^\perp}(y).$$

Note that if $x \in H$, then

$$\pi_{H^\perp}^{-1}(\pi_{H^\perp}(x)) = H.$$

We thus know that $y \in \pi_{H^\perp}^{-1}(\pi_{H^\perp}(x)) = H$. At the same time, $y \in K \subset C$. So $y \in C \cap H = \{x\}$. Hence, $y = x$, and we proved that the exposed points of C are also in K. Because K is totally disconnected and the set of all exposed points of C is a subset of K, we have that the set of exposed points is also totally disconnected. ■

Consequently, it is impossible for convex sets with a smooth boundary and everywhere positive Gaussian curvature, like the Euclidean ball, to be 1-dimensional fractal-decomposable using totally disconnected sets. Fractal decomposability remains an open question for those convex sets with totally disconnected exposed points, but that are not polytopes.

7.2. Higher dimensional fractal-decomposability

If we now try to project onto two or higher dimensional subspaces, total disconnectedness can no longer be obtained even for convex polytopes. The following proposition works on $\mathbb{R}^3$, and it clearly works on any greater dimensions.

Proposition 7.2. *Suppose that a closed convex polytope C on $\mathbb{R}^3$ is 2-dimensional fractal decomposable with a compact set K. Then all edges of the polytope must be in the set K. Consequently, K cannot be totally disconnected.*

Proof. Note that for each edge E, we can always find a supporting hyperplane H such that $H \cap C = E$. If C is 2-dimensional fractal-decomposable with compact set K, then every line passing through C must also pass through K. However, for each $x \in E$, we can take a line ℓ in H that is orthogonal to E such that $\{x\} = \ell \cap E$. Then $\ell \cap K = \{x\}$, meaning that $E \subset K$. This completes the proof. ∎

As a result, we do not have 2-dimensional fractal-decomposability with a totally disconnected compact set K for a convex polytope. We need to replace total disconnectedness with some other type of connectivity. On the other hand, the unit cube $[0, 1]^3$ admits a trivial solution for 2-dimensional fractal-decomposability. One can subdivide the unit cube into n^3 smaller cubes of sidelength $1/n$ and choose cubes that intersect the boundary of $[0, 1]^3$. The self-similar fractal contains the boundary of $[0, 1]^3$ which is trivially 2-dimensional fractal-decomposable. A simple question to ask here which avoids a trivial answer is the following.

Question. Is it possible to construct a self-similar fractal K on $[0, 1]^3$ such that $[0, 1]^3$ is 2-dimensional fractal-decomposable with K and

$$\dim_H(P \cap K) < 2$$

for all affine hyperplanes P that pass through $[0, 1]^3$?

In some sense, we expect that K should be "plane-free". Another question worth considering is that if K is totally disconnected, how many hyperplanes can we choose to guarantee that the projection of K is equal to that of the convex hull on those hyperplanes?

Finally, our Theorem 1.4 (1) showed that every polytope is 1-dimensional fractal-decomposable using a finite union of self-affine sets. We also showed that triangles also admit self-similar solutions. It is interesting to see if every convex polytope is 1-dimensional fractal-decomposable using only one totally disconnected self-similar or self-affine sets. It is also unclear if higher dimensional simplices admit a self-similar solution as simplices themselves are no longer self-similar on higher dimensions (see, e.g., [10]).

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