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The packing measure of the linear Gauss system

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Abstract. For every $k \in \mathbb{N}$, let $f_k : [\frac{1}{k+1}, \frac{1}{k}] \to [0,1]$ be decreasing, linear functions such that $f_k(\frac{1}{k+1}) = 1$ and $f_k(\frac{1}{k}) = 0$, $k = 1, 2, \ldots$. We define the iterated function system S_n by limiting the collection of functions f_k to first n, meaning $S_n = \{f_k\}_{k=1}^n$. Let J_n denote the limit set of S_n. Then, $\lim_{n\to\infty} \mathcal{P}_{h_n}(J_n) = 2$, where h_n is the packing dimension of J_n and $\mathcal{P}_{h_n}$ is the corresponding packing measure.

1. Introduction

Let $g_k(x) = \rho_k x + b_k$, $(k = 1, 2, \ldots, m)$ be a collection of linear contractions defined on the interval $[0, 1]$ and such that $g_k([0,1]) \subset [0,1]$. Assume additionally that this collection satisfies the Open Set Condition (OSC; defined in Definition 2.9). It is well known that the packing dimension of the limit set (defined as in Definition 2.2) is equal to α, the unique positive solution of the implicit equation $\sum_{k=1}^m \rho_k^\alpha = 1$; the α-dimensional packing measure $\mathcal{P}_\alpha(K)$ and the α-dimensional Hausdorff measure $\mathcal{H}^\alpha(K)$ are both finite and positive. In 1999 E. Ayer and R. S. Strichartz [1] provided an algorithm for calculating the Hausdorff measure of the limit set of the iterated function system (IFS) consisting of maps g_k. In 2003 D. Feng [4] provided similar formula for the packing measure in the same setup.

On the other hand, let $f_k : [\frac{1}{k+1}, \frac{1}{k}] \to [0,1]$, $f_k(x) = \{\frac{1}{x}\}$ define the well-known Gauss map. Then, let $g_k : [0,1] \to [\frac{1}{k+1}, \frac{1}{k}]$ be a collection of inverse maps $g_k = f_k^{-1}$. For each, we define an IFS S_n consisting of the maps g_k, $k = 1, \ldots, n$. Let J_n be the Julia set (limit set) generated by S_n. The asymptotics of the Hausdorff dimension of J_n was estimated first in 1929 by V. Jarnik [6], and then more precisely in 1992 by Doug Hensley [5]. In 2016 Mariusz Urbański and Anna Zdunik in [9] proved using Hensley's result, which states that for the previously mentioned sets, we have continuity of the Hausdorff measure in the Hausdorff dimension, meaning

$$\lim_{n\to\infty} H_{h_n}(J_n) = 1,$$

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where H_h denotes the numerical value of Hausdorff measure in dimension h. In this paper, we combine these two ideas by considering linear, decreasing functions

$$f_k\Big(\frac{1}{k+1}\Big) = 1 \quad \text{and} \quad f_k\Big(\frac{1}{k}\Big) = 0$$

and their inverses $g_k = f_k^{-1}$. Then, we define an IFS S_n consisting of the maps g_k, $k = 1, \ldots, n$. Let J_n be the limit set generated by S_n. We prove that the packing measure is continuous, meaning

Theorem 5.16. *Let S_n be the IFS defined in Definition 2.1. Then,*

$$\lim_{n\to\infty} \mathcal{P}_{h_n}(J_n) = 2,$$

where J_n is the limit set of the IFS S_n and P_h denotes packing measure in packing dimension h.

The proof splits into two main parts: one is to estimate the packing measure from below, which is done in Section 4. This is the easier part of the proof. Then, in Section 5 we estimate the lower limit of the n-th density (see Definition 2.15) on different families of sets: starting with sets of the form $[0, r]$ and slowly expanding families up to a point from which we can conclude that the lower limit of the n-th density on all subintervals of $[0, 1]$ centered at J_n is at least $\frac{1}{2}$.

2. Notation and definitions

Let $f_k(x) : [\frac{1}{k+1}, \frac{1}{k}] \to [0, 1]$ for $k \in \{1, 2, \ldots\}$ be a linear analogue of the Gauss map, that is, a linear, decreasing function such that

$$f_k\Big(\frac{1}{k+1}\Big) = 1 \quad \text{and} \quad f_k\Big(\frac{1}{k}\Big) = 0.$$

Definition 2.1. An IFS S_n is defined by limiting the collection of functions f_k to the first n, meaning $S_n = \{f_k\}_{k=1}^n$.

By g_k we will denote the inverse map f_k^{-1}.

Notation 2.2. *By J_n we will denote the limit set created by the IFS S_n,*

$$J_n = \bigcap_{l=1}^{\infty} \bigcup_{q_1,q_2,\ldots,q_l \in \{1,2,\ldots,n\}^l} g_{q_1} \circ g_{q_2} \circ \cdots \circ g_{q_l}([0, 1]).$$

Definition 2.3. The δ-packing of a given set $E \subset \mathbb{R}^n$ is a countable family of pairwise disjoint closed balls of radii at most δ and with centers in E.

Now, we will formulate the classic definition of the packing measure, which can be found in [7, Definition 5.10]. For $s \geq 0$, the s-dimensional packing premeasure of E is defined as follows:

Definition 2.4.

$$P_s^{\text{premeasure}}(E) = \inf_{\delta>0} \{P_\delta^s\},$$

where $P_\delta^s = \sup\{\sum_{B_i \in \mathcal{R}} |B_i|^s : \mathcal{R} \text{ is a } \delta - \text{packing of E}\}$ and $|B_i|$ denotes the diameter of B_i.

Definition 2.5. The s-dimensional packing measure of E is defined as

$$\mathcal{P}_s(E) = \inf\Big\{\sum_{i=1}^{\infty} P_s^{\text{premeasure}}(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i\Big\}.$$

Definition 2.6. The packing dimension of E is by definition the quantity

$$\dim_P(E) := \inf\big\{s \geq 0 : \mathcal{P}_s(E) = 0\big\} = \sup\big\{s \geq 0 : \mathcal{P}_s(E) = \infty\big\}.$$

Notation 2.7. *We will denote Hausdorff dimension of the set J_n by h_n and the Hausdorff measure of the set A in dimension h by $H_h(A)$.*

Notation 2.8. *Analogously, by $P_{h^p}^A$ we denote the packing measure of set A in the packing dimension h^p. By h_n^p we denote the packing dimension of set J_n.*

We denote by $\operatorname{diam}(F)$, or $|F|$, the diameter of the set F.

Definition 2.9. We say that an IFS composed of contractions $\{\phi_i\}_{i=1}^n$ fulfills the Open Set Condition (OSC) if there exists an open set V such that the following two conditions hold:

$$\bigcup_{i=1}^{n} \phi_i(V) \subseteq V \tag{2.1}$$

and the sets $\phi_i(V)$ are pairwise disjoint.

Based on the fact that an IFS S_n fulfills the OSC, it is well known that the packing dimension and Hausdorff dimension are equal (see [2, Theorem 2.7]). Due to this fact, we will be using h_n to denote the packing and Hausdorff dimension of the set J_n. We know that h_n is a unique solution to the following equation:

$$\sum_{k=1}^{n} \Big(\frac{1}{k} - \frac{1}{k+1}\Big)^{h_n} = 1.$$

Proof of this well-known fact can be found in [3]. It follows from this equation that $\lim_{n\to\infty} h_n = 1$ and $0 < h_n < h_{n+1} < 1$.

Definition 2.10. Let $\mathcal{P}^{h_n}$ denote the packing measure in the packing dimension h_n. If $0 < \mathcal{P}^{h_n}(X) < \infty$, then we denote by m_n the normalized $\mathcal{P}^{h_n}$ packing measure

$$m_n(A) := \frac{\mathcal{P}^{h_n}(A \cap X)}{\mathcal{P}^{h_n}(X)}.$$

Let A be a Borel subset of $[0, 1]$. Then, for all $k \in \{1, 2, \dots, n\}$,

$$m_n(g_k(A)) = |g_k'|^{h_n} \cdot m_n(A);$$

thus, m_n is the h_n-conformal measure for the set J_n. It is worth noting that m_n is a unique h_n-conformal measure on the set J_n. This implies that the following holds true:

$$m_n(A) = \frac{H^{h_n}(A \cap J_n)}{H^{h_n}(J_n)}.$$

Using the fact that m_n is conformal measure, we immediately get the following lemma, which we will use throughout our proof:

Lemma 2.11. *Let A be Borel subset of* $[0, 1]$. *Then, for all $k \in \{1, 2, \dots, n\}$, the following holds:*

$$\frac{m_n(A)}{(\mathrm{diam} A)^{h_n}} = \frac{m_n(g_k(A))}{(\mathrm{diam}(g_k(A)))^{h_n}}.$$

We can develop this further.

Lemma 2.12. *Let B be Borel subset of $(\frac{1}{k+1}, \frac{1}{k})$ for some $k \in \{1, 2, \dots, n\}$ intersecting J_n. Then, there exists a set $\widehat{B}$ that is a Borel subset of* $[0, 1]$ *such that*

$$\frac{m_n(B)}{(\mathrm{diam} B)^{h_n}} = \frac{m_n(\widehat{B})}{(\mathrm{diam} \widehat{B})^{h_n}}$$

and $\frac{1}{j} \in \widehat{B}$ for some $j \in \{1, 2, \dots, n\}$.

Proof. If $B \subset [\frac{1}{k+1}, \frac{1}{k}]$ for some $k \leq n$, then $B = g_k(A)$ for some $A \subset [0, 1]$. We can apply Lemma 2.11 with $g_k(A) = B$, giving us

$$\frac{m_n(B)}{(\mathrm{diam} B)^{h_n}} = \frac{m_n(g_k(A))}{(\mathrm{diam}(g_k(A)))^{h_n}} = \frac{m_n(A)}{(\mathrm{diam} A)^{h_n}}.$$

Now notice that $2 \cdot \mathrm{diam} B \leq \mathrm{diam} A$, and either there exists $j \in \{1, 2, \dots, n\}$ such that $\frac{1}{j} \in A$ and then we set $\widehat{B} = A$, or we can apply this procedure again to the set A, expanding it until it intersects the set $\{\frac{1}{j}\}_{j=1}^{n}$, ending the proof. ∎

[8, Theorem 8.6.2], called a Frostman-type lemma by the authors, states the following: let μ be a Borel probability measure on $\mathbb{R}^n$ and A be a bounded subset of $\mathbb{R}^n$. Then, if there exists $C \in (0, \infty]$ such that

(a) for all $x \in A$,

$$\liminf_{r\to 0} \frac{\mu(B(x,r))}{r^s} \le C < \infty$$

and

(b) for all $x \in A$,

$$\liminf_{r\to 0} \frac{\mu(B(x,r))}{r^s} \ge C^{-1},$$

then $0 < \mathcal{P}_s(E) < \infty$ for each Borel set $E \subset A$.

We know that $\mu = m_n$ fulfills this assumption. In fact, the following well-known proposition holds:

Proposition 2.13. *Let m_n be the normalized packing measure on the set J_n. Then, there exist constants $C_n > 0$ such that*

$$C_n^{-1} \cdot r^{h_n} \le m_n(B(x,r)) \le C_n \cdot r^{h_n}$$

for each $x \in J_n$ and $r > 0$, and thus, we know that $0 < \mathcal{P}_{h_n}(J_n) < \infty$.

Proof of this proposition can be found in [3, Theorem 9.3].

Definition 2.14. Let S_n be the IFS generated by f_k, $k = 1, \dots, n$. We denote by $\mathcal{F}_l^n$ the l-th generation of intervals generated by S_n,

$$\mathcal{F}_l^n = \{g_{i_1} \circ g_{i_2} \circ \dots \circ g_{i_l}([0,1]) : i_1, i_2, \dots \in \{1, 2, \dots, n\}\}.$$

Definition 2.15. Let us denote $d_n(J) = \frac{m_n(J)}{(\mathrm{diam}(J))^{h_n}}$. We call this value the n-th density of the interval J.

Lemma 2.16. *Let $[0,1] \supseteq I = I_1 \cup I_2$, where I_1 and I_2 are closed adjacent intervals. Then,*

$$d_n(I) \ge \min\{d_n(I_1), d_n(I_2)\}.$$

Proof. Note that the measure m_n is atomless. This is a direct consequence of Proposition 2.13. Using this fact together with $h_n < 1$, we obtain

$$\begin{aligned} d_n(J) &= \frac{m_n(J)}{(\mathrm{diam}(J))^{h_n}} = \frac{m_n(I_1 \cup I_2)}{(\mathrm{diam}(I_1 \cup I_2))^{h_n}} \ge \frac{m_n(I_1) + m_n(I_2)}{\mathrm{diam}(I_1)^{h_n} + \mathrm{diam}(I_2)^{h_n}} \\ &\ge \min\Big\{\frac{m_n(I_1)}{\mathrm{diam}(I_1)^{h_n}}, \frac{m_n(I_2)}{\mathrm{diam}(I_2)^{h_n}}\Big\} = \min\{d_n(I_1), d_n(I_2)\}. \end{aligned}$$ ■

Lemma 2.17. *Let c be a real number with $0 < c < 1$ and let $a_j \in (0, 1)$ for $j \in 1, 2, \dots, k$. Then,*

$$\sum_{j=1}^{k} a_j^c \ge \Big[\sum_{j=1}^{k} a_j\Big]^c.$$

3. Density theorems for the packing measure

For a Borel measure in $\mathbb{R}^n$, let the lower density be defined as follows:

$$\Theta^\alpha_*(\mu, x) := \liminf_{r\to 0} \frac{\mu(B(x,r))}{(2r)^\alpha}.$$

In particular, if $n = 1$, then

$$\Theta^\alpha_*(\mu, x) := \liminf_{r\to 0} \frac{\mu([x-r, x+r])}{(2r)^\alpha}.$$

P. Mattila in [7, Theorem 6.10] provides a proof of the following theorem:

Theorem 3.1. *Suppose $A \subset \mathbb{R}^n$ with $\mathcal{P}_\alpha(A) < \infty$. Then,*

$$\Theta^\alpha_*(\mathcal{P}_\alpha|_A, x) = 1$$

for $\mathcal{P}_\alpha$ almost all $x \in A$.

Based on this theorem, Feng in [4] observed the following:

Theorem 3.2. *Let μ be the normalized packing measure on the limit set $K \subset \mathbb{R}$ of the IFS consisting of finitely many linear, orientation-preserving contractions, satisfying the OSC. Then, for μ-almost all $x \in \mathbb{R}$, $\Theta^\alpha_*(\mu, x) = d_{\min}$, where $d_{\min}$ is defined as*

$$d_{\min} = \inf\big\{d(J) : J \text{ a closed interval centered in } K \text{ with } J \subset [0,1]\big\},$$

where $d(J) = \frac{\mu(J)}{|J|^\alpha}$ and α denotes the packing dimension of K.

This theorem is easily adapted to the case where the IFS consists of finitely many linear, changing-orientation contractions satisfying the OSC. An immediate consequence of this fact is the following: denote by $\mathcal{P}_\alpha|_K$ the restriction of the α–dimensional packing measure on K, that is, $\mathcal{P}_\alpha|_K = \mathcal{P}_h(A \cap K)$ for any Borel set $A \subset \mathbb{R}$. Since $\mu = c \cdot \mathcal{P}_\alpha|_K$ with $c = 1/\mathcal{P}_\alpha(K)$, we have

$$\Theta^\alpha_*(\mu, x) = \frac{1}{\mathcal{P}_\alpha(K)} \Theta^\alpha_*(\mathcal{P}_\alpha|_K, x) \tag{3.1}$$

for all $x \in \mathbb{R}$. With those tools in hand, Feng noticed the following theorem, which is essential to his paper:

Theorem 3.3. $\mathcal{P}_\alpha(K) = d_{\min}^{-1}$.

Proof. From observation (3.1), we get $\Theta^\alpha_*(\mathcal{P}_\alpha|_K, x) = \Theta^\alpha_*(\mu, x)\mathcal{P}_\alpha(K)$ for any $x \in \mathbb{R}$. However, using the lower density theorem (Theorem 3.1), we have that $\Theta^\alpha_*(\mathcal{P}_\alpha|_K, x) = 1$ for $\mathcal{P}_\alpha|_K$-almost all $x \in \mathbb{R}$, which implies the result. ■

Adapting this theorem to our case, we get the following result:

Corollary 3.4. *The following equality holds:*

$$\mathcal{P}_{h_n}(J_n) = \sup_{F \text{ centered at} J_n \, F \subseteq [0,1]} \frac{(\operatorname{diam}(F))^{h_n}}{m_n(F)} = \Big(\inf_{F \text{ centered at} J_n \, F \subseteq [0,1]} d_n(F) \Big)^{-1},$$

where $\mathcal{P}_{h_n}$ denotes the packing measure with dimension h_n, $\operatorname{diam}(F)$ *is the diameter of the interval F, m_n is the normalized packing measure, and $d_n(F)$ denotes n-th density of the interval F.*

4. Estimating the packing measure from below

In 2003 De-Jun Feng showed in [4] the following result regarding comparing the Hausdorff and packing measures on the real line:

Theorem. *For each* $0 < s < 1$, *define*

$$c(s) = \inf_E \frac{\mathcal{P}_s(E)}{H_s(E)}$$

where $\mathcal{P}_s$, H_s denote the s-dimensional packing and Hausdorff measure, respectively, and the infimum is taken over all sets $E \subset \mathbb{R}$ with $0 < H_s(E) < \infty$. *Then,*

$$2^s(1 + v(s))^s \le c(s) \le 2^s(2^{\frac{1}{s}} - 1)^s$$

for each $0 < s < 1$ *and* $v(s) = \min\{16^{-\frac{1}{1-s}}, 8^{-\frac{1}{(1-s)^2}}\}$.

This result is a general estimate of the quotient of the packing and Hausdorff measure. One can deduce from [9] that $H_{h_n}(J_n) \to 1$ as n tends to infinity, where J_n is the limit set of IFS S_n and h_n is its Hausdorff dimension. This result implies the following estimate:

$$\liminf_{n\to\infty} \mathcal{P}_{h_n}(J_n) \ge 2.$$

However, we are going to provide a short and direct proof of this fact for our case.

Theorem 4.1. *Let S_n be IFS defined in* (2.1). *By J_n, we denote the limit set of the IFS S_n. We denote by $\mathcal{P}_h$ the packing measure in the packing dimension h and by h_n the packing dimension of J_n. Then,*

$$\liminf_{n\to\infty} \mathcal{P}_{h_n}(J_n) \ge 2.$$

Proof. From definition of g_k, we know that

$$g_1(x) = -\frac{1}{2}x + 1$$

and

$$g_n(x) = \Big(\frac{1}{n+1} - \frac{1}{n}\Big)x + \frac{1}{n}$$

for $x \in [0,1]$. The leftmost point in J_n is a stationary point such that $g_n \circ g_1(x_n) = x_n$. A short computation yields that

$$x_n = \frac{2n}{2n^2+2n-1}.$$

Let us define an interval $I_n = [a_n, b_n]$ such that it is centered at x_n, and $b_n = \frac{1}{n}$. Then, because x_n is the left endpoint of the set J_n, the following holds:

$$m_n([a_n, b_n]) = m_n\Big(\Big[\frac{1}{n+1}, \frac{1}{n}\Big]\Big) = \Big|\frac{1}{n} - \frac{1}{n+1}\Big|^{h_n}$$

and

$$|b_n - a_n| = 2 \cdot |b_n - x_n| = 2 \cdot \Big(\frac{1}{n} - \frac{2n}{2n^2+2n-1}\Big) = 2 \cdot \frac{2n-1}{n(2n^2+2n-1)}.$$

Hence,

$$d_n(I_n) = \frac{m_n([a_n, b_n])}{|b_n - a_n|^{h_n}} = \frac{\Big|\frac{1}{n} - \frac{1}{n+1}\Big|^{h_n}}{\Big(2 \cdot \frac{2n-1}{n(2n^2+2n-1)}\Big)^{h_n}} = \Big(\frac{1}{2}\Big)^{h_n} \cdot \Big(\frac{2n^2+2n-1}{2n^2+n-1}\Big)^{h_n}.$$

We showed that for each J_n, there is an interval I_n centered at J_n with density equal to $(\frac{1}{2})^{h_n} \cdot (\frac{2n^2+2n-1}{2n^2+n-1})^{h_n}$, and thus, based on Corollary 3.4, we get

$$\mathcal{P}_{h_n}(J_n) \geq \Big[\Big(\frac{1}{2}\Big)^{h_n} \cdot \Big(\frac{2n^2+2n-1}{2n^2+n-1}\Big)^{h_n}\Big]^{-1},$$

which implies

$$\liminf_{n\to\infty} \mathcal{P}_{h_n}(J_n) \geq 2. \qquad \blacksquare$$

5. Estimating the packing measure from above

We will show that the function that assigns to every number $n \in \mathbb{N}$ the packing measure of the J_n in its packing dimension h_n has the limit equal to 2 when n tends to infinity. We do this in several steps: first, by showing that the lower limit of the densities of the intervals $[0, r]$ is at least $\frac{1}{2}$, then expanding the family of intervals with this property up to a family of intervals that has a right endpoint in the set $\{\frac{1}{k}\}$, $k = 0, 1, \dots$, in Propositions 5.3–5.6. Then, we deduce the same about the intervals with a left endpoint in set $\{\frac{1}{k}\}$, $k = 0, 1, \dots$, in Propositions 5.7–5.9. Then, in the final propositions we use previously obtained estimates to show that any interval centered at J_n that intersects set $\{\frac{1}{k}\}, k = 0, 1, \ldots$ has the limit of the densities at least $\frac{1}{2}$. From those estimates, we will be able to deduce that all intervals centered at J_n have this property.

5.1. Density on intervals $[0, r]$ and $[\frac{1}{k}, \frac{1}{l}]$

Proposition 5.1. *The following estimate holds:*

$$\liminf_{n\to\infty} \inf\Big\{d_n([0,r]) : r \in [0,1], \text{ and interval } [0,r] \text{ is centered at } J_n\Big\} \geq \frac{1}{2}.$$

Proof. Notice that there exists $k \in \{1, 2, \ldots, n\}$ such that $\frac{1}{k+1} < r \leq \frac{1}{k}$. Such k exists, because $[0, r]$ is centered at J_n. Moreover, the fact that $[0, r]$ is centered at J_n implies that $r/2 > \frac{1}{n+1}$, and thus, $\frac{1}{k} > \frac{2}{n+1}$, giving $k < (n+1)/2$. Then, by Lemma 2.17,

$$d_n([0,r]) = \frac{m_n([0,r])}{(\mathrm{diam}([0,r]))^{h_n}} \geq \frac{\sum_{j=k+1}^{n} (1/j - 1/(j+1))^{h_n}}{(1/k)^{h_n}} \geq \frac{\left|\frac{1}{k+1} - \frac{1}{n+1}\right|^{h_n}}{\left|\frac{1}{k}\right|^{h_n}}$$
$$\geq \left[\frac{n-k}{n+1} \cdot \frac{k}{k+1}\right]^{h_n}.$$

Because $k < \frac{n+1}{2}$, we can see that the minimum value of this expression is attained at $k = 1$ or $k = \frac{n+1}{2}$. Indeed,

$$\frac{\partial}{\partial k} \frac{n-k}{n+1} \cdot \frac{k}{k+1} = \frac{n - k(k+2)}{(n+1)(k+1)^2}$$

is equal to zero if and only if $n - k(k+2) = 0$, and hence, $k = \sqrt{n+1} - 1$ is the value at which this expression attains maximum. Thus, the minimum values are attained at the edge of domain of the function. The value of the expression for $k = 1$ is equal to

$$\left[\frac{n-1}{n+1} \cdot \frac{1}{2}\right]^{h_n}$$

and for $k = \frac{n+1}{2}$,

$$\left[\frac{n - \frac{n+1}{2}}{n+1} \cdot \frac{\frac{n+1}{2}}{\frac{n+1}{2} + 1}\right]^{h_n} = \left[\frac{1}{2} - \frac{2}{n+3}\right]^{h_n}.$$

Hence,

$$\liminf_{n\to\infty} \inf\Big\{d_n([0,r]) : r \in [0,1], \text{ and interval } [0,r] \text{ is centered at } J_n\Big\} \geq \frac{1}{2}. \qquad \blacksquare$$

Note that Proposition 5.1 requires only $r > \frac{2}{n+1}$. Hence, we will be using the following corollary in the next steps:

Corollary 5.2. *The following estimate holds:*

$$\liminf_{n\to\infty} \inf\Big\{d_n([0,r]) : r \in \Big[\frac{2}{n+1}, 1\Big]\Big\} \geq \frac{1}{2}.$$

Now, we move to showing similar property for intervals with endpoints in the set $\{\frac{1}{k}\}$, $k = 1, 2, \ldots, n$.

Proposition 5.3. *The following estimate holds:*

$$\lim_{n\to\infty} \inf\Big\{d_n\Big(\Big[\frac{1}{k+l}, \frac{1}{k}\Big]\Big) : k \in \{1, 2, \ldots, n\},$$
$$l + k \in \{k+1, k+2, \ldots, n+1\}\Big\} \geq 1.$$

Proof. Notice that, using Lemma 2.17, we get

$$d_n\Big(\Big[\frac{1}{k+l}, \frac{1}{k}\Big]\Big) = \frac{m_n\big(\big[\frac{1}{k+l}, \frac{1}{k}\big]\big)}{\big|\big[\frac{1}{k+l}, \frac{1}{k}\big]\big|^{h_n}} \geq \frac{\Big|\sum_{j=k}^{k+l-1} \frac{1}{j} - \frac{1}{j+1}\Big|^{h_n}}{\big|\frac{1}{k} - \frac{1}{k+l}\big|^{h_n}} = \frac{\big|\frac{1}{k} - \frac{1}{k+l}\big|^{h_n}}{\big|\frac{1}{k} - \frac{1}{k+l}\big|^{h_n}} = 1.$$

Hence, obviously

$$\liminf_{n\to\infty} \inf\Big\{d_n\Big(\Big[\frac{1}{k+l}, \frac{1}{k}\Big]\Big) : k \in \{1, 2, \ldots, n\},$$
$$l + k \in \{k+1, k+2, \ldots, n+1\}\Big\} \geq 1. \qquad \blacksquare$$

5.2. Density on intervals with a right endpoint in set $\{\frac{1}{k}\}$, $k \in \mathbb{N}$

Now, we will prove that the lower limit of the densities of intervals contained in $[\frac{1}{k+1}, \frac{1}{k}]$ for some $k = 1, 2, \ldots$ and having the right endpoint equal to $\frac{1}{k}$ is at least $\frac{1}{2}$.

Proposition 5.4. *The following estimate holds:*

$$\liminf_{n\to\infty} \inf\Big\{d_n\Big(\Big[r, \frac{1}{k}\Big]\Big) : k \in \{1, 2, \ldots, n\},$$
$$r \in \Big[\frac{1}{k+1}, \frac{1}{k}\Big), \text{interval is centered at } J_n\Big\} \geq \frac{1}{2}.$$

Proof. Note that $[r, \frac{1}{k}] \subseteq [\frac{1}{k+1}, \frac{1}{k}]$, and thus, applying Lemma 2.11, we get

$$d_n\Big(\Big[r, \frac{1}{k}\Big]\Big) = d_n([0, \hat{r}]),$$

where $r = g_k(\hat{r})$. Now we notice that centers of the intervals are transformed under g_k to the centers of the intervals, and the center of $[r, \frac{1}{k}]$ is in J_n. Thus, the interval $[0, \hat{r}]$ is also centered at J_n and based on Proposition 5.1, we get

$$\liminf_{n\to\infty} \inf\Big\{d_n\Big(\Big[r, \frac{1}{k}\Big]\Big) : k \in \{1, 2, \ldots, n\},\\ r \in \frac{1}{k+1}, \frac{1}{k}\Big), \text{ interval is centered at } J_n\Big\} \geq \frac{1}{2}. \qquad \blacksquare$$

Similarly to Corollary 5.2, notice that the proof of this theorem only requires $r > |\frac{1}{k} - \frac{1}{k+1}| \cdot \frac{2}{n+1}$. This yields another corollary, used later on.

Corollary 5.5. *The following estimate holds:*

$$\liminf_{n\to\infty} \inf\Big\{d_n\Big(\Big[r, \frac{1}{k}\Big]\Big) : k \in \{1, 2, \ldots, n\}, r \in \Big[\frac{1}{k+1}, \frac{1}{k}\Big),\\ \Big|\frac{1}{k} - r\Big| > \Big|\frac{1}{k} - \frac{1}{k+1}\Big| \cdot \frac{2}{n+1}\Big\} \geq \frac{1}{2}.$$

To expand this further, we will show that the lower limit of the densities of the intervals with a right endpoint equal to $\{\frac{1}{k}\}$, $k \in \mathbb{N}$ and containing interval $[\frac{1}{k+1}, \frac{1}{k}]$ is at least $\frac{1}{2}$.

Proposition 5.6. *The following holds:*

$$\liminf_{n\to\infty} \inf\Big\{d_n\Big(\Big[r, \frac{1}{k}\Big]\Big) : k \in \{1, 2, \ldots, n\}, r \in \Big(0, \frac{1}{k+1}\Big), \textit{centered at } J_n\Big\} \geq \frac{1}{2}.$$

Proof. Let $l \in \mathbb{N}$ be a number such that $\frac{1}{k+l+1} \leq r < \frac{1}{k+l}$.

Part (A). First, assume that $k + l + 1 \leq n + 1$. Then, by Lemma 2.17,

$$\begin{aligned} d_n\Big(\Big[r, \frac{1}{k}\Big]\Big) &= \frac{m_n([r, \frac{1}{k}])}{|r - \frac{1}{k}|^{h_n}} \geq \frac{m_n([\frac{1}{k+l}, \frac{1}{k}])}{|\frac{1}{k+l+1} - \frac{1}{k}|^{h_n}} = \frac{\sum_{j=k}^{k+l-1} |\frac{1}{j} - \frac{1}{j+1}|^{h_n}}{|\frac{1}{k+l+1} - \frac{1}{k}|^{h_n}} \\ &\geq \frac{\Big|\sum_{j=k}^{k+l-1} \frac{1}{j} - \frac{1}{j+1}\Big|^{h_n}}{|\frac{1}{k+l+1} - \frac{1}{k}|^{h_n}} = \frac{|\frac{1}{k+l} - \frac{1}{k}|^{h_n}}{|\frac{1}{k+l+1} - \frac{1}{k}|^{h_n}} \\ &= \Big[\frac{l}{k(k+l)} \cdot \frac{k(k+l+1)}{l+1}\Big]^{h_n} = \Big[\frac{l}{l+1} \cdot \frac{k+l+1}{k+l}\Big]^{h_n} \geq \Big(\frac{1}{2}\Big)^{h_n}. \end{aligned}$$

Note that in Part (A) of this theorem, we are not using the assumption that the interval $[r, \frac{1}{k}]$ is centered. This assumption is used later on, in Part (B).

Part (B). Now, assume that $r < \frac{1}{n+1}$. From the fact that the interval $[r, \frac{1}{k}]$ is centered at J_n, we know that the point c, the center of $[r, \frac{1}{k}]$, must be located to the right of the point $\frac{1}{n+1}$. Using this observation and Lemma 2.17, we get

$$d_n\left(\left[r, \frac{1}{k}\right]\right) = \frac{m_n\left(\left[r, \frac{1}{k}\right]\right)}{\left|r - \frac{1}{k}\right|^{h_n}} = \frac{m_n\left(\left[\frac{1}{n+1}, \frac{1}{k}\right]\right)}{\left|r - \frac{1}{k}\right|^{h_n}} = \frac{\sum_{j=k}^{n} \left|\frac{1}{j} - \frac{1}{j+1}\right|^{h_n}}{\left|r - \frac{1}{k}\right|^{h_n}}$$

$$\geq \frac{\left|\frac{1}{k} - \frac{1}{n+1}\right|^{h_n}}{\left|r - \frac{1}{k}\right|^{h_n}} \geq \frac{\left|\frac{1}{k} - c\right|^{h_n}}{\left|r - \frac{1}{k}\right|^{h_n}} = \left(\frac{1}{2}\right)^{h_n}.$$

Thus, we get

$$\liminf_{n\to\infty} \inf \left\{ d_n\left(\left[r, \frac{1}{k}\right]\right) : k \in \{1, 2, \ldots, n\}, r \in \left(0, \frac{1}{k+1}\right) \right\} \geq \frac{1}{2}. \qquad \blacksquare$$

5.3. Density on intervals with a left endpoint in the set $\{\frac{1}{k}\}, k = 0, 1, \ldots$

Now we move to the case where we analyze the intervals with a left endpoint in $\{\frac{1}{k+1}\}$, for some $k \leq n$, and contained in $[\frac{1}{k+1}, \frac{1}{k}]$.

Proposition 5.7. *The following estimate holds:*

$$\liminf_{n\to\infty} \inf \left\{ d_n\left(\left[\frac{1}{k+1}, r\right]\right) : k \in \{1, 2, \ldots, n\}, \right.$$
$$\left. r \in \left(\frac{1}{k+1}, \frac{1}{k}\right], \textit{interval centered at } J_n \right\} \geq \frac{1}{2}$$

Proof. Notice that the interval $[\frac{1}{k+1}, r]$ is contained in $[\frac{1}{k+1}, \frac{1}{k}]$. Thus,

$$d_n\left(\left[\frac{1}{k+1}, r\right]\right) = d_n\left(f_k\left(\left[\frac{1}{k+1}, r\right]\right)\right) = d_n([\hat{r}, 1])$$

for $\hat{r} = f_k(r)$. If $|\hat{r} - 1| > \frac{1}{2}$, then invoking Proposition 5.6 with $k = 1$, we get our thesis. If $|\hat{r} - 1| \leq \frac{1}{2}$, then we can apply the map f_1 to the interval $[\hat{r}, 1]$. Further, notice that because interval $[\hat{r}, 1]$ was centered at J_n, then so must be the interval $f_1([\hat{r}, 1]) = [0, \hat{\hat{r}}]$. Moreover, the intervals $[\hat{r}, 1]$ and $[0, \hat{\hat{r}}]$ have the same density based on Lemma 2.11 and so we can apply Proposition 5.1, which ends the proof. $\blacksquare$

Notice that similarly to the proof of Proposition 5.1, the only assumption needed in the proof of Proposition 5.7 is that $|[\frac{1}{k+1}, r]| > |\frac{1}{k} - \frac{1}{k+1}| \cdot \frac{1}{2} \cdot \frac{2}{n+1}$. This is due to the fact that in the first part of the proof, where we assume that $|\hat{r} - 1| > \frac{1}{2}$, we can use Part (A) of the Proposition 5.6, which does not assume being centered at J_n. As for the other case, when $|\hat{r} - 1| < \frac{1}{2}$, we can invoke Corollary 5.2 to end the proof. This observation gives another corollary.

Corollary 5.8. *The following estimate holds:*

$$\liminf_{n\to\infty} \inf \Big\{ d_n\Big(\Big[\frac{1}{k+1}, r\Big]\Big) : k \in \{1,2,\dots,n\}, r \in \Big(\frac{1}{k+1}, \frac{1}{k}\Big], \Big|\Big[\frac{1}{k+1}, r\Big]\Big| > \Big|\frac{1}{k} - \frac{1}{k+1}\Big| \cdot \frac{1}{2} \cdot \frac{2}{n+1} \Big\} \geq \frac{1}{2}.$$

Proposition 5.9. *The following estimate holds:*

$$\liminf_{n\to\infty} \inf \Big\{ d_n\Big(\Big[\frac{1}{k+1}, r\Big]\Big) : k,l \in \mathbb{N}, k+l < n+1, l > 0, r \in \Big(\frac{1}{k+l}, 1\Big] \text{ and } \frac{1}{k+l} < r \leq \frac{1}{k} \Big\} \geq \frac{1}{2}.$$

Proof. As a side note, one can notice that this case is symmetric to the one in Proposition 5.6, although the method used in the proof is different. First, let us assume that $|\frac{1}{k+1} - r| \geq |\frac{1}{k+1} - \frac{1}{k}| \cdot \frac{1}{2} \cdot \frac{2}{n+1}$. Additionally, we can split the interval $[\frac{1}{k+l+1}, r]$ into two: $[\frac{1}{k+l+1}, \frac{1}{k+1}]$ and $[\frac{1}{k+1}, r]$. Using Lemma 2.16, we get

$$\begin{aligned} d_n\Big(\Big[\frac{1}{k+l+1}, r\Big]\Big) &\geq \min\Big\{ d_n\Big(\Big[\frac{1}{k+l+1}, \frac{1}{k+1}\Big]\Big), d_n\Big(\Big[\frac{1}{k+1}, r\Big]\Big)\Big\} \\ &\geq \min\Big\{1, d_n\Big(\Big[\frac{1}{k+1}, r\Big]\Big)\Big\}, \end{aligned}$$

and using Proposition 5.3 and Corollary 5.8 gives us the result. Now if

$$\Big|\frac{1}{k+1} - r\Big| \leq \Big|\frac{1}{k} - \frac{1}{k+1}\Big| \cdot \frac{1}{2} \cdot \frac{2}{n+1},$$

then by Lemma 2.17,

$$\begin{aligned} d_n\Big(\Big[\frac{1}{k+l+1}, r\Big]\Big) &\geq \frac{\sum_{j=k+1}^{k+l} \left|\frac{1}{j} - \frac{1}{j+1}\right|^{h_n}}{\left|\frac{1}{k+l+1} - \frac{1}{k+1} + \left|\frac{1}{k} - \frac{1}{k+1}\right| \cdot \frac{1}{2} \cdot \frac{2}{n+1}\right|^{h_n}} \\ &\geq \frac{\left|\sum_{j=k+1}^{k+l} \frac{1}{j} - \frac{1}{j+1}\right|^{h_n}}{\left|\frac{1}{k+l+1} - \frac{1}{k+1} + \left|\frac{1}{k} - \frac{1}{k+1}\right| \cdot \frac{1}{2} \cdot \frac{2}{n+1}\right|^{h_n}} \\ &= \frac{\left|\frac{1}{k+l+1} - \frac{1}{k+1}\right|^{h_n}}{\left|\frac{1}{k+l+1} - \frac{1}{k+1} + \left|\frac{1}{k+1} - \frac{1}{k}\right| \cdot \frac{1}{2} \cdot \frac{2}{n+1}\right|^{h_n}} \\ &= \frac{1}{\left|1 + \frac{1}{n+1} \frac{\left|\frac{1}{k+1} - \frac{1}{k}\right|}{\left|\frac{1}{k+l+1} - \frac{1}{k+1}\right|}\right|^{h_n}} \geq \Big(\frac{1}{2}\Big)^{h_n} \end{aligned}$$

for sufficiently large n and all $k, l \in \mathbb{N}$ such that $k + l < n + 1$. ∎

5.4. Density on intervals intersecting the set $\{\frac{1}{k}\}$, $k = 0, 1, \ldots$

Now we move to the final step of the proof. This step is split into three parts. The next two propositions require two auxiliary lemmas, which we will formulate here and prove later on in Section 5.6.

Lemma 5.17. *The following estimate holds:*

$$\liminf_{n\to\infty}\left(\inf_{d>\frac{1}{n}}\left\{\frac{m_n\left(\left[\frac{1}{n+1}, d\right]\right)}{\left|d-\frac{1}{n+1}\right|^{h_n}}\right\}\right) \geq 1.$$

It is worth noting this lemma cannot be replaced by Proposition 5.9. This is due to the fact that in the Proposition 5.9, we have the estimate of the lower limit of the densities by $\frac{1}{2}$, whereas here we need a stronger estimate.

Lemma 5.18. *The following estimate holds:*

$$\liminf_{n\to\infty}\left(\inf_{d<\frac{1}{2}}\left\{\frac{m_n\left([d, 1]\right)}{|1-d|^{h_n}}\right\}\right) \geq 1.$$

It is worth noting this lemma cannot be replaced by Proposition 5.6. In Proposition 5.6 we have the estimate of the lower limit of the densities by $\frac{1}{2}$ and the intervals are required to be centered, whereas here we need a stronger estimate on all intervals.

Lemma 5.17 is used in the next proposition, however Lemma 5.18 is only utilized in Proposition 5.11, but because those lemmas are similar, we formulate them here. Now, let us start with the first proposition.

Proposition 5.10. *The following estimate holds:*

$$\liminf_{n\to\infty}\inf\Big\{d_n([a,b]) : 0 < a < \frac{1}{l+1} < \frac{1}{l} < b \leq 1, \\ l \in \{2, 3, \ldots, n\},\ [a,b] \textit{ centered at } J_n\Big\} \geq \frac{1}{2}.$$

Proof. Let k and l be unique integer such that $[a,b] = [a, \frac{1}{k}] \cup [\frac{1}{k}, \frac{1}{l}] \cup [\frac{1}{l}, b]$ for some integer $l \leq k$ and $\frac{1}{k+1} \leq a < \frac{1}{k} \leq \frac{1}{l} \leq b < \frac{1}{l-1}$. First, assume that $k \geq n+1$. Note that this implies that $a \notin J_n$. Let $c = \frac{a+b}{2}$ be the center of the interval $[a,b]$. Because the interval $[a,b]$ is centered at J_n, $c \geq \frac{1}{n+1}$, and thus

$$\frac{|\frac{1}{n+1} - a|}{|b-a|} < \frac{1}{2},$$

which directly implies that

$$\frac{|b - \frac{1}{n+1}|}{|b-a|} \geq \frac{1}{2}.$$

Now, fix $\varepsilon > 0$. Using Lemma 5.17 with $d = b$, fix n large enough such that

$$\frac{m_n\left(\left[\frac{1}{n+1}, b\right]\right)}{\left|b - \frac{1}{n+1}\right|^{h_n}} \geq \left(1 - \frac{\varepsilon}{2}\right)$$

for all $b \geq \frac{1}{n}$. Hence,

$$\begin{aligned} d_n([a,b]) &= \frac{m_n([a,b])}{|b-a|^{h_n}} = \frac{m_n\left(\left[\frac{1}{n+1}, b\right]\right)}{|b-a|^{h_n}} = \frac{m_n\left(\left[\frac{1}{n+1}, b\right]\right)}{\left|b - \frac{1}{n+1}\right|^{h_n}} \cdot \frac{\left|b - \frac{1}{n+1}\right|^{h_n}}{|b-a|^{h_n}} \\ &\geq \left(1 - \frac{\varepsilon}{2}\right) \cdot \left(\frac{1}{2}\right)^{h_n} \geq \frac{1}{2} - \varepsilon, \end{aligned}$$

where last inequality comes from the fact that $h_n \to 1$ when $n \to \infty$ for all n large enough.

From now on, we will assume that $k \leq n$. Let $J_1 = [a, \frac{1}{k}]$, $J_2 = [\frac{1}{k}, \frac{1}{l}]$ and $J_3 = [\frac{1}{l}, b]$.

We say that J_1 is long when

$$|J_1| = \left|a - \frac{1}{k}\right| \geq \left|\frac{1}{k} - \frac{1}{k+1}\right| \cdot \frac{2}{n+1}; \tag{5.1}$$

otherwise, we say that J_1 is short. We say that J_3 is long when

$$|J_3| = \left|\frac{1}{l} - b\right| \geq \left|\frac{1}{l-1} - \frac{1}{l}\right| \cdot \frac{1}{2} \cdot \frac{2}{n+1}; \tag{5.2}$$

otherwise, we say it is short. We will split this proof into four parts, based on the length of the intervals J_1 and J_3. The cases are as follows:

(1) Both intervals J_1 and J_3 are long.

(2) J_1 is long and J_3 is short.

(3) J_1 is short and J_3 is long.

(4) Both J_1 and J_3 are short.

First, assume both J_1 and J_3 are long. Then, by using Lemma 2.16, we notice that

$$d_n([a,b]) = d_n(J_1 \cup J_2 \cup J_3) \geq \min\left\{d_n(J_1), d_n(J_2), d_n(J_3)\right\}.$$

And thus, using Proposition 5.3, Corollary 5.5, and 5.8, we get our thesis. Now, moving to the second case, we assume that J_1 is long and J_3 is short. Now, we split our interval into two J_1 and $J_2 \cup J_3$, out of which the first one is in the form of the intervals from Corollary 5.5 and the other one is in form of the ones from Proposition 5.9. Utilizing both of those theorems and Lemma 2.16, we get

$$d_n([a,b]) \geq \min\left\{d_n(J_1), d_n(J_2 \cup J_3)\right\} \geq \frac{1}{2}.$$

In the third case, when J_1 is short and J_3 is long, we can split our interval into two: $J_1 \cup J_2$ and J_3. Then, the interval $J_1 \cup J_2$ fulfills the assumptions of the Proposition 5.6 Part (A), whereas J_3 fulfills the assumptions of Corollary 5.8. This together with Lemma 2.16 gives us

$$d_n([a,b]) \geq \min\left\{d_n(J_1 \cup J_2), d_n(J_3)\right\} \geq \frac{1}{2}.$$

In Case 4, assume that both of the intervals J_1 and J_3 are short, that is, they do not satisfy (5.1) and (5.2). By Lemma 2.17,

$$\begin{aligned}
d_n([a,b]) &\geq \frac{\sum\limits_{j=l}^{k-1} \left|\frac{1}{j} - \frac{1}{j+1}\right|^{h_n}}{\left|\left|\frac{1}{k} - \frac{1}{k+1}\right| \cdot \frac{2}{n+1} + \sum\limits_{j=l}^{k-1} \left|\frac{1}{j} - \frac{1}{j+1}\right| + \left|\frac{1}{l-1} - \frac{1}{l}\right| \cdot \frac{1}{2} \cdot \frac{2}{n+1}\right|^{h_n}} \\
&\geq \frac{\left|\sum\limits_{j=l}^{k-1} \frac{1}{j} - \frac{1}{j+1}\right|^{h_n}}{\left|\left|\frac{1}{k} - \frac{1}{k+1}\right| \cdot \frac{2}{n+1} + \sum\limits_{j=l}^{k-1} \left|\frac{1}{j} - \frac{1}{j+1}\right| + \left|\frac{1}{l-1} - \frac{1}{l}\right| \cdot \frac{1}{2} \cdot \frac{2}{n+1}\right|^{h_n}} \\
&\geq \frac{\left|\frac{1}{l} - \frac{1}{k}\right|^{h_n}}{\left|\frac{1}{l} - \frac{1}{k} + \frac{2}{n+1} \cdot \left(\frac{1}{k(k+1)} + \frac{1}{2} \cdot \frac{1}{l(l-1)}\right)\right|^{h_n}} \\
&\geq \frac{1}{\left|1 + \frac{2}{n+1} \cdot \frac{\left(\frac{1}{k(k+1)} + \frac{1}{2} \cdot \frac{1}{l(l-1)}\right)}{\left|\frac{1}{l} - \frac{1}{k}\right|}\right|^{h_n}} \geq \frac{1}{2},
\end{aligned}$$

for sufficiently large n and all $l \in \{2, 3, \ldots, n\}$ and $k \in \{l+1, l+2, \ldots, n+1\}$. This ends the proof of the Proposition 5.10. ∎

Now, we focus on the case where there is no whole interval of the first generation in the interval $[a,b]$.

Proposition 5.11. *The following estimate holds:*

$$\liminf_{n\to\infty} \inf \left\{d_n([a,b]) : \frac{1}{k+1} < a < \frac{1}{k} < b < \frac{1}{k-1} \leq 1,\right.$$
$$\left. k \in \{2, 3, \ldots, n+1\},\ [a,b] \text{ centered at } J_n\right\} \geq \frac{1}{2}.$$

Proof. Assume first that $k \leq n$. The case when $k = n+1$ will be analyzed at the end of the proof. Let $J_1 = [a, \frac{1}{k}]$ and $J_2 = [\frac{1}{k}, b]$. As in the proof of the Proposition 5.10, we say that J_1 is long when

$$|J_1| = \left|a - \frac{1}{k}\right| \geq \left|\frac{1}{k} - \frac{1}{k+1}\right| \cdot \frac{2}{n+1}, \tag{5.3}$$

and we call it short otherwise. We say that J_2 is long when

$$|J_2| = \left|\frac{1}{k} - b\right| \geq \left|\frac{1}{k-1} - \frac{1}{k}\right| \cdot \frac{1}{2} \cdot \frac{2}{n+1}, \tag{5.4}$$

It is worth noting that

$$\frac{1}{k} - \left|\frac{1}{k} - \frac{1}{k+1}\right| \cdot \frac{1}{n+1} = g_k\Big(\frac{1}{n+1}\Big)$$

and

$$\frac{1}{k} + \left|\frac{1}{k-1} - \frac{1}{k}\right| \cdot \frac{1}{2} \cdot \frac{1}{n+1} = g_{k-1} \circ g_1\Big(\frac{1}{n+1}\Big).$$

The immediate consequence of this fact is the following observation:

Observation 5.12. *Let c be the center of the interval $[a,b]$. Assume that $[a,b]$ is centered at J_n. Moreover, assume that J_1 is short. Then, c must be to the right of the set $\{\frac{1}{k}\}$.*

Proof. Note that by c_1 we denote the center of the interval J_1. Because

$$\frac{1}{k} - \left|\frac{1}{k} - \frac{1}{k+1}\right| \cdot \frac{1}{n+1} = g_k\Big(\frac{1}{n+1}\Big)$$

(see Figure 1) and J_1 is short (i.e., does not fulfill (5.3)), then the center of the interval J_1 must be to the right of $g_k(\frac{1}{n+1})$. This means that $g_k(\frac{1}{n+1}) < c_1$. But the center of the interval $[a,b]$ lies to the right of c_1, that is, $c_1 < c$, thus $g_k(\frac{1}{n+1}) < c$. Now, notice that $J_n \cap [g_k(\frac{1}{n+1}), \frac{1}{k}] = \emptyset$. This, together with $g_k(\frac{1}{n+1}) < c$, implies that $\frac{1}{k} < c$. ■

Now, we will formulate a symmetric observation regarding J_2.

Observation 5.13. *Let c be the center of the interval $[a,b]$. Assume that $[a,b]$ is centered at J_n. Moreover, assume that J_2 is short. Then, c must be to the left of $\{\frac{1}{k}\}$.*

Proof. Note by c_2 the center of the interval J_2. Because $\frac{1}{k} + |\frac{1}{k-1} - \frac{1}{k}| \cdot \frac{1}{2} \cdot \frac{1}{n+1} = g_{k-1} \circ g_1(\frac{1}{n+1})$ (see Figure 1) and J_2 is short (i.e. does not fulfill (5.4)), then the center of the interval J_2 must be to the left of $g_{k-1} \circ g_1(\frac{1}{n+1})$. This means that $c_2 < g_{k-1} \circ g_1(\frac{1}{n+1})$. But the center of the interval $[a,b]$ lies to the left of c_2 i.e. $c < c_2$, thus $c < g_{k-1} \circ g_1(\frac{1}{n+1})$. Now, notice that $J_n \cap [\frac{1}{k}, g_{k-1} \circ g_1(\frac{1}{n+1})] = \emptyset$. This, together with $c < g_{k-1} \circ g_1(\frac{1}{n+1})$ implies that $c < \frac{1}{k}$. ■

Now, we will split the proof of the Proposition 5.11 into four cases based on the length of intervals J_1 and J_2.

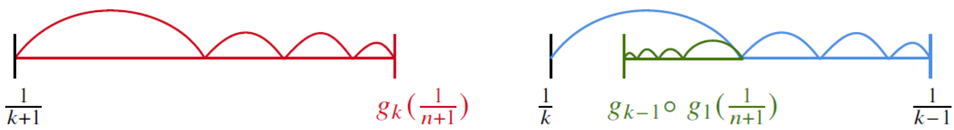

Figure 1. Visualization of J_n.

Case 1. Assume that J_1 and J_2 are long. Then, utilizing Lemma 2.16 yields

$$d_n([a,b]) \geq \min\left\{d_n(J_1), d_n(J_2)\right\} \geq \frac{1}{2},$$

based on Proposition 5.4 and Corollary 5.8.

Case 2. Now, assume that J_1 is long and J_2 is short (i.e., J_1 fulfills (5.3) and J_2 does not fulfill (5.4)). Fix $\varepsilon > 0$. Let $c = \text{center}([a,b]) = (a+b)/2$ be the center of the interval $[a,b]$. Recall that, according to the assumption, the interval $[a,b]$ is centered at J_n. By Observation 5.13, we have $c < \frac{1}{k}$. Moreover, c cannot be in the interval $[g_k(\frac{1}{n+1}), \frac{1}{k}]$, because the interval $[g_k(\frac{1}{n+1}), \frac{1}{k}]$ does not intersect J_n. This implies that $|[a,c]| \leq |[a, g_k(\frac{1}{n+1})]|$. Let $I_1 = [a,c]$ and $I_2 = [c,b]$. Because c is the middle point of $[a,b]$, $|I_1| = |I_2|$. From Lemma 5.17, we can find n to be large enough for the following to hold:

$$\frac{m_n\left(\left[\frac{1}{n+1}, d\right]\right)}{\left|d - \frac{1}{n+1}\right|^{h_n}} \geq \left(1 - \frac{\varepsilon}{2}\right)$$

for all $d \geq \frac{1}{n}$. Because J_1 is long, $g_k^{-1}(a) > \frac{1}{n}$. Now, applying Lemma 2.11 to the interval $[a, g_k(\frac{1}{n+1})]$, we get

$$\frac{m_n\left(\left[a, g_k\left(\frac{1}{n+1}\right)\right]\right)}{\left|g_k(\frac{1}{n+1}) - a\right|^{h_n}} \geq \left(1 - \frac{\varepsilon}{2}\right). \tag{5.5}$$

Using this fact and $|[a,c]| \leq |[a, g_k(\frac{1}{n+1})]|$, we get

$$\begin{aligned} d_n([a,b]) &\geq \frac{m_n\left(\left[a, \frac{1}{k}\right]\right)}{|b-a|^{h_n}} \geq \frac{m_n\left(\left[a, g_k\left(\frac{1}{n+1}\right)\right]\right)}{|b-a|^{h_n}} \\ &= \frac{\frac{m_n([a, g_k(\frac{1}{n+1})])}{|g_k(\frac{1}{n+1})-a|^{h_n}} \cdot \left|g_k(\frac{1}{n+1}) - a\right|^{h_n}}{|b-a|^{h_n}} \geq \frac{|g_k(\frac{1}{n+1}) - a|^{h_n} \cdot (1 - \frac{\varepsilon}{2})}{|b-a|^{h_n}} \\ &\geq \frac{|c-a|^{h_n} \cdot (1-\frac{\varepsilon}{2})}{|b-a|^{h_n}} = \frac{|I_1|^{h_n} \cdot (1-\frac{\varepsilon}{2})}{|I_1 + I_2|^{h_n}} = \left(\frac{1}{2}\right)^{h_n} \cdot \left(1 - \frac{\varepsilon}{2}\right) \geq \frac{1}{2} - \varepsilon \end{aligned} \tag{5.6}$$

for sufficiently large n.

Case 3. For the second-to-last case, assume that J_1 is short (i.e., does not fulfill (5.3)) and J_2 is long (i.e., fulfills (5.4)). Fix $\varepsilon > 0$. Let c be the center of the interval $[a,b]$. From Observation 5.12, we know that $\frac{1}{k} < c$. Moreover, the interval $[\frac{1}{k}, g_{k-1} \circ g_1(\frac{1}{n+1})]$ does not intersect J_n. Additionally, $[a,b]$ is centered at J_n, and thus, $g_{k-1} \circ g_1(\frac{1}{n+1}) \leq c$. Let $I_1 = [a,c]$ and $I_2 = [c,b]$. Because c is the middle point of $[a,b]$, $|I_1| = |I_2|$. If $|\frac{1}{k} - b| \geq |\frac{1}{k-1} - \frac{1}{k}| \cdot \frac{1}{2}$, then using Lemma 5.18, we can find n large enough such that

$$\frac{m_n([d,1])}{|1-d|^{h_n}} \geq 1 - \frac{\varepsilon}{2}$$

for all $d \leq \frac{1}{2}$. Since $|\frac{1}{k} - b| \geq |\frac{1}{k-1} - \frac{1}{k}| \cdot \frac{1}{2}$, we can apply Lemma 2.11 and obtain

$$\frac{m_n\big(\big[g_{k-1} \circ g_1\big(\frac{1}{n+1}\big), b\big]\big)}{\big|b - g_{k-1} \circ g_1\big(\frac{1}{n+1}\big)\big|^{h_n}} \geq 1 - \frac{\varepsilon}{2}.$$

Applying this fact together with observation that $|[c,b]| \leq |[g_{k-1} \circ g_1\big(\frac{1}{n+1}\big), b]|$, we get

$$\begin{aligned}
d_n([a,b]) &\geq \frac{m_n\big(\big[\frac{1}{k}, b\big]\big)}{|a-b|^{h_n}} = \frac{m_n\big(\big[g_{k-1} \circ g_1\big(\frac{1}{n+1}\big), b\big]\big)}{|b-a|^{h_n}} \\
&= \frac{m_n\big(\big[g_{k-1} \circ g_1\big(\frac{1}{n+1}\big), b\big]\big)}{\big|b - g_{k-1} \circ g_1\big(\frac{1}{n+1}\big)\big|^{h_n}} \cdot \frac{\big|b - g_{k-1} \circ g_1\big(\frac{1}{n+1}\big)\big|^{h_n}}{|b-a|^{h_n}} \\
&\geq \frac{\big(1 - \frac{\varepsilon}{2}\big) \cdot |b-c|^{h_n}}{|b-a|^{h_n}} \geq \frac{\big|I_2\big|^{h_n} \cdot \big(1 - \frac{\varepsilon}{2}\big)}{|I_1 + I_2|^{h_n}} \\
&= \left(\frac{1}{2}\right)^{h_n} \cdot \left(1 - \frac{\varepsilon}{2}\right) \geq \left(\frac{1}{2} - \varepsilon\right)
\end{aligned} \tag{5.7}$$

for sufficiently large n.

Now if $|\frac{1}{k} - b| < |\frac{1}{k-1} - \frac{1}{k}| \cdot \frac{1}{2}$, then from Lemma 5.17, we find n large enough to get

$$\frac{m_n\big(\big[\frac{1}{n+1}, d\big]\big)}{\big|d - \frac{1}{n+1}\big|^{h_n}} \geq 1 - \frac{\varepsilon}{2}$$

for all $d \geq \frac{1}{n}$. Now apply Lemma 2.11 twice, first to the interval $[g_{k-1} \circ g_1(\frac{1}{n+1}), b]$, resulting in $d_n([g_{k-1} \circ g_1(\frac{1}{n+1}), b]) = d_n([g_{k-1}^{-1}(b), g_1(\frac{1}{n+1})]) = d_n([w, 1])$ with $w = g_{k-1}^{-1}(b)$. Then, from the fact that $|\frac{1}{k} - b| < |\frac{1}{k-1} - \frac{1}{k}| \cdot \frac{1}{2}$, we know that $|1 - w| < \frac{1}{2}$, and thus we can apply Lemma 2.11 a second time, which results in $d_n([w, g_1(\frac{1}{n+1})]) = d_n([\frac{1}{n+1}, d])$ with $d = g_1^{-1}(w)$. This together with the fact that J_2 is long results in $d \geq \frac{2}{n+1} > \frac{1}{n}$. This results in the following:

$$\frac{m_n\left(\left[g_{k-1} \circ g_1\left(\frac{1}{n+1}\right), b\right]\right)}{\left|b - g_{k-1} \circ g_1\left(\frac{1}{n+1}\right)\right|^{h_n}} \geq 1 - \frac{\varepsilon}{2}.$$

Using this result yields

$$\begin{aligned} d_n([a,b]) &\geq \frac{m_n\left(\left[\frac{1}{k}, b\right]\right)}{|a-b|^{h_n}} = \frac{m_n\left(\left[g_{k-1} \circ g_1\left(\frac{1}{n+1}\right), b\right]\right)}{|b-a|^{h_n}} \\ &\geq \frac{m_n\left(\left[g_{k-1} \circ g_1\left(\frac{1}{n+1}\right), b\right]\right)}{\left|b - g_{k-1} \circ g_1\left(\frac{1}{n+1}\right)\right|^{h_n}} \cdot \frac{\left|b - g_{k-1} \circ g_1\left(\frac{1}{n+1}\right)\right|^{h_n}}{|b-a|^{h_n}} \\ &\geq \frac{\left(1 - \frac{\varepsilon}{2}\right) \cdot |b-c|^{h_n}}{|b-a|^{h_n}} \geq \frac{|I_2|^{h_n} \cdot \left(1 - \frac{\varepsilon}{2}\right)}{|I_1 + I_2|^{h_n}} = \left(\frac{1}{2}\right)^{h_n} \cdot \left(1 - \frac{\varepsilon}{2}\right) \geq \left(\frac{1}{2} - \varepsilon\right) \end{aligned}$$

for sufficiently large n, which ends the proof for this case.

Case 4. For the last case, assume that both J_1 and J_2 are short. We assumed that the interval $[a,b]$ must be centered at J_n. As before, denote by c the center of the interval $[a,b]$. Because J_1 is short, we can invoke Observation 5.12 and deduce that $c > \frac{1}{k}$. The interval J_2 is short as well, and invoking Observation 5.13 yields $c < \frac{1}{k}$. This implies that $[a,b]$ cannot be centered at J_n.

The only thing left is to consider case when $k = n+1$. This case is even simpler, because the only possible case is Case 3. When $k = n+1$, then $\frac{1}{k} = \frac{1}{n+1}$ and thus, $[a, \frac{1}{k}] \cap J_n = \emptyset$. Since the interval $[a,b]$ is centered at J_n, this implies that the center of the interval $[a,b]$ is in the interval $[\frac{1}{n+1}, b]$. The proof for this situation is exactly the proof for the Case 3 of this proposition.

This ends the proof of Proposition 5.11 ∎

5.5. Finalizing the proof

As the last part of our proof, we put previously obtained results to get the following:

Theorem 5.14. *The following estimate holds:*

$$\liminf_{n\to\infty} \inf\left\{d_n([a,b]) : 0 < a < b \leq 1,\ [a,b] \text{ centered at } J_n\right\} \geq \frac{1}{2}.$$

Proof. If there does not exist $k \in \{1, 2, \ldots, n+1\}$ such that $\frac{1}{k} \in (a,b)$, then we can apply Lemma 2.12 to the interval $[a,b]$. This yields the interval $[\widehat{a}, \widehat{b}]$, with the same density, such that at least one $\frac{1}{k}$ belongs to $[\widehat{a}, \widehat{b}]$, $k \in \{1, 2, \ldots, n+1\}$. Then, by the results of Propositions 5.1, 5.3, 5.4, 5.6, 5.7, 5.9, 5.10, and 5.11 we get the thesis. ∎

This directly implies, using the definition of the packing measure from 2.5 and Corollary 3.4, that the upper limit of the packing measure is at most one, meaning

Theorem 5.15. *Let S_n be IFS defined in 2.1. Then,*

$$\limsup_{n\to\infty} \mathcal{P}_{h_n}(J_n) \leq 2,$$

where J_n is the limit set of the IFS S_n and P_h denotes packing measure in the packing dimension h.

This result along with Theorem 4.1 proves our main result.

Theorem 5.16. *Let S_n be IFS defined in 2.1. Then,*

$$\lim_{n\to\infty} \mathcal{P}_{h_n}(J_n) = 2,$$

where J_n is the limit set of the IFS S_n and P_h denotes the packing measure in the packing dimension h.

5.6. Proofs of the auxiliary lemmas

Now, we will provide the proofs for auxiliary Lemmas 5.17 and 5.18.

Lemma 5.17. *The following estimate holds:*

$$\liminf_{n\to\infty}\left(\inf_{d>\frac{1}{n}}\left\{\frac{m_n\left(\left[\frac{1}{n+1},d\right]\right)}{\left|d-\frac{1}{n+1}\right|^{h_n}}\right\}\right) \geq 1.$$

Proof. Fix some $n > 0$. Take arbitrary $d > \frac{1}{n}$. We claim that in order to estimate the ratio

$$\frac{m_n\left(\left[\frac{1}{n+1},d\right]\right)}{\left|d-\frac{1}{n+1}\right|^{h_n}} \tag{5.8}$$

from below, one can assume that $d \in J_n$, $d > \frac{1}{n}$. Indeed, if $d \notin J_n$ (i.e., d is in some "gap" of the Cantor set J_n), then we can replace d by $\widehat{d} := \inf\{x \in J_n : x > d\}$ (the right endpoint of the "gap"). Then, $m_n([\frac{1}{n+1}, d]) = m_n([\frac{1}{n+1}, \widehat{d}])$ and $|d - \frac{1}{n+1}| \leq |\widehat{d} - \frac{1}{n+1}|$ and, clearly, $\widehat{d} \geq d > \frac{1}{n}$. So

$$\frac{m_n\left(\left[\frac{1}{n+1},d\right]\right)}{\left|d-\frac{1}{n+1}\right|^{h_n}} \geq \frac{m_n\left(\left[\frac{1}{n+1},\widehat{d}\right]\right)}{\left|\widehat{d}-\frac{1}{n+1}\right|^{h_n}}.$$

So, from now on we assume that $d \in J_n$, $d > \frac{1}{n}$.

Since $d \in J_n$, there exists a unique sequence of integers $(q_j)_{j=1}^{\infty}$, $q_j \leq n$ such that

$$\{d\} = \bigcap_{k=1}^{\infty} g_{q_1} \circ \cdots \circ g_{q_k}([0,1]).$$

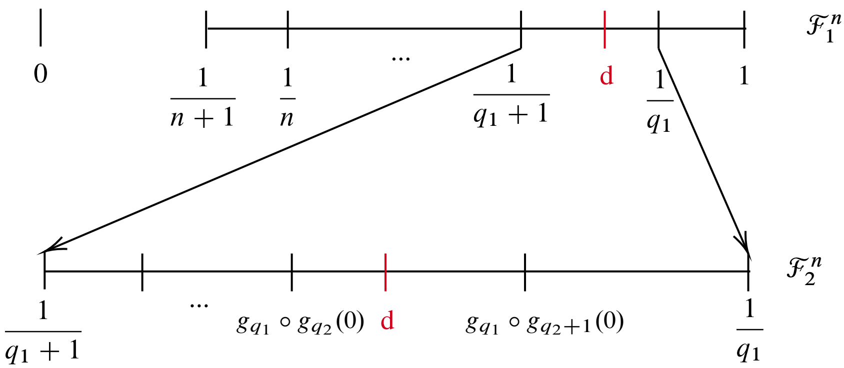

Figure 2. Length and measure of the intervals to the left of d.

The measure $m_n([\frac{1}{n+1}, d])$ can be expressed as the sum of the measures of the cylinder sets (i.e., intervals from the collection $\mathcal{F}_l^n$ located to the left of d). Summing up first the measures of the cylinder set of the first generation (see Figure 2), we obtain

$$\sum_{k=q_1+1}^{n} \Big(\frac{1}{k} - \frac{1}{k+1}\Big)^{h_n}.$$

Now, looking at the cylinder set containing d (i.e., the interval $[\frac{1}{q_1+1}, \frac{1}{q_1}]$), we see that the cylinder sets of the second generations (i.e., the elements of the collection $\mathcal{F}_2^n$) contained in $[\frac{1}{q_1+1}, \frac{1}{q_1}]$ and located to the left of d have length

$$\Big(\frac{1}{q_1} - \frac{1}{q_1+1}\Big) \cdot \Big(\frac{1}{k} - \frac{1}{k+1}\Big),$$

$k = 1, \dots, q_2 - 1$, and measure

$$\Big(\frac{1}{q_1} - \frac{1}{q_1+1}\Big)^{h_n} \cdot \Big(\frac{1}{k} - \frac{1}{k+1}\Big)^{h_n}.$$

Proceeding by induction, we easily conclude that the measure $m_n([\frac{1}{n+1}, d])$ can be expressed in the following form:

$$m_n\Big(\Big[\frac{1}{n+1}, d\Big]\Big) = \sum_{k=q_1+1}^{n} \Big(\frac{1}{k} - \frac{1}{k+1}\Big)^{h_n} + \Big(\frac{1}{q_1} - \frac{1}{q_1+1}\Big)^{h_n} \cdot \sum_{k=1}^{q_2-1} \Big(\frac{1}{k} - \frac{1}{k+1}\Big)^{h_n}$$

$$+\prod_{i=1}^{2}\Big(\frac{1}{q_i}-\frac{1}{q_i+1}\Big)^{h_n}\cdot\sum_{k=q_3+1}^{n}\Big(\frac{1}{k}\frac{1}{k+1}\Big)^{h_n}$$
$$+\prod_{i=1}^{3}\Big(\frac{1}{q_i}-\frac{1}{q_i+1}\Big)^{h_n}\cdot\sum_{k=1}^{q_4-1}\Big(\frac{1}{k}-\frac{1}{k+1}\Big)^{h_n}+\dots \quad (5.9)$$

Analogously, the value $|d-\frac{1}{n+1}|^{h_n}$ can be expressed as follows:

$$\Big|d-\frac{1}{n+1}\Big|^{h_n}=\Big[\sum_{k=q_1+1}^{n}\Big(\frac{1}{k}-\frac{1}{k+1}\Big)+\Big(\frac{1}{q_1}-\frac{1}{q_1+1}\Big)\cdot\sum_{k=1}^{q_2-1}\Big(\frac{1}{k}-\frac{1}{k+1}\Big)$$
$$+\prod_{i=1}^{2}\Big(\frac{1}{q_i}-\frac{1}{q_i+1}\Big)\cdot\sum_{k=q_3+1}^{\infty}\Big(\frac{1}{k}-\frac{1}{k+1}\Big)$$
$$+\prod_{i=1}^{3}\Big(\frac{1}{q_i}-\frac{1}{q_i+1}\Big)\cdot\sum_{k=1}^{q_4-1}\Big(\frac{1}{k}-\frac{1}{k+1}\Big)+\dots\Big]^{h_n}.$$

Now, clearly each summand

$$\prod_{i=1}^{2k}\Big(\frac{1}{q_i}-\frac{1}{q_i+1}\Big)\cdot\sum_{k=q_{2k+1}+1}^{\infty}\Big(\frac{1}{k}-\frac{1}{k+1}\Big)$$

can be divided into two sums:

$$\prod_{i=1}^{2k}\Big(\frac{1}{q_i}-\frac{1}{q_i+1}\Big)\cdot\sum_{k=q_{2k+1}+1}^{n}\Big(\frac{1}{k}-\frac{1}{k+1}\Big)$$
$$+\prod_{i=1}^{2k}\Big(\frac{1}{q_i}-\frac{1}{q_i+1}\Big)\cdot\sum_{k=n+1}^{\infty}\Big(\frac{1}{k}-\frac{1}{k+1}\Big).$$

Note that the first sum corresponds to the sum appearing in the expression for $m_n([\frac{1}{n+1},d])$. Grouping expressions that occur also in the formula for $m_n([\frac{1}{n+1},d])$ yield the following expression for $|d-\frac{1}{n+1}|^{h_n}$

$$\Big|d-\frac{1}{n+1}\Big|^{h_n}=\Big[\sum_{k=q_1+1}^{n}\Big(\frac{1}{k}-\frac{1}{k+1}\Big)+\Big(\frac{1}{q_1}-\frac{1}{q_1+1}\Big)\cdot\sum_{k=1}^{q_2-1}\Big(\frac{1}{k}-\frac{1}{k+1}\Big)$$
$$+\prod_{i=1}^{2}\Big(\frac{1}{q_i}-\frac{1}{q_i+1}\Big)\cdot\sum_{k=q_3+1}^{n}\Big(\frac{1}{k}-\frac{1}{k+1}\Big)$$
$$+\prod_{i=1}^{3}\Big(\frac{1}{q_i}-\frac{1}{q_i+1}\Big)\cdot\sum_{k=1}^{q_4-1}\Big(\frac{1}{k}-\frac{1}{k+1}\Big)+\dots$$

$$+\sum_{i=1}^{\infty}\prod_{j=1}^{2i}\Big(\frac{1}{q_j}-\frac{1}{q_j+1}\Big)\cdot\sum_{k=n+1}^{\infty}(\frac{1}{k}-\frac{1}{k+1})\Big]^{h_n}. \tag{5.10}$$

Now, using Lemma 2.17 with $m_n([\frac{1}{n+1},d])$ and taking the quotient, we get

$$\begin{aligned}
&\frac{\left|d-\frac{1}{n+1}\right|^{h_n}}{m_n\left(\left[\frac{1}{n+1},d\right]\right)}\\
&\le\left[1+\frac{\sum_{i=1}^{\infty}\prod_{j=1}^{2i}\left(\frac{1}{q_j}-\frac{1}{q_j+1}\right)\cdot\sum_{k=n+1}^{\infty}(\frac{1}{k}-\frac{1}{k+1})}{\sum_{k=q_1+1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)+\sum_{i=1}^{\infty}\prod_{j=1}^{2i}\left(\frac{1}{q_j}-\frac{1}{q_j+1}\right)\cdot\left(\frac{1}{q_{2i+1}}-\frac{1}{n+1}\right)+\sum_{i=1}^{\infty}\prod_{j=1}^{2i+1}\left(\frac{1}{q_j}-\frac{1}{q_j+1}\right)\cdot\left(1-\frac{1}{q_{2i+2}}\right)}\right]^{h_n}\\
&\le\left[1+\frac{\sum_{i=1}^{\infty}\prod_{j=1}^{2i}\left(\frac{1}{q_j}-\frac{1}{q_j+1}\right)\cdot\left(\frac{1}{n+1}\right)}{\sum_{k=q_1+1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)+\sum_{i=1}^{\infty}\prod_{j=1}^{2i}\left(\frac{1}{q_j}-\frac{1}{q_j+1}\right)\cdot\left(\frac{1}{q_{2i+1}}-\frac{1}{n+1}\right)+\sum_{i=1}^{\infty}\prod_{j=1}^{2i+1}\left(\frac{1}{q_j}-\frac{1}{q_j+1}\right)\cdot\left(1-\frac{1}{q_{2i+2}}\right)}\right]^{h_n}\\
&\le\left[1+\frac{\left(\frac{1}{n+1}\right)\left(\frac{1}{q_1}-\frac{1}{q_1+1}\right)\left(\frac{1}{q_2}-\frac{1}{q_2+1}\right)\cdot\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i+1}}{\sum_{k=q_1+1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)+\sum_{i=1}^{\infty}\prod_{j=1}^{2i}\left(\frac{1}{q_j}-\frac{1}{q_j+1}\right)\cdot\left(\frac{1}{q_{2i+1}}-\frac{1}{n+1}\right)+\sum_{i=1}^{\infty}\prod_{j=1}^{2i+1}\left(\frac{1}{q_j}-\frac{1}{q_j+1}\right)\cdot\left(1-\frac{1}{q_{2i+2}}\right)}\right]^{h_n}\\
&\le\left[1+\frac{2}{n}\right]^{h_n}
\end{aligned} \tag{5.11}$$

where inequality (5.11) comes from the fact that $\frac{1}{q_i}-\frac{1}{q_{i+1}}\le\frac{1}{2}$ for every $i=1,2,\ldots$. This implies that the numerator is limited from above by the following expression $\frac{1}{n+1}\cdot(\frac{1}{q_1}-\frac{1}{q_1+1})\cdot(\frac{1}{q_2}-\frac{1}{q_2+1})\cdot\sum_{k=1}^{\infty}(\frac{1}{2})^k$. The last inequality follows from the fact that $d>\frac{1}{q_1+1}$, and thus, the denominator of the expression is limited from below by the first summand $\frac{1}{q_1+1}-\frac{1}{n+1}$. From this, we get

$$\frac{m_n\left(\left[\frac{1}{n+1},d\right]\right)}{\left|d-\frac{1}{n+1}\right|^{h_n}}\ge\frac{1}{\left[1+\frac{2}{n}\right]^{h_n}}\ge\frac{1}{1+\frac{2}{n}}=\frac{n}{n+2}=1-\frac{2}{n+2},$$

which concludes the proof. ∎

Very similar reasoning can be used to show the following lemma:

Lemma 5.18. *The following estimate holds:*

$$\liminf_{n\to\infty}\left(\inf_{d<\frac{1}{2}}\left\{\frac{m_n([d,1])}{|1-d|^{h_n}}\right\}\right)\geq 1$$

Proof. Proof of this lemma uses the same techniques as one of the Lemma 5.17, with the only difference being that now we look at the intervals to the right of d instead of left. ∎

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