

Discontinuous solutions for the Navier–Stokes equations with density-dependent viscosity

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Abstract. We prove the existence of a unique global-in-time weak solution of the Navier–Stokes equations that govern the motion of a compressible viscous fluid with density-dependent viscosity in two-dimensional space. The initial velocity belongs to the Sobolev space $H^1(\mathbb{R}^2)$, and the initial fluid density is α -Hölder continuous on both sides of a $C^{1+\alpha}$ -regular interface with some geometrical assumption. We prove that this configuration persists over time: the initial interface is transported by the flow to an interface that maintains the same regularity as the initial one. Our result generalizes a previous known result of Hoff [Comm. Pure Appl. Math. 55 (2002), 1365–140] and Hoff and Santos [Arch. Ration. Mech. Anal. 188 (2008), 509–543] concerning the propagation of regularity for discontinuity surfaces by allowing a more general non-linear pressure law and density-dependent viscosity. Moreover, it supplements the work by Danchin, Fanelli, and Paicu [Anal. PDE 13 (2020), 275–316] with global-in-time well-posedness, even for density-dependent viscosity, and we achieve uniqueness in a large space.

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1. Introduction

1.1. Presentation of the model

In this paper we study the problem of existence and uniqueness of global-in-time weak solutions with intermediate regularity for the Navier–Stokes equations describing the motion of compressible fluid with density-dependent viscosity in \mathbb{R}^2 . Our main interest is to generalize the result by Hoff [21] and Hoff and Santos [22] concerning the propagation

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of discontinuous surfaces by allowing non-linear pressure law and density-dependent viscosity. We aim to supplement the work by Danchin, Fanelli, Paicu [6] with global-in-time well-posedness, even for density-dependent viscosity. Indeed, we consider the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \operatorname{div}(2\mu(\rho)\mathbb{D}u) + \nabla(\lambda(\rho) \operatorname{div} u), \end{cases} \quad (1.1)$$

describing the motion of a compressible viscous fluid at constant temperature. Above, $\rho = \rho(t, x) \geq 0$, and $u = u(t, x) \in \mathbb{R}^2$ are respectively the density, and the velocity of the fluid and they are the unknowns of the problem. Meanwhile $P = P(\rho)$, $\mu = \mu(\rho)$, $\lambda = \lambda(\rho)$ are respectively the pressure, dynamic, and kinematic viscosity law of the fluid and they are given \mathcal{C}^2 -regular functions of the density. The equations (1.1) are supplemented with initial data

$$\rho|_{t=0} = \rho_0 \in L^\infty(\mathbb{R}^2) \quad \text{and} \quad u|_{t=0} = u_0 \in H^1(\mathbb{R}^2). \quad (1.2)$$

We assume there exists $\tilde{\rho} > 0$ such that

$$\rho_0 - \tilde{\rho} \in L^2(\mathbb{R}^2), \quad \text{and we define } \tilde{P} = P(\tilde{\rho}), \quad \tilde{\mu} = \mu(\tilde{\rho}). \quad (1.3)$$

Additionally, we suppose that the initial density ρ_0 is upper bounded, bounded away from zero:

$$0 < \rho_{*,0} := \inf_{x \in \mathbb{R}^2} \rho_0(x) \leq \sup_{x \in \mathbb{R}^2} \rho_0(x) =: \rho_0^* < \infty, \quad \mu_{0,*} := \inf_{x \in \mathbb{R}^2} \mu(\rho_0(x)) > 0, \quad (1.4)$$

and α -Hölder continuous on both sides of a $\mathcal{C}^{1+\alpha}$ -regular non-self-intersecting curve $\mathcal{C}(0)$, which is the boundary of an open, bounded, and simply connected domain $D(0) \subset \mathbb{R}^2$. The latter regularity is defined as follows (based on [36]).

Definition 1.1. We introduce the following definitions and notation:

- (1) We say that an interface \mathcal{C} is $\mathcal{C}^{1+\alpha}$ -regular and non-self-intersecting if we have the following conditions:

- **$\mathcal{C}^{1+\alpha}$ -regularity:** There exist intervals $V_j \subset \mathbb{R}$, $j \in \llbracket 1, J \rrbracket$ and maps

$$\gamma_j: V_j \mapsto \mathbb{R}^2 \in \mathcal{C}^{1+\alpha} \quad \text{such that } \mathcal{C} \subset \bigcup_{j=1}^J \gamma_j(V_j),$$

with well-defined normal vector fields.

- **Non-self-intersection condition:** There exists $c_\gamma > 0$ such that

$$\forall j \in \llbracket 1, J \rrbracket, (s, s') \in V_j \times V_j, \quad \text{we have } |\gamma_j(s) - \gamma_j(s')| \geq c_\gamma^{-1} |s - s'|. \quad (1.5)$$

- (2) Consider an open, bounded, and simply connected domain D in \mathbb{R}^2 . Assume that D is $\mathcal{C}^{1+\alpha}$ -regular. There exists a function $\varphi: \mathbb{R}^2 \mapsto \mathbb{R} \in \mathcal{C}^{1+\alpha}$ such that

$$D = \{x \in \mathbb{R}^2: \varphi(x) > 0\} \quad \text{and} \quad |\nabla \varphi|_{\inf} := \inf_{x \in \partial D} |\nabla \varphi(x)| > 0.$$

We refer to [23, Section 3.1] for the construction of such a level-set function. We then define

$$\ell_\varphi = \min\left\{1, \left(\frac{|\nabla\varphi|_{\inf}}{\|\nabla\varphi\|_{\dot{C}^\alpha}}\right)^{1/\alpha}\right\}. \quad (1.6)$$

(3) Given that

$$\mathbb{R}^2 = D \cup \mathcal{C} \cup (\mathbb{R}^2 \setminus \bar{D}),$$

we define the space of piecewise α -Hölder continuous functions with respect to \mathcal{C} as follows:

$$\dot{\mathcal{C}}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2) := \dot{C}^\alpha(\bar{D}) \cap \dot{C}^\alpha(\mathbb{R}^2 \setminus D), \quad \|g\|_{\dot{\mathcal{C}}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2)} = \|g\|_{\dot{C}^\alpha(\bar{D})} + \|g\|_{\dot{C}^\alpha(\mathbb{R}^2 \setminus D)},$$

and the non-homogeneous space

$$\mathcal{C}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2) := L^\infty(\mathbb{R}^2) \cap \dot{\mathcal{C}}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2), \quad \|g\|_{\mathcal{C}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2)} = \|g\|_{L^\infty(\mathbb{R}^2)} + \|g\|_{\dot{\mathcal{C}}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2)}.$$

These spaces strictly contain the Hölder space $C^\alpha(\mathbb{R}^2)$.

(4) Given a function $g \in \mathcal{C}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2)$, the jump $\llbracket g \rrbracket$ and the average $\langle g \rangle$ of g are defined as follows: for all $\sigma \in \mathcal{C}$,

$$\begin{cases} \llbracket g \rrbracket(\sigma) = \lim_{r \rightarrow 0} [g(\sigma + rn_x(\sigma)) - g(\sigma - rn_x(\sigma))], \\ \langle g \rangle(\sigma) = \frac{1}{2} \lim_{r \rightarrow 0} [g(\sigma + rn_x(\sigma)) + g(\sigma - rn_x(\sigma))]. \end{cases} \quad (1.7)$$

Above, n_x denotes the normal vector of \mathcal{C} .

(5) Consider a time-dependent interface $\mathcal{C} = \mathcal{C}(t)$, with a local parameterization γ . We assume that $\gamma_j \in \mathcal{C}(I, C^{1+\alpha}(V_j))$ and for all $t \in I$, $\mathcal{C}(t)$ is a $C^{1+\alpha}$ -regular, non-self-intersecting hypersurface that forms the boundary of an open, bounded, and simply connected domain $D(t)$. We define the following space:

$$\begin{aligned} & L^p(I, \mathcal{C}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2)) \\ & := \left\{ g = g(t, x) : \begin{cases} \int_I \|g(t)\|_{\mathcal{C}_{\text{pw},\gamma(t)}^\alpha(\mathbb{R}^2)}^p dt < \infty & \text{if } 1 \leq p < \infty, \\ \sup_{t \in I} \text{ess} \|g(t)\|_{\mathcal{C}_{\text{pw},\gamma(t)}^\alpha(\mathbb{R}^2)} < \infty & \text{if } p = \infty. \end{cases} \right\} \end{aligned}$$

(6) Finally, we define the functional

$$\mathfrak{F}_{\gamma(t)} = (1 + |\mathcal{C}(t)|) \mathfrak{F}(\|\nabla\gamma(t)\|_{L^\infty} + c_{\gamma(t)} \|\nabla\gamma(t)\|_{\dot{C}^\alpha}), \quad (1.8)$$

where $c_{\gamma(t)}$ satisfies (1.5). Here, \mathfrak{F} is a polynomial that is larger than those provided by Proposition 2.5 below for specific second-order Riesz operators.

In addition to the assumption on the initial density ρ_0 in (1.3)–(1.4), we assume the existence of an open, bounded, and simply connected set $D(0) \subset \mathbb{R}^2$ such that $\mathcal{C}(0) = \partial D(0)$ (with parameterization γ_0) is a $C^{1+\alpha}$ -regular and non-self-intersecting curve and

$$\rho_0 \in \mathcal{C}_{\text{pw},\gamma_0}^\alpha(\mathbb{R}^2). \quad (1.9)$$

The purpose of this paper is to establish the existence of a unique global-in-time weak solution to the system (1.1) with initial data (1.2)–(1.3)–(1.4)–(1.9) in the spirit of the works by Hoff [21] and Hoff and Santos [22]. The regularity of the velocity helps to propagate the $\mathcal{C}^{1+\alpha}$ and the non-self-intersection (1.5) regularities of the initial curve all over time. We find that discontinuities in the initial density persist over time, with the jump decaying exponentially in time. The extension to density-dependent viscosity is not trivial and the analysis of the model is more subtle. We obtain uniqueness in a large space. On the other hand, this result supplements the work by Danchin, Fanelli, and Paicu [6] with global well-posedness even with density-dependent viscosity.

We will now proceed with the review of known results on the propagation of discontinuity surfaces in the mathematical analysis of the Navier–Stokes equations for compressible fluids.

1.2. Review of known results

Since its definitive formulation in the mid-19th century, the Navier–Stokes equations have consistently captivated the attention of numerous mathematicians. The inaugural achievement in this realm is attributed to Nash [31], who proved a local well-posedness of strong solution in the whole \mathbb{R}^3 . The density belongs to $\mathcal{C}^{1+\alpha}$ while the velocity belongs to $\mathcal{C}^{2+\alpha}$ for $\alpha \in (0, 1)$. We also refer to Solonnikov’s work [34], in which the system (1.1) is considered in a \mathcal{C}^2 -regular bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$. The initial density is bounded away from vacuum and belongs to $W^{1,p}(\Omega)$ for some $p > d$, whereas the initial velocity belongs to the Sobolev–Slobodetskii space $W_q^{2,1}(\Omega)$. Nash’s work considers heat-conducting fluids with viscosity laws that may depend on density or temperature. In contrast, Solonnikov did not account for temperature, and the viscosities are constant. The first global-in-time result is obtained by Matsumura and Nishida [29, 30] for small initial data. For constant viscosity, the initial data needs to be small in $H^3(\mathbb{R}^3)$, while for non-constant viscosity, smallness in $H^4(\mathbb{R}^3)$ is required. Later, the regularity requirements for the initial data were relaxed in [3, 4, 18], allowing for small initial data in the critical Besov space in the case of constant viscosity. However, the density remains a continuous function in space.

In the constant viscosity setting, the classical solutions constructed in the referenced papers, come with the following energy balance:

$$E(t) + \mu \int_0^t \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + (\mu + \lambda) \int_0^t \|\operatorname{div} u\|_{L^2(\mathbb{R}^d)}^2 = E(0) =: E_0. \quad (1.10)$$

Above, $E(t)$ is the energy functional defined by

$$E(t) = \int_{\mathbb{R}^d} \rho(t, x) \left[\frac{|u(t, x)|^2}{2} + \int_{\tilde{\rho}}^{\rho(t, x)} s^{-2} (P(s) - \tilde{P}) ds \right] dx.$$

In the particular case when the pressure law is of the form $P(\rho) = a\rho^\gamma$, global weak solutions were obtained for the first time by Lions [28], and Feireisl, Novotný, Petzeltová [13]

with some restriction on the adiabatic constant γ . The initial data is assumed to have finite initial energy, that is, $E_0 < \infty$, and the solutions verify (1.10) with inequality. Bresch and Desjardins [2] established the existence of a global weak solution for the Navier–Stokes equation with density-dependent viscosity. *However, their result requires certain Sobolev regularity assumptions on the initial density, which do not apply to our framework, as we assume the density is discontinuous across a hypersurface, with its weak gradient containing Dirac masses.*

In the last three decades, there has been significant interest in studying the propagation of discontinuity surfaces in models derived from fluid mechanics, such as the Euler or Navier–Stokes equations. These discontinuity surfaces are sets of singularity points of certain quantities, such as vorticity for the incompressible Euler equations or density for the Navier–Stokes equations. For instance, we refer to the so-called density-patch problem proposed by Lions [27] for the incompressible Navier–Stokes equations: *assuming $\rho_0 = \mathbb{1}_{D_0}$ for some domain $D_0 \subset \mathbb{R}^2$, the question is whether or not for any time $t > 0$, the density is $\rho(t) = \mathbb{1}_{D(t)}$, with $D(t)$ a domain with the same regularity as the initial one.* This problem has been addressed in [7–9, 12, 14, 15, 24–26, 32], where satisfactory solutions were provided for different regularities of D_0 , including cases with density-dependent viscosity. As far as we know, there are not enough results in the literature concerning the analogous density-patch problem for the Navier–Stokes equations for compressible fluids. On the one hand, classical solutions are too regular and do not account for discontinuous initial density. On the other hand, while the weak solutions constructed by Lions [28] or Feireisl, Novotný, Petzeltová [13] allow for discontinuous initial density, the associated velocity is too weak to track down density discontinuities. As explained, for instance in [15], an effective approach to tracking discontinuous surfaces is to construct weak solutions for the full model within a class that allows for the study of its dynamics.

The initial result in this area is credited to Hoff’s 2002 study [21], which is a logical follow-up to his previous results [19, 20]. Indeed, in his pioneer work [19], Hoff provided bounds for the following functionals (with $d \in \{2, 3\}$):

$$\begin{cases} \mathcal{A}_1^H(t) = \sup_{[0,t]} \sigma \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + \int_0^t \sigma \|\sqrt{\rho} \dot{u}\|_{L^2(\mathbb{R}^d)}^2, \\ \mathcal{A}_2^H(t) = \sup_{[0,t]} \sigma^d \|\sqrt{\rho} \dot{u}\|_{L^2(\mathbb{R}^d)}^2 + \int_0^t \sigma^d \|\nabla \dot{u}\|_{L^2(\mathbb{R}^d)}^2, \end{cases} \quad (1.11)$$

owing to some smallness condition on the initial data. Here, \dot{v} denotes the material derivative of v , while σ represents a time weight. They are defined as follows:

$$\dot{v} := \partial_t v + (u \cdot \nabla) v \quad \text{and} \quad \sigma(t) := \min\{1, t\}.$$

He observed that the so-called effective flux defined by

$$F^H := (2\mu + \lambda) \operatorname{div} u - P(\rho) + \tilde{P}$$

and the vorticity $\text{curl } u$ solve the following elliptic equations:

$$\Delta F^H = \text{div}(\rho \dot{u}) \quad \text{and} \quad \mu \Delta \text{curl } u = \text{curl}(\rho \dot{u}).$$

By using the regularity of \dot{u} provided by the functionals \mathcal{A}_1^H and \mathcal{A}_2^H in (1.11), he obtained the fact that the effective flux and vorticity belong, at least, to $L^{8/3}((1, \infty), L^\infty(\mathbb{R}^d))$. This finding enables the propagation of the $L^\infty(\mathbb{R}^d)$ -norm of the density. As a result, he proved the existence of a global weak solution for the Navier–Stokes equations with a linear pressure law in an initial paper [19], and later extended this to a non-linear pressure law (gamma law) in a subsequent paper [20]. These weak solutions have lower regularity compared to the unique global classical solution constructed by Matsumura and Nishida [30]; however, they exhibit higher regularity than the solutions with finite initial energy constructed by Lions [28] or by Feireisl, Novotný, and Petzeltová [13]. Specifically, discontinuous initial densities are allowed, and the regularity of the velocity at positive times aids in tracking down the discontinuities in the density. For instance in 2008, Hoff and Santos [22] explored the Lagrangian structure of these weak solutions. Basically, they write the velocity as the sum of two terms,

$$\begin{aligned} u &= -\left(\frac{1}{2\mu + \lambda}(-\Delta)^{-1}\nabla F^H + (-\Delta)^{-1}\nabla \cdot \text{curl } u\right) + \frac{1}{2\mu + \lambda}(-\Delta)^{-1}\nabla(P(\rho) - \tilde{P}) \\ &=: u_F + u_P. \end{aligned} \tag{1.12}$$

The first term is at least Lipschitz at positive times, while the second one is less regular in space than the first one. Specifically, its gradient belongs to $L^\infty((0, \infty), \text{BMO}(\mathbb{R}^d))$. To lower the initial time singularity of the first term, they require the initial velocity to be slightly more regular ($u_0 \in H^s(\mathbb{R}^d)$ for $s > 0$ in $d = 2$ and $s > 1/2$ in $d = 3$). As a result, the velocity gradient belongs to $L^1_{\text{loc}}([0, \infty), \text{BMO}(\mathbb{R}^d))$, which is sufficient to define a continuous flow map for the velocity field u . Consequently, continuous manifolds preserve their regularity over time. However, for initially \mathcal{C}^α -regular interfaces, one can only ensure $\mathcal{C}^{\alpha(t)}$ -regularity, with $\alpha(t)$ decaying exponentially to zero. It is worth noting that they constructed a solution to the heat equation with specific initial velocity $u_0 \in H^s(\mathbb{R}^3)$ for $s < 1/2$, which has infinitely many integral curves approaching $x = 0$ as t goes to 0. This exponential loss of interface regularity was also observed in [10]. In that paper, the authors constructed global-in-time solutions with large data under the nearly incompressible assumption: the velocity divergence is assumed to be small. The velocity is relatively weak ($\nabla u \in L^1_{\text{loc}}([0, \infty), \text{BMO}(\mathbb{R}^d))$), leading to the exponential-in-time loss of interface regularity. They proved uniqueness only for the linear pressure law case, even though the velocity field is not Lipschitz.

The decomposition of the velocity (1.12) was previously used by Hoff [21] to propagate the regularity of discontinuity surfaces in \mathbb{R}^2 . Specifically, for an initial velocity in $H^\beta(\mathbb{R}^2)$, both the effective flux and vorticity belong to $L^1_{\text{loc}}([0, \infty), \mathcal{C}^\alpha(\mathbb{R}^2))$, $0 < \alpha < \beta < 1$, and as a result, the gradient of the regular part of the velocity does as well. Assuming that initially the density is Hölder continuous on both sides of a $\mathcal{C}^{1+\alpha}$ -regular interface

(with geometrical assumption (1.5)) across which it is discontinuous, the author showed that the second part of the velocity is at least Lipschitz. In fact, its gradient is Hölder continuous along the tangential direction of the transported interface, guaranteeing that the latter retains the same regularity as the initial one.

It is worth noting that the approximate density sequence is constructed within a large space that precludes any non-linear pressure law. Recently, in [36], we established the existence of local-in-time weak solutions for the two-fluid model with density-dependent viscosity and discontinuous initial data. Notably, the regularity of these local solutions is sufficient to maintain the interface’s regularity. These solutions accommodate general non-linear pressure laws and will serve as a block for the construction of global-in-time solutions in this paper.

In their 2020 paper, Danchin, Fanelli, and Paicu [6] proved that, assuming the initial density has tangential regularity, the less regular part of the velocity is Lipschitz. They specifically showed that $W^{2,p}$ -regular hypersurfaces retain their regularity up to a finite time. However, they were unable to establish global-in-time propagation of the interface regularity, even for small initial data.

All of the above results pertain to the case of constant viscosity. When the dynamic viscosity is constant and $\lambda(\rho) = \rho^\beta$, with $\beta > 3$, the existence of a global-in-time weak solution, with no small assumption, was pioneered by Vařgant and Kazhikhov [35]. Their framework allows discontinuous density, and although not explicitly stated, the propagation of Hölder interface regularity with exponential-in-time loss also holds, with the analysis being very similar to that in [22]. However, when the viscosity μ depends on the density, there is no clear notion of effective flux, the analysis becomes more complicated, and it is not even clear how can one propagate the L^∞ -norm of the density. In what follows, we will present some observations showing that when the viscosity μ is discontinuous, the effective flux and the vorticity lack the smoothness observed in the constant viscosity case. Specifically, we will show that these quantities are continuous at the interface only where the viscosity μ is continuous.

We first apply the divergence and the rotational operators to the momentum equation to express

$$\begin{aligned} F &:= (2\mu(\rho) + \lambda(\rho)) \operatorname{div} u - P(\rho) + \tilde{P} = -(-\Delta)^{-1} \operatorname{div}(\rho \dot{u}) + [K, \mu(\rho) - \tilde{\mu}] \mathbb{D}u, \\ \mu(\rho) \operatorname{curl} u &= -(-\Delta)^{-1} \operatorname{curl}(\rho \dot{u}) + [K', \mu(\rho) - \tilde{\mu}] \mathbb{D}u. \end{aligned}$$

We refer to Section 3.1 for technical details leading to these expressions. Above, K and K' are second-order Riesz operators well known to linearly map $L^p(\mathbb{R}^d)$ into itself for $1 < p < \infty$. At the end point they map $L^\infty(\mathbb{R}^d)$ into the $\operatorname{BMO}(\mathbb{R}^d)$ space.

It is straightforward to derive from the mass equation (see the computations leading to (3.19) below)

$$\partial_t f(\rho) + u \cdot \nabla f(\rho) + P(\rho) - \tilde{P} = -F, \quad \text{where } f(\rho) = \int_{\tilde{\rho}}^{\rho} \frac{2\mu(s) + \lambda(s)}{s} ds,$$

with the pressure term on the left-hand side interpreted as a damping term. Hence, estimates for the lower and upper bounds of the density stem from L^∞ -norm boundedness of F . A priori energy estimates (see for example (1.11) above) provide regularity for \dot{u} , which translates to $L^\infty(\mathbb{R}^2)$ -norm boundedness for the first term of F . In contrast, the second term of F , which vanishes for constant viscosity, is actually of the same order as ∇u . Worse, given that K is not continuous over $L^\infty(\mathbb{R}^2)$, it is less clear whether the last term of the expression of F is bounded, with the only information that $\nabla u, \mu(\rho) \in L^\infty(\mathbb{R}^2)$. This issue precludes the $L^\infty(\mathbb{R}^2)$ -norm propagation of the density as has been done for the isotropic case by Hoff [19].

The second observation involves computing the jump of the effective flux, vorticity, and velocity at the interface. The discussion follows the same lines as in [33], where the case of constant viscosity is analyzed. First, we observe that there is a balance of forces applied to the interface, which suggests the continuity of the stress tensor in the normal direction, that is,

$$\llbracket \Pi^j \rrbracket \cdot n_x = 0, \quad \text{where } \Pi^{jk} = 2\mu(\rho)\mathbb{D}^{jk}u + (\lambda(\rho)\operatorname{div}u - P(\rho) + \tilde{P})\delta^{jk} \quad (1.13)$$

is the stress tensor, n_x is the outward normal vector field of the interface, and $\llbracket g \rrbracket$ denotes the jump of g at the interface (see (1.7) above). Since the velocity is continuous in the whole space, and its gradient is continuous on both sides of the interface, then the velocity gradient is also continuous in the tangential direction of the interface. Basically, discontinuities in ∇u can only occur in the normal direction of the interface. In other words, there exists a vector field $a = a(t, \sigma) \in \mathbb{R}^2$ such that

$$\llbracket \nabla u \rrbracket = a \cdot n_x^t. \quad (1.14)$$

From this, one easily deduces that such a vector field reads

$$a = (a \cdot \tau_x)\tau_x + (a \cdot n_x)n_x = \llbracket \operatorname{curl} u \rrbracket \tau_x + \llbracket \operatorname{div} u \rrbracket n_x, \quad (1.15)$$

where τ_x is the tangential vector field of the interface. Using (1.14), we rewrite (1.13) as follows:

$$\langle \mu(\rho) \rangle (a^j + a \cdot n_x n_x^j) + 2\llbracket \mu(\rho) \rrbracket \langle \mathbb{D}^{jk}u \rangle n_x^k + \llbracket \lambda(\rho)\operatorname{div}u - P(\rho) \rrbracket n_x^j = 0. \quad (1.16)$$

Next we multiply the above by n_x^j before summing over j to obtain

$$2\langle \mu(\rho) \rangle a \cdot n_x + 2\llbracket \mu(\rho) \rrbracket \langle \mathbb{D}^{jk}u \rangle n_x^j n_x^k + \llbracket \lambda(\rho)\operatorname{div}u - P(\rho) \rrbracket = 0,$$

and since $a \cdot n_x = \llbracket \operatorname{div} u \rrbracket$, the jump of the effective flux reads

$$\llbracket (2\mu(\rho) + \lambda(\rho))\operatorname{div}u - P(\rho) \rrbracket = 2\llbracket \mu(\rho) \rrbracket (\langle \operatorname{div}u \rangle - \langle \mathbb{D}^{jk}u \rangle n_x^j n_x^k). \quad (1.17)$$

As above, we take the scalar product of (1.16) with the tangential vector τ_x and use the fact that $\llbracket \operatorname{curl} u \rrbracket = a \cdot \tau_x$ to obtain

$$\llbracket \mu(\rho)\operatorname{curl}u \rrbracket = \llbracket \mu(\rho) \rrbracket (\langle \operatorname{curl}u \rangle - 2\langle \mathbb{D}^{jk}u \rangle n_x^k \tau_x^j). \quad (1.18)$$

It turns out that when the dynamic viscosity is continuous at the interface, for instance when it is constant, the effective flux and vorticity are also continuous at the interface. Another condition for these quantities to be continuous is that the terms in brackets vanish; this seems not to hold in general.

In view of all the above observations, it is less clear whether the effective flux and the vorticity are continuous at the interface. However, their jumps are “proportional” to the jump in viscosity $\mu(\rho)$. As we will see in Section 3.2 below, the viscous damping of the density will cause the jump in viscosity $\mu(\rho)$ to decrease exponentially in time. Consequently, the jumps in effective flux, vorticity, and velocity gradient will also decay exponentially in time, as observed in [21, 22, 33] in the case of constant viscosity.

We are now in position to state our main result.

1.3. Statement of the main result

We consider the classical Navier–Stokes equations (1.1) in two-dimensional space with initial data (1.2) that fulfills (1.3)–(1.4)–(1.9). The parameterization γ_0 of the interface $\mathcal{C}(0)$ fulfills the condition (1.5) with a constant c_{γ_0} . Next we introduce some energy functionals:

- Classical energy functional:

$$E(t) = \int_{\mathbb{R}^2} \left[\rho \frac{|u|^2}{2} + \rho \int_{\tilde{\rho}}^{\rho} s^{-2} (P(s) - \tilde{P}) ds \right] (t, x) dx.$$

- Higher-order energy functionals:

$$\begin{cases} \mathcal{A}_1(t) = \sup_{[0,t]} \|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \|\sqrt{\rho} \dot{u}\|_{L^2(\mathbb{R}^2)}^2, \\ \mathcal{A}_2(t) = \sup_{[0,t]} \sigma \|\sqrt{\rho} \dot{u}\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \sigma \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^2, \\ \mathcal{A}_3(t) = \sup_{[0,t]} \sigma^2 \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \sigma^2 \|\sqrt{\rho} \ddot{u}\|_{L^2(\mathbb{R}^2)}^2, \end{cases} \quad (1.19)$$

where $\dot{u} = \partial_t u + (u \cdot \nabla)u$ and $\ddot{u} = \partial_t \dot{u} + (u \cdot \nabla)\dot{u}$ and $\sigma(t) = \min\{1, t\}$.

- Piecewise Hölder regularity functional:

$$\begin{aligned} \theta(t) &= \sup_{[0,t]} \|f(\rho)\|_{C_{\text{pw},\gamma}^\alpha(\mathbb{R}^2)}^4 \\ &+ \int_0^t [\|f(\rho(\tau))\|_{C_{\text{pw},\gamma(\tau)}^\alpha(\mathbb{R}^2)}^4 + \sigma^{r_\alpha}(\tau) \|\nabla u(\tau)\|_{C_{\text{pw},\gamma(\tau)}^\alpha(\mathbb{R}^2)}^4] d\tau. \end{aligned} \quad (1.20)$$

Above, $\gamma(\cdot) = X(\cdot)\gamma_0$, where X is the flow of u ,

$$r_\alpha = 1 + 2\alpha, \quad \text{and} \quad f(\rho) = \int_{\tilde{\rho}}^{\rho} \frac{2\mu(s) + \lambda(s)}{s} ds. \quad (1.21)$$

Recall that P , μ , and λ are \mathcal{C}^2 -regular functions of the density. Additionally, we assume the existence of $a_* \in (0, \rho_{*,0}/4)$ and $a^* \in (4\rho_0^*, \infty)$ (see (1.4) for the definitions of $\rho_{0,*}$ and ρ_0^*) such that

$$P'(\rho) > 0, \quad \mu(\rho) > 0, \quad \text{and} \quad \lambda(\rho) \geq 0, \quad \text{for all } \rho \in [a_*, a^*]. \quad (1.22)$$

The smallness of the initial data will be measured in the following norms:

$$c_0 := \|u_0\|_{H^1(\mathbb{R}^2)}^2 + \|\rho_0 - \tilde{\rho}\|_{L^2(\mathbb{R}^2) \cap \mathcal{C}_{pw, \gamma_0}^\alpha(\mathbb{R}^2)}^2 + \|\llbracket \rho_0 \rrbracket\|_{L^4(\mathcal{E}(0)) \cap L^\infty(\mathcal{E}(0))}^2. \quad (1.23)$$

Our result reads as follows.

Theorem 1.2. *Let (ρ_0, u_0) be the initial data associated with the Navier–Stokes equations (1.1) and satisfying conditions (1.3), (1.4), and (1.9). Additionally, assume that condition (1.22) holds for the pressure and viscosity laws.*

There exist constants $c > 0$ and $[\mu]_0 > 0$ such that if

$$c_0 \leq c \quad \text{and} \quad \|\mu(\rho_0) - \tilde{\mu}\|_{\mathcal{C}_{pw, \gamma_0}^\alpha(\mathbb{R}^2)} \leq [\mu]_0,$$

then there exists a unique solution (ρ, u) for the Cauchy problem associated with (1.1) and initial data (ρ_0, u_0) , satisfying

$$E(t) + \mathcal{A}_1(t) + \mathcal{A}_2(t) + \mathcal{A}_3(t) + \sqrt{\theta(t)} \leq C c_0, \quad \text{for all } t \in (0, \infty).$$

Above, the constant C depends non-linearly on α , $\rho_{0,}$, ρ_0^* , $\mu_{0,*}$, c_{γ_0} , $\|\nabla \gamma_0\|_{\mathcal{C}^\alpha}$, $\|\nabla \varphi_0\|_{\mathcal{C}^\alpha}$, and $|\nabla \varphi_0|_{\inf}$.*

The proof of Theorem 1.2 is presented in Section 3.5 below.

Remark 1.3. A primary challenge in this paper is deriving a Lipschitz bound for the velocity (see the functional θ in (1.20) above). To tackle this, we express the velocity gradient as the sum of four terms (we refer to the computations leading to (2.20) below):

$$\begin{aligned} \tilde{\mu} \nabla u &= -(-\Delta)^{-1} \nabla(\rho \dot{u}) + \nabla(-\Delta)^{-1} \nabla \left(\frac{\tilde{\mu} + \lambda(\rho)}{2\mu(\rho) + \lambda(\rho)} F \right) \\ &\quad - \nabla(-\Delta)^{-1} \nabla \left(\frac{2\mu(\rho) - \tilde{\mu}}{2\mu(\rho) + \lambda(\rho)} (P(\rho) - \tilde{P}) \right) + \nabla(-\Delta)^{-1} \operatorname{div}((2\mu(\rho) - \tilde{\mu}) \mathbb{D}u) \\ &= \nabla u_* + \nabla u_F + \nabla u_P + \nabla u_\delta. \end{aligned} \quad (1.24)$$

- (1) The energy functionals \mathcal{A}_1 and \mathcal{A}_2 , as defined in (1.19) above, provide sufficient regularity for \dot{u} , ensuring that the first term of the expression above, namely ∇u_* , belongs to $L_{\text{loc}}^1([0, \infty), \mathcal{C}^\alpha(\mathbb{R}^2))$. In fact, we are not able to obtain a uniform-in-time estimate for the $L^1((1, t), L^\infty(\mathbb{R}^2))$ -norm of ∇u_* .
- (2) The other terms are less regular, and Proposition 2.5 below is crucial for obtaining their piecewise Hölder regularity. With the help of Proposition 2.5, we obtain that the last term, ∇u_δ , is small compared to the left-hand side as long as the viscosity $\mu(\rho)$ is a small perturbation of $\tilde{\mu}$.

- (3) The piecewise Hölder norm of F can be derived similarly to the two previous steps as F reads (see (2.11) below)

$$F = -(-\Delta)^{-1} \operatorname{div}(\rho \dot{u}) + [K, \mu(\rho) - \tilde{\mu}] \mathbb{D}u.$$

Then we make use of Proposition 2.5 to convert this bound for F into a piecewise Hölder norm for ∇u_F .

- (4) Given that

$$\partial_t f(\rho) + u \cdot \nabla f(\rho) + P(\rho) - \tilde{P} = -F,$$

we convert the previously obtained bound on F into a piecewise Hölder norm for the pressure, and subsequently for the remaining terms of (1.24).

- (5) Finally, we use a bootstrap argument to close all of these estimates, and uniform-in-time bounds are required. As mentioned in the first step, ∇u_* lacks adequate time decay, and it is unclear whether an $L^2((1, t), \mathcal{C}_{\text{pw}, \gamma}^\alpha(\mathbb{R}^2))$ -norm estimate, uniform with respect to $t > 1$, can be established for ∇u_P . However, we succeed in obtaining a uniform-in-time estimate for ∇u_P in $L^4((0, t), \mathcal{C}_{\text{pw}, \gamma}^\alpha(\mathbb{R}^2))$. Hence, we achieve higher time-integrability for ∇u_* by providing a bound for the functional \mathcal{A}_3 , since this ensures $\nabla \dot{u} \in L^\infty((1, \infty), L^2(\mathbb{R}^2))$. A similar functional was derived in [11] in the context of the incompressible Navier–Stokes model.

Remark 1.4. The key findings of our work are as follows:

- (1) Theorem 1.2 generalizes the works by Hoff [21] and Hoff and Santos [22] by allowing a non-linear pressure law and density-dependent viscosity. We also extend the work of Danchin, Fanelli, and Paicu [6] by achieving global-in-time propagation of interface regularity.
- (2) Our result accounts for viscosity of the form $1 + \mu(\rho)$, which falls outside the Bresch–Desjardins framework and is relevant for suspension models; see for example [16].

Remark 1.5. Since $r_\alpha < 3$, the velocity gradient belongs to $L_{\text{loc}}^1([0, \infty), \mathcal{C}_{\text{pw}, \gamma}^\alpha(\mathbb{R}^2))$, which is sufficient to propagate the regularity of the initial curve. Indeed, the characteristics of the interface $\gamma(t)$ exhibit exponential growth over time,

$$|\nabla \varphi(t)|_{\text{inf}}^{-1} + c_{\gamma(t)} + \|\nabla \gamma(t), \nabla \varphi(t)\|_{\mathcal{C}^\alpha} \leq C e^{Ct^{3/4}},$$

although this growth is slower than that obtained in [21].

Remark 1.6 (Exponential-in-time decay of jumps). In Section 3.2 below, we derive that $f(\rho)$, as defined in (1.21) above, verifies

$$\begin{aligned} \llbracket f(\rho(t, \gamma(t, s))) \rrbracket &= \llbracket f(\rho_0, \gamma_0(s)) \rrbracket \\ &\times \exp \left[\int_0^t \left[-g(\tau, s) - 2h(\tau, s) \langle (\operatorname{div} u(\tau, \gamma(\tau, s))) \right. \right. \\ &\quad \left. \left. - \langle \mathbb{D}^{jk} u(\tau, \gamma(\tau, s)) \rangle (n_x^j n_x^k)(\tau, s) \right] d\tau \right], \end{aligned}$$

where g and h are given by

$$g(t, s) = \frac{\llbracket P(\rho(t, \gamma(t, s))) \rrbracket}{\llbracket f(\rho(t, \gamma(t, s))) \rrbracket} \quad \text{and} \quad h(t, s) = \frac{\llbracket \mu(\rho(t, \gamma(t, s))) \rrbracket}{\llbracket f(\rho(t, \gamma(t, s))) \rrbracket}.$$

- (1) For constant viscosity, f is the logarithm function, $h = 0$, and the exponential decay rate is immediate when the pressure is an increasing function of the density. Indeed, the increasing assumption on the pressure law ensures that g is lower bounded away from zero. This observation was made by Hoff [21] and Hoff and Santos [22].
- (2) In our context, although h is no longer zero, it remains upper bounded. Therefore, by applying Young's inequality and using the $L^4((1, \infty), L^\infty(\mathbb{R}^2))$ -norm estimate for the velocity gradient, we obtain the exponential-in-time decay for $\llbracket f(\rho) \rrbracket$. This results in the exponential decay over time of the pressure and viscosity jump, given that g and h are upper bounded. This leads to the exponential decay of the vorticity and effective flux jumps over time; see (1.17)–(1.18). As a result, the vector field a , defined in (1.15), decays exponentially over time, and therefore so does the jump of the velocity gradient, as can be deduced from the expressions (1.14) and (1.15).
- (3) Notably, if the density is initially continuous at a point $\gamma_0(s)$ of the interface, then the density, effective flux, vorticity, and velocity gradient at time t are continuous at $\gamma(t, s)$ for $t > 0$.

Outline of the paper. The rest of this paper is structured as follows. In the next section, Section 2, we derive a priori estimates for local-in-time solutions. In Section 3 we provide the proofs of the lemmas presented in the aforementioned section. The proof of the main theorem, which is a consequence of Section 2, is the focus of Section 3.5.

2. Sketch of the proof of the main result

In this section, we derive a priori estimates for weak solutions for the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \operatorname{div}(2\mu(\rho)\mathbb{D}u) + \nabla(\lambda(\rho) \operatorname{div} u). \end{cases} \quad (2.1)$$

The existence and uniqueness of such a local solution is the purpose of our recent contribution [36], which is summarized as follows.

Theorem 2.1. *Let (ρ_0, u_0) be an initial data associated with the system (2.1) that satisfies the conditions (1.3)–(1.4)–(1.9). Assume that the viscosity and pressure laws satisfy (1.22), and additionally, assume the compatibility condition*

$$(\rho \dot{u})|_{t=0} = \operatorname{div}(\Pi)|_{t=0} \in L^2(\mathbb{R}^2). \quad (2.2)$$

There exists a positive constant $[\mu] > 0$ depending only on α and $\tilde{\mu}$ such that if¹

$$\begin{aligned} & [1 + \|\lambda(\rho_0)\|_{\dot{\mathcal{C}}_{\text{pw},\gamma_0}^\alpha(\mathbb{R}^2)} + (\mathfrak{B}_{\gamma_0} + \ell_{\varphi_0}^{-\alpha}) \|\llbracket \lambda(\rho_0) \rrbracket\|_{L^\infty(\mathcal{C}(0))}] \|\mu(\rho_0) - \tilde{\mu}\|_{\mathcal{C}_{\text{pw},\gamma_0}^\alpha(\mathbb{R}^2)} \\ & + (\mathfrak{B}_{\gamma_0} + \ell_{\varphi_0}^{-\alpha}) \left[\|\llbracket \mu(\rho_0) \rrbracket\|_{L^\infty(\mathcal{C}(0))} \right. \\ & \quad \left. + \|\llbracket \mu(\rho_0) \rrbracket, \llbracket \lambda(\rho_0) \rrbracket\|_{L^\infty(\mathcal{C}(0))}] \left\| 1 - \frac{\tilde{\mu}}{\langle \mu(\rho_0) \rangle} \right\|_{L^\infty(\mathcal{C}(0))} \right] \\ & \leq \frac{[\mu]}{4}, \end{aligned}$$

then there exist a time $T > 0$ and a unique solution (ρ, u) of the system (2.1) with initial data (ρ_0, u_0) , which satisfy the following:

- (1) $P(\rho) - \tilde{P} \in \mathcal{C}([0, T], L^2(\mathbb{R}^2) \cap \mathcal{C}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2))$, where $\gamma = \gamma(t)$ is a parameterization of a $\mathcal{C}^{1+\alpha}$ -regular and non-self-intersecting interface $\mathcal{C}(t)$;
- (2) $u \in \mathcal{C}([0, T], H^1(\mathbb{R}^2)) \cap L^\infty((0, T), \dot{W}^{1,6}(\mathbb{R}^2)) \cap L^{16}((0, T), \dot{W}^{1,8}(\mathbb{R}^2))$ and $\sigma^{\bar{r}/4} \nabla u \in L^4((0, T), \mathcal{C}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2))$ for

$$\bar{r} = \max\left\{\frac{1}{3}, 2\alpha\right\};$$

- (3) $\dot{u} \in \mathcal{C}([0, T], L^2(\mathbb{R}^2)) \cap L^2((0, T), \dot{H}^1(\mathbb{R}^2))$, $\sqrt{\sigma} \nabla \dot{u} \in L^\infty((0, T), L^2(\mathbb{R}^2))$, $\sqrt{\sigma} \nabla \dot{u} \in L^4((0, T) \times \mathbb{R}^2)$;
- (4) $\sqrt{\sigma} \ddot{u} \in L^2((0, T) \times \mathbb{R}^2)$, $\sigma \ddot{u} \in L^\infty((0, T), L^2(\mathbb{R}^2)) \cap L^2((0, T), \dot{H}^1(\mathbb{R}^2))$.

Remark 2.2. The velocity exhibits additional regularity. Indeed, the proof of Theorem 2.1 (specifically [36, Remark 1.4, items 3 and 5]) shows that not only is u continuous throughout the entire space, but its material derivative is as well. Furthermore, \ddot{u} is at least continuous across the interface γ . Additionally, both ∇u and $\nabla \dot{u}$ are Hölder continuous on both sides of the interface γ .

Theorem 2.1 comes with the following blow-up criterion.

Corollary 2.3 (Blow-up criterion). *Let (ρ, u) be the solution constructed in Theorem 2.1 defined up to a maximal time T^* . Assume the following:*

- (1) *Continuation. There exists a time-dependent function $C = C(t)$ such that for all $t \in [0, T^*)$,*

$$\begin{aligned} & c_{\gamma(t)} + \|\nabla \gamma(t)\|_{\mathcal{C}^\alpha} + \left\| \frac{1}{\rho(t)}, \frac{1}{\mu(\rho(t), c(t))} \right\|_{L^\infty(\mathbb{R}^2)} \\ & + \|u(t)\|_{H^1(\mathbb{R}^2)} + \|(\rho \dot{u})(t)\|_{L^2(\mathbb{R}^2)} + \|P(\rho(t)) - \tilde{P}\|_{\mathcal{C}_{\text{pw},\gamma(t)}^\alpha(\mathbb{R}^2)} \\ & \leq C(t). \end{aligned} \tag{2.3}$$

¹We refer to (1.6)–(1.8) for the definition of ℓ_{φ_0} and \mathfrak{B}_{γ_0} .

(2) *Smallness of the viscosity fluctuation. For all $t \in [0, T^*)$,*

$$\begin{aligned}
& (1 + \|\lambda(\rho(t))\|_{\dot{C}_{pw,\gamma(t)}^\alpha(\mathbb{R}^2)}) \|\mu(\rho(t)) - \tilde{\mu}\|_{C_{pw,\gamma(t)}^\alpha(\mathbb{R}^2)} \\
& + (\mathfrak{B}_{\gamma(t)} + \ell_{\varphi(t)}^{-\alpha}) \|\llbracket \lambda(\rho(t)) \rrbracket\|_{L^\infty(\mathcal{E}(t))} \|\mu(\rho(t)) - \tilde{\mu}\|_{C_{pw,\gamma(t)}^\alpha(\mathbb{R}^2)} \\
& + (\mathfrak{B}_{\gamma(t)} + \ell_{\varphi(t)}^{-\alpha}) \left[\|\llbracket \mu(\rho(t)) \rrbracket\|_{L^\infty(\mathcal{E}(t))} \right. \\
& \quad \left. + \|\llbracket \mu(\rho(t)) \rrbracket, \llbracket \lambda(\rho(t)) \rrbracket\|_{L^\infty(\mathcal{E}(t))} \left\| 1 - \frac{\tilde{\mu}}{\langle \mu(\rho(t)) \rangle} \right\|_{L^\infty(\mathcal{E}(t))} \right] \\
& < [\mu].
\end{aligned} \tag{2.4}$$

Then $T^* = \infty$.

Theorem 2.1 and Corollary 2.3 lay the groundwork for constructing the global-in-time solution in Theorem 1.2. The regularity of u is sufficient in order to use u , \dot{u} , $\sigma \dot{u}$, and $\sigma^2 \ddot{u}$ as a test function in the following computations. Also, the a priori estimates we will derive will involve lower regularity on the initial data; in particular, the compatibility condition (2.2) will not be required.

2.1. Basic energy functional

The basic energy balance is derived by taking the scalar product of the momentum equation (2.1)₂ with the velocity u , and then integrating over time and space. By doing so, we obtain

$$E(t) + \int_0^t \int_{\mathbb{R}^2} [2\mu(\rho)|\mathbb{D}u|^2 + \lambda(\rho)|\operatorname{div} u|^2] = E(0) = E_0. \tag{2.5}$$

Here, E represents the energy functional defined as

$$E(t) = \int_{\mathbb{R}^2} \left(\rho \frac{|u|^2}{2} + H_1(\rho) \right) (t, x) dx,$$

where H_1 stands for the potential energy that solves the ODE

$$\rho H_1'(\rho) - H_1(\rho) = P(\rho) - \tilde{P}, \quad \text{and given by } H_1(\rho) = \rho \int_{\tilde{\rho}}^{\rho} s^{-2} (P(s) - \tilde{P}) ds.$$

More generally, as in [1], we define the potential energy H_l , $l \in (1, \infty)$, as the solution to the ODE

$$\rho H_l'(\rho) - H_l(\rho) = |P(\rho) - \tilde{P}|^{l-1} (P(\rho) - \tilde{P}),$$

which reads

$$H_l(\rho) = \rho \int_{\tilde{\rho}}^{\rho} s^{-2} |P(s) - \tilde{P}|^{l-1} (P(s) - \tilde{P}) ds.$$

These potential energies satisfy

$$\partial_t H_l(\rho) + \operatorname{div}(H_l(\rho)u) + |P(\rho) - \tilde{P}|^{l-1} (P(\rho) - \tilde{P}) \operatorname{div} u = 0, \tag{2.6}$$

and they will help in deriving the $L^{l+1}(\mathbb{R}^2)$ -norm estimate for the pressure in the subsequent step.

2.2. Estimates for the functionals \mathcal{A}_1 and \mathcal{A}_2

This subsection is devoted to providing bounds for the functionals \mathcal{A}_1 and \mathcal{A}_2 . These functionals yield estimates for the material acceleration \dot{u} and, consequently, for the velocity. We always assume the following bounds for the density and viscosity:

$$0 < \underline{\rho} \leq \rho(t, x) \leq \bar{\rho}, \quad 0 < \underline{\mu} \leq \mu(\rho(t, x)) \leq \bar{\mu}, \quad \text{and} \quad 0 \leq \underline{\lambda} \leq \lambda(\rho(t, x)) \leq \bar{\lambda}, \quad (2.7)$$

and we define the viscosity fluctuation

$$\delta(t) := \frac{1}{\underline{\mu}} \sup_{[0, t]} \|\mu(\rho) - \tilde{\mu}\|_{L^\infty(\mathbb{R}^2)}.$$

We denote by C_* any constant that depends on the bounds of the density and viscosity and $\delta(t)$, and by C_0 any constant that depends polynomially on

$$\|u_0\|_{H^1(\mathbb{R}^2)}^2 + \|\rho_0 - \tilde{\rho}\|_{L^2(\mathbb{R}^2)}^2.$$

These constants may change from one line to another. We derive the following estimates for functionals \mathcal{A}_1 and \mathcal{A}_2 under smallness of δ .

Lemma 2.4. *Suppose (2.7) holds. There exists a positive function $\hat{\kappa} = \hat{\kappa}(l)$, $l \in (1, \infty)$, such that if*

$$\delta(t) \left(\frac{2\underline{\mu} + \underline{\lambda}}{2\bar{\mu} + \bar{\lambda}} \right)^{-\frac{1}{l+1}} < \hat{\kappa}(l), \quad (2.8)$$

for $l \in \{2, 3\}$, then we have

$$\mathcal{A}_1(t) \leq C_*(C_0 + \mathcal{A}_1(t)^2), \quad (2.9)$$

$$\mathcal{A}_2(t) \leq C_*[C_0 + \mathcal{A}_1(\sigma(t)) + \mathcal{A}_1(t)(C_0 + \mathcal{A}_1(t))]. \quad (2.10)$$

The proof of Lemma 2.4 is given in Section 3.1 below. The functionals \mathcal{A}_1 and \mathcal{A}_2 are under control as long as the density is bounded away from vacuum and upper bounded, and the dynamic viscosity is a small perturbation of the constant state $\tilde{\mu}$. Achieving this control is the purpose of the subsequent steps.

2.3. Discussion on the propagation of the $L^\infty(\mathbb{R}^2)$ -norm of the density

In this section we show where the difficulty in propagating the L^∞ -norm of the density, as done by Hoff [19], arises and how we circumvent this difficulty. In particular, we achieve exponential-in-time decay of jumps, which compensates for the exponential-in-time growth of the interface characteristics.

First, we use mass equation (2.1) to derive

$$\partial_t \log \rho + u \cdot \nabla \log \rho + \operatorname{div} u = 0.$$

Multiplying by $2\mu(\rho) + \lambda(\rho)$ and using the expression of the effective flux

$$F = (2\mu(\rho) + \lambda(\rho)) \operatorname{div} u - P(\rho) + \tilde{P}$$

we find (see (1.20) for the definition of f)

$$\partial_t f(\rho) + u \cdot \nabla f(\rho) + P(\rho) - \tilde{P} = -F,$$

where the last term on the left-hand side is understood as a damping term. As derived in (3.4) below, F can also be expressed as

$$F = -(-\Delta)^{-1} \operatorname{div}(\rho \dot{u}) + [K, \mu(\rho) - \tilde{\mu}] \mathbb{D}u, \quad (2.11)$$

where K is a combination of second-order Riesz operators. In general, the $L^\infty(\mathbb{R}^2)$ -norm estimate of the density, or equivalently of $f(\rho)$, follows as long as we have an $L^\infty(\mathbb{R}^2)$ -norm estimate for F . However, as explained in [1], the algebraic structure of the Navier–Stokes equations with density-dependent viscosity does not allow for such an estimate for F , as is often done in the isotropic case (see, for example [19]). The issue stems from the last term of F (see (2.11) above), which is of the same order as the velocity gradient due to the roughness of $\mu(\rho)$. Indeed, it is not clear whether the commutator $[K, \mu(\rho) - \tilde{\mu}]$ is continuous over $L^\infty(\mathbb{R}^2)$ when the viscosity is discontinuous across a hypersurface. In fact, this term is discontinuous even for regular velocity, as its jump corresponds exactly to the right-hand side of (1.17) above. In order to control this term we use the following result which establishes that even-order singular operators are continuous over $\mathcal{C}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2)$.

Proposition 2.5. *Let the hypotheses in Definition 1.1 (1) hold for an interface \mathcal{C} and consider a Calderón–Zygmund-type singular integral operator \mathcal{T} of even order and let $p \in [1, \infty)$. There exists a constant $C = C(\alpha, p)$ such that for $g \in L^p(\mathbb{R}^2) \cap \mathcal{C}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2)$ we have*

$$\begin{aligned} \|\mathcal{T}(g)\|_{\mathcal{C}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2)} &\leq C(\|g\|_{L^p(\mathbb{R}^2)} + \|g\|_{\mathcal{C}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2)} + \ell_\varphi^{-\frac{1}{p}} \|\llbracket g \rrbracket\|_{L^p(\mathcal{C})}) \\ &\quad + C \|\llbracket g \rrbracket\|_{L^\infty(\mathcal{C})} (\ell_\varphi^{-\alpha} + (1 + |\mathcal{C}|) \mathfrak{B}^{\mathcal{T}} (\|\nabla \gamma\|_{L^\infty} + c_\gamma) \|\nabla \gamma\|_{\dot{\mathcal{C}}^\alpha}). \end{aligned}$$

Above, $\mathfrak{B}^{\mathcal{T}}$ is a polynomial depending on the kernel of \mathcal{T} .

The proof of Proposition 2.5 follows directly from [36, Lemmas A.1 & A.2], so we do not present it here. Applying the result above with

$$\mathcal{C}(t) = X(t)C(0) \quad \text{and} \quad \varphi(t) = \varphi_0(X^{-1}(t)), \quad (2.12)$$

we obtain

$$\begin{aligned} &\| [K, \mu(\rho) - \tilde{\mu}] \mathbb{D}u(t) \|_{\mathcal{C}_{\text{pw},\gamma(t)}^\alpha(\mathbb{R}^2)} \\ &\leq C \|\mu(\rho(t)) - \tilde{\mu}\|_{\mathcal{C}_{\text{pw},\gamma(t)}^\alpha(\mathbb{R}^2)} (\|\nabla u(t)\|_{L^4(\mathbb{R}^2)} + \|\nabla u(t)\|_{\mathcal{C}_{\text{pw},\gamma(t)}^\alpha(\mathbb{R}^2)}) \\ &\quad + C \|\mu(\rho(t)) - \tilde{\mu}\|_{\mathcal{C}_{\text{pw},\gamma(t)}^\alpha(\mathbb{R}^2)} (\ell_{\varphi(t)}^{-\frac{1}{4}} \|\llbracket \nabla u(t) \rrbracket\|_{L^p(\mathcal{C}(t))} \\ &\quad \quad \quad + (\ell_{\varphi(t)}^{-\alpha} + \mathfrak{B}_{\gamma(t)}) \|\llbracket \nabla u(t) \rrbracket\|_{L^\infty(\mathcal{C}(t))}) \end{aligned}$$

$$\begin{aligned}
 &+ C \|\nabla u(t)\|_{L^\infty(\mathbb{R}^2)} (\ell_{\varphi(t)}^{-\frac{1}{4}} \|\llbracket \mu(\rho(t)) \rrbracket\|_{L^p(\mathcal{E}(t))} \\
 &\quad + (\ell_{\varphi(t)}^{-\alpha} + \mathfrak{B}_{\gamma(t)}) \|\llbracket \mu(\rho(t)) \rrbracket\|_{L^\infty(\mathcal{E}(t))}), \quad (2.13)
 \end{aligned}$$

for a.e. t . The proof of (2.13) is the purpose of the first part of Section 3.2. Recall the definitions of $\ell_{\varphi(t)}$ and $\mathfrak{B}_{\gamma(t)}$ given in (1.6)–(1.8), which we will estimate in the next step.

From (2.12), it is straightforward to derive

$$\begin{cases}
 |\mathcal{E}(t)| \leq |\mathcal{E}(0)| \exp\left(\int_0^t \|\nabla u\|_{L^\infty(\mathbb{R}^2)}\right), \\
 \|\nabla \gamma(t)\|_{L^\infty} \leq \|\nabla \gamma_0\|_{L^\infty} \exp\left(\int_0^t \|\nabla u\|_{L^\infty(\mathbb{R}^2)}\right), \\
 c_{\gamma(t)} \leq c_{\gamma_0} \exp\left(\int_0^t \|\nabla u\|_{L^\infty(\mathbb{R}^2)}\right), \\
 |\nabla \varphi(t)|_{\text{inf}} \geq |\nabla \varphi_0|_{\text{inf}} \exp\left(-\int_0^t \|\nabla u\|_{L^\infty(\mathbb{R}^2)}\right),
 \end{cases} \quad (2.14)$$

and

$$\begin{cases}
 \|\nabla \gamma(t)\|_{\dot{C}^\alpha} \leq \left(\|\nabla \gamma_0\|_{\dot{C}^\alpha} + \|\nabla \gamma_0\|_{L^\infty}^{1+\alpha} \int_0^t \|\nabla u(\tau)\|_{\dot{C}^\alpha_{\text{pw},\gamma(\tau)}} d\tau \right) \\
 \quad \times \exp\left((2+\alpha) \int_0^t \|\nabla u\|_{L^\infty(\mathbb{R}^2)}\right), \\
 \|\nabla \varphi(t)\|_{\dot{C}^\alpha} \leq \left(\|\nabla \varphi_0\|_{\dot{C}^\alpha} + \|\nabla \varphi_0\|_{L^\infty}^{1+\alpha} \int_0^t \|\nabla u(\tau)\|_{\dot{C}^\alpha_{\text{pw},\gamma(\tau)}} d\tau \right) \\
 \quad \times \exp\left((2+\alpha) \int_0^t \|\nabla u\|_{L^\infty(\mathbb{R}^2)}\right),
 \end{cases} \quad (2.15)$$

resulting in the fact that the parameters $\mathfrak{B}_{\gamma(t)}$, $\ell_{\varphi(t)}^{-1}$ appearing in (2.13) grow exponentially with respect to

$$\int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau. \quad (2.16)$$

We are unable to obtain a uniform-in-time estimate for (2.16), and this results in exponential growth over time of $\mathfrak{B}_{\gamma(t)}$ and $\ell_{\varphi(t)}^{-1}$. To counterbalance the growth of the interface characteristics over time, the exponential-in-time decay of the viscosity and velocity gradient jumps is crucial. This leads to the following lemma.

Lemma 2.6. *Suppose (2.7) holds. There are constants $0 < \underline{\nu} < \bar{\nu}$ depending only on $\underline{\rho}$, $\underline{\mu}$ and $\bar{\rho}$, $\bar{\mu}$, $\bar{\lambda}$ such that the following hold true:*

$$\begin{aligned}
 \|\llbracket f(\rho(t)) \rrbracket\|_{L^p(\mathcal{E}(t))} &\leq \|\llbracket f(\rho_0) \rrbracket\|_{L^p(\mathcal{E}(0))} \\
 &\quad \times \exp\left(-\underline{\nu}t + (6\bar{\nu} + 1/p) \int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau\right), \quad (2.17)
 \end{aligned}$$

$$\begin{aligned} \|\llbracket \nabla u(t) \rrbracket\|_{L^p(\mathcal{E}(t))} &\leq C_* \|\llbracket f(\rho_0) \rrbracket\|_{L^p(\mathcal{E}(0))} (1 + \|\nabla u(t)\|_{L^\infty(\mathbb{R}^2)}) \\ &\quad \times \exp\left(-\underline{\nu} t + (6\bar{\nu} + 1/p) \int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau\right), \end{aligned} \quad (2.18)$$

for all $1 \leq p \leq \infty$.

The proof of Lemma 2.6 is given in the second part of Section 3.2. It shows that when the pressure and viscosity laws are proportional, the constants $\underline{\nu}$ and $\bar{\nu}$ do not depend on $\underline{\rho}$, $\underline{\mu}$ or $\bar{\rho}$, $\bar{\mu}$, $\bar{\lambda}$.

The exponential-in-time decay of the pressure and velocity gradient jump follows immediately as long as we have a uniform-in-time $L^q((0, t), L^\infty(\mathbb{R}^2))$ estimate for $\sigma^s \nabla u$, with some $q < \infty$ and $sq' < 1$. Indeed, Hölder's and Young's inequalities yield

$$\int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau \leq \frac{\varepsilon}{1 - sq'} t + \frac{1}{q(\varepsilon q')^{q-1}} \int_0^t \sigma^{sq} \|\nabla u\|_{L^\infty(\mathbb{R}^2)}^q d\tau, \quad (2.19)$$

for all $\varepsilon > 0$, and by virtue of (2.17)–(2.18), we can take

$$\varepsilon = (1 - sq') \frac{\underline{\nu}}{4(6\bar{\nu} + 1/p)}.$$

Achieving such a uniform-in-time estimate for the velocity gradient is the purpose of the next section.

2.4. Final estimates

In this section we establish bounds for the functionals \mathcal{A}_3 and θ , and we conclude by closing all the estimates. First of all, we observe that (2.1) can be rewritten as

$$\begin{aligned} \tilde{\mu} \Delta u &= \rho \dot{u} - \nabla((\tilde{\mu} + \lambda(\rho)) \operatorname{div} u - P(\rho) + \tilde{P}) - \operatorname{div}((2\mu(\rho) - \tilde{\mu}) \mathbb{D}u) \\ &= \rho \dot{u} - \nabla\left(\frac{\tilde{\mu} + \lambda(\rho)}{2\mu(\rho) + \lambda(\rho)} F\right) + \nabla\left(\frac{2\mu(\rho) - \tilde{\mu}}{2\mu(\rho) + \lambda(\rho)} (P(\rho) - \tilde{P})\right) \\ &\quad - \operatorname{div}((2\mu(\rho) - \tilde{\mu}) \mathbb{D}u), \end{aligned}$$

and therefore

$$\begin{aligned} \tilde{\mu} \nabla u &= -(-\Delta)^{-1} \nabla(\rho \dot{u}) + \nabla(-\Delta)^{-1} \nabla\left(\frac{\tilde{\mu} + \lambda(\rho)}{2\mu(\rho) + \lambda(\rho)} F\right) \\ &\quad - \nabla(-\Delta)^{-1} \nabla\left(\frac{2\mu(\rho) - \tilde{\mu}}{2\mu(\rho) + \lambda(\rho)} (P(\rho) - \tilde{P})\right) + \nabla(-\Delta)^{-1} \operatorname{div}((2\mu(\rho) - \tilde{\mu}) \mathbb{D}u) \\ &=: \nabla u_* + \nabla u_F + \nabla u_P + \nabla u_\delta. \end{aligned} \quad (2.20)$$

The last three terms above are second-order Riesz transforms of discontinuous functions, and Proposition 2.5 plays a key role in establishing their piecewise Hölder norm estimates. The regularity of \dot{u} , as provided by the functionals \mathcal{A}_1 and \mathcal{A}_2 , ensures that the remaining term ∇u_* is Hölder continuous in the whole space.

On the other hand, it is unclear whether a uniform-in-time $L^q((0, t), L^\infty(\mathbb{R}^2))$ -norm estimate, with $q = 2$, can be established for $\sigma^s \nabla u_*$ and $\sigma^s \nabla u_P$. However, such an estimate is possible for $q = 4$, which explains the time integral in the definition of θ ; see (1.20) above. At the same time, the functionals \mathcal{A}_1 and \mathcal{A}_2 do not provide sufficient time integrability for the material derivative \dot{u} to control the $L^4((0, t), \mathcal{C}_{pw, \gamma}^\alpha(\mathbb{R}^2))$ -norm for $\sigma^s \nabla u_*$. This leads us to perform another estimate with the goal of obtaining better integrability for $\nabla \dot{u}$.

Lemma 2.7. *Assume that the density and viscosities are bounded as in (2.7), and suppose that (2.8) holds for $l \in \{2, 3, 5\}$. Then the following estimate holds for the functional \mathcal{A}_3 , defined in (1.19):*

$$\mathcal{A}_3(t) \leq C_* [C_0 + \mathcal{A}_3(t)^2 + \mathcal{A}_1(t)(1 + \mathcal{A}_1(t)^3) + \mathcal{A}_2(t)(1 + \mathcal{A}_2(t)^3)]. \quad (2.21)$$

The proof of Lemma 2.7 is the focus of Section 3.3 below. The functional \mathcal{A}_3 provides us with $\sigma \nabla \dot{u} \in L^\infty((0, t), L^2(\mathbb{R}^2))$, and whence

$$\dot{u} \in L^\infty((\sigma(t), t), L^p(\mathbb{R}^2)), \quad \text{for all } p \in (2, \infty).$$

In [21, Section 2.3], the author obtained the conclusion above using a different approach. Although the viscosities are constant in his analysis, he assumed a smallness condition on the kinematic viscosity λ , which does not apply in this paper. We now proceed to derive an estimate for the functional θ .

Lemma 2.8. *Assume that (2.7) holds, and consider the functional θ , as defined in (1.20). Then we have*

$$\begin{aligned} \int_0^t \sigma^{r\alpha} \|F\|_{\mathcal{C}_{pw, \gamma}^\alpha(\mathbb{R}^2)}^4 &\leq C_* [C_0 + \mathcal{A}_1(t)^2(1 + \mathcal{A}_1(t)^2) + \mathcal{A}_2(t)^2 + \mathcal{A}_3(t)^2 + \theta(t)^2] \\ &\quad + C_* K_0 e^{C_* \theta(t)} \|\llbracket f(\rho_0) \rrbracket\|_{L^4 \cap L^\infty(\mathcal{E}(0))}^4 \end{aligned}$$

and

$$\begin{aligned} \theta(t) &\leq C_* \left[C_0 + \mathcal{A}_1(t)^2 + K_0 e^{C_* \theta(t)} \|\llbracket f(\rho_0) \rrbracket\|_{L^4 \cap L^\infty(\mathcal{E}(0))}^4 \int_0^t \sigma^{r\alpha} \|F\|_{\mathcal{C}_{pw, \gamma}^\alpha(\mathbb{R}^2)}^4 \right] \\ &\quad + C_* e^{C_* \theta(t)} \left(\|f(\rho_0)\|_{\mathcal{C}_{pw, \gamma_0}^\alpha(\mathbb{R}^2)}^4 + K_0 \|\llbracket f(\rho_0) \rrbracket\|_{L^4 \cap L^\infty(\mathcal{E}(0))}^4 \right. \\ &\quad \left. + \int_0^t \sigma^{r\alpha} \|F\|_{\mathcal{C}_{pw, \gamma}^\alpha(\mathbb{R}^2)}^4 \right). \end{aligned}$$

Above, K_0 is a constant that depends on c_{γ_0} , $\|\nabla \gamma_0\|_{\mathcal{C}^\alpha}$, $\|\nabla \varphi_0\|_{\mathcal{C}^\alpha}$, and $|\nabla \varphi_0|_{\text{inf}}$.

The proof of this proposition is given in Section 3.4 below. We finally close the estimates in Lemmas 2.4, 2.7, and 2.8 with the help of a bootstrap argument similar to the one in [1], and we do not present the proof.

Lemma 2.9. *Let (ρ, u) be a local solution to the equations (2.1) with initial data (ρ_0, u_0) that verifies (1.3)–(1.4)–(1.9) and the compatibility condition*

$$\operatorname{div}\{2\mu(\rho_0)\mathbb{D}u_0 + (\lambda(\rho_0)\operatorname{div}u_0 - P(\rho_0) + \tilde{P})I_2\} \in L^2(\mathbb{R}^2).$$

Assume that the solution (ρ, u) is defined up to a maximal time T^ . There exist constants $c > 0$ and $[\mu]_0 > 0$, such that if*

$$C_0 := \|u_0\|_{H^1(\mathbb{R}^2)}^2 + \|\rho_0 - \tilde{\rho}\|_{L^2(\mathbb{R}^2) \cap C_{pw,\gamma_0}^\alpha(\mathbb{R}^2)}^2 + \|[\rho_0]\|_{L^2(\mathcal{C}(0)) \cap L^\infty(\mathcal{C}(0))}^2 \leq c$$

and

$$\|\mu(\rho_0) - \tilde{\mu}\|_{C_{pw,\gamma_0}^\alpha(\mathbb{R}^2)} \leq [\mu]_0,$$

then we have (see (1.22) for the definition of a_* and a^*)

$$\begin{cases} 0 < a_* \leq \inf_{x \in \mathbb{R}^2} \rho(t, x) \leq \sup_{x \in \mathbb{R}^2} \rho(t, x) \leq a^*, \\ E(t) + \mathcal{A}_1(t) + \mathcal{A}_2(t) + \mathcal{A}_3(t) + \sqrt{\theta(t)} \leq CC_0, \end{cases}$$

for all $t \in (0, T^*)$.

Above, C is a constant that depends on α , $\rho_{0,*}$, ρ_0^* , $\mu_{0,*}$, c_{γ_0} , $\|\nabla\gamma_0\|_{C^\alpha}$, $\|\nabla\varphi_0\|_{C^\alpha}$, and $|\nabla\varphi_0|_{\inf}$.

This concludes this section. The proof of Theorem 1.2 is postponed to Section 3.5 below.

3. Proofs

3.1. Proof of Lemma 2.4

Proof. In this section we prove (2.9)–(2.10).

Preliminary estimates. The functional \mathcal{A}_1 appears while testing the momentum equation, written in the form

$$\rho \dot{u}^j = \operatorname{div} \Pi^j, \quad (3.1)$$

with \dot{u} . By doing so, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} \rho |\dot{u}^j|^2 + \frac{d}{dt} \int_{\mathbb{R}^2} \left[\mu(\rho) |\mathbb{D}u|^2 + \frac{\lambda(\rho)}{2} |\operatorname{div} u|^2 \right] \\ &= - \int_{\mathbb{R}^2} 2\mu(\rho) \mathbb{D}^{jk} u \partial_k u^l \partial_l u^j \\ & \quad + \int_{\mathbb{R}^2} (\rho \mu'(\rho) - \mu(\rho)) |\mathbb{D}u|^2 \operatorname{div} u + \frac{1}{2} \int_{\mathbb{R}^2} (\rho \lambda'(\rho) - \lambda(\rho)) (\operatorname{div} u)^3 \\ & \quad - \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} u \nabla u^l \cdot \partial_l u + \frac{d}{dt} \int_{\mathbb{R}^2} \operatorname{div} u (P(\rho) - \tilde{P}) \\ & \quad + \int_{\mathbb{R}^2} \nabla u^l \cdot \partial_l u (P(\rho) - \tilde{P}) + \int_{\mathbb{R}^2} (\operatorname{div} u)^2 (\rho P'(\rho) - P(\rho) + \tilde{P}). \end{aligned}$$

Integrating the above in time, we find

$$\begin{aligned} \mathcal{A}_1(t) \leq C_* \left(C_0 + \sup_{[0,t]} \|P(\rho) - \tilde{P}\|_{L^2(\mathbb{R}^2)}^2 \right. \\ \left. + \int_0^t \|\nabla u\|_{L^3(\mathbb{R}^2)}^3 + \int_0^t \|P(\rho) - \tilde{P}\|_{L^p(\mathbb{R}^2)} \|\nabla u\|_{L^{2p'}(\mathbb{R}^2)}^2 \right), \end{aligned} \quad (3.2)$$

where we have used the classical energy (2.5) and where $p \geq 3$. On the other hand, we take the material derivative, $\partial_t \cdot + \operatorname{div}(\cdot u)$, of the momentum equation (3.1), then multiply the resulting equation by \dot{u} , yielding

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \rho |\dot{u}^j|^2 + \int_{\mathbb{R}^2} (2\mu(\rho) |\mathbb{D}^{jk} \dot{u}|^2 + \lambda(\rho) |\operatorname{div} \dot{u}|^2) \\ = \int_{\mathbb{R}^2} \partial_k \dot{u}^j (\mu(\rho) \partial_j u^l \partial_l u^k + \mu(\rho) \partial_k u^l \partial_l u^j + 2\rho \mu'(\rho) \mathbb{D}^{jk} u \operatorname{div} u) \\ + \int_{\mathbb{R}^2} \operatorname{div} \dot{u} (\lambda(\rho) \nabla u^l \partial_l u + \rho \lambda'(\rho) (\operatorname{div} u)^2 - \rho P'(\rho) \operatorname{div} u) \\ - \int_{\mathbb{R}^2} \partial_k \dot{u}^j \Pi^{jk} \operatorname{div} u + \int_{\mathbb{R}^2} \partial_l \dot{u}^j \partial_k u^l \Pi^{jk}. \end{aligned}$$

The computations leading to the above equality can be found in Appendix A.1 below. Next we multiply the above by $\sigma(t) = \min(1, t)$ before integrating in time; then, applying Hölder's and Young's inequalities yields

$$\mathcal{A}_2(t) \leq C_* \left(C_0 + \mathcal{A}_1(\sigma(t)) + \int_0^t \sigma \|\nabla u\|_{L^4(\mathbb{R}^2)}^4 + \int_0^t \sigma \|P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^2)}^4 \right). \quad (3.3)$$

The remainder of this section is devoted to estimating the $L^p(\mathbb{R}^2)$ -norms of the gradients of the velocity and pressure as they appear in (3.2) and (3.3). To do this, we begin by expressing the effective flux F and vorticity $\operatorname{curl} u$ in terms of singular operators.

Expression of the effective flux and vorticity. We apply the divergence operator to the momentum equation (3.1), resulting in an elliptic equation

$$\operatorname{div}(\rho \dot{u}) = \operatorname{div} \operatorname{div}(2\mu(\rho) \mathbb{D}u) + \Delta(\lambda(\rho) \operatorname{div} u - P(\rho) + \tilde{P})$$

from which we deduce

$$\begin{aligned} \lambda(\rho) \operatorname{div} u - P(\rho) + \tilde{P} &= (-\Delta)^{-1} \operatorname{div} \operatorname{div}(2\mu(\rho) \mathbb{D}u) - (-\Delta)^{-1} \operatorname{div}(\rho \dot{u}) \\ &= [(-\Delta)^{-1} \operatorname{div} \operatorname{div}, 2\mu(\rho)] \mathbb{D}u + 2\mu(\rho) (-\Delta)^{-1} \operatorname{div} \operatorname{div} \mathbb{D}u - (-\Delta)^{-1} \operatorname{div}(\rho \dot{u}) \\ &= -2\mu(\rho) \operatorname{div} u + [(-\Delta)^{-1} \operatorname{div} \operatorname{div}, 2(\mu(\rho) - \tilde{\mu})] \mathbb{D}u - (-\Delta)^{-1} \operatorname{div}(\rho \dot{u}). \end{aligned}$$

Hence, we have the following expression for the effective flux:

$$F = (2\mu(\rho) + \lambda(\rho)) \operatorname{div} u - P(\rho) + \tilde{P} = -(-\Delta)^{-1} \operatorname{div}(\rho \dot{u}) + [K, \mu(\rho) - \tilde{\mu}] \mathbb{D}u. \quad (3.4)$$

To express $\operatorname{curl} u$, we apply the rotational operator to the momentum equation (3.1) to obtain

$$\operatorname{curl}_{jk}(\rho\dot{u}) = \operatorname{curl}_{jk} \operatorname{div}(2\mu(\rho)\mathbb{D}u),$$

from which we deduce

$$\begin{aligned} (-\Delta)^{-1} \operatorname{curl}_{jk}(\rho\dot{u}) &= (-\Delta)^{-1} \operatorname{curl}_{jk} \operatorname{div}(2\mu(\rho)\mathbb{D}u) \\ &= [(-\Delta)^{-1} \operatorname{curl}_{jk} \operatorname{div}, 2\mu(\rho)]\mathbb{D}u + 2\mu(\rho)(-\Delta)^{-1} \operatorname{curl}_{jk} \operatorname{div} \mathbb{D}u \\ &= [(-\Delta)^{-1} \operatorname{curl}_{jk} \operatorname{div}, 2(\mu(\rho) - \tilde{\mu})]\mathbb{D}u - \mu(\rho) \operatorname{curl}_{jk} u. \end{aligned}$$

It then holds that the vorticity reads

$$\mu(\rho) \operatorname{curl}_{jk} u = -(-\Delta)^{-1} \operatorname{curl}_{jk}(\rho\dot{u}) + [K'_{jklm}, \mu(\rho) - \tilde{\mu}]\mathbb{D}^{lm}u. \quad (3.5)$$

Above, K and K' are combinations of second-order Riesz operators, and commutators with BMO functions are known to be continuous on $L^p(\mathbb{R}^2)$ for $p \in (1, \infty)$; see [17]. This aids in obtaining L^p -norm estimates for the velocity gradient and pressure in the next step.

L^p -norm estimates for velocity gradient and pressure. With the help of the expressions (3.4)–(3.5) above, we derive

$$\left\{ \begin{array}{l} \|F\|_{L^p(\mathbb{R}^2)} \leq \kappa(p) \|\mu(\rho) - \tilde{\mu}\|_{L^\infty(\mathbb{R}^2)} \|\mathbb{D}u\|_{L^p(\mathbb{R}^2)} \\ \quad + \|(-\Delta)^{-1} \operatorname{div}(\rho\dot{u})\|_{L^p(\mathbb{R}^2)}, \\ \|\mu(\rho) \operatorname{curl} u\|_{L^p(\mathbb{R}^2)} \leq \kappa(p) \|\mu(\rho) - \tilde{\mu}\|_{L^\infty(\mathbb{R}^2)} \|\mathbb{D}u\|_{L^p(\mathbb{R}^2)} \\ \quad + \|(-\Delta)^{-1} \operatorname{curl}(\rho\dot{u})\|_{L^p(\mathbb{R}^2)}, \end{array} \right. \quad (3.6)$$

for all $1 < p < \infty$. Consequently,

$$\begin{aligned} \|\nabla u\|_{L^p(\mathbb{R}^2)} &\leq \kappa(p) (\|\operatorname{div} u\|_{L^p(\mathbb{R}^2)} + \|\operatorname{curl} u\|_{L^p(\mathbb{R}^2)}) \\ &\leq \kappa(p) \left(\frac{1}{2\underline{\mu} + \underline{\lambda}} \|F + P(\rho) - \tilde{P}\|_{L^p(\mathbb{R}^2)} + \frac{1}{\underline{\mu}} \|\mu(\rho) \operatorname{curl} u\|_{L^p(\mathbb{R}^2)} \right) \\ &\leq \frac{\kappa(p)^2}{\underline{\mu}} \|\mu(\rho) - \tilde{\mu}\|_{L^\infty(\mathbb{R}^2)} \|\mathbb{D}u\|_{L^p(\mathbb{R}^2)} + \frac{\kappa(p)}{2\underline{\mu} + \underline{\lambda}} \|P(\rho) - \tilde{P}\|_{L^p(\mathbb{R}^2)} \\ &\quad + \frac{\kappa(p)}{\underline{\mu}} (\|(-\Delta)^{-1} \operatorname{div}(\rho\dot{u})\|_{L^p(\mathbb{R}^2)} + \|(-\Delta)^{-1} \operatorname{curl}(\rho\dot{u})\|_{L^p(\mathbb{R}^2)}). \end{aligned}$$

Assuming that

$$\delta(t) = \frac{1}{\underline{\mu}} \sup_{[0,t]} \|\mu(\rho) - \tilde{\mu}\|_{L^\infty(\mathbb{R}^2)} < \frac{1}{\kappa(p)^2}, \quad (3.7)$$

the first term of the right-hand side of the above inequality can be absorbed into the left-hand side, yielding

$$\begin{aligned} \|\nabla u\|_{L^p(\mathbb{R}^2)} &\leq \frac{\kappa(p)}{\underline{\mu}(1 - \delta\kappa(p)^2)} (\|(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})\|_{L^p(\mathbb{R}^2)} + \|(-\Delta)^{-1} \operatorname{curl}(\rho \dot{u})\|_{L^p(\mathbb{R}^2)}) \\ &\quad + \frac{\kappa(p)}{(2\underline{\mu} + \underline{\lambda})(1 - \delta\kappa(p)^2)} \|P(\rho) - \tilde{P}\|_{L^p(\mathbb{R}^2)}. \end{aligned} \quad (3.8)$$

We turn to estimating the L^p -norm of the pressure. We recall that the potential energy H_l satisfies (see (2.6) above)

$$\frac{d}{dt} \int_{\mathbb{R}^2} H_l(\rho) + \int_{\mathbb{R}^2} |P(\rho) - \tilde{P}|^{l-1} (P(\rho) - \tilde{P}) \operatorname{div} u = 0.$$

We substitute the divergence of the velocity in the expression of the effective flux (3.4) as

$$\operatorname{div} u = (2\mu(\rho) + \lambda(\rho))^{-1} (F + P(\rho) - \tilde{P})$$

to obtain, after Hölder's inequality,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} H_l(\rho) + \int_{\mathbb{R}^2} (2\mu(\rho) + \lambda(\rho))^{-1} |P(\rho) - \tilde{P}|^{l+1} \\ = - \int_{\mathbb{R}^2} (2\mu(\rho) + \lambda(\rho))^{-1} |P(\rho) - \tilde{P}|^{l-1} (P(\rho) - \tilde{P}) F \\ \leq \frac{l}{l+1} \int_{\mathbb{R}^2} (2\mu(\rho) + \lambda(\rho))^{-1} |P(\rho) - \tilde{P}|^{l+1} \\ + \frac{1}{l+1} \int_{\mathbb{R}^2} (2\mu(\rho) + \lambda(\rho))^{-1} |F|^{l+1}. \end{aligned}$$

The first term in the right-hand side can be absorbed into the left-hand side and it follows that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} H_l(\rho) + \frac{1}{l+1} \int_{\mathbb{R}^2} (2\mu(\rho) + \lambda(\rho))^{-1} |P(\rho) - \tilde{P}|^{l+1} \\ \leq \frac{1}{l+1} \int_{\mathbb{R}^2} (2\mu(\rho) + \lambda(\rho))^{-1} |F|^{l+1}. \end{aligned} \quad (3.9)$$

To estimate the L^{l+1} -norm of the effective flux, we go back to (3.6) and make use of (3.8) to obtain

$$\begin{aligned} \int_{\mathbb{R}^2} (2\mu(\rho) + \lambda(\rho))^{-1} |F|^{l+1} \\ \leq \frac{1}{2\underline{\mu} + \underline{\lambda}} \|[K, \mu(\rho) - \tilde{\mu}]\mathbb{D}u\|_{L^{l+1}(\mathbb{R}^2)}^{l+1} + \frac{1}{2\underline{\mu} + \underline{\lambda}} \int_{\mathbb{R}^2} |(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})|^{l+1} \\ \leq \frac{\kappa(l+1)^{l+1}}{2\underline{\mu} + \underline{\lambda}} \|\mu(\rho) - \tilde{\mu}\|_{L^\infty(\mathbb{R}^2)}^{l+1} \|\mathbb{D}u\|_{L^{l+1}(\mathbb{R}^2)}^{l+1} \\ + \frac{1}{2\underline{\mu} + \underline{\lambda}} \int_{\mathbb{R}^2} |(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})|^{l+1} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\kappa(l+1)^{l+1}}{2\underline{\mu} + \underline{\lambda}} \left(\frac{\kappa(l+1)}{2\underline{\mu} + \underline{\lambda}} \frac{\|\mu(\rho) - \tilde{\mu}\|_{L^\infty(\mathbb{R}^2)}}{1 - \delta\kappa(l+1)^2} \right)^{l+1} \|P(\rho) - \tilde{P}\|_{L^{l+1}(\mathbb{R}^2)}^{l+1} \\
&\quad + \frac{\kappa(l+1)^{l+1}}{2\underline{\mu} + \underline{\lambda}} \left(\frac{\kappa(l+1)}{\underline{\mu}} \frac{\|\mu(\rho) - \tilde{\mu}\|_{L^\infty(\mathbb{R}^2)}}{1 - \delta\kappa(l+1)^2} \right)^{l+1} \\
&\quad \quad \times \left(\|(-\Delta)^{-1} \operatorname{div}(\rho\dot{u})\|_{L^{l+1}(\mathbb{R}^2)}^{l+1} + \|(-\Delta)^{-1} \operatorname{curl}(\rho\dot{u})\|_{L^{l+1}(\mathbb{R}^2)}^{l+1} \right) \\
&\quad + \frac{1}{2\underline{\mu} + \underline{\lambda}} \|(-\Delta)^{-1} \operatorname{div}(\rho\dot{u})\|_{L^{l+1}(\mathbb{R}^2)}^{l+1}.
\end{aligned}$$

We replace the above estimate in (3.9) and absorb the first term of the right-hand side above into the left-hand side of (3.9). To achieve this, we require the following smallness assumption:

$$\delta(t) \left(\frac{2\underline{\mu} + \underline{\lambda}}{2\underline{\mu} + \underline{\lambda}} \right)^{-\frac{1}{l+1}} < \frac{1}{3\sqrt{(l+1)\kappa(l+1)^2}}.$$

In conclusion, for all $l > 1$ there exists a constant $\hat{\kappa} = \hat{\kappa}(l)$ such that if

$$\delta \left(\frac{2\underline{\mu} + \underline{\lambda}}{2\underline{\mu} + \underline{\lambda}} \right)^{-\frac{1}{l+1}} < \hat{\kappa}(l), \quad (3.10)$$

then (3.8) holds for $p = l + 1$ and additionally

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^2} H_l(\rho) + \|P(\rho) - \tilde{P}\|_{L^{l+1}(\mathbb{R}^2)}^{l+1} \\
&\quad \leq C_* \left(\|(-\Delta)^{-1} \operatorname{div}(\rho\dot{u})\|_{L^{l+1}(\mathbb{R}^2)}^{l+1} + \|(-\Delta)^{-1} \operatorname{curl}(\rho\dot{u})\|_{L^{l+1}(\mathbb{R}^2)}^{l+1} \right). \quad (3.11)
\end{aligned}$$

To close the estimates for functionals \mathcal{A}_1 and \mathcal{A}_2 , we will only need the smallness condition (3.10) for $l \in \{2, 3\}$. We take $l = 2$ in (3.11) and integrate in time to obtain

$$\begin{aligned}
&\sup_{[0,t]} \int_{\mathbb{R}^2} H_2(\rho) + \int_0^t \|P(\rho) - \tilde{P}\|_{L^3(\mathbb{R}^2)}^3 \\
&\quad \leq C_* \int_{\mathbb{R}^2} H_2(\rho_0) \\
&\quad \quad + C_* \int_0^t \left[\|(-\Delta)^{-1} \operatorname{div}(\rho\dot{u})\|_{L^3(\mathbb{R}^2)}^3 + \|(-\Delta)^{-1} \operatorname{curl}(\rho\dot{u})\|_{L^3(\mathbb{R}^2)}^3 \right]. \quad (3.12)
\end{aligned}$$

Next we take $l = 3$ in (3.11), then multiply by σ before integrating in time to obtain

$$\begin{aligned}
&\sup_{[0,t]} \sigma \int_{\mathbb{R}^2} H_3(\rho) + \int_0^t \sigma \|P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^2)}^4 \\
&\quad \leq C_* \int_0^{\sigma(t)} \int_{\mathbb{R}^2} H_3(\rho) \\
&\quad \quad + C_* \int_0^t \sigma \left[\|(-\Delta)^{-1} \operatorname{div}(\rho\dot{u})\|_{L^4(\mathbb{R}^2)}^4 + \|(-\Delta)^{-1} \operatorname{curl}(\rho\dot{u})\|_{L^4(\mathbb{R}^2)}^4 \right].
\end{aligned}$$

Combining the above estimate with (3.8), Gagliardo–Nirenberg’s inequality, and with the fact that

$$H_3(\rho) + |P(\rho) - \tilde{P}|^2 \leq C_* H_1(\rho),$$

we find

$$\int_0^t \sigma [\|\nabla u\|_{L^4(\mathbb{R}^2)}^4 + \|P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^2)}^4] \leq C_* [C_0 + \mathcal{A}_1(t)(C_0 + \mathcal{A}_1(t))]. \quad (3.13)$$

Similarly, (3.8)–(3.12) imply

$$\int_0^{\sigma(t)} (\|\nabla u\|_{L^3(\mathbb{R}^2)}^3 + \|P(\rho) - \tilde{P}\|_{L^3(\mathbb{R}^2)}^3) \leq C_* [C_0 + \mathcal{A}_1(t)^{1/2}(C_0 + \mathcal{A}_1(t))]. \quad (3.14)$$

With (3.13)–(3.14) in hand, we can finally close the estimates for the functionals \mathcal{A}_1 and \mathcal{A}_2 .

Final estimates. We return to (3.3) from which we deduce

$$\mathcal{A}_2(t) \leq C_* (C_0 + \mathcal{A}_1(\sigma(t)) + \mathcal{A}_1(t)(C_0 + \mathcal{A}_1(t))).$$

We recall the following estimate for \mathcal{A}_1 (see (3.2)):

$$\mathcal{A}_1(t) \leq C_* \left(C_0 + \int_0^t \|\nabla u\|_{L^3(\mathbb{R}^2)}^3 + \int_0^t \|P(\rho) - \tilde{P}\|_{L^p(\mathbb{R}^2)} \|\nabla u\|_{L^{2p'}(\mathbb{R}^2)}^2 \right)$$

for some $p \geq 3$. The time integral is split into two parts:

$$\int_0^t = \int_0^{\sigma(t)} + \int_{\sigma(t)}^t.$$

To bound the first term, we take $p = 3$ and apply Hölder’s and Young’s inequalities to obtain (3.14). For the second part, we take $p = 4$, and similar arguments together with (3.13) yield

$$\begin{aligned} & \int_{\sigma(t)}^t \|\nabla u\|_{L^3(\mathbb{R}^2)}^3 + \int_{\sigma(t)}^t \|P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^2)} \|\nabla u\|_{L^{8/3}(\mathbb{R}^2)}^2 \\ & \leq \int_{\sigma(t)}^t \|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + \int_{\sigma(t)}^t \|\nabla u, P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^2)}^4 \\ & \leq C_* (C_0 + \mathcal{A}_1(t)(C_0 + \mathcal{A}_1(t))). \end{aligned} \quad (3.15)$$

We finally end up with

$$\mathcal{A}_1(t) \leq C_* C_0 + C_* \mathcal{A}_1(t)^{1/2} (1 + \mathcal{A}_1(t)^{1/2}) (C_0 + \mathcal{A}_1(t)),$$

and (2.9) follows from Young’s inequality. This ends the proof of Lemma 2.4. \blacksquare

3.2. Proofs of (2.13) and Lemma 2.6

Proof of (2.13). We consider

$$\mathcal{C}(t) = X(t)\mathcal{C}(0) \quad \text{and} \quad \varphi(t, x) = \varphi_0(X^{-1}(t, x)),$$

where X is the flow associated with the velocity u . It is clear that

$$\begin{aligned} & \| [K, \mu(\rho) - \tilde{\mu}] \mathbb{D}u \|_{\mathcal{C}_{pw, \gamma}^\alpha(\mathbb{R}^2)} \\ & \leq \| K((\mu(\rho) - \tilde{\mu}) \mathbb{D}u) \|_{\mathcal{C}_{pw, \gamma}^\alpha(\mathbb{R}^2)} + \| (\mu(\rho) - \tilde{\mu}) K(\mathbb{D}u) \|_{\mathcal{C}_{pw, \gamma}^\alpha(\mathbb{R}^2)} \\ & \leq \| K((\mu(\rho) - \tilde{\mu}) \mathbb{D}u) \|_{\mathcal{C}_{pw, \gamma}^\alpha(\mathbb{R}^2)} + \| \mu(\rho) - \tilde{\mu} \|_{\mathcal{C}_{pw, \gamma}^\alpha(\mathbb{R}^2)} \| K(\mathbb{D}u) \|_{\mathcal{C}_{pw, \gamma}^\alpha(\mathbb{R}^2)}, \end{aligned} \quad (3.16)$$

and Proposition 2.5 provides us with

$$\begin{aligned} & \| K(\mathbb{D}u)(t) \|_{\mathcal{C}_{pw, \gamma(t)}^\alpha(\mathbb{R}^2)} \\ & \leq C (\| \nabla u(t) \|_{L^4(\mathbb{R}^2)} + \| \nabla u(t) \|_{\mathcal{C}_{pw, \gamma(t)}^\alpha(\mathbb{R}^2)} + \ell_{\varphi(t)}^{-\frac{1}{4}} \| \llbracket \nabla u(t) \rrbracket \|_{L^4(\mathcal{C}(t))}) \\ & \quad + C \| \llbracket \nabla u(t) \rrbracket \|_{L^\infty(\mathcal{C}(t))} (\ell_{\varphi(t)}^{-\alpha} + \mathfrak{P}_{\gamma(t)}), \end{aligned} \quad (3.17)$$

where $\mathfrak{P}_{\gamma(t)}$ and $\ell_{\varphi(t)}$ are defined in (1.8)–(1.6) and associated with $\mathcal{C}(t)$. We notice that \mathfrak{P} is a polynomial which satisfies $\mathfrak{P}^K \leq \mathfrak{P}$. Similarly, we estimate the first term on the right-hand side of (3.16) as

$$\begin{aligned} & \| K((\mu(\rho) - \tilde{\mu}) \mathbb{D}u)(t) \|_{\mathcal{C}_{pw, \gamma(t)}^\alpha(\mathbb{R}^2)} \leq C \| \mu(\rho(t)) - \tilde{\mu} \|_{L^\infty(\mathbb{R}^2)} \| \nabla u(t) \|_{L^4(\mathbb{R}^2)} \\ & \quad + C \| \mu(\rho(t)) - \tilde{\mu} \|_{\mathcal{C}_{pw, \gamma(t)}^\alpha(\mathbb{R}^2)} \| \nabla u(t) \|_{\mathcal{C}_{pw, \gamma(t)}^\alpha(\mathbb{R}^2)} \\ & \quad + C \ell_{\varphi(t)}^{-\frac{1}{4}} \| \mu(\rho(t)) - \tilde{\mu} \|_{L^\infty(\mathbb{R}^2)} \| \llbracket \nabla u(t) \rrbracket \|_{L^p(\mathcal{C}(t))} \\ & \quad + C \ell_{\varphi(t)}^{-\frac{1}{4}} \| \llbracket \mu(\rho(t)) \rrbracket \|_{L^p(\mathcal{C}(t))} \| \nabla u(t) \|_{L^\infty(\mathbb{R}^2)} \\ & \quad + C (\ell_{\varphi(t)}^{-\alpha} + \mathfrak{P}_{\gamma(t)}) \| \mu(\rho(t)) - \tilde{\mu} \|_{L^\infty(\mathbb{R}^2)} \| \llbracket \nabla u(t) \rrbracket \|_{L^\infty(\mathcal{C}(t))} \\ & \quad + C (\ell_{\varphi(t)}^{-\alpha} + \mathfrak{P}_{\gamma(t)}) \| \llbracket \mu(\rho(t)) \rrbracket \|_{L^\infty(\mathcal{C}(t))} \| \nabla u(t) \|_{L^\infty(\mathbb{R}^2)}. \end{aligned} \quad (3.18)$$

Finally, (2.13) follows by summing (3.16)–(3.17)–(3.18). \blacksquare

Proof of Lemma 2.6. This section aims at proving (2.17)–(2.18). We first rewrite the mass equation (2.1)₁ as

$$\partial_t \log \rho + u \cdot \nabla \log \rho + \operatorname{div} u = 0.$$

Then we multiply the above by $2\mu(\rho) + \lambda(\rho)$ and substitute the last term with the help of the effective flux (see (3.4)) to obtain

$$\partial_t f(\rho) + u \cdot \nabla f(\rho) + P(\rho) - \tilde{P} = -F, \quad (3.19)$$

where $f(\rho)$ is

$$f(\rho) = \int_{\tilde{\rho}}^{\rho} \frac{2\mu(s) + \lambda(s)}{s} ds.$$

Then, for all $x \in \mathbb{R}^2$, we have

$$\frac{d}{dt} f(\rho(t, X(t, x))) + P(\rho(t, X(t, x))) - \tilde{P} = -F(t, X(t, x)). \quad (3.20)$$

In particular, along the interface $\mathcal{C}(t)$, which is parameterized by $\gamma(t, s) = X(t, \gamma_0(s))$, we have

$$\frac{d}{dt} f(\rho(t, \gamma(t, s))) + P(\rho(t, \gamma(t, s))) - \tilde{P} = -F(t, \gamma(t, s)),$$

and then, by taking the jump at $\gamma(t, s)$, it holds that

$$\begin{aligned} \frac{d}{dt} \llbracket f(\rho(t, \gamma(t, s))) \rrbracket + g(t, s) \llbracket f(\rho(t, \gamma(t, s))) \rrbracket &= -\llbracket F(t, \gamma(t, s)) \rrbracket \\ &= -2h(t, s) \llbracket f(\rho(t, \gamma(t, s))) \rrbracket (\langle \operatorname{div} u \rangle - \langle \mathbb{D}^{jk} u \rangle n_x^j n_x^k)(t, \gamma(t, s)). \end{aligned}$$

Above, we utilized the expression for the effective flux jump as derived in (1.17) above. Additionally, the functions g and h are defined as

$$g(t, s) := \frac{\llbracket P(\rho(t, \gamma(t, s))) \rrbracket}{\llbracket f(\rho(t, \gamma(t, s))) \rrbracket} \quad \text{and} \quad h(t, s) := \frac{\llbracket \mu(\rho(t, \gamma(t, s))) \rrbracket}{\llbracket f(\rho(t, \gamma(t, s))) \rrbracket}.$$

To achieve an exponential-in-time decay for the jump of $f(\rho)$, and subsequently for the pressure jump, we require that g is both upper bounded and bounded away from zero, while h needs simply to be upper bounded. Specifically, there might exist two constants $\underline{\nu}$ and $\bar{\nu}$, potentially dependent on ρ , $\underline{\mu}$ and $\bar{\rho}$, $\bar{\mu}$, $\bar{\lambda}$, such that for all $0 < \underline{\rho} \leq \rho, \rho' \leq \bar{\rho}$

$$0 < \underline{\nu} \leq \frac{P(\rho) - P(\rho')}{f(\rho) - f(\rho')} \leq \bar{\nu} \quad \text{and} \quad \left| \frac{\mu(\rho) - \mu(\rho')}{f(\rho) - f(\rho')} \right| \leq \bar{\nu}. \quad (3.21)$$

We observe that when the pressure and viscosity laws are proportional, the constants $\underline{\nu}$ and $\bar{\nu}$ in (3.21) do not depend on the bounds of the density. The (strict) positivity of $\underline{\nu}$ arises from the fact that both the pressure and f are increasing functions of ρ . It then follows that

$$\begin{aligned} \frac{d}{dt} \left[e^{\int_0^t g(\tau, s) d\tau} \llbracket f(\rho(t, \gamma(t, s))) \rrbracket \right] &= -2h(t, s) e^{\int_0^t g(\tau, s) d\tau} \llbracket f(\rho(t, \gamma(t, s))) \rrbracket \\ &\quad \times (\langle \operatorname{div} u \rangle - \langle \mathbb{D}^{jk} u \rangle n_x^j n_x^k)(t, \gamma(t, s)), \end{aligned}$$

and whence

$$\left| \llbracket f(\rho(t, \gamma(t, s))) \rrbracket \right| \leq \left| \llbracket f(\rho_0(\gamma_0(s))) \rrbracket \right| \exp\left(-\underline{\nu}t + 6\bar{\nu} \int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau\right).$$

Therefore,

$$\|\llbracket f(\rho(t)) \rrbracket\|_{L^\infty(\mathcal{C}(t))} \leq \|\llbracket f(\rho_0) \rrbracket\|_{L^\infty(\mathcal{C}(0))} \exp\left(-\underline{\nu}t + 6\bar{\nu} \int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau\right),$$

and furthermore, for all $1 \leq p < \infty$, we have

$$\begin{aligned} \|\llbracket f(\rho(t)) \rrbracket\|_{L^p(\mathcal{E}(t))} &\leq \|\llbracket f(\rho_0) \rrbracket\|_{L^p(\mathcal{E}(0))} \\ &\times \exp\left(-\underline{\nu}t + (6\bar{\nu} + 1/p) \int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau\right). \end{aligned} \quad (3.22)$$

Given (3.21), a similar estimate applies to the jumps in viscosity $\mu(\rho)$ and pressure $P(\rho)$. Assuming that λ also satisfies the same condition as μ in (3.21), a similar estimate to (3.22) also applies to $\llbracket \lambda(\rho) \rrbracket$. This proves (2.17). Returning to (1.17)–(1.18), we express the jumps in $\operatorname{div} u$ and $\operatorname{curl} u$ as follows:

$$\begin{aligned} (2\mu(\rho) + \lambda(\rho))\llbracket \operatorname{div} u \rrbracket &= \llbracket P(\rho) \rrbracket - \llbracket \lambda(\rho) \rrbracket \langle \operatorname{div} u \rangle - 2\llbracket \mu(\rho) \rrbracket \langle \mathbb{D}^{jk} u \rangle n_x^j n_x^k, \\ \langle \mu(\rho) \rrbracket \llbracket \operatorname{curl} u \rrbracket &= -2\llbracket \mu(\rho) \rrbracket \langle \mathbb{D}^{jk} u \rangle n_x^k \tau_x^j. \end{aligned}$$

From these expressions and (3.22), we deduce an estimate of the $L^p(\mathcal{E}(t))$ -norm of $\llbracket \operatorname{div} u \rrbracket$ and $\llbracket \operatorname{curl} u \rrbracket$ before using (1.14)–(1.15) to derive

$$\begin{aligned} \|\llbracket \nabla u(t) \rrbracket\|_{L^p(\mathcal{E}(t))} &\leq C_* \|\llbracket f(\rho_0) \rrbracket\|_{L^p(\mathcal{E}(0))} (1 + \|\nabla u(t)\|_{L^\infty(\mathbb{R}^2)}) \\ &\times \exp\left(-\underline{\nu}t + (6\bar{\nu} + 1/p) \int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau\right), \quad p \in [1, \infty]. \end{aligned}$$

This ends the proof of Lemma 2.6. ■

3.3. Proof of Lemma 2.7

Proof. In this section we will provide an estimate for \mathcal{A}_3 as defined in (1.19).

Preliminary estimates. We apply the material derivative $\partial_t \cdot + \operatorname{div}(\cdot u)$ to the momentum equations (2.1)₂ and find that \dot{u} satisfies

$$\partial_t(\rho \dot{u}^j) + \operatorname{div}(\rho \dot{u}^j u) = \partial_k(\dot{\Pi}^{jk}) + \partial_k(\Pi^{jk} \operatorname{div} u) - \operatorname{div}(\partial_k u \Pi^{jk}). \quad (3.23)$$

We then use $\sigma^2 \ddot{u}$ as a test function to obtain

$$\begin{aligned} &\int_0^t \sigma^2 \|\sqrt{\rho} \ddot{u}\|_{L^2(\mathbb{R}^2)}^2 + \sigma^2(t) \int_{\mathbb{R}^2} \left[\mu(\rho) |\mathbb{D} \dot{u}|^2 + \frac{\lambda(\rho)}{2} |\operatorname{div} \dot{u}|^2 \right] \\ &= 2 \int_0^{\sigma(t)} \sigma \int_{\mathbb{R}^2} \left[\mu(\rho) |\mathbb{D} \dot{u}|^2 + \frac{\lambda(\rho)}{2} |\operatorname{div} \dot{u}|^2 \right] - \sigma^2 \int_{\mathbb{R}^2} \rho P'(\rho) \operatorname{div} \dot{u} \operatorname{div} u \\ &\quad + \int_0^{\sigma(t)} \sigma \int_{\mathbb{R}^2} \rho P'(\rho) \operatorname{div} \dot{u} \operatorname{div} u + \int_0^t \sigma^2 \int_{\mathbb{R}^2} \rho P'(\rho) (\operatorname{div} \dot{u})^2 \\ &\quad + \int_0^t \sigma^2 \int_{\mathbb{R}^2} \operatorname{div} \dot{u} (\operatorname{div} u)^2 (\rho P'(\rho) - P(\rho) + \tilde{P}) \end{aligned}$$

$$\begin{aligned}
 & - \int_0^t \sigma^2 \int_{\mathbb{R}^2} \partial_l \dot{u}^j \operatorname{div} u \partial_j u^l (\rho P'(\rho) - P(\rho) + \tilde{P}) \\
 & + \sigma^2(t) I_1(t) - 2 \int_0^{\sigma(t)} \sigma I_1(s) ds + \int_0^t \sigma^2 I_2(s) ds + \int_0^t \sigma^2 I_3(s) ds, \quad (3.24)
 \end{aligned}$$

where the terms I_1, I_2, I_3 are

$$I_1 = \int_{\mathbb{R}^2} \varphi(\rho) \partial_{j_1} \dot{u}^{j_2} \partial_{j_3} u^{j_4} \partial_{j_5} u^{j_6} + \int_{\mathbb{R}^2} \partial_{j_1} \dot{u}^{j_2} \partial_{j_3} u^{j_4} (P(\rho) - \tilde{P}), \quad (3.25)$$

$$\begin{aligned}
 I_2 &= \int_{\mathbb{R}^2} \varphi(\rho) \partial_{j_1} \dot{u}^{j_2} \partial_{j_3} \dot{u}^{j_4} \partial_{j_5} u^{j_6} + \int_{\mathbb{R}^2} \partial_{j_1} \dot{u}^{j_2} \partial_{j_3} \dot{u}^{j_4} (P(\rho) - \tilde{P}) \\
 &+ \int_{\mathbb{R}^2} \psi(\rho) \partial_{j_1} \dot{u}^{j_2} \partial_{j_3} u^{j_4} \partial_{j_5} \dot{u}^{j_6}, \quad (3.26)
 \end{aligned}$$

$$I_3 = \int_{\mathbb{R}^2} \varphi(\rho) \partial_{j_1} \dot{u}^{j_2} \partial_{j_3} u^{j_4} \partial_{j_5} u^{j_6} \partial_{j_7} u^{j_8} + \int_{\mathbb{R}^2} \partial_{j_1} \dot{u}^{j_2} \partial_{j_3} u^{j_4} \partial_{j_5} u^{j_6} (P(\rho) - \tilde{P}), \quad (3.27)$$

and where φ is either the viscosity $\mu, \lambda, \rho\mu', \rho\lambda', \rho^2\mu'',$ or $\rho^2\lambda''$, whereas ψ is either $\rho P'$ or $\rho^2 P''$. The computations leading to (3.24) can be found in Appendix A.2. In the following, we will estimate the terms appearing on the left-hand side of (3.24).

Estimates for the lower-order terms. The first term on the right-hand side of (3.24) is bounded by

$$2 \int_0^{\sigma(t)} \sigma \int_{\mathbb{R}^2} \left[\mu(\rho) |\mathbb{D}\dot{u}|^2 + \frac{\lambda(\rho)}{2} |\operatorname{div} \dot{u}|^2 \right] \leq C_* \mathcal{A}_2(\sigma(t)),$$

while the next term is estimated as

$$\sigma^2(t) \left| \int_{\mathbb{R}^2} \rho P'(\rho) \operatorname{div} \dot{u} \operatorname{div} u \right| \leq \eta \mathcal{A}_3(t) + \frac{C_*}{\eta} \mathcal{A}_1(t),$$

where η is a small positive constant. Next, the third and fourth terms on the right-hand side of (3.24) are controlled by

$$\left| \int_0^{\sigma(t)} \sigma \int_{\mathbb{R}^2} \rho P'(\rho) \operatorname{div} \dot{u} \operatorname{div} u \right| + \left| \int_0^t \sigma^2 \int_{\mathbb{R}^2} \rho P'(\rho) (\operatorname{div} \dot{u})^2 \right| \leq E_0 + C_* \mathcal{A}_2(t),$$

and the next two terms are bounded by

$$\begin{aligned}
 & \left| \int_0^t \sigma^2 \int_{\mathbb{R}^2} \operatorname{div} \dot{u} (\operatorname{div} u)^2 (\rho P'(\rho) - P(\rho) + \tilde{P}) \right| \\
 & + \left| \int_0^t \sigma^2 \int_{\mathbb{R}^2} \partial_l \dot{u}^j \operatorname{div} u \partial_j u^l (\rho P'(\rho) - P(\rho) + \tilde{P}) \right| \\
 & \leq C_* \left[\int_0^t \sigma \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^2 \right]^{1/2} \left[\int_0^t \sigma \|\nabla u\|_{L^4(\mathbb{R}^2)}^4 \right]^{1/2} \\
 & \leq C_* \mathcal{A}_2(t)^{1/2} (C_0 + \mathcal{A}_1(t) (C_0 + \mathcal{A}_1(t)))^{1/2} \\
 & \leq C_* (C_0 + \mathcal{A}_2(t)) + C_* \mathcal{A}_1(t) (C_0 + \mathcal{A}_1(t)),
 \end{aligned}$$

where we have used (3.13). Similar argument leads to

$$\left| \int_0^{\sigma(t)} s I_1(s) ds \right| \leq C_*(C_0 + \mathcal{A}_2(t)) + C_* \mathcal{A}_1(t)(C_0 + \mathcal{A}_1(t)).$$

Estimates for $\sigma^2(t)I_1(t)$, $\int_0^t \sigma^2(s)I_j(s) ds$, $j \in \{2, 3\}$. Hölder's and Young's inequalities imply

$$|\sigma^2(t)I_1(t)| \leq \eta \sigma^2(t) \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^2 + \frac{C_*}{\eta} \sigma^2(t) (\|\nabla u\|_{L^4(\mathbb{R}^2)}^4 + \|P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^2)}^4).$$

From $|P(\rho) - \tilde{P}|^4 \leq C_* H_1(\rho)$, classical energy balance (2.5), (3.8), and Gagliardo–Nirenberg's inequality, we have

$$\begin{aligned} & \sup_{[0,t]} \sigma (\|\nabla u\|_{L^4(\mathbb{R}^2)}^4 + \|P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^2)}^4) \\ & \leq C_* C_0 + C_* \sup_{[0,t]} \sigma (\|(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})\|_{L^4(\mathbb{R}^2)}^4 + \|(-\Delta)^{-1} \operatorname{curl}(\rho \dot{u})\|_{L^4(\mathbb{R}^2)}^4) \\ & \leq C_* C_0 + C_* \sup_{[0,t]} \sigma \|\rho \dot{u}\|_{L^2(\mathbb{R}^2)}^2 (\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + \|P(\rho) - \tilde{P}\|_{L^2(\mathbb{R}^2)}^2) \\ & \leq C_* C_0 + C_* \mathcal{A}_2(t)(C_0 + \mathcal{A}_1(t)), \end{aligned} \quad (3.28)$$

yielding

$$|\sigma^2(t)I_1(t)| \leq \eta \mathcal{A}_3(t) + \frac{C_*}{\eta} (C_0 + \mathcal{A}_2(t)(C_0 + \mathcal{A}_1(t))).$$

Next, interpolation, the Hölder inequality, and (3.8) yield

$$\begin{aligned} & \left| \int_0^t \sigma^2 I_2(s) ds \right| \leq C_* \int_0^t \sigma^2 \|\nabla \dot{u}\|_{L^{8/3}(\mathbb{R}^2)}^2 (\|\nabla u\|_{L^4(\mathbb{R}^2)} + \|P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^2)}) \\ & \leq C_* \int_0^t \sigma^2 \|\nabla \dot{u}\|_{L^3(\mathbb{R}^2)}^{3/2} \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^{1/2} (\|\nabla u\|_{L^4(\mathbb{R}^2)} + \|P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^2)}) \\ & \leq C_* \left[\int_0^t \sigma (\|\nabla u\|_{L^4(\mathbb{R}^2)}^4 + \|P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^2)}^4) \right]^{1/4} \\ & \quad \times \left[\int_0^t \sigma \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^2 \right]^{1/4} \left[\int_0^t \sigma^3 \|\nabla \dot{u}\|_{L^3(\mathbb{R}^2)}^3 \right]^{1/2} \\ & \leq C_* \mathcal{A}_2(t)^{1/4} (C_0 + \mathcal{A}_1(t)(C_0 + \mathcal{A}_1(t)))^{1/4} \left[\int_0^t \sigma^3 \|\nabla \dot{u}\|_{L^3(\mathbb{R}^2)}^3 \right]^{1/2} \\ & \leq C_* (C_0 + \mathcal{A}_2(t)) + C_* \mathcal{A}_1(t)(C_0 + \mathcal{A}_1(t)) \\ & \quad + C_* \int_0^t \sigma^3 \|\nabla \dot{u}\|_{L^3(\mathbb{R}^2)}^3. \end{aligned} \quad (3.29)$$

Finally, owing to Hölder's inequality the remaining term is bounded as

$$\begin{aligned}
 \left| \int_0^t \sigma^2 I_3(s) ds \right| &\leq C_* \int_0^t \sigma^2 \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)} (\|\nabla u\|_{L^6(\mathbb{R}^2)}^3 + \|P(\rho) - \tilde{P}\|_{L^6(\mathbb{R}^2)}^3) \\
 &\leq C_* \left[\int_0^t \sigma \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^2 \right]^{1/2} \left[\int_0^t \sigma^3 (\|\nabla u\|_{L^6(\mathbb{R}^2)}^6 + \|P(\rho) - \tilde{P}\|_{L^6(\mathbb{R}^2)}^6) \right]^{1/2} \\
 &\leq C_* \mathcal{A}_2(t) + C_* \int_0^t \sigma^3 (\|\nabla u\|_{L^6(\mathbb{R}^2)}^6 + \|P(\rho) - \tilde{P}\|_{L^6(\mathbb{R}^2)}^6).
 \end{aligned}$$

Assuming the smallness condition for the viscosity in (3.10) holds for $l = 5$, we multiply (3.11) by σ , integrate over time, and using (3.8), we arrive at

$$\begin{aligned}
 &\int_0^t \sigma (\|\nabla u\|_{L^6(\mathbb{R}^2)}^6 + \|P(\rho) - \tilde{P}\|_{L^6(\mathbb{R}^2)}^6) \\
 &\leq C_* C_0 + C_* \int_0^t \sigma (\|(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})\|_{L^6(\mathbb{R}^2)}^6 + \|(-\Delta)^{-1} \operatorname{curl}(\rho \dot{u})\|_{L^6(\mathbb{R}^2)}^6) \\
 &\leq C_* C_0 + C_* \int_0^t \sigma \|\rho \dot{u}\|_{L^2(\mathbb{R}^2)}^4 (\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + \|P(\rho) - \tilde{P}\|_{L^2(\mathbb{R}^2)}^2) \\
 &\leq C_* C_0 + C_* \mathcal{A}_1(t) \mathcal{A}_2(t) (\mathcal{A}_1(t) + C_0). \tag{3.30}
 \end{aligned}$$

With this, the paragraph is concluded, and the next step is to derive an $L^3((0, t) \times \mathbb{R}^2)$ -norm estimate for $\sigma \nabla \dot{u}$ as it appears in (3.29).

$L^3((0, t) \times \mathbb{R}^2)$ -norm estimate for $\sigma \nabla \dot{u}$. The approach is similar to what was done previously to estimate the $L^4((0, T) \times \mathbb{R}^2)$ -norm of $\sigma^{1/4} \nabla u$. We rewrite (3.23):

$$\rho \ddot{u}^j = \partial_k (\dot{\Pi}^{jk}) + \partial_k (\Pi^{jk} \operatorname{div} u) - \operatorname{div}(\partial_k u \Pi^{jk}), \tag{3.31}$$

and by applying the divergence operator, we express F_* , defined as

$$F_* := (2\mu(\rho) + \lambda(\rho)) \operatorname{div} \dot{u} - \lambda(\rho) \nabla u^l \partial_l u - \rho \lambda'(\rho) (\operatorname{div} u)^2 + \rho P'(\rho) \operatorname{div} u$$

in the following form:

$$\begin{aligned}
 F_* &= -(-\Delta)^{-1} \operatorname{div}(\rho \ddot{u}) + (-\Delta)^{-1} \partial_{jk} (\Pi^{jk} \operatorname{div} u) - (-\Delta)^{-1} \partial_j \operatorname{div}(\partial_k u \Pi^{jk}) \\
 &\quad - (-\Delta)^{-1} \partial_{jk} (\mu(\rho) \partial_j u^l \partial_l u^k + \mu(\rho) \partial_k u^l \partial_l u^j + 2\rho \mu'(\rho) \mathbb{D}^{jk} u \operatorname{div} u) \\
 &\quad + [K, \mu(\rho) - \tilde{\mu}] \mathbb{D} \dot{u}. \tag{3.32}
 \end{aligned}$$

On the other hand, by applying the rotational operator to (3.31), we obtain that $\operatorname{curl} \dot{u}$ reads

$$\begin{aligned}
 \mu(\rho) \operatorname{curl} \dot{u} &= -(-\Delta)^{-1} \operatorname{curl}(\rho \ddot{u}) + K' \{(\mu(\rho) - \tilde{\mu}) \mathbb{D} \dot{u}\} \\
 &\quad + (-\Delta)^{-1} \operatorname{curl} \partial_k (\Pi^{jk} \operatorname{div} u) - (-\Delta)^{-1} \operatorname{curl} \operatorname{div}(\partial_k u \Pi^{jk}) \\
 &\quad - (-\Delta)^{-1} \operatorname{curl} \partial_k (\mu(\rho) \partial_j u^l \partial_l u^k + \mu(\rho) \partial_k u^l \partial_l u^j \\
 &\quad \quad \quad + 2\rho \mu'(\rho) \mathbb{D}^{jk} u \operatorname{div} u). \tag{3.33}
 \end{aligned}$$

Similarly to the argument leading to (3.8), we deduce

$$\begin{aligned} \|\nabla \dot{u}\|_{L^3(\mathbb{R}^2)} &\leq C_* \|(-\Delta)^{-1} \operatorname{div}(\rho \ddot{u}), (-\Delta)^{-1} \operatorname{curl}(\rho \ddot{u})\|_{L^3(\mathbb{R}^2)} \\ &\quad + C_* \|\nabla u, P(\rho) - \tilde{P}\|_{L^6(\mathbb{R}^2)}^2 + C_* \|\nabla u\|_{L^3(\mathbb{R}^2)}, \end{aligned}$$

provided that (3.7) holds true for $p = 3$, and hence

$$\begin{aligned} \int_0^t \sigma^{5/2} \|\nabla \dot{u}\|_{L^3(\mathbb{R}^2)}^3 &\leq C_* \int_0^t \sigma^{5/2} \|(-\Delta)^{-1} \operatorname{div}(\rho \ddot{u}), (-\Delta)^{-1} \operatorname{curl}(\rho \ddot{u})\|_{L^3(\mathbb{R}^2)}^3 \\ &\quad + C_* \int_0^t \sigma^{5/2} \|\nabla u\|_{L^3(\mathbb{R}^2)}^3 \\ &\quad + C_* \int_0^t \sigma^{5/2} (\|\nabla u\|_{L^6(\mathbb{R}^2)}^6 + \|P(\rho) - \tilde{P}\|_{L^6(\mathbb{R}^2)}^6). \end{aligned}$$

Then the last two terms in the inequality above are bounded in (3.30)–(3.14)–(3.15). We use Gagliardo–Nirenberg’s inequality along with (3.32) and (3.33), to estimate $\|(-\Delta)^{-1} \operatorname{div}(\rho \ddot{u})\|_{L^2(\mathbb{R}^2)}$ and $\|(-\Delta)^{-1} \operatorname{curl}(\rho \ddot{u})\|_{L^2(\mathbb{R}^2)}$, as well as (3.13)–(3.28), we obtain

$$\begin{aligned} &\int_0^t \sigma^{5/2} \|(-\Delta)^{-1} \operatorname{div}(\rho \ddot{u}), (-\Delta)^{-1} \operatorname{curl}(\rho \ddot{u})\|_{L^3(\mathbb{R}^2)}^3 \\ &\leq C_* \int_0^t \sigma^{5/2} \|\rho \ddot{u}\|_{L^2(\mathbb{R}^2)} \|(-\Delta)^{-1} \operatorname{div}(\rho \ddot{u}), (-\Delta)^{-1} \operatorname{div}(\rho \ddot{u})\|_{L^2(\mathbb{R}^2)}^2 \\ &\leq C_* \int_0^t \sigma^{5/2} \|\rho \ddot{u}\|_{L^2(\mathbb{R}^2)} (\|\nabla \dot{u}, \nabla u\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla u, P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^2)}^4) \\ &\leq \eta \int_0^t \sigma^2 \|\sqrt{\rho} \ddot{u}\|_{L^2(\mathbb{R}^2)}^2 + \frac{C_*}{\eta} \int_0^t \sigma^3 (\|\nabla \dot{u}, \nabla u\|_{L^2(\mathbb{R}^2)}^4 + \|\nabla u, P(\rho) - \tilde{P}\|_{L^4(\mathbb{R}^2)}^8) \\ &\leq \eta \int_0^t \sigma^2 \|\sqrt{\rho} \ddot{u}\|_{L^2(\mathbb{R}^2)}^2 + \frac{C_*}{\eta} (\mathcal{A}_3(t) \mathcal{A}_2(t) + C_0 \mathcal{A}_1(t)) \\ &\quad + \frac{C_*}{\eta} (C_0 + \mathcal{A}_2(t)(C_0 + \mathcal{A}_1(t)))(C_0 + \mathcal{A}_1(t)(C_0 + \mathcal{A}_1(t))). \end{aligned}$$

Finally, (2.21) is derived by summing all the preceding computations and choosing η small. \blacksquare

3.4. Proof of Lemma 2.8

Proof. In this section we will derive an estimate for the functional θ . We start by estimating the characteristics of the interface.

Estimates for the $\ell_{\varphi(t)}^{-1}$ and $\mathfrak{B}_\gamma(t)$. We recall the definition of $\ell_{\varphi(t)}$:

$$\ell_{\varphi(t)} = \min \left\{ 1, \left(\frac{|\nabla \varphi(t)|_{\inf}}{\|\nabla \varphi(t)\|_{\dot{C}^\alpha}} \right)^{1/\alpha} \right\},$$

where the level-set function $\varphi = \varphi(t)$ satisfies

$$\begin{cases} \partial_t \varphi + u \cdot \nabla \varphi = 0, \\ \varphi|_{t=0} = \varphi_0. \end{cases}$$

From this, we directly derive (2.14)₄ and (2.15)₂. Moreover, (2.19) implies

$$|\nabla \varphi(t)|_{\inf} \geq |\nabla \varphi_0|_{\inf} \exp\left(-\varepsilon t - \frac{C}{\varepsilon} \theta(t)\right),$$

and

$$\begin{aligned} \|\nabla \varphi(t)\|_{\dot{c}^\alpha} &\leq \left[\|\nabla \varphi_0\|_{\dot{c}^\alpha} + \|\nabla \varphi_0\|_{L^\infty} \theta(t)^{\frac{1}{4}} \left(\int_0^t \sigma^{-\frac{r\alpha}{3}} \right)^{3/4} \right] \exp\left(\varepsilon t + \frac{C}{\varepsilon} \theta(t)\right) \\ &\leq \left[\|\nabla \varphi_0\|_{\dot{c}^\alpha} + \|\nabla \varphi_0\|_{L^\infty} \left(\varepsilon t + \frac{C}{\varepsilon} \theta(t)\right) \right] \exp\left(\varepsilon t + \frac{C}{\varepsilon} \theta(t)\right) \\ &\leq \|\nabla \varphi_0\|_{c^\alpha} \exp\left(2\varepsilon t + \frac{2C}{\varepsilon} \theta(t)\right). \end{aligned}$$

In total, there exists a constant $C_{\varphi_0} > 0$ depending on the regularity of φ_0 , such that

$$\ell_{\varphi(t)}^{-\alpha} \leq C_{\varphi_0} \exp\left(3\varepsilon t + \frac{3C}{\varepsilon} \theta(t)\right). \quad (3.34)$$

This completes the estimate for $\ell_{\varphi(t)}^{-1}$, and we now proceed to the estimate for $\mathfrak{F}_{\gamma(t)}$. First, recall that

$$\mathfrak{F}_{\gamma(t)} = (1 + |\mathcal{C}(t)|) \mathfrak{F}(\|\nabla \gamma(t)\|_{L^\infty} + c_{\gamma(t)}) \|\nabla \gamma(t)\|_{\dot{c}^\alpha},$$

where \mathfrak{F} is a given polynomial. By combining (2.14)–(2.15) with the computations leading to (3.34), we obtain

$$\begin{aligned} \mathfrak{F}_{\gamma(t)} &\leq (1 + |\mathcal{C}(0)|) \mathfrak{F}(\|\nabla \gamma_0\|_{L^\infty} + c_{\gamma_0}) \\ &\quad \times \left(\|\nabla \gamma_0\|_{\dot{c}^\alpha} + \|\nabla \gamma_0\|_{L^\infty}^{1+\alpha} \int_0^t \|\nabla u(\tau)\|_{\dot{c}_{pw,\gamma(\tau)}^\alpha} d\tau \right) \exp\left(C \int_0^t \|\nabla u\|_{L^\infty(\mathbb{R}^2)}\right), \end{aligned}$$

which simplifies to

$$\mathfrak{F}_{\gamma(t)} \leq C_{\gamma_0} \exp\left(3\varepsilon t + \frac{3C}{\varepsilon} \theta(t)\right), \quad (3.35)$$

where C_{γ_0} is a constant that depends on the regularity of γ_0 . We now turn to the estimate for the effective flux.

Estimate for F . Let us begin by recalling

$$F = -(-\Delta)^{-1} \operatorname{div}(\rho \dot{u}) + [K, \mu(\rho) - \tilde{\mu}] \mathbb{D}u, \quad (3.36)$$

along with the estimate for the last term above derived in (2.13), which implies

$$\begin{aligned} \int_0^t \sigma^{r\alpha} \|[K, \mu(\rho) - \tilde{\mu}]\mathbb{D}u\|_{\mathcal{C}_{pw,\gamma}^\alpha(\mathbb{R}^2)}^4 &\leq C_*\theta(t) \left(\theta(t) + \int_0^t \sigma \|\nabla u\|_{L^4(\mathbb{R}^2)}^4 \right) \\ &+ C_*\theta(t) \int_0^t [\ell_{\varphi(\tau)}^{-1} \|\nabla u(\tau)\|_{L^4(\mathcal{E}(\tau))}^4 + (\ell_{\varphi(\tau)}^{-4\alpha} + \mathfrak{F}_{\gamma(\tau)}^4) \|\nabla u(\tau)\|_{L^\infty(\mathcal{E}(\tau))}^4] d\tau \\ &+ C\theta(t) \sup_{[0,t]} [\ell_{\varphi}^{-1} \|\mu(\rho)\|_{L^4(\mathcal{E})}^4 + (\ell_{\varphi}^{-4\alpha} + \mathfrak{F}_{\gamma}^4) \|\mu(\rho)\|_{L^\infty(\mathcal{E})}^4]. \end{aligned} \quad (3.37)$$

From (2.17), we find

$$\begin{cases} \|\mu(\rho(\tau))\|_{L^4(\mathcal{E}(\tau))} \leq C_* \|f(\rho_0)\|_{L^4(\mathcal{E}(0))} \exp\left(-\frac{\nu}{2}\tau + C_*\theta(\tau)\right), \\ \|\mu(\rho(\tau))\|_{L^\infty(\mathcal{E}(\tau))} \leq C_* \|f(\rho_0)\|_{L^\infty(\mathcal{E}(0))} \exp\left(-\frac{\nu}{2}\tau + C_*\theta(\tau)\right), \end{cases}$$

and hence (see (3.34)–(3.35))

$$\begin{aligned} &[\ell_{\varphi(\tau)}^{-1} \|\mu(\rho(\tau))\|_{L^4(\mathcal{E}(\tau))}^4 + (\ell_{\varphi(\tau)}^{-4\alpha} + \mathfrak{F}_{\gamma(\tau)}^4) \|\mu(\rho(\tau))\|_{L^\infty(\mathcal{E}(\tau))}^4] \\ &\leq C_* C_{\varphi_0}^{\frac{1}{\alpha}} \|f(\rho_0)\|_{L^4(\mathcal{E}(0))}^4 \exp\left[\left(\frac{3\varepsilon}{\alpha} - 2\nu\right)\tau + \frac{C_*}{\varepsilon}\theta(\tau)\right] \\ &\quad + C_*(C_{\varphi_0}^4 + C_{\gamma_0}^4) \|f(\rho_0)\|_{L^\infty(\mathcal{E}(0))}^4 \exp\left[(12\varepsilon - 2\nu)\tau + \frac{C_*}{\varepsilon}\theta(\tau)\right]. \end{aligned}$$

By setting $\varepsilon = \alpha\nu/6$, it follows that

$$\begin{aligned} &\sup_{[0,t]} [\ell_{\varphi}^{-1} \|\mu(\rho)\|_{L^4(\mathcal{E})}^4 + (\ell_{\varphi}^{-4\alpha} + \mathfrak{F}_{\gamma}^4) \|\mu(\rho)\|_{L^\infty(\mathcal{E})}^4] \\ &\leq C_* K_0 e^{C_*\theta(t)} \|f(\rho_0)\|_{L^4 \cap L^\infty(\mathcal{E}(0))}^4, \end{aligned} \quad (3.38)$$

where K_0 depends polynomially on C_{γ_0} and C_{φ_0} . Following the same computations, we find from (2.18)–(3.34)–(3.35),

$$\begin{aligned} &\int_0^t [\ell_{\varphi(\tau)}^{-1} \|\nabla u(\tau)\|_{L^4(\mathcal{E}(\tau))}^4 + (\ell_{\varphi(\tau)}^{-4\alpha} + \mathfrak{F}_{\gamma(\tau)}^4) \|\nabla u(\tau)\|_{L^\infty(\mathcal{E}(\tau))}^4] d\tau \\ &\leq C_* K_0 e^{C_*\theta(t)} \|f(\rho_0)\|_{L^4 \cap L^\infty(\mathcal{E}(0))}^4. \end{aligned} \quad (3.39)$$

Summing (3.37)–(3.38)–(3.39) and (3.13), we have the following estimate for the last term of (3.36):

$$\begin{aligned} &\int_0^t \sigma^{r\alpha} \|[K, \mu(\rho) - \tilde{\mu}]\mathbb{D}u\|_{\mathcal{C}_{pw,\gamma}^\alpha(\mathbb{R}^2)}^4 \\ &\leq C_*\theta(t)(C_0 + \theta(t) + \mathcal{A}_1(t)^2) + C_* K_0 \theta(t) e^{C_*\theta(t)} \|f(\rho_0)\|_{L^4 \cap L^\infty(\mathcal{E}(0))}^4. \end{aligned} \quad (3.40)$$

Next we estimate the first term of (3.36) in terms of the functionals \mathcal{A}_1 , \mathcal{A}_2 , and \mathcal{A}_3 . Indeed, the embedding inequality implies

$$\begin{aligned} \int_0^t \sigma^{1+2\alpha} \|(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})\|_{\dot{C}^\alpha(\mathbb{R}^2)}^4 &\leq C \int_0^t \sigma^{1+2\alpha} \|\rho \dot{u}\|_{L^{2/(1-\alpha)}(\mathbb{R}^2)}^4 \\ &\leq C_* \int_0^t \sigma^{1+2\alpha} \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^{4\alpha} \|\dot{u}\|_{L^2(\mathbb{R}^2)}^{4(1-\alpha)} \\ &\leq C_* \left(\int_0^t \sigma^3 \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^4 \right)^\alpha \left(\int_0^t \sigma \|\dot{u}\|_{L^2(\mathbb{R}^2)}^4 \right)^{1-\alpha}, \end{aligned}$$

and whence

$$\int_0^t \sigma^{1+2\alpha} \|(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})\|_{\dot{C}^\alpha(\mathbb{R}^2)}^4 \leq C_* (\mathcal{A}_1(t)^2 + \mathcal{A}_2(t)^2 + \mathcal{A}_3(t)^2). \quad (3.41)$$

Secondly, we have

$$\begin{aligned} \int_0^t \sigma \|(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})\|_{L^\infty(\mathbb{R}^2)}^4 &\leq C \int_0^t \sigma \|\rho \dot{u}\|_{L^3(\mathbb{R}^2)}^3 \|(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})\|_{L^2(\mathbb{R}^2)} \\ &\leq C_* \int_0^t \sigma \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)} \|\dot{u}\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u, P(\rho) - \tilde{P}\|_{L^2(\mathbb{R}^2)} \\ &\leq C_* \left[\int_0^t \sigma \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)}^2 \right]^{\frac{1}{2}} \left[\int_0^t \sigma \|\dot{u}\|_{L^2(\mathbb{R}^2)}^4 \|\nabla u, P(\rho) - \tilde{P}\|_{L^2(\mathbb{R}^2)}^2 \right]^{\frac{1}{2}} \\ &\leq C_* (C_0 + \mathcal{A}_1(t)^2 + \mathcal{A}_2(t)^2). \end{aligned} \quad (3.42)$$

Summing (3.36)–(3.40)–(3.41)–(3.42), we conclude that for $r_\alpha = 1 + 2\alpha$,

$$\begin{aligned} \int_0^t \sigma^{r_\alpha} \|F\|_{\dot{C}^{\alpha, \gamma}(\mathbb{R}^2)}^4 &\leq C_* K_0 e^{C_* \theta(t)} \|f(\rho_0)\|_{L^4 \cap L^\infty(\mathcal{E}(0))}^4 \\ &\quad + C_* [C_0 + \mathcal{A}_1(t)^2 + \mathcal{A}_2(t)^2 + \mathcal{A}_3(t)^2 + \theta(t)(C_0 + \theta(t) + \mathcal{A}_1(t)^2)]. \end{aligned} \quad (3.43)$$

This completes the estimate for the effective flux, from which we derive the pressure estimate.

Estimate for the pressure. We begin by recalling equation (3.20), which provides us with the following expression:

$$f(\rho(\tau, X(\tau, x))) = f(\rho_0(x)) e^{-\int_0^\tau g_1(\tau', x) d\tau'} - \int_0^\tau e^{-\int_\tau^\tau g_1(\tau'', x) d\tau''} F(\tau', X(\tau', x)) d\tau',$$

where g_1 is defined as

$$g_1(t, x) := \frac{P(\rho(t, X(t, x))) - \tilde{P}}{f(\rho(t, X(t, x)))} \in [\underline{\nu}, \bar{\nu}].$$

This leads to the following bound:

$$\|f(\rho(\tau))\|_{L^\infty(\mathbb{R}^2)} \leq e^{-\underline{\nu}\tau} \|f(\rho_0)\|_{L^\infty(\mathbb{R}^2)} + \int_0^\tau e^{\underline{\nu}(\tau'-\tau)} \|F(\tau')\|_{L^\infty(\mathbb{R}^2)} d\tau'.$$

We express the last term above as

$$\begin{aligned} \int_0^\tau e^{\nu(\tau'-\tau)} \|F(\tau')\|_{L^\infty(\mathbb{R}^2)} d\tau' &= \int_0^\tau e^{\nu(\tau'-\tau)} \|F(\tau')\|_{L^\infty(\mathbb{R}^2)} \mathbb{1}_{\{\tau' < 1\}} d\tau' \\ &\quad + \int_0^\tau e^{\nu(\tau'-\tau)} \|F(\tau')\|_{L^\infty(\mathbb{R}^2)} \mathbb{1}_{\{\tau' > 1\}} d\tau', \end{aligned}$$

where the first term is bounded as

$$\int_0^\tau e^{\nu(\tau'-\tau)} \|F(\tau')\|_{L^\infty(\mathbb{R}^2)} \mathbb{1}_{\{\tau' < 1\}} d\tau' \leq \left(\frac{3}{3-r_\alpha}\right)^{3/4} \|\sigma^{r_\alpha/4} F\|_{L^4((0,\sigma(t)), L^\infty(\mathbb{R}^2))},$$

and the second term is bounded as

$$\int_0^\tau e^{\nu(\tau'-\tau)} \|F(\tau')\|_{L^\infty(\mathbb{R}^2)} \mathbb{1}_{\{\tau' > 1\}} d\tau' \leq \left(\frac{3}{4\nu}\right)^{3/4} \|F\|_{L^4((\sigma(t),t), L^\infty(\mathbb{R}^2))}.$$

Thus, we obtain

$$\sup_{[0,t]} \|f(\rho)\|_{L^\infty(\mathbb{R}^2)}^4 \leq \|f(\rho_0)\|_{L^\infty(\mathbb{R}^2)}^4 + C_* \int_0^t \sigma^{r_\alpha} \|F\|_{L^\infty(\mathbb{R}^2)}^4. \quad (3.44)$$

Moreover, by applying Young's inequality for convolution, we have

$$\begin{aligned} \left\| \int_0^\tau e^{\nu(\tau'-\tau)} \|F(\tau')\|_{L^\infty(\mathbb{R}^2)} \mathbb{1}_{\{\tau' < 1\}} d\tau' \right\|_{L^4(0,t)} \\ \leq \left(\frac{1}{4\nu}\right)^{1/4} \left(\frac{3}{3-r_\alpha}\right)^{3/4} \|\sigma^{r_\alpha/4} F\|_{L^4((0,\sigma(t)), L^\infty(\mathbb{R}^2))}, \end{aligned}$$

and

$$\left\| \int_0^\tau e^{\nu(\tau'-\tau)} \|F(\tau')\|_{L^\infty(\mathbb{R}^2)} \mathbb{1}_{\{\tau' > 1\}} d\tau' \right\|_{L^4(0,t)} \leq \frac{1}{\nu} \|F\|_{L^4((\sigma(t),t), L^\infty(\mathbb{R}^2))}.$$

As a result, we obtain

$$\int_0^t \|f(\rho(\tau))\|_{L^\infty(\mathbb{R}^2)}^4 d\tau \leq C_* \left(\|f(\rho_0)\|_{L^\infty(\mathbb{R}^2)}^4 + \int_0^t \sigma^{r_\alpha} \|F(\tau)\|_{L^\infty(\mathbb{R}^2)}^4 d\tau \right). \quad (3.45)$$

With the $L^\infty(\mathbb{R}^2)$ -norm estimate for $f(\rho)$ now complete, we proceed to estimating the $\dot{C}_{\text{pw},\gamma}^\alpha(\mathbb{R}^2)$ -norm of $f(\rho)$. To this end, we consider two points x_j^0 , $j \in \{1, 2\}$, located on the same side of the interface, and define $x_j(t) = X(t, x_j^0)$. We infer from (3.20) that

$$\begin{aligned} \frac{d}{dt} f(\rho(t, x_j(t))) \Big|_{j=1}^{j=2} + f(\rho(t, x_j(t))) \Big|_{j=1}^{j=2} g_2(t, x_1(t), x_2(t)) \\ = -F(t, x_j(t)) \Big|_{j=1}^{j=2}, \end{aligned}$$

where g_2 is

$$g_2(t, x, y) := \frac{P(\rho(t, x)) - P(\rho(t, y))}{f(\rho(t, x)) - f(\rho(t, y))} \in [\underline{\nu}, \bar{\nu}].$$

Integrating this differential equation yields

$$\begin{aligned} f(\rho(\tau, x_j(\tau))) \Big|_{j=1}^{j=2} &= f(\rho_0(x_j^0)) \Big|_{j=1}^{j=2} e^{-\int_0^\tau g_2(\tau', x_2(\tau'), x_1(\tau')) d\tau'} \\ &\quad - \int_0^\tau e^{-\int_{\tau'}^\tau g_2(\tau'', x_2(\tau''), x_1(\tau'')) d\tau''} F(\tau', x_j(\tau')) \Big|_{j=1}^{j=2} d\tau'. \end{aligned} \quad (3.46)$$

It is straightforward to obtain, for all $0 \leq \tau' \leq \tau$,

$$|x_2(\tau') - x_1(\tau')| \leq e^{\int_{\tau'}^\tau \|\nabla u(\tau'')\|_{L^\infty(\mathbb{R}^2)} d\tau''} |x_2(\tau) - x_1(\tau)|,$$

and hence (3.46) implies

$$\begin{aligned} \|f(\rho(\tau))\|_{\dot{C}_{pw, \gamma(\tau)}^\alpha(\mathbb{R}^2)} &\leq e^{-\underline{\nu}\tau + \int_0^\tau \|\nabla u(\tau')\|_{L^\infty(\mathbb{R}^2)} d\tau'} \|f(\rho_0)\|_{\dot{C}_{pw, \gamma_0}^\alpha(\mathbb{R}^2)} \\ &\quad + \int_0^\tau e^{-\underline{\nu}(\tau-\tau') + \int_{\tau'}^\tau \|\nabla u(\tau'')\|_{L^\infty(\mathbb{R}^2)} d\tau''} \|F(\tau')\|_{\dot{C}_{pw, \gamma(\tau')}^\alpha(\mathbb{R}^2)} d\tau'. \end{aligned}$$

Given that for all $0 \leq s' < s$, the inequality

$$\int_{s'}^s \|\nabla u(\tau'')\|_{\dot{C}_{pw, \gamma(\tau'')}^\alpha(\mathbb{R}^2)} d\tau'' \leq \frac{1}{2\underline{\nu}}(s - s') + C_*\theta(s),$$

holds, we infer

$$\begin{aligned} \|f(\rho(\tau))\|_{\dot{C}_{pw, \gamma(\tau)}^\alpha(\mathbb{R}^2)} &\leq e^{C_*\theta(\tau)} \left[e^{-\frac{\underline{\nu}}{2}\tau} \|f(\rho_0)\|_{\dot{C}_{pw, \gamma_0}^\alpha(\mathbb{R}^2)} + \int_0^\tau e^{-\frac{\underline{\nu}}{2}(\tau-\tau')} \|F(\tau')\|_{\dot{C}_{pw, \gamma(\tau')}^\alpha(\mathbb{R}^2)} d\tau' \right]. \end{aligned}$$

From this, we deduce, following the computations leading to (3.44) and (3.45), that

$$\begin{aligned} \sup_{[0, t]} \|f(\rho)\|_{\dot{C}_{pw, \gamma}^\alpha(\mathbb{R}^2)}^4 &+ \int_0^t \|f(\rho)\|_{\dot{C}_{pw, \gamma}^\alpha(\mathbb{R}^2)}^4 \\ &\leq C_* e^{C_*\theta(t)} \left(\|f(\rho_0)\|_{\dot{C}_{pw, \gamma_0}^\alpha(\mathbb{R}^2)}^4 + \int_0^t \sigma^{r\alpha} \|F\|_{\dot{C}_{pw, \gamma}^\alpha(\mathbb{R}^2)}^4 \right). \end{aligned} \quad (3.47)$$

Finally, the estimate for the pressure follows from (3.44), (3.45), and (3.47), and we now turn to the final step devoted to the velocity gradient.

Final estimates. We start with the following expression for the velocity gradient derived in (2.20):

$$\nabla u = \nabla u_* + \nabla u_F + \nabla u_P + \nabla u_\delta.$$

The estimates derived for $(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})$ in (3.41)–(3.42) and for $[K, \mu(\rho) - \tilde{\mu}] \mathbb{D}u$ in (3.40) apply to ∇u_δ and ∇u_* , respectively. We now focus on the $L^4((0, t), \mathcal{C}_{\text{pw}, \gamma}^\alpha(\mathbb{R}^2))$ -norm estimates of ∇u_P and ∇u_F . We first recall

$$\nabla u_P = -\nabla(-\Delta)^{-1} \nabla(\hat{\psi}_1(\rho)(P(\rho) - \tilde{P})), \quad \text{with } \hat{\psi}_1(\rho) = \frac{2\mu(\rho) - \tilde{\mu}}{2\mu(\rho) + \lambda(\rho)}.$$

By applying Proposition 2.5, we obtain

$$\begin{aligned} \|\nabla u_P\|_{\mathcal{C}_{\text{pw}, \gamma}^\alpha(\mathbb{R}^2)} &\leq C(\|\hat{\psi}_1(\rho)(P(\rho) - \tilde{P})\|_{L^4(\mathbb{R}^2)} + \|\hat{\psi}_1(\rho)(P(\rho) - \tilde{P})\|_{\mathcal{C}_{\text{pw}, \gamma}^\alpha(\mathbb{R}^2)}) \\ &\quad + C\ell_\varphi^{-\frac{1}{4}}\|\llbracket \hat{\psi}_1(\rho)(P(\rho) - \tilde{P}) \rrbracket\|_{L^4(\mathcal{E})} \\ &\quad + C\|\llbracket \hat{\psi}_1(\rho)(P(\rho) - \tilde{P}) \rrbracket\|_{L^\infty(\mathcal{E})}(\ell_\varphi^{-\alpha} + \mathfrak{B}_\gamma). \end{aligned}$$

Next, using (3.13) along with the previous step, we obtain

$$\begin{aligned} &\int_0^t \sigma^{r_\alpha} [\|\hat{\psi}_1(\rho)(P(\rho) - \tilde{P})\|_{L^4(\mathbb{R}^2)}^4 + \|\hat{\psi}_1(\rho)(P(\rho) - \tilde{P})\|_{\mathcal{C}_{\text{pw}, \gamma}^\alpha(\mathbb{R}^2)}^4] d\tau \\ &\leq C_*(C_0 + \mathcal{A}_1(t)^2) \\ &\quad + C_*(1 + \theta(t))e^{C_*\theta(t)} \left(\|f(\rho_0)\|_{\mathcal{C}_{\text{pw}, \gamma_0}^\alpha(\mathbb{R}^2)}^4 + \int_0^t \sigma^{r_\alpha} \|F\|_{\mathcal{C}_{\text{pw}, \gamma}^\alpha(\mathbb{R}^2)}^4 \right) \\ &\leq C_* \left[C_0 + \mathcal{A}_1(t)^2 + e^{C_*\theta(t)} \left(\|f(\rho_0)\|_{\mathcal{C}_{\text{pw}, \gamma_0}^\alpha(\mathbb{R}^2)}^4 + \int_0^t \sigma^{r_\alpha} \|F\|_{\mathcal{C}_{\text{pw}, \gamma}^\alpha(\mathbb{R}^2)}^4 \right) \right]. \end{aligned}$$

Following the computations leading to (3.39), we have

$$\begin{aligned} &\int_0^t \sigma^{r_\alpha} [\ell_\varphi^{-1} \|\llbracket \hat{\psi}_1(\rho)(P(\rho) - \tilde{P}) \rrbracket\|_{L^4(\mathcal{E})}^4 + \|\llbracket \hat{\psi}_1(\rho)(P(\rho) - \tilde{P}) \rrbracket\|_{L^\infty(\mathcal{E})}^4 (\ell_\varphi^{-4\alpha} + \mathfrak{B}_\gamma^4)] \\ &\leq C_* K_0 e^{C_*\theta(t)} \|\llbracket f(\rho_0) \rrbracket\|_{L^4 \cap L^\infty(\mathcal{E}(0))}^4. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} &\int_0^t \sigma^{r_\alpha} \|\nabla u_P(\tau)\|_{\mathcal{C}_{\text{pw}, \gamma(\tau)}^\alpha(\mathbb{R}^2)}^4 d\tau \leq C_*(C_0 + \mathcal{A}_1(t)^2) + e^{C_*\theta(t)} \|f(\rho_0)\|_{\mathcal{C}_{\text{pw}, \gamma_0}^\alpha(\mathbb{R}^2)}^4 \\ &\quad + C_* e^{C_*\theta(t)} \left(K_0 \|\llbracket f(\rho_0) \rrbracket\|_{L^4 \cap L^\infty(\mathcal{E}(0))}^4 + \int_0^t \sigma^{r_\alpha} \|F\|_{\mathcal{C}_{\text{pw}, \gamma}^\alpha(\mathbb{R}^2)}^4 \right). \quad (3.48) \end{aligned}$$

We now proceed to estimate ∇u_F , which is given by

$$\nabla u_F = \nabla(-\Delta)^{-1} \nabla(\hat{\psi}_2(\rho)F), \quad \text{with } \hat{\psi}_2(\rho) = \frac{\tilde{\mu} + \lambda(\rho)}{2\mu(\rho) + \lambda(\rho)}.$$

Once again, we apply Proposition 2.5 to obtain

$$\begin{aligned} \|\nabla u_F\|_{\mathcal{C}_{\text{pw}, \gamma}^\alpha(\mathbb{R}^2)} &\leq C(\|\hat{\psi}_2(\rho)F\|_{L^4(\mathbb{R}^2)} + \|\hat{\psi}_2(\rho)F\|_{\mathcal{C}_{\text{pw}, \gamma}^\alpha(\mathbb{R}^2)}) \\ &\quad + C\ell_\varphi^{-\frac{1}{4}}\|\llbracket \hat{\psi}_2(\rho)F \rrbracket\|_{L^4(\mathcal{E})} + C\|\llbracket \hat{\psi}_2(\rho)F \rrbracket\|_{L^\infty(\mathcal{E})}(\ell_\varphi^{-\alpha} + \mathfrak{B}_\gamma). \end{aligned}$$

Straightforwardly, we derive

$$\begin{aligned} & \int_0^t \sigma^{r\alpha} [\|\hat{\psi}_2(\rho)F\|_{L^4(\mathbb{R}^2)}^4 + \|\hat{\psi}_2(\rho)F\|_{\mathbb{C}_{pw,\gamma}^\alpha(\mathbb{R}^2)}^4] \\ & \leq C_* \left[C_0 + \mathcal{A}_1(t)^2 + (1 + \theta(t)) \int_0^t \sigma^{r\alpha} \|F\|_{\mathbb{C}_{pw,\gamma}^\alpha(\mathbb{R}^2)}^4 \right], \end{aligned}$$

and (see the expression of $\llbracket F \rrbracket$ in (1.17))

$$\|\llbracket \hat{\psi}_2(\rho)F \rrbracket\|_{L^4 \cap L^\infty(\mathcal{E})} \leq C_* \|\llbracket f(\rho) \rrbracket\|_{L^4 \cap L^\infty(\mathcal{E})} (\|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|F\|_{L^\infty(\mathbb{R}^2)}).$$

Following the computations leading to (3.38), we obtain

$$\begin{aligned} & \int_0^t \sigma^{r\alpha} [\ell_\varphi^{-1} \|\llbracket \hat{\psi}_2(\rho)F \rrbracket\|_{L^4(\mathcal{E})}^4 + \|\llbracket \hat{\psi}_2(\rho)F \rrbracket\|_{L^\infty(\mathcal{E})}^4 (\ell_\varphi^{-4\alpha} + \mathfrak{R}_\gamma^4)] \\ & \leq C_* K_0 e^{C_* \theta(t)} \|\llbracket f(\rho_0) \rrbracket\|_{L^4 \cap L^\infty(\mathcal{E}(0))}^4 \left(\theta(t) + \int_0^t \sigma^{r\alpha} \|F\|_{L^\infty(\mathbb{R}^2)}^4 \right), \end{aligned}$$

and finally

$$\begin{aligned} & \int_0^t \sigma^{r\alpha} \|\nabla u_F\|_{\mathbb{C}_{pw,\gamma}^\alpha(\mathbb{R}^2)}^4 \\ & \leq C_* (C_0 + \mathcal{A}_1(t)^2 + K_0 e^{C_* \theta(t)} \|\llbracket f(\rho_0) \rrbracket\|_{L^4 \cap L^\infty(\mathcal{E}(0))}^4) \\ & \quad + C_* [1 + \theta(t) + K_0 e^{C_* \theta(t)} \|\llbracket f(\rho_0) \rrbracket\|_{L^4 \cap L^\infty(\mathcal{E}(0))}^4] \int_0^t \sigma^{r\alpha} \|F\|_{\mathbb{C}_{pw,\gamma}^\alpha(\mathbb{R}^2)}^4. \end{aligned} \tag{3.49}$$

Lemma 2.8 follows by summing (3.43), (3.48), and (3.49). \blacksquare

3.5. Proof of Theorem 1.2

This section is devoted to the proof of the main result of this paper. It is structured into two steps: the first part, Section 3.5.1, focuses on the construction of a solution (ρ, u) to the Navier–Stokes equations (1.1), while the second part, Section 3.5.2, establishes uniqueness within a large space.

3.5.1. Proof of the existence. This section is dedicated to constructing a solution for the Navier–Stokes equations (1.1). It starts with the construction of an approximate sequence (ρ^δ, u^δ) and goes up to the convergence of this sequence to a limit (ρ, u) that solves the equations (1.1). Usually, (ρ^δ, u^δ) corresponds to the solution to the Cauchy problem (2.1) with initial data $(\rho_0^\delta, u_0^\delta)$, where $(\rho_0^\delta, u_0^\delta)$ is obtained by smoothing (ρ_0, u_0) . In our case, this does not seem a good idea, as smoothing the initial data would result in the loss of the density discontinuity. This motivates our local-in-time well-posedness result in [36]. Although a compatibility condition is required, the solution exhibits similar regularity to that of Theorem 1.2. In particular, the density and velocity gradients are discontinuous. We now begin the proof of the existence by constructing the sequence of initial data $(\rho_0^\delta, u_0^\delta)$.

Step 1: Construction of $(\rho_0^\delta, u_0^\delta)$. We initiate by identifying which initial quantity needs to be smoothed and which not. The initial interface γ_0 , density, and velocity, possess the necessary regularity as outlined in the local-in-time result, Theorem 2.1. Therefore, there is no need to smooth these quantities. However, the stress tensor at the initial time only fulfills

$$\operatorname{div}(\Pi_0) = \operatorname{div}(2\mu(\rho_0)\mathbb{D}u_0 + (\lambda(\rho_0)\operatorname{div}u_0 - P(\rho_0) + \tilde{P})I_2) \in H^{-1}(\mathbb{R}^2).$$

To preserve the discontinuity in the initial density, we consider (ρ_0, u_0^δ) (namely $\rho_0^\delta = \rho_0$) as a sequence of initial data, where u_0^δ solves the following elliptic equation:

$$-\operatorname{div}(2\mu(\rho_0)\mathbb{D}u_0^\delta + (\lambda(\rho_0)\operatorname{div}u_0^\delta - P(\rho_0) + \tilde{P})I_2) + c^\delta u_0^\delta = -\operatorname{div}(w_\delta * \Pi_0). \quad (3.50)$$

Above, $w_\delta := \delta^{-2}w(\cdot/\delta)$, $\delta \in (0, 1)$, where w is a smooth non-negative function supported in the unit ball centered at the origin, and whose integral equals 1. Additionally, the constant c^δ is defined as

$$c^\delta := \|w_\delta * \Pi_0 - \Pi_0\|_{L^2(\mathbb{R}^2)}, \quad \text{and satisfies } c^\delta \xrightarrow{\delta \rightarrow 0} 0.$$

Since the viscosity $2\mu(\rho_0) + \lambda(\rho_0)$ is bounded away from vacuum and from above, and the pressure $P(\rho_0) - \tilde{P}$ belongs to $L^2(\mathbb{R}^2)$, the existence of a unique solution $u_0^\delta \in H^1(\mathbb{R}^2)$ of (3.50) follows from the Lax–Milgram theorem. Moreover, the sequence $(u_0^\delta)_\delta$ satisfies the following estimate:

$$C_*^{-1} \|\nabla u_0^\delta\|_{L^2(\mathbb{R}^2)}^2 + c^\delta \|u_0^\delta\|_{L^2(\mathbb{R}^2)}^2 \leq C_*(\|\Pi_0\|_{L^2(\mathbb{R}^2)}^2 + \|P(\rho_0) - \tilde{P}\|_{L^2(\mathbb{R}^2)}^2). \quad (3.51)$$

We now move on to proving that $(u_0^\delta)_\delta$ converges strongly to u_0 in $H^1(\mathbb{R}^2)$.

We add $\operatorname{div}(\Pi_0)$ to both sides of (3.50) to obtain

$$-\operatorname{div}(2\mu(\rho_0)\mathbb{D}(u_0^\delta - u_0) + \lambda(\rho_0)\operatorname{div}(u_0^\delta - u_0)I_2) + c^\delta u_0^\delta = -\operatorname{div}(w_\delta * \Pi_0 - \Pi_0).$$

Using $u_0^\delta - u_0$ as a test function yields

$$\begin{aligned} & 2 \int_{\mathbb{R}^2} \mu(\rho_0) |\mathbb{D}(u_0^\delta - u_0)|^2 + \int_{\mathbb{R}^2} \lambda(\rho_0) |\operatorname{div}(u_0^\delta - u_0)|^2 + c^\delta \int_{\mathbb{R}^2} u_0^\delta (u_0^\delta - u_0) \\ &= \int_{\mathbb{R}^2} \nabla(u_0^\delta - u_0) : (w_\delta * \Pi_0 - \Pi_0), \end{aligned}$$

from which, with the help of Hölder's and Young's inequalities, we deduce

$$\begin{aligned} & C_*^{-1} \|\nabla(u_0^\delta - u_0)\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} c^\delta \|u_0^\delta\|_{L^2(\mathbb{R}^2)}^2 \\ & \leq C_* \|w_\delta * \Pi_0 - \Pi_0\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} c^\delta \|u_0\|_{L^2(\mathbb{R}^2)}^2. \end{aligned} \quad (3.52)$$

Now, given that $c^\delta \xrightarrow{\delta \rightarrow 0} 0$ (see (3.51) above), we immediately obtain

$$u_0^\delta \xrightarrow{\delta \rightarrow 0} u_0, \quad \text{in } \dot{H}^1(\mathbb{R}^2).$$

Furthermore, (3.52) implies that

$$\limsup_{\delta \rightarrow 0} \|u_0^\delta\|_{L^2(\mathbb{R}^2)}^2 \leq \|u_0\|_{L^2(\mathbb{R}^2)}^2,$$

and whence

$$u_0^\delta \xrightarrow{\delta \rightarrow 0} u_0, \quad \text{in } L^2(\mathbb{R}^2).$$

This proves the strong convergence of the sequence $(u_0^\delta)_\delta$ to u_0 in $H^1(\mathbb{R}^2)$. Additionally, as intended, we obtain from (3.50),

$$\begin{aligned} & \operatorname{div}(2\mu(\rho_0)\mathbb{D}u_0^\delta + (\lambda(\rho_0)\operatorname{div}u_0^\delta - P(\rho_0) + \tilde{P})I_2) \\ &= -c^\delta u_0^\delta + \operatorname{div}(w_\delta * \Pi_0) \in L^2(\mathbb{R}^2). \end{aligned} \quad (3.53)$$

Finally, considering the regularity of the initial density and interface γ_0 , the small perturbation assumption of the initial viscosity $\mu(\rho_0)$ around the constant state $\tilde{\mu}$, along with the regularity of u_0^δ and (3.53), Theorem 2.1 ensures the existence of a unique solution (ρ^δ, u^δ) for equations (2.1) with the initial conditions

$$\rho|_{t=0} = \rho_0 \quad \text{and} \quad u|_{t=0} = u_0^\delta.$$

The solution is defined up to a maximal time $T_\delta > 0$ and enjoys the regularity in Theorem 2.1. This regularity is sufficient for the computations carried out in the previous sections to make sense. As a consequence, Lemma 2.9 holds true for the solution (ρ^δ, u^δ) , as well as the first condition of the blow-up criterion, (2.3). For the second condition, (2.4), we use the exponential decay in time for jumps to derive, for all $t \in (0, T_\delta)$,

$$\begin{aligned} & (1 + \|\lambda(\rho(t))\|_{\dot{C}_{\text{pw},\gamma(t)}^\alpha(\mathbb{R}^2)}) \|\mu(\rho(t)) - \tilde{\mu}\|_{C_{\text{pw},\gamma(t)}^\alpha(\mathbb{R}^2)} \\ &+ (\mathfrak{B}_{\gamma(t)} + \ell_{\varphi(t)}^{-\alpha}) \|\llbracket \lambda(\rho(t)) \rrbracket\|_{L^\infty(\mathcal{E}(t))} \|\mu(\rho(t)) - \tilde{\mu}\|_{C_{\text{pw},\gamma(t)}^\alpha(\mathbb{R}^2)} \\ &+ (\mathfrak{B}_{\gamma(t)} + \ell_{\varphi(t)}^{-\alpha}) \|\llbracket \mu(\rho(t)) \rrbracket\|_{L^\infty(\mathcal{E}(t))} \\ &+ \|\llbracket \mu(\rho(t)) \rrbracket, \llbracket \lambda(\rho(t)) \rrbracket\|_{L^\infty(\mathcal{E}(t))} \left\| 1 - \frac{\tilde{\mu}}{\langle \mu(\rho(t)) \rangle} \right\|_{L^\infty(\mathcal{E}(t))} \\ &\leq C_* K_0 e^{C_* \theta(t)} \|\llbracket f(\rho_0) \rrbracket\|_{L^\infty(\mathcal{E}(0))} \\ &+ C_* (1 + \theta(t)^{\frac{1}{4}} + K_0 e^{C_* \theta(t)} \|\llbracket f(\rho_0) \rrbracket\|_{L^\infty(\mathcal{E}(0))}) \theta(t)^{\frac{1}{4}}. \end{aligned}$$

Due to the smallness of c_0 (see (1.23) above), (2.4) is satisfied, leading to $T_\delta = +\infty$. We now proceed to the final step, which focuses on showing that the sequence (ρ^δ, u^δ) converges to a pair (ρ, u) that solves (1.1).

Step 2: Convergence of the approximate sequence (ρ^δ, u^δ) . We recall that for all $\delta > 0$, the pair (ρ^δ, u^δ) satisfies the system

$$\begin{cases} \partial_t \rho^\delta + \operatorname{div}(\rho^\delta u^\delta) = 0, \\ \partial_t(\rho^\delta u^\delta) + \operatorname{div}(\rho^\delta u^\delta \otimes u^\delta) + \nabla P(\rho^\delta) = \operatorname{div}(2\mu(\rho^\delta)\mathbb{D}u^\delta) + \nabla(\lambda(\rho^\delta)\operatorname{div}u^\delta). \end{cases} \quad (3.54)$$

Additionally, for all $T \in (0, \infty)$ and $\delta \in (0, 1)$, we have

$$\begin{aligned} & \|\rho^\delta - \tilde{\rho}\|_{L^\infty((0,T),L^2(\mathbb{R}^2))}^2 + \|\rho^\delta - \tilde{\rho}\|_{L^\infty((0,T)\times\mathbb{R}^2)}^2 + \|u^\delta\|_{L^\infty((0,T),H^1(\mathbb{R}^2))}^2 \\ & + \|\nabla u^\delta\|_{L^3((0,T)\times\mathbb{R}^2)}^2 + \|\dot{u}^\delta\|_{L^2((0,T)\times\mathbb{R}^2)}^2 \leq C_{*,0}. \end{aligned} \quad (3.55)$$

Hereafter, $C_{*,0}$ is a positive constant that depends on C_* and c_0 . Sometimes we write $C_{*,0}(T)$ (resp. $C_{*,0}(n)$) to emphasize the additional dependence on $T > 0$ (resp. $n \in \mathbb{N}^*$). From (3.55), there exist $\rho - \tilde{\rho} \in L^\infty((0, \infty), L^2(\mathbb{R}^2)) \cap L^\infty((0, \infty) \times \mathbb{R}^2)$ and $u \in L^\infty((0, \infty), H^1(\mathbb{R}^2))$ such that

$$\begin{cases} \rho^\delta - \tilde{\rho} \rightharpoonup^* \rho - \tilde{\rho} & \text{in } L^\infty((0, T) \times \mathbb{R}^2), \\ \rho^\delta - \tilde{\rho} \rightarrow \rho - \tilde{\rho} & \text{strongly in } \mathcal{C}([0, T], L_w^2(\mathbb{R}^2)), \\ u^\delta \rightharpoonup^* u & \text{in } L^\infty((0, T), H^1(\mathbb{R}^2)), \\ u^\delta \rightarrow u & \text{strongly in } \mathcal{C}([0, T], L_{loc}^2(\mathbb{R}^2)). \end{cases}$$

Additionally, by interpolation we have

$$u^\delta \rightarrow u \text{ strongly in } L_{loc}^\infty((0, \infty), L_{loc}^p(\mathbb{R}^2)), \text{ for every } p \in [2, \infty).$$

The initial interface $\mathcal{C}(0)$ is transported by the flow X^δ associated with the velocity u^δ into an interface

$$\mathcal{C}^\delta(t) = X^\delta(t)\mathcal{C}(0), \quad \text{with parameterization } \gamma^\delta(t, s) = X^\delta(t, \gamma_0(s)). \quad (3.56)$$

Given that the velocity sequence (u^δ) satisfies

$$\sup_{(0, \infty)} \sigma \|\nabla u^\delta\|_{L^4(\mathbb{R}^2)}^4 + \int_0^\infty \sigma^{r_\alpha} \|\nabla u^\delta\|_{\mathcal{C}_{pw, \gamma^\delta}^\alpha(\mathbb{R}^2)}^4 \leq C_{*,0}, \quad (3.57)$$

we infer, together with (3.55), that for any $T > 0$ and $n \in \mathbb{N} \setminus \{0\}$,

$$\sup_{[0, T]} \|\nabla \gamma^\delta\|_{L^\infty} \leq C_{*,0}(T) \quad \text{and} \quad \sup_{[1/n, T]} \|\partial_t \gamma^\delta\|_{L^\infty} \leq C_{*,0}(n), \quad \text{for all } \delta > 0.$$

Hence, up to a subsequence, $(\gamma^\delta)_\delta$ converges uniformly on $[0, T] \times V$ to some $\gamma \in W^{1, \infty}((0, T) \times V)$. From (3.55) and the embedding $W^{1,3}(\mathbb{R}^2) \hookrightarrow \mathcal{C}^{1/3}(\mathbb{R}^2)$, the velocity sequence satisfies

$$\sup_\delta \|u^\delta\|_{L^3((0, \infty), \mathcal{C}^{1/3}(\mathbb{R}^2))} \leq C_{*,0}.$$

Therefore, taking the limit as $\delta \rightarrow 0$ in (3.56), we find that the limit parameterization γ satisfies

$$\gamma(t, s) = \gamma_0(s) + \int_0^t u(\tau, \gamma(\tau, s)) d\tau. \quad (3.58)$$

Furthermore, from (3.56)–(3.57), we obtain

$$\sup_\delta \int_0^T \sigma^{r_\alpha} \|\partial_t \nabla \gamma^\delta\|_{\mathcal{C}^\alpha}^4 \leq C_{*,0}(T).$$

Since $r_\alpha < 3$, for all $q \in (1, 4/(1 + r_\alpha))$ we have

$$\sup_\delta \int_0^T \|\partial_t \nabla \gamma^\delta\|_{\mathcal{C}^\alpha}^q \leq C_{*,0}(T, q).$$

Therefore, the lower semi-continuity of norms implies that $\partial_t \nabla \gamma \in L_{\text{loc}}^q([0, \infty), \mathcal{C}^\alpha(V))$, and as a result,

$$\gamma \in \mathcal{C}([0, \infty), \mathcal{C}^{1+\alpha}(V)).$$

Once more, (3.57) combined with the lower semi-continuity of norms implies that $\nabla u \in L_{\text{loc}}^q([0, \infty), L^\infty(\mathbb{R}^2))$. This guarantees the uniqueness of γ that satisfies (3.58). We now move on to the proof of the strong convergence of the density sequence $(\rho^\delta)_\delta$.

Arguing as above, we obtain that the sequence $(X^\delta)_\delta$ converges uniformly on compact sets in $[0, \infty) \times \mathbb{R}^2$ to $X \in \mathcal{C}([0, \infty), \mathcal{C}^{1+\alpha}(\mathbb{R}^2))$, the flow of the limit velocity u . We observe that the a priori estimate yields the following bound:

$$\sup_\delta \|\rho^\delta - \bar{\rho}\|_{L^\infty((0, \infty), \mathcal{C}_{\text{pw}, \gamma^\delta}^\alpha(\mathbb{R}^2))}^4 \leq C_{*,0}.$$

From this, combined with (3.57), we infer that for any $T > 0$,

$$\sup_\delta \|\varrho^\delta - \bar{\rho}\|_{L^\infty((0, T), \mathcal{C}_{\text{pw}, \gamma_0}^\alpha(\mathbb{R}^2))}^4 + \sup_\delta \|\partial_t \varrho^\delta\|_{L^q((0, T), L^\infty(\mathbb{R}^2))}^4 \leq C_{*,0}(T, q),$$

where ϱ^δ is given by

$$\varrho^\delta(t, x) = \rho^\delta(t, X^\delta(t, x)).$$

Since $\mathcal{C}_{\text{pw}, \gamma_0}^\alpha(\mathbb{R}^2)$ embeds compactly in $L_{\text{loc}}^\infty(\mathbb{R}^2)$, we deduce from the Aubin–Lions lemma that $(\varrho^\delta)_\delta$ converges uniformly on compact sets in $\{(t, x) : x \in \mathbb{R}^2 \setminus \mathcal{C}\{0\}\}$ to some $\varrho \in \mathcal{C}([0, \infty), \mathcal{C}_{\text{pw}, \gamma_0}^\alpha(\mathbb{R}^2))$. The final step is to prove that $(\rho^\delta)_\delta$ converges strongly to $\bar{\rho}$ in $L_{\text{loc}}^2([0, \infty) \times \mathbb{R}^2)$, where

$$\bar{\rho}(t, x) = \varrho(t, X^{-1}(t, x)).$$

Above, $X^{-1} \in \mathcal{C}([0, \infty), \mathcal{C}^{1+\alpha}(\mathbb{R}^2))$ satisfies

$$\begin{aligned} X^{-1}(t, x) &= x - \int_0^t u(\tau, X^{-1}(\tau, x)) d\tau, \\ X(t, X^{-1}(t, x)) &= x, \\ X^{-1}(t, X(t, x)) &= x. \end{aligned} \tag{3.59}$$

Let $T > 0$ and B be an arbitrary bounded subset of \mathbb{R}^2 . We have

$$\begin{aligned} &\int_0^T \int_B |\rho^\delta(t, x) - \bar{\rho}(t, x)|^2 dt dx \\ &= \int_0^T \int_{X^\delta(t)B} |\rho^\delta(t, X^\delta(t, x)) - \bar{\rho}(t, X^\delta(t, x))|^2 J^\delta(t, x) dt dx \end{aligned}$$

$$\begin{aligned}
&\leq C_* \int_0^T \int_{X^\delta(t)B} |\varrho^\delta(t, x) - \varrho(t, x)|^2 dt dx \\
&\quad + C_* \int_0^T \int_{X^\delta(t)B} |\bar{\varrho}(t, X^\delta(t, x)) - \varrho(t, x)|^2 dt dx, \tag{3.60}
\end{aligned}$$

where J^δ comes from the change of variable $x \mapsto X^\delta(t, x)$ and satisfies (see [19, Lemma 3.2])

$$\sup_{[0, T]} \|J^\delta\|_{L^\infty(\mathbb{R}^2)} \leq C_*.$$

- Given that $(X^\delta)_\delta$ converges to X uniformly on compact sets in $[0, \infty) \times \mathbb{R}^2$, there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$, we have

$$X^\delta(t)B \subset X(t)B + B(0, 1).$$

Furthermore, because ϱ is uniformly continuous on both sides of $\mathcal{C}(0)$, and using (3.59), we obtain

$$\bar{\varrho}(t, X^\delta(t, x)) = \varrho(t, X^{-1}(t, X^\delta(t, x))) \xrightarrow{\delta \rightarrow 0} \varrho(t, x) \quad \text{a.e } x \in \mathbb{R}^2.$$

Therefore, we conclude

$$\lim_{\delta \rightarrow 0} \int_0^T \int_{X^\delta(t)B} |\bar{\varrho}(t, X^\delta(t, x)) - \varrho(t, x)|^2 dt dx = 0. \tag{3.61}$$

- Additionally, since the sequence $(\varrho^\delta)_\delta$ converges to ϱ uniformly on compact sets, we have

$$\lim_{\delta \rightarrow 0} \int_0^T \int_{X^\delta(t)B} |\varrho^\delta(t, x) - \varrho(t, x)|^2 dt dx = 0. \tag{3.62}$$

Finally, the strong convergence of $(\rho^\delta)_\delta$ to $\bar{\varrho}$ in $L^2_{\text{loc}}((0, \infty) \times \mathbb{R}^2)$ follows from (3.60), (3.61), and (3.62). In particular, we have $\rho = \bar{\varrho} \in \mathcal{C}([0, \infty), \mathcal{C}^{\alpha}_{\text{pw}, \gamma}(\mathbb{R}^2))$. Additionally, by interpolation, it follows that $(\rho^\delta)_\delta$ converges to ρ strongly in $L^p_{\text{loc}}((0, \infty) \times \mathbb{R}^2)$ for any $1 \leq p < \infty$.

By combining the strong and weak convergences of $(\rho^\delta, u^\delta)_\delta$ with standard arguments, we take the limit as $\delta \rightarrow 0$ in (3.54) and we conclude that (ρ, u) satisfies (1.1). This completes the current section. We now turn to the proof of uniqueness.

3.5.2. Proof of the uniqueness. The uniqueness of the solution constructed above is immediately implied by the following result.

Proposition 3.1. *Consider the system (1.1) and assume that the pressure and viscosity laws are $W^{1, \infty}$ -regular functions of the density. To simplify, we assume that λ is non-negative on $[0, \infty)$. Let (ρ_0, u_0) be the initial data associated with (1.1), satisfying the following conditions:*

$$\rho_0, \quad \frac{1}{\rho_0} \in L^\infty(\mathbb{R}^2), \quad \text{and} \quad u_0 \in L^2(\mathbb{R}^2). \tag{3.63}$$

Let $T > 0$. On the time interval $[0, T]$, there exists at most one solution to the Cauchy problem associated with (1.1) and initial data (ρ_0, u_0) satisfying

$$\begin{cases} \frac{1}{\mu(\rho)} & \in L^\infty((0, T) \times \mathbb{R}^2), \\ \nabla u & \in L^1((0, T), L^\infty(\mathbb{R}^2)), \\ \sqrt{\sigma} \nabla u & \in L^2((0, T), L^\infty(\mathbb{R}^2)). \end{cases} \quad (3.64)$$

Proposition 3.1 establishes uniqueness for the system (1.1) within a broader framework than that of Theorem 1.2. In particular, neither piecewise Hölder continuity for the density or velocity gradient, nor smallness conditions on the initial data or viscosity fluctuations are required.

Proof of Proposition 3.1. Let (ρ, u) and (ϱ, v) be two solutions to the Cauchy problem associated with (1.1) and with initial data (ρ_0, u_0) satisfying (3.63). Additionally, we assume that (ρ, u) and (ϱ, v) satisfy the conditions in (3.64). As a consequence, for any $k_0 \in (0, 1)$, there exists a time $T_0 > 0$ such that

$$\int_0^{T_0} \|\nabla u\|_{L^\infty(\mathbb{R}^2)} < k_0 \quad \text{and} \quad \int_0^{T_0} \|\nabla v\|_{L^\infty(\mathbb{R}^2)} < k_0.$$

The regularity of u, v is sufficient to recast the equations they satisfy in Lagrangian coordinates,

$$\begin{cases} \rho_0 \partial_t \bar{u} - \operatorname{div}[\operatorname{Adj}(DX_u)(2\mu(\rho_0 J_u^{-1})\mathbb{D}_{A_u} \bar{u} + (\lambda(\rho_0 J_u^{-1}) \operatorname{div}_{A_u} \bar{u})I)] \\ \quad = -\operatorname{div}(\operatorname{Adj}(DX_u)(P(\rho_0 J_u^{-1}) - \tilde{P})), \\ \rho_0 \partial_t \bar{v} - \operatorname{div}[\operatorname{Adj}(DX_v)(2\mu(\rho_0 J_v^{-1})\mathbb{D}_{A_v} \bar{v} + (\lambda(\rho_0 J_v^{-1}) \operatorname{div}_{A_v} \bar{v})I)] \\ \quad = -\operatorname{div}(\operatorname{Adj}(DX_v)(P(\rho_0 J_v^{-1}) - \tilde{P})). \end{cases} \quad (3.65)$$

Here, X_w is the flow associated with the velocity w . We define \bar{w} by

$$\bar{w}(t, y) = w(t, X_w(t, y)), \quad \text{so that } X_w(t, y) = y + \int_0^t \bar{w}(\tau, y) d\tau.$$

The Jacobian matrix of X_w is denoted by DX_w , and we define $J_w = \det(DX_w)$. By (3.64), the matrix DX_w is invertible, with its inverse denoted by A_w . The matrix of cofactors of DX_w , also known as the adjugate matrix, is denoted by $\operatorname{Adj}(DX_w)$. Finally, the operators \mathbb{D}_{A_w} and div_{A_w} are defined as

$$\mathbb{D}_{A_w} z = \frac{1}{2}(Dz \cdot A_w + A_w^\top \cdot \nabla z) \quad \text{and} \quad \operatorname{div}_{A_w} z = Dz : A_w = A_w^\top : \nabla z.$$

The computations leading to (3.65) are standard and can be found for instance in [5]. Additionally, with a slight modification, the following bounds can be derived from [5, Lemmas A.3 and A.4].

Lemma 3.2. *There exists a constant C_{k_0} , depending only on k_0 , such that the following estimates hold for all $p \in [1, \infty]$, $t \in [0, T_0]$, and $w \in \{u, v\}$:*

$$\begin{cases} \|\text{Adj}(DX_w(t))\mathbb{D}_{A_w(t)}z - \mathbb{D}z\|_{L^p(\mathbb{R}^2)} \leq C_{k_0} \|\nabla w\|_{L^1((0,t),L^\infty(\mathbb{R}^2))} \|Dz\|_{L^p(\mathbb{R}^2)}, \\ \|\text{Adj}(DX_w(t))\text{div}_{A_w(t)}z - \text{div}zI_2\|_{L^p(\mathbb{R}^2)} \leq C_{k_0} \|\nabla w\|_{L^1((0,t),L^\infty(\mathbb{R}^2))} \|Dz\|_{L^p(\mathbb{R}^2)}, \end{cases}$$

and

$$\begin{cases} \|A_u(t) - A_v(t)\|_{L^p(\mathbb{R}^2)} \leq C_{k_0} \|\nabla \delta \bar{u}\|_{L^1((0,t),L^p(\mathbb{R}^2))}, \\ \|\text{Adj}(DX_u(t)) - \text{Adj}(DX_v(t))\|_{L^p(\mathbb{R}^2)} \leq C_{k_0} \|\nabla \delta \bar{u}\|_{L^1((0,t),L^p(\mathbb{R}^2))}, \\ \|J_u^{\pm 1}(t) - J_v^{\pm 1}(t)\|_{L^p(\mathbb{R}^2)} \leq C_{k_0} \|\nabla \delta \bar{u}\|_{L^1((0,t),L^p(\mathbb{R}^2))}, \end{cases}$$

where $\delta \bar{u} = \bar{u} - \bar{v}$.

We now take the difference in (3.65) to obtain

$$\begin{aligned} & \rho_0 \partial_t \delta \bar{u} - \text{div}(2\mu(\rho_0 J_u^{-1})\mathbb{D}\delta \bar{u}) - \nabla(\lambda(\rho_0 J_u^{-1})\text{div}\delta \bar{u}) \\ & = \text{div}(\mathcal{I}_1) + \text{div}(\mathcal{I}_2) + \text{div}(\mathcal{I}_3), \end{aligned} \quad (3.66)$$

where

$$\begin{cases} \mathcal{I}_1 = (\mu(\rho_0 J_u^{-1}) - \mu(\rho_0 J_v^{-1}))(\text{Adj}(DX_u)\mathbb{D}_{A_u}\bar{u} - \mathbb{D}\bar{u}) \\ \quad + \mu(\rho_0 J_v^{-1})(\text{Adj}(DX_u)\mathbb{D}_{A_u}\delta \bar{u} - \mathbb{D}\delta \bar{u}) \\ \quad + \mu(\rho_0 J_v^{-1})(\text{Adj}(DX_u) - \text{Adj}(DX_v))\mathbb{D}_{A_u}\bar{v} \\ \quad + \mu(\rho_0 J_v^{-1})\text{Adj}(DX_v)(\mathbb{D}_{A_u} - \mathbb{D}_{A_v})\bar{v}, \\ \mathcal{I}_2 = (\lambda(\rho_0 J_u^{-1}) - \lambda(\rho_0 J_v^{-1}))(\text{Adj}(DX_u)\text{div}_{A_u}\bar{u} - \text{div}\bar{u}I_2) \\ \quad + \lambda(\rho_0 J_v^{-1})(\text{Adj}(DX_u)\text{div}_{A_u}\delta \bar{u} - \text{div}\delta \bar{u}I_2) \\ \quad + \lambda(\rho_0 J_v^{-1})(\text{Adj}(DX_u) - \text{Adj}(DX_v))\text{div}_{A_u}\bar{v} \\ \quad + \lambda(\rho_0 J_v^{-1})\text{Adj}(DX_v)(\text{div}_{A_u} - \text{div}_{A_v})\bar{v}, \\ \mathcal{I}_3 = (P(\rho_0 J_v^{-1}) - \tilde{P})(\text{Adj}(DX_v) - \text{Adj}(DX_u)) \\ \quad + \text{Adj}(DX_u)(P(\rho_0 J_v^{-1}) - P(\rho_0 J_u^{-1})). \end{cases}$$

We fix $k_0 \in (0, 1)$ and denote by $C_{k_0}^*$ a constant that may depend on C_{k_0} (see Lemma 3.2 above), as well as on the lower and upper bounds of the density and viscosity (see (3.63)–(3.64) above). Note that this constant may change from one line to the next. We now perform an energy estimate for (3.66): we use $\delta \bar{u}$ as a test function and it follows that (recall $\delta \bar{u}|_{t=0} = 0$)

$$\begin{aligned} & \|\sqrt{\rho_0}\delta \bar{u}(t)\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \int_{\mathbb{R}^2} [2\mu(\rho_0 J_u^{-1})|\mathbb{D}\delta \bar{u}|^2 + \lambda(\rho_0 J_u^{-1})(\text{div}\delta \bar{u})^2] \\ & \leq \int_0^t \|\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\|_{L^2(\mathbb{R}^2)} \|\nabla \delta \bar{u}\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Next, applying Lemma 3.2, we derive the following estimates:

$$\begin{cases} \|\mathcal{I}_1(t), \mathcal{I}_2(t)\|_{L^2(\mathbb{R}^2)} \leq C_{k_0}^* (\|\nabla \delta \bar{u}\|_{L^1((0,t),L^2(\mathbb{R}^2))} \|\nabla u(t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla \delta \bar{u}(t)\|_{L^2(\mathbb{R}^2)}), \\ \|\mathcal{I}_3(t)\|_{L^2(\mathbb{R}^2)} \leq C_{k_0}^* \|\nabla \delta \bar{u}\|_{L^1((0,t),L^2(\mathbb{R}^2))}, \end{cases}$$

which leads to

$$\begin{aligned} & \|\sqrt{\rho_0}\delta\bar{u}(t)\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \|\nabla(\delta\bar{u})\|_{L^2(\mathbb{R}^2)}^2 \\ & \leq C_{k_0}^* \int_0^t [1 + \tau(1 + \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)}^2)] \left[\int_0^\tau \|\nabla\delta\bar{u}\|_{L^2(\mathbb{R}^2)}^2 \right] d\tau. \end{aligned}$$

As a result, we have

$$\mathcal{E}(t) \leq C_{k_0}^* \int_0^t [1 + \tau(1 + \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)}^2)] \mathcal{E}(\tau) d\tau,$$

where

$$\mathcal{E}(t) = \|\sqrt{\rho_0}\delta\bar{u}(t)\|_{L^2(\mathbb{R}^2)}^2 + \int_0^t \|\nabla(\delta\bar{u})\|_{L^2(\mathbb{R}^2)}^2.$$

Invoking the assumption (3.64) and applying Gronwall’s lemma, we conclude that $\mathcal{E} \equiv 0$ on $[0, T_0]$, which ensures uniqueness on $[0, T_0]$. The uniqueness on $[0, T]$ follows by a standard continuation argument, thereby proving Proposition 3.1 and, ultimately, Theorem 1.2. \blacksquare

A. Energy computations

In this section we provide details of the energy computations for the solution (ρ, u) to the system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = \operatorname{div} \Pi, \\ \llbracket \Pi \rrbracket \cdot n = 0 \text{ on } \mathcal{C}, \end{cases} \quad (\text{A.1})$$

where we recall

$$\Pi = 2\mu(\rho)\mathbb{D}u + (\lambda(\rho)\operatorname{div}u - P(\rho) + \tilde{P})I_2.$$

A.1. Second Hoff energy

We start by investigating the Rankine–Hugoniot condition for \dot{u} at the surface of discontinuity \mathcal{C} . We first notice that after applying the derivative $\partial_t \cdot + \operatorname{div}(\cdot u)$ to the momentum equation, we obtain that \dot{u} solves the equation

$$\partial_t(\rho\dot{u}^j) + \operatorname{div}(\rho\dot{u}^j u) = \partial_k(\dot{\Pi}^{jk}) + \partial_k(\Pi^{jk}\operatorname{div}u) - \operatorname{div}(\partial_k u^l \Pi^{jk}). \quad (\text{A.2})$$

Implicitly, the Rankine–Hugoniot condition yields

$$\llbracket \rho\dot{u}^j \rrbracket n_t + \llbracket \rho\dot{u}^j u^k \rrbracket n_x^k = \llbracket \dot{\Pi}^{jk} \rrbracket n_x^k + \llbracket \Pi^{jk}\operatorname{div}u \rrbracket n_x^k - \llbracket \partial_k u^l \Pi^{jk} \rrbracket n_x^l.$$

Furthermore, from the mass equation (A.1)₁ and the Rankine–Hugoniot condition, we deduce that

$$\llbracket \rho \rrbracket n_t + \llbracket \rho u^k \rrbracket n_x^k = 0,$$

and since \dot{u} is continuous on the whole space, we finally obtain

$$\llbracket \dot{\Pi}^{jk} \rrbracket n_x^k + \llbracket \Pi^{jk} \operatorname{div} u \rrbracket n_x^k - \llbracket \partial_k u^l \Pi^{jk} \rrbracket n_x^l = 0. \quad (\text{A.3})$$

This relation will be used in the subsequent computations.

We recall that the second Hoff estimate consists in multiplying (A.2) by the material derivative of the velocity before integrating in space. By doing so, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} \dot{u}^j [\partial_t (\rho \dot{u}^j) + \operatorname{div} (\rho \dot{u}^j u)] \\ &= \int_{\mathbb{R}^2} \dot{u}^j [\partial_k (\dot{\Pi}^{jk}) + \partial_k (\Pi^{jk} \operatorname{div} u) - \operatorname{div} (\partial_k u \Pi^{jk})]. \end{aligned} \quad (\text{A.4})$$

The right-hand side of the above equality is

$$\begin{aligned} & \int_{\mathbb{R}^2} \dot{u}^j [\partial_t (\rho \dot{u}^j) + \operatorname{div} (\rho \dot{u}^j u)] \\ &= \int_{\mathbb{R}^2} \partial_t (\rho |\dot{u}^j|^2) - \int_{\mathbb{R}^2} (\rho \dot{u}^j \partial_t \dot{u}^j) + \int_{\mathbb{R}^2} \operatorname{div} (\rho |\dot{u}^j|^2 u) - \int_{\mathbb{R}^2} \rho \dot{u}^j u \cdot \nabla \dot{u}^j \\ &= \frac{1}{2} \int_{\mathbb{R}^2} \partial_t (\rho |\dot{u}^j|^2) + \frac{1}{2} \int_{\mathbb{R}^2} |\dot{u}^j|^2 \partial_t \rho + \frac{1}{2} \int_{\mathbb{R}^2} \operatorname{div} (\rho |\dot{u}^j|^2 u) + \frac{1}{2} \int_{\mathbb{R}^2} |\dot{u}^j|^2 \operatorname{div} (\rho u) \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \rho |\dot{u}^j|^2, \end{aligned} \quad (\text{A.5})$$

where we have used Liouville's transport theorem and the mass equation (A.1)₁. We turn now to the computations of the right-hand side of (A.4):

$$\begin{aligned} & \int_{\mathbb{R}^2} \dot{u}^j [\partial_k (\dot{\Pi}^{jk}) + \partial_k (\Pi^{jk} \operatorname{div} u) - \operatorname{div} (\partial_k u \Pi^{jk})] \\ &= \int_{\mathbb{R}^2} [\partial_k (\dot{u}^j \dot{\Pi}^{jk}) + \partial_k (\dot{u}^j \Pi^{jk} \operatorname{div} u) - \partial_l (\dot{u}^j \Pi^{jk} \partial_k u^l)] \\ &\quad - \int_{\mathbb{R}^2} \partial_k \dot{u}^j \dot{\Pi}^{jk} - \int_{\mathbb{R}^2} \partial_k \dot{u}^j \Pi^{jk} \operatorname{div} u + \int_{\mathbb{R}^2} \partial_l \dot{u}^j \partial_k u^l \Pi^{jk}. \end{aligned}$$

We calculate

$$\begin{aligned} \dot{\Pi}^{jk} &= 2\mu(\rho) \mathbb{D}^{jk} \dot{u} - \mu(\rho) \partial_j u^l \partial_l u^k - \mu(\rho) \partial_k u^l \partial_l u^j - 2\rho \mu'(\rho) \mathbb{D}^{jk} u \operatorname{div} u \\ &\quad + (\lambda(\rho) \operatorname{div} \dot{u} - \lambda(\rho) \nabla u^l \partial_l u - \rho \lambda'(\rho) (\operatorname{div} u)^2 + \rho P'(\rho) \operatorname{div} u) \delta^{jk}, \end{aligned} \quad (\text{A.6})$$

and hence, the terms in the right-hand side of (A.4) are

$$\begin{aligned} & \int_{\mathbb{R}^2} \dot{u}^j [\partial_k (\dot{\Pi}^{jk}) + \partial_k (\Pi^{jk} \operatorname{div} u) - \operatorname{div} (\partial_k u \Pi^{jk})] \\ &= - \int_{\mathbb{R}^2} 2\mu(\rho) |\mathbb{D}^{jk} \dot{u}|^2 \\ &\quad + \int_{\mathbb{R}^2} [\partial_k (\dot{u}^j \dot{\Pi}^{jk}) + \partial_k (\dot{u}^j \Pi^{jk} \operatorname{div} u) - \partial_l (\dot{u}^j \Pi^{jk} \partial_k u^l)] \end{aligned}$$

$$\begin{aligned}
 & + \int_{\mathbb{R}^2} \partial_k \dot{u}^j [\mu(\rho) \partial_j u^l \partial_l u^k + \mu(\rho) \partial_k u^l \partial_l u^j + 2\rho\mu'(\rho) \mathbb{D}^{jk} u \operatorname{div} u] \\
 & - \int_{\mathbb{R}^2} \lambda(\rho) |\operatorname{div} \dot{u}|^2 + \int_{\mathbb{R}^2} \operatorname{div} \dot{u} [\lambda(\rho) \nabla u^l \partial_l u + \rho\lambda'(\rho) (\operatorname{div} u)^2 - \rho P'(\rho) \operatorname{div} u] \\
 & + \int_{\mathbb{R}^2} \partial_l \dot{u}^j \partial_k u^l \Pi^{jk}. \tag{A.7}
 \end{aligned}$$

We combine (A.4), (A.5), and (A.7) to obtain

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \rho |\dot{u}^j|^2 + \int_{\mathbb{R}^2} [2\mu(\rho) |\mathbb{D}^{jk} \dot{u}|^2 + \lambda(\rho) |\operatorname{div} \dot{u}|^2] \\
 & = \int_{\mathbb{R}^2} \partial_k \dot{u}^j [\mu(\rho) \partial_j u^l \partial_l u^k + \mu(\rho) \partial_k u^l \partial_l u^j + 2\rho\mu'(\rho) \mathbb{D}^{jk} u \operatorname{div} u] \\
 & - \int_{\mathbb{R}^2} \partial_k \dot{u}^j \Pi^{jk} \operatorname{div} u + \int_{\mathbb{R}^2} \operatorname{div} \dot{u} [\lambda(\rho) \nabla u^l \partial_l u + \rho\lambda'(\rho) (\operatorname{div} u)^2 - \rho P'(\rho) \operatorname{div} u] \\
 & + \int_{\mathbb{R}^2} \partial_l \dot{u}^j \partial_k u^l \Pi^{jk}.
 \end{aligned}$$

A.2. Third Hoff estimate

One notices that the material derivative of the velocity solves a parabolic equation like the velocity; see (A.2). The goal is to perform the first Hoff energy to the aforementioned equation, by using \ddot{u} as a test function. For this purpose, we write (A.2) as

$$\rho \ddot{u}^j = \partial_k (\dot{\Pi}^{jk}) + \partial_k (\Pi^{jk} \operatorname{div} u) - \operatorname{div} (\partial_k u \Pi^{jk}).$$

We multiply the above by \ddot{u}^j to obtain

$$\begin{aligned}
 \int_{\mathbb{R}^2} \rho |\ddot{u}|^2 & = \int_{\mathbb{R}^2} \ddot{u}^j \partial_k (\dot{\Pi}^{jk}) + \int_{\mathbb{R}^2} \ddot{u}^j \partial_k (\Pi^{jk} \operatorname{div} u) - \int_{\mathbb{R}^2} \ddot{u}^j \operatorname{div} (\partial_k u \Pi^{jk}) \\
 & = \int_{\mathcal{E}} \{ \llbracket \ddot{u}^j (\dot{\Pi}^{jk} + \Pi^{jk} \operatorname{div} u) \rrbracket n_x^k - \llbracket \ddot{u}^j \partial_k u^l \Pi^{jk} \rrbracket n_x^l \} \\
 & - \int_{\mathbb{R}^2} \partial_k \ddot{u}^j \dot{\Pi}^{jk} - \int_{\mathbb{R}^2} \partial_k \ddot{u}^j \Pi^{jk} \operatorname{div} u + \int_{\mathbb{R}^2} \partial_l \ddot{u}^j \partial_k u^l \Pi^{jk}. \tag{A.8}
 \end{aligned}$$

The first term on the right-hand side above vanishes since \ddot{u} is continuous through the interface and due to (A.3). Next, the second term is, thanks to (A.6),

$$\begin{aligned}
 & - \int_{\mathbb{R}^2} \partial_k \ddot{u}^j \dot{\Pi}^{jk} \\
 & = - \int_{\mathbb{R}^2} \partial_k \ddot{u}^j [2\mu(\rho) \mathbb{D}^{jk} \dot{u} - \mu(\rho) \partial_j u^l \partial_l u^k - \mu(\rho) \partial_k u^l \partial_l u^j - 2\rho\mu'(\rho) \mathbb{D}^{jk} u \operatorname{div} u \\
 & \quad + (\lambda(\rho) \operatorname{div} \dot{u} - \lambda(\rho) \nabla u^l \partial_l u - \rho\lambda'(\rho) (\operatorname{div} u)^2 \\
 & \quad + \rho P'(\rho) \operatorname{div} u) \delta^{jk}]. \tag{A.9}
 \end{aligned}$$

The first term on the right-hand side above is

$$\begin{aligned}
& - \int_{\mathbb{R}^2} 2\mu(\rho)\partial_k\ddot{u}^j\mathbb{D}^{jk}\dot{u} \\
& = -2 \int_{\mathbb{R}^2} \mu(\rho)\partial_{tk}\dot{u}^j\mathbb{D}^{jk}\dot{u} - 2 \int_{\mathbb{R}^2} \mu(\rho)u^l\partial_{lk}\dot{u}^j\mathbb{D}^{jk}\dot{u} - 2 \int_{\mathbb{R}^2} \mu(\rho)\partial_k u^l\partial_l\dot{u}^j\mathbb{D}^{jk}\dot{u} \\
& = - \int_{\mathbb{R}^2} [\partial_t(\mu(\rho)|\mathbb{D}^{jk}\dot{u}|^2) + \operatorname{div}(\mu(\rho)u|\mathbb{D}^{jk}\dot{u}|^2)] - 2 \int_{\mathbb{R}^2} \mu(\rho)\partial_k u^l\partial_l\dot{u}^j\mathbb{D}^{jk}\dot{u} \\
& \quad + \int_{\mathbb{R}^2} |\mathbb{D}^{jk}\dot{u}|^2(\partial_t\mu(\rho) + \operatorname{div}(\mu(\rho)u)) \\
& = -\frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho)|\mathbb{D}^{jk}\dot{u}|^2 - 2 \int_{\mathbb{R}^2} \mu(\rho)\partial_k u^l\partial_l\dot{u}^j\mathbb{D}^{jk}\dot{u} \\
& \quad + \int_{\mathbb{R}^2} |\mathbb{D}^{jk}\dot{u}|^2(\rho\mu'(\rho) - \mu(\rho)) \operatorname{div} u.
\end{aligned}$$

As for the second term on the right-hand side of (A.9), one has

$$\begin{aligned}
& \int_{\mathbb{R}^2} \mu(\rho)\partial_k\ddot{u}^j\partial_j u^l\partial_l u^k \\
& = \int_{\mathbb{R}^2} \partial_t[\mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l u^k] + \int_{\mathbb{R}^2} \partial_m[\mu(\rho)u^m\partial_k\dot{u}^j\partial_j u^l\partial_l u^k] \\
& \quad + \int_{\mathbb{R}^2} \mu(\rho)\partial_k u^m\partial_m\dot{u}^j\partial_j u^l\partial_l u^k + \int_{\mathbb{R}^2} (\rho\mu'(\rho) - \mu(\rho))\partial_k\dot{u}^j \operatorname{div} u\partial_j u^l\partial_l u^k \\
& \quad - \int_{\mathbb{R}^2} \mu(\rho)\partial_k\dot{u}^j\partial_j\dot{u}^l\partial_l u^k + \int_{\mathbb{R}^2} \mu(\rho)\partial_k\dot{u}^j\partial_j u^m\partial_m u^l\partial_l u^k \\
& \quad - \int_{\mathbb{R}^2} \mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l\dot{u}^k + \int_{\mathbb{R}^2} \mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l u^m\partial_m u^k \\
& = \frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l u^k + \int_{\mathbb{R}^2} \mu(\rho)\partial_k u^m\partial_m\dot{u}^j\partial_j u^l\partial_l u^k \\
& \quad + \int_{\mathbb{R}^2} (\rho\mu'(\rho) - \mu(\rho))\partial_k\dot{u}^j \operatorname{div} u\partial_j u^l\partial_l u^k - \int_{\mathbb{R}^2} \mu(\rho)\partial_k\dot{u}^j\partial_j\dot{u}^l\partial_l u^k \\
& \quad - \int_{\mathbb{R}^2} \mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l\dot{u}^k + \int_{\mathbb{R}^2} \mu(\rho)\partial_k\dot{u}^j\partial_j u^m\partial_m u^l\partial_l u^k \\
& \quad + \int_{\mathbb{R}^2} \mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l u^m\partial_m u^k.
\end{aligned}$$

The third term on the right-hand side of (A.9) can be derived from the previous results by swapping j and k . For the fourth term, we have

$$\begin{aligned}
& 2 \int_{\mathbb{R}^2} \rho\mu'(\rho)\partial_k\ddot{u}^j\mathbb{D}^{jk}u \operatorname{div} u \\
& = 2\frac{d}{dt} \int_{\mathbb{R}^2} \rho\mu'(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}u \operatorname{div} u + 2 \int_{\mathbb{R}^2} \rho\mu'(\rho)\partial_k u^m\partial_m\dot{u}^j\mathbb{D}^{jk}u \operatorname{div} u \\
& \quad + 2 \int_{\mathbb{R}^2} \rho^2\mu''(\rho)\partial_k\dot{u}^j \operatorname{div} u\mathbb{D}^{jk}u \operatorname{div} u - 2 \int_{\mathbb{R}^2} \rho\mu'(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}\dot{u} \operatorname{div} u
\end{aligned}$$

$$\begin{aligned}
 & + \int_{\mathbb{R}^2} \rho \mu'(\rho) \partial_k \dot{u}^j (\partial_j u^m \partial_m u^k + \partial_k u^m \partial_m u^j) \operatorname{div} u \\
 & - 2 \int_{\mathbb{R}^2} \rho \mu'(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} u \operatorname{div} \dot{u} + 2 \int_{\mathbb{R}^2} \rho \mu'(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} u \nabla u^m \cdot \partial_m u.
 \end{aligned}$$

Next, the fifth term on the right-hand side of (A.9) is computed as

$$\begin{aligned}
 - \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \ddot{u} \operatorname{div} \dot{u} &= -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \lambda(\rho) |\operatorname{div} \dot{u}|^2 - \int_{\mathbb{R}^2} \lambda(\rho) \nabla u^m \cdot \partial_m \dot{u} \operatorname{div} \dot{u} \\
 & - \frac{1}{2} \int_{\mathbb{R}^2} (\rho \lambda'(\rho) - \lambda(\rho)) |\operatorname{div} \dot{u}|^2 \operatorname{div} u.
 \end{aligned}$$

The sixth term is

$$\begin{aligned}
 \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \ddot{u} \partial_k u^l \partial_l u^k &= \frac{d}{dt} \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} \partial_k u^l \partial_l u^k + \int_{\mathbb{R}^2} \lambda(\rho) \nabla u^m \partial_m \dot{u} \partial_k u^l \partial_l u^k \\
 & + \int_{\mathbb{R}^2} (\rho \lambda'(\rho) - \lambda(\rho)) \operatorname{div} u \operatorname{div} \dot{u} \partial_k u^l \partial_l u^k - \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} \partial_k \dot{u}^l \partial_l u^k \\
 & + \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} \partial_k u^m \partial_m u^l \partial_l u^k - \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} \partial_k u^l \partial_l \dot{u}^k \\
 & + \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} \partial_k u^l \partial_l u^m \partial_m u^k.
 \end{aligned}$$

The next-to-last term is

$$\begin{aligned}
 \int_{\mathbb{R}^2} \operatorname{div} \ddot{u} \rho \lambda'(\rho) (\operatorname{div} u)^2 &= \frac{d}{dt} \int_{\mathbb{R}^2} \rho \lambda'(\rho) \operatorname{div} \dot{u} (\operatorname{div} u)^2 + \int_{\mathbb{R}^2} \rho \lambda'(\rho) \partial_j u^m \partial_m \dot{u}^j (\operatorname{div} u)^2 \\
 & + \int_{\mathbb{R}^2} \rho^2 \lambda''(\rho) \operatorname{div} \dot{u} (\operatorname{div} u)^3 - 2 \int_{\mathbb{R}^2} \rho \lambda'(\rho) \operatorname{div} \dot{u} \operatorname{div} \dot{u} \operatorname{div} u \\
 & + 2 \int_{\mathbb{R}^2} \rho \lambda'(\rho) \operatorname{div} \dot{u} \partial_j u^m \partial_m u^j \operatorname{div} u
 \end{aligned}$$

and the last term is

$$\begin{aligned}
 - \int_{\mathbb{R}^2} \rho P'(\rho) \operatorname{div} \ddot{u} \operatorname{div} u &= -\frac{d}{dt} \int_{\mathbb{R}^2} \rho P'(\rho) \operatorname{div} \dot{u} \operatorname{div} u - \int_{\mathbb{R}^2} \rho P'(\rho) \nabla u^m \partial_m \dot{u} \operatorname{div} u \\
 & - \int_{\mathbb{R}^2} \rho^2 P''(\rho) \operatorname{div} \dot{u} (\operatorname{div} u)^2 + \int_{\mathbb{R}^2} \rho P'(\rho) (\operatorname{div} \dot{u})^2 \\
 & - \int_{\mathbb{R}^2} \rho P'(\rho) \operatorname{div} \dot{u} \nabla u^m \partial_m u.
 \end{aligned}$$

This concludes the computation of the second term on the right-hand side of (A.8). We now proceed to the computation of the third term, which we express as follows:

$$\begin{aligned}
 & - \int_{\mathbb{R}^2} \partial_k \ddot{u}^j \Pi^{jk} \operatorname{div} u \\
 & = - \int_{\mathbb{R}^2} \partial_k \ddot{u}^j [2\mu(\rho) \mathbb{D}^{jk} u + (\lambda(\rho) \operatorname{div} u - P(\rho) + \tilde{P}) \delta^{jk}] \operatorname{div} u. \quad (\text{A.10})
 \end{aligned}$$

The first term on the right-hand side above is

$$\begin{aligned}
& -2 \int_{\mathbb{R}^2} \partial_k \ddot{u}^j \mu(\rho) \mathbb{D}^{jk} u \operatorname{div} u \\
& = -2 \frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} u \operatorname{div} u - 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_k u^m \partial_m \dot{u}^j \mathbb{D}^{jk} u \operatorname{div} u \\
& \quad - 2 \int_{\mathbb{R}^2} (\rho \mu'(\rho) - \mu(\rho)) \partial_k \dot{u}^j \mathbb{D}^{jk} u (\operatorname{div} u)^2 + 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} \dot{u} \operatorname{div} u \\
& \quad - \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j (\partial_j u^m \partial_m u^k + \partial_k u^m \partial_m u^j) \operatorname{div} u \\
& \quad + 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} u \operatorname{div} \dot{u} - 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} u \nabla u^m \partial_m u.
\end{aligned}$$

Regarding the second term on the right-hand side of (A.10), one has

$$\begin{aligned}
- \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \ddot{u} (\operatorname{div} u)^2 & = - \frac{d}{dt} \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} (\operatorname{div} u)^2 - \int_{\mathbb{R}^2} \lambda(\rho) \nabla u^m \partial_m \dot{u} (\operatorname{div} u)^2 \\
& \quad - \int_{\mathbb{R}^2} (\rho \lambda'(\rho) - \lambda(\rho)) \operatorname{div} \dot{u} (\operatorname{div} u)^3 \\
& \quad + 2 \int_{\mathbb{R}^2} \lambda(\rho) (\operatorname{div} \dot{u})^2 \operatorname{div} u - 2 \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} \nabla u^m \partial_m u \operatorname{div} u,
\end{aligned}$$

and finally the last term is

$$\begin{aligned}
& \int_{\mathbb{R}^2} \operatorname{div} \ddot{u} (P(\rho) - \tilde{P}) \operatorname{div} u \\
& = \frac{d}{dt} \int_{\mathbb{R}^2} \operatorname{div} \dot{u} (P(\rho) - \tilde{P}) \operatorname{div} u + \int_{\mathbb{R}^2} \nabla u^m \partial_m \dot{u} (P(\rho) - \tilde{P}) \operatorname{div} u \\
& \quad + \int_{\mathbb{R}^2} \operatorname{div} \dot{u} (\operatorname{div} u)^2 (\rho P'(\rho) - P(\rho) + \tilde{P}) - \int_{\mathbb{R}^2} \operatorname{div} \dot{u} (P(\rho) - \tilde{P}) \operatorname{div} \dot{u} \\
& \quad + \int_{\mathbb{R}^2} \operatorname{div} \dot{u} (P(\rho) - \tilde{P}) \nabla u^m \partial_m u.
\end{aligned}$$

This completes the computation of the terms in expression (A.10). We now turn to the computation of the final term in (A.8), which we express as

$$\begin{aligned}
& \int_{\mathbb{R}^2} \partial_l \ddot{u}^j \partial_k u^l \Pi^{jk} \\
& = \int_{\mathbb{R}^2} \partial_l \ddot{u}^j \partial_k u^l [2\mu(\rho) \mathbb{D}^{jk} u + (\lambda(\rho) \operatorname{div} u - P(\rho) + \tilde{P}) \delta^{jk}]. \tag{A.11}
\end{aligned}$$

The first term on the right-hand side above is

$$\begin{aligned}
& \int_{\mathbb{R}^2} 2\mu(\rho) \partial_l \ddot{u}^j \partial_k u^l \mathbb{D}^{jk} u \\
& = 2 \frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) \partial_l \dot{u}^j \partial_k u^l \mathbb{D}^{jk} u + 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_l u^m \partial_m \dot{u}^j \partial_k u^l \mathbb{D}^{jk} u
\end{aligned}$$

$$\begin{aligned}
 & + 2 \int_{\mathbb{R}^2} (\rho \mu'(\rho) - \mu(\rho)) \operatorname{div} u \partial_l \dot{u}^j \partial_k u^l \mathbb{D}^{jk} u - 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_l \dot{u}^j \partial_k \dot{u}^l \mathbb{D}^{jk} u \\
 & + 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_l \dot{u}^j \partial_k u^m \partial_m u^l \mathbb{D}^{jk} u - 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_l \dot{u}^j \mathbb{D}^{jk} \dot{u} \partial_k u^l \\
 & + \int_{\mathbb{R}^2} \mu(\rho) \partial_l \dot{u}^j \partial_k u^l [\partial_j u^m \partial_m u^k + \partial_k u^m \partial_m u^j],
 \end{aligned}$$

and the second term is

$$\begin{aligned}
 & \int_{\mathbb{R}^d} \lambda(\rho) \partial_l \ddot{u}^j \partial_j u^l \operatorname{div} u \\
 & = \frac{d}{dt} \int_{\mathbb{R}^2} \lambda(\rho) \partial_l \dot{u}^j \partial_j u^l \operatorname{div} u + \int_{\mathbb{R}^2} \lambda(\rho) \partial_l u^m \partial_m \dot{u}^j \partial_j u^l \operatorname{div} u \\
 & + \int_{\mathbb{R}^2} (\rho \lambda'(\rho) - \lambda(\rho)) (\operatorname{div} u)^2 \partial_l \dot{u}^j \partial_j u^l - \int_{\mathbb{R}^2} \lambda(\rho) \partial_l \dot{u}^j \partial_j \dot{u}^l \operatorname{div} u \\
 & + \int_{\mathbb{R}^2} \lambda(\rho) \partial_l \dot{u}^j \partial_j u^m \partial_m u^l \operatorname{div} u - \int_{\mathbb{R}^2} \lambda(\rho) \partial_l \dot{u}^j \operatorname{div} \dot{u} \partial_j u^l \\
 & + \int_{\mathbb{R}^2} \lambda(\rho) \partial_l \dot{u}^j \nabla u^m \partial_m u \partial_j u^l.
 \end{aligned}$$

The last term of (A.11) is

$$\begin{aligned}
 & - \int_{\mathbb{R}^d} \partial_l \ddot{u}^j \partial_j u^l (P(\rho) - \tilde{P}) \\
 & = - \frac{d}{dt} \int_{\mathbb{R}^2} \partial_l \dot{u}^j \partial_j u^l (P(\rho) - \tilde{P}) - \int_{\mathbb{R}^2} \partial_l u^m \partial_m \dot{u}^j \partial_j u^l (P(\rho) - \tilde{P}) \\
 & - \int_{\mathbb{R}^2} \partial_l \dot{u}^j \operatorname{div} u \partial_j u^l (\rho P'(\rho) - P(\rho) + \tilde{P}) + \int_{\mathbb{R}^2} \partial_l \dot{u}^j (P(\rho) - \tilde{P}) \partial_j \dot{u}^l \\
 & - \int_{\mathbb{R}^2} \partial_l \dot{u}^j (P(\rho) - \tilde{P}) \partial_j u^m \partial_m u^l.
 \end{aligned}$$

This completes the computations of the third Hoff energy:

$$\begin{aligned}
 & \int_{\mathbb{R}^2} \rho |\ddot{u}|^2 + \frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) |\mathbb{D}^{jk} \dot{u}|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \lambda(\rho) |\operatorname{div} \dot{u}|^2 \\
 & = -2 \int_{\mathbb{R}^2} \mu(\rho) \partial_k u^l \partial_l \dot{u}^j \mathbb{D}^{jk} \dot{u} + \int_{\mathbb{R}^2} |\mathbb{D}^{jk} \dot{u}|^2 (\rho \mu'(\rho) - \mu(\rho)) \operatorname{div} u \\
 & + \frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j \partial_j u^l \partial_l u^k + \int_{\mathbb{R}^2} \mu(\rho) \partial_k u^m \partial_m \dot{u}^j \partial_j u^l \partial_l u^k \\
 & - \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j \partial_j \dot{u}^l \partial_l u^k + \int_{\mathbb{R}^2} (\rho \mu'(\rho) - \mu(\rho)) \partial_k \dot{u}^j \operatorname{div} u \partial_j u^l \partial_l u^k \\
 & - \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j \partial_j u^l \partial_l \dot{u}^k + \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j \partial_j u^m \partial_m u^l \partial_l u^k \\
 & + \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j \partial_j u^l \partial_l u^m \partial_m u^k + \frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) \partial_j \dot{u}^k \partial_k u^l \partial_l u^j
 \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^2} \mu(\rho) \partial_j u^m \partial_m \dot{u}^k \partial_k u^l \partial_l u^j - \int_{\mathbb{R}^2} \mu(\rho) \partial_j \dot{u}^k \partial_k \dot{u}^l \partial_l u^j \\
& + \int_{\mathbb{R}^2} (\rho \mu'(\rho) - \mu(\rho)) \partial_j \dot{u}^k \operatorname{div} u \partial_k u^l \partial_l u^j - \int_{\mathbb{R}^2} \mu(\rho) \partial_j \dot{u}^k \partial_k u^l \partial_l \dot{u}^j \\
& + \int_{\mathbb{R}^2} \mu(\rho) \partial_j \dot{u}^k \partial_k u^m \partial_m u^l \partial_l u^j + \int_{\mathbb{R}^2} \mu(\rho) \partial_j \dot{u}^k \partial_k u^l \partial_l u^m \partial_m u^j \\
& + 2 \frac{d}{dt} \int_{\mathbb{R}^2} \rho \mu'(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} u \operatorname{div} u + 2 \int_{\mathbb{R}^2} \rho \mu'(\rho) \partial_k u^m \partial_m \dot{u}^j \mathbb{D}^{jk} u \operatorname{div} u \\
& + 2 \int_{\mathbb{R}^2} \rho^2 \mu''(\rho) \partial_k \dot{u}^j \operatorname{div} u \mathbb{D}^{jk} u \operatorname{div} u - 2 \int_{\mathbb{R}^2} \rho \mu'(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} \dot{u} \operatorname{div} u \\
& + \int_{\mathbb{R}^2} \rho \mu'(\rho) \partial_k \dot{u}^j (\partial_j u^m \partial_m u^k + \partial_k u^m \partial_m u^j) \operatorname{div} u - 2 \int_{\mathbb{R}^2} \rho \mu'(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} u \operatorname{div} \dot{u} \\
& + 2 \int_{\mathbb{R}^2} \rho \mu'(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} u \nabla u^m \cdot \partial_m u - \int_{\mathbb{R}^2} \lambda(\rho) \nabla u^m \cdot \partial_m \dot{u} \operatorname{div} \dot{u} \\
& - \frac{1}{2} \int_{\mathbb{R}^2} (\rho \lambda'(\rho) - \lambda(\rho)) |\operatorname{div} \dot{u}|^2 \operatorname{div} u + \frac{d}{dt} \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} \partial_k u^l \partial_l u^k \\
& + \int_{\mathbb{R}^2} \lambda(\rho) \nabla u^m \partial_m \dot{u} \partial_k u^l \partial_l u^k + \int_{\mathbb{R}^2} (\rho \lambda'(\rho) - \lambda(\rho)) \operatorname{div} u \operatorname{div} \dot{u} \partial_k u^l \partial_l u^k \\
& - \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} \partial_k \dot{u}^l \partial_l u^k + \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} \partial_k u^m \partial_m u^l \partial_l u^k \\
& - \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} \partial_k u^l \partial_l \dot{u}^k \\
& + \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} \partial_k u^l \partial_l u^m \partial_m u^k + \frac{d}{dt} \int_{\mathbb{R}^2} \rho \lambda'(\rho) \operatorname{div} \dot{u} (\operatorname{div} u)^2 \\
& + \int_{\mathbb{R}^2} \rho \lambda'(\rho) \partial_j u^m \partial_m \dot{u}^j (\operatorname{div} u)^2 + \int_{\mathbb{R}^2} \rho^2 \lambda''(\rho) \operatorname{div} \dot{u} (\operatorname{div} u)^3 \\
& - 2 \int_{\mathbb{R}^2} \rho \lambda'(\rho) \operatorname{div} \dot{u} \operatorname{div} \dot{u} \operatorname{div} u + 2 \int_{\mathbb{R}^2} \rho \lambda'(\rho) \operatorname{div} \dot{u} \partial_j u^m \partial_m u^j \operatorname{div} u \\
& - \frac{d}{dt} \int_{\mathbb{R}^2} \rho P'(\rho) \operatorname{div} \dot{u} \operatorname{div} u - \int_{\mathbb{R}^2} \rho P'(\rho) \nabla u^m \partial_m \dot{u} \operatorname{div} u \\
& - \int_{\mathbb{R}^2} \rho^2 P''(\rho) \operatorname{div} \dot{u} (\operatorname{div} u)^2 + \int_{\mathbb{R}^2} \rho P'(\rho) (\operatorname{div} \dot{u})^2 - \int_{\mathbb{R}^2} \rho P'(\rho) \operatorname{div} \dot{u} \nabla u^m \partial_m u \\
& - 2 \frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} u \operatorname{div} u - 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_k u^m \partial_m \dot{u}^j \mathbb{D}^{jk} u \operatorname{div} u \\
& - 2 \int_{\mathbb{R}^2} (\rho \mu'(\rho) - \mu(\rho)) \partial_k \dot{u}^j \mathbb{D}^{jk} u (\operatorname{div} u)^2 + 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} \dot{u} \operatorname{div} u \\
& - \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j (\partial_j u^m \partial_m u^k + \partial_k u^m \partial_m u^j) \operatorname{div} u + 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} u \operatorname{div} \dot{u} \\
& - 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_k \dot{u}^j \mathbb{D}^{jk} u \nabla u^m \partial_m u - \frac{d}{dt} \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} (\operatorname{div} u)^2
\end{aligned}$$

$$\begin{aligned}
 & - \int_{\mathbb{R}^2} \lambda(\rho) \nabla u^m \partial_m \dot{u} (\operatorname{div} u)^2 + 2 \int_{\mathbb{R}^2} \lambda(\rho) (\operatorname{div} \dot{u})^2 \operatorname{div} u \\
 & - \int_{\mathbb{R}^2} (\rho \lambda'(\rho) - \lambda(\rho)) \operatorname{div} \dot{u} (\operatorname{div} u)^3 - 2 \int_{\mathbb{R}^2} \lambda(\rho) \operatorname{div} \dot{u} \nabla u^m \partial_m u \operatorname{div} u \\
 & + \frac{d}{dt} \int_{\mathbb{R}^2} \operatorname{div} \dot{u} (P(\rho) - \tilde{P}) \operatorname{div} u + \int_{\mathbb{R}^2} \nabla u^m \partial_m \dot{u} (P(\rho) - \tilde{P}) \operatorname{div} u \\
 & + \int_{\mathbb{R}^2} \operatorname{div} \dot{u} (\operatorname{div} u)^2 (\rho P'(\rho) - P(\rho) + \tilde{P}) - \int_{\mathbb{R}^2} \operatorname{div} \dot{u} (P(\rho) - \tilde{P}) \operatorname{div} \dot{u} \\
 & + \int_{\mathbb{R}^2} \operatorname{div} \dot{u} (P(\rho) - \tilde{P}) \nabla u^m \partial_m u + 2 \frac{d}{dt} \int_{\mathbb{R}^2} \mu(\rho) \partial_l \dot{u}^j \partial_k u^l \mathbb{D}^{jk} u \\
 & + 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_l u^m \partial_m \dot{u}^j \partial_k u^l \mathbb{D}^{jk} u + 2 \int_{\mathbb{R}^2} (\rho \mu'(\rho) - \mu(\rho)) \operatorname{div} u \partial_l \dot{u}^j \partial_k u^l \mathbb{D}^{jk} u \\
 & - 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_l \dot{u}^j \partial_k \dot{u}^l \mathbb{D}^{jk} u + 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_l \dot{u}^j \partial_k u^m \partial_m u^l \mathbb{D}^{jk} u \\
 & - 2 \int_{\mathbb{R}^2} \mu(\rho) \partial_l \dot{u}^j \mathbb{D}^{jk} \dot{u} \partial_k u^l + \int_{\mathbb{R}^2} \mu(\rho) \partial_l \dot{u}^j \partial_k u^l (\partial_j u^m \partial_m u^k + \partial_k u^m \partial_m u^j) \\
 & + \frac{d}{dt} \int_{\mathbb{R}^2} \lambda(\rho) \partial_l \dot{u}^j \partial_j u^l \operatorname{div} u + \int_{\mathbb{R}^2} \lambda(\rho) \partial_l u^m \partial_m \dot{u}^j \partial_j u^l \operatorname{div} u \\
 & + \int_{\mathbb{R}^2} (\rho \lambda'(\rho) - \lambda(\rho)) (\operatorname{div} u)^2 \partial_l \dot{u}^j \partial_j u^l - \int_{\mathbb{R}^2} \lambda(\rho) \partial_l \dot{u}^j \partial_j \dot{u}^l \operatorname{div} u \\
 & + \int_{\mathbb{R}^2} \lambda(\rho) \partial_l \dot{u}^j \partial_j u^m \partial_m u^l \operatorname{div} u - \int_{\mathbb{R}^2} \lambda(\rho) \partial_l \dot{u}^j \operatorname{div} \dot{u} \partial_j u^l \\
 & + \int_{\mathbb{R}^2} \lambda(\rho) \partial_l \dot{u}^j \nabla u^m \partial_m u \partial_j u^l - \frac{d}{dt} \int_{\mathbb{R}^2} \partial_l \dot{u}^j \partial_j u^l (P(\rho) - \tilde{P}) \\
 & - \int_{\mathbb{R}^2} \partial_l u^m \partial_m \dot{u}^j \partial_j u^l (P(\rho) - \tilde{P}) - \int_{\mathbb{R}^2} \partial_l \dot{u}^j \operatorname{div} u \partial_j u^l (\rho P'(\rho) - P(\rho) + \tilde{P}) \\
 & + \int_{\mathbb{R}^2} \partial_l \dot{u}^j (P(\rho) - \tilde{P}) \partial_j \dot{u}^l - \int_{\mathbb{R}^2} \partial_l \dot{u}^j (P(\rho) - \tilde{P}) \partial_j u^m \partial_m u^l.
 \end{aligned}$$

Many terms appearing on the left-hand side above can be grouped into three categories: I_1 , I_2 , and I_3 , each having the respective forms (3.25), (3.26), and (3.27).

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