



Lucas Affonso · Rodrigo Bissacot · Eric O. Endo · Satoshi Handa

Long-range Ising models: Contours, phase transitions and decaying fields

Received 31 July 2022; revised 5 June 2024

Abstract. Inspired by Fröhlich–Spencer and subsequent authors who introduced the notion of contour for long-range systems, we provide a definition of contour and a direct proof for the phase transition for ferromagnetic long-range Ising models on \mathbb{Z}^d , $d \geq 2$. The argument, which is based on a multiscale analysis, works for the sharp region $\alpha > d$ and improves previous results obtained by Park for $\alpha > 3d + 1$, and by Ginibre, Grossmann and Ruelle for $\alpha > d + 1$, where α is the power of the coupling constant. The key idea is to avoid a large number of small contours. As an application, we prove the persistence of the phase transition when we add a polynomially decaying magnetic field with power $\delta > 0$ as $h^*|x|^{-\delta}$, where $h^* > 0$. For $d < \alpha < d + 1$, the phase transition occurs when $\delta > \alpha - d$, and when h^* is small enough over the critical line $\delta = \alpha - d$. For $\alpha \geq d + 1$, $\delta > 1$ is enough to prove the phase transition, and for $\delta = 1$ we have to ask h^* small. The natural conjecture is that this region is also sharp for the phase transition problem when we have a decaying field.

Keywords: long-range Ising model, phase transitions, multiscale analysis, decaying fields.

1. Introduction

Most of our knowledge about statistical mechanics of lattice systems comes from short-range interactions, and one of the main problems in the area is deciding whether a model presents a phase transition or not. The standard strategy to prove the phase transition is

Lucas Affonso: Institute of Mathematics and Statistics (IME-USP), University of São Paulo, Rua do Matão, 1010, Butantã, 05508-090 São Paulo, Brazil; affonso@ime.usp.br, lucas.affonso.pereira@gmail.com

Rodrigo Bissacot (corresponding author): Institute of Mathematics and Statistics (IME-USP), University of São Paulo, Rua do Matão, 1010, Butantã, 05508-090 São Paulo, Brazil; bissacot@usp.br, rodrigo.bissacot@gmail.com

Eric O. Endo: NYU-ECNU Institute of Mathematical Sciences, New York University Shanghai, 567 West Yangsi Road, 200124 Shanghai, P. R. China; eo2@nyu.edu, ericossamiendo@gmail.com

Satoshi Handa: Fujitsu Laboratories Ltd., 4-1-1 Kamikodanaka, Nakahara-ku, Kawasaki 211-8588, Japan; handa.math@gmail.com

Mathematics Subject Classification 2020: 82B05 (primary); 82B26, 82B20 (secondary).

to define a notion of contour for the model to apply the *Peierls* argument [38]. However, many important examples have long-range interactions. In this paper, we consider the ferromagnetic long-range Ising model with an external field in dimension $d \geq 2$. The Hamiltonian of the model is given formally by

$$H = - \sum_{x,y \in \mathbb{Z}^d} J_{xy} \sigma_x \sigma_y - \sum_{x \in \mathbb{Z}^d} h_x \sigma_x, \quad (1.1)$$

where $J_{xy} = \tilde{J}(|x - y|) = J|x - y|^{-\alpha}$, $J > 0$, $\alpha > d$ and $h_x \in \mathbb{R}$. The case considered originally was the zero magnetic field, that is, $h_x = 0$ for every $x \in \mathbb{Z}^d$.

Part of the results for ferromagnetic long-range Ising models in dimension one ($d = 1$) can be summarized as follows: Kac and Thompson conjectured in [31] that the model exhibits a phase transition at low temperatures when $\alpha \in (1, 2]$. The conjecture was proved in 1969 by Dyson in [20] for $\alpha \in (1, 2)$. Dyson's approach applies correlation inequalities between a model that is known nowadays as *hierarchical model*. In a private communication with Dyson, Thompson told that there was no phase transition for $\alpha = 2$ (see the references in [20]). Still, in 1982, Fröhlich and Spencer [23] introduced a notion of one-dimensional contour and proved the phase transition based on a Peierls-type argument.

The contours presented in [23] were inspired by techniques introduced by the same authors in [22] to study two-dimensional systems with continuous symmetries, where breakthroughs were made in the study of the Berezin–Kosterlitz–Thouless transition. The idea consists in organizing the spin flips in (not necessarily connected) contours that satisfy a condition related to their distance between each other (in [24], it is called *Condition D*).

Cassandro, Ferrari, Merola and Presutti [12] attempted to extend the contour argument to different exponents for the interactions introducing a more geometric approach in terms of triangle configurations to the problem of the phase transition. Their modification comes with an additional condition on the coupling for nearest neighbors, namely, $\tilde{J}(1) \gg 1$. The authors showed the phase transition assuming that $\alpha \in (2 - \alpha^+, 2]$, where $\alpha^+ = \log(3)/\log(2) - 1$. Many results were obtained following either the contour approach of Fröhlich and Spencer or Cassandro, Ferrari, Merola and Presutti, such as cluster expansion [14, 28, 29], phase separation [13], phase transition for the one-dimensional long-range model with a random field [15] and condensation for the corresponding lattice gas model [30].

Recently, Bissacot, Endo, van Enter, Kimura and Ruszel [9], based on the contour argument of [12], considered the model with a presence of the decaying magnetic field $(h_x)_{x \in \mathbb{Z}}$. Moreover, restricting the region of α to $(2 - \alpha^*, 2]$, where $\alpha^+ < \alpha^*$ satisfies $\sum_{n \geq 1} n^{-\alpha^*} = 2$, they removed the condition of the nearest neighbor's coupling $\tilde{J}(1) \gg 1$.

For multidimensional models ($d \geq 2$), one of the only notions of contour available beyond the short-range case was proposed by Park [36, 37], on an extension of Pirogov–Sinai theory. Park's arguments work for pair interactions and finite state space, but the interaction's exponent has to be assumed $\alpha > 3d + 1$. Before this, in [26], Ginibre, Grossmann and Ruelle extended the usual Peierls argument for a class of interactions

more general than nearest-neighbor. Their result implies that, if we add a long-range perturbation to the nearest-neighbor model of the form $K_{x,y} = K|x - y|^{-\alpha}$, with a real constant K , the cost of erasing a contour γ from a family of contours Γ containing γ is bounded below by $2(J - |K|c_\alpha)|\gamma|$, where $c_\alpha = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k|^{1-\alpha} - 2d$. Notice that this lower bound is positive for K small enough and $\alpha > d + 1$. Indeed, their result holds in more generality: the constant K may be replaced by a family of real constants $K_{x,y}$ with $\sup_{\{x,y\} \in \mathbb{Z}^d} |K_{x,y}|$ small enough. Notice that the sign of $K_{x,y}$ may change, making their result applicable for antiferromagnetic long-range perturbations, for instance. Since the methods of Ginibre–Grossmann–Ruelle hold for antiferromagnetic perturbations, the results by van Enter [43] and Biskup, Chayes and Kivelson [5] show that their enhanced Peierls argument is not generalizable for $d < \alpha \leq d + 1$, since an arbitrarily small antiferromagnetic long-range interaction with this decay forces the magnetization to be zero.

One of our main contributions in this paper is to present a proof for the phase transition at low temperature for the *ferromagnetic long-range Ising model on the lattice \mathbb{Z}^d* with $d \geq 2$. Our proof combines ideas from [12, 22, 24, 36, 37, 41], and we are able to obtain the Peierls argument in the case of zero magnetic field for the sharp region $\alpha > d$. Notice that the positivity of the magnetization follows from the GKS inequalities and the Peierls argument for the nearest neighbor Ising model. The novelty in our results resides in the fact that we have a contour argument applicable to models where the correlation inequalities do not imply the phase transition or even do not hold, such as the model with a decaying field in the former case and to random magnetic fields [2], in the later.

It is well known that ferromagnetic Ising models do not present phase transition in the presence of a uniform magnetic field; see Lee and Yang [34, 44]. When the field is not constant, the situation drastically changes. There is some literature about models with fields, including the famous case of the random field. See, for instance, [3, 4, 8, 10, 11, 16, 27, 40].

The problem of studying the phase diagram with a decaying field was introduced in [7]. Since the pressure of the Ising model with a decaying magnetic field $(h_x)_{x \in \mathbb{Z}^d}$ is equal to the pressure of the Ising model with zero magnetic field, it may induce the belief that these two models should present the same phase diagram. In [6], the authors studied the phase diagram of the ferromagnetic nearest-neighbor Ising model on the lattice \mathbb{Z}^d in the presence of the spatially dependent magnetic field $(h_x)_{x \in \mathbb{Z}^d}$ given by $h_x = h^*|x|^{-\delta}$, when $x \neq 0$, and $h_0 = h^*$, where h^* and δ are positive constants. They studied the behavior of the model at low temperatures according to the exponent of the decaying field: the model undergoes a phase transition at low temperature for $\delta > 1$, and we have uniqueness for $0 < \delta < 1$. Afterward, Cioletti and Vila [17] closed the gap, concluding the uniqueness of the Gibbs measure when $0 < \delta < 1$ for all temperatures, the argument uses the Fortuin–Kasteleyn representation.

When we add a decaying magnetic field, it is not possible to obtain the phase transition for the long-range Ising model from the short-range case, so the natural strategy is to use the Peierls argument, as we see in [6, 7, 9]. In fact, these models show that arguments using contours are sharp with respect to the decaying of the magnetic field.

In the present paper, we extended the analysis in [6] from the short-range to the long-range case considering decaying external fields in Hamiltonian (1.1). By a similar approach of Fröhlich and Spencer, we define a notion of contour on this model to show the phase transition at low temperature when $d < \alpha < d + 1$ and $\delta > \alpha - d$, and when $\alpha \geq d + 1$ and $\delta > 1$.

To understand the heuristics of our result on phase transitions, let us consider the configurations

$$\sigma_x = \begin{cases} +1 & \text{if } x \in B_R(z), \\ -1 & \text{otherwise,} \end{cases}$$

where $B_R(z) \subset \mathbb{Z}^d$ is the closed ball in the ℓ_1 -norm centered in z with radius $R \geq 0$. Let Ω_c be the collection of all such configurations together with the configuration that is -1 in all \mathbb{Z}^d and, for fixed $\Lambda \Subset \mathbb{Z}^d$, let $\Omega_{c,\Lambda}$ be the subset of configurations where $B_R(z) \subset \Lambda$ together with the configuration that is -1 in all \mathbb{Z}^d . Then, we have

$$\mu_{\beta,\Lambda}^-(\sigma_0 = +1 | \Omega_{c,\Lambda}) \leq \sum_{R \geq 1} R^d \exp\left(-\beta\left(cR^{d-1} + F_{B_R} - \sum_{x \in B_R(0)} h_x\right)\right), \quad (1.2)$$

where the quantity F_{B_R} is defined as

$$F_{B_R} := \sum_{\substack{x \in B_R(0) \\ y \in B_R(0)^c}} J_{xy}.$$

One can understand this quantity as a surface energy term, and it has different asymptotics depending on the parameters α and d . Denoting by $f \approx g$ the fact that given two functions $f, g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, there exist positive constants $A' := A'(\alpha, d)$, $A := A(\alpha, d)$ such that $A'f(x) \leq g(x) \leq Af(x)$ for every $x > 0$ large enough, we have

$$F_{B_R} \approx \begin{cases} R^{2d-\alpha} & \text{if } d < \alpha < d + 1, \\ R^{d-1} \log(R) & \text{if } \alpha = d + 1, \\ R^{d-1} & \text{if } \alpha > d + 1. \end{cases}$$

For the proof, see [4, Propositions 3.1 and 3.4]. For our purposes, we will need estimates for more general subsets than balls, see Lemma 4.4. Now, inequality (1.2) shows us that the phase transition occurs when $\delta > \alpha - d$. To see this, observe that the surface energy term should be larger than the contribution of field, which is given by

$$\sum_{x \in B_R(0)} h_x = O(R^{d-\delta}).$$

Our main result can be summarized by the Figure 1.

This paper is divided as follows. In Section 2, we give some basic definitions and define the model. In Section 3, we introduce the notion of contour for the long-range Ising model and estimates for the entropy of the contours. In Section 4, we present the proof of the phase transition for the region indicated in Figure 1.

This paper is contained in the Ph.D. thesis of Lucas Affonso [1].

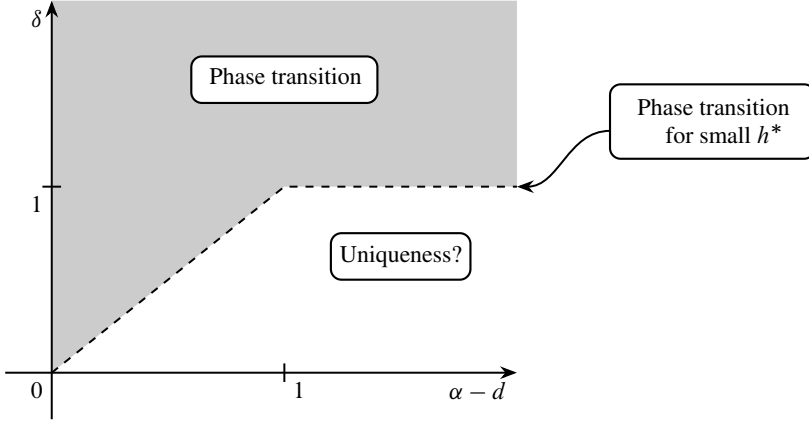


Fig. 1. The phase diagram for the long-range Ising model depending on α and δ .

2. Definitions and notations

2.1. The model

Although most of the definitions and results cited in this section are available in greater generality, we choose to consider only the case where the lattice is \mathbb{Z}^d for $d \geq 2$ and the state space is $E = \{-1, +1\}$. Our configuration space will be $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$. We will denote by σ^+ (resp. by σ^-) the configuration which assume the same value equal to $+1$ (resp. -1) in all vertices, i.e., $\sigma_x^\pm = \pm 1$ for all $x \in \mathbb{Z}^d$. We write $\Lambda \Subset \mathbb{Z}^d$ for finite subsets of the lattice. Fixing such Λ , we define the local configuration space $\Omega_\Lambda = \{-1, +1\}^\Lambda$ and, given $\omega \in \Omega$, we define $\Omega_\Lambda^\omega \subset \Omega$ as the subset of configurations such that $\sigma_{\Lambda^c} = \omega_{\Lambda^c}$, i.e., $\sigma_x = \omega_x$ for all $x \in \Lambda^c$. We say that σ has $+$ -boundary condition (resp. $-$ -boundary condition) when $\sigma \in \Omega_\Lambda^+$ (resp. $\sigma \in \Omega_\Lambda^-$).

For each $\Lambda \Subset \mathbb{Z}^d$, $\omega \in \Omega$, and $\mathbf{h} = (h_x)_{x \in \mathbb{Z}^d}$ a collection of real numbers, the *Hamiltonian function of the long-range Ising model* $H_{\Lambda, \mathbf{h}}^\omega: \Omega_\Lambda^\omega \rightarrow \mathbb{R}$ is given by

$$H_{\Lambda, \mathbf{h}}^\omega(\sigma) = - \sum_{\{x, y\} \subset \Lambda} J_{xy} \sigma_x \sigma_y - \sum_{\substack{x \in \Lambda \\ y \in \Lambda^c}} J_{xy} \sigma_x \omega_y - \sum_{x \in \Lambda} h_x \sigma_x, \quad (2.1)$$

where, for $\alpha > d$, $J > 0$ and $x, y \in \mathbb{Z}^d$ with $x \neq y$, the *coupling constant* J_{xy} is given by

$$J_{xy} = \begin{cases} \frac{J}{|x - y|^\alpha} & \text{if } x \neq y, \\ 0 & \text{otherwise,} \end{cases} \quad (2.2)$$

where $|z|$ is the ℓ^1 -norm of $z \in \mathbb{Z}^d$. For $\delta, h^* > 0$, the *magnetic field* $(h_x)_{x \in \mathbb{Z}^d}$ is given by

$$h_x = \begin{cases} h^* & \text{if } x = 0, \\ \frac{h^*}{|x|^\delta} & \text{if } x \neq 0. \end{cases} \quad (2.3)$$

For a subset $\Lambda \subset \mathbb{Z}^d$, consider the σ -algebra \mathcal{F}_Λ generated by the cylinder sets supported on Λ . The basic object of the study in the classical statistical mechanics is the collection of probability measures in $(\Omega, \mathcal{F}_{\mathbb{Z}^d})$ called *finite volume Gibbs measures* defined by

$$\mu_{\beta, \mathbf{h}, \Lambda}^\omega(\sigma) := \frac{e^{-\beta H_{\Lambda, \mathbf{h}}^\omega(\sigma)}}{Z_{\beta, \mathbf{h}, \Lambda}^\omega} \quad \text{if } \sigma_{\Lambda^c} = \omega_{\Lambda^c},$$

and $\mu_{\beta, \mathbf{h}, \Lambda}^\omega(\sigma) = 0$ otherwise. Here $\beta > 0$ is the inverse temperature, and the normalization factor $Z_{\beta, \mathbf{h}, \Lambda}^\omega$, called *partition function*, is defined by

$$Z_{\beta, \mathbf{h}, \Lambda}^\omega := \sum_{\sigma \in \Omega_\Lambda} e^{-\beta H_{\Lambda, \mathbf{h}}^\omega(\sigma)}.$$

The Gibbs measure with $+$ -boundary (resp. $-$ -boundary) condition is when $\omega_x = +1$ (resp. -1) for every $x \in \mathbb{Z}^d$. We say that a sequence of finite subsets $(\Lambda_n)_{n \in \mathbb{N}}$ *invades* \mathbb{Z}^d , denoted by $\Lambda_n \nearrow \mathbb{Z}^d$, if for every finite subset Λ there exists $N > 0$ such that $\Lambda \subset \Lambda_n$ for all $n \geq N$. Since the configuration space Ω is compact, the space of probability measures is a weak* compact set. Therefore, the net defined by the collection of finite volume Gibbs measures has a convergent subsequence. We define the set \mathcal{G}_β as the closed convex hull of all the limits obtained by this procedure, i.e.,

$$\mathcal{G}_\beta := \overline{\text{conv}} \left\{ \mu_\beta : \mu_\beta = w^* \text{-} \lim_{\Lambda' \nearrow \mathbb{Z}^d} \mu_{\beta, \Lambda'}^\omega, \Lambda' \Subset \mathbb{Z}^d \text{ invades the lattice} \right\}.$$

The set \mathcal{G}_β is always nonempty in our case by a simple application of the Banach–Alaoglu theorem. We say that the model has *uniqueness at β* if $|\mathcal{G}_\beta| = 1$ and it undergoes to a *phase transition at β* if $|\mathcal{G}_\beta| > 1$.

3. Contours

Contours are geometric objects first introduced in a seminal paper of Peierls [38]. The technique is known nowadays as *Peierls' argument*. Many attempts were made to extend the ideas of the Peierls argument to the other systems [18, 19, 36, 39, 45]. The most successful generalization was made by Pirogov and Sinai in [39], and later improved by Zahradník [45]. Their work is known as *Pirogov–Sinai theory*, and it is applied to models with short-range interactions that may not have symmetries.

Park, in [36, 37], extended the theory of Pirogov–Sinai to systems with two-body long-range interactions that satisfy a condition equivalent to (2.2) having decay $\alpha > 3d + 1$. Inspired by [23], in this section we will introduce new contours more suitable to study long-range two-body interactions.

Definition 3.1. Given $\sigma \in \Omega$, a point $x \in \mathbb{Z}^d$ is called $+$ (resp. $-$) *correct* if $\sigma_y = +1$, (resp. -1) for all points y in $B_1(x)$. The *boundary* of σ , denoted by $\partial\sigma$, is the set of all points in \mathbb{Z}^d that are neither $+$ nor $-$ correct.

We recall the reader that $B_1(x)$ is the ball of radius 1 with respect to the ℓ^1 -norm. Note that the boundary can be an infinite subset of \mathbb{Z}^d . Indeed, if we take $\sigma \in \Omega$ defined by

$$\sigma_x = \begin{cases} +1 & \text{if } |x| \text{ is even,} \\ -1 & \text{otherwise,} \end{cases}$$

it is easy to see that every point in \mathbb{Z}^d is incorrect with respect to the configuration σ , and thus $\partial\sigma = \mathbb{Z}^d$. This situation can be avoided by restricting our attention only to configurations with finite boundary set. By definition of incorrectness, for a configuration σ to have a finite boundary set, it must satisfy $\sigma \in \Omega_\Lambda^+$ or Ω_Λ^- for some subset $\Lambda \Subset \mathbb{Z}^d$.

For each subset $\Lambda \Subset \mathbb{Z}^d$, we call $\Lambda^{(0)}$ the unique unbounded connected component of Λ^c . Then, we define the *volume* by $V(\Lambda) = (\Lambda^{(0)})^c$. Note that this set is a union of simply connected sets that contains Λ and is the smallest one in the subset order. The *interior* is defined by $I(\Lambda) = \Lambda^c \setminus \Lambda^{(0)}$.

In Pirogov–Sinai theory, the construction of the contours starts by considering first the connected subsets of the boundary $\partial\sigma$. This procedure is troublesome for long-range models since each point of the lattice interacts with all the other points. Thus, the contours always have a nonvanishing interaction between themselves. To avoid this problem, we will divide the boundary of a configuration in a way where the interaction between them will be negligible in a sense to be specified later. Inspired by Fröhlich and Spencer [23], we introduce the following definition.

Definition 3.2. Fix real numbers $M, a, r > 0$. For each configuration $\sigma \in \Omega$ with finite boundary $\partial\sigma$, a set $\Gamma(\sigma) := \{\bar{\gamma} : \bar{\gamma} \subset \partial\sigma\}$ is called an (M, a, r) -*partition* when the following conditions are satisfied:

- (A) They form a partition of $\partial\sigma$, i.e., $\bigcup_{\bar{\gamma} \in \Gamma(\sigma)} \bar{\gamma} = \partial\sigma$ and $\bar{\gamma} \cap \bar{\gamma}' = \emptyset$ for distinct elements of $\Gamma(\sigma)$.
- (B) For all $\bar{\gamma} \in \Gamma(\sigma)$, there exist $1 \leq n \leq 2^r - 1$ and a family of subsets $(\bar{\gamma}_k)_{1 \leq k \leq n}$ satisfying
 - (B1) $\bar{\gamma} = \bigcup_{1 \leq k \leq n} \bar{\gamma}_k$,
 - (B2) for all distinct $\bar{\gamma}, \bar{\gamma}' \in \Gamma(\sigma)$,

$$\text{dist}(\bar{\gamma}, \bar{\gamma}') > M \min \left\{ \max_{1 \leq k \leq n} \text{diam}(\bar{\gamma}_k), \max_{1 \leq j \leq n'} \text{diam}(\bar{\gamma}'_j) \right\}^a, \quad (3.1)$$

where $(\bar{\gamma}'_j)_{1 \leq j \leq n'}$ is the family given by item (B1) for $\bar{\gamma}'$.

Note that the sets $\bar{\gamma} \in \Gamma(\sigma)$ may be disconnected. We also stress that in condition (B1) the sets $(\bar{\gamma}_k)_{1 \leq k \leq n}$ may not be disjoint. Some of the next results are true for any $M, a, r > 0$, as the existence of (M, a, r) -partition for any configuration σ with finite boundary $\partial\sigma$, see Proposition 3.7.

Remark 3.3. For the main purposes of this paper, which is the proof of the phase transition, the constant a is chosen as $a := a(\alpha, d) = \max\{(d + 1 + \varepsilon)/(\alpha - d), d + 1 + \varepsilon\}$, for some $\varepsilon > 0$ fixed and r given by $r = \lceil \log_2(a + 1) \rceil + d + 1$, where $\lceil x \rceil$ is the smallest

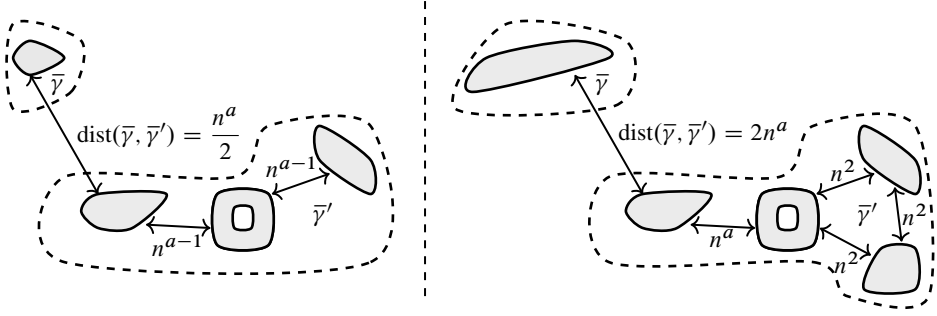


Fig. 2. Consider $M = 1$, $r = 2$, and $a > 2$. For the image on the left, consider that all the connected components (grey regions) of $\bar{\gamma}'$ have diameters equal to n and $\text{diam}(\bar{\gamma}) = 3n$. In this case, there is no family of subsets of $\bar{\gamma}'$ satisfying condition (B). A possible (M, a, r) -partition for this case is $\Gamma(\sigma) = \{\bar{\gamma} \cup \bar{\gamma}'\}$. For the figure on the right, consider that all the connected components of $\bar{\gamma}'$ have diameter n and $\text{diam}(\bar{\gamma}) = 3n^2$. Notice that, in this case, the family of subsets of $\bar{\gamma}'$ satisfying inequality (3.1) must have $n' > 2^r - 1$.

integer greater than or equal to x . The motivation for these choices will be clear in the proofs. The specific value of $M := M(\alpha, d)$ will be chosen later, but we will assume for now that $M \geq 1$. In Figure 2, we have examples of (M, a, r) -partitions and a discussion about conditions that forbid the subsets to be a (M, a, r) -partition.

In general, there are many possible (M, a, r) -partitions for a given $\sigma \in \Omega$ with finite boundary and fixed $M, a, r > 0$. If we have two (M, a, r) -partitions Γ and Γ' , we say that Γ is finer than Γ' if for every $\bar{\gamma} \in \Gamma$, there is $\bar{\gamma}' \in \Gamma'$ with $\bar{\gamma} \subseteq \bar{\gamma}'$. The finest (M, a, r) -partition also satisfies stronger separation properties than what is stated in condition (A). It will actually satisfy the following condition (see also Figure 3):

(A1) For any $\bar{\gamma}, \bar{\gamma}' \in \Gamma(\sigma)$, $\bar{\gamma}'$ is contained in only one connected component of $(\bar{\gamma})^c$.

The proposition below is important in the proof that there is a finest (M, a, r) -partition.

Proposition 3.4. *Let σ be a configuration with finite $\partial\sigma$, and let Γ, Γ' be two (M, a, r) -partitions. Then the set defined as*

$$\Gamma \cap \Gamma' := \{\gamma \cap \gamma' : \gamma \in \Gamma, \gamma' \in \Gamma', \gamma \cap \gamma' \neq \emptyset\}$$

is also an (M, a, r) -partition. Moreover, if one of them satisfies condition (A1), then $\Gamma \cap \Gamma'$ also satisfies condition (A1).

Proof. The existence of nontrivial (M, a, r) -partitions will be given in the next section. Consider two (M, a, r) -partitions Γ and Γ' . We will show that we can build an (M, a, r) -partition that is finer than Γ and Γ' . We will prove that $\Gamma \cap \Gamma'$ is an (M, a, r) -partition. It is easy to see that $\Gamma \cap \Gamma'$ forms a partition of $\partial\sigma$. For condition (B), consider $\bar{\gamma}_i \cap \bar{\gamma}'_i \in \Gamma \cap \Gamma'$ and choose as a family of subsets

$$(\bar{\gamma}_i \cap \bar{\gamma}'_i)_k := \bar{\gamma}_i \cap \bar{\gamma}'_{i,k}$$

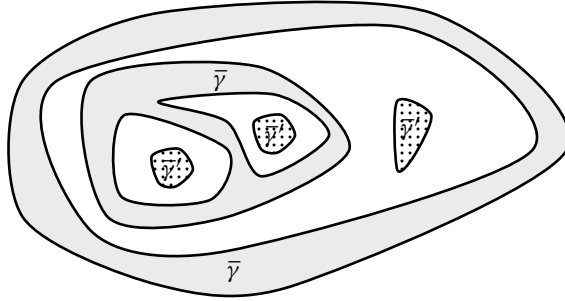


Fig. 3. To illustrate how condition (A1) works, consider the figure above. In this case, the connected components of $\bar{\gamma}'$ are dotted and the connected components of $\bar{\gamma}$ are grey. One can readily see that there is a connected component of $(\bar{\gamma})^c$ that has a nonempty intersection with $V(\bar{\gamma}')$ but does not fully contain it. In order to fix such a problem, one should separate $\bar{\gamma}'$ in three different sets.

for all $k \in \{1, \dots, n'\}$ where the intersection is not empty. Here $(\bar{\gamma}'_{i,k})_{1 \leq k \leq n'}$ is the family of subsets for $\bar{\gamma}'$ given by condition (B). We have

$$\begin{aligned} \text{dist}(\bar{\gamma}_1 \cap \bar{\gamma}'_1, \bar{\gamma}_2 \cap \bar{\gamma}'_2) &\geq \text{dist}(\bar{\gamma}'_1, \bar{\gamma}'_2) > M \min \left\{ \max_{1 \leq k \leq n'} \text{diam}(\bar{\gamma}'_{1,k}), \max_{1 \leq j \leq m'} \text{diam}(\bar{\gamma}'_{2,j}) \right\}^a \\ &\geq M \min \left\{ \max_{1 \leq k \leq n'} \text{diam}((\bar{\gamma}_1 \cap \bar{\gamma}'_1)_k), \max_{1 \leq j \leq m'} \text{diam}((\bar{\gamma}_2 \cap \bar{\gamma}'_2)_j) \right\}^a. \end{aligned}$$

Let $\bar{\gamma}_i \in \Gamma$ and $\bar{\gamma}'_i \in \Gamma'$ be such that $\bar{\gamma}_i \cap \bar{\gamma}'_i \in \Gamma \cap \Gamma'$ for $i = 1, 2$. Suppose that Γ satisfies condition (A1). Then, there is a connected component A of $(\bar{\gamma}_2)^c$ that contains $\bar{\gamma}_1$. Hence,

$$\bar{\gamma}_1 \cap \bar{\gamma}'_1 \subset A \subset (\bar{\gamma}_2 \cap \bar{\gamma}'_2)^c. \quad \blacksquare$$

Remark 3.5. Since there are a finite number of (M, a, r) -partitions for a finite boundary set $\partial\sigma$, we can construct the finest one by intersecting all of them, following the above construction. If we had two (M, a, r) -partitions that have the property of being the finest, we could produce another (M, a, r) -partition finer than both, yielding a contradiction.

Remark 3.6. In their one-dimensional paper [23], Fröhlich and Spencer assumed that they would choose the finest partition of spin flips satisfying their Condition D, but the uniqueness of this partition would not be a problem, since you just need to fix a partition for the phase transition argument to hold. It was in [28] where Imbrie settled this question and our proof is inspired by his argument.

3.1. The definition and a discussion about contours on long-range Ising models

For a fixed configuration σ with finite boundary $\partial\sigma$, the (M, a, r) -partitions will be the *support of the contours*, subsets of \mathbb{Z}^d where every point is incorrect. The starting point for contour-based phase transition arguments for $d \geq 2$ long-range models was Park's extension of Pirogov–Sinai theory, developed in [36, 37]. His methods applied to the

problem in consideration in this work only allowed us to study long-range interactions with $\alpha > 3d + 1$. At this point, although we are strongly inspired by the papers of Fröhlich and Spencer [22–24] (see also [28, 29]) and the previous authors, we implemented slight modifications that allow us to cover all the region $\alpha > d$.

The first point is that, differently from the original papers of Fröhlich and Spencer, we have no arithmetic condition over the (M, a, r) -partitions, which means we do not ask the sum of the spins should be zero over the support of the contours. In particular, there are no constraints over the size of the contours, they could have an even or odd number of points of \mathbb{Z}^d . The definition is stated only in terms of the distances among subsets of \mathbb{Z}^d .

In order to control the entropy (the quantity) of contours, we needed to introduce a parameter r . It is worthwhile to stress that in the original works of Fröhlich and Spencer [22–24] the parameter $r = 1$.

The exponent a plays an important role in our arguments. When α is close to d , the interaction is stronger, and the contours should be far from each other. In the original papers of Fröhlich and Spencer, a is chosen as a fixed number or it belongs to a finite interval as follows. In the first paper [22], for two-dimensional models, we have $3/2 < a < 2$. In the paper about the Dyson model when $d = 1$ and $\alpha = 2$, we have $a = 3/2$. In the paper about multidimensional random Schrödinger operators, they assumed $1 \leq a < 2$, in [24, Section 4]. These papers use the idea of multiscale analysis to study different problems, which strongly inspired us in the definition and construction of our contours. A brief summary mentioning the multiscale methods and its power is in [42] and at the references therein.

After this, Cassandro, Ferrari, Merola, Presutti [12] defined the contours with a more geometrical approach, using triangles. Their energy bounds for erasing a contour depend on these triangles. They choose $a = 3$ for one-dimensional long-range Ising models with $\alpha \in (2 - \alpha^+, 2]$, where $\alpha^+ = \log(3)/\log(2) - 1$. This is an important point; while our arguments work for any $2 \leq d < \alpha$, Littin and Picco proved that in the one-dimensional case, it is impossible to produce a direct proof of the phase transition using the Peierls argument and the definition of contour in [12], although they prove the phase transition for the entire region with $\alpha \in (1, 2]$. The papers [12, 35] also assume the extra assumption $J(1) \gg 1$, which means that first neighbor interaction should be big enough, and the model simulates a short-range behavior. Recently, it was proved that this extra assumption could be removed [9, 33].

The following proposition guarantees the existence of an (M, a, r) -partition for each configuration σ with finite boundary.

Proposition 3.7. *Fix real numbers $M, a, r > 0$. For every $\sigma \in \Omega$ with finite boundary, there is an (M, a, r) -partition $\Gamma(\sigma)$. Moreover, it satisfies condition (A1).*

Proof. For each $x \in \mathbb{Z}^d$ and $n > 0$, we define an n -cube $C_n(x) \subset \mathbb{Z}^d$ as

$$C_n(x) := \left(\prod_{i=1}^d [2^{n-1}x_i - 2^{n-1}, 2^{n-1}x_i + 2^{n-1}] \right) \cap \mathbb{Z}^d. \quad (3.2)$$

These cubes have side length 2^n and center at the point $2^{n-1}x$. For $n = 0$, we establish the convention that $C_0(x) = x$, for any point $x \in \mathbb{Z}^d$. For each $\Lambda \Subset \mathbb{Z}^d$ and $n \geq 0$, we define $\mathcal{C}_n(\Lambda)$ as a minimal cover of Λ by n -cubes.

For each cover $n \geq 0$, we define the graph $G_n(\Lambda) = (v(G_n(\Lambda)), e(G_n(\Lambda)))$ with vertices $v(G_n(\Lambda)) := \mathcal{C}_n(\Lambda)$ and edges $e(G_n(\Lambda)) := \{(C_n(x), C_n(y)) : \text{dist}(C_n(x), C_n(y)) \leq Md^a 2^{an}\}$. Note that $d2^n$ is the diameter in the ℓ^1 -norm of an n -cube. Let $\mathcal{G}_n(\Lambda)$ be the set of all connected components of the graph $G_n(\Lambda)$ and, for each $G \in \mathcal{G}_n(\Lambda)$, define

$$\gamma_G := \bigcup_{C_n(x) \in v(G)} (\Lambda \cap C_n(x)).$$

We are ready to establish the existence of an (M, a, r) -partition for the boundary $\partial\sigma$ of a configuration σ . Set $\partial\sigma_0 := \partial\sigma$ and

$$\mathcal{P}_0 := \{G \in \mathcal{G}_0(\partial\sigma_0) : |v(G)| \leq 2^r - 1\}.$$

Notice that this set separates all points that are distant by at least Md^a . Define inductively, for $n \geq 1$,

$$\mathcal{P}_n := \{G \in \mathcal{G}_n(\partial\sigma_n) : |v(G)| \leq 2^r - 1\},$$

where $\partial\sigma_n := \partial\sigma_{n-1} \setminus \bigcup_{G \in \mathcal{P}_{n-1}} \gamma_G$, for $n \geq 1$. Since the n -cubes invade the lattice, when we continue increasing n , there exists $N \geq 0$ such that $\partial\sigma_n = \emptyset$ for every $n \geq N$. In this case, we define $\mathcal{P}_n = \emptyset$. Let $\mathcal{P} := \bigcup_{n \geq 0} \mathcal{P}_n$. We are going to show that the family $\Gamma(\sigma) := \{\gamma_G : G \in \mathcal{P}\}$ is an (M, a, r) -partition.

In order to show that condition (B) is satisfied, we will construct families of subsets, with at most $2^r - 1$ elements, where inequality (3.1) is verified. We will write $G_n := G_n(\partial\sigma_n)$ to simplify our notation. Take distinct $\gamma_G, \gamma_{G'} \in \Gamma(\sigma)$. There are positive integers n, m , with $n \leq m$, such that $G \in \mathcal{P}_n$ and $G' \in \mathcal{P}_m$. Let G'' be the subgraph of G_n such that $v(G'')$ covers $\gamma_{G'}$ and it is minimal in the sense that all other subgraphs G''' of G_n satisfying this property have $|v(G''')| \geq |v(G'')|$. Thus, defining $\gamma_k = \gamma_G \cap C_n(x_k)$, for each $C_n(x_k) \in v(G)$, we have

$$\text{dist}(\gamma_G, C_n(z)) = \min_{1 \leq k \leq |v(G)|} \text{dist}(\gamma_k, C_n(z)) \geq \min_{1 \leq k \leq |v(G)|} \text{dist}(C_n(x_k), C_n(z)) \quad (3.3)$$

for each $C_n(z) \in v(G'')$. There is no edge between the subgraph G'' and the connected component G , by construction. Thus,

$$\text{dist}(C_n(x_k), C_n(z)) > Md^a 2^{an}.$$

Consider, also, the sets $\gamma'_j = \gamma_{G'} \cap C_m(y_j)$, where $C_m(y_j) \in v(G')$. Then

$$\begin{aligned} \text{dist}(\gamma_G, \gamma_{G'}) &= \min_{1 \leq j \leq |v(G')|} \min_{\substack{C_n(z) \in v(G'') \\ C_n(z) \cap \gamma'_j \neq \emptyset}} \text{dist}(\gamma_G, \gamma'_j \cap C_n(z)) \\ &\geq \min_{C_n(z) \in v(G'')} \text{dist}(\gamma_G, C_n(z)). \end{aligned}$$

Using inequality (3.3), we arrive at the inequality $\text{dist}(\gamma_G, \gamma_{G'}) > Md^a 2^{an}$. Note that, by construction, both $\{\gamma_k\}_{1 \leq k \leq |v(G)|}$ and $\{\gamma'_j\}_{1 \leq j \leq |v(G')|}$ satisfy

$$\max_{1 \leq k \leq |v(G)|} \text{diam}(\gamma_k) \leq d2^n \quad \text{and} \quad \max_{1 \leq j \leq |v(G')|} \text{diam}(\gamma'_j) \leq d2^m,$$

respectively. Since we assumed that $n \leq m$, we get

$$\min \left\{ \max_{1 \leq k \leq |v(G)|} \text{diam}(\gamma_k), \max_{1 \leq j \leq |v(G')|} \text{diam}(\gamma'_j) \right\}^a \leq d^a 2^{an},$$

and we proved that the family $\Gamma(\sigma)$ satisfies condition (B).

In order to establish condition (A), we first note that the equality $\partial\sigma = \bigcup_{G \in \mathcal{P}} \gamma_G$ follows by construction. The elements of $\Gamma(\sigma)$ are pairwise disjoint since inequality (3.1) is satisfied.

Eventually, let us present the proof of condition (A1) for our constructed partition. Let $\gamma_G, \gamma_{G'} \in \Gamma(\sigma)$, with $V(\gamma_G) \cap V(\gamma_{G'}) \neq \emptyset$. There are positive integers n, m satisfying $n \leq m$ such that $G \in \mathcal{P}_n$ and $G' \in \mathcal{P}_m$. Consider G'' , as before, the minimal subgraph of G_n that covers $\gamma_{G'}$. Since $\gamma_G \cap \gamma_{G'} = \emptyset$, it holds that $\gamma_G \subset (\gamma_{G'})^c$. We will show that γ_G must be contained in only one connected component of $(\gamma_{G'})^c$. Every n -cube $C_n(x) \in v(G)$ cannot have a nonempty intersection with $\gamma_{G'}$, since the last one is covered by the n -cubes in $v(G'')$ and there is no edge between G and G'' . This is sufficient to conclude that each n -cube in $v(G)$ is in only one connected component of $(\gamma_{G'})^c$.

If $(\gamma_{G'})^c$ has only one connected component or $|v(G)| = 1$, there is nothing to prove. Suppose, by contradiction, that there exist two n -cubes $C_n(x), C_n(x') \in v(G)$ in different connected components of $(\gamma_{G'})^c$. We claim

$$\text{dist}(C_n(x), C_n(x')) \geq 2Md^a 2^{an}. \quad (3.4)$$

Indeed, take two points $z \in C_n(x)$ and $z' \in C_n(x')$ such that $\text{dist}(C_n(x), C_n(x')) = |z - z'|$. Let $\lambda_{z,z'}$ be a minimal path in \mathbb{Z}^d starting at z and ending at z' . Note that $|\lambda_{z,z'}| = |z - z'|$. Since $C_n(x)$ and $C_n(x')$ are in different connected components of $(\gamma_{G'})^c$, there must exist $y \in \lambda_{z,z'} \cap \gamma_{G'}$. We can break $\lambda_{z,z'}$ as the union of minimal paths $\lambda_{z,y}$ and $\lambda_{y,z'}$. This fact implies

$$\begin{aligned} \text{dist}(C_n(x), C_n(x')) &= |z - y| + |y - z'| \geq \min_{y' \in \gamma_{G'}} [|z - y'| + |y' - z'|] \\ &\geq \min_{y' \in \gamma_{G'}} [\text{dist}(C_n(x), y') + \text{dist}(C_n(x'), y')] \\ &\geq \min_{C_n(z) \in v(G'')} [\text{dist}(C_n(x), C_n(z)) + \text{dist}(C_n(x'), C_n(z))] \\ &\geq 2Md^a 2^{an}, \end{aligned}$$

where the last inequality is due to the fact that the subgraphs G and G'' have no edge between them. Inequality (3.4) is valid for any pair of n -cubes in different connected components of $(\gamma_{G'})^c$, thus our discussion implies that $C_n(x)$ and $C_n(x')$ are vertices of two different connected components. This cannot happen since G is connected, we arrive at a contradiction. ■

Definition 3.8. For $\Lambda \subset \mathbb{Z}^d$, we define the *inner boundary*

$$\partial_{\text{in}}\Lambda := \{x \in \Lambda : \inf_{y \in \Lambda^c} |x - y| = 1\},$$

and the *edge boundary* as

$$\partial\Lambda := \{\{x, y\} \subset \mathbb{Z}^d : |x - y| = 1, x \in \Lambda, y \in \Lambda^c\}.$$

Remark 3.9. The usual isoperimetric inequality says $2d|\Lambda|^{1-1/d} \leq |\partial\Lambda|$. The inner boundary and the edge boundary are related by $|\partial_{\text{in}}\Lambda| \leq |\partial\Lambda| \leq 2d|\partial_{\text{in}}\Lambda|$, yielding us the inequality $|\Lambda|^{1-1/d} \leq |\partial_{\text{in}}\Lambda|$, which we will use in the rest of the paper.

In order to define the label of a contour, we must be careful since the inner boundary of a set $\bar{\gamma}$ may have different signs; see Figure 4.

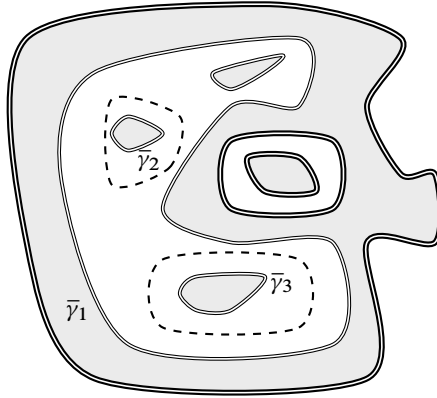


Fig. 4. An example of $\Gamma(\sigma) = \{\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3\}$, with $\bar{\gamma}_1$ having regions in the inner boundary with different signs. In the figure, the grey region are for incorrect points, the thin and thick border corresponds to, respectively, +1 and -1 labels.

For a set $\bar{\gamma} \in \Gamma(\sigma)$, let $\bar{\gamma}^{(1)}, \dots, \bar{\gamma}^{(n)}$ be its connected components. To define the label of $\bar{\gamma}$, we must introduce the following concept. A connected component $\bar{\gamma}^{(k)}$ of $\bar{\gamma}$ is called *external* if for any other connected component $\bar{\gamma}^{(k')}$ with $V(\bar{\gamma}^{(k')}) \cap V(\bar{\gamma}^{(k)}) \neq \emptyset$, we have $V(\bar{\gamma}^{(k')}) \subset V(\bar{\gamma}^{(k)})$.

Lemma 3.10. Any configuration σ with finite boundary is constant on $\partial_{\text{in}}V(\bar{\gamma})$ for each $\bar{\gamma} \in \Gamma(\sigma)$.

Proof. Note that $V(\bar{\gamma})$ is the union of $V(\bar{\gamma}^{(k)})$ for all its external connected components $\bar{\gamma}^{(k)}$. Suppose there are $\bar{\gamma}^{(k)}, \bar{\gamma}^{(j)}$ external connected components of $\bar{\gamma}$ such that the sign of σ on $\partial_{\text{in}}V(\bar{\gamma}^{(k)})$ is different from the sign on $\partial_{\text{in}}V(\bar{\gamma}^{(j)})$. Then, the configuration σ must change sign inside $V(\bar{\gamma}^{(k)})^c \cap V(\bar{\gamma}^{(j)})^c$. Since σ is constant outside some finite set Λ , either $\bar{\gamma}^{(k)}$ or $\bar{\gamma}^{(j)}$ must be surrounded by a different region of incorrect points, let

us call it $\bar{\gamma}^{(l)}$. We can assume that the connected component surrounded by $\bar{\gamma}^{(l)}$ is $\bar{\gamma}^{(k)}$. The set $\bar{\gamma}^{(l)}$ cannot be a connected component of $\bar{\gamma}$, otherwise the set $\bar{\gamma}^{(k)}$ would not be external. If $\bar{\gamma}^{(l)}$ is a connected component of another element $\bar{\gamma}' \in \Gamma(\sigma)$, then $\bar{\gamma}$ have a nonempty intersection with at least two connected components of $(\bar{\gamma}')^c$, contradiction to condition (A). ■

The *label* of $\bar{\gamma}$ is defined as the function $\text{lab}_{\bar{\gamma}} : \{I(\bar{\gamma})^{(0)}, I(\bar{\gamma})^{(1)}, \dots, I(\bar{\gamma})^{(n)}\} \rightarrow \{-1, +1\}$, where $\text{lab}_{\bar{\gamma}}(I(\bar{\gamma})^{(k)})$ is the sign of the configuration σ in $\partial_{\text{in}} V(I(\bar{\gamma})^{(k)})$, for $k \geq 1$, and $\text{lab}_{\bar{\gamma}}(I(\bar{\gamma})^{(0)})$ is the sign of σ in $\partial_{\text{in}} V(\bar{\gamma})$.

Definition 3.11. Given a configuration σ with finite boundary, its *contours* γ are pairs $(\bar{\gamma}, \text{lab}_{\bar{\gamma}})$, where $\bar{\gamma} \in \Gamma(\sigma)$ and $\text{lab}_{\bar{\gamma}}$ is the label function defined previously. The *support of the contour* γ is defined as $\text{sp}(\gamma) := \bar{\gamma}$ and its *size* is given by $|\gamma| := |\text{sp}(\gamma)|$.

Another important concept for our analysis of phase transition is the interior of a contour. The following sets will be useful:

$$I_{\pm}(\gamma) = \bigcup_{\substack{k \geq 1, \\ \text{lab}_{\bar{\gamma}}(I(\text{sp}(\gamma))^{(k)}) = \pm 1}} I(\text{sp}(\gamma))^{(k)},$$

$$I(\gamma) = I_+(\gamma) \cup I_-(\gamma), \quad V(\gamma) = \text{sp}(\gamma) \cup I(\gamma),$$

where $I(\text{sp}(\gamma))^{(k)}$ are the connected components of $I(\text{sp}(\gamma))$. Notice that the interior of contours in Pirogov–Sinai theory are at most unions of simple connected sets. In our case, they are only connected, i.e., they may have holes. One feature that our contours share with Pirogov–Sinai theory is the absence of a bijective correspondence between contours and configurations (see [21, Section 7.2.6] and [36]). Usually, there is more than one configuration giving the same boundary set. Also, it is not true that for all families of contours $\Gamma := \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ there is a configuration σ whose contours are exactly Γ . This happens because they may not form an (M, a, r) -partition and, even if this is the case, their labels may not be compatible, see Figure 5 below. When such a configuration exists, we say that the family of contours Γ is *compatible*.

3.2. Entropy bounds

The proofs in this section are highly inspired by Fröhlich–Spencer [24, Section 4], the one-dimensional case studied in Fröhlich–Spencer [23] and by Cassandro, Ferrari, Merola and Presutti [12]. We say that a contour γ in a family Γ is *external* if its external connected components are not contained in any other $V(\gamma')$, for $\gamma' \in \Gamma \setminus \{\gamma\}$. For each $\Lambda \subset \mathbb{Z}^d$, let us define the set of all external compatible families of contours Γ with labels \pm contained in Λ by

$$\mathcal{E}_{\Lambda}^{\pm} := \{\Gamma = \{\gamma_1, \dots, \gamma_n\} : \Gamma \text{ is compatible, } \gamma_i \text{ is external, } \text{lab}(\gamma_i) = \pm 1, V(\Gamma) \subset \Lambda\},$$

where $V(\Gamma) = \bigcup_{1 \leq i \leq n} V(\gamma_i)$ and $\text{lab}(\gamma_i)$ is the value of the corresponding configuration compatible configuration at a point $x \in \partial_{\text{in}} V(\gamma_i)$. When we write $\gamma \in \mathcal{E}_{\Lambda}^{\pm}$, we mean

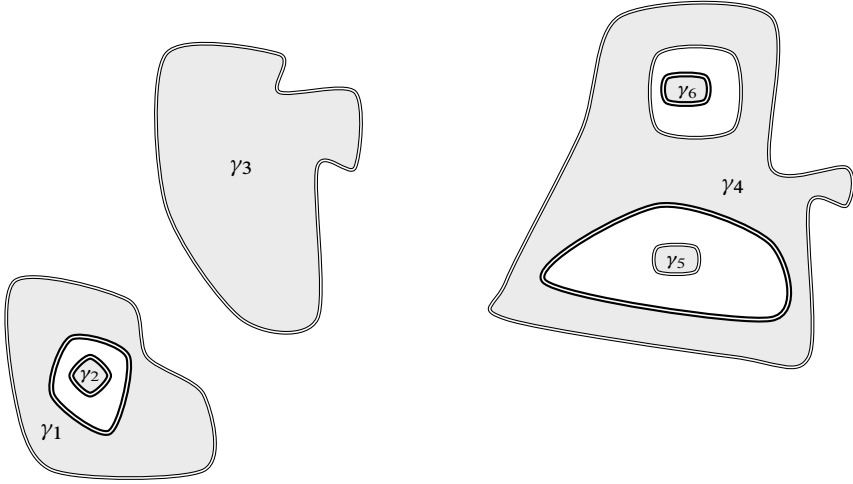


Fig. 5. Above we have two situations where incompatibility happens. In the first case, we have γ_1 and γ_2 , two contours that are close, thus they should not be separated. In the case $\gamma_4, \gamma_5, \gamma_6$, we have the usual problem of not matching labels.

$\{\gamma\} \in \mathcal{E}_\Lambda^\pm$. We also write $\mathcal{E}_{\mathbb{Z}^d}^\pm := \mathcal{E}^\pm$. To hold a Peierls-type argument, it is important to control the number of contours with a given fixed size. Hence, we need to find an upper bound of the number of contours with fixed size $|\gamma|$ containing a given point, meaning $0 \in V(\gamma)$. To do so, we will need some auxiliary results.

Since a contour does not need to be connected, the counting argument is different from the short-range case. Let us denote by \mathcal{C}_n an arbitrary collection of n -cubes. For $n, m \geq 0$ with $n \leq m$, we say that \mathcal{C}_n is *subordinated* to \mathcal{C}_m , denoted by $\mathcal{C}_n \leq \mathcal{C}_m$, if \mathcal{C}_m is a minimal cover of $\bigcup_{C_n(x) \in \mathcal{C}_n} C_n(x)$. For each $n, m \geq 1$, with $n \leq m$, define

$$N(\mathcal{C}_m, V_n) := |\{\mathcal{C}_n : \mathcal{C}_n \leq \mathcal{C}_m, |\mathcal{C}_n| = V_n\}|$$

as the number of collections of n -cubes \mathcal{C}_n subordinated to a given \mathcal{C}_m such that $|\mathcal{C}_n| = V_n$. The following two propositions are straightforward generalizations of [24, Theorem 4.2 and Proposition 4.3]. The original proofs correspond to our case when $r = 1$, and are included here for the benefit of the reader.

Proposition 3.12. *Let $r, n \geq 1$ be integers, and let \mathcal{C}_{rn} be a fixed collection of rn -cubes. Then there exists a constant $c := c(d, r) > 0$ such that*

$$N(\mathcal{C}_{rn}, V_{r(n-1)}) \leq e^{cV_{r(n-1)}}. \quad (3.5)$$

Proof. For each rn -cube $C_{rn}(x) \in \mathcal{C}_{rn}$, let $N_{C_{rn}(x)}$ be the number of cubes in a collection of $r(n-1)$ -cubes $\mathcal{C}_{r(n-1)}$ that are covered by $C_{rn}(x)$. Fix $(n_{C_{rn}(x)})_{C_{rn}(x) \in \mathcal{C}_{rn}}$, with $n_{C_{rn}(x)} \geq 1$, an integer solution to the inequality

$$V_{r(n-1)} \leq \sum_{C_{rn}(x) \in \mathcal{C}_{rn}} n_{C_{rn}(x)} \leq 2dV_{r(n-1)}, \quad (3.6)$$

and define $N(\mathcal{C}_{rn}, V_{r(n-1)}, (n_{C_{rn}(x)})_{C_{rn}(x) \in \mathcal{C}_{rn}})$ to be the number of collections $\mathcal{C}_{r(n-1)}$ of $r(n-1)$ -cubes subordinated to \mathcal{C}_{rn} such that $|\mathcal{C}_{r(n-1)}| = V_{r(n-1)}$ and $N_{C_{rn}(x)} = n_{C_{rn}(x)}$ for each $C_{rn}(x) \in \mathcal{C}_{rn}$. We get

$$N(\mathcal{C}_{rn}, V_{r(n-1)}) = \sum_{(n_{C_{rn}(x)})_{C_{rn}(x) \in \mathcal{C}_{rn}}} N(\mathcal{C}_{rn}, V_{r(n-1)}, (n_{C_{rn}(x)})_{C_{rn}(x) \in \mathcal{C}_{rn}}).$$

The number of positions that an $r(n-1)$ -cube can be arranged inside an rn -cube is at most $(2^{r+1} - 1)^d$. Since the number of combinations that $n_{C_{rn}(x)} r(n-1)$ -cubes can be arranged in the cube $C_{rn}(x)$ is at most $\binom{(2^{r+1}-1)^d}{n_{C_{rn}(x)}}$, we get

$$N(\mathcal{C}_{rn}, V_{r(n-1)}, (n_{C_{rn}(x)})_{C_{rn}(x) \in \mathcal{C}_{rn}}) \leq \prod_{C_{rn}(x) \in \mathcal{C}_{rn}} \binom{(2^{r+1} - 1)^d}{n_{C_{rn}(x)}}.$$

The number of solutions to (3.6) is bounded by $2^{2^d V_{r(n-1)} + 1}$, concluding that inequality (3.5) holds for

$$c = (2d + 1) \log(2) + d \log(2^{r+1} - 1). \quad \blacksquare$$

Given a subset $\Lambda \Subset \mathbb{Z}^d$ and integers $r \geq 1$ and $n \geq 0$, define the *total volume* by

$$V_r(\Lambda) := \sum_{n=0}^{n_r(\Lambda)} |\mathcal{C}_{rn}(\Lambda)|,$$

where $n_r(\Lambda) = \lceil \log_{2^r}(\text{diam}(\Lambda)) \rceil$ and $\mathcal{C}_{rn}(\Lambda)$ is a *minimal cover* of Λ with rn -cubes $C_{rn}(x)$ defined in (3.2). Observe that $|\mathcal{C}_0(\Lambda)| = |\Lambda|$. Let $V \geq 1$ be a positive integer and \mathcal{F}_V be the set defined by

$$\mathcal{F}_V := \{\Lambda \Subset \mathbb{Z}^d : V_r(\Lambda) = V, 0 \in \Lambda\}.$$

By using Proposition 3.12, let us show that the number of elements in \mathcal{F}_V is exponentially bounded by V .

Proposition 3.13. *There exists $b := b(d, r) > 0$ such that*

$$|\mathcal{F}_V| \leq e^{bV}. \quad (3.7)$$

Proof. For each $\Lambda \in \mathcal{F}_V$, there is a family of minimal covers $(\mathcal{C}_{rn}(\Lambda))_{0 \leq n \leq n_r(\Lambda)}$ characterizes uniquely the subset $\Lambda \in \mathcal{F}_V$ since the minimal cover $\mathcal{C}_0(\Lambda)$ is a cover by points in \mathbb{Z}^d . Moreover, the minimal covers $\mathcal{C}_{rn}(\Lambda)$ can always be chosen in a way that \mathcal{C}_{rn_1} is subordinated to \mathcal{C}_{rn_2} whenever $n_1 \leq n_2$, since, in order to compute the total volume $V_r(\Lambda)$, we only need to know the size of each minimal cover $\mathcal{C}_{rn}(\Lambda)$. Fix $(V_{rn})_{0 \leq n \leq n_r(\Lambda)-1}$, a solution to the equation

$$\sum_{n=0}^{n_r(\Lambda)-1} V_{rn} = V - 1. \quad (3.8)$$

We can estimate $|\mathcal{F}_V|$ by counting the number of families $(\mathcal{C}_{rn}(\Lambda))_{0 \leq n \leq n_r(\Lambda)}$, where the last cover $\mathcal{C}_{rn_r(\Lambda)}$ consists of a unique cube $C_{rn_r(\Lambda)}(x)$ containing 0. Let $\mathcal{F}_{V,m} = \{\Lambda \in \mathcal{F}_V : n_r(\Lambda) = m\}$. Then,

$$|\mathcal{F}_V| \leq \sum_{m=1}^V |\mathcal{F}_{V,m}|$$

since $n_r(\Lambda) \leq V_r(\Lambda) = V$. Now,

$$\begin{aligned} |\mathcal{F}_{V,m}| &\leq \sum_{(V_{rn})_{0 \leq n \leq m-1}} \left| \{(\mathcal{C}_{rn})_{0 \leq n \leq m} : \mathcal{C}_{rn} \preceq \mathcal{C}_{r(n+1)}, |\mathcal{C}_{rn}| = V_{rn}, \right. \\ &\quad \left. \mathcal{C}_{rm} = \{C_{rm}(x)\}, 0 \in C_{rm}(x)\} \right| \\ &= \sum_{(V_{rn})_{0 \leq n \leq m-1}} \sum_{C_{rm}(x) \ni 0} \sum_{\substack{\mathcal{C}_{r(m-1)} \\ |\mathcal{C}_{r(m-1)}| = V_{r(m-1)} \\ \mathcal{C}_{r(m-1)} \preceq \mathcal{C}_{rm}}} \cdots \sum_{\substack{\mathcal{C}_r \\ |\mathcal{C}_r| = V_r \\ \mathcal{C}_r \preceq \mathcal{C}_{2r}}} N(\mathcal{C}_r, V_0). \end{aligned}$$

Iterating inequality (3.5), we get

$$|\mathcal{F}_{V,m}| \leq |\{C_{rm}(x) : 0 \in C_{rm}(x)\}| \sum_{(V_{rn})_{0 \leq n \leq m-1}} \prod_{n=0}^{m-1} e^{cV_{rn}}. \quad (3.9)$$

We have at most 2^V solutions for equation (3.8), thus inequality (3.9) together with the fact that the number of rm -cubes containing 0 is bounded by 3^d yield us that

$$|\mathcal{F}_V| \leq 3^d V 2^V e^{cV}.$$

Therefore, inequality (3.7) holds for $b = d \log(3) + \log(2) + c + 1$. ■

We are able to prove Proposition 3.16 once we show that a fixed configuration σ with $\Gamma(\sigma) = \{\gamma\}$ and a fixed volume $|\gamma| = m$ imply that the total volume $V_r(\text{sp}(\gamma))$ is finite. We need the following auxiliary result about graphs, which is a generalization of [32, Claim 4.2].

Proposition 3.14. *Let $k \geq 1$ and let G be a finite nonempty connected simple graph. Then, G can be covered by $\lceil |v(G)|/k \rceil$ connected subgraphs of size at most $2k$.*

Proof. Since we can always consider a spanning tree from a connected graph G , it is sufficient to prove the proposition when G is a tree. If either $k = 1$ or $|v(G)| \leq 2k$, our statement is trivially true, so we suppose $k \geq 2$ and $|v(G)| \geq 2k + 1$, and proceed by induction on $\lceil |v(G)|/k \rceil$.

Choose a vertex $r \in v(G)$ to be our root. For every vertex u of G , let $\text{dep}(u)$ be the depth of the vertex u , i.e., the distance in the graph between r and u . We say that a vertex w is a descendant of u if there is a path $u_1 = w, u_2, \dots, u_{n-1}, u_n = u$ in G with $\text{dep}(u_i) > \text{dep}(u)$, for all $1 \leq i \leq n - 1$, and let $\text{desc}(u)$ be the number of descendants of u . Take a vertex u^* from $\{u \in v(G) : \text{desc}(u) \geq k\}$ that is not empty since

$\text{desc}(r) \geq 2k$, with highest depth, i.e., such that for any other $u \in v(G)$ with $\text{desc}(u) \geq k$ we have $\text{dep}(u^*) \geq \text{dep}(u)$. Let u_1, \dots, u_t be the children of the vertex u^* , and define $a_i = \text{desc}(u_i) + 1$. By definition of u^* , we have that $a_i \leq k$, for $1 \leq i \leq t$, and $a_1 + \dots + a_t \geq k$. Hence, there must be $s \in [1, t]$ for which $k \leq a_1 + \dots + a_s < 2k$. Therefore, we can consider the subtree T whose vertex set is composed by u^*, u_1, \dots, u_s and their descendants. By construction, it holds that $k + 1 \leq |v(T)| \leq 2k$. The induced subgraph $H := G[(v(G) \setminus v(T)) \cup \{u^*\}]$ is connected and satisfies $|v(H)| \leq |v(G)| - k$. Using the induction hypothesis, H can be covered by $\lceil |v(H)|/k \rceil \leq \lceil |v(G)|/k \rceil - 1$ connected subgraphs. Adding T to this cover completes the proof. \blacksquare

The importance of the choice $r = \lceil \log_2(a + 1) \rceil + d + 1$ in the (M, a, r) -partition will be seen in the next proposition, where we will show that the total volume can be bounded by the size of the contour.

Proposition 3.15. *There exists a constant $\kappa := \kappa(d, M, r) > 0$ such that, for any contour $\gamma \in \mathcal{E}_{\Lambda}^{\pm}$,*

$$V_r(\text{sp}(\gamma)) \leq \kappa |\gamma|.$$

Proof. Define $g: \mathbb{N} \rightarrow \mathbb{Z}$ by

$$g(n) = \left\lfloor \frac{n - 2 - \log_{2^r}(2Md^a)}{a} \right\rfloor.$$

We are going to prove that

$$|\mathcal{C}_{rn}(\text{sp}(\gamma))| \leq \frac{1}{2^{r-d-1}} |\mathcal{C}_{rg(n)}(\text{sp}(\gamma))|, \quad (3.10)$$

whenever $g(n) > 0$, and either the graph $G_{rg(n)}(\text{sp}(\gamma))$ defined in Proposition 3.7 has at least two connected components or it has only one connected component with at least 2^r vertices. Remember that $\mathcal{G}_{rg(n)}(\text{sp}(\gamma))$ is the set of all connected components of the graph $G_{rg(n)}(\text{sp}(\gamma))$. Note that

$$|\mathcal{C}_{rg(n)}(\text{sp}(\gamma))| = 2^r \sum_{G \in \mathcal{G}_{rg(n)}(\text{sp}(\gamma))} \frac{|v(G)|}{2^r}. \quad (3.11)$$

Proposition 3.14 states that we can cover the vertex set $v(G)$ with a family of connected graphs G_i with $1 \leq i \leq \lceil |v(G)|/2^r \rceil$ and $|v(G_i)| \leq 2^{r+1}$. Using the inequality

$$\text{diam}(\Lambda \cup \Lambda') \leq \text{diam}(\Lambda) + \text{diam}(\Lambda') + \text{dist}(\Lambda, \Lambda'), \quad \text{for all } \Lambda, \Lambda' \in \mathbb{Z}^d,$$

and the fact that we can always extract a vertex of a finite connected graph in a way that the induced subgraph is still connected, by removing a leaf of a spanning tree, we can bound the diameter of $B_{G_i} = \bigcup_{C_{rg(n)}(x) \in v(G_i)} C_{rg(n)}(x)$ by

$$\begin{aligned} \text{diam}(B_{G_i}) &\leq \sum_{C_{rg(n)}(x) \in v(G_i)} \text{diam}(C_{rg(n)}(x)) + |v(G_i)| M d^a 2^{arg(n)} \\ &\leq d 2^{r(g(n)+1)+1} + M d^a 2^{r(ag(n)+1)+1} \leq 2^{rn}. \end{aligned} \quad (3.12)$$

The third inequality holds since $M, a, r \geq 1$. Therefore, each graph G_i can be covered by one cube of side length 2^{rn} with center in \mathbb{Z}^d . We claim that every cube of side length 2^{rn} with an arbitrary center in \mathbb{Z}^d can be covered by at most 2^d rn -cubes $C_{rn}(x)$. Note that it is enough to consider the case where the cube has the form

$$\prod_{i=1}^d [q_i - 2^{rn-1}, q_i + 2^{rn-1}] \cap \mathbb{Z}^d,$$

where

$$q_i \in \{0, 1, \dots, 2^{rn-1} - 1\} \quad \text{for } 1 \leq i \leq d.$$

It is easy to see that

$$[q_i - 2^{rn-1}, q_i + 2^{rn-1}] \subset [-2^{rn-1}, 2^{rn-1}] \cup [0, 2^{rn}].$$

Taking products for all $1 \leq i \leq d$, it concludes our claim. This reasoning allows us to conclude that the maximum number of rn -cubes required to cover each connected component G of $\mathcal{G}_{rg(n)}(\text{sp}(\gamma))$ is at most $2^d \lceil |v(G)|/2^r \rceil$, yielding us that

$$\begin{aligned} |\mathcal{C}_{rn}(\text{sp}(\gamma))| &\leq \sum_{G \in \mathcal{G}_{rg(n)}(\text{sp}(\gamma))} \left| \mathcal{C}_{rn} \left(\bigcup_{1 \leq i \leq \lceil |v(G)|/2^r \rceil} B_{G_i} \right) \right| \\ &\leq \sum_{G \in \mathcal{G}_{rg(n)}(\text{sp}(\gamma))} 2^d \left\lceil \frac{|v(G)|}{2^r} \right\rceil. \end{aligned} \quad (3.13)$$

If $|\mathcal{G}_{rg(n)}(\text{sp}(\gamma))| \geq 2$, since $\gamma \in \mathcal{E}_\Lambda^\pm$, each connected component $G \in \mathcal{G}_{rg(n)}(\text{sp}(\gamma))$ satisfies $|v(G)| \geq 2^r$. Indeed, if $|v(G)| \leq 2^r - 1$, by our construction in Proposition 3.7 the set $v(G)$ would be separated into another element of the (M, a, r) -partition. By hypothesis, if $|\mathcal{G}_{rg(n)}(\text{sp}(\gamma))| = 1$, it already satisfies $|v(G_{rg(n)})| \geq 2^r$. Together with

$$\frac{1}{2} \left\lceil \frac{|v(G)|}{2^r} \right\rceil \leq \frac{|v(G)|}{2^r},$$

equality (3.11) and inequality (3.13) yield that

$$|\mathcal{C}_{rn}(\text{sp}(\gamma))| \leq 2^{d+1} \sum_{G \in \mathcal{G}_{rg(n)}(\text{sp}(\gamma))} \frac{|v(G)|}{2^r} = \frac{2^{d+1}}{2^r} |\mathcal{C}_{rg(n)}(\text{sp}(\gamma))|.$$

So, inequality (3.10) is proved. Let us define two auxiliary quantities

$$\begin{aligned} l_1(n) &:= \max\{m \geq 0 : g^m(n) \geq 0\}, \\ l_2(n) &:= \max\{m \geq 0 : |\mathcal{G}_{rg^m(n)}(\text{sp}(\gamma))| = 1, |v(G_{rg^m(n)})| \leq 2^r - 1\}. \end{aligned}$$

For the set $\mathcal{G}_{rg^m(n)}(\text{sp}(\gamma))$ to be well defined, we must have $g^m(n) \geq 0$, thus $l_2(n) \leq l_1(n)$. Moreover, knowing that $|\mathcal{C}_n(\Lambda)| \leq |\Lambda|$ for any $n \geq 0$,

$$|\mathcal{C}_{rn}(\text{sp}(\gamma))| \leq |\mathcal{C}_{rg^{l_2(n)}}(\text{sp}(\gamma))| \leq \frac{1}{2^{(r-d-1)(l_1(n)-l_2(n))}} |\gamma|. \quad (3.14)$$

We claim

$$l_1(n) \geq \begin{cases} 0 & \text{if } 0 \leq n \leq n_0, \\ \left\lfloor \frac{\log_2(n) - \log_2(n_0)}{\log_2(a)} \right\rfloor & \text{if } n > n_0, \end{cases} \quad (3.15)$$

where $n_0 = (a + 2 + \log_{2^r}(2Md^a))(a - 1)^{-1}$. The first bound is trivial. Let $n > n_0$ and consider the function

$$\tilde{g}(n) = \frac{n - a - 2 - \log_{2^r}(2Md^a)}{a}.$$

From the fact that $g(n) \geq \tilde{g}(n)$ and both functions are increasing, we have

$$g^m(n) \geq \tilde{g}^m(n) \quad \text{for all } m \geq 1,$$

which implies that $\max_{\tilde{g}^m(n) \geq 0} m \leq l_1(n)$. Thus, we need to compute a lower bound for $\max_{\tilde{g}^m(n) \geq 0} m$. Since

$$\tilde{g}^m(n) = \frac{n}{a^m} - b \left(\frac{a^m - 1}{a^{m-1}(a - 1)} \right),$$

is sufficient to have

$$\frac{n}{a^m} > \frac{ab}{a - 1}, \quad (3.16)$$

where $b = (a + 2 + \log_{2^r}(2Md^a))a^{-1}$. We get the desired bound after taking the logarithm with respect to base two in both sides of inequality (3.16). To finish our calculation, we will analyze two cases depending if $l_2(n)$ is zero or not. First, let us consider the case where $l_2(n) = 0$. Using inequality (3.14), we get

$$V_r(\text{sp}(\gamma)) \leq |\gamma| \sum_{n=0}^{\infty} \frac{1}{2^{(r-d-1)l_1(n)}}.$$

To finish, notice that equation above can be bounded in the following way:

$$\sum_{n=0}^{\infty} \frac{1}{2^{(r-d-1)l_1(n)}} \leq n_0 + 1 + 2^{r-d-1} n_0^{(r-d-1)/(\log_2(a))} \zeta \left(\frac{r-d-1}{\log_2(a)} \right).$$

Now consider the case $l_2(n) \neq 0$. A similar bound as (3.12), the fact that $\mathcal{C}_{r_g^m(n)}(\text{sp}(\gamma))$ is a cover for the set $\text{sp}(\gamma)$ and assuming that

$$m \in \{k \geq 0 : |\mathcal{G}_{r_g^k(n)}| = 1, |v(G_{r_g^k(n)})| \leq 2^r - 1\}$$

gives that

$$\begin{aligned} \text{diam}(\text{sp}(\gamma)) &\leq \text{diam}(B_{G_{r_g^m(n)}(\text{sp}(\gamma))}) \\ &\leq (d2^{rg^m(n)} + Md^a 2^{arg^m(n)}) |v(G_{r_g^m(n)}(\text{sp}(\gamma)))| \\ &\leq 2Md^a 2^{arg^m(n)+r}. \end{aligned}$$

The inequality above yields

$$\log_{2^r}(\text{diam}(\text{sp}(\gamma))) \leq \log_{2^r}(2Md^a) + ag^m(n) + 1 \leq \log_{2^r}(2Md^a) + \frac{n}{a^{m-1}} + 1.$$

Let us assume that $\text{diam}(\text{sp}(\gamma)) > 2^{2r+1} M d^a$. Isolating the term depending on m and taking the logarithm with respect to base two in both sides of the inequality above, it gives us

$$m \leq 1 + \frac{\log_2(n) - \log_2(\log_{2^r}(\text{diam}(\text{sp}(\gamma))) - \log_{2^r}(2M d^a) - 1)}{\log_2(a)}.$$

The inequality above is valid for all $m \in \{k \geq 0 : |\mathcal{G}_{r g^k(n)}| = 1, |v(G_{r g^k(n)})| \leq 2^r - 1\}$. Thus, together with the lower bound (3.15), we get for $n > n_0$ that

$$l_1(n) - l_2(n) \geq \frac{\log_2(\log_{2^r}(\text{diam}(\text{sp}(\gamma))) - \log_{2^r}(2M d^a) - 1) - \log_2(n_0)}{\log_2(a)} - 2.$$

Inequality (3.14) together with the inequality above yields

$$\begin{aligned} V_r(\text{sp}(\gamma)) &\leq (n_0 + 1)|\gamma| + |\gamma| \frac{2^{2(r-d-1)} n_0^{(r-d-1)/(\log_2(a))} (n_r(\text{sp}(\gamma)) - n_0)}{(\log_{2^r}(\text{diam}(\text{sp}(\gamma))) - \log_{2^r}(2M d^a) - 1)^{(r-d-1)/(\log_2(a))}} \\ &\leq (n_0 + 1 + 2^{2(r-d-1)} n_0^{(r-d-1)/(\log_2(a))} (2 + \log_{2^r}(2M d^a))) |\gamma|, \end{aligned}$$

where the last inequality is due to the fact that $x/(x-w) \leq 1+w$ for any constant $x \geq w+1$ and the fact that $M \geq 1$ implies that $n_0 \geq 1$. If $\text{diam}(\text{sp}(\gamma)) \leq 2^{2r+1} M d^a$, we have

$$V_r(\text{sp}(\gamma)) \leq (n_r(\text{sp}(\gamma)) + 1)|\gamma| \leq (4 + \log_{2^r}(2M d^a)) |\gamma|.$$

Taking

$$\begin{aligned} \kappa = \max \left\{ 4 + \log_{2^r}(2M d^a), n_0 + 1 + 2^{2(r-d-1)} n_0^{(r-d-1)/(\log_2(a))} (2 + \log_{2^r}(2M d^a)), \right. \\ \left. n_0 + 1 + 2^{r-d-1} n_0^{(r-d-1)/(\log_2(a))} \zeta \left(\frac{r-d-1}{\log_2(a)} \right) \right\} \end{aligned}$$

concludes the desired result. \blacksquare

We are ready to show Proposition 3.14 that bounds exponentially the number of contours containing the origin with a fixed size.

Proposition 3.16. *Let $m \geq 1$, $d \geq 2$, and let the set $\mathcal{C}_0(m)$ be defined as*

$$\mathcal{C}_0(m) := \{\text{sp}(\gamma) \in \mathbb{Z}^d : \gamma \in \mathcal{E}^-, 0 \in V(\gamma), |\gamma| = m\}.$$

There exists $c_1 := c_1(d, M, r) > 0$ such that

$$|\mathcal{C}_0(m)| \leq e^{c_1 m}.$$

Proof. For a given contour $\gamma \in \mathcal{E}^-$, define the set \mathcal{C}_γ by

$$\mathcal{C}_\gamma := \{\text{sp}(\gamma') \in \mathcal{C}_0(m) : \exists x \in \mathbb{Z}^d \text{ s.t. } \text{sp}(\gamma') = \text{sp}(\gamma) + x\}.$$

Thus, we can partition the set $\mathcal{C}_0(m)$ into

$$\mathcal{C}_0(m) = \bigcup_{\substack{0 \in \text{sp}(\gamma) \\ |\gamma| = m}} \mathcal{C}_\gamma.$$

Given a contour $\gamma \in \mathcal{E}^-$, there are at most $|V(\gamma)|$ possibilities for the position of the point 0. Then,

$$|\mathcal{C}_0(m)| \leq \sum_{\substack{0 \in \text{sp}(\gamma), \gamma \in \mathcal{E}^- \\ |\gamma|=m}} |\mathcal{C}_\gamma| \leq \sum_{\substack{0 \in \text{sp}(\gamma), \gamma \in \mathcal{E}^- \\ |\gamma|=m}} |V(\gamma)|. \quad (3.17)$$

Using the isoperimetric inequality and the fact $\partial_{\text{in}} V(\gamma) \subset \text{sp}(\gamma)$, we obtain

$$\sum_{\substack{0 \in \text{sp}(\gamma), \gamma \in \mathcal{E}^- \\ |\gamma|=m}} |V(\gamma)| \leq m^{1+1/(d-1)} |\{\text{sp}(\gamma) \Subset \mathbb{Z}^d : \gamma \in \mathcal{E}^-, 0 \in \text{sp}(\gamma), |\gamma| = m\}|. \quad (3.18)$$

By Proposition 3.15, and since not all the finite sets with bounded total volume are contours, we have

$$\{\text{sp}(\gamma) \Subset \mathbb{Z}^d : \gamma \in \mathcal{E}^-, 0 \in \text{sp}(\gamma), |\gamma| = m\} \subset \{\Lambda \Subset \mathbb{Z}^d : 0 \in \Lambda, V_r(\Lambda) \leq \kappa m\}. \quad (3.19)$$

Proposition 3.13 yields

$$|\{\Lambda \Subset \mathbb{Z}^d : 0 \in \Lambda, V_r(\Lambda) \leq \kappa m\}| = \sum_{V=1}^{\lfloor \kappa m \rfloor} |\mathcal{F}_V| \leq \frac{e^{b(\kappa m+1)}}{e^b - 1}. \quad (3.20)$$

Substituting inequalities (3.18), (3.19) and (3.20) into inequality (3.17), we conclude

$$|\mathcal{C}_0(m)| \leq m^{1+\frac{1}{d-1}} \frac{e^{2b\kappa m+1}}{e^b - 1} \leq e^{c_1 m},$$

for $c_1 = 2b\kappa + 1 + (d-1)^{-1}$. ■

4. Phase transition

In this section, we prove that the long-range Ising model with decaying field undergoes a phase transition at low temperature when $\min\{\alpha - d, 1\} < \delta < d$. When the magnetic field decays with power $\delta \geq d$, the result is straightforward. In fact, for $\delta > d$, the magnetic field is summable and, by a general result of Georgii (see [25, Example 7.32 and Theorem 7.33]), there is an affine bijection between the Gibbs measures of the Ising model with $h = 0$. Then, the phase transition is already known in the summable case. For $\delta = d$, the sum $\sum_{x \in \Lambda} h_x$ can be bounded by $\log |\Lambda|$. This implies that $\sum_{x \in \Lambda} h_x = o(|\Lambda|^\varepsilon)$ for any $\varepsilon > 0$. Thus, if we prove the phase transition for $\delta < d$, it is easy to extend to this case.

Theorem 4.1. *For a fixed $d \geq 2$, suppose that $\alpha > d$ and $\delta > 0$. There exists $\beta_c := \beta_c(\alpha, d) > 0$ such that, for every $\beta > \beta_c$, the long-range Ising model with coupling constant (2.2) and magnetic field (2.3) undergoes a phase transition at inverse of temperature β when*

- $d < \alpha < d + 1$ and $\delta > \alpha - d$;
- $d < \alpha < d + 1$ and $\delta = \alpha - d$ if h^* is small enough;

- $\alpha \geq d + 1$ and $\delta > 1$;
- $\alpha \geq d + 1$ and $\delta = 1$ if h^* is small enough.

For a fixed $x \in \mathbb{Z}^d$, define the function $\Theta_x: \Omega \rightarrow \mathbb{R}$ by

$$\Theta_x(\sigma) = \prod_{\substack{y \in \mathbb{Z}^d \\ |x-y| \leq 1}} \mathbb{1}_{\{\sigma_y = +1\}} - \prod_{\substack{y \in \mathbb{Z}^d \\ |x-y| \leq 1}} \mathbb{1}_{\{\sigma_y = -1\}}.$$

The function Θ_x assumes the value $+1$ if the point x is $+$ -correct, -1 if the point is $-$ -correct, and 0 otherwise. By the definition of contours, given a finite $\Lambda \Subset \mathbb{Z}^d$ and a configuration $\sigma \in \Omega_\Lambda^-$, it may happen that a contour γ associated to it have volume outside Λ . To avoid this problem, consider the probability measure defined as

$$\nu_{\beta, \mathbf{h}, \Lambda}^-(A) := \mu_{\beta, \mathbf{h}, \Lambda}^-(A | \Theta_x = -1, x \in \partial_{\text{in}} \Lambda)$$

for every measurable set A . The Markov property implies that the probability measures $\nu_{\beta, \mathbf{h}, \Lambda}^-$ are the finite volume Gibbs measure in a subset of Λ , and working with them is advantageous since we can study important quantities in terms of contours. Fixed $\Lambda \Subset \mathbb{Z}^d$, for each $\Lambda' \subset \Lambda$, let the restricted partition functions be

$$Z_{\beta, \mathbf{h}}^-(\Lambda') := \sum_{\substack{\Gamma \in \mathcal{E}_\Lambda^- \\ V(\Gamma) \subset \Lambda'}} \sum_{\sigma \in \Omega(\Gamma)} e^{-\beta H_{\Lambda, \mathbf{h}}^-(\sigma)},$$

where, given $\Gamma \in \mathcal{E}_\Lambda^-$, the space of configurations is $\Omega(\Gamma) := \{\sigma \in \Omega_\Lambda^- : \Gamma \subset \Gamma(\sigma)\}$. Note that we will abuse the notation and denote $\Gamma(\sigma)$ as the set of contours associated to a configuration σ instead of the (M, a, r) -partition. Define the map $\tau_\Gamma: \Omega(\Gamma) \rightarrow \Omega_\Lambda^-$ as

$$\tau_\Gamma(\sigma)_x = \begin{cases} \sigma_x & \text{if } x \in \text{L}(\Gamma) \cup V(\Gamma)^c, \\ -\sigma_x & \text{if } x \in \text{I}_+(\Gamma), \\ -1 & \text{if } x \in \text{sp}(\Gamma). \end{cases}$$

The map τ_Γ erases a family of compatible contours, since the spin flip preserves incorrect points but transforms $+$ -correct points into $-$ -correct points. Given $\Gamma \in \mathcal{E}_\Lambda^-$ and a configuration $\sigma \in \Omega(\Gamma)$, we will calculate the energy cost to extract one of its elements. We can start with bounding the number of integer points in the ℓ^1 -sphere and continue with giving a lower bound for the diameter of a finite subset of \mathbb{Z}^d .

Lemma 4.2. *Let $s_d(n)$ be the cardinality of integer points in the ℓ^1 -sphere, centered at the origin and with radius n . Then, for any $n \geq d$, we have*

$$s_d(n) = \sum_{k=0}^{d-1} 2^{d-k} \binom{d}{k} \binom{n-1}{d-k-1}.$$

If $n < d$, the sum above starts in $k = d - n$.

This lemma allows us to show that

$$c_d n^{d-1} \leq s_d(n) \leq 2^{2d-1} e^{d-1} n^{d-1}$$

for all $n \geq d$ and $c_d = (d-1)^{-(d-1)}$.

Lemma 4.3. *There exists $k_d > 0$ such that for every $\Lambda \in \mathbb{Z}^d$, it holds*

$$\text{diam}(\Lambda) \geq k_d |\Lambda|^{1/d}. \quad (4.1)$$

Proof. Consider a closed ball with positive integer radius n . Lemma 4.2 implies that the diameter satisfies

$$\text{diam}(B_n(x)) = 2n \geq C_d |B_n(x)|^{1/d},$$

where $C_d = (2^{2d} e^{d-1} / d)^{-1/d}$. Let Λ be any finite subset of \mathbb{Z}^d . If we take $x^*, y^* \in \Lambda$ such that $\text{diam}(\Lambda) = |x^* - y^*|$, we have

$$2\text{diam}(\Lambda) = \text{diam}(B_{|x^*-y^*|}(x^*)) \geq C_d |\Lambda|^{1/d}.$$

Inequality (4.1) follows by choosing the constant $k_d = C_d/2$. ■

In the next proposition, we will give a lower bound for the cost of extracting a contour from a given configuration. The main difference in our results compared to the short-range case is that we have an additional surface order term, defined as

$$F_\Lambda := \sum_{\substack{x \in \Lambda \\ y \in \Lambda^c}} J_{xy}$$

for every finite set $\Lambda \in \mathbb{Z}^d$. First, let us give a lower bound to the surface energy term, that will be useful to the proof of phase transition.

Lemma 4.4. *Given $\alpha > d$, there exists $K_\alpha := K_\alpha(\alpha, d) > 0$ such that for every $\Lambda \in \mathbb{Z}^d$ it holds*

$$F_\Lambda \geq K_\alpha \max\{|\Lambda|^{2-\alpha/d}, |\partial\Lambda|\}.$$

Proof. Since all the edges of $\partial\Lambda$ are present in the surface energy term F_Λ , we have the bound $F_\Lambda \geq J|\partial\Lambda|$. Fix $x \in \Lambda$. If we set $R = \lceil (dc_d^{-1}|\Lambda|)^{1/d} \rceil$, then using the fact that $\sum_{y \in \mathbb{Z}^d} J_{xy} < \infty$, we get

$$\sum_{y \in \Lambda^c} J_{xy} - \sum_{y \in B_R(x)^c} J_{xy} = \sum_{y \in B_R(x)} J_{xy} - \sum_{y \in \Lambda} J_{xy} \geq \frac{J(|B_R(x)| - |\Lambda|)}{R^\alpha} \geq 0.$$

Lemma 4.2 yields us

$$\sum_{y \in B_R(x)^c} J_{xy} = J \sum_{n \geq R+1} \frac{s_d(n)}{n^\alpha} \geq Jc_d \sum_{n \geq R+1} \frac{1}{n^{\alpha-d+1}}.$$

The right-hand side can be bounded below by an integral, and we take

$$K_\alpha = Jc_d(\alpha - d)^{-1}((dc_d^{-1})^{1/d} + 2)^{d-\alpha}. \quad \blacksquare$$

In the next proposition, we will denote by H_Λ^- the Hamiltonian function in (2.1) when the field $h_x = 0$ for all $x \in \mathbb{Z}^d$.

Proposition 4.5. *For M large enough, there are constants $c_i := c_i(\alpha, d) > 0$, $i = 2, 3, 4$, such that for $\Lambda \Subset \mathbb{Z}^d$, any fixed contour $\gamma \in \mathcal{E}_\Lambda^-$, and $\sigma \in \Omega(\gamma)$, it holds*

$$H_\Lambda^-(\sigma) - H_\Lambda^-(\tau(\sigma)) \geq c_2|\gamma| + c_3F_{I_+(\gamma)} + c_4F_{\text{sp}(\gamma)}.$$

Proof. Fix some $\sigma \in \Omega(\gamma)$. We will denote $\tau_\gamma(\sigma) := \tau$ and $\Gamma(\sigma) := \Gamma$ throughout this proposition. The difference between the Hamiltonians is

$$\begin{aligned} H_\Lambda^-(\sigma) - H_\Lambda^-(\tau) &= \sum_{\{x,y\} \subset V(\Gamma)} J_{xy}(\tau_x \tau_y - \sigma_x \sigma_y) + \sum_{\substack{x \in V(\Gamma) \\ y \in V(\Gamma)^c}} J_{xy}(\sigma_x - \tau_x) \\ &= \sum_{\substack{x \in A(\gamma) \\ y \in B(\gamma)}} J_{xy}(\tau_x \tau_y - \sigma_x \sigma_y) + \sum_{\{x,y\} \subset A(\gamma)} J_{xy}(\tau_x \tau_y - \sigma_x \sigma_y), \end{aligned}$$

where $A(\gamma) = I_+(\gamma) \cup \text{sp}(\gamma)$ and $B(\gamma) = I_-(\gamma) \cup V(\gamma)^c$. Using the definition of the map τ_γ and decomposing the sets $A(\gamma)$ and $B(\gamma)$ in a suitable manner, we get

$$\begin{aligned} H_\Lambda^-(\sigma) - H_\Lambda^-(\tau) &= \sum_{\substack{x \in \text{sp}(\gamma) \\ y \in \mathbb{Z}^d}} J_{xy} \mathbb{1}_{\{\sigma_x \neq \sigma_y\}} + \sum_{\substack{x \in \text{sp}(\gamma) \\ y \in \text{sp}(\gamma)^c}} J_{xy} \mathbb{1}_{\{\sigma_x \neq \sigma_y\}} - 2 \sum_{\substack{x \in I_+(\gamma) \\ y \in B(\gamma)}} J_{xy} \sigma_x \sigma_y \\ &\quad - 2 \sum_{\substack{x \in \text{sp}(\gamma) \\ y \in B(\gamma)}} J_{xy} \mathbb{1}_{\{\sigma_y = +1\}} - 2 \sum_{\substack{x \in \text{sp}(\gamma) \\ y \in I_+(\gamma)}} J_{xy} \mathbb{1}_{\{\sigma_y = -1\}}. \end{aligned} \quad (4.2)$$

We need to analyze each negative term of the equality above carefully. We start with the terms depending on $\text{sp}(\gamma)$. Notice that the characteristic functions on $B(\gamma)$ and $I_+(\gamma)$ can only be different from zero at the other contours volumes. Thus,

$$\sum_{\substack{x \in \text{sp}(\gamma) \\ y \in B(\gamma)}} J_{xy} \mathbb{1}_{\{\sigma_y = +1\}} + \sum_{\substack{x \in \text{sp}(\gamma) \\ y \in I_+(\gamma)}} J_{xy} \mathbb{1}_{\{\sigma_y = -1\}} \leq \sum_{\substack{x \in \text{sp}(\gamma) \\ y \in V(\Gamma')}} J_{xy}, \quad (4.3)$$

where $\Gamma' := \Gamma(\tau)$. Let

$$\gamma = \bigcup_{1 \leq k \leq n} \gamma_k \quad \text{and} \quad \gamma' = \bigcup_{1 \leq j \leq n'} \gamma'_j$$

for each $\gamma' \in \Gamma'$ be the subsets given to us by condition (B). We will divide the right-hand side of equation (4.3) into two terms depending on the sets

$$\Upsilon_1 = \{\gamma' \in \Gamma' : \max_{1 \leq k \leq n} \text{diam}(\gamma_k) \leq \max_{1 \leq j \leq n'} \text{diam}(\gamma'_j)\} \quad \text{and} \quad \Upsilon_2 = \Gamma' \setminus \Upsilon_1.$$

On the first sum, condition (B2) implies

$$\sum_{\substack{x \in \text{sp}(\gamma) \\ y \in V(\Upsilon_1)}} J_{xy} \leq \sum_{\substack{x \in \text{sp}(\gamma) \\ y \in B_R(x)^c}} J_{xy},$$

where $R = M \max_{1 \leq k \leq n} \text{diam}(\gamma_k)^a$. Using condition (B1), it holds

$$\begin{aligned} \sum_{\substack{x \in \text{sp}(\gamma) \\ y \in B_R(x)^c}} J_{xy} &\leq \frac{J 2^{d+\alpha-1} e^{d-1} |\gamma|}{(\alpha-d) M^{\alpha-d}} \max_{1 \leq k \leq n} \text{diam}(\gamma_k)^{a(d-\alpha)} \\ &\leq \frac{J 2^{d+\alpha-1} e^{d-1} (2^r - 1)}{(\alpha-d) k_d^{a(\alpha-d)} M^{\alpha-d}}. \end{aligned} \quad (4.4)$$

We turn our attention to the sum depending on Υ_2 . We divide the set Υ_2 into sets $\Upsilon_{2,m}$ consisting of contours of Γ' where the maximum diameter of its partition is m . Thus, for any x in $\text{sp}(\gamma)$ and $\gamma' \in \Gamma'$, there is a point $y_{\gamma',x} \in V(\gamma')$ such that $|x - y_{\gamma',x}| = \text{dist}(\{x\}, \gamma')$. Then, for every $1 \leq m < \max_{1 \leq k \leq n} \text{diam}(\gamma_k)$,

$$\sum_{\substack{x \in \text{sp}(\gamma) \\ y \in V(\Upsilon_{2,m})}} J_{xy} \leq \sum_{\substack{x \in \text{sp}(\gamma) \\ \gamma' \in \Upsilon_{2,m}}} |V(\gamma')| J_{xy_{\gamma',x}}.$$

For each $\gamma' \in \Upsilon_{2,m}$, define the graph $G_{\gamma'}$ with vertex set $v(G_{\gamma'}) = \{\gamma'_j\}_{1 \leq j \leq n'}$ and an edge is placed when $\text{dist}(\gamma'_j, \gamma'_i) \leq 1$. Let G_j be the maximal connected component of $G_{\gamma'}$ such that γ'_j is an element of its vertex set. Also, let $V(G_j) \subset V(\gamma')$ be the subset of all connected components of $V(\gamma')$ that have a nonempty intersection with the vertices of G_j . Using Lemma 4.3, we have

$$|V(\gamma')| \leq \frac{1}{k_d^d} \sum_{j=1}^{n'} \text{diam}(V(G_j))^d.$$

The diameter of $V(G_j)$ is realized by the distance between two points, namely x^* and y^* , that must be into $\gamma'_i, \gamma'_l \in v(G_j)$. We can make a minimal path λ_1 in the graph G_j between γ'_i and γ'_l since it is connected. Thus, using the path λ_1 , we can construct a minimal path λ_2 in \mathbb{Z}^d connecting x^* and y^* that passes through every vertex that is visited by the path λ_1 . Let λ_3 be a minimal path realizing the distance between x^* and y^* . Since the path is minimal, we have

$$\text{diam}(V(G_j)) = |\lambda_3| \leq |\lambda_2| \leq \sum_{\gamma'_i \in v(G_j)} [\text{diam}(\gamma'_i) + 1].$$

Hence,

$$\frac{1}{k_d^d} \sum_{j=1}^{n'} \text{diam}(V(G_j))^d \leq \frac{1}{k_d^d} \sum_{j=1}^{n'} \left(\sum_{\gamma'_i \in v(G_j)} [\text{diam}(\gamma'_i) + 1] \right)^d. \quad (4.5)$$

The number of elements in $v(G_j)$ is at most $2^r - 1$ by condition (B), thus

$$\sum_{\substack{x \in \text{sp}(\gamma) \\ y \in V(\Upsilon_{2,m})}} J_{xy} \leq \frac{(2(2^r - 1))^{d+1}}{k_d^d} m^d \sum_{\substack{x \in \text{sp}(\gamma) \\ \gamma' \in \Upsilon_{2,m}}} J_{xy_{\gamma',x}}. \quad (4.6)$$

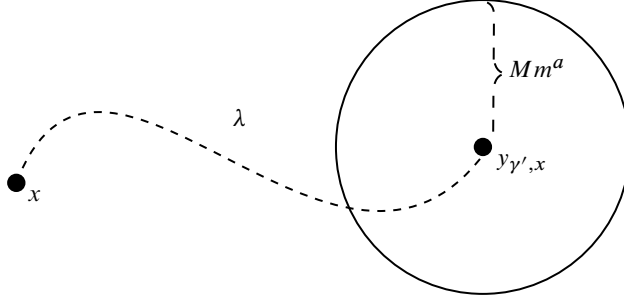


Fig. 6. Minimal path λ between x and $y_{\gamma',x}$.

We know that there is no other point $y_{\gamma'',x}$ at least in a ball of radius Mm^a centered at $y_{\gamma',x}$. Thus, taking balls of radius $Mm^a/3$ guarantees that they become disjoint by condition (B2). Also, if λ is the minimal path realizing the distance between x and $y_{\gamma',x}$, we know that it must have at least Mm^a points (see Figure 6). Thus,

$$\sum_{\substack{x \in \text{sp}(\gamma) \\ \gamma' \in \Upsilon_{2,m}}} J_{xy_{\gamma',x}} \leq \frac{3}{Mm^a} F_{\text{sp}(\gamma)}. \quad (4.7)$$

Inequalities (4.7), (4.6), (4.4) plugged into inequality (4.3) yield

$$\sum_{\substack{x \in \text{sp}(\gamma) \\ y \in B(\gamma)}} J_{xy} \mathbb{1}_{\{\sigma_y = +1\}} + \sum_{\substack{x \in \text{sp}(\gamma) \\ y \in I_+(\gamma)}} J_{xy} \mathbb{1}_{\{\sigma_y = -1\}} \leq \frac{k_\alpha^{(1)}}{M^{(\alpha-d) \wedge 1}} F_{\text{sp}(\gamma)}, \quad (4.8)$$

where

$$k_\alpha^{(1)} = 2 \max \left\{ \frac{J 2^{d+\alpha-1} e^{d-1} (2^r - 1)}{(\alpha - d) k_d^{a(\alpha-d)}}, \frac{(2(2^r - 1))^{d+1} 3\zeta(a-d)}{k_d^d} \right\},$$

and $(\alpha - d) \wedge 1 = \min\{\alpha - d, 1\}$.

The remaining term in our analysis is the one involving the interaction between $I_+(\gamma)$ and $B(\gamma)$. Recall that $\Gamma(\tau) = \Gamma'$ is the set of external contours of Γ after γ is removed and define $\Gamma_1 \subset \Gamma'$ as the set of external contours that are contained in $I_+(\gamma)$ and $\Gamma_2 = \Gamma' \setminus \Gamma_1$. We have

$$\begin{aligned} \sum_{\substack{x \in I_+(\gamma) \\ y \in B(\gamma)}} J_{xy} \sigma_x \sigma_y &= \sum_{\substack{x \in V(\Gamma_1) \\ y \in V(\Gamma_2)}} J_{xy} + \sum_{\substack{x \in I_+(\gamma) \setminus V(\Gamma_1) \\ y \in V(\Gamma_2)}} 2J_{xy} \mathbb{1}_{\{\sigma_y = +1\}} \\ &+ \sum_{\substack{x \in V(\Gamma_1) \\ y \in B(\gamma) \setminus V(\Gamma_2)}} 2J_{xy} \mathbb{1}_{\{\sigma_x = -1\}} - \sum_{\substack{x \in I_+(\gamma) \setminus V(\Gamma_1) \\ y \in V(\Gamma_2)}} J_{xy} \\ &- \sum_{\substack{x \in V(\Gamma_1) \\ y \in V(\Gamma_2)}} 2J_{xy} \mathbb{1}_{\{\sigma_x \neq \sigma_y\}} - \sum_{\substack{x \in I_+(\gamma) \\ y \in B(\gamma) \setminus V(\Gamma_2)}} J_{xy}. \end{aligned} \quad (4.9)$$

We start our analysis with the first two terms on right-hand side of (4.9). Note that

$$\sum_{\substack{x \in V(\Gamma_1) \\ y \in V(\Gamma_2)}} J_{xy} + \sum_{\substack{x \in I_+(\gamma) \setminus V(\Gamma_1) \\ y \in V(\Gamma_2)}} 2J_{xy} \mathbb{1}_{\{\sigma_y = +1\}} \leq 2 \sum_{\substack{x \in I_+(\gamma) \\ y \in V(\Gamma_2)}} J_{xy}. \quad (4.10)$$

Consider the two sets

$$\Upsilon_3 = \{\gamma' \in \Gamma_2 : \max_{1 \leq k \leq n} \text{diam}(\gamma_k) \leq \max_{1 \leq j \leq n'} \text{diam}(\gamma'_j)\} \quad \text{and} \quad \Upsilon_4 = \Gamma_2 \setminus \Upsilon_3.$$

Now we proceed as in the previous case for Υ_1 ,

$$\sum_{\substack{x \in I_+(\gamma) \\ y \in V(\Upsilon_3)}} J_{xy} \leq \frac{J 2^{d+\alpha-1} e^{d-1}}{(\alpha-d) M^{\alpha-d}} |I_+(\gamma)| \max_{1 \leq k \leq n} \text{diam}(\gamma_k)^{a(d-\alpha)}. \quad (4.11)$$

Notice that by using the isoperimetric inequality together with Lemma 4.3, we get that

$$\begin{aligned} |I_+(\gamma)| \max_{1 \leq k \leq n} \text{diam}(\gamma_k)^{a(d-\alpha)} &\leq \frac{|\partial_{\text{in}} \gamma|^{d/(d-1)}}{k_d^{a(\alpha-d)}} \max_{1 \leq k \leq n} |\gamma_k|^{a(d-\alpha)/d} \\ &\leq \frac{(2^r - 1)}{k_d^{a(\alpha-d)}} |\partial_{\text{in}} \gamma|^{1/(d-1)} \max_{1 \leq k \leq n} |\gamma_k|^{a(d-\alpha)/d+1}. \end{aligned}$$

Plugging the above inequality into (4.11) and using our choice of the constant a , we get

$$\sum_{\substack{x \in I_+(\gamma) \\ y \in V(\Upsilon_3)}} J_{xy} \leq \frac{k_\alpha^{(1)}}{M^{\alpha-d}} |\partial_{\text{in}} \gamma|^{1/(d-1)} \leq \frac{k_\alpha^{(1)}}{M^{\alpha-d}} |\gamma|. \quad (4.12)$$

For the sum depending on the contours in Υ_4 , we will need to break, as before, into sets $\Upsilon_{4,m}$ whose contours have maximum diameter equals to m . An argument similar to the one employed in (4.8) holds, hence

$$\sum_{\substack{x \in I_+(\gamma) \\ y \in V(\Upsilon_4)}} J_{xy} \leq \frac{(2(2^r - 1))^{d+1} 3\zeta(a-d)}{k_d^d M} \sum_{\substack{x \in I_+(\gamma) \\ y \in I_-(\gamma) \cup V(\gamma)^c}} J_{xy} \leq \frac{k_\alpha^{(1)}}{M} F_{I_+(\gamma)}. \quad (4.13)$$

For the next term, since we have $B(\gamma) \setminus V(\Gamma_2) \subset I_+(\gamma)^c$, we get

$$\sum_{\substack{x \in V(\Gamma_1) \\ y \in B(\gamma) \setminus V(\Gamma_2)}} J_{xy} \leq \sum_{\substack{x \in V(\Gamma_1) \\ y \in I_+(\gamma)^c}} J_{xy}.$$

We claim that for any $\gamma' \in \Gamma_1$, $\max_{1 \leq j \leq n'} \text{diam}(\gamma'_j) < \max_{1 \leq k \leq n} \text{diam}(\gamma_k)$ for $M > (2(2^r - 1))^{d+1} / k_d^d$. Indeed, by condition (A), $\text{sp}(\gamma')$ is contained in only one connected component of $I(\gamma)$, let us call it $I_+(\gamma)^{(1)}$. By similar reasonings as the one that gave us inequality (4.5), we have

$$|I_+(\gamma)^{(1)}| \leq \frac{(2(2^r - 1))^{d+1}}{k_d^d} \max_{1 \leq k \leq n} \text{diam}(\gamma_k)^d.$$

Assume by contradiction that $\max_{1 \leq j \leq n'} \text{diam}(\gamma'_j) \geq \max_{1 \leq k \leq n} \text{diam}(\gamma_k)$, then condition (B2) implies that $\text{dist}(\gamma', \gamma) \geq M \max_{1 \leq k \leq n} \text{diam}(\gamma_k)^\alpha$. Therefore, $|I_+(\gamma)|$ must have at least $M \max_{1 \leq k \leq n} \text{diam}(\gamma_k)^\alpha$ points inside it, which is a contradiction with our choice of M .

Thus, let us break Γ_1 into layers $\Gamma_{1,m}$ where $\max_{1 \leq j \leq n'} \text{diam}(\gamma'_j) = m$. For each $y \in I_+(\gamma)^c$ and $\gamma' \in \Gamma_1$, there is $x_{\gamma',y} \in V(\gamma')$ that realizes the distance between $V(\gamma')$ and y . Hence,

$$\begin{aligned} \sum_{\substack{x \in V(\Gamma_1) \\ y \in B(\gamma) \setminus V(\Gamma_2)}} J_{xy} &\leq \sum_{m=1}^{N-1} \sum_{\substack{x \in V(\Gamma_{1,m}) \\ y \in I_+(\gamma)^c}} J_{xy} \leq \sum_{m=1}^{N-1} \frac{(2(2^r - 1))^{d+1}}{k_d^d} m^d \sum_{\substack{\gamma' \in \Gamma_{1,m} \\ y \in I_+(\gamma)^c}} J_{x_{\gamma',y} y} \\ &\leq \frac{k_\alpha^{(1)}}{M} F_{I_+(\gamma)}, \end{aligned} \quad (4.14)$$

where $N := \max_{1 \leq k \leq n} \text{diam}(\gamma_k)$. We turn our attention to the term containing $\mathbb{1}_{\{\sigma_x \neq \sigma_y\}}$ in the right-hand side of inequality (4.9). The triangle inequality implies that the following inequality holds:

$$J_{xy} \geq \frac{1}{(2d+1)2^\alpha} \sum_{|x-x'| \leq 1} J_{x'y} \quad (4.15)$$

for every distinct pair of points $x, y \in \mathbb{Z}^d$. Thus, we have that

$$\sum_{\substack{x \in V(\Gamma_1) \\ y \in V(\Gamma_2)}} J_{xy} \mathbb{1}_{\{\sigma_x \neq \sigma_y\}} \geq \frac{1}{(2d+1)2^\alpha} \sum_{\substack{x \in V(\Gamma_1)_0 \\ y \in V(\Gamma_2)}} J_{xy}, \quad (4.16)$$

where $V(\Gamma_1)_0 = \{x \in V(\Gamma_1) : \Theta_x(\sigma) = 0\}$. Plugging inequalities (4.12), (4.13), (4.14), (4.16) into equation (4.9), we get

$$\begin{aligned} \sum_{\substack{x \in I_+(\gamma) \\ y \in B(\gamma)}} J_{xy} \sigma_x \sigma_y &\leq \frac{4k_\alpha^{(1)}}{M} F_{I_+(\gamma)} + \frac{2(2d)^{1/(d-1)} k_\alpha^{(1)}}{M^{\alpha-d}} |\gamma| - \frac{1}{(2d+1)2^{\alpha-1}} \sum_{\substack{x \in V(\Gamma_1)_0 \\ y \in V(\Gamma_2)}} J_{xy} \\ &\quad - \sum_{\substack{x \in I_+(\gamma) \\ y \in B(\gamma) \setminus V(\Gamma_2)}} J_{xy} - \sum_{\substack{x \in I_+(\gamma) \setminus V(\Gamma_1) \\ y \in V(\Gamma_2)}} J_{xy}. \end{aligned}$$

We must add the regions with correct points into the sum depending on $V(\Gamma_1)_0$. But this is a simple task, since we have

$$\sum_{\substack{x \in V(\Gamma_1) \setminus V(\Gamma_1)_0 \\ y \in V(\Gamma_2)}} J_{xy} \leq \sum_{\substack{x \in V(\Gamma_1) \\ y \in V(\Gamma_2)}} J_{xy},$$

and proceeding as we did in (4.10), we arrive at the following inequality:

$$\sum_{\substack{x \in I_+(\gamma) \\ y \in B(\gamma)}} J_{xy} \sigma_x \sigma_y \leq \frac{5k_\alpha^{(1)}}{M} F_{I_+(\gamma)} + \frac{3(2d)^{1/(d-1)} k_\alpha^{(1)}}{M^{\alpha-d}} |\gamma| - \frac{2^{1-\alpha}}{2d+1} \sum_{\substack{x \in I_+(\gamma) \\ y \in B(\gamma)}} J_{xy}. \quad (4.17)$$

Also, inequality (4.15) implies that

$$\begin{aligned} & \sum_{\substack{x \in \text{sp}(\gamma) \\ y \in \mathbb{Z}^d}} J_{xy} \mathbb{1}_{\{\sigma_x \neq \sigma_y\}} + \sum_{\substack{x \in \text{sp}(\gamma) \\ y \in \text{sp}(\gamma)^c}} J_{xy} \mathbb{1}_{\{\sigma_x \neq \sigma_y\}} \geq \frac{1}{(2d+1)2^\alpha} (Jc_\alpha |\gamma| + F_{\text{sp}(\gamma)}) \\ & \geq \frac{Jc_\alpha}{(2d+1)2^\alpha} |\gamma| + \frac{1}{(2d+1)2^{\alpha+1}} \left(F_{\text{sp}(\gamma)} + \sum_{\substack{x \in \text{sp}(\gamma) \\ y \in \mathbb{1}_+(\gamma)}} J_{xy} \right), \end{aligned} \quad (4.18)$$

where $c_\alpha = \sum_{y \neq 0 \in \mathbb{Z}^d} 1/|y|^\alpha$. Joining inequalities (4.8), (4.17) and (4.18) into (4.2) yields

$$\begin{aligned} H_\Lambda^-(\sigma) - H_\Lambda^-(\tau) & \geq \left(\frac{Jc_\alpha}{(2d+1)2^\alpha} - \frac{6(2d)^{1/(d-1)}k_\alpha^{(1)}}{M^{\alpha-d}} \right) |\gamma| \\ & \quad + \left(\frac{1}{(2d+1)2^{\alpha+1}} - \frac{10k_\alpha^{(1)}}{M} \right) F_{\mathbb{1}_+(\gamma)} \\ & \quad + \left(\frac{1}{(2d+1)2^{\alpha+1}} - \frac{2k_\alpha^{(1)}}{M^{(\alpha-d) \wedge 1}} \right) F_{\text{sp}(\gamma)}. \end{aligned}$$

Letting

$$M > \max \left\{ \frac{(2(2^r-1))^{d+1}}{k_d^d}, M_1, M_2 \right\},$$

where $Jc_\alpha M_1^{\alpha-d} := 36d(2d)^{1/(d-1)}k_\alpha^{(1)}2^{\alpha+2}$ and $M_2^{(\alpha-d) \wedge 1} := 30dk_\alpha^{(1)}2^{\alpha+2}$, we arrive at the desired result. \blacksquare

The following lemma is necessary to study the competition between the magnetic field and the long-range interaction.

Lemma 4.6. *There exists $c_5 := c_5(d, \delta, h^*) > 0$ such that for any $\Lambda \Subset \mathbb{Z}^d$,*

$$\sum_{x \in \Lambda} h_x \leq c_5 |\Lambda|^{1-\delta/d}, \quad (4.19)$$

where $(h_x)_{x \in \mathbb{Z}^d}$ is the magnetic field as in (2.3) and $\delta < d$.

Proof. Fix $\Lambda \Subset \mathbb{Z}^d$. In order to prove inequality (4.19), we will show that the sum in the left-hand side is always upper bounded by the sum of the magnetic field h_x in some ball $B_R(0)$ with R large enough. In fact, the magnetic field satisfies $h_x \geq h^*/R^\delta$ for $x \in B_R(0)$, and $h_x < h^*/R^\delta$ for $x \in \Lambda \setminus B_R(0)$. Then, we have

$$\sum_{x \in B_R(0)} h_x - \sum_{x \in \Lambda} h_x \geq \frac{h^*}{R^\delta} (|B_R(0)| - |\Lambda|). \quad (4.20)$$

By using Lemma 4.2, we find that the volume of ball satisfies $|B_R(0)| \geq d^{-1}c_d R^d$. Thus, if we choose $R := \lceil (dc_d^{-1}|\Lambda|)^{1/d} \rceil$, the right-hand side of inequality (4.20) is

nonnegative. We can bound the sum of the magnetic field in a ball $B_R(0)$ in the following way:

$$\sum_{x \in B_R(0)} h_x \leq h^* + h^* 2^{2d-1} e^{d-1} \sum_{n=1}^R n^{d-1-\delta}.$$

The result is obtained taking $c_5 = h^* 2^{2d} e^{d-1} (d - \delta)^{-1} ((dc_d^{-1})^{1/d} + 2)^{d-\delta}$. \blacksquare

As in the usual Peierls argument, Theorem 4.1 will follow once we prove the following proposition.

Proposition 4.7. *Let $\alpha > d$ and $\delta > 0$. For β large enough, it holds that*

$$v_{\beta, \mathbf{h}, \Lambda}^-(\sigma_0 = +1) < \frac{1}{2}$$

for every $\Lambda \Subset \mathbb{Z}^d$ when

- $d < \alpha < d + 1$ and $\delta > \alpha - d$;
- $d < \alpha < d + 1$ and $\delta = \alpha - d$ if h^* is small enough;
- $\alpha \geq d + 1$ and $\delta > 1$;
- $\alpha \geq d + 1$ and $\delta = 1$ if h^* is small enough.

Proof. Let $R > 0$ and $(\hat{h}_x)_{x \in \mathbb{Z}^d}$ be the truncated magnetic field

$$\hat{h}_x = \begin{cases} 0 & \text{if } |x| < R, \\ h_x & \text{if } |x| \geq R. \end{cases}$$

The constant R will be chosen later. Existence of phase transition under the presence of the truncated field implies phase transition for the model with the decaying field (see [25, Theorem 7.33] for a more general statement). If $\sigma_0 = +1$, there must exist a contour γ such that $0 \in V(\gamma)$. Hence,

$$v_{\beta, \hat{\mathbf{h}}, \Lambda}^-(\sigma_0 = +1) \leq \sum_{\substack{\gamma \in \mathcal{G}_{\Lambda}^- \\ 0 \in V(\gamma)}} v_{\beta, \hat{\mathbf{h}}, \Lambda}^-(\Omega(\gamma)).$$

Using Proposition 4.5, we know that the Hamiltonian $H_{\Lambda, \hat{\mathbf{h}}}^-$ satisfies

$$H_{\Lambda, \hat{\mathbf{h}}}^-(\sigma) - H_{\Lambda, \hat{\mathbf{h}}}^-(\tau(\sigma)) \geq c_2 |\gamma| + c_3 F_{1+(\gamma)} - 2 \sum_{x \in 1+(\gamma) \cup \text{sp}(\gamma)} \hat{h}_x, \quad (4.21)$$

where $\tau_{\gamma}(\sigma) = \tau(\sigma)$. Notice that

$$\sum_{x \in \text{sp}(\gamma)} \hat{h}_x \leq \frac{h^* |\gamma|}{R^{\delta}}. \quad (4.22)$$

The choice $R^{\delta} > 4h^*/c_2$ is sufficient to guarantee that the term $c_2 |\gamma|$ is larger than the field contribution term (4.22) in the right-hand side of inequality (4.21). The next step is

to analyze when the term $c_3 F_{I_+(\gamma)} - 2 \sum_{x \in I_+(\gamma)} \hat{h}_x$ is nonnegative. If $I_+(\gamma) = \emptyset$, there is nothing to do since the bound is trivial. Otherwise, we must analyze the competition of the decaying field with the different regimes of decay for the couplings constants J_{xy} .

(i) *Case $d < \alpha < d + 1$.* By Lemmas 4.4 and 4.6, we have

$$c_3 F_{I_+(\gamma)} - 2 \sum_{x \in I_+(\gamma)} \hat{h}_x \geq c_3 K_\alpha |I_+(\gamma)|^{2-\alpha/d} - 2c_5 |I_+(\gamma)|^{1-\delta/d}. \quad (4.23)$$

Thus, if $\delta > \alpha - d$ and $|I_+(\gamma)| \geq c'_\alpha := (2c_5/(c_3 K_\alpha))^{d/(\delta-(\alpha-d))}$, we have that the right-hand side of inequality (4.23) is nonnegative. In order to get a positive difference for all sizes of $I_+(\gamma)$, we need to consider $R^\delta > R_1^\delta := 2h^* c'_\alpha / (c_3 K_\alpha)$. For the case $\delta = \alpha - d$, we must take h^* small enough since the exponents in (4.23) will be equal.

(ii) *Case $\alpha \geq d + 1$.* By Lemmas 4.4 and 4.6, we have

$$c_3 F_{I_+(\gamma)} - 2 \sum_{x \in I_+(\gamma)} \hat{h}_x \geq c_3 K_\alpha |\partial I_+(\gamma)| - 2c_5 |I_+(\gamma)|^{1-\delta/d}. \quad (4.24)$$

Thus, if $\delta > 1$ and $|I_+(\gamma)| \geq b_\alpha := (c_5/(dc_3 K_\alpha))^{d/(\delta-1)}$, we have that the right-hand side of inequality (4.24) is nonnegative. In order to get a positive difference for all sizes of $I_+(\gamma)$, we need to consider $R^\delta > R_2^\delta := h^* b_\alpha / (dc_3 K_\alpha)$. In the case where $\delta = 1$, we must take h^* small enough and use the isoperimetric inequality in inequality (4.24).

It is clear that by taking $R = \max\{(4h^*/c_2)^{1/\delta}, R_1, R_2\}$ together with (4.21), we get

$$H_{\Lambda, \hat{\mathbf{h}}}^-(\sigma) - H_{\Lambda, \hat{\mathbf{h}}}^-(\tau(\sigma)) \geq \frac{c_2}{2} |\gamma|,$$

which implies

$$v_{\beta, \hat{\mathbf{h}}, \Lambda}^-(\Omega(\gamma)) \leq \frac{e^{-\beta c_2 |\gamma|/2}}{Z_{\beta, \hat{\mathbf{h}}}^-(\Lambda)} \sum_{\sigma \in \Omega(\gamma)} e^{-\beta H_{\Lambda, \hat{\mathbf{h}}}^-(\tau(\sigma))}. \quad (4.25)$$

Using the decomposition

$$\Omega(\gamma) = \bigcup_{\Gamma: \Gamma \cup \{\gamma\} \in \mathcal{E}_\Lambda^-} \{\sigma \in \Omega_\Lambda^- : \Gamma(\sigma) = \Gamma \cup \{\gamma\}\},$$

together with the fact that we may create new external contours when we erase the contour γ in the decomposition $\Omega(\gamma)$, but note that $V(\Gamma(\tau(\sigma))) \subset \Lambda \setminus \text{sp}(\gamma)$. Hence, the right-hand side of inequality (4.25) can be bounded as follows:

$$\begin{aligned} \sum_{\sigma \in \Omega(\gamma)} e^{-\beta H_{\Lambda, \hat{\mathbf{h}}}^-(\tau(\sigma))} &\leq \sum_{\Gamma \in \mathcal{E}_\Lambda^-} \sum_{\Gamma' \in \mathcal{E}_\Lambda^-} \sum_{\tau(\sigma)} \sum_{\omega: \tau(\omega) = \tau(\sigma)} e^{-\beta H_{\Lambda, \hat{\mathbf{h}}}^-(\tau(\sigma))} \\ &\leq |\{\sigma \in \Omega_{\text{sp}(\gamma)} : \Theta_x(\sigma) = 0, \text{ for each } x \in \text{sp}(\gamma)\}| \sum_{\Gamma \in \mathcal{E}_\Lambda^-} \sum_{\sigma \in \Omega(\Gamma)} e^{-\beta H_{\Lambda, \hat{\mathbf{h}}}^-(\sigma)}. \end{aligned}$$

Since the number of configurations that are incorrect in $\text{sp}(\gamma)$ are bounded by $2^{|\gamma|}$, we get

$$\nu_{\beta, \hat{h}, \Lambda}^{-}(\Omega(\gamma)) \leq \frac{Z_{\beta, \hat{h}}^{-}(\Lambda \setminus \text{sp}(\gamma))e^{(\log(2) - \beta c_2/2)|\gamma|}}{Z_{\beta, \hat{h}}^{-}(\Lambda)}.$$

Summing over all contours yields, together with Proposition 3.16,

$$\begin{aligned} \nu_{\beta, \hat{h}, \Lambda}^{-}(\sigma_0 = +1) &\leq \sum_{\substack{\gamma \in \mathcal{E}_{\Lambda}^{-} \\ 0 \in V(\gamma)}} \frac{Z_{\beta, \hat{h}}^{-}(\Lambda \setminus \text{sp}(\gamma))e^{(\log(2) - \beta c_2/2)|\gamma|}}{Z_{\beta, \hat{h}}^{-}(\Lambda)} \\ &\leq \sum_{m \geq 1} |\mathcal{C}_0(m)| e^{(\log(2) - \beta c_2/2)m} \\ &\leq \sum_{m \geq 1} e^{(c_1 + \log(2) - \beta c_2/2)m} < \frac{1}{2}, \end{aligned}$$

for β large enough. ■

5. Concluding remarks

In this paper, we developed a contour argument for phase transition in long-range Ising models when $d \geq 2$. As an application, we showed that the ferromagnetic Ising model with a decaying field presents a phase transition. Since for the borderline case $\delta = \alpha - d$ we need to consider h^* small, this is an indication that phase transition should not hold further into the region of the exponents. Some piece of evidence of this phenomenon is given by the nearest-neighbor case studied in [6], where uniqueness happened whenever $\delta < 1$.

Another natural question is to investigate if we can extend the argument to more general interactions in a sharp region of the exponents. One of such models is the ferromagnet nearest-neighbor Ising model with a competing long-range antiferromagnet interaction, as considered in [5]. As stated in that paper, a zero magnetization does not imply the absence of phase transition, and maybe some of the techniques developed here could be helpful to investigate the problem.

Moreover, our contours make sense for more general state spaces, so one could try to study other systems such as the Potts model, extending the results of [36, 37]. We did that and also studied the cluster expansion and decay of correlations, as was done in [28, 29] for the one-dimensional case, in a separate paper which will appear soon.

Acknowledgments. LA, RB and EE thank Bruno Kimura for providing us with his Ph.D. thesis [33] and many helpful discussions concerning his work in one-dimensional long-range Ising models, to Aernout van Enter for many references and help with the literature. We thank Christian Táfula for suggesting a simpler proof for Proposition 3.12. A sincere thank to João Maia for his diligent proofreading of this paper. RB thanks Marcelo Disconzi for his support and his friendship over the years. In addition, we thank the referee for the careful reading, which improved the paper.

Funding. LA is supported by FAPESP Grants 2017/18152-2 and 2020/14563-0. RB is supported by CNPq Grant 312294/2018-2 and FAPESP Grant 16/25053-8. SH is supported by the Ministry of Education, Culture, Sports, Science and Technology through the Program for Leading Graduate Schools (Hokkaido University Ambitious Leader's Program).

References

- [1] Affonso, L.: Multidimensional contours á la Fröhlich–Spencer and boundary conditions for quantum spin systems. Ph.D. thesis, University of São Paulo, <https://doi.org/10.11606/T.45.2023.tde-25092023-062451>, visited on 16 October 2024 (2023)
- [2] Affonso, L., Bissacot, R., Maia, J.: Phase transitions in multidimensional long-range random field Ising models. arXiv:2307.14150 (2024)
- [3] Bakchich, A., Benyoussef, A., Laanait, L.: Phase diagram of the Potts model in an external magnetic field. *Ann. Inst. H. Poincaré Phys. Théor.* **50**, 17–35 (1989) Zbl 0664.60111 MR 994041
- [4] Biskup, M., Borgs, C., Chayes, J. T., Kotecký, R.: Gibbs states of graphical representations of the Potts model with external fields. 41, 1170–1210 (2000) Zbl 0977.82012 MR 1757955
- [5] Biskup, M., Chayes, L., Kivelson, S. A.: On the absence of ferromagnetism in typical 2D ferromagnets. *Comm. Math. Phys.* **274**, 217–231 (2007) Zbl 1206.82014 MR 2318853
- [6] Bissacot, R., Cassandro, M., Cioletti, L., Presutti, E.: Phase transitions in ferromagnetic Ising models with spatially dependent magnetic fields. *Comm. Math. Phys.* **337**, 41–53 (2015) Zbl 1326.82008 MR 3324154
- [7] Bissacot, R., Cioletti, L.: Phase transition in ferromagnetic Ising models with non-uniform external magnetic fields. *J. Stat. Phys.* **139**, 769–778 (2010) Zbl 1197.82038 MR 2639889
- [8] Bissacot, R., Endo, E. O., van Enter, A. C. D.: Stability of the phase transition of critical-field Ising model on Cayley trees under inhomogeneous external fields. *Stochastic Process. Appl.* **127**, 4126–4138 (2017) Zbl 1373.82017 MR 3718108
- [9] Bissacot, R., Endo, E. O., van Enter, A. C. D., Kimura, B., Ruszel, W. M.: Contour methods for long-range Ising models: Weakening nearest-neighbor interactions and adding decaying fields. *Ann. Henri Poincaré* **19**, 2557–2574 (2018) Zbl 1398.82016 MR 3830223
- [10] Bovier, A.: *Statistical mechanics of disordered systems. A mathematical perspective.* *Camb. Ser. Stat. Probab. Math.* 18, Cambridge University Press, Cambridge (2006) Zbl 1108.82002 MR 2252929
- [11] Bricmont, J., Kupiainen, A.: Phase transition in the 3d random field Ising model. *Comm. Math. Phys.* **116**, 539–572 (1988) MR 943702
- [12] Cassandro, M., Ferrari, P. A., Merola, I., Presutti, E.: Geometry of contours and Peierls estimates in $d = 1$ Ising models with long range interactions. *J. Math. Phys.* **46**, article no. 053305 (2005) Zbl 1110.82006 MR 2143008
- [13] Cassandro, M., Merola, I., Picco, P.: Phase separation for the long range one-dimensional Ising model. *J. Stat. Phys.* **167**, 351–382 (2017) Zbl 1376.82016 MR 3626633
- [14] Cassandro, M., Merola, I., Picco, P., Rozikov, U.: One-dimensional Ising models with long range interactions: Cluster expansion, phase-separating point. *Comm. Math. Phys.* **327**, 951–991 (2014) Zbl 1297.82006 MR 3192055
- [15] Cassandro, M., Orlandi, E., Picco, P.: Phase transition in the 1d random field Ising model with long range interaction. *Comm. Math. Phys.* **288**, 731–744 (2009) Zbl 1173.82011 MR 2500998
- [16] Chayes, L., Machta, J., Redner, O.: Graphical representations for Ising systems in external fields. *J. Stat. Phys.* **93**, 17–32 (1998) Zbl 0963.82009 MR 1656363
- [17] Cioletti, L., Vila, R.: Graphical representations for Ising and Potts models in general external fields. *J. Stat. Phys.* **162**, 81–122 (2016) Zbl 1336.82001 MR 3439332

- [18] Datta, N., Fernández, R., Fröhlich, J.: *Low-temperature phase diagrams of quantum lattice systems. I. Stability for quantum perturbations of classical systems with finitely-many ground states*. *J. Stat. Phys.* **84**, 455–534 (1996) Zbl 1081.82506 MR 1400181
- [19] Dobrushin, R. L., Zahradník, M.: *Phase diagrams for continuous-spin models: An extension of the Pirogov–Sinai theory*. In: *Mathematical problems of statistical mechanics and dynamics*, *Math. Appl. (Soviet Ser.)* 6, Reidel, Dordrecht, 1–123 (1986) Zbl 0611.58025 MR 875430
- [20] Dyson, F. J.: *Existence of a phase-transition in a one-dimensional Ising ferromagnet*. *Comm. Math. Phys.* **12**, 91–107 (1969) Zbl 1306.47082 MR 436850
- [21] Friedli, S., Velenik, Y.: *Statistical mechanics of lattice systems*. Cambridge University Press, Cambridge (2018) Zbl 1407.82001 MR 3752129
- [22] Fröhlich, J., Spencer, T.: *The Kosterlitz–Thouless transition in two-dimensional abelian spin systems and the Coulomb gas*. *Comm. Math. Phys.* **81**, 527–602 (1981) MR 634447
- [23] Fröhlich, J., Spencer, T.: *The phase transition in the one-dimensional Ising model with $1/r^2$ interaction energy*. *Comm. Math. Phys.* **84**, 87–101 (1982) Zbl 1110.82302 MR 660541
- [24] Fröhlich, J., Spencer, T.: *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*. *Comm. Math. Phys.* **88**, 151–184 (1983) Zbl 0519.60066 MR 696803
- [25] Georgii, H.-O.: *Gibbs measures and phase transitions*. 2nd ed., De Gruyter Stud. Math. 9, Walter de Gruyter, Berlin (2011) Zbl 1225.60001 MR 2807681
- [26] Ginibre, J., Grossmann, A., Ruelle, D.: *Condensation of lattice gases*. *Comm. Math. Phys.* **3**, 187–193 (1966) MR 1552496
- [27] González-Navarrete, M., Pechersky, E., Yambartsev, A.: *Phase transition in ferromagnetic Ising model with a cell-board external field*. *J. Stat. Phys.* **162**, 139–161 (2016) Zbl 1338.82011 MR 3439334
- [28] Imbrie, J. Z.: *Decay of correlations in the one-dimensional Ising model with $J_{ij} = |i - j|^{-2}$* . *Comm. Math. Phys.* **85**, 491–515 (1982) MR 677994
- [29] Imbrie, J. Z., Newman, C. M.: *An intermediate phase with slow decay of correlations in one-dimensional $1/|x - y|^2$ percolation, Ising and Potts models*. *Comm. Math. Phys.* **118**, 303–336 (1988) MR 956170
- [30] Johansson, K.: *Condensation of a one-dimensional lattice gas*. *Comm. Math. Phys.* **141**, 41–61 (1991) Zbl 0737.60091 MR 1133259
- [31] Kac, M., Thompson, C. J.: *Critical behavior of several lattice models with long-range interaction*. *J. Math. Phys.* **10**, 1373–1386 (1969) Zbl 1306.82004 MR 250627
- [32] Kharash, V., Peled, R.: *The Fröhlich–Spencer proof of the Berezinskii–Kosterlitz–Thouless transition*. arXiv:1711.04720 (2017)
- [33] Kimura, B.: *Critical behavior of Ising spin systems*. Ph.D. thesis, Delft University of Technology, <https://doi.org/10.4233/uuid:62197dda-cb0e-402f-afa6-bc1a7d69ad3d>, visited on 16 October 2024 (2019)
- [34] Lee, T. D., Yang, C. N.: *Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model*. *Phys. Rev.* **87**, 410–419 (1952) Zbl 0048.43401 MR 53029
- [35] Littin, J., Picco, P.: *Quasi-additive estimates on the Hamiltonian for the one-dimensional long range Ising model*. *J. Math. Phys.* **58**, article no. 073301 (2017) Zbl 1370.82022 MR 3680332
- [36] Park, Y. M.: *Extension of Pirogov–Sinai theory of phase transitions to infinite range interactions. I. Cluster expansion*. *Comm. Math. Phys.* **114**, 187–218 (1988) Zbl 0663.60075 MR 928225
- [37] Park, Y. M.: *Extension of Pirogov–Sinai theory of phase transitions to infinite range interactions. II. Phase diagram*. *Comm. Math. Phys.* **114**, 219–241 (1988) Zbl 0663.60076 MR 928226
- [38] Peierls, R. E.: *On Ising’s model of ferromagnetism*. *Proc. Camb. Philos. Soc.* **32**, 477–481 (1936) Zbl 0014.33604

-
- [39] Pirogov, S. A., Sinai, Ya. G.: [Phase diagrams of classical lattice systems](#). Theoret. and Math. Phys. **25**, 1185–1192 (1975) MR [676316](#)
 - [40] Preston, C. J.: [Gibbs states on countable sets](#). Cambridge Tracts in Math. 68, Cambridge University Press, London (1974) Zbl [0297.60054](#) MR [474556](#)
 - [41] Presutti, E.: [Scaling limits in statistical mechanics and microstructures in continuum mechanics](#). Theoret. Math. Phys., Springer, Berlin (2009) Zbl [1156.82001](#) MR [2460018](#)
 - [42] Simon, B.: [A celebration of Jürg and Tom](#). J. Stat. Phys. **134**, 809–812 (2009) Zbl [1166.82300](#)
 - [43] van Enter, A. C. D.: [A note on the stability of phase diagrams in lattice systems](#). Comm. Math. Phys. **79**, 25–32 (1981) MR [609225](#)
 - [44] Yang, C. N., Lee, T. D.: [Statistical theory of equations of state and phase transitions. I. Theory of condensation](#). Phys. Rev. **87**, 404–409 (1952) Zbl [0048.43305](#) MR [53028](#)
 - [45] Zahradník, M.: [An alternate version of Pirogov–Sinai theory](#). Comm. Math. Phys. **93**, 559–581 (1984) MR [763759](#)