Rend. Sem. Mat. Univ. Padova (Online first)
DOI 10.4171/RSMUP/172

Published by EMS Press

On the average degree sum of proper subgroups of a finite group

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Abstract – In this paper, the structure of a finite group G when all proper subgroups H of G satisfy that the average degree sum of H is ≤ 2 is determined.

Mathematics Subject Classification 2020 – 20C15 (primary); 20D06 (secondary).

Keywords – simple group, character degree sum, proper subgroup.

1. Introduction

All groups are finite in this note. Let G be a finite group and let $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of G, say $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \dots, \chi_s\}$. Let $\mathrm{cd}(G) = \{\chi_i(1) : \chi_i \in \mathrm{Irr}(G)\}$. Write

$$\mathrm{acd}(G) = \frac{1}{|\mathrm{Irr}(G)|} \sum_{\chi \in \mathrm{Irr}(G)} \chi(1).$$

Then $\mathrm{acd}(G)$ is the average (irreducible) character degree of G. Some scholars have shown the relation between $\mathrm{acd}(G)$ and group structure. For instance, a group G is solvable when $\mathrm{acd}(G) \le 3$ [10, Theorem A] or $\mathrm{acd}(G) < \frac{16}{5}$ [1, 16] or $\mathrm{acd}(G) \le 2$ [15, Theorem 1.4] or $\mathrm{acd}(G) < \frac{2(r^2-1)}{r^2}$ [18, Proposition 5.4]; a group G is p-nilpotent when $\mathrm{acd}_{p'}(G) < 2p(p+1)$, $p \ge 3$ [17, Theorem A] or [6]; G has a normal Sylow p-subgroup when $\mathrm{acd}_p(G) < \frac{2\mathfrak{l}(p)p}{\mathfrak{l}(p)p+1}$ [7, Theorem A].

Some researchers have determined the structure of groups by using some subgroup properties. For example, a group G is determined when the Sylow p-subgroup is a

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T. I. set [12] or when proper subgroups have prime power degrees [13] or when proper subgroups have only square-free character degrees [14].

In this paper, we consider the connection between proper subgroup properties and group structure. In order to argue briefly, we introduce the following definition.

Definition 1.1. Let $\mathbf{prop}(G)$ be the set of the proper subgroups of a finite group G. A group G is called an SAD-group if for each $H \in \mathbf{prop}(G)$, $\mathrm{acd}(H) \leq 2$.

We know that a group G is abelian if and only if $\mathrm{acd}(G) = 1$. Corresponding to the results of [10, Theorem A] and [15, Theorem A], we prove the following result.

Theorem 1.2. *An* SAD-*group is either solvable or isomorphic to* $\mathrm{PSL}_2(7)$ *or* $\mathrm{PSL}_2(2^p)$ *with* p *a prime or* $\mathrm{SL}_2(5)$.

As an application of Theorem 1.2, we also can show the following result.

Corollary 1.3. *Let* G *be a finite group. If for all* $H \in \mathbf{prop}(G)$, $\mathrm{acd}(H) < 1.5$, *then* G *is solvable.*

Note that in Corollary 1.3 the bound "$\mathrm{acd}(H) < 1.5$ for each $H \in \mathbf{prop}(G)$" is best possible. In fact, A_5 has A_4, D_{10} and S_3 as its maximal subgroups (see [4, p. 2]), and so

$$\begin{aligned}
\mathrm{acd}(A_4) &= \frac{3 \cdot 1 + 1 \cdot 3}{4} = 1.5,\\
\mathrm{acd}(D_{10}) &= \frac{2 \cdot 1 + 2 \cdot 2}{4} = 1.5,\\
\mathrm{acd}(S_3) &= \frac{2 \cdot 1 + 1 \cdot 2}{3} = \frac{4}{3} \approx 1.33.
\end{aligned}$$

Now we will introduce the structure of this paper. In Section 2, some properties of average character degree are given and also some other information is obtained which will be used to control the group structure. In Section 3, we will prove Theorem 1.2 and also Corollary 1.3. The notation and notions are standard; see [4] and [9].

2. Some needed results

In this section, some wanted results are shown. First the basic properties of the average degree of irreducible character are given.

Let $\exp(G, d)$ be the number of all ordinary irreducible characters of G having the same degree d and let C_n be a cyclic group of order n. The following two lemmas are given since they can control the structures of groups.

LEMMA 2.1. *Let G be a dihedral group D_{2n} of order $2n$. Then*

$$\mathrm{acd}(G) = \begin{cases} \dfrac{2n+2}{n+3}, & n \text{ odd}, \\ \dfrac{2n+4}{n+6}, & n \text{ even}. \end{cases}$$

PROOF. We divide the proofs into two cases in light of n.

Case 1: n is odd. Then $G' = C_n$, so for each $\chi \in \mathrm{Irr}(G)$, $\chi(1) \big| |G/G'| = 2$, and $\chi(1) = 1$ or $\chi(1) = 2$. Now the equation

$$|G/G'| \cdot 1^2 + \exp(G, 2) \cdot 2^2 = 2n$$

gives that $\exp(G, 2) = \frac{n-1}{2}$. It follows that

$$\mathrm{acd}(G) = \frac{|G/G'| + \exp(G,2) \cdot 2}{|G/G'| + \exp(G,2)} = \frac{2 + \frac{n-1}{2} \cdot 2}{2 + \frac{n-1}{2}} = \frac{2n+2}{n+3}.$$

Case 2: n is even. Now $G' = C_{n/2}$ and $|G/G'| = 4$. In this case, for every $\chi \in \mathrm{Irr}(G)$, $\chi(1)^2 \big| |G/G'| = 4$, so $\mathrm{cd}(G) = \{1, 2\}$ and we have

$$4 \cdot 1^2 + \exp(G, 2) \cdot 2^2 = 2n.$$

It follows that $\exp(G, 2) = \frac{n-2}{2}$ and so $\mathrm{acd}(G) = \frac{2n+4}{n+6}$.

The lemma is complete. ■

Let E_q be an elementary abelian $\pi(q)$-group of order q.

LEMMA 2.2. *Let G be a Frobenius group $E_q : C_n$ or $(q : n)$ with kernel E_q and complement C_n respectively. Then* $\mathrm{acd}(G) = \frac{n^2+n(q-1)}{n^2+q-1}$.

PROOF. Theorem 13.7 of [5] shows that $G' = E_q$, $\mathrm{cd}(G) = \{1, n\}$ and so the equation

$$|G/G'| + \exp(G, n) \cdot n^2 = qn$$

implies that $\exp(G, n) = \frac{q-1}{n}$. Thus $\mathrm{acd}(G) = \frac{n^2+n(q-1)}{n^2+q-1}$. ■

We also need the following result due to Thompson [20].

LEMMA 2.3 ([20, Corollary 1]). *Every minimal simple group is isomorphic to one of the following minimal simple groups:*

(1) $\mathrm{PSL}_2(2^p)$ *for p a prime;*

(2) $\mathrm{PSL}_2(3^p)$ *for p an odd prime;*

(3) $\mathrm{PSL}_2(p)$, *for* p *any prime exceeding* 3 *such that* $p^2 + 1 \equiv 0 \pmod 5$;

(4) $\mathrm{Sz}(2^p)$ *for* p *an odd prime;*

(5) $\mathrm{PSL}_3(3)$.

3. Nonsolvable SAD-groups

In this section, first the structures of minimal nonabelian simple SAD-groups are given, then those of nonsolvable SAD-groups are determined and finally the proof of Corollary 1.3 is shown.

LEMMA 3.1. *Let* G *be a nonabelian simple group. If* G *is an* SAD-*group, then* G *is isomorphic to* $\mathrm{PSL}_2(7)$ *or* $\mathrm{PSL}_2(2^p)$ *with* p *a prime.*

Let $\max G$ be the set of maximal proper subgroups of a group G subject to subgroup-order divisibility.

PROOF OF LEMMA 3.1. Let $N \in \mathbf{prop}(G)$. Then N is solvable as $\mathrm{acd}(N) \le 2 < 3$. It follows that a nonabelian simple group G is nonsolvable but its proper subgroups are all solvable, so G is a minimal simple group. Now Lemma 2.3 gives the possibilities for G, so, the following cases are dealt with.

Case 1: $\mathrm{PSL}_2(2^p)$, p a prime. In this case, $k = 1$, and by Table 1,

$$\max \mathrm{PSL}_2(2^p) = \left\{E_{2^p} : C_{2^p-1}, D_{2(2^p-1)}, D_{2(2^p+1)}\right\}$$

and for $H \in \max \mathrm{PSL}_2(2^p)$, $\mathrm{acd}(H) \le 2$ by hypothesis. In fact, we know from Lemmas 2.1 and 2.2 that

$$\begin{aligned}
\mathrm{acd}(E_{2^p} : C_{2^p-1}) &= \frac{(2^p-1)^2 + (2^p-1)(2^p-1)}{(2^p-1)^2 + 2^p - 1} \le 2,\\
\mathrm{acd}(D_{2(2^p-1)}) &= \frac{2(2^p-1)+2}{2^p-1+3} \le 2,\\
\mathrm{acd}(D_{2(2^p+1)}) &= \frac{2(2^p+1)+2}{2^p+1+3} \le 2,
\end{aligned}$$

so for all $H \in \mathbf{prop}(\mathrm{PSL}_2(2^p))$, $\mathrm{acd}(H) \le 2$. Thus $\mathrm{PSL}_2(2^p)$ is an SAD-group and so G is isomorphic to $\mathrm{PSL}_2(2^p)$.

Case 2: $\mathrm{PSL}_2(3^p)$, p an odd prime. Now $k = 2$ and $E_{3^p} : C_{(3^p-1)/2} \in \max \mathrm{PSL}_2(3^p)$ by Table 1, so the hypothesis shows that $\mathrm{acd}(E_{3^p} : C_{(3^p-1)/2}) \le 2$. By Lemma 2.2,

	max $\mathrm{PSL}_2(q)$	Condition
$\mathscr{C}_1$	$E_q : C_{(q-1)/k}$	$k = \gcd(q-1, 2)$
$\mathscr{C}_2$	$D_{2(q-1)/k}$	$q \notin \{5, 7, 9, 11\}$
C_3	$D_{2(q+1)/k}$	$q \notin \{7, 9\}$
$\mathscr{C}_5$	$\mathrm{PSL}_2(q_0).(k, b)$	$q = q_0^b$, b a prime, $q_0 \neq 2$
$\mathscr{C}_6$	S_4	$q = p \equiv \pm 1 \pmod 8$
	A_4	$q = p \equiv 3, 5, 13, 27, 37 \pmod{40}$
$\mathscr{S}$	A_5	$q \equiv \pm 1 \pmod{10}$, $F_q = F_p[\sqrt{5}]$

TABLE 1. $\mathrm{PSL}_2(q)$, $q \geq 5$ [8, Chapter II, Theorem 8.27]

the equation

$$\operatorname{acd}(E_{3^p} : C_{(3^p-1)/2}) = \frac{(\frac{3^p-1}{2})^2 + \frac{3^p-1}{2}(3^p-1)}{(\frac{3^p-1}{2})^2 + 3^p - 1} \leq 2$$

shows $p = 2$, a contradiction.

Case 3: $\mathrm{PSL}_2(p)$, $p > 3$ an odd prime with $p^2 + 1 \equiv 0 \pmod 5$. Also $k = 2$ and by Table 1, $E_p : C_{(p-1)/2} \in \max \mathrm{PSL}_2(p)$. Thus $\operatorname{acd}(E_p : C_{(p-1)/2}) \leq 2$ by assumption. From Lemma 2.2, we obtain that

$$\operatorname{acd}(E_p : C_{(p-1)/2}) = \frac{(\frac{p-1}{2})^2 + \frac{p-1}{2}(p-1)}{(\frac{p-1}{2})^2 + p - 1} \leq 2$$

$$(*) \qquad \iff \left(\frac{p-1}{2}\right)^2 + (p-1)\left(1 - \frac{p-1}{2}\right) \geq 0$$

Now equation (*) has $p \leq 9$. Note that $p^2 + 1 \equiv 0 \pmod 5$ and $p > 3$, so $p = 7$. By [4, p. 3], $\max \mathrm{PSL}_2(7) = \{S_4, 7 : 3\}$. From [3], we obtain that

$$\operatorname{acd}(S_4) = \frac{1+1+2+3+3}{5} = 2 \quad \text{and} \quad \operatorname{acd}(7:3) = \frac{3 + 2\cdot 3}{5} = \frac{9}{5} < 2,$$

so for $H \in \mathbf{prop}(\mathrm{PSL}_2(7))$, $\operatorname{acd}(H) \leq 2$. Thus G is isomorphic to $\mathrm{PSL}_2(7)$.

Case 4: $\mathrm{Sz}(2^p)$, p an odd prime. Let $q = 2^p$. By [2, p. 385], we get $E_q^{1+1} : C_{q-1} \in \max \mathrm{Sz}(2^p)$. Let $H = E_q^{1+1} : C_{q-1}$. Then C_{q-1} acts fixed-point-freely on E_q^{1+1}, so $q - 1 \in \operatorname{cd}(H)$ and $\exp(H, q-1) = 1$. Now by [19, Lemma 1], $Z(E_q^{1+1})$ is abelian of order q, so $\frac{E_q^{1+1}}{Z(E_q^{1+1})}$ is abelian of order q. We know that for a nonlinear character $\chi \in \operatorname{Irr}(E_q^{1+1})$, $\chi(1)^2 \mid q$. Say $p = 2m + 1$, then $\chi(1) = 2^m$. By [5, Theorem 13.7],

$2^m(2^{2m+1}-1) \in \mathrm{cd}(H)$. Note that $G' = E_q^{1+1}$, so

$$(2^{2m+1}-1) + (2^{2m+1}-1)^2 + \exp(G, 2^m(2^{2m+1}-1)) \cdot (2^m(2^{2m+1}-1))^2 \\ = (2^{2m+1})^2(2^{2m+1}-1).$$

It follows that $\exp(G, 2^m(2^{2m+1}-1)) = 2$. Now $\mathrm{acd}(H) = \frac{2(q-1)+2\sqrt{q/2}(q-1)}{q+2}$. Since $H \in \mathbf{prop}(\mathrm{Sz}(q))$, $\mathrm{acd}(H) \leq 2$ by assumption, and so, $\frac{2(q-1)+2\sqrt{q/2}(q-1)}{q+2} \leq 2$, i.e.,

$$(**) \qquad \sqrt{q/2}(q-1) \leq 3.$$

But since $q \geq 8$, (**) has no solution in $\mathbf{N}$, the set of nature integers.

Case 5: $\mathrm{PSL}_3(3)$. By [4, p. 13], $13:3 \in \max \mathrm{PSL}_3(3)$ and so, $\mathrm{acd}(13:3) \leq 2$. By Lemma 2.2 or [3], $\mathrm{acd}(13:3) = \frac{3^2+3(13-1)}{3^2+13-1} = \frac{15}{7} \nleq 2$, a contradiction. ∎

Lemma 3.2. *Let* $G = \mathrm{PSL}_2(q)$ *with* $q = 2^p$. *Then* $\mathrm{acd}(\mathrm{PSL}_2(q)) = \frac{q^2}{1+q}$.

Proof. By [11], we obtain

$$\mathrm{acd}(\mathrm{PSL}_2(q)) = \frac{1\cdot 1 + (q-1)\cdot \frac{1}{2}q + q\cdot 1 + (q+1)\cdot(\frac{1}{2}q - 1)}{1 + \frac{1}{2}q + 1 + \frac{1}{2}q - 1} = \frac{q^2}{1+q},$$

the desired result. ∎

Note that $A_5 \cong \mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4)$.

Theorem 3.3. *Let G be an almost simple* SAD-*group. Then G is isomorphic to* $\mathrm{PSL}_2(7)$ *or* $\mathrm{PSL}_2(2^p)$ *with p a prime.*

Proof. Since G is an almost simple group, we have that G is isomorphic to either $\mathrm{PSL}_2(7)$ or $\mathrm{PSL}_2(7).2$ or $\mathrm{PSL}_2(2^p)$ or $\mathrm{PSL}_2(2^p).p$. If G is isomorphic to $\mathrm{PSL}_2(7)$ or $\mathrm{PSL}_2(2^p)$, we get the desired result. If $G \cong \mathrm{PSL}_2(7).2$, then $\mathrm{PSL}_2(7) \in \mathbf{prop}(G)$ and by [4, p. 3],

$$\mathrm{acd}(\mathrm{PSL}_2(7)) = \frac{1+3+3+6+7+8}{6} = \frac{28}{6} \approx 4.6 \nless 2,$$

a contradiction. If G is isomorphic to $\mathrm{PSL}_2(2^p).p$, then, $\mathrm{PSL}_2(2^p) \in \mathbf{prop}(G)$ and by hypothesis, $\mathrm{acd}(\mathrm{PSL}_2(2^p)) \leq 2$. Thus by Lemma 3.2,

$$\frac{q^2}{1+q} \leq 2, \quad \text{i.e., } q^2 - \sqrt{q} - \sqrt{2} \leq 0.$$

Consider the function $f(x) = x^2 - \sqrt{2}x - \sqrt{2}$, $f'(x) = 2x - \sqrt{2} > 0$ as $x \geq 4$, and $f(x) \geq f(4) = 16 - 4 \cdot \sqrt{2} - \sqrt{2} \approx 8.9 > 0$. It follows that the inequality $\frac{q^2}{1+q} \leq 2$ has no solution in $\mathbf{N}$ as $q \geq 4$.

Thus G is isomorphic to $\mathrm{PSL}_2(7)$ or $\mathrm{PSL}_2(2^p)$ with p a prime. ∎

LEMMA 3.4. *Let G be an* SAD*-group and assume that G/H is isomorphic to* $\mathrm{PSL}_2(5)$. *Then G is isomorphic to* $\mathrm{SL}_2(5)$ *or* $\mathrm{PSL}_2(5)$.

PROOF. Since $G/H \cong \mathrm{PSL}_2(5)$, we have $G'/H \cong \mathrm{PSL}_2(5)$. Notice that $\mathrm{PSL}_2(5)$ has Schur multiplier with order 2 (see [4, p. xvi]), so $[G', H] \leq C_2$.

If $[G', H] \leq 1$, then G is isomorphic to $\mathrm{PSL}_2(5) \times H$. If $H > 1$, then $\mathrm{PSL}_2(5) \in \mathbf{prop}(G)$, and so $\mathrm{PSL}_2(5)$ is solvable since $\mathrm{acd}(\mathrm{PSL}_2(5)) \leq 2$, a contradiction. Thus $H = 1$ and $G \cong \mathrm{PSL}_2(5)$, as desired.

If $[G', H] = C_2$, then $G \cong \mathrm{SL}_2(5) * H$, where $A * B$ denotes the central product of two groups A and B. If $H > C_2$, then $\mathrm{SL}_2(5) \in \mathbf{prop}(G)$. By [3], we have

$$\mathrm{acd}(\mathrm{SL}_2(5)) = \frac{1+2+2+3+3+4+4+5+6}{9} = \frac{30}{9} \approx 3.33 \not< 2,$$

a contradiction. Thus $H = C_2$. Now by [2, p. 377], $\max \mathrm{SL}_2(5) = \{E_5 : C_4, D_{12}, 2.A_4\}$, and by Lemmas 2.1 and 2.2,

$$\mathrm{acd}(E_5 : C_4) = \frac{4^2 + 4 \cdot (5-1)}{4^2 + 5 - 1} = \frac{32}{20} = 1.6,$$
$$\mathrm{acd}(D_{12}) = \frac{4}{3} \approx 1.3,$$
$$\mathrm{acd}(2.A_4) = \frac{1 \cdot 3 + 3 \cdot 1 + 2 \cdot 3}{3+1+3} = \frac{12}{7} \approx 1.7.$$

Thus for every $H \in \mathbf{prop}(\mathrm{SL}_2(5))$, $\mathrm{acd}(H) \leq 2$ by hypothesis. Now $\mathrm{SL}_2(5)$ is an SAD-group and so G is isomorphic to $\mathrm{SL}_2(5)$. ∎

LEMMA 3.5. *Let G be an* SAD*-group and assume that G/H is isomorphic to* $\mathrm{PSL}_2(7)$ *or* $\mathrm{PSL}_2(2^p)$ *with $p > 2$ a prime. Then G is isomorphic to* $\mathrm{PSL}_2(7)$ *or* $\mathrm{PSL}_2(2^p)$ *with $p > 2$ a prime.*

PROOF. If $G/H \cong \mathrm{PSL}_2(2^p)$ with $p > 2$ a prime, then $G \cong \mathrm{PSL}_2(2^p) \times H$ as

$$\mathrm{SL}_2(2^p) \cong \mathrm{PSL}_2(2^p) \cong \mathrm{GL}_2(2^p).$$

If $H > 1$, then $\mathrm{PSL}_2(2^p) \in \mathbf{prop}(G)$ and by Lemma 3.2, $\mathrm{acd}(\mathrm{PSL}_2(2^p)) = \frac{2^{2p}}{1+2^p} \leq 2$. So $(2^p)^2 - 2 \cdot 2^p - 2 \leq 0$. Consider the function $f(x) = x^2 - 2x - 2$. Then $f'(x) = 2x - 2$, and $f''(x) = 2$. We know that $x \geq 8$, so $f'(x) \geq f'(8) = 14 > 0$ and $f(x) \geq$

$f(8) = 46 > 0$. It follows that $(2^p)^2 - 2 \cdot 2^p - 2 > 0$, a contradiction. Now $H = 1$ and G is isomorphic to $\mathrm{PSL}_2(2^p)$ with $p > 2$ a prime, the desired result.

If $G/H \cong \mathrm{PSL}_2(7)$, then G'/H is isomorphic to $\mathrm{PSL}_2(7)$. We see that the Schur multiplier of $\mathrm{PSL}_2(7)$ has order 2, so $[G', H] \leq C_2$.

If $[G', H] = 1$, then $G \cong H \times \mathrm{PSL}_2(7)$. If $H > 1$, then $\mathrm{PSL}_2(7) \in \mathbf{prop}(G)$ and $\mathrm{acd}(\mathrm{PSL}_2(7)) \leq 2$ by hypothesis. But by [4, p. 3],

$$\mathrm{acd}(\mathrm{PSL}_2(7)) = \frac{1+3+3+6+7+8}{6} = \frac{28}{6} \approx 4.66 \not< 2,$$

a contradiction. It follows that $H = 1$ and G is isomorphic to $\mathrm{PSL}_2(7)$.

If $[G', H] = C_2$, then G is isomorphic to $\mathrm{SL}_2(7) * H$. Similarly, if $H > C_2$, then $\mathrm{SL}_2(7) \in \mathbf{prop}(G)$ shows that $\mathrm{acd}(\mathrm{SL}_2(7)) \leq 2$. But by [3],

$$\mathrm{acd}(\mathrm{SL}_2(7)) = \frac{1+3+3+4+4+6+6+6+7+8+8}{11} = \frac{56}{11} \approx 5.09,$$

a contradiction. So $H = C_2$ and G is isomorphic to $\mathrm{SL}_2(7)$. By [2, p. 377], $2.S_4 \in \max \mathrm{SL}_2(7)$. By [3], we have

$$\mathrm{acd}(2.S_4) = \frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1}{8} = 2.25 \not< 2,$$

a contradiction. Thus $\mathrm{SL}_2(7)$ is not an SAD-group. ■

In the proof of Theorem 3.7, we can get the result $k = 1$ by the following lemma.

Lemma 3.6. $\mathrm{acd}(H \times K) = \mathrm{acd}(H)\,\mathrm{acd}(K)$.

Proof. Let $\chi_i \in \mathrm{Irr}(H)$ and $\eta_j \in \mathrm{Irr}(K)$. Then by [9, Theorem 4.21], $\chi_i \times \eta_j \in \mathrm{Irr}(H \times K)$. Thus

$$\begin{aligned}
\mathrm{acd}(H \times K) &= \frac{\sum_{i,j} (\chi_i \times \eta_j)(1)}{|\chi_i \times \eta_j : \chi_i \in \mathrm{Irr}(H), \eta_j \in \mathrm{Irr}(K)|} \\
&= \frac{\sum_{i,j} \chi_i(1)\eta_j(1)}{k(H)k(K)} \\
&= \frac{\sum_i \chi_i(1)(\eta_1(1) + \eta_2(1) + \cdots + \eta_k(1))}{k(H)k(K)} \\
&= \frac{\sum_i \chi_i(1)\mathrm{T}(K)}{k(H)k(K)} \\
&= \frac{\mathrm{T}(H)\mathrm{T}(K)}{k(H)k(K)} \\
&= \mathrm{acd}(H)\,\mathrm{acd}(K),
\end{aligned}$$

the desired result. ■

Theorem 3.7. *Let G be a nonsolvable* SAD*-group. Then G is isomorphic to* $\mathrm{PSL}_2(2^p)$ *with $p > 2$ a prime or* $\mathrm{PSL}_2(7)$ *or* $\mathrm{SL}_2(5)$.

Proof. Let N be a minimal normal subgroup of G. If N is nonabelian, then, as G is nonsolvable,

$$N \cong S_1 \times \cdots \times \cdots S_k,$$

where S_i is isomorphic to S for all $i \in \{1, 2, \ldots, k\}$. Let $k \geq 2$. Then $S^{k-1} \in \mathbf{prop}(N)$ is solvable, a contradiction. It follows that $k = 1$ and $N \cong S$ is isomorphic to $\mathrm{PSL}_2(2^p)$ with p a prime or $\mathrm{PSL}_2(7)$.

Note that G cannot be an almost simple group by Lemma 3.3, and so, $G = N$ is isomorphic to $\mathrm{PSL}_2(2^p)$ with p a prime or $\mathrm{PSL}_2(7)$.

Now we consider that N is abelian. Then there is a normal subgroup H such that H/N is isomorphic to $\mathrm{PSL}_2(2^p)$ with p a prime or $\mathrm{PSL}_2(7)$. By Lemmas 3.4 and 3.5, H is isomorphic to $\mathrm{PSL}_2(2^p)$ with $p > 2$ a prime or $\mathrm{PSL}_2(7)$ or $\mathrm{SL}_2(5)$. If $H < G$, then $\mathrm{acd}(H) \leq 2$, and so H is solvable, a contradiction. It follows that $H = G$.

This completes the proof. ■

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Assume that G is nonsolvable. Then by Theorem 3.7, we get the desired result. ■

We will prove Corollary 1.3.

Proof of Corollary 1.3. Let G be a counterexample with minimal possible order. Then G is nonsolvable. In particular, as $\mathrm{acd}(G) < 1.5 \leq 2$, G is an SAD-group and so, by Theorem 1.2, G is isomorphic to $\mathrm{PSL}_2(2^p)$ with $p > 2$ a prime or $\mathrm{PSL}_2(7)$ or $\mathrm{SL}_2(5)$. So three cases are done with in the following.

Case 1: $\mathrm{PSL}_2(2^p)$ with $p > 2$ a prime. By Table 1, $E_{2^p} : C_{2^p-1} \in \max \mathrm{PSL}_2(2^p)$ and by Lemma 2.2 and hypothesis, $\mathrm{acd}(E_{2^p} : C_{2^p-1}) = \frac{2(2^p-1)}{2^p} < 1.5$. Let $f(x) = 2x - 1.5x - 2 = 0.5x - 2$. Then $f(x) \geq 0$ as $2^p - 1 \geq 3$, so $\frac{2(2^p-1)}{2^p} \geq 1.5$, a contradiction.

Case 2: $\mathrm{PSL}_2(7)$. We see from [4, p. 3], $E_7 : C_3 \in \max \mathrm{PSL}_2(7)$ and by Lemma 2.2, $\mathrm{acd}(E_7 : C_3) = \frac{3^2+3(7-1)}{3^2+7-1} = \frac{9}{5} = 1.8 \not< 1.5$, a contradiction.

Case 3: $\mathrm{SL}_2(5)$. We know that $2.S_4 \in \max \mathrm{SL}_2(5)$ and $\mathrm{acd}(2.S_4) = 2.25 \not< 1.5$, a contradiction.

These contradictions complete the proof. ■

Acknowledgments – The author is very grateful for the helpful suggestions of the referee.

Funding – S. Liu is supported by the NSF of China (Grant No. 11871360), by the Research Center of Mathematics and Finance (Grant No. SCMF202303), by the Teaching Reform of Sichuan University of Arts and Science (Grant No. 2023JG048T) and by the Science and Technology project of Guangxi (Guike AD21220114) (Grant No. CAMG2022A04).

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Manoscritto pervenuto in redazione il 24 aprile 2024.