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Polynomial log-volume growth and the GK-dimensions of twisted homogeneous coordinate rings

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Abstract. Let f be a zero entropy automorphism of a compact Kähler manifold X. We study the polynomial log-volume growth $\mathrm{Plov}(f)$ of f in light of the dynamical filtrations introduced in our previous work with T.-C. Dinh. We obtain new upper bounds and lower bounds of $\mathrm{Plov}(f)$. As a corollary, we completely determine $\mathrm{Plov}(f)$ when $\dim X = 3$, extending a result of Artin–Van den Bergh for surfaces. When X is projective, $\mathrm{Plov}(f) + 1$ coincides with the Gelfand–Kirillov dimensions $\mathrm{GKdim}(X, f)$ of the twisted homogeneous coordinate rings associated to (X, f). Reformulating these results for $\mathrm{GKdim}(X, f)$, we improve Keeler's bounds of $\mathrm{GKdim}(X, f)$ and provide effective upper bounds of $\mathrm{GKdim}(X, f)$ which only depend on $\dim X$.

1. Introduction

1.1. Zero entropy automorphisms

Let X be a compact Kähler manifold and let $f : X \circlearrowleft$ be an automorphism (i.e., biholomorphic self-map) of X. The topological entropy $h_{\mathrm{top}}(f)$ is an invariant measuring the complexity of the dynamical system $f : X \circlearrowleft$. Thanks to Gromov–Yomdin's theorem [12, 23], we have

$$h_{\mathrm{top}}(f) = \log r(f) \geq 0, \tag{1.1}$$

where $r(f)$ is the spectral radius of $f^* : H^\bullet(X, \mathbb{C}) \circlearrowleft$.

This paper focuses on automorphisms f with *zero entropy* $h_{\mathrm{top}}(f) = 0$ (cf. Lemma 2.8). In the context of complex dynamics of compact Kähler manifolds, they have recently been investigated in various works (see, e.g., [4,5,8,11,18]). In these works, more refined invariants of them are studied, such as the polynomial entropy, the polynomial log-volume growth $\mathrm{Plov}(f)$ [5], and *the polynomial growth* $k(f)$ of the pullbacks [8, 18],

$$\|(f^m)^* : H^{1,1}(X) \circlearrowleft \| \asymp_{m\to\infty} m^{k(f)}.$$

New structures of $f^* : H^\bullet(X, \mathbb{C}) \circlearrowleft$ have also been discovered such as the dynamical filtrations [8, §3]. Below is one such consequence, which is also relevant to the present work.

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Theorem 1.1 ([8, Theorem 1.1, Remark 3.9.(1)] and Corollary 3.7). *Let $f \in \mathrm{Aut}(X)$ be an automorphism of zero entropy. Assume that $d := \dim X \geq 1$. Then $k(f)$ is an even integer which satisfies*

$$k(f) \leq 2(d-1) \tag{1.2}$$

and

$$k(f) \leq 2(d-\kappa(X)), \tag{1.3}$$

where $\kappa(X)$ is the Kodaira dimension of X. Moreover, these estimates are optimal.

The upper bound (1.2) is the most essential part and was proven in [8, Theorem 1.1]. We will prove the refinement (1.3) in Corollary 3.7, based on the approach developed in [8].

1.2. Polynomial log-volume growths

The main goal of this paper is to study the *polynomial log-volume growth* $\mathrm{Plov}(f)$ of an automorphism $f : X \circlearrowleft$. We first recall its definition. For every $n \geq 1$, let $\Gamma_f(n) \subset X^{n+1}$ be the graph of

$$f \times f^2 \times \cdots \times f^n : X \to X^n$$

and let $\mathrm{Vol}_\omega(\Gamma_f(n))$ be the volume of $\Gamma_f(n)$ defined with respect to a Kähler form ω on X. We then define

$$\mathrm{Plov}(f) := \mathrm{Plov}(X, f) := \limsup_{n\to\infty} \frac{\log \mathrm{Vol}_\omega(\Gamma_f(n))}{\log n} \in [0, \infty].$$

This invariant of f is independent of the choice of ω (Lemma 2.1).

We will study upper bounds and lower bounds of $\mathrm{Plov}(f)$ in terms of $d = \dim X$ and $k(f)$ introduced in Section 1.1. Using dynamical filtrations, we obtain the following estimates.

Theorem 1.2. *Let X be a compact Kähler manifold of dimension d and let $f \in \mathrm{Aut}(X)$ be a zero entropy automorphism.*

(0) *(Corollary 2.19)* $\mathrm{Plov}(f) = d$ *if and only if* $k(f) = 0$.

(1) *(Proposition 4.1 and Theorem 5.1) Suppose that* $k(f) > 0$. *Then we have*

$$d + 2k(f) - 2 \leq \mathrm{Plov}(f) \leq k(f)(d-1) + d.$$

(2) *(Theorem 4.2) Suppose that* $k(f) > 0$ *and* $d \geq 3$. *Then we have*

$$\mathrm{Plov}(f) \leq k(f)(d-1) + d - 2.$$

By Theorem 1.1, we have

$$k(f) \in \{0, 2, \dots, 2d-2\}.$$

Also, $\mathrm{Plov}(f)$ has the same parity as $\dim X$; see Corollary 2.20. These together with Theorem 1.2 immediately determine $\mathrm{Plov}(f)$ when $d = 2, 3$.

Corollary 1.3. (1) *If $d = 2$, then*

$$\mathrm{Plov}(f) = \begin{cases} 2 & \text{if } k(f) = 0, \\ 4 & \text{if } k(f) = 2. \end{cases}$$

(2) *If $d = 3$, then*

$$\mathrm{Plov}(f) = \begin{cases} 3 & \text{if } k(f) = 0, \\ 5 & \text{if } k(f) = 2, \\ 9 & \text{if } k(f) = 4. \end{cases}$$

Together with Theorem 1.1, Theorem 1.2 implies that $\mathrm{Plov}(f) \le 2d^2 - 3d$ whenever $d \ge 3$. When $d \ge 4$, we will further improve this upper bound to

$$\mathrm{Plov}(f) \le 2d^2 - 3d - 2; \tag{1.4}$$

see Proposition 4.4.

1.3. A conjectural upper bound

When X is a complex torus, we determine $\mathrm{Plov}(f)$ in terms of the pullback

$$f^* : H^{1,0}(X) \circlearrowleft .$$

Theorem 1.4. *Let X be a complex torus of dimension d and $f \in \mathrm{Aut}(X)$ an automorphism of zero entropy. Assume that the Jordan canonical form of $f^* : H^{1,0}(X) \circlearrowleft$ consists of Jordan blocks of sizes $k_1, \ldots, k_p$, counted with multiplicities. Then*

$$\mathrm{Plov}(f) = \sum_{i=1}^{p} k_i^2 .$$

In particular, we have $\mathrm{Plov}(f) \le d^2$, and this upper bound is optimal among complex tori.

Theorem 1.4 also shows that the quadratic *order* of the upper bounds with respect to d in (1.4) is optimal. We will also compute $\mathrm{Plov}(X, f)$ for other examples including threefolds; see Section 7. As we fail to construct examples of $f : X \circlearrowleft$ such that $d^2 < \mathrm{Plov}(f) < \infty$ where $d = \dim X$, presumably the upper bound in (1.4) when $d \ge 4$ is still not optimal. Taking Corollary 1.3 and Theorem 1.4 into account, it seems reasonable to ask the following questions.

Question 1.5. Let X be a compact Kähler manifold of dimension $d \ge 1$. Let $f \in \mathrm{Aut}(X)$ be a zero entropy automorphism.

(1) Is $\mathrm{Plov}(f) \le d^2$?

(2) More precisely, are possible values of $\mathrm{Plov}(X, f)$ always realizable by d-dimensional tori?

Question 1.5 (1) is the analogous question to [5, Question 4.1], which asked for polynomial entropy by Cantat–Paris-Romaskevich. By [5, (2.7)], a positive answer to Question 1.5 (1) also answers [5, Question 4.1] in the affirmative.

The following partial answer to Question 1.5 is a direct consequence of the above theorems.

Corollary 1.6 (See Section 5). *Let X be a compact Kähler manifold of dimension $d \geq 3$ and let $f \in \mathrm{Aut}(X)$ be a zero entropy automorphism.*

(1) *If $k(f) \leq d$, then $\mathrm{Plov}(f) \leq d^2 - 2$. In particular, $\mathrm{Plov}(f) \leq d^2 - 2$, whenever $\kappa(X) \geq d/2$.*

(2) *Question 1.5 has positive answers when $\dim X \leq 3$.*

1.4. Gelfand–Kirillov dimension

When X is projective, the polynomial log-volume growth $\mathrm{Plov}(f)$ actually coincides with some known invariant of f studied in noncommutative algebra. The following identification is implicit in the seminal work of Keeler [15].

Theorem 1.7. *Let X be a smooth projective variety defined over an algebraically closed field, and let $f \in \mathrm{Aut}(X)$ be a zero entropy automorphism. Then*

$$\mathrm{GKdim} B(X, f, L) - 1 = \mathrm{Plov}(X, f).$$

Here, $\mathrm{GKdim} B(X, f, L)$ is the Gelfand–Kirillov dimension (or GK-dimension for short) of the twisted homogeneous coordinate ring $B(X, f, L)$ associated to $f : X \circlearrowleft$ and any ample line bundle L.

We refer to Section 8 for the definition and basic properties of $\mathrm{GKdim} B(X, f, L)$, as well as the proof of Theorem 1.7. In this regard, two of our results are not new for projective varieties. The first one is the upper bound in Theorem 1.2 (1), as the estimate

$$\mathrm{GKdim} B(X, f, L) - 1 \leq k(f)(d-1) + d$$

has already been proven in [15, Lemma 6.13]. The second one is Corollary 1.3 (1), due to Artin–Van den Bergh [1, Theorem 1.7]. Our approach based on dynamical filtrations is however completely different, and extends both results in a non-trivial way.

Thanks to Theorem 1.7, the main results we prove for $\mathrm{Plov}(f)$ also translate to new results about the GK-dimension of $B(X, f, L)$; see Corollary 8.5 for some instances. So far, the GK-dimension has been studied mostly by specialists in noncommutative algebras. We hope that the dynamical properties of (X, f) might provide a better understanding of the algebraic structure of $B(X, f, L)$, and *vice versa*.

1.5. Organization of the paper and a few remarks to the readers

We start with Section 2, proving basic properties of polynomial log-volume growth (see, e.g., Proposition 2.5). In Section 3, we recall the construction of quasi-nef sequences and dynamical filtrations, together with their fundamental properties. We also prove several auxiliary results related to them, which will be useful in the study of upper and lower bounds of $\mathrm{Plov}(f)$. Section 3 also contains a proof of the statement in Theorem 1.1 involving the Kodaira dimension. Section 4 and Section 5 are devoted to upper and lower bounds of $\mathrm{Plov}(f)$ respectively, all together implying Theorem 1.2 and Corollary 1.6. In Section 6 and Section 7, we study explicit examples, which contain complete descriptions of $\mathrm{Plov}(f)$ for tori (Theorem 1.4). Section 8 starts with a brief review of twisted homogeneous coordinate rings and their GK-dimensions. We recall some fundamental results proven in [1,15] (Theorem 8.1) and derive Theorem 1.7 as a direct consequence. We finish Section 8 by Corollary 8.5, translating results from $\mathrm{Plov}(f)$ to GK-dimensions.

Notations and conventions

All manifolds are assumed to be connected. Let X be a compact Kähler manifold. Write

$$H^{i,i}(X,\mathbb{R}) := H^{i,i}(X) \cap H^{2i}(X,\mathbb{R}).$$

For every $\alpha \in H^{i,i}(X,\mathbb{R})$, if $\alpha \cdot H^{1,1}(X,\mathbb{R})^{d-i} = 0$ (where $d = \dim X$), we write

$$\alpha \equiv 0.$$

We follow [6] for the basic terminology, like positive classes and cones.

2. Polynomial log-volume growth

2.1. Definition and basic properties

Let X be a compact Kähler manifold of dimension $d \geq 1$ and let $f \in \mathrm{Aut}(X)$. Let $\omega \in H^{1,1}(X,\mathbb{R})$ be a Kähler class. For every integer $n \geq 1$, the volume of the graph $\Gamma_f(n) \subset X^{n+1}$ of

$$f \times f^2 \times \cdots \times f^n : X \to X^n$$

with respect to any Kähler metric in the class ω is equal to

$$\mathrm{Vol}_\omega(\Gamma_f(n)) = \int_{\Gamma_f(n)} \frac{1}{d!}\left(\sum_{i=1}^{n+1} \mathrm{pr}_i^*\omega\right)^d = \frac{1}{d!}\Delta_n(f,\omega)^d,$$

where $\mathrm{pr}_i : X^{n+1} \to X$ is the projection to the i-th factor and

$$\Delta_n(f,\omega) := \sum_{i=0}^{n} (f^i)^*\omega \in H^{1,1}(X,\mathbb{R}).$$

Note that the class $\Delta_n(f,\omega)$ and the invariant $\mathrm{Vol}_\omega(\Gamma_f(n))$ are defined more generally for any $\omega \in H^{1,1}(X,\mathbb{R})$. But in order to define $\mathrm{Plov}(f,\omega)$ below, the class ω needs to satisfy $\Delta_n(f,\omega)^d \geq 0$. A natural sufficient condition is that ω is nef.

Lemma 2.1. *For every nef* $\alpha \in \mathrm{Nef}(X) \subset H^{1,1}(X,\mathbb{R})$*, define*

$$\mathrm{Plov}(f,\alpha) := \limsup_{n\to\infty} \frac{\log \mathrm{Vol}_\alpha(\Gamma_f(n))}{\log n} = \limsup_{n\to\infty} \frac{\log \Delta_n(f,\alpha)^d}{\log n} \in \mathbb{R} \cup \{-\infty, \infty\},$$

where we set $\log 0 := -\infty$*. Then* $\mathrm{Plov}(f,\omega)$ *is independent of* ω *whenever* ω *is nef and big, and we have* $\mathrm{Plov}(f,\omega) \ge 1$*.*

Lemma 2.1 justifies the well-definedness of the polynomial logarithmic volume growth of f in the introduction, which is defined to be

$$\mathrm{Plov}(f) := \mathrm{Plov}(X, f) := \mathrm{Plov}(f,\omega),$$

where ω is any nef and big class. We refer to Corollary 2.7 for an improvement.

To prove Lemma 2.1, we need the following easy but useful result.

Lemma 2.2. *Let* X *be a compact Kähler manifold of dimension* d*, and let*

$$L_1, \dots, L_d, M_1, \dots, M_d$$

be nef classes in $H^{1,1}(X,\mathbb{R})$ *such that* $M_i \ge L_i$*, i.e.,* $M_i - L_i$ *is pseudo-effective. Then*

$$(M_1 \cdots M_d) \ge (L_1 \cdots L_d).$$

In particular, $(M_1^d) \ge (L_1^d)$*.*

Proof. Inductively, we have

$$(M_1 \cdots M_d) \ge (L_1 \cdot M_2 \cdots M_d) \ge \cdots \ge (L_1 \cdots L_j \cdot M_{j+1} \cdots M_d) \ge \cdots \ge (L_1 \cdots L_d),$$

which proves Lemma 2.2. ∎

Proof of Lemma 2.1. Let ω and ω' be two nef and big classes. Then there exists some $\varepsilon > 0$ such that $\omega - \varepsilon\omega'$ is pseudo-effective. Accordingly, $\Delta_n(f,\omega) - \varepsilon\Delta_n(f,\omega')$ is pseudo-effective, so $\Delta_n(f,\omega)^d \ge \varepsilon^d \Delta_n(f,\omega')^d$ by Lemma 2.2, and therefore $\mathrm{Plov}(f,\omega) \ge \mathrm{Plov}(f,\omega')$. By symmetry, we have $\mathrm{Plov}(f,\omega) = \mathrm{Plov}(f,\omega')$.

Finally, since ω is big and nef, we have

$$\Delta_n(f,\omega)^d = \left(\sum_{i=0}^{n} (f^i)^*\omega\right)^d \ge \sum_{i=1}^{n} ((f^i)^*\omega)^d = n\omega^d > 0.$$

Hence,

$$\mathrm{Plov}(f,\omega) = \limsup_{n\to\infty} \frac{\log \Delta_n(f,\omega)^d}{\log n} \ge \limsup_{n\to\infty} \frac{\log(\omega^d) + \log n}{\log n} = 1.$$ ∎

The following is an immediate consequence of Lemma 2.1.

Corollary 2.3. *Let X and Y be compact Kähler manifolds with automorphisms $f \in \mathrm{Aut}(X)$ and $g \in \mathrm{Aut}(Y)$. Suppose that there exists a $\mathbb{C}$-linear isomorphism*

$$\phi : H^\bullet(X, \mathbb{C}) \xrightarrow{\sim} H^\bullet(Y, \mathbb{C})$$

of the cohomology rings such that the following conditions are satisfied:

(i) $\phi \circ f^* = g^* \circ \phi$;

(ii) *there exists a Kähler class $\omega \in H^{1,1}(X)$ on X such that $\phi(\omega)$ is Kähler on Y.*

Then

$$\mathrm{Plov}(f) = \mathrm{Plov}(g).$$

The similar statement holds if ϕ is replaced by a $\mathbb{C}$-linear isomorphism of the subalgebras

$$\psi : \bigoplus_i H^{i,i}(X) \xrightarrow{\sim} \bigoplus_i H^{i,i}(X').$$

The same argument as in the proof of Lemma 2.2 proves the following.

Lemma 2.4. *For every nef class $\alpha \in H^{1,1}(X, \mathbb{R})$, we have*

$$\mathrm{Plov}(f, \alpha) \le \mathrm{Plov}(f).$$

Proof. Take a Kähler class ω such that $\omega \ge \alpha$. By Lemma 2.2, we have $\Delta_n(f, \omega)^d \ge \Delta_n(f, \alpha)^d$ for every integer $n \ge 0$. Hence $\mathrm{Plov}(f, \alpha) \le \mathrm{Plov}(f)$. ∎

Now we prove some basic dynamical properties of $\mathrm{Plov}(X, f)$ summarized in the following, which will be frequently used in this paper.

Proposition 2.5. *Let $f : X \circlearrowleft$ be an automorphism of a compact Kähler manifold.*

(1) *(Independence of the metric and positivity; Lemma 2.1) The invariant $\mathrm{Plov}(f)$ is independent of $\omega \in H^{1,1}(X, \mathbb{R})$ whenever ω is nef and big, and we have $\mathrm{Plov}(f) \ge 1$.*

(2) *(Finiteness and integrality; Lemmas 2.8 and 2.16) We have $\mathrm{Plov}(f) < \infty$ if and only if $f^* : H^{1,1}(X) \circlearrowleft$ is quasi-unipotent. In this case, $\mathrm{Plov}(f)$ is an integer.*

(3) *(Finite index; Lemma 2.6) We have $\mathrm{Plov}(f) = \mathrm{Plov}(f^N)$ for any integer $N \ne 0$.*

(4) *(Product; Lemma 2.18) Let X_i $(i = 1, 2)$ be compact Kähler manifolds and let $f_i \in \mathrm{Aut}(X_i)$. Then*

$$\mathrm{Plov}(f_1 \times f_2) = \mathrm{Plov}(f_1) + \mathrm{Plov}(f_2)$$

for the product automorphism $f_1 \times f_2 \in \mathrm{Aut}(X_1 \times X_2)$.

(5) *(Invariance under generically finite maps; Lemma 2.9) Let X and Y be compact Kähler manifolds and $f_X \in \mathrm{Aut}(X)$ and $f_Y \in \mathrm{Aut}(Y)$. Let $\phi : X \dashrightarrow Y$ be a generically finite dominant meromorphic map such that $f_Y \circ \phi = \phi \circ f_X$. Then*

$$\mathrm{Plov}(f_X) = \mathrm{Plov}(f_Y).$$

(6) *(Restriction; see Lemma 2.10, also for the precise definition of* $\mathrm{Plov}(f_{|W})$ *when* W *is not smooth) Let* $W \subset X$ *be a closed subvariety such that* $f(W) = W$. *Then* $\mathrm{Plov}(f|_W) \le \mathrm{Plov}(f)$ *for the automorphism* $f|_W \in \mathrm{Aut}(W)$ *induced from* f *by restriction.*

First, we prove that $\mathrm{Plov}(X, f)$ is invariant under taking finite iterations.

Lemma 2.6. *For every integer* $N \neq 0$, *we have* $\mathrm{Plov}(f^N) = \mathrm{Plov}(f)$.

Proof. Since

$$\left(\sum_{i=0}^{n}(f^{-i})^*\omega\right)^d = \left((f^{-n})^*\sum_{i=0}^{n}(f^{i})^*\omega\right)^d = \left(\sum_{i=0}^{n}(f^{i})^*\omega\right)^d,$$

we have $\mathrm{Plov}(f^{-1}) = \mathrm{Plov}(f)$. So it suffices to prove Lemma 2.6 for $N > 0$.

For every integers r and $j > 0$ such that $0 \le r < N$, consider the Kähler form $\omega_{r,j} := \sum_{i=r}^{r+j-1}(f^i)^*\omega$. Then

$$\omega_{r,(m+1)N} \le \omega_{0,r} + \omega_{r,(m+1)N} = \omega_{0,r+(m+1)N} \le \omega_{r-N,(m+2)N}.$$

So

$$\mathrm{Vol}_{\omega_{r,(m+1)N}}(X) \le \mathrm{Vol}_{\omega_{0,r+(m+1)N}}(X) \le \mathrm{Vol}_{\omega_{r-N,(m+2)N}}(X)$$

by Lemma 2.2, and thus

$$\mathrm{Vol}_{\sum_{j=0}^{N-1}(f^{r+j})^*\omega}(\Gamma_{f^N}(m)) \le \mathrm{Vol}_{\omega}(\Gamma_f(r+mN)) \le \mathrm{Vol}_{\sum_{j=1}^{N}(f^{r-j})^*\omega}(\Gamma_{f^N}(m+1)).$$

By Lemma 2.1, we have

$$\limsup_{m\to\infty}\frac{\log \mathrm{Vol}_{\sum_{j=0}^{N-1}(f^{r+j})^*\omega}(\Gamma_{f^N}(m))}{\log m} = \mathrm{Plov}(f^N) = \limsup_{m\to\infty}\frac{\log \mathrm{Vol}_{\sum_{j=1}^{N}(f^{r-j})^*\omega}(\Gamma_{f^N}(m+1))}{\log(m+1)},$$

so for every integer r such that $0 \le r < N$, we have

$$\limsup_{m\to\infty}\frac{\log \mathrm{Vol}_{\omega}(\Gamma_f(r+mN))}{\log(r+mN)} = \limsup_{m\to\infty}\frac{\log \mathrm{Vol}_{\omega}(\Gamma_f(r+mN))}{\log(m)} = \mathrm{Plov}(f).$$

Hence, $\mathrm{Plov}(f) = \mathrm{Plov}(f^N)$. ∎

Corollary 2.7. *Let* $\alpha \in \mathrm{Nef}(X)$. *We have*

$$\mathrm{Plov}(f,\alpha) = \mathrm{Plov}(f) \in [1,\infty]$$

as long as $\mathrm{Plov}(f,\alpha) \neq -\infty$.

Proof. Suppose that $\mathrm{Plov}(f,\alpha) \neq -\infty$. Then $\Delta_N(f,\alpha)^d > 0$ for some integer $N \geq 0$. Since $\omega := \Delta_N(f,\alpha)$ is nef, it is thus big. Using Lemma 2.6, we have

$$\begin{aligned}\mathrm{Plov}(f,\alpha) &= \limsup_{n\to\infty} \frac{\log \Delta_n(f,\alpha)^d}{\log n} \\ &\geq \limsup_{k\to\infty} \frac{\log \Delta_{Nk-1}(f,\alpha)^d}{\log(Nk-1)} = \limsup_{k\to\infty} \frac{\log \Delta_k(f^N,\omega)^d}{\log k} \\ &= \mathrm{Plov}(f^N) = \mathrm{Plov}(f).\end{aligned}$$

It follows from Lemma 2.4 that $\mathrm{Plov}(f,\alpha) = \mathrm{Plov}(f)$. ■

We can characterize whether a holomorphic automorphism $f \in \mathrm{Aut}(X)$ has zero entropy based on the finiteness of $\mathrm{Plov}(f)$.

Lemma 2.8. *Let X be a compact Kähler manifold of dimension $d \geq 1$ and let $f \in$ $\mathrm{Aut}(X)$. Then the following conditions are equivalent.*

(1) *$f^* : H^\bullet(X,\mathbb{C}) \circlearrowleft$ is quasi-unipotent, i.e., a positive power of it is unipotent.*

(2) *$f^* : H^{1,1}(X) \circlearrowleft$ is quasi-unipotent.*

(3) *The first dynamical degree $d_1(f) = 1$.*

(4) *The topological entropy $h_{\mathrm{top}}(f) = 0$.*

(5) *The growth of $\mathrm{Vol}_\omega(\Gamma_f(n))$ for any Kähler class ω is sub-exponential, namely*

$$\limsup_{n\to\infty} \mathrm{Vol}_\omega(\Gamma_f(n))^{1/n} = 1.$$

(6) $\mathrm{Plov}(f) < \infty$. *In other words, the growth of $\mathrm{Vol}_\omega(\Gamma_f(n))$ for any Kähler class ω is polynomial.*

Here we recall that for $1 \leq i \leq d$, the i-th *dynamical degree* of f is defined as

$$d_i(f) := \lim_{n\to\infty} \left(\omega^{d-i} \cdot (f^n)^*\omega^i\right)^{1/n}, \tag{2.1}$$

where $\omega \in H^{1,1}(X)$ is a Kähler class [10]; these $d_i(f)$ are independent of ω.

Proof. The equivalence of the first five conditions is well known and is obtained as follows. By Gromov–Yomdin's theorem (cf. [12, 23]; see also [20, Theorem 3.6]), we have

$$h_{\mathrm{top}}(f) = \mathrm{lov}(f) = \log r(f^*) = \log(\max_{1\leq i\leq d}\{d_i(f)\}),$$

where $r(f^*)$ is the spectral radius of $f^* : H^\bullet(X,\mathbb{C}) \circlearrowleft$, and

$$\mathrm{lov}(f) := \limsup_{n\to\infty} \frac{\log \mathrm{Vol}_\omega(\Gamma_f(n))}{n}.$$

Together with the log concavity of dynamical degrees $d_i(f)$ (which follows from Khovanskii–Teissier's inequality), this implies that $h_{\mathrm{top}}(f) > 0$ if and only if $d_i(f) > 1$

for some (and hence all) $i \in \{1, \dots, d-1\}$. Thus the equivalence of the first five assertions follows from Kronecker's theorem. Also, since

$$\frac{\log n}{n} \cdot \mathrm{Plov}(f) \geq \mathrm{lov}(f) = h_{\mathrm{top}}(f) \geq 0$$

for all $n > 1$, (6) implies these assertions.

To see that (2) implies (6), recall that in order to compute $\mathrm{Plov}(f)$, by Lemma 2.6 we can replace f by some iteration of it, so that $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent. Hence $\mathrm{Plov}(f) < \infty$ is a consequence of Lemma 2.16 below. ∎

Next, we prove the invariance of $\mathrm{Plov}(f)$ under generically finite meromorphic maps.

Lemma 2.9. *Let X and Y be compact Kähler manifolds of dimension $d \geq 1$ and $f_X \in \mathrm{Aut}(X)$ and $f_Y \in \mathrm{Aut}(Y)$. Let $\phi : X \dashrightarrow Y$ be a generically finite dominant meromorphic map such that $f_Y \circ \phi = \phi \circ f_X$. Then*

$$\mathrm{Plov}(f_X) = \mathrm{Plov}(f_Y).$$

Proof. First we reduce to the case where ϕ is holomorphic. Let Γ be the Zariski closure of the graph of ϕ in $X \times Y$. Let $p_X : \Gamma \to X$ and $p_Y : \Gamma \to Y$ be the projection. Since $f_X \in \mathrm{Aut}(X)$ and $f_Y \in \mathrm{Aut}(Y)$, it follows that

$$f_\Gamma := (f_X \times f_Y)|_\Gamma \in \mathrm{Aut}(\Gamma)$$

and f_Γ and f_X (resp. f_Γ and f_Y) are equivariant with respect to a generically finite surjective morphism p_X (resp. p_Y). By the existence of functorial resolution of singularities [2] (see also [17, Theorem 3.45]), there exists a Kähler desingularization $\nu : \widetilde{\Gamma} \to \Gamma$ such that $f_\Gamma \circ \nu = \nu \circ f_{\widetilde{\Gamma}}$ for some $f_{\widetilde{\Gamma}} \in \mathrm{Aut}(\widetilde{\Gamma})$. If Lemma 2.9 holds whenever ϕ is holomorphic, then $\mathrm{Plov}(f_X) = \mathrm{Plov}(f_{\widetilde{\Gamma}}) = \mathrm{Plov}(f_Y)$.

For every Kähler class ω on Y, since

$$\left(\sum_{i=0}^{n} (f_X^i)^*(\phi^*\omega)\right)^d = \phi^*\left(\sum_{i=0}^{n} (f_Y^i)^*\omega\right)^d = \deg(\phi) \cdot \left(\sum_{i=0}^{n} (f_Y^i)^*\omega\right)^d,$$

we have $\mathrm{Plov}(f_X, \phi^*\omega) = \mathrm{Plov}(f_Y, \omega) = \mathrm{Plov}(f_Y)$. As $\phi^*\omega$ is nef and big, it follows from Lemma 2.1 that $\mathrm{Plov}(f_X) = \mathrm{Plov}(f_Y)$. ∎

Lemma 2.10. *Let $W \subset X$ be a closed subvariety such that $f(W) = W$. Then $\mathrm{Plov}(f|_W) \leq \mathrm{Plov}(f)$ for the automorphism $f|_W \in \mathrm{Aut}(W)$ induced from f by restriction. Here we define*

$$\mathrm{Plov}(f|_W) := \mathrm{Plov}(\widetilde{W}, \widetilde{f}),$$

where $\tau : \widetilde{W} \to W$ is any Kähler desingularization of W such that $f|_W \circ \tau = \tau \circ \widetilde{f}$ for some $\widetilde{f} \in \mathrm{Aut}(\widetilde{W})$, which does not depend on the choice of $\widetilde{W}$ by Lemma 2.9.

Proof. Let $\nu : \widetilde{W} \to X$ be the composition of τ with the inclusion $W \hookrightarrow X$. Let $d := \dim X$ and $e := \dim W$. Let ω be a Kähler class of X. Up to replacing ω by some positive multiple of it, we can assume that

$$(\omega^{d-e} - [W]) \cdot \beta \geq 0$$

for every β in the closed convex cone generated by products of e Kähler classes.

Since $\sum_{i=0}^{n} (f^i)^*(\omega) - \omega$ is Kähler when $n \geq 1$, by Lemma 2.2 we have,

$$\begin{aligned}\left(\sum_{i=0}^{n} (\widetilde{f}^i)^*(\nu^*\omega)\right)^e &= \left(\left(\sum_{i=0}^{n} (f^i)^*(\omega)\right)^e \cdot [W]\right) \\ &\leq \left(\left(\sum_{i=0}^{n} (f^i)^*(\omega)\right)^e \cdot \omega^{d-e}\right) \leq \left(\sum_{i=0}^{n} (f^i)^*(\omega)\right)^d. \end{aligned} \tag{2.2}$$

So $\mathrm{Plov}(\widetilde{f}, \nu^*\omega) \leq \mathrm{Plov}(f, \omega) = \mathrm{Plov}(f)$. As $\nu^*\omega$ is nef and big, we have $\mathrm{Plov}(\widetilde{f}) = \mathrm{Plov}(\widetilde{f}, \nu^*\omega)$ by Lemma 2.1. Hence $\mathrm{Plov}(f|_W) \leq \mathrm{Plov}(f)$. ■

2.2. Cohomological polynomial growth $k(f)$

Assume that $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent. The operator

$$N := f^* - \mathrm{Id} : H^{1,1}(X) \to H^{1,1}(X)$$

is thus nilpotent, and we define

$$k(f) := \max\{k \in \mathbb{Z} \mid (f^* - \mathrm{Id})^k \neq 0\}.$$

Equivalently, $k(f) + 1$ is the maximal size of the Jordan blocks of the Jordan canonical form of $f^* : H^{1,1}(X) \circlearrowleft$. If $f^* : H^{1,1}(X) \circlearrowleft$ is quasi-unipotent, we define

$$k(f) := k(f^M),$$

where M is a positive integer such that $(f^*)^M$ is unipotent; this invariant is independent of M. Finally, if $f^* : H^{1,1}(X) \circlearrowleft$ is not quasi-unipotent, we set $k(f) = \infty$.

The following result implies in particular that $k(f)$ is invariant under bimeromorphic modifications.

Proposition 2.11. *Let $\pi : X \dashrightarrow Y$ be a dominant, generically finite meromorphic map between compact Kähler manifolds. Let $f_X \in \mathrm{Aut}(X)$ and $f_Y \in \mathrm{Aut}(Y)$ be automorphisms such that*

$$\pi \circ f_X = f_Y \circ \pi.$$

Then

$$k(f_X) = k(f_Y).$$

We shall also prove the following.

Lemma 2.12. *Let X and Y be compact Kähler manifolds. Let $f_X \in \operatorname{Aut}(X)$ and $f_Y \in \operatorname{Aut}(Y)$. Then we have*

$$k(f_X \times f_Y) = \max\{k(f_X), k(f_Y)\}.$$

To prove both Proposition 2.11 and Lemma 2.12, we need the following result in linear algebra.

Lemma 2.13. *Let V be a finite dimensional vector space over $\mathbb{R}$ and let $\phi \in \mathrm{GL}(V)$ be a unipotent operator. Let $N := \phi - \mathrm{Id}_V$ and let k denote the largest integer such that $N^k \neq 0$. Assume that ϕ preserves a closed salient convex cone $\mathcal{C} \subset V$ with nonempty interior. Then for every $v \in \operatorname{Int}(\mathcal{C})$, the following assertions hold.*

(1) *We have $N^k(v) \in \mathcal{C}\backslash\{0\}$ and*

$$\phi^n(v) \sim_{n\to\infty} C_v n^k \cdot N^k(v)$$

for some $C_v > 0$.

(2) *For every linear form $\chi : V \to \mathbb{R}$ such that $\chi(\mathcal{C}\backslash\{0\}) > 0$, we have*

$$\chi(\phi^n(v)) \sim_{n\to\infty} C'_v n^k$$

for some $C'_v > 0$.

Proof. We can assume $\phi \neq \mathrm{Id}$. Then $\ker N^k \neq V$, and for every $w \in V\backslash \ker N^k$, developing $\phi^n(w) = (\mathrm{Id}_V + N)^n(w)$ shows that

$$\phi^n(w)/n^k \sim_{n\to\infty} C_w N^k(w)$$

for some $C_w > 0$. If moreover $w \in \mathcal{C}$, then $\phi(\mathcal{C}) \subset \mathcal{C}$ and $\mathcal{C}$ being closed, imply $N^k(w) \in \mathcal{C}$. Assume the contrary that there exists some $x \in \operatorname{Int}(\mathcal{C})$ such that $N^k(x) = 0$. Then there exists some $\varepsilon \in V$ such that

$$x \pm \varepsilon \in \mathcal{C} \quad \text{and} \quad N^k(x \pm \varepsilon) \neq 0.$$

As $x \pm \varepsilon \in \mathcal{C}$ and $\phi(\mathcal{C}) \subset \mathcal{C}$, both $N^k(x \pm \varepsilon) = \pm N^k(\varepsilon)$ are limits of elements in $\mathcal{C}$, which contradicts the assumptions that $\mathcal{C}$ is closed and salient. This proves (1).

Since $N^k(v) \in \mathcal{C}\backslash\{0\}$, we have $\chi(N^k(v)) > 0$. Thus (2) follows from (1). ∎

Proof of Lemma 2.12. Assume that $k(f_X) = \infty$ or $k(f_Y) = \infty$. Then Lemma 2.12 follows from the product formula of the first dynamical degree ([9, Theorem 1.1] together with Lemma 2.8.

Assume that both $k(f_X)$ and $k(f_Y)$ are finite. By Lemma 2.8, up to replacing f_X and f_Y by a common positive power, we can assume

$$f_X^* : H^{1,1}(X, \mathbb{R}) \circlearrowleft, \quad f_Y^* : H^{1,1}(Y, \mathbb{R}) \circlearrowleft, \quad \text{and} \quad (f_X \times f_Y)^* : H^{1,1}(X \times Y, \mathbb{R}) \circlearrowleft$$

are unipotent. Fix Kähler classes $\omega_X \in H^{1,1}(X,\mathbb{R})$ and $\omega_Y \in H^{1,1}(Y,\mathbb{R})$. Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the projections. Applying Lemma 2.13 to $H^{1,1}(\bullet,\mathbb{R})$ and the nef cone therein shows that $k(f_X \times f_Y)$ (resp. $k(f_X)$ and $k(f_Y)$) is the polynomial growth rate of

$$((f_X \times f_Y)^*)^n(p_X^*\omega_X + p_Y^*\omega_Y) = p_X^*(f_X^*)^n(\omega_X) + p_Y^*(f_Y^*)^n(\omega_Y)$$

(resp. $(f_X^*)^n(\omega_X)$ and $(f_Y^*)^n(\omega_Y)$).

Hence $k(f_X \times f_Y) = \max\{k(f_X), k(f_Y)\}$. ■

Proof of Proposition 2.11. As in Lemma 2.9, up to replacing X by an equivariant desingularization of the graph of π, we can assume that π is holomorphic.

By Lemmas 2.9 and 2.8, we have $k(f_X) = \infty$ if and only if $k(f_Y) = \infty$. Thus we can assume that both $f_X^* : H^{1,1}(X) \circlearrowleft$ and $f_Y^* : H^{1,1}(Y) \circlearrowleft$ are quasi-unipotent. Up to replacing f_X and f_Y by some positive iterations, we can assume that the above actions are both unipotent.

Applying Lemma 2.13 to the nef cone in $H^{1,1}(X,\mathbb{R})$, we see that for every pair of Kähler classes ω, η on X, we have

$$(f_X^*)^n(\omega) \cdot \eta^{\dim X - 1} \sim_{n\to\infty} Cn^{k(f_X)} \tag{2.3}$$

for some $C > 0$. Similarly, for every pair of Kähler classes ω', η' on Y, we have

$$(f_X^*)^n(\pi^*\omega') \cdot \pi^*\eta'^{\dim X-1} = \deg(\pi) \cdot (f_Y^*)^n(\omega') \cdot \eta'^{\dim Y-1} \sim_{n\to\infty} C'n^{k(f_Y)} \tag{2.4}$$

for some $C' > 0$. Since the classes $\omega, \eta, \pi^*\omega', \pi^*\eta'$ are all nef and big, with the notation of Lemma 2.2 we have

$$c_1\pi^*\omega' \le \omega \le c_2\pi^*\omega' \ \text{ and } \ c_3\pi^*\eta' \le \eta \le c_4\pi^*\eta'$$

for some positive real numbers c_i. It follows from Lemma 2.2 that the growth rates of (2.3) and (2.4) have the same polynomial order. Hence $k(f_X) = k(f_Y)$. ■

2.3. Bounding the polynomial log-volume growth

From now on until the end of Section 2, we assume that

$$f^* : H^{1,1}(X) \circlearrowleft \ \text{ is unipotent,}$$

unless otherwise specified.

For every $\alpha \in H^{1,1}(X,\mathbb{R})$, recall that

$$\Delta_n(f,\alpha) := \sum_{i=0}^{n} (f^i)^*\omega \in H^{1,1}(X,\mathbb{R}).$$

The following lemma shows that $\Delta_n(f,\alpha)$ has polynomial expressions in n for both ranges $n \in \mathbb{Z}_{\ge 0}$ and $n \in \mathbb{Z}_{\le 0}$ (but these two polynomials are usually different).

Lemma 2.14. *We have*

$$\Delta_n(f,\alpha) = \begin{cases} \Delta_n^+(f,\alpha) & \text{if } n \geq 0, \\ -\Delta_{n-1}^+(f,\alpha) & \text{if } n \leq 0, \end{cases} \tag{2.5}$$

where

$$\Delta_n^+(f,\alpha) := \sum_{j=0}^{k(f)} \binom{n+1}{j+1} N^j \alpha.$$

Proof. By definition of $k := k(f)$, for every $\alpha \in H^{1,1}(X,\mathbb{R})$ and every $n \in \mathbb{Z}$,

$$\Delta_n(f,\alpha) = \sum_{i=0}^{n} (f^i)^*(\alpha) = \sum_{i=0}^{n} \sum_{j=0}^{k} \binom{i}{j} N^j \alpha = \sum_{j=0}^{k} \sum_{i=0}^{n} \binom{i}{j} N^j \alpha. \tag{2.6}$$

If $n \geq 0$, then

$$\sum_{i=0}^{n} \binom{i}{j} = \binom{n+1}{j+1}$$

by the hockey-stick identity. Similarly, if $n \leq 0$, then

$$\sum_{i=0}^{n} \binom{i}{j} = (-1)^j \sum_{i=0}^{n} \binom{j-i-1}{j} = (-1)^j \binom{j-n}{j+1} = -\binom{n}{j+1}.$$

Hence Lemma 2.14 follows. ∎

The following lemma will be useful to prove results on lower bounds of $\mathrm{Plov}(f)$ in this paper. In the projective setting, this lemma was due to Keeler [15, Lemma 6.5 (4)] and was applied in his work to prove his lower bound.

Lemma 2.15. *For every integer $i \in [1,d]$, let*

$$P_{f,\omega,i}(n) := \Delta_n^+(f,\omega)^i \omega^{d-i} = \left(\sum_{j=0}^{k} \binom{n+1}{j+1} N^j \omega \right)^i \omega^{d-i},$$

which is a polynomial in n of degree $\deg_n P_{f,\omega,i}(n)$. Then we have

$$\deg_n P_{f,\omega,i}(n) > \deg_n P_{f,\omega,i-1}(n).$$

Proof. For every non-negative integer m, define

$$\begin{aligned} P_{f,\omega,i,m}(n) &:= \Delta_n^+(f,\omega)^{i-1} \cdot (f^m)^*\omega \cdot \omega^{d-i} \\ &= \left(\sum_{j=0}^{k} \binom{n+1}{j+1} N^j \omega \right)^{i-1} \cdot (\mathrm{Id} + N)^m(\omega) \cdot \omega^{d-i}, \end{aligned}$$

which is a polynomial in n of degree $\deg_n P_{f,\omega,i,m}(n)$. Note that since both ω and $(f^m)^*\omega$ are Kähler, we have

$$C_1\omega \le (f^m)^*\omega \le C_2\omega$$

for some $C_1, C_2 > 0$, so

$$C_1 P_{f,\omega,i,0}(n) \le P_{f,\omega,i,m}(n) \le C_2 P_{f,\omega,i,0}(n)$$

by Lemma 2.2 and therefore

$$\deg_n P_{f,\omega,i,m}(n) = \deg_n P_{f,\omega,i,0}(n) = \deg_n P_{f,\omega,i-1}(n).$$

In particular, $\deg_n P_{f,\omega,i,m}(n)$ is independent of m.

For every m, since $P_{f,\omega,i,m}(n) > 0$, the leading coefficient $C_{f,\omega,i}(m)$ of the polynomial $P_{f,\omega,i,m}$ satisfies $C_{f,\omega,i}(m) > 0$. As $C_{f,\omega,i}(m)$ is a polynomial in m (because N is nilpotent), the minimum of

$$\left\{C_{f,\omega,i}(m) \mid m \in \mathbb{Z}_{\ge 0}\right\}$$

exists; let $\ell \in \mathbb{Z}_{\ge 0}$ such that $C_{f,\omega,i}(\ell)$ is the minimum.

By construction, we have

$$\lim_{n\to\infty} \frac{P_{f,\omega,i,m}(n)}{P_{f,\omega,i,\ell}(n)} = \frac{C_{f,\omega,i}(m)}{C_{f,\omega,i}(\ell)} \ge 1$$

for every $m \in \mathbb{Z}$. So

$$\frac{P_{f,\omega,i}(n)}{P_{f,\omega,i,\ell}(n)} = \sum_{m=0}^{n} \frac{P_{f,\omega,i,m}(n)}{P_{f,\omega,i,\ell}(n)} \succeq_{n\to\infty} \gamma,$$

for any $\gamma > 0$, which shows that $\deg_n P_{f,\omega,i,\ell}(n) < \deg_n P_{f,\omega,i}(n)$. Hence

$$\deg_n P_{f,\omega,i-1}(n) = \deg_n P_{f,\omega,i,\ell}(n) < \deg_n P_{f,\omega,i}(n). \qquad \blacksquare$$

Lemma 2.16. Plov(f) *is equal to the degree of the polynomial*

$$n \mapsto P_{f,\omega}(n) := P_{f,\omega,d}(n) = \Delta_n^+(f,\omega)^d = \left(\sum_{j=0}^{k(f)} \binom{n+1}{j+1} N^j\omega\right)^d$$

for any Kähler class ω on X. As a consequence, Plov(f) *is a positive integer satisfying*

$$k(f) + d \le \mathrm{Plov}(f) \le d + \max\left\{\sum_{j=1}^{d} i_j \mid i_j \in \mathbb{Z}_{\ge 0},\ (N^{i_1}\omega)\cdots(N^{i_d}\omega) \ne 0\right\}, \quad (2.7)$$

where $d = \dim X$. Also, the limit superior defining Plov(f,ω) *in Lemma 2.1 for any nef and big class ω is actually a limit.*

Proof. The first claim and the last statement about the limit superior are clear by Lemma 2.14 and the definition of $\mathrm{Plov}(f)$. Then, the upper bound of $\mathrm{Plov}(f)$ is clear by equation (2.6).

For the lower bound, by Lemma 2.15 with the notations therein, we have

$$\mathrm{Plov}(f) = \deg_n P_{f,\omega,d}(n) > \deg_n P_{f,\omega,d-1}(n) > \cdots > \deg_n P_{f,\omega,1}(n).$$

As

$$N^{k(f)}\omega = k(f)! \cdot \lim_{m\to\infty} \frac{(f^*)^m(\omega)}{m^{k(f)}}$$

is nef and nonzero by definition of $k(f)$, we have $(N^{k(f)}\omega)\cdot\omega^{d-1} \neq 0$. So $\deg_n P_{f,\omega,1}(n) \geq k(f)+1$, which shows that $\mathrm{Plov}(f) \geq k(f)+d$. ■

Remark 2.17. Based on $\mathrm{Plov}(f,\omega) = \mathrm{Plov}(f,(f^*)^i\omega)$ for any integer i, the last statement in Lemma 2.16 regarding the limit superior still holds if $f^* : H^{1,1}(X) \circlearrowleft$ is quasi-unipotent. We do not know whether it continues to hold without the quasi-unipotence assumption.

Now we can prove that the polynomial logarithmic volume growth is also compatible with product.

Lemma 2.18. *Let X_i ($i = 1, 2$) be compact Kähler manifolds and let $f_i \in \mathrm{Aut}(X_i)$ (without assuming that $f_i^* : H^{1,1}(X_i) \circlearrowleft$ is unipotent). Then*

$$\mathrm{Plov}(f_1 \times f_2) = \mathrm{Plov}(f_1) + \mathrm{Plov}(f_2).$$

Proof. Let ω_i be a Kähler metric on X_i and let $\mathrm{pr}_i : X_1 \times X_2 \to X_i$ be the projection to the i-th factor. Then

$$\mathrm{Vol}_{\mathrm{pr}_1^*\omega_1+\mathrm{pr}_2^*\omega_2}(\Gamma_{f_1\times f_2}(n)) = \mathrm{Vol}_{\omega_1}(\Gamma_{f_1}(n))\mathrm{Vol}_{\omega_2}(\Gamma_{f_2}(n)), \tag{2.8}$$

which proves Lemma 2.18 in the case where $\mathrm{Plov}(f_1) = \infty$ or $\mathrm{Plov}(f_2) = \infty$.

Assume that both $\mathrm{Plov}(f_1)$ and $\mathrm{Plov}(f_2)$ are finite, then $\mathrm{Plov}(f_1 \times f_2)$ is also finite by the equivalence (1) $\Leftrightarrow$ (6) in Lemma 2.8 and the Künneth formula. To prove Lemma 2.18, by Lemma 2.6 we can replace f_1 and f_2 by some common power. Thus by Lemma 2.8 again, we can assume that the actions of f_1, f_2, and $f_1 \times f_2$ acting on the cohomology rings of X_1, X_2, and $X_1 \times X_2$ respectively are unipotent. It follows from Lemma 2.16 that the limits superior in the definitions of $\mathrm{Plov}(f_1)$, $\mathrm{Plov}(f_2)$, and $\mathrm{Plov}(f_1 \times f_2)$ are all limits. Hence Lemma 2.18 for finite $\mathrm{Plov}(f_1)$ and $\mathrm{Plov}(f_2)$ follows again from (2.8). ■

Corollary 2.19. *Let X be a compact Kähler manifold of dimension d and let $f \in \mathrm{Aut}(X)$ be a zero entropy automorphism. Then $\mathrm{Plov}(f) = d$ if and only if $k(f) = 0$.*

Proof. Since f has zero entropy, by Lemmas 2.6 and 2.8 we can assume that $f^* :$ $H^{1,1}(X)$ is unipotent. Thus $k(f) = 0$ implies $N = 0$, and $\mathrm{Plov}(f) = d$ by Lemma 2.16. Again by Lemma 2.16, $\mathrm{Plov}(f) = d$ implies $k(f) = 0$. ■

Another consequence of Lemma 2.16 is the following.

Corollary 2.20. $\mathrm{Plov}(f)$ *has the same parity as* $d = \dim X$.

Proof. Since $\Delta_n(f,\omega)$ is Kähler, we have $\Delta_n(f,\omega)^d > 0$ for all $n \in \mathbb{Z}$. So by Lemma 2.14, we have

$$\begin{cases} \Delta_n^+(f,\omega)^d > 0 & \text{for } n \gg 0, \\ (-1)^d \Delta_{n-1}^+(f,\omega)^d > 0 & \text{for } n \ll 0. \end{cases} \tag{2.9}$$

It follows that the degree of the polynomial $n \mapsto \Delta_n^+(f,\omega)^d$, which is also $\mathrm{Plov}(f)$ by Lemma 2.16, has the same parity as d. ∎

The following lemma provides another way to compute $\mathrm{Plov}(f)$, and turns out to be useful. Define

$$\Delta'_n(f,\omega) := \sum_{i=0}^{n} ((f^i)^* + (f^{-i})^*)(\omega).$$

Lemma 2.21. $\mathrm{Plov}(f)$ *is also the degree of the polynomial*

$$n \mapsto \Delta'_n(f,\omega)^d = \left(\omega + \sum_{i=-n}^{n} (f^i)^*\omega\right)^d.$$

Proof. Recall that $\mathrm{Plov}(f)$ is defined as the polynomial degree of $n \mapsto (\sum_{i=0}^{n}(f^i)^*\omega)^d$, which is also the polynomial degree of $n \mapsto (\sum_{i=0}^{2n}(f^i)^*\omega)^d$ as well as the one of $n \mapsto (\sum_{i=0}^{2n} 2(f^i)^*\omega)^d$. Hence Lemma 2.21 follows from

$$\begin{aligned} \left(\omega + \sum_{i=-n}^{n} (f^i)^*\omega\right)^d &= \left((f^{-n})^*\left((f^n)^*\omega + \sum_{i=0}^{2n}(f^i)^*\omega\right)\right)^d \\ &= \left((f^n)^*\omega + \sum_{i=0}^{2n}(f^i)^*\omega\right)^d \end{aligned}$$

and

$$\left(\sum_{i=0}^{2n}(f^i)^*\omega\right)^d \le \left((f^n)^*\omega + \sum_{i=0}^{2n}(f^i)^*\omega\right)^d \le \left(\sum_{i=0}^{2n} 2(f^i)^*\omega\right)^d.$$ ∎

As $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent, $(f^{-1})^* : H^{1,1}(X) \circlearrowleft$ is also unipotent. Set

$$N' := (f^{-1})^* - \mathrm{Id} : H^{1,1}(X) \to H^{1,1}(X)$$

and let

$$N_m := N^m + N'^m.$$

We have an analogous statement of Lemma 2.16 with N^m replaced by N_m.

Lemma 2.22. *We have*

$$\mathrm{Plov}(f) \leq d + \max\left\{\sum_{j=1}^{d} i_j \mid i_j \in \mathbb{Z}_{\geq 0},\ (N_{i_1}\omega)\cdots(N_{i_d}\omega) \neq 0\right\}.$$

Proof. Lemma 2.22 follows from Lemma 2.21 together with

$$\Delta'_n(f,\omega) = \sum_{i=0}^{n}(f^i)^*\omega + \sum_{i=-n}^{0}(f^i)^*\omega = \sum_{i=0}^{k(f)}\binom{n+1}{i+1}(N^i + N'^i)(\omega)$$
$$= \sum_{i=0}^{k(f)}\binom{n+1}{i+1}N_i(\omega).$$ ■

Lemma 2.23. *For every integer $i \in [0, d]$, let*

$$P'_{f,\omega,i}(n) := \Delta'_n(f,\omega)^i\omega^{d-i} = \left(\sum_{j=0}^{k}\binom{n+1}{j+1}N_j\omega\right)^i\omega^{d-i},$$

which is a polynomial in n. Then we have

$$\deg P'_{f,\omega,i} > \deg P'_{f,\omega,i-1}.$$

Proof. As in the proof of Lemma 2.15, for every non-negative integer m, we define

$$P'_{f,\omega,i,m}(n) := \Delta'_n(f,\omega)^{i-1}\cdot((f^m)^*\omega + (f^{-m})^*\omega)\cdot\omega^{d-i}.$$

The same argument in Lemma 2.15 shows that

$$\deg_n P'_{f,\omega,i,m}(n) = \deg_n P'_{f,\omega,i,0}(n) = \deg_n P'_{f,\omega,i-1}(n)$$

for every m, and there exists $\ell \in \mathbb{Z}_{>0}$ such that the leading coefficient $C_{f,\omega,i}(\ell) > 0$ of $P'_{f,\omega,i,\ell}$ is minimum among all $\ell \in \mathbb{Z}_{>0}$. Since $\Delta'_n(f,\omega)^{i-1}\cdot\omega\cdot\omega^{d-i} > 0$ (because ω is Kähler), it follows that

$$\frac{P'_{f,\omega,i}(n)}{P'_{f,\omega,i,\ell}(n)} = \sum_{m=0}^{n}\frac{P'_{f,\omega,i,m}(n)}{P'_{f,\omega,i,\ell}(n)} \succeq_{n\to\infty} \gamma,$$

for any $\gamma > 0$, and we conclude the proof as in Lemma 2.15. ■

Let $\omega \in H^{1,1}(X)$. For all integer $0 \leq p \leq d$, consider the following polynomial in n with coefficients in $H^{p,p}(X)$:

$$Q_{f,\omega,p} : n \mapsto ((f^n)^*\omega + (f^{-n})^*\omega)^p = \left(\sum_{i=0}^{k}\binom{n}{i}N_i(\omega)\right)^p.$$

Let $\lambda_p(f,\omega)$ denote the polynomial degree of $Q_{f,\omega,p}(n)$.

Remark 2.24. Note that for any product $\Omega \in H^{d-p,d-p}(X)$ of $d-p$ Kähler classes, $\lambda_p(f,\omega)$ is also the polynomial degree of

$$n \mapsto \Omega \cdot ((f^n)^*\omega + (f^{-n})^*\omega)^p.$$

The same argument proving Lemma 2.1 shows that the polynomial degree $\lambda_p(f,\omega)$ is independent of the choice of ω whenever ω is nef and big.

We will use the next lemma in the proof of Lemma 4.5.

Lemma 2.25. *For every integer p, we have*

$$\lambda_p(f,\omega) \le \max\{r_i \in \mathbb{Z} \mid \|(f^n)^* \circlearrowleft H^{i,i}(X)\| \sim_{n\to\infty} C_i n^{r_i} \text{ for some } C_i > 0, 0 \le i \le p\}.$$

In particular,

$$\lambda_p(f,\omega) \le k(f)\left\lfloor \frac{d}{2} \right\rfloor \le \frac{k(f)d}{2} \le d(d-1).$$

Proof. The first statement follows from

$$\begin{aligned}((f^n)^*\omega + (f^{-n})^*\omega)^p &= \sum_{j=0}^{p} \binom{p}{j} (f^n)^*\omega^j \cdot (f^{-n})^*\omega^{p-j} \\ &= \sum_{j=0}^{p} \binom{p}{j} ((f^{2n})^*\omega^j) \cdot \omega^{p-j}.\end{aligned}$$

For the last statement, the first inequality follows from [7, Proposition 5.8] and the last inequality from Theorem 1.1. ∎

3. Quasi-nef sequences and dynamical filtrations

3.1. Dynamical filtrations and proof of the upper bound (1.3) in Theorem 1.1

First we recall the definitions and basic properties of quasi-nef sequences and dynamical filtrations. We then prove some useful lemmas, and finally the optimal upper bound (1.3) in Theorem 1.1 (see Corollary 3.7).

Let X be a compact Kähler manifold of dimension $d \ge 1$. For every $\alpha \in H^{i,i}(X,\mathbb{R})$, if $\alpha \cdot H^{1,1}(X,\mathbb{R})^{d-i} = 0$, we write

$$\alpha \equiv 0$$

as in Notations. Let $\mathcal{K}^i(X) \subset H^{i,i}(X,\mathbb{R})$ be the closed convex cone generated by classes of smooth positive (i,i)-forms. We have $\mathcal{K}^1(X) = \mathrm{Nef}(X)$, which is the nef cone of X. For every $\alpha \in \mathcal{K}^i(X)$, define

$$\mathrm{Nef}(\alpha) := \overline{\alpha \cdot \mathrm{Nef}(X)} \subset H^{i+1,i+1}(X,\mathbb{R}).$$

As $\mathrm{Nef}(X)$ is a convex cone, so is $\mathrm{Nef}(\alpha)$. Since $\mathrm{Nef}(\alpha) \subset \mathcal{K}^{i+1}(X)$ and $\mathcal{K}^{i+1}(X)$ is salient, $\mathrm{Nef}(\alpha)$ is a closed salient cone.

Construction 3.1 (Quasi-nef sequence [24]). Let $f \in \mathrm{Aut}(X)$ be an automorphism of X such that $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent. A *quasi-nef sequence* (with respect to f) is a sequence

$$M_1, \dots, M_d \in H^{1,1}(X, \mathbb{R})$$

constructed recursively as follows. Suppose that $M_1, \dots, M_i \in H^{1,1}(X, \mathbb{R})$ are constructed, then $M_{i+1} \in H^{1,1}(X, \mathbb{R})$ is an element such that

- $f^*(M_1 \cdots M_{i+1}) = M_1 \cdots M_{i+1} \neq 0$,
- $M_1 \cdots M_i M_{i+1} \in \mathrm{Nef}(M_1 \cdots M_i)$.

Since $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent, the existence of M_{i+1} follows from Birkhoff's Perron–Frobenius theorem [3] applied to $\mathrm{Nef}(M_1 \cdots M_i)$. See also [16, Theorem 1.1] for a generalization. We set $L_0 := 1 \in H^0(X, \mathbb{R})$ and define $L_i := M_1 \cdots M_i \in H^{i,i}(X, \mathbb{R})$.

Note that $M_1, \dots, M_d \in H^{1,1}(X, \mathbb{R})$ is also a quasi-nef sequence with respect to f^{-1}. ∎

Given a quasi-nef sequence $M_1, \dots, M_d \in H^{1,1}(X, \mathbb{R})$ with respect to an automorphism $f \in \mathrm{Aut}(X)$ such that $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent, define

$$F_i := \left\{\alpha \in H^{1,1}(X, \mathbb{R}) \mid L_i \alpha \equiv 0\right\}$$

and let F'_i be the subspace of F_i spanned by

$$\left\{\alpha \in F_i \mid L_{i-1}\alpha \equiv \beta \text{ for some } \beta \in \mathrm{Nef}(L_{i-1})\right\}.$$

Recall from [8] that these subspaces form an f^*-stable filtration

$$0 = F_0 \subset F'_1 \subset F_1 \subset \cdots \subset F'_{d-1} \subset F_{d-1} \subset F'_d = H^{1,1}(X, \mathbb{R}). \tag{3.1}$$

We note that the filtration (3.1) *depends on the choice* of a quasi-nef sequence $M_1, \dots, M_d \in H^{1,1}(X, \mathbb{R})$. Here are some fundamental properties of these filtrations proven in [8].

Proposition 3.2 ([8, Theorem 1.3]). (1) *We have* $\dim(F'_i/F_{i-1}) \leq 1$ *and*

$$F'_i = \{\gamma \in F_i \mid L_{i-1}\gamma^2 \equiv 0\}.$$

Moreover, the following conditions are equivalent:

(i) $F_{i-1} \neq F'_i$;

(ii) $F'_i = F_{i-1} \oplus (\mathbb{R} \cdot M_i)$;

(iii) $L_{i-1} M_i^2 = 0$.

(2) *There exist an integer* $r \in [1, d-1]$ *and a strictly decreasing sequence of integers*

$$d - 1 \geq s_1 > \cdots > s_r \geq 1$$

such that for every Kähler class $\omega \in H^{1,1}(X, \mathbb{R})$ *and every integer* $j \in [1, r]$,

$$(f^* - \mathrm{Id})^{2j-1}\omega \in F_{s_j} \backslash F'_{s_j} \quad \text{and} \quad (f^* - \mathrm{Id})^{2j}\omega \in F'_{s_j} \backslash F_{s_j - 1},$$

and $(f^* - \mathrm{Id})^{2r+1}\omega = 0$. *In particular,* $(f^* - \mathrm{Id})^{2r+1} = 0 \in \mathrm{End}(H^{1,1}(X, \mathbb{R}))$.

The sequence $s_1 > \cdots > s_r$ in Proposition 3.2 (2) depends on f and is unique for a given quasi-nef sequence. The inverse f^{-1} defines the same sequence with respect to the same quasi-nef sequence by the next lemma.

Lemma 3.3. *Let $s_1 > \cdots > s_r$ be the sequence in Proposition 3.2 (2) associated to f. Then for every Kähler class $\omega \in H^{1,1}(X, \mathbb{R})$ and every integer $j \in [1, r]$, we have*

$$((f^{-1})^* - \mathrm{Id})^{2j-1}\omega \in F_{s_j} \backslash F'_{s_j} \quad \text{and} \quad ((f^{-1})^* - \mathrm{Id})^{2j}\omega \in F'_{s_j} \backslash F_{s_j-1},$$

and $((f^{-1})^* - \mathrm{Id})^{2r+1}\omega = 0$.

Proof. Since both F_{s_j} and F'_{s_j} are f^*-invariant, we have

$$((f^{-1})^* - \mathrm{Id})^{2j-1}\omega = (-1)^{2j-1}(f^{1-2j})^*(f^* - \mathrm{Id})^{2j-1}\omega \in F_{s_j} \backslash F'_{s_j}.$$

The same argument shows that $((f^{-1})^* - \mathrm{Id})^{2j}\omega \in F'_{s_j} \backslash F_{s_j-1}$ and $((f^{-1})^* - \mathrm{Id})^{2r+1}\omega = 0$. ∎

The following two lemmas are both consequences of Proposition 3.2 (1).

Lemma 3.4. *For $i \in [1, d] \cap \mathbb{Z}$, take $\eta_i \in F'_i$. Let $p \in [1, d] \cap \mathbb{Z}$ and $j \in [0, p] \cap \mathbb{Z}$. Then:*

(1) *There exists some $C \in \mathbb{R}$ such that*

$$L_j \eta_{j+1} \cdots \eta_p \equiv C L_p.$$

(2) *For any $\eta \in F_p$, we have*

$$L_j \eta_{j+1} \cdots \eta_p \eta \equiv 0.$$

Proof. Since either $F'_p = F_{p-1}$ or F'_p / F_{p-1} is a line spanned by $M_p + F_{p-1}$ by Proposition 3.2 (1), there exists some $C_p \in \mathbb{R}$ such that $\eta_p - C_p M_p \in F_{p-1}$. As $L_{p-1} F_{p-1} \equiv 0$, we have

$$L_{p-1}\eta_p \equiv C_p L_{p-1} M_p = C_p L_p.$$

Induction proves that $L_j \eta_{j+1} \cdots \eta_p \equiv C L_p$ for some $C \in \mathbb{R}$.

Since $L_p F_p \equiv 0$, (2) follows from (1) and the definition of F_p. ∎

Lemma 3.5. *Assume that $M_1 = \cdots = M_i \in H^{1,1}(X, \mathbb{R})$. Then*

$$F'_j = F_{j-1}$$

for every $j \leq i - 1$.

Proof. Assume to the contrary that $F'_j \neq F_{j-1}$ for some $j \leq i-1$. By Proposition 3.2 (1), we would have

$$L_{j+1} = L_{j-1} M_j M_{j+1} = L_{j-1} M_j^2 = 0,$$

which is impossible. Hence $F'_j = F_{j-1}$ for every $j \leq i - 1$. ∎

As a consequence of these results, we obtain the following refinements of Theorem 1.1.

Corollary 3.6. *Let $\phi : X \to B$ be a surjective morphism with connected fibers between compact Kähler manifolds. Let $f \in \mathrm{Aut}(X)$ such that $f^* : H^{1,1}(X,\mathbb{R}) \circlearrowleft$ is unipotent and $\phi^*\omega_B$ is f^*-invariant for some Kähler class $\omega_B \in H^{1,1}(B,\mathbb{R})$. Then*

$$k(f) \le 2(\dim X - \dim B).$$

Here, we recall that $k(f)+1$ is the maximal size of the Jordan blocks of the Jordan canonical form of the unipotent $f^*|_{H^{1,1}(X,\mathbb{R})}$.

Proof. Let $m := \dim B$. As $\phi^*\omega_B$ is an f^*-invariant nef class and $\phi^*\omega_B^m \not\equiv 0$, we can complete

$$M_1 = \cdots = M_m = \phi^*\omega_B$$

to a quasi-nef sequence $M_1, \ldots, M_d$. By Lemma 3.5, we have $F'_j = F_{j-1}$ for every $j \le m-1$. So according to Proposition 3.2 (2) and the notation therein, necessarily $s_r \ge m$, so $r \le \dim X - \dim B$. Hence

$$(f^* - \mathrm{Id})^{2(\dim X - \dim B)+1}(\omega) = 0$$

for every $\omega \in H^{1,1}(X,\mathbb{R})$. ■

Corollary 3.7. *Let X be a compact Kähler manifold of dimension $d \ge 1$ and of Kodaira dimension $\kappa(X)$. Let $f \in \mathrm{Aut}(X)$ be an automorphism of zero entropy.*

(1) *We have*

$$k(f) \le 2(\dim X - \kappa(X)).$$

In other words,

$$\|(f^m)^* : H^{1,1}(X) \circlearrowleft \| = O(m^{2(d-\kappa(X))})$$

as $m \to \infty$ for any norm of $\mathrm{End}_{\mathbb{C}}(H^{1,1}(X))$.

(2) *The estimate in (1) is optimal, in the sense that for every $d \ge 1$ and $\kappa \ge 1$, there exist some X and $f \in \mathrm{Aut}(X)$ such that $\dim(X) = d$, $\kappa(X) = \kappa$, and*

$$\|(f^m)^* : H^{1,1}(X) \circlearrowleft \| \sim_{m\to\infty} C m^{2(d-\kappa(X))}$$

for some $C > 0$.

We prove first Corollary 3.7 (1). We will prove Corollary 3.7 (2) in Section 7 by constructing explicit examples.

Proof of Corollary 3.7 (1). By an equivariant Kähler desingularization, there exists a bimeromorphic morphism $\nu : \widetilde{X} \to X$ from a compact Kähler manifold $\widetilde{X}$ such that f lifts to an automorphism $\widetilde{f} \in \mathrm{Aut}(\widetilde{X})$ and that $\widetilde{X}$ admits a surjective morphism $\phi : \widetilde{X} \to B$ to a projective manifold as a model of its Iitaka fibration. As ϕ is an Iitaka fibration of $\widetilde{X}$, $\widetilde{f}$ descends to a bimeromorphic self-map of B of finite order by [19, Theorem A]. Up to replacing f by a finite iteration of it, we can assume that ϕ is $\widetilde{f}$-invariant. In particular, $\phi^*\omega_B$ is $\widetilde{f}^*$-invariant for every Kähler class $\omega_B \in H^{1,1}(X,\mathbb{R})$.

Since f has zero entropy, we have $d_1(\widetilde{f}) = 1$ by Lemma 2.8 and [9, Theorem 1.1] for the invariance under a generically finite map. Replacing f by its finite iteration, we can assume that $\widetilde{f}^* : H^{1,1}(\widetilde{X},\mathbb{C}) \circlearrowleft$ is unipotent by Lemma 2.8. Thus by Corollary 3.6, we have

$$\|(\widetilde{f}^m)^* : H^{1,1}(\widetilde{X},\mathbb{C}) \circlearrowleft \| =_{m\to\infty} O(m^{2(d-\kappa(X))}).$$

As $H^{1,1}(X,\mathbb{C}) \hookrightarrow H^{1,1}(\widetilde{X},\mathbb{C})$ is $\widetilde{f}^*$-stable and the restriction of $\widetilde{f}^* : H^{1,1}(\widetilde{X},\mathbb{C}) \circlearrowleft$ to $H^{1,1}(X,\mathbb{C})$ is $f^* : H^{1,1}(X,\mathbb{C}) \circlearrowleft$, we have

$$\|(f^m)^* : H^{1,1}(X,\mathbb{C}) \circlearrowleft \| =_{m\to\infty} O(m^{2(d-\kappa(X))}). \qquad \blacksquare$$

3.2. Some vanishing lemmas

From now on till the end of Section 3, $f \in \mathrm{Aut}(X)$ is an automorphism such that f^* : $H^{1,1}(X) \circlearrowleft$ is unipotent. Under this assumption, $(f^{-1})^* : H^{1,1}(X) \circlearrowleft$ is also unipotent. Recall that in Section 2, we have defined

$$N := f^* - \mathrm{Id} \in \mathrm{End}(H^{1,1}(X,\mathbb{R})) \quad \text{and} \quad N' := (f^{-1})^* - \mathrm{Id} \in \mathrm{End}(H^{1,1}(X,\mathbb{R})),$$

and also $N_m := N^m + N'^m$.

In this subsection, we will prove some vanishing results of intersections of $(1,1)$-classes which are images of N_m or N^m. Let us start with the following lemma.

Lemma 3.8. *Let $\alpha \in H^{1,1}(X,\mathbb{R})$.*

(1) *Let $d-1 \geq s_1 > \cdots > s_r \geq 1$ be the sequence associated to f as in Proposition 3.2 (2). Then we have*

$$N_{2i-1}(\omega), N_{2i}(\omega) \in F'_{s_i} \backslash F_{s_i-1}$$

for any Kähler class ω. In particular,

$$N_{2i-1}(\alpha), N_{2i}(\alpha) \in F'_{d-i}.$$

(2) *Both $N_{k(f)-1}(\alpha)$ and $N_{k(f)}(\alpha)$ are f^*-invariant.*

(3) *If ω is nef, then both $N_{k(f)}(\omega)$ and $N_{k(f)-1}(\omega)$ are nef.*

Proof. First we prove (1). Note that $N + N' = -NN'$, so

$$N_{2i-1} = N^{2i-1} + N'^{2i-1} = (N+N')\sum_{j=0}^{2i-2}(-1)^j N^j N'^{2i-2-j}$$

$$= \sum_{j=0}^{2i-2}(-1)^{j+1} N^{j+1} N'^{2i-1-j}$$

$$= \sum_{j=0}^{2i-2} (-1)^{j+1} (f^* - \mathrm{Id})^{j+1} ((f^{-1})^* - \mathrm{Id})^{2i-j-1}$$
$$= \sum_{j=0}^{2i-2} (f^* - \mathrm{Id})^{2i} (f^{-2i+j+1})^* = \sum_{j=0}^{2i-2} N^{2i} \circ (f^{-2i+j+1})^*.$$

Since $\sum_{j=0}^{2i-2} (f^{-2i+j+1})^* \omega$ is Kähler, Proposition 3.2 (2) implies

$$N_{2i-1}(\omega) = N^{2i} \left(\sum_{j=0}^{2i-2} (f^{-2i+j+1})^* \omega \right) \in F'_{s_i} \backslash F_{s_i - 1}.$$

By Proposition 3.2 (2) and Lemma 3.3, we have $N^{2i}(\omega), N'^{2i}(\omega) \in F'_{s_i}$, so $N_{2i}(\omega) \in F'_{s_i}$. Since $N^{2i}\omega \in F'_{s_i}$, we have

$$L_{s_i - 1} N^{2i} \omega \equiv C L_{s_i}$$

for some $C \in \mathbb{R}$ by Lemma 3.4. Since $N^{2i}\omega \notin F_{s_i - 1}$, we have $C \neq 0$. Moreover, as $L_{s_i-1} N^p \omega \equiv 0$ for every $p > 2i$ by Proposition 3.2 (2), we have

$$L_{s_i-1} N^{2i} \omega = (2i)! \cdot \lim_{m \to \infty} L_{s_i - 1} \frac{(f^*)^m(\omega)}{m^{2i}} \in \mathcal{K}^{s_i}(X)/\equiv$$

where $\mathcal{K}^{s_i}(X)/\equiv$ denotes the image of $\mathcal{K}^{s_i}(X)$ in $H^{s_i, s_i}(X, \mathbb{R})/\equiv$. Since $L_{s_i} \in \mathcal{K}^{s_i}(X)$, necessarily $C > 0$. Since $s_1 > \cdots > s_r$ is also the sequence associated to f^{-1} by Lemma 3.3, the same argument shows that there exists $C' > 0$ such that

$$L_{s_i - 1} N'^{2i} \omega \equiv C' L_{s_i}.$$

Hence

$$L_{s_i - 1} N_{2i}(\omega) \equiv (C + C') L_{s_i} \not\equiv 0,$$

namely $N_{2i}(\omega) \notin F_{s_i - 1}$. The last part follows from $F'_{s_i} \subset F'_{d-i}$, noting that $s_i \le d - i$.

For (2), recall that $k(f)$ is an even number (Theorem 1.1) so we can write $k(f) = 2i$. Since $N^{2i+1} = 0$ and $f^* = \mathrm{Id} + N$, we have

$$N^{2i}/(2i)! = \lim_{m \to \infty} (f^*)^m / m^{2i}$$

whose image is hence f^*-invariant. Since f^* commutes with N, and $N' = -N(f^{-1})^*$, we have $N'^{2i} = N^{2i}(f^{-2i})^*$ whose image is hence f^*-invariant, too. Thus the images of $N_{2i} = N^{2i} + N'^{2i}$ and $N_{2i-1} = \sum_{j=0}^{2i-2} N^{2i} \circ (f^{-2i+j+1})^*$ are also f^*-invariant.

For (3),

$$N^{2i}(\omega) = (2i)! \lim_{m \to \infty} (f^*)^m(\omega)/m^{2i}, \quad N_{2i-1}(\omega) = \sum_{j=0}^{2i-2} N^{2i}((f^{-2i+j+1})^*(\omega))$$

are clearly all nef. ∎

Corollary 3.9. *Let ω be a Kähler class. Assume that $k(f) = 2d - 2$. Then for every integer $\ell \geq 2$, we have*

$$N_{i_1}(\omega) \cdots N_{i_\ell}(\omega) \equiv 0$$

whenever

$$i_j \geq 2(d-j) - 1 \quad \text{for all } j \leq \ell - 2, \quad \text{and} \quad i_{\ell-1}, i_\ell \geq 2(d - \ell + 1) - 1.$$

Moreover, whenever

$$i_j \in \{2(d-j), 2(d-j) - 1\} \quad \text{for all } j,$$

there exists some $C \in \mathbb{R}$ such that

$$N_{i_1}(\omega) \cdots N_{i_j}(\omega) \equiv C N_{2d-2}(\omega) \cdots N_{2(d-j)}(\omega).$$

Proof. Corollary 3.9 follows directly from Lemma 3.4 and Lemma 3.8. Indeed, by the assumption and Lemma 3.8, we have

$$N_{i_1}(\omega) \in F_1', \quad N_{i_2}(\omega) \in F_2', \ldots, N_{i_{l-2}}(\omega) \in F_{l-2}', \quad N_{i_{l-1}}(\omega), N_{i_l}(\omega) \in F_{l-1}'.$$

Thus the first assertion follows from Lemma 3.4 (2).

Similarly, by the assumption and Lemma 3.8, we have

$$N_{i_1}(\omega), N_{2d-2}(\omega) \in F_1', \quad N_{i_2}(\omega), N_{2d-4}(\omega) \in F_2', \ldots, N_{i_j}(\omega), N_{2(d-j)}(\omega) \in F_j'.$$

Thus the second assertion follows from Lemma 3.4 (1). ∎

Lemma 3.10. *Let m be a positive integer and let*

$$\Sigma := \left\{ N^{k(f)}(\omega), N_{k(f)}(\omega), N_{k(f)-1}(\omega) \mid \omega \in H^{1,1}(X, \mathbb{R}) \text{ Kähler} \right\}.$$

Then the following conditions are equivalent:

(1) *$M_1 \cdots M_m \not\equiv 0$ for some $M_1, \ldots, M_m \in \Sigma$.*

(2) *$M_1 \cdots M_m \not\equiv 0$ for every $M_1, \ldots, M_m \in \Sigma$.*

Proof. Fix a positive integer m. It suffices to prove that (1) implies (2). To this end, it suffices to prove that given $M_1, \ldots, M_m, M_m' \in \Sigma$,

$$M_1 \cdots M_{m-1} M_m \not\equiv 0 \quad \text{implies} \quad M_1 \cdots M_{m-1} M_m' \not\equiv 0.$$

Then we can replace each factor of $M_1 \cdots M_m$ by any choice of m elements $M_1', \ldots, M_m' \in \Sigma$ one by one and obtain $M_1' \cdots M_m' \not\equiv 0$.

Since every element of Σ is nef and f^*-invariant by Lemma 3.8, the sequence $M_1, \ldots, M_{m-1}$ can be completed to a quasi-nef sequence. Since $L_{m-1} M_m = M_1 \cdots M_{m-1} M_m \not\equiv 0$ and $M_m' \in \Sigma$ by assumption, Proposition 3.2 (2) and Lemmas 3.3 and 3.8 (1) then imply that

$$M_1 \cdots M_{m-1} M_m' = L_{m-1} M_m' \not\equiv 0.$$ ∎

As for when we have $(N^{k(f)}\omega)^i = 0$, we have the following.

Lemma 3.11. $(N^{k(f)}\omega)^i = 0$ *whenever* $2i > d$.

Proof. Let $j \in \mathbb{Z}_{\geq 0}$. Since $\|(f^n)^* : H^{1,1}(X) \circlearrowleft \| = O(n^{k(f)})$, we have

$$\|(f^n)^* : H^{j,j}(X) \circlearrowleft \| = O(n^{jk(f)})$$

by [7, Proposition 5.8]. Suppose that $(N^{k(f)}\omega)^j \neq 0$, then

$$\|(f^n)^* : H^{j,j}(X) \circlearrowleft \| \sim Cn^{jk(f)}.$$

As

$$\|(f^n)^* : H^{j,j}(X) \circlearrowleft \| \sim \|(f^n)^* : H^{d-j,d-j}(X) \circlearrowleft \|,$$

necessarily $(N^{k(f)}\omega)^i = 0$ whenever $2i > d$. ∎

Corollary 3.12. *Let m be a non-negative integer such that $N^{k(f)}(\omega_0)^m \equiv 0$ (or equivalently $N_{k(f)}(\omega_0)^m \equiv 0$ by Lemma 3.10) for some Kähler class ω_0. Then for every $\omega \in H^{1,1}(X, \mathbb{R})$, we have*

$$N_{k(f)}(\omega)^i N_{k(f)-1}(\omega)^j \equiv 0$$

whenever $i + j \geq m$.

As a consequence, for every $\omega \in H^{1,1}(X, \mathbb{R})$ and every pair of non-negative integers i and j such that $2i + 2j > \min(d, 2d - k(f))$, we have

$$N_{k(f)}(\omega)^i N_{k(f)-1}(\omega)^j \equiv 0.$$

Proof. Since the vanishing $N_{k(f)}(\omega)^i N_{k(f)-1}(\omega)^j \equiv 0$ is a Zariski closed condition for $\omega \in H^{1,1}(X, \mathbb{R})$ and since the Kähler cone is Zariski dense in $H^{1,1}(X, \mathbb{R})$, we can assume that $\omega \in H^{1,1}(X, \mathbb{R})$ is Kähler. Then the first statement follows from Lemma 3.10.

Now we prove the second statement. Once again, we can assume that ω is Kähler. Recall that $k(f) = 2\ell$ is an even number (see, e.g., Theorem 1.1). By the first statement, it suffices to show that

$$N_{k(f)}(\omega)^{d-\ell+1} \equiv 0,$$

as we already know that $N_{k(f)}(\omega)^i \equiv 0$ if $2i > d$ by Lemma 3.11. To this end, we can assume that $N_{k(f)}(\omega)^{d-\ell} \not\equiv 0$ and complete

$$M_1 = \cdots = M_{d-\ell} := N_{k(f)}(\omega)$$

to a quasi-nef sequence. Then Lemma 3.8 implies that

$$N_{k(f)}(\omega) = N_{2\ell}(\omega) \in F'_{d-\ell} \subset F_{d-\ell}.$$

Hence,

$$N_{k(f)}(\omega)^{d-\ell+1} = L_{d-\ell} N_{k(f)}(\omega) \equiv 0.$$ ∎

4. Upper bounds of Plov(f): Beginning of the proof of Theorem 1.2

Let us first prove Keeler's upper bound in Theorem 1.2 (1).

Proposition 4.1. *Let X be a compact Kähler manifold of dimension d and let $f \in \mathrm{Aut}(X)$ be a zero entropy automorphism. Suppose that $k(f) > 0$. Then we have*

$$\mathrm{Plov}(f) \leq k(f)(d-1) + d.$$

We will first provide a sketch of Keeler's original proof, then an alternative proof using Corollary 3.12.

Keeler's proof of Proposition 4.1. Recall that $\mathrm{Plov}(f)$ is the degree of the polynomial $P_{f,\omega}(n)$ which is the same as the polynomial $(\Delta_n(f, L)^d)$ in Theorem 8.1 (4) if we replace the ample class L by the Kähler class ω. Therefore, by setting $D = \omega$ and $P = f^*$ in the proof of [15, Lemma 6.13], the purely cohomological proof of [15, Lemma 6.13] works without any further change, which proves the result. ∎

Second proof of Proposition 4.1. By Lemmas 2.6 and 2.8, we can assume that $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent. Let $\alpha \in H^{1,1}(X, \mathbb{R})$ and let

$$k(f) \geq i_1 \geq \cdots \geq i_d \geq 0$$

be d integers such that

$$\sum_{j=1}^{d} i_j > k(f)(d-1).$$

Write the product $N_{i_1}(\alpha) \cdots N_{i_d}(\alpha)$ in the form

$$\Pi := N_{k(f)}(\alpha)^a N_{k(f)-1}(\alpha)^b N_{i_{a+b+1}}(\alpha) \cdots N_{i_d}(\alpha)$$

with $i_{a+b+1} \leq k(f) - 2$. Then $2a + 2b > 2d - k(f)$ by the assumption. It follows from Corollary 3.12 that $\Pi = 0$. Thus $\mathrm{Plov}(f) \leq k(f)(d-1) + d$ by Lemma 2.22. ∎

The main result of this section is the following sharpened upper bound of $\mathrm{Plov}(f)$.

Theorem 4.2. *Let X be a compact Kähler manifold of dimension d and let $f \in \mathrm{Aut}(X)$ be a zero entropy automorphism. Assume that $d \geq 3$ and $k(f) > 0$. Then*

$$\mathrm{Plov}(f) \leq k(f)(d-1) + d - 2.$$

When $k(f) = 2$, we have the optimal upper bound

$$\mathrm{Plov}(f) \leq \begin{cases} 2d & \text{if } d \text{ is even;} \\ 2d-1 & \text{if } d \text{ is odd.} \end{cases}$$

The above inequality for $k(f) = 2$ was originally due to F. Hu with a different proof[1]. We will prove Theorem 4.2 based on results in Section 3 about the dynamical filtrations. Let us first prove Theorem 4.2 when $k(f) = 2$.

Proof of Theorem 4.2 when $k(f) = 2$. By Lemmas 2.6 and 2.8, we can assume that $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent. Let ω be a Kähler class and let i be the largest integer such that $(N^2\omega)^i \neq 0$. By Lemma 3.11, we have $i \leq \lfloor d/2 \rfloor$. Since $(N^2\omega)^{i+1} = 0$, it follows from Corollary 3.12 that

$$(N_2\omega)^a (N_1\omega)^b \equiv 0$$

whenever $a + b > i$. Hence by Lemma 2.22,

$$\mathrm{Plov}(f) \leq d + 2i \leq d + 2\lfloor d/2 \rfloor.$$

For optimal examples, let S be any compact Kähler surface and $f \in \mathrm{Aut}(S)$ any automorphism with $k(f) = 2$ (see, e.g., [8, §4.1] for an example where S is a torus). Then $\mathrm{Plov}(f) = 4$ by Corollary 1.3. If $d = 2m$, then $k(f^{\times m}) = 2$ for $f^{\times m} \in \mathrm{Aut}(S^m)$ by Lemma 2.12 and $\mathrm{Plov}(f^{\times m}) = 4m = 2d$ by Lemma 2.18. If $d = 2m + 1$, then we consider $f^{\times m} \times \mathrm{Id}_C \in \mathrm{Aut}(S^m \times C)$ where C is any smooth projective curve. ■

The proof of Theorem 4.2 when $k(f) > 2$ follows from a different argument. In Lemmas 4.3 and 4.5 below, let X be a compact Kähler manifold of dimension $d \geq 1$ and $f \in \mathrm{Aut}(X)$ an automorphism such that $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent.

Lemma 4.3. *Assume $k(f) > 0$. Let (a, b) be a pair of non-negative integers such that $2a + 2b \geq 2d - k(f)$. Let $i_1 \geq \cdots \geq i_{d'} \geq 0$ be d' integers. When $2a + 2b = 2d - k(f)$ we assume*

$$\sum_{j=1}^{d'} i_j > (k(f) - 4)d' + 2.$$

Then

$$N_{k(f)}(\alpha)^a N_{k(f)-1}(\alpha)^b N_{i_1}(\alpha) \cdots N_{i_{d'}}(\alpha) \equiv 0$$

for every $\alpha \in H^{1,1}(X, \mathbb{R})$.

Proof. If $2a + 2b > 2d - k(f)$, then we already have

$$N_{k(f)}(\alpha)^a N_{k(f)-1}(\alpha)^b \equiv 0$$

by Corollary 3.12. So we can assume that $2a + 2b = 2d - k(f)$, and that $i_1 \leq k(f) - 2$.

We can also assume that $N_{k(f)}(\alpha)^a N_{k(f)-1}(\alpha)^b \not\equiv 0$ and that $\alpha \in H^{1,1}(X, \mathbb{R})$ is Kähler. By Lemma 3.8, both $N_{k(f)}(\alpha)$ and $N_{k(f)-1}(\alpha)$ are f^*-invariant nef, so we can complete

$$M_1 = \cdots = M_a := N_{k(f)}(\alpha), \quad M_{a+1} = \cdots = M_{a+b} := N_{k(f)-1}(\alpha)$$

to a quasi-nef sequence.

[1] F. Hu, private communication.

Since $\sum_{j=1}^{d'} i_j > (k(f)-4)d'+2$, we have

$$i_1, i_2 \in [k(f)-3, k(f)-2].$$

Indeed, otherwise we would have $i_2 \leq k(f)-4$ and $\sum_{j=1}^{d'} i_j \leq (k(f)-2)+(d'-1)(k(f)-4) = (k(f)-4)d'+2$. It follows from Lemma 3.8 that

$$N_{i_1}(\alpha), N_{i_2}(\alpha) \in F'_{d-\frac{k(f)}{2}+1} = F'_{a+b+1}.$$

So

$$N_{k(f)}(\alpha)^a N_{k(f)-1}(\alpha)^b N_{i_1}(\alpha) N_{i_2}(\alpha) = L_{a+b} N_{i_1}(\alpha) N_{i_2}(\alpha) \equiv 0$$

by Lemma 3.4 (2), which proves Lemma 4.3. ∎

End of proof of Theorem 4.2. Recall that $k(f)$ is an even number (Theorem 1.1), and we already proved the statement for $k(f)=2$. It remains to prove Theorem 4.2 for $k(f) \geq 4$.

By Lemmas 2.6 and 2.8, we can assume that $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent. Let $\alpha \in H^{1,1}(X, \mathbb{R})$ and let

$$k(f) \geq i_1 \geq \cdots \geq i_d \geq 0$$

be d integers such that

$$\sum_{j=1}^{d} i_j > k(f)(d-1)-2.$$

Then the product $N_{i_1}(\alpha) \cdots N_{i_d}(\alpha)$ is of the form

$$\Pi := N_{k(f)}(\alpha)^a N_{k(f)-1}(\alpha)^b N_{i_{a+b+1}}(\alpha) \cdots N_{i_d}(\alpha)$$

with $i_{a+b+1} \leq k(f)-2$. We now show that $\Pi = 0$. We have

$$(a+b)k(f) + \sum_{j=a+b+1}^{d} i_j \geq \sum_{j=1}^{a+b} i_j + \sum_{j=a+b+1}^{d} i_j = \sum_{j=1}^{d} i_j > k(f)(d-1)-2.$$

So if $d' := d-a-b$, then

$$d'(k(f)-2) \geq \sum_{j=a+b+1}^{d} i_j > k(f)(d-a-b-1)-2 = k(f)d'-k(f)-2,$$

which implies $k(f)+2 > 2d'$. As $k(f)$ is even, we have $k(f) \geq 2d'$, namely $2a+2b \geq 2d-k(f)$. Assume that $2a+2b = 2d-k(f)$, namely $2d' = k(f)$, then since $2d' = k(f) \geq 4$ by assumption, we have

$$\sum_{j=a+b+1}^{d} i_j > k(f)d'-k(f)-2 \geq (k(f)-4)d'+2.$$

It follows from Lemma 4.3 that $\Pi = 0$, and thus Theorem 4.2 follows from Lemma 2.22. ∎

We finish this section by the following upper bound of $\mathrm{Plov}(f)$ when $d \geq 4$, which improves the upper bound $\mathrm{Plov}(f) \leq 2d^2 - 3d$ obtained by combining Theorem 4.2 and $k(f) \leq 2d - 2$ in Theorem 1.1.

Proposition 4.4. *Let X be a compact Kähler manifold of dimension $d \geq 4$ and let $f \in \mathrm{Aut}(X)$ such that $d_1(f) = 1$. Then*

$$\mathrm{Plov}(f) \leq 2d^2 - 3d - 2.$$

We first prove the following.

Lemma 4.5. *Assume that $k := k(f) = 2d - 2$ and $d \geq 4$. Take d integers*

$$k \geq i_1 \geq \cdots \geq i_d \geq 0$$

such that

$$\sum_{j=1}^{d} i_j \geq (k-2)d - 1.$$

Then for every $\alpha \in H^{1,1}(X)$, we have

$$N_{i_1}(\alpha) \cdots N_{i_d}(\alpha) = 0. \tag{4.1}$$

Proof. First we assume that $i_1 \geq k - 1$. We have

$$\begin{aligned} \sum_{j=2}^{d} i_j = \left(\sum_{j=1}^{d} i_j\right) - i_1 &> (k-2)d - 1 - k = 2d^2 - 6d + 1 \\ &\geq 2d^2 - 8d + 8 = (k-4)(d-1) + 2, \end{aligned} \tag{4.2}$$

where the second inequality follows from $d \geq 4$. So $N_{i_1}(\alpha) \cdots N_{i_d}(\alpha) = 0$ by Lemma 4.3.

Assume that $i_1 \leq k - 2$. Since $\sum_{j=1}^{d} i_j \geq (k-2)d - 1$ and the sequence i_j is decreasing, necessarily

$$i_1 = \cdots = i_{d-1} = k - 2 \quad \text{and} \quad i_d = k - 2 \text{ or } k - 3.$$

Since we have already proven that $N_{j_1}(\alpha) \cdots N_{j_d}(\alpha) = 0$ whenever $j_1 \geq k - 1$, in particular, whenever

$$\sum_{l=1}^{d} j_l > (k-2)d,$$

we have

$$
\begin{aligned}
&Q_{f,\alpha,d}(n) \\
&\quad := \left(\sum_{i=0}^{k} \binom{n}{i} N_i(\alpha)\right)^d \\
&\quad =_{n\to\infty} \binom{n}{k-2}^d N_{k-2}(\alpha)^d + d\binom{n}{k-2}^{d-1}\binom{n}{k-3} N_{k-2}(\alpha)^{d-1} N_{k-3}(\alpha) \\
&\qquad + O(n^{(k-2)d-2}). \qquad (4.3)
\end{aligned}
$$

Recall that $\deg(Q_{f,\alpha,d}) \le d(d-1)$ by Lemma 2.25. Since $d(d-1) \le (k-2)d-2$ (because $d \ge 4$), it follows from (4.3) that $N_{k-2}(\alpha)^d = 0$ and then $N_{k-2}(\alpha)^{d-1}N_{k-3}(\alpha) = 0$, which proves Lemma 4.5. ∎

Proof of Proposition 4.4. By Theorem 1.1, we have $k(f) \le 2d-2$ and $k(f)$ is even. Since $d \ge 4$, Proposition 4.4 in the case $k(f) < 2d-2$ (resp. $k(f) = 2d-2$) follows from Theorem 1.2 (resp. Lemmas 2.22 and 4.5). ∎

5. A refined lower bound: End of the proof of Theorem 1.2 and Corollary 1.6

In this section, we prove the following lower bound of $\mathrm{Plov}(f)$. At the end we will conclude the proof of Theorem 1.2 together with Corollary 1.6.

Theorem 5.1. *Let X be a compact Kähler manifold of dimension $d > 0$ and let $f \in \mathrm{Aut}(X)$ be a zero entropy automorphism. Then we have*

$$\mathrm{Plov}(f) \ge d + 2k(f) - 2.$$

Proof. We can assume that $\dim X \ge 2$, otherwise $k(f) = 0$, and Theorem 5.1 holds trivially.

By Lemmas 2.6 and 2.8, we can assume that $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent. Let ω be a Kähler class. Recall that we have

$$\Delta'_n(f,\omega) := \sum_{i=0}^{n}((f^i)^*\omega + (f^{-i})^*\omega) = \sum_{i=0}^{k(f)} \binom{n+1}{i+1} N_i(\omega)$$

from the computation in the proof of Lemma 2.22. By Lemma 2.23, and using the notations therein, we have

$$\mathrm{Plov}(f) = \deg_n P'_{f,\omega,d}(n) > \deg_n P'_{f,\omega,d-1}(n) > \cdots > \deg_n P'_{f,\omega,2}(n).$$

Therefore it suffices to show that

$$\deg_n P'_{f,\omega,2}(n) \ge 2k(f).$$

Recall that

$$P'_{f,\omega,2}(n) = \Delta'_n(f,\omega)^2\omega^{d-2} = \left(\sum_{j=0}^{k(f)} \binom{n+1}{j+1} N_j\omega\right)^2 \omega^{d-2}. \tag{5.1}$$

Assume that $(N_{k(f)}\omega)^2 \neq 0$. Since $N_{k(f)}\omega$ is nef by Lemma 3.8 (3), we have $(N_{k(f)}\omega)^2 \cdot \omega^{d-2} \neq 0$. Hence $\deg_n P'_{f,\omega,2}(n) \geq 2k(f) + 2$ by (5.1).

Now assume that $(N_{k(f)}\omega)^2 = 0$. Then

$$(N_{k(f)-1}\omega)^2 \equiv 0, \quad (N_{k(f)}\omega)(N_{k(f)-1}\omega) \equiv 0 \tag{5.2}$$

by Lemma 3.10. Since $N_{k(f)}\omega$ is nef and f^*-invariant by Lemma 3.8, we can construct a quasi-nef sequence $M_1, \dots, M_d$ with $M_1 = N_{k(f)}\omega$. Suppose that $(N_{k(f)}\omega)(N_{k(f)-2}\omega) \equiv 0$. Then $(N_{k(f)-2}\omega) \in F_1$, and we would have $(N_{k(f)}\omega) \in F_0 = 0$ by Lemma 3.8 (1), which contradicts the assumption that $N_{k(f)}\omega \neq 0$. Hence $(N_{k(f)}\omega)(N_{k(f)-2}\omega) \not\equiv 0$. Together with the vanishings (5.2) and (5.1), we obtain $\deg_n P'_{f,\omega,2}(n) = 2k(f)$. ∎

Proof of Theorem 1.2. The upper bound and lower bound of $\mathrm{Plov}(f)$ in Theorem 1.2 follows from Theorems 4.2 and 5.1 respectively. ∎

Proof of Corollary 1.6. The main statement of Corollary 1.6 follows from Theorem 1.2 (0) (resp. Theorem 1.2 (1)) when $k(f) > 0$ (resp. $k(f) = 0$). Together with Theorem 1.1, it follows that $\kappa(X) \geq d/2$ implies $\mathrm{Plov}(f) \leq d^2 - 2$. ∎

6. Complex tori: Proof of Theorem 1.4 and a few remarks

In this section, we prove Theorem 1.4; see Remark 6.5 for further discussion.

Proof of Theorem 1.4. First we perform some reduction. By Lemmas 2.6 and 2.8, up to replacing f by some finite iteration of it, we can assume that $f^* : H^{1,0}(X) \circlearrowleft$ is unipotent. Fix a basis

$$dz_{1,1}, dz_{1,2}, \dots, dz_{1,k_1}, \dots, dz_{p,1}, \dots, dz_{p,k_p}$$

of $H^{1,0}(X)$ such that for every $i = 1, \dots, p$,

$$f^*dz_{i,j} = \begin{cases} dz_{i,1} & \text{if } j = 1, \\ dz_{i,j} + dz_{i,j-1} & \text{if } 2 \leq j \leq k_i. \end{cases}$$

As the f^*-action on $H^{1,0}(X)$ determines the f^*-action on $H^\bullet(X, \mathbb{C})$ when X is a torus (because $H^\bullet(X, \mathbb{C})$ is generated by $H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus \overline{H^{1,0}(X)}$), by Corollary 2.3 we can assume that $X = E^d$ with E being an elliptic curve, or even $E = \mathbb{C}/\mathbb{Z}[\sqrt{-1}]$, and $(z_{i,j})_{1 \leq i \leq p, 2 \leq j \leq k_i}$ are the global coordinates of E^d, so that

$$f : E^d = \prod_{i=1}^p E^{k_i} \to \prod_{i=1}^p E^{k_i} = E^d$$

is the product of $E^{k_i} \to E^{k_i}$ defined by the unipotent Jordan matrix of size k_i.

By the product formula (Proposition 2.5 (4)), it suffices to prove Theorem 1.4 for the case $p = 1$. So, from now on until the end of the proof, we assume that $p = 1$. Namely, $f^* : H^{1,0}(X) \circlearrowleft$ has only one Jordan block.

Set $e_i = dz_i$ and $\overline{e}_i = d\overline{z}_i$. For every $\sigma = \sum_{i,j} a_{ij} e_i \wedge \overline{e}_j \in H^{1,1}(X, \mathbb{C}) \setminus \{0\}$, define

$$w(\sigma) := \max\{i + j \mid a_{ij} \neq 0\},$$

and for every $p = 2, \ldots, 2d$, define

$$\sigma(p) := \sum_{i+j=p} a_{ij} e_i \wedge \overline{e}_j.$$

Note that $\sum_{i=1}^{d} w_i(\sigma_i) \leq d(d+1)$ by definition. We need the following.

Lemma 6.1. *Let $\sigma_1, \ldots, \sigma_d \in H^{1,1}(X, \mathbb{C}) \setminus \{0\}$ and let $w_i := w(\sigma_i)$.*

(1) *If $\sum_{i=1}^{d} w_i < d(d+1)$, then $\sigma_1 \wedge \cdots \wedge \sigma_d = 0$.*

(2) *If $\sum_{i=1}^{d} w_i = d(d+1)$, then $\sigma_1 \wedge \cdots \wedge \sigma_d = \sigma_1(w_1) \wedge \cdots \wedge \sigma_d(w_d)$.*

Proof. By multi-linearity of $\sigma_1 \wedge \cdots \wedge \sigma_d$, it is clear that (1) implies (2), and that it suffices to prove (1) for $\sigma_1, \ldots, \sigma_d$ of the form $\sigma_i = e_{i_1} \wedge \overline{e}_{j_1}, \ldots, \sigma_d = e_{i_d} \wedge \overline{e}_{j_d}$. If $\sigma_1 \wedge \cdots \wedge \sigma_d \neq 0$, then necessarily

$$\{i_1, \ldots, i_d\} = \{1, \ldots, d\} = \{j_1, \ldots, j_d\},$$

so $\sum_{i=1}^{d} w_i = d(d+1)$. ∎

We return to the proof of Theorem 1.4. Let $N := f^* - \mathrm{Id}$ and let

$$\omega := \sqrt{-1} \sum_{i=1}^{d} e_i \wedge \overline{e}_i,$$

which is a Kähler class on X. For every $q = 0, \ldots, 2d-2$, by induction on q we have

$$(N^q \omega)(p) = 0$$

for every $p > 2d - q$ and

$$\begin{aligned}(N^q \omega)(2d - q) &= \sqrt{-1}(N^q(e_d \wedge \overline{e}_d))(2d - q) \\ &= \sqrt{-1} \sum_{i+j=q} \binom{q}{i} e_{d-q+i} \wedge \overline{e}_{d-q+j} \neq 0.\end{aligned}$$

Therefore,

$$w(N^q \omega) = 2d - q.$$

Let $q_1, \ldots, q_d \geq 0$ be non-negative integers. If $\sum_{i=1}^{d} q_i > d^2 - d$, then by Lemma 6.1

$$(N^{q_1} \omega) \cdots (N^{q_d} \omega) = 0,$$

so $\mathrm{Plov}(f) \leq d^2$ by (2.7).

It remains to prove that $\mathrm{Plov}(f) \geq d^2$. Note that since $\omega(2d) = \sqrt{-1} e_d \wedge \overline{e}_d$ is nef, by Lemma 2.4 we have $\mathrm{Plov}(f) \geq \mathrm{Plov}(f, \omega(2d))$. Until the end of the proof, we formally define $e_i \wedge \overline{e}_j = 0$ whenever i and j are integers such that $i \notin [1, d]$ or $j \notin [1, d]$.

Claim 6.2. We have

$$N^q(e_d \wedge \overline{e}_d) = \sum_{i+j \leq q} \binom{q}{i, j, q-i-j} e_{d-q+i} \wedge \overline{e}_{d-q+j}.$$

Proof. Let $V := \mathbb{C}[X, Y]/(X^d, Y^d)$. We have an isomorphism of $\mathbb{C}$-vector spaces $V \simeq H^{1,1}(X)$ sending each $X^i Y^j$ to $e_{d-i} \wedge \overline{e}_{d-j}$. Under this isomorphism, $N : H^{1,1}(X) \to H^{1,1}(X)$ becomes

$$\begin{aligned} N : V &\to V, \\ P &\mapsto (XY + X + Y)P \mod (X^d, Y^d), \end{aligned}$$

so

$$N^q(1) = (XY + X + Y)^q = \sum_{i+j \leq q} \binom{q}{i, j, q-i-j} X^{q-i} Y^{q-j} \mod (X^d, Y^d). \tag{6.1}$$

Translating (6.1) back to $N : H^{1,1}(X) \to H^{1,1}(X)$ proves the claim. ■

For every integer $n > 0$, by Claim 6.2 we have

$$\begin{aligned} \Omega &:= \sum_{q=0}^{2d-2} \binom{n}{q+1} N^q(\omega(2d)) \\ &= \sqrt{-1} \sum_{q=0}^{2d-2} \sum_{i+j \leq q} \binom{n}{q+1} \binom{q}{i, j, q-i-j} e_{d-q+i} \wedge \overline{e}_{d-q+j}. \end{aligned}$$

For each pair of integers $1 \leq i, j \leq d$, define the polynomial $P_{i,j}(n)$ in n by

$$\Omega = \sqrt{-1} \sum_{1 \leq i,j \leq d} P_{i,j}(n) e_i \wedge \overline{e}_j. \tag{6.2}$$

Claim 6.3. The polynomial $P_{d-i,d-j}(n)$ in n has degree $i + j + 1$ and leading coefficient

$$\frac{1}{(i+j+1)!} \binom{i+j}{i}.$$

Proof. As $e_{d-i} \wedge \overline{e}_{d-j} = e_{d-q+(q-i)} \wedge \overline{e}_{d-q+(q-j)}$, by construction we have (with q varying in the sum)

$$P_{d-i,d-j}(n) = \sum_{(q-i)+(q-j) \leq q} \binom{n}{q+1} \binom{q}{q-i, q-j, i+j-q}.$$

So the degree and the leading coefficient of $P_{d-i,d-j}$, are equal to those of the polynomial $\binom{n}{q+1}\binom{q}{q-i,q-j,i+j-q}$ in n when q is maximal and satisfying $(q-i)+(q-j) \le q$ (that is, when $q = i + j$). This proves the claim. ∎

By (6.2), we have

$$\Omega^d = (\sqrt{-1})^d d! P(n)(e_1 \wedge \overline{e}_1) \wedge \cdots \wedge (e_d \wedge \overline{e}_d)$$

where $P(n)$ is the determinant of the matrix $(P_{i,j}(n))_{1\le i,j\le d}$. Further, by Claim 6.3, we have $\deg_n(P(n)) \le d^2$ and the coefficient in front of n^{d^2} is $\det M$, where $M = (M_{i+1,j+1})_{0\le i,j\le d-1}$ is the $(d \times d)$ matrix defined by

$$M_{i+1,j+1} = \frac{1}{(i+j+1)!}\binom{i+j}{i} = \frac{1}{i!j!}\cdot\frac{1}{(i+j+1)}, \quad 0 \le i,j \le d-1.$$

We have

$$\det M = \frac{1}{(\prod_{p=0}^{d-1} p!)^2}\det\left(\frac{1}{(i+j+1)}\right)_{0\le i,j\le d-1} = \frac{\prod_{p=0}^{d-1} p!}{\prod_{p=d}^{2d-1} p!} \neq 0,$$

where the second equality follows from the determinant of the Hilbert matrix (see, e.g., [14, (1.1)]). Since $\mathrm{Plov}(f, \omega(2d)) = \deg_n(P(n))$ by (2.6) and the definition of Ω, it thus follows that

$$\mathrm{Plov}(f) \ge \mathrm{Plov}(f, \omega(2d)) = \deg_n(P(n)) = d^2.$$

This completes the proof of the main statement of Theorem 1.4. The optimality of the upper bound is provided by Example 6.4 below. ∎

Example 6.4. Let E be a complex elliptic curve and let $X = E^d$. Define $f : X \to X$ by

$$(x_1, \ldots, x_d) \mapsto (x_1, x_2 + x_1, \ldots, x_d + x_{d-1}). \tag{6.3}$$

Then $f^* : H^{1,0}(X) \circlearrowleft$ is represented by the $(d \times d)$-Jordan matrix, and $\mathrm{Plov}(X, f) = d^2$ as a consequence of the main statement of Theorem 1.4.

Remark 6.5. (1) Consider $f \in \mathrm{Aut}(E^3)$ with $f(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_3)$. Then $f^* : H^{1,0}(E^3) \circlearrowleft$ has two Jordan blocks, of sizes 2 and 1 respectively. By Theorem 1.4, we have $\mathrm{Plov}(f) = 2^2 + 1^2 = 5$, which is also consistent with [15, Example 6.14].

(2) The upper bound in Theorem 1.4 is also asserted in the proof of [5, Proposition 4.3] (without optimality). However, the estimates [5, (4.6)–(4.7)] using ℓ_1 in their proof have to be suitably modified, otherwise as we can see that if f is the identity, the estimate $\mathrm{Vol}(\Gamma(n)) \le Cn^{\ell_1^2}$ in [5, (4.6)–(4.7)] would imply that $\mathrm{Vol}(\Gamma(n))$ grows at most linearly in n, which contradicts the equality $\mathrm{Plov}(X) = d$.

Finally, note that in this paper, whenever we prove that $\mathrm{Plov}(f)$ is bounded from above by some constant C for an automorphism $f \in \mathrm{Aut}(X)$ such that $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent, we actually prove that the right hand side of the inequality in Lemma 2.16 or Lemma 2.22 is bounded by C. In view of Question 1.5, we ask the following.

Question 6.6. Let X be a compact Kähler manifold of dimension $d \geq 1$ and let $f \in \mathrm{Aut}(X)$ such that $f^* : H^{1,1}(X) \circlearrowleft$ is unipotent. For every $\omega \in H^{1,1}(X)$, do we have

$$(N^{i_1}\omega) \cdots (N^{i_d}\omega) = 0$$

and

$$(N_{i_1}\omega) \cdots (N_{i_d}\omega) = 0$$

whenever the i_j are non-negative integers satisfying $\sum_{j=1}^{d} i_j > d(d-1)$?

7. Some explicit examples

We know that $\mathrm{Plov}(f) = 1$ for a compact Riemann surface, $\mathrm{Plov}(X, f) = 2$ or 4 for a compact Kähler surface by Corollary 1.3, and $\mathrm{Plov}(X, f) = d$ for a projective variety X of dimension d whose desingularization $\widetilde{X}$ is of general type (as $|\mathrm{Bir}(\widetilde{X})| < \infty$ by [22, Corollary 14.3]). Besides complex tori and these three cases, we can also determine $\mathrm{Plov}(f)$ for some other classes of compact Kähler manifolds X (Proposition 7.1). We also prove Corollary 3.7 (2) in this section.

Proposition 7.1. (1) *Let X be a compact hyper-Kähler manifold of dimension $2d$ and let $f \in \mathrm{Aut}(X)$ such that $d_1(f) = 1$. Then $\mathrm{Plov}(f) = 2d$ if f is of finite order, and $\mathrm{Plov}(f) = 4d$ if f is of infinite order; both cases are realizable, with X projective.*

(2) *Let X be a smooth projective variety whose nef cone is a finite rational polyhedral cone. Let $\dim X = d$ and $f \in \mathrm{Aut}(X)$. Then f is quasi-unipotent and $\mathrm{Plov}(f) = d$. In particular, this is the case when X is a Mori dream space, especially when X is a toric variety or a Fano manifold.*

Proof. (1) The reader is referred to [13] for basics about compact hyper-Kähler manifolds. Note that a compact hyper-Kähler manifold has no global vector field other than 0. Hence $\mathrm{Aut}(X)$ is discrete. Thus f is of finite order if and only if $f^* : H^{1,1}(X, \mathbb{R}) \circlearrowleft$ is of finite order.

So, replacing f by its power and using Proposition 2.5 (2), we can assume that $f = \mathrm{Id}_X$ or $f^* : H^{1,1}(X, \mathbb{R}) \circlearrowleft$ is unipotent of infinite order. The result is clear when $f = \mathrm{Id}_X$. In the rest, we will assume that $f^* : H^{1,1}(X, \mathbb{R}) \circlearrowleft$ is unipotent of infinite order.

Let $q_X(x)$ be Beauville–Bogomolov–Fujiki's quadratic form on $H^{1,1}(X, \mathbb{R})$. The signature of $q_X(x)$ is $(1, h^{1,1}(X) - 1)$.

Let ω be a Kähler class on X. Then the degree of the polynomial $q_X(P_{f,\omega}(n))$ in Lemma 2.16, with respect to n is $2^2 = 4$ by [1, Lemma 5.4]. The first part of (1) then follows from Fujiki's relation below (with positive constant $c_X > 0$):

$$(x^{2d})_X = c_X(q_X(x))^d.$$

For the realization part of (1), let $\varphi : S \to \mathbb{P}^1$ be a projective elliptic K3 surface whose Mordell–Weil group $\mathrm{MW}(\varphi)$ has an element of infinite order, say f. There are plenty of such K3 surfaces. Then $f \in \mathrm{Aut}(S)$ and it induces an automorphism $f^{[d]} \in \mathrm{Aut}(\mathrm{Hilb}^d(S)/\mathbb{P}^d)$ of infinite order. Here the Hilbert scheme $X := \mathrm{Hilb}^d(S)$ is a projective hyper-Kähler manifold of dimension $2d$ with the Lagrangian fibration $\mathrm{Hilb}^d(S) \to \mathbb{P}^d$ induced by φ. Hence, $d_1(f^{[d]}) = 1$ as it preserves the pullback h of the hyperplane class of $\mathbb{P}^d$, which is a nonzero nef class on X such that $q_X(h) = 0$. Thus $(X, f^{[d]})$ provides an example such that $\mathrm{Plov}(f^{[d]}) = 2\dim X = 4d$. This completes the proof of (1).

(2) By the assumption, $f^* : N^1(X) \circlearrowleft$ is always of finite order (even though the order of f itself can be often infinite). Thus we have $\mathrm{Plov}(f) = d$ by Theorem 1.2. ∎

We finish this section with proofs of Corollary 3.7 (2) by constructing explicit examples. The examples that we will construct also appear in other complex dynamical contexts [8, 21].

Proof of Corollary 3.7 (2). Let $X_d = E^d$ $(d \geq 2)$ be the d-fold self-product of an elliptic curve E and f_d the automorphism of X_d defined by

$$f_d(x_1, x_2, \ldots, x_d) = (x_1, x_2 + x_1, \ldots, x_d + x_{d-1}),$$

as in Example 6.4. We have $k(f_d) = 2d - 2$ [8, §4.1].

Let C be a smooth projective curve of genus $g(C) \geq 2$ with a surjective morphism $\pi : C \to E$. Let $Y_d := C \times E^{d-1}$. Then Y_d is a smooth projective variety with $\dim Y_d = d$ and Kodaira dimension $\kappa(Y_d) = 1$. We define $g_d \in \mathrm{Aut}(Y_d)$ by

$$g_d(P, x_2, x_3, \ldots, x_d) = (P, x_2 + \pi(P), x_3 + x_2 \ldots, x_d + x_{d-1}).$$

We also define

$$p : Y_d \to X_d; \quad (P, x_2, x_3, \ldots, x_d) \mapsto (\pi(P), x_2, x_3, \ldots, x_d).$$

Then p is a finite surjective morphism such that $f_d \circ p = p \circ g_d$, so $k(g_d) = k(f_d) = 2d - 2$ by Proposition 2.11. Finally, for every smooth projective variety V with $\kappa(V) = \dim V$, let $V_d := Y_d \times V$ and consider $\phi_d := g_d \times \mathrm{Id}_V \in \mathrm{Aut}(V_d)$. We have

$$2(d-1) = k(g_d) \leq k(g_d \times \mathrm{Id}_V) \leq 2(\dim V_d - \kappa(V_d)) = 2(d-1),$$

where the second inequality follows from the first statement of Corollary 3.7. So

$$k(\phi_d) = 2(\dim V_d - \kappa(V_d)).$$

When d and V vary, any pair of positive integers $\dim V_d \geq 1$ and $\kappa(V_d) \geq 1$ is realizable, which finishes the proof. ∎

8. Twisted homogeneous coordinate rings and GK-dimensions: Proofs of Theorem 1.7 and Corollary 8.5

In this section, we first relate the polynomial log-volume growth $\mathrm{Plov}(f)$ to the GK-dimensions $\mathrm{GKdim}(X, f)$ of twisted homogeneous coordinate rings through Keeler's work [15]. Then we prove Theorem 1.7 and Corollary 8.5, explaining how the results of $\mathrm{Plov}(f)$ imply the analogous statements for $\mathrm{GKdim}(X, f)$.

8.1. Recollection of Keeler's work [15]

Following [15], we recall the definition of twisted homogeneous coordinate rings and related notions, together with the fundamental properties proven in [1] and [15].

Let X be an irreducible projective variety defined over an algebraically closed field $\mathbf{k}$ of characteristic 0. Let $f \in \mathrm{Aut}(X)$ be an automorphism. We say that a line bundle L on X is *f-ample* if for any coherent sheaf F on X, there is a positive integer m_F such that

$$H^q(X, F \otimes L \otimes f^*L \otimes \cdots \otimes (f^m)^*L) = 0$$

for any integer $q > 0$ and for any integer $m > m_F$. A Cartier divisor D is called f-ample if $\mathcal{O}(D)$ is f-ample.

Let $f \in \mathrm{Aut}(X)$ and let L be a line bundle on X. For any integer $n \in \mathbb{Z}_{\geq 0}$, define

$$\Delta_n(f, L) := L \otimes f^*L \otimes \cdots \otimes (f^n)^*L,$$

and

$$B_{n+1}(X, f, L) := H^0(X, \Delta_n(f, L)), \quad B_0 = H^0(X, \mathcal{O}_X) = \mathbf{k}.$$

The *twisted homogeneous coordinate ring* of X associated to (f, L) is the (noncommutative) associative graded $\mathbf{k}$-algebra

$$B(X, f, L) := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} B_n(X, f, L).$$

The study of $B(X, f, L)$ was initiated by Artin and Van den Bergh [1]. Together with the seminal work of Keeler [15], here are some fundamental properties they proved. In the statements, $N^1(X) := \mathrm{NS}(X)/(\mathrm{torsion})$.

Theorem 8.1 (Keeler, Artin–Van den Bergh). *Let X be a projective variety of dimension $d > 0$ and let $f \in \mathrm{Aut}(X)$.*

(1) *f-ample line bundles exist if and only if $f^* : N^1(X) \circlearrowleft$ is quasi-unipotent.*

In the following, we assume that $f^* : N^1(X) \circlearrowleft$ is quasi-unipotent.

(2) *L is f-ample if and only if there exists an integer $n > 0$ such that $\Delta_n(f, L)$ is ample. In particular, any ample line bundle is f-ample.*

(3) *The GK-dimension* GKdim$B$ *is independent of the choice of an* f*-ample line bundle* L*. We therefore define the GK-dimension as*

$$\mathrm{GKdim}(X, f) := \mathrm{GKdim} B(X, f, L)$$

for any choice of f*-ample line bundle* L*.*

(4) *The GK-dimension is a positive integer. More precisely, after replacing* f *by its suitable positive power so that* $f^* : N^1(X) \circlearrowleft$ *is unipotent, the self intersection number* $(\Delta_n(f, L)^d)$ *is a polynomial in* n *and its degree satisfies*

$$\mathrm{rGKdim}(X, f) := \mathrm{GKdim}(X, f) - 1 = \deg_n(\Delta_n(f, L)^d).$$

We call rGKdim(X, f) *the reduced GK-dimension of* (X, f)*.*

Proof. Let us just indicate the references where these statements are proven. Statement (1) is contained in [15, Theorem 1.2]. Statements (2), (3), and (4) follow from [1, Lemma 4.1], and [15, Proposition 6.11, Theorem 6.1 (1)], respectively. ∎

When X is a complex projective manifold, Keeler's work implies Theorem 1.7 as an immediate corollary that the reduced GK-dimension of (X, f) coincides with the polynomial log-volume growth of f. Together with Theorem 8.1, this suggests unexpected relations between noncommutative algebra and complex dynamics of automorphisms of zero entropy.

Proof of Theorem 1.7. Since $f^* : N^1(X) \circlearrowleft$ is quasi-unipotent, by Theorem 8.1 (4) and the definition of Plov(f), given any ample line bundle L on X, we have

$$\mathrm{rGKdim} B = \deg_n(\Delta_n(f, L)^d) = \limsup_{n\to\infty} \frac{\log \Delta_n(f, c_1(L))^d}{\log n} = \mathrm{Plov}(f). \qquad ∎$$

8.2. From Kähler to projective

Let X be a projective variety over $\mathbb{C}$ and let $f \in \mathrm{Aut}(X)$. Let $\nu : \widetilde{X} \to X$ be a projective desingularization of X such that

$$f \circ \nu = \nu \circ \widetilde{f}$$

for some $\widetilde{f} \in \mathrm{Aut}(\widetilde{X})$ (see, e.g., [17, Theorem 3.45] for the existence). Before we prove Corollary 8.5, first we identify some dynamical properties and invariants of (X, f) as a projective variety with those of $(\widetilde{X}, \widetilde{f})$ as a compact Kähler manifold.

Lemma 8.2. *The following conditions are equivalent.*

(1) $f^* : N^1(X) \circlearrowleft$ *is quasi-unipotent.*

(2) $\widetilde{f}^* : N^1(\widetilde{X}) \circlearrowleft$ *is quasi-unipotent.*

(3) $\widetilde{f}$ *has zero entropy.*

Proof. We define $d_1(f)$ as in (2.1) but replacing ω by an ample divisor. Then the same proof of [19, Proposition A.2 and Lemma A.7] says that in the definition $d_1(f)$, we can assume that ω is a nef and big divisor instead, and $d_1(f)$ is the spectral radius of $f^* : N^1(X) \circlearrowleft$. Note that $d_1(f) = 1$ if and only if $f^* : N^1(X) \circlearrowleft$ is quasi-unipotent by Kronecker's theorem. Then the projection formula shows $d_1(f) = d_1(\widetilde{f})$, hence (1) and (2) are equivalent. The equivalence between (2) and (3) follows from Lemma 2.8, as we can compute $d_1(\widetilde{f})$ using ample classes (which also lie in $N^1(\widetilde{X})$). ∎

Assume that $f^* : N^1(X) \circlearrowleft$ is quasi-unipotent. Then

$$\|(f^m)^* : N^1(X) \otimes \mathbb{C} \circlearrowleft \| \sim_{m\to\infty} C m^{k_{\mathrm{NS}}(f)}$$

for some $k_{\mathrm{NS}}(f) \in \mathbb{Z}_{\geq 0}$ and $C > 0$.

Lemma 8.3. *We have* $k_{\mathrm{NS}}(f) = k(\widetilde{f})$.

Proof. First of all, the same argument proving Proposition 2.11 shows that $k_{\mathrm{NS}}(f) = k_{\mathrm{NS}}(\widetilde{f})$. It suffices to show that $k_{\mathrm{NS}}(\widetilde{f}) = k(\widetilde{f})$.

Since $k(\widetilde{f})$ is invariant under finite iterations, by Lemma 8.2 and Lemma 2.8 we can assume that $\widetilde{f}^* : H^{1,1}(\widetilde{X}) \circlearrowleft$ is unipotent. As the ample cone of $\widetilde{X}$ spans $\mathrm{NS}(\widetilde{X})_{\mathbb{R}}$ we can thus find an ample class ω of $\widetilde{X}$ such that

$$(\widetilde{f}^* - \mathrm{Id})^{k_{\mathrm{NS}}(\widetilde{f})}(\omega) \neq 0 \quad \text{and} \quad (\widetilde{f}^* - \mathrm{Id})^{k_{\mathrm{NS}}(\widetilde{f})+1}(\omega) = 0.$$

Hence $k_{\mathrm{NS}}(\widetilde{f}) = k(\widetilde{f})$ by Proposition 3.2 (2). ∎

Lemma 8.4. *We have*

$$\mathrm{GKdim}(X, f) = \mathrm{GKdim} B(X, f, L)$$

for any big and nef line bundle L. *As a consequence,*

$$\mathrm{rGKdim}(X, f) = \mathrm{rGKdim}(\widetilde{X}, \widetilde{f}) = \mathrm{Plov}(\widetilde{f})$$

and $\mathrm{GKdim}(X, f)$ *is a birational invariant.*

Proof. Since $\mathrm{GKdim}(X, f)$ is the polynomial degree of $n \mapsto \deg_n(\Delta_n(f, L)^d)$ by Theorem 8.1 (4), the same argument of Lemma 2.1 proves the first assertion of Lemma 8.4.

Let L be an ample line bundle on X, since

$$\left(\sum_{i=0}^{n} (\widetilde{f}^i)^*(\nu^* c_1(L))\right)^d = \nu^* \left(\sum_{i=0}^{n} (f^i)^* c_1(L)\right)^d = \left(\sum_{i=0}^{n} (f^i)^* c_1(L)\right)^d,$$

by Theorem 8.1 (4) we have

$$\mathrm{GKdim} B(\widetilde{X}, \widetilde{f}, \nu^* L) = \mathrm{GKdim} B(X, f, L) = \mathrm{GKdim}(X, f).$$

As $\nu^* L$ is nef and big, it follows from the first statement that $\mathrm{GKdim} B(\widetilde{X}, \widetilde{f}, \nu^* L) = \mathrm{GKdim}(\widetilde{X}, \widetilde{f})$, which finishes the proof of Lemma 8.4. ∎

In the following corollary, $k(f)$ is defined with $f^* : H^{1,1}(X) \circlearrowleft$ replaced by $f^* : N^1(X) \circlearrowleft$ (denoted as $k_{\mathrm{NS}}(f)$ in Section 8.2).

Corollary 8.5. *Let X be a projective variety of dimension $d > 0$ over $\mathbf{k}$, not necessarily smooth, and let $f \in \mathrm{Aut}(X)$ be an automorphism such that $f^* : N^1(X) \circlearrowleft$ is quasi-unipotent. Then the analogous statements of Theorem 1.2, Corollary 1.3, inequality (1.4) (Proposition 4.4), etc., hold with $\mathrm{Plov}(f)$ replaced by $\mathrm{rGKdim}(X, f)$ under the same assumptions on d and $k(f)$. In particular, we have*

$$\mathrm{rGKdim}(X, f) \in \{3, 5, 9\}$$

if $d = 3$ (by Corollary 1.3), and

$$\mathrm{rGKdim}(X, f) \leq 2d^2 - 3d - 2$$

whenever $d \geq 4$ (by Proposition 4.4).

Proof. By the Lefschetz principle, we can assume that the pair (X, f) is defined over $\mathbf{k} = \mathbb{C}$. Corollary 8.5 then follows from the existence of the equivariant projective desingularization (see, e.g., [17, Theorem 3.45]), together with the comparison results Lemma 8.4 and Lemma 8.3. ∎

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References

[1] M. Artin and M. Van den Bergh, Twisted homogeneous coordinate rings. *J. Algebra* **133** (1990), no. 2, 249–271 Zbl 0717.14001 MR 1067406

[2] E. Bierstone and P. D. Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. Math.* **128** (1997), no. 2, 207–302 Zbl 0896.14006 MR 1440306

[3] G. Birkhoff, Linear transformations with invariant cones. *Amer. Math. Monthly* **74** (1967), 274–276 Zbl 0192.26703 MR 0214605

[4] S. Cantat, Automorphisms and dynamics: a list of open problems. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*, pp. 619–634, World Sci. Publ., Hackensack, NJ, 2018 Zbl 1441.14139 MR 3966782

[5] S. Cantat and O. Paris-Romaskevich, Automorphisms of compact Kähler manifolds with slow dynamics. *Trans. Amer. Math. Soc.* **374** (2021), no. 2, 1351–1389 Zbl 1458.37056 MR 4196396

[6] J.-P. Demailly, *Analytic methods in algebraic geometry*. Surv. Mod. Math. 1, International Press, Somerville, MA; Higher Education Press, Beijing, 2012 Zbl 1271.14001 MR 2978333

[7] T.-C. Dinh, Suites d'applications méromorphes multivaluées et courants laminaires. *J. Geom. Anal.* **15** (2005), no. 2, 207–227 Zbl 1085.37039 MR 2152480

[8] T.-C. Dinh, H.-Y. Lin, K. Oguiso, and D.-Q. Zhang, Zero entropy automorphisms of compact Kähler manifolds and dynamical filtrations. *Geom. Funct. Anal.* **32** (2022), no. 3, 568–594 Zbl 1509.14088 MR 4431123

[9] T.-C. Dinh, V.-A. Nguyên, and T. T. Truong, On the dynamical degrees of meromorphic maps preserving a fibration. *Commun. Contemp. Math.* **14** (2012), no. 6, article no. 1250042 MR 2989646

[10] T.-C. Dinh and N. Sibony, Regularization of currents and entropy. *Ann. Sci. École Norm. Sup. (4)* **37** (2004), no. 6, 959–971 Zbl 1074.53058 MR 2119243

[11] Y.-W. Fan, L. Fu, and G. Ouchi, Categorical polynomial entropy. *Adv. Math.* **383** (2021), article no. 107655 Zbl 1467.37019 MR 4233273

[12] M. Gromov, On the entropy of holomorphic maps. *Enseign. Math. (2)* **49** (2003), no. 3-4, 217–235 Zbl 1080.37051 MR 2026895

[13] D. Huybrechts, Compact hyperkähler manifolds. In *Calabi–Yau manifolds and related geometries (Nordfjordeid, 2001)*, pp. 161–225, Universitext, Springer, Berlin, 2003 MR 1963562

[14] M. Ishikawa, S. Okada, H. Tagawa, and J. Zeng, Generalizations of Cauchy's determinant and Schur's Pfaffian. *Adv. in Appl. Math.* **36** (2006), no. 3, 251–287 Zbl 1134.15005 MR 2219947

[15] D. S. Keeler, Criteria for σ-ampleness. *J. Amer. Math. Soc.* **13** (2000), no. 3, 517–532 Zbl 0952.14002 MR 1758752

[16] J. Keum, K. Oguiso, and D.-Q. Zhang, Conjecture of Tits type for complex varieties and theorem of Lie–Kolchin type for a cone. *Math. Res. Lett.* **16** (2009), no. 1, 133–148 Zbl 1172.14025 MR 2480567

[17] J. Kollár, *Lectures on resolution of singularities*. Ann. of Math. Stud. 166, Princeton University Press, Princeton, NJ, 2007 Zbl 1113.14013 MR 2289519

[18] F. Lo Bianco, On the cohomological action of automorphisms of compact Kähler threefolds. *Bull. Soc. Math. France* **147** (2019), no. 3, 469–514 Zbl 1444.14073 MR 4030548

[19] N. Nakayama and D.-Q. Zhang, Building blocks of étale endomorphisms of complex projective manifolds. *Proc. Lond. Math. Soc. (3)* **99** (2009), no. 3, 725–756 Zbl 1185.14012 MR 2551469

[20] K. Oguiso, Some aspects of explicit birational geometry inspired by complex dynamics. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pp. 695–721, Kyung Moon Sa, Seoul, 2014 Zbl 1373.14016 MR 3728634

[21] K. Oguiso and T. T. Truong, Explicit examples of rational and Calabi–Yau threefolds with primitive automorphisms of positive entropy. *J. Math. Sci. Univ. Tokyo* **22** (2015), no. 1, 361–385 Zbl 1349.14055 MR 3329200

[22] K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*. Lect. Notes Math. 439, Springer, Berlin-New York, 1975 Zbl 0299.14007 MR 0506253

[23] Y. Yomdin, Volume growth and entropy. *Israel J. Math.* **57** (1987), no. 3, 285–300 Zbl 0641.54036 MR 0889979

[24] D.-Q. Zhang, A theorem of Tits type for compact Kähler manifolds. *Invent. Math.* **176** (2009), no. 3, 449–459 Zbl 1170.14029 MR 2501294

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