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Whether p-conductive homogeneity holds depends on p

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Abstract. We introduce two fractals, in Euclidean spaces of dimension two and three, respectively, such that the 2-conductive homogeneity holds but there is some $\varepsilon \in (0, 1)$ so that the p-conductive homogeneity fails for every $p \in (1, 1 + \varepsilon)$. In addition, these two fractals have Ahlfors regular conformal dimension within the interval $(1, 2)$ and $(2, 3)$, respectively.

1. Introduction

Some new progress [4, 5, 7, 10] has been made in recent years on the construction of p-energies and therefore, on Sobolev space $\mathcal{W}^{1,p}$ on Sierpiński-like fractals for $p \in (1, \infty)$, based on the framework of Kusuoka and Zhou [8]. The idea is to define the p-energy forms as the Γ-limit of discrete p-energies on graph partitions of the fractals. To show the existence of a good limit, a crucial step is to verify certain inequalities of effective conductances, which are called conditions (B1) and (B2) in Kusuoka and Zhou [8] when $p = 2$. In a recent work, Kigami [7, Definition 3.4] introduced a p-conductive homogeneity condition for $p > 1$ as the p-counterpart of [8, conditions (B1) and (B2)]; see the paragraph following [7, Definition 1.2, p. 6]. This p-conductive homogeneity condition plays an important role for some key properties of the Sobolev spaces $\mathcal{W}^{1,p}$ defined in [7].

It is a natural question if p-conductive homogeneity of a compact metric space K holds for some $p \in (1, \infty)$, then it holds for all $p \in (1, \infty)$. It is shown recently by Murugan and Shimizu [9, Theorem C.28] that p-conductive homogeneity holds for the standard planar Sierpiński carpet equipped with the self-similar measure with the equal weight for any $p \in (1, \infty)$, where the associated covering system is chosen to be the set of all pairs of cells of the same level that share a common border line. In this paper, we show, however, that this is not true for general compact metric spaces. We show that there are two fractals $F^{(2)}$ and $F^{(3)}$ in dimension 2 and 3, respectively, so that "p-conductive homogeneity" holds for $p = 2$ but fails for $p \in (1, 1 + \varepsilon)$ for

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some $\varepsilon \in (0,1)$ in the sense of Remark 1.2. The fractal $F^{(3)}$ has Ahlfors regular conformal dimension strictly larger than 2. To circumvent the issue about the correct definition of neighbor disparity constants which depend on the covering system used, we use capacity (effective conductance) estimates to show that for each of these two fractals, there is some $\varepsilon \in (0,1)$ so that p-conductive homogeneity cannot hold for $p \in (1, 1+\varepsilon)$ and for any of its covering systems. See [7, Definition 2.29] for the definition of covering system.

We now describe these two fractals $F^{(2)}$ and $F^{(3)}$ in detail. Fractal $F^{(2)}$ is an example of unconstrained planar Sierpiński carpets considered in Cao and Qiu [4], while $F^{(3)}$ is an example of unconstrained Sierpiński carpets in $\mathbb{R}^3$ studied in Cao and Qiu [5].

For $d \geq 1$, let $F_0^{(d)} := [0,1]^d$ be the unit cube in $\mathbb{R}^d$ and set $\mathcal{Q}_0^{(d)} := \{F_0^{(d)}\}$. For each integer $n \geq 1$, divide $F_0^{(d)}$ into 5^{nd} identical non-overlapping sub-cubes with side length 5^{-n}. Denote the collection of such cubes by $\mathcal{Q}_n^{(d)}$:

$$\mathcal{Q}_n^{(d)} := \left\{ \prod_{i=1}^{d} \left[(l_i - 1)/5^n, l_i/5^n \right] : 1 \leq l_i \leq 5^n, i = 1, \ldots, d \right\}. \tag{1.1}$$

For each $A \subset \mathbb{R}^d$ and $n \geq 0$, define

$$\mathcal{Q}_n^{(d)}(A) := \left\{ Q \in \mathcal{Q}_n^{(d)} : \mathrm{int}(Q) \cap A \neq \emptyset \right\}, \tag{1.2}$$

where $\mathrm{int}(Q)$ stands for the interior of the closed cube Q in $\mathbb{R}^d$.

Next, define $F_1^{(d)}$ by erasing from $F_0^{(d)}$ all cubes in $\mathcal{Q}_n^{(d)}$ that are attached to the center cube $[2/5, 3/5]^d$ with a $d-1$ dimensional face:

$$F_1^{(d)} := F_0^{(d)} \setminus \left(\bigcup_{i=1}^{d} (2/5, 3/5)^{i-1} \times \big((1/5, 2/5) \cup (3/5, 4/5) \big) \times (2/5, 3/5)^{d-i} \right).$$

See Figure 1 for the picture of $F_1^{(2)}$. Define $F_n^{(d)} := \bigcup_{Q \in \mathcal{Q}_1^{(d)}(F_1^{(d)})} \Psi_Q(F_{n-1}^{(d)})$ for $n \geq 2$, where, for each $Q \in \bigcup_{n=0}^{\infty} \mathcal{Q}_n^{(d)}$, Ψ_Q is the orientation preserving affine map from $F_0^{(d)}$ onto Q. The fractals that we are interested in are

$$F^{(d)} := \bigcap_{n=0}^{\infty} F_n^{(d)}$$

with $d = 2, 3$. See Figure 1 for a picture of an approximation of $F^{(2)}$. Note that $F^{(d)}$ is not a generalized Sierpiński carpet in the sense of [1, §2.2] as the interior connectedness condition (H2) there are not satisfied. Moreover, $F^{(2)}$ is not strongly connected in the sense of [7, Definition 10.2].

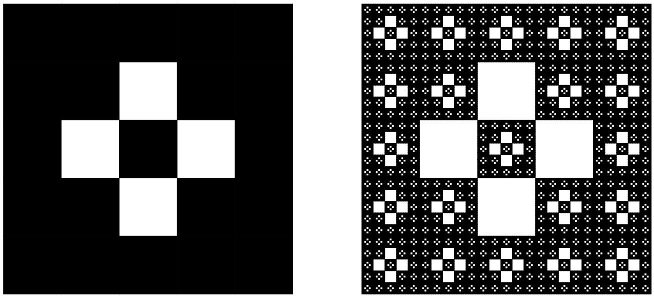

Figure 1. $F_1^{(2)}$ and $F^{(2)}$

Under the Euclidean metric, $F^{(d)}$ is a compact α-Ahlfors regular set with respect to the Hausdorff measure on $F^{(d)}$, where α is the Hausdorff dimension of $F^{(d)}$, that is,

$$\alpha = \dim_H F^{(d)} = \frac{\log(5^d - 2d)}{\log 5}. \tag{1.3}$$

We now introduce a natural partition of the metric measure space $(F^{(d)}, \mu)$, where μ is the normalized Hausdorff measure on $F^{(d)}$ so that $\mu(F^{(d)}) = 1$.

Partition of $F^{(d)}$. There is a natural partition in the sense of Kigami [7, Definition 2.3], explained as follows. Let $T = \bigcup_{n=0}^{\infty} \mathcal{Q}_n^{(d)}(F^{(d)})$, and let $\mathcal{A}$ be the subset of $T \times T$ such that $(Q, Q') \in \mathcal{A}$ if and only if $Q \subset Q'$ with $Q \in \mathcal{Q}_{n+1}^{(d)}(F^{(d)}), Q' \in \mathcal{Q}_n^{(d)}(F^{(d)})$ for some $n \geq 0$, or $Q' \subset Q$ with $Q \in \mathcal{Q}_n^{(d)}(F^{(d)})$, $Q' \in \mathcal{Q}_{n+1}^{(d)}(F^{(d)})$ for some $n \geq 0$. Then, $(T, \mathcal{A}, F_0^{(d)})$ is a rooted tree, where T is the set of vertices, $\mathcal{A}$ is the set of edges and $F_0^{(d)}$ is the root. We assign each $Q \in T$ the subset $\Psi_Q(F^{(d)})$ of $F^{(d)}$. One can check that $\{\Psi_Q(F^{(d)}); Q \in T\}$ is a partition of $F^{(d)}$ that satisfies [7, Assumption 2.15] with $M_* = M_0 = 1$.

We further define the discrete p-energy forms for $p \in (1, \infty)$ and effective p-conductances.

p-energy forms. For integers $d \geq 2$ and $n \geq 1$, define the discrete p-energy forms on $l(\mathcal{Q}_n^{(d)}(F^{(d)}))$ by

$$\mathcal{E}_p^n(f) = \frac{1}{2} \sum_{\substack{Q, Q' \in \mathcal{Q}_n^{(d)}(F^{(d)}) \\ Q \cap Q' \neq \emptyset}} \left| f(Q) - f(Q') \right|^p \quad \text{for each } f \in l\big(\mathcal{Q}_n^{(d)}(F^{(d)})\big).$$

Effective p-conductances. For each $n, m \geq 0$ and $A \subset \mathcal{Q}_n^{(d)}(F^{(d)})$, define

$$S^m(A) := \big\{ Q \in \mathcal{Q}_{n+m}^{(d)}(F^{(d)}) : Q \subset Q' \text{ for some } Q' \in A \big\}. \tag{1.4}$$

For $n \geq 1$ and disjoint $A_1, A_2 \subset \mathcal{Q}_n^{(d)}(F^{(d)})$, define

$$\mathcal{E}_{p,m}(A_1, A_2) := \inf\left\{\mathcal{E}_p^{n+m}(f) : f \in l\left(\mathcal{Q}_{n+m}^{(d)}(F^{(d)})\right),\ f|_{S^m(A_1)} = 1,\ f|_{S^m(A_2)} = 0\right\}.$$

For short, if $A_1 = \{Q\}$ for some $Q \in \mathcal{Q}_n^{(d)}(F^{(d)})$ and $A_2 \subset \mathcal{Q}_n^{(d)}(F^{(d)})$, we write $\mathcal{E}_{p,m}(Q, A_2)$ for $\mathcal{E}_{p,m}(\{Q\}, A_2)$.

We note that [7, Assumption 2.15(5)] is just [7, Assumption 2.7], and (1)–(4) of [7, Assumption 2.15] imply Assumptions 2.6, 2.10 and 2.12 of [7] by [7, Proposition 2.16]. As $F^{(d)}$ satisfies [7, Assumption 2.15] with partition $\{\Psi_Q(F^{(d)}); Q \in T\}$, we have the following from [7, Theorem 3.30].

Lemma 1.1. *For $p > 1$, if $F^{(d)}$ is p-conductive homogeneous with respect to some covering system in the sense of* [7, Definition 3.4]*, then the following holds.*
($\mathbf{A}_p$). *There exist some positive constants $\sigma > 0$ and $c_1, c_2 > 0$ so that, for each $n \geq 1, m \geq 0$ and $Q \in \mathcal{Q}_n^{(d)}(F^{(d)})$,*

$$c_1 \sigma^{-m} \leq \mathcal{E}_{p,m}(Q, \Gamma(Q)^c) \leq c_2 \sigma^{-m},$$

where $\Gamma(Q) := \{Q' \in \mathcal{Q}_n^{(d)}(F^{(d)}) : Q' \cap Q \neq \emptyset\}$ and $\Gamma(Q)^c := \mathcal{Q}_n^{(d)}(F^{(d)}) \setminus \Gamma(Q)$.

We remark that in the notation of [7], $\mathcal{E}_{1,p,m}(Q, \mathcal{Q}_n^{(d)}(F^{(d)})) = \mathcal{E}_{p,m}(Q, \Gamma(Q)^c)$ for each $n \geq 1$, $m \geq 0$ and $Q \in \mathcal{Q}_n^{(d)}(F^{(d)})$.

Remark 1.2. In [7, Definition 3.4], the definition of p-conductive homogeneity of a compact metric space (K, ρ) involves the class of neighbor disparity constants that depends on the covering system $\mathcal{J}$ used; cf. [7, Definition 2.29, p. 35].

In this paper, we say that p-conductive homogeneity fails for a compact metric space (K, ρ) if for any covering system $\mathcal{J}$ (in the sense of [7, Definition 2.29]), the corresponding p-conductive homogeneity condition for (K, ρ) fails. Otherwise, we say that p-conductive homogeneity condition holds for (K, ρ). ■

The following theorem is the main result of this paper, whose proof is given in the next section.

Theorem 1.3. *On $F^{(d)}$, property ($\mathbf{A}_p$) fails for $p \in \left(1, \frac{\log 10}{\log 5}\right)$ when $d = 2$, and fails for $p \in \left(1, \frac{\log 16}{\log 5}\right)$ when $d = 3$.*

Corollary 1.4. *For fractal $F^{(2)}$, the p-conductive homogeneity condition holds for $p > \dim_{AR}(F^{(2)}, \rho)$ with the covering system*

$$\mathcal{J}^{(2)} = \left\{\{Q, Q'\} : \{Q, Q'\} \subset \mathcal{Q}_n^{(2)}(F^{(2)}) \text{ for some } n \geq 1,\ Q \neq Q',\ Q \cap Q' \neq \emptyset\right\},$$

however p-conductive homogeneity condition fails for $p \in \left(1, \frac{\log 10}{\log 5}\right)$.

For $F^{(3)}$, p-conductive homogeneity condition holds for $p = 2$ with the covering system

$$\mathcal{J}^{(3)} = \big\{\{Q, Q'\} : \{Q, Q'\} \subset \mathcal{Q}_n^{(3)}(F^{(3)}) \text{ for some } n \geq 1,\ Q \neq Q',\ \#(Q \cap Q') > 1\big\},$$

however p-conductive homogeneity condition fails for $p \in \big(1, \frac{\log 16}{\log 5}\big)$.

Moreover,

$$\frac{\ln 10}{\ln 5} \leq \dim_{AR}(F^{(2)}, \rho) \leq \frac{\log 21}{\log 5} \quad \text{and} \quad \frac{\ln 80}{\ln 5} \leq \dim_{AR}(F^{(3)}, \rho) \leq \frac{\log 119}{\log 5}, \tag{1.5}$$

where $\dim_{AR}\big(F^{(d)}, \rho\big)$ stands for the Ahlfors regular conformal dimension of the metric space $F^{(d)}$ equipped with Euclidean metric ρ; see Remark 2.2 for its definition.

Proof. For $F^{(2)}$, the first claim is due to [4, Condition (B) and its proof in Section 4] for $p = 2$ and the same proof of [4] also works for $p > \dim_{AR}(F^{(2)}, \rho)$, while the second claim is a consequence of Lemma 1.1 and Theorem 1.3.

For $F^{(3)}$, the first claim is due to [5, Theorem 8.1], while the second claim is a consequence of Lemma 1.1 and Theorem 1.3.

Assertion (1.5) is proved in Remark 2.2. ■

Remark 1.5. When $d \geq 3$, we think the proof of [4] can be suitably modified to show that the p-conductive homogeneity holds for every $p > \dim_{AR}(F^{(d)}, \rho)$ with the covering system

$$\mathcal{J}^{(3)} = \big\{\{Q, Q'\} : \{Q, Q'\} \subset \mathcal{Q}_n^{(3)}(F^{(3)}) \text{ for some } n \geq 1,\ Q \neq Q',\ \#(Q \cap Q') > 1\big\}.$$

However, we do not pursue this extension in this paper. ■

Remark 1.6. The fractal $F^{(3)}$, in particular, gives an example of a compact metric space for which the p-conductive homogeneity condition holds for some $p = 2 < \dim_{AR}(F^{(3)}, \rho)$ and fails for any other $p \in \big(1, \frac{\log 16}{\log 5}\big)$, which is also smaller than $\dim_{AR}(F^{(3)}, \rho)$.

We can say more about the p-conductive homogeneity condition for $F^{(2)}$ by comparing it with a closely related generalized Sierpiński carpet $\widetilde{F}^{(2)}$ in $\mathbb{R}^2$ to be defined below. Let $\widetilde{F}_0^{(2)} = [0, 1]^2$,

$$\widetilde{F}_1^{(2)} = [0,1]^2 \setminus \left(\left(\frac{2}{5}, \frac{3}{5}\right) \times \left(\frac{1}{5}, \frac{4}{5}\right) \bigcup \left(\frac{2}{5}, \frac{3}{5}\right) \times \left(\frac{1}{5}, \frac{4}{5}\right)\right) \subset F_1^{(2)},$$

and let $\widetilde{F}_n^{(2)} = \bigcup_{Q \in \mathcal{Q}_1^{(2)}(\widetilde{F}_1^{(2)})} \Psi_Q(\widetilde{F}_{n-1}^{(2)})$ for $n \geq 2$. Here, as in the above, Ψ_Q is the orientation preserving affine map from $\widetilde{F}_0^{(2)}$ onto Q. Then $\widetilde{F}^{(2)} := \bigcap_{n=1}^{\infty} F_n^{(2)}$ is a generalized Sierpinski carpet in the sense of [1, §2.2] having Hausdorff dimension $\dim_H(\widetilde{F}^{(2)}) = \frac{\log 20}{\log 5}$. See Figure 2 for $\widetilde{F}^{(2)}$ and $\widetilde{F}_1^{(2)}$.

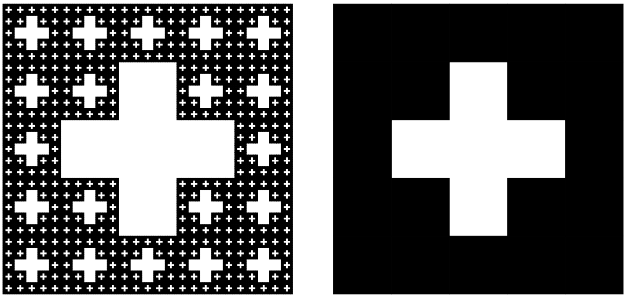

Figure 2. The generalized carpet F and its first level approximating F_1

Proposition 1.7. *Property* $(\mathbf{A}_p)$ *fails for* $F^{(2)}$ *for* $1 < p < \dim_{AR}(\widetilde{F}^{(2)}, \rho)$. *Consequently, the* p*-conductive homogeneity fails for* $F^{(2)}$ *for* $1 < p < \dim_{AR}(\widetilde{F}^{(2)}, \rho)$.

The proof of Proposition 1.7 is given in Section 2. We conclude this section with two open questions.

(i) It can be shown that $\dim_{AR}(F^{(2)}, \rho) \geq \dim_{AR}(\widetilde{F}^{(2)}, \rho)$ but we do not know if they are the same or not. We suspect they are. If they are, then Proposition 1.7 combined with Corollary 1.4 would imply that the p-conductive homogeneity holds on $F^{(2)}$ for $p > \dim_{AR}(F^{(2)}, \rho)$ but fails for $1 < p < \dim_{AR}(F^{(2)}, \rho)$.

(ii) Corollary 1.4 and Proposition 1.7 raise a natural question: if the p-conductive homogeneity holds on a compact metric space (K, ρ) for some $p > 1$, does the q-conductive homogeneity hold on (K, ρ) for any $q > p$? This looks quite plausible but we do not have a solution for it. The second part of Corollary 1.4 shows that on a compact metric space (K, ρ) the p-conductive homogeneity fails for some $p > 1$, the smallest q that the q-conductive homogeneity holds on (K, ρ) is in general different from the Ahlfors regular conformal dimension $\dim_{AR}(K, \rho)$ of K.

2. p-conductive homogeneity

In this section, we present the proof for Theorem 1.3 and Proposition 1.7. In the following two lemmas, we consider two cells Q_1 and Q_2, and deduce some estimates of the effective p-conductances. For $Q \in \mathcal{Q}_n^{(d)}(F^{(d)})$, recall the definition of $\Gamma(Q)$ from Lemma 1.1. For each $n, m \geq 0$ and $A \subset \mathcal{Q}_n^{(d)}(F^{(d)})$, recall the definition of $S^m(A)$ from (1.4).

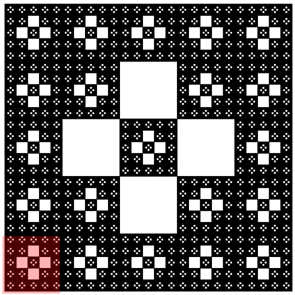

Figure 3. Q_1 marked red

Lemma 2.1. *Let $Q_1 = [0, 1/5]^d$ (see Figure 3 for an illustration). Then*

$$\mathcal{E}_{p,m}(Q_1, \Gamma(Q_1)^c) \geq 2^m (5^m + 1)^{1-p} \qquad \text{for } d = 2 \text{ and } m \geq 1,$$
$$\mathcal{E}_{p,m}(Q_1, \Gamma(Q_1)^c) \geq 16^m (5^m + 1)^{1-p} \qquad \text{for } d = 3 \text{ and } m \geq 1.$$

Proof. For $d \geq 1$, we define $G_1^{(d)}$ as

$$G_1^{(d)} := \bigcup \{Q \in \mathcal{Q}_1^{(d)}(F_1^{(d)}) : Q \cap \partial F_0^{(d)} \neq \emptyset\},$$

and inductively, define for $n \geq 2$,

$$G_n^{(d)} = \bigcup_{Q \in \mathcal{Q}_1^{(d)}(G_1^{(d)})} \Psi_Q(G_{n-1}^{(d)}) \text{ and } G^{(d)} = \bigcap_{n=0}^{\infty} G_n^{(d)}.$$

Note that when $d = 1$, $G^{(d)}$ is a Cantor set; when $d = 2$, $G^{(d)}$ is a generalized Sierpiński carpet in the sense of [1, §2.2]. Moreover,

$$[0, 1] \times G^{(d-1)} \subset F^{(d)} \quad \text{for } d \geq 2.$$

Let $h \in l(\mathcal{Q}_{m+1}^{(d)}(F^{(d)}))$ be a function that satisfies

$$h|_{S^m(Q_1)} = 1 \quad \text{and} \quad h|_{S^m(\Gamma(Q_1)^c)} = 0.$$

For each $\widetilde{Q} \in \mathcal{Q}_{m+1}^{(d-1)}(G^{(d-1)} \cap [0, 1/5]^{d-1})$, consider the path of cells

$$Q_{\widetilde{Q},i} = \left[\frac{5^m - 1 + i}{5^{m+1}}, \frac{5^m + i}{5^{m+1}}\right] \times \widetilde{Q} \quad \text{for } i = 0, 1, \dots, 5^m + 1.$$

In particular, one can check that $Q_{\widetilde{Q},0} \in S^m(Q_1)$, $Q_{\widetilde{Q},5^m+1} \in S^m(\Gamma(Q_1)^c)$ and $Q_{\widetilde{Q},i} \in \mathcal{Q}_{m+1}^{(d)}([0, 1] \times G^{(d-1)}) \subset \mathcal{Q}_{m+1}^{(d)}(F^{(d)})$ for each $i = 0, 1, \dots, 5^{m+1}$. Then, we have

$$\begin{aligned}\mathcal{E}_p^{m+1}(h) &\geq \sum_{\widetilde{Q}\in\mathcal{Q}_{m+1}^{(d-1)}(G^{(d-1)}\cap[0,1/5]^{d-1})} \sum_{i=0}^{5^m} \left|h(Q_{\widetilde{Q},i+1}) - h(Q_{\widetilde{Q},i})\right|^p \\ &\geq \#\mathcal{Q}_{m+1}^{(d-1)}(G^{(d-1)}\cap[0,1/5]^{d-1})\cdot(5^m+1)^{1-p},\end{aligned}$$

where in the second inequality we used that $h(Q_{\widetilde{Q},0}) = 1$ and $h(Q_{\widetilde{Q},5^m+1}) = 0$ for each $\widetilde{Q} \in \mathcal{Q}_{m+1}^{(d-1)}(G^{(d-1)} \cap [0,1/5]^{d-1})$ and Hölder's inequality that

$$(M+1)^{-1}\Big|\sum_{k=0}^{M} a_k\Big| \leq \Big(\sum_{k=0}^{M} |a_k|^p/(M+1)\Big)^{1/p}.$$

The lemma follows after noticing that $\#\mathcal{Q}_{m+1}^{(d-1)}(G^{(d-1)} \cap [0,1/5]^{d-1}) = 2^m$ when $d = 2$, and $\#\mathcal{Q}_{m+1}^{(d-1)}(G^{(d-1)} \cap [0,1/5]^{d-1}) = 16^m$ when $d = 3$. ∎

Remark 2.2. Lemma 2.1 has the following geometric implication. By [6, Theorems 4.7.6 and 4.9.1], for $F^{(d)}$ with $d = 2, 3$ and the Euclidean metric ρ, we know that

$$\limsup_{m\to\infty} \max_{Q\in\bigcup_{n\geq 1}\mathcal{Q}_n^{(d)}(F^{(d)})} \big(\mathcal{E}_{p,m}(Q,\Gamma(Q)^c)\big)^{1/m} < 1 \quad \text{iff} \quad p > \dim_{AR}(F^{(d)},\rho),$$

where $\dim_{AR}(F^{(d)},\rho)$ is the Ahlfors regular conformal dimension of the metric space $(F^{(d)},\rho)$, that is,

$$\begin{aligned}\dim_{AR}(F^{(d)},\rho) := \inf\{\alpha :\ &\text{there exists a metric } \rho' \text{ on } F^{(d)} \text{ that is quasi-symmetric} \\ &\text{to } \rho \text{ and a Borel regular measure } \mu' \text{ that is } \alpha\text{-Ahlfors} \\ &\text{regular with respect to } \rho'\}\end{aligned}$$

This, together with Lemma 2.1, implies that

$$\dim_{AR}(F^{(2)},\rho) \geq 1 + \frac{\ln 2}{\ln 5} = \frac{\ln 10}{\ln 5} \quad \text{and} \quad \dim_{AR}(F^{(3)},\rho) \geq 1 + \frac{\ln 16}{\ln 5} = \frac{\ln 80}{\ln 5}.$$

The lower bounds above can also be proved by using the fact

$$\dim_{AR}([0,1]\times Z,\rho) = \dim_H([0,1]\times Z,\rho)$$

for any Borel $Z \subset \mathbb{R}^{d-1}$, $d \geq 2$, see [2, Remark 1].

On the hand, since $(F^{(d)},\rho)$ is $\dim_H F^{(d)}$-Ahlfors regular with the Hausdorff measure on $F^{(d)}$, we have by the definition of Ahlfors regular conformal dimension and (1.3) that

$$\dim_{AR}(F^{(d)},\rho) \leq \dim_H(F^{(d)},\rho) = \frac{\log(5^d - 2d)}{\log 5}.$$

Consequently, we get

$$\frac{\ln 10}{\ln 5} \le \dim_{AR}(F^{(2)}, \rho) \le \frac{\log 21}{\log 5} \quad \text{and} \quad \frac{\ln 80}{\ln 5} \le \dim_{AR}(F^{(3)}, \rho) \le \frac{\log 119}{\log 5}. \quad \blacksquare$$

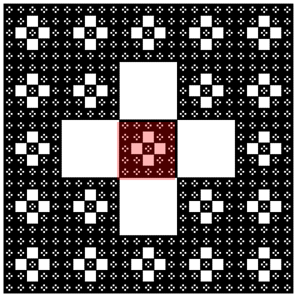

Figure 4. Q_2 marked red

Lemma 2.3. *Let $Q_2 = [2/5, 3/5]^d$ (see Figure 4 for an illustration). We have*

$$\mathcal{E}_{p,m}(Q_2, \Gamma(Q_2)^c) \le 4 \qquad \text{for } d = 2, \text{ and } m \ge 1,$$
$$\mathcal{E}_{p,m}(Q_2, \Gamma(Q_2)^c) \le 7(12 \cdot 5^m - 16) \qquad \text{for } d = 3 \text{ and } m \ge 1.$$

Proof. Define $f \in l(\mathcal{Q}^{(d)}_{m+1}(F^{(d)}))$ by

$$f(Q) = \begin{cases} 1 & \text{if } Q \in \mathcal{Q}^{(d)}_{m+1}(F^{(d)} \cap Q_2), \\ 0 & \text{if } Q \in \mathcal{Q}^{(d)}_{m+1}(F^{(d)} \setminus Q_2). \end{cases}$$

Then

$$\mathcal{E}_{p,m}(Q_2, \Gamma(Q_2)^c) \le \mathcal{E}_p^{m+1}(f) = \sum_{Q \in S^m(Q_2)} \sum_{\substack{Q' \in S^m(\Gamma(Q_2)^c) \\ Q \cap Q' \ne \emptyset}} \left| f(Q) - f(Q') \right|^p.$$

Furthermore, for $d = 2$, there are only 4 $Q \in S^m(Q_2)$ such that $Q' \cap Q \ne \emptyset$ for some $Q' \in S^m(\Gamma(Q_2)^c)$, and each Q intersects exactly with one such Q', so we have $\mathcal{E}_{p,m}(Q, \Gamma(Q)^c) \le \mathcal{E}_p^{m+1}(f) = 4$; for $d = 3$, there are $12 \cdot 5^m - 16$ $Q \in S^m(Q_2)$ such that $Q' \cap Q \ne \emptyset$ for some $Q' \in S^m(\Gamma(Q_2)^c)$ (that is all the cells attached to the boundary edges of the Q), and each Q intersects with at most $7 = 2^3 - 1$ Q', so we have $\mathcal{E}_{p,m}(Q_2, \Gamma(Q_2)^c) \le \mathcal{E}_p^{m+1}(f) \le 7(12 \cdot 5^m - 16)$. $\blacksquare$

Proof of Theorem 1.3. For $d = 2$, when $1 < p < 1 + \frac{\log 2}{\log 5}$, we have

$$\lim_{m \to \infty} \frac{\mathcal{E}_{p,m}(Q_1, \Gamma(Q_1)^c)}{\mathcal{E}_{p,m}(Q_2, \Gamma(Q_2)^c)} = \infty, \tag{2.1}$$

where $Q_1, Q_2 \in \mathcal{Q}_1^{(2)}(F^{(2)})$ are the cells in the statements of Lemmas 2.1 and 2.3, respectively. Hence, $(\mathbf{A}_p)$ cannot hold for $p \in (1, \frac{\log 10}{\log 5})$. For $d = 3$, when $1 < p < \frac{\log 16}{\log 5}$, we have that equation (2.1) holds, where $Q_1, Q_2 \in \mathcal{Q}_1^{(3)}(F^{(3)})$ are the cells of Lemmas 2.1 and 2.3. Hence, $(\mathbf{A}_p)$ cannot hold for $p \in (1, \frac{\log 16}{\log 5})$. ■

Proof of Proposition 1.7. For each $p > 1$, define the discrete p-energy form $\widetilde{\mathcal{E}}_p^m$ on $\mathcal{Q}_m^{(2)}(\widetilde{F}^{(2)})$ by

$$\widetilde{\mathcal{E}}_p^m(f) = \sum_{\substack{Q,Q' \in \mathcal{Q}_m^{(2)}(\widetilde{F}^{(2)}) \\ Q \cap Q' \neq \emptyset}} \left|f(Q) - f(Q')\right|^p.$$

Let ν be the normalized Hausdorff measure on $\widetilde{F}^{(2)}$ such that $\nu(\widetilde{F}^{(2)}) = 1$. For $n \geq 0$ and $f \in L^p(\widetilde{F}^{(2)}; \nu)$, define $P_n f \in l(\mathcal{Q}_n^{(2)}(\widetilde{F}^{(2)}))$ by

$$P_n f(Q) = \frac{1}{\nu\big(Q \cap \widetilde{F}^{(2)}\big)} \int_{\widetilde{F}^{(2)} \cap Q} f(w)\nu(dw) \quad \text{for each } Q \in \mathcal{Q}_n^{(2)}(\widetilde{F}^{(2)}).$$

For each $n \geq 1$, $m \geq 0$ and $Q \in \mathcal{Q}_n^{(2)}(\widetilde{F}^{(2)})$, we define

$$\begin{aligned}\widetilde{\mathcal{E}}_{p,m}\big(Q, \Gamma(Q)^c\big) := \inf\Big\{&\widetilde{\mathcal{E}}_p^{m+n}(P_{m+n} f) : f \in L^p(\widetilde{F}^{(2)}; \nu),\ f|_Q = 1,\ f|_{Q'} = 0, \\ &\text{for each } Q' \in \mathcal{Q}_n^{(2)}(\widetilde{F}^{(2)}) \text{ such that } Q' \cap Q = \emptyset\Big\}.\end{aligned}$$

According to [9, Theorem 10.2 and Remark 10.20], there is some $\widetilde{\sigma}_p > 0$ so that

$$\widetilde{\mathcal{E}}_{p,m}\big(Q, \Gamma(Q)^c\big) \asymp \widetilde{\sigma}_p^{-m} \text{ for every } m \geq 1,\ Q \in \mathcal{Q}_n^{(2)}(\widetilde{F}^{(2)}). \tag{2.2}$$

Moreover, for $Q_1 = [0, 1/5] \times [0, 1/5]$ and $p > 1$, by [9, Theorem 6.17, Theorem 10.2 and Remark 10.20], we can find $f_p \in C(\widetilde{F}^{(2)})$ such that $f_p|_{Q_1} = 1,\ f_p|_{Q'} = 0$ for each $Q' \in \mathcal{Q}_1^{(2)}(\widetilde{F}^{(2)})$ with $Q' \cap Q_1 = \emptyset$ and that

$$\widetilde{\mathcal{E}}_p^m(P_m f_p) \asymp \widetilde{\sigma}_p^{-m} \quad \text{for all } m \geq 1.$$

As a consequence, we have for each $p > p' > 1$,

$$\frac{\widetilde{\sigma}_p^{-m}}{\widetilde{\sigma}_{p'}^{-m}} \lesssim \frac{\widetilde{\mathcal{E}}_p^m(P_m f_{p'})}{\widetilde{\mathcal{E}}_{p'}^m(P_m f_{p'})} \leq \sup_{\substack{Q,Q' \in \mathcal{Q}_m^{(2)}(\widetilde{F}^{(2)}) \\ Q \cap Q' \neq \emptyset}} |P_m f_{p'}(Q) - P_m f_{p'}(Q')|^{p-p'} \to 0$$

as $m \to \infty$. This implies that $\widetilde{\sigma}_p$ is strictly increasing in $p \in (1, \infty)$. Moreover, we can easily check that $\widetilde{F}^{(2)}$ satisfies [7, Assumption 2.15] with $M_* = 1$. Hence, by (2.2) and [7, Proposition 3.3], we know that $\widetilde{\sigma}_p > 1$ if and only if $p > \dim_{AR}(\widetilde{F}^{(2)}, \rho)$, which together with the fact that $\widetilde{\sigma}_p$ is strictly increasing yields $\widetilde{\sigma}_p < 1$ if $p < \dim_{AR}(\widetilde{F}^{(2)}, \rho)$.

Noticing that $\mathcal{Q}^{(2)}_{m+1}(\widetilde{F}^{(2)}) \subset \mathcal{Q}^{(2)}_{m+1}(F^{(2)})$ for each $m \geq 0$, we observe that for $1 < p < \dim_{AR}(\widetilde{F}^{(2)}, \rho)$,

$$\lim_{m\to\infty} \mathcal{E}_{p,m}\big(Q_1, \Gamma(Q_1)\big) \geq \lim_{m\to\infty} \widetilde{\mathcal{E}}_{p,m}\big(Q_1, \Gamma(Q_1)\big) \gtrsim \lim_{m\to\infty} \widetilde{\sigma}_p^{-m} = \infty. \tag{2.3}$$

This result, together with Lemma 2.3, implies that property ($\mathbf{A}_p$) fails for $F^{(2)}$ when $1 < p < \dim_{AR}(\widetilde{F}^{(2)}, \rho)$. ∎

Remark 2.4. Based on the same idea, we can introduce many more examples similar to $\widetilde{F}^{(2)}$. For instance, the compact space $\mathcal{X} := [-1, 0]^2 \cup [0, 1]^2$ equipped with the Euclidean metric and the Lebesgue measure is p-conductively homogeneous for any $p > 2$ if we assign it with the natural partition into $2 \cdot 3^n$ squares for $n \geq 0$, but it is not p-conductively homogeneous for $p < 2$. Such a metric space is a typical example of bow-ties, which are metric spaces constructed by gluing of two metric spaces at a point. A characterization of (p, q)-Poincaré inequalities on bow-ties in terms of their validity on component metric spaces is given in [3]. In particular, on the compact space $\mathcal{X} = [-1, 0]^2 \cup [0, 1]^2$ equipped with the Lebesgue measure, we know by [3, Theorem 1.1] (via (i), (ii) and (iii’) there) that $(1, p)$-Poincaré inequality holds when $p > 2$, and fails when $p \in (1, 2]$. This critical phenomenon is similar in spirit to the validity of p-conductive homogeneity on this bow-tie $\mathcal{X}$.

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