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A perturbed-Alexander invariant

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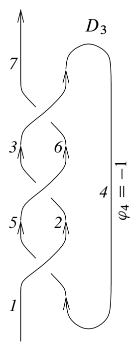
Abstract. In this note, we give concise formulas, which lead to a simple and fast computer program that computes a powerful knot invariant. This invariant ρ_1 is not new; yet our formulas are by far the simplest and fastest. Given a knot, we write one of the standard matrices, A, whose determinant is its Alexander polynomial; yet instead of computing the determinant, we consider a certain quadratic expression in the entries of A^{-1}. The proximity of our formulas to the Alexander polynomial suggests that they should have a topological explanation, which we do not have yet.

Dedicated to the memory of V. F. R. Jones, 1952–2020, a friend and a mentor

1. The formulas

One of the selling points for this article is that the formulas in it are concise. Thus, we start by running through these formulas for a knot invariant ρ_1 as quickly as we can. In Section 2, we turn the formulas into a short, yet very fast computer program. In Section 3, we give a partial interpretation of the formulas in terms of car traffic on a knot diagram and use it to prove the invariance of ρ_1, and in Section 4, we quickly sketch the context: Alexander, Burau, Jones, Melvin, Morton, Rozansky, Overbay, and our own prior work. This article accompanies two talks [9, 10] (videos and handouts available).



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Given an oriented n-crossing knot K, we draw it in the plane as a long knot diagram D in such a way that the two strands intersecting at each crossing are pointed up (that is always possible because we can always rotate crossings as needed) and so that at its beginning and at its end the knot is oriented upward. We call such a diagram an *upright knot diagram.* An example of an upright knot diagram D_3 is shown above.

We then label each edge of the diagram with two integer labels: a running index k, which runs from 1 to $2n+1$, and a "rotation number" φ_k, the geometric rotation number of that edge (the signed number of times the tangent to the edge is horizontal and heading right, with cups counted with $+1$ signs and caps with -1; this number is well defined because at their ends, all edges are headed up). On the right, the running index runs from 1 to 7, and the rotation numbers for all edges are 0 (and hence are omitted) except for φ_4, which is -1.

A technicality. Some Reidemeister moves create or lose an edge, and to avoid the need for renumbering, it is beneficial to also allow labelling the edges with non-consecutive labels. Hence, we allow that and write i^+ for the successor of the label i along the knot and i^{++} for the successor of i^+. (These are $i+1$ and $i+2$ if the labelling is by consecutive integers.) Also, "1" will always refer to the label of the first edge, and "$2n+1$" will always refer to the label of the last.

We let A be the $(2n+1)\times(2n+1)$ matrix of Laurent polynomials in the formal variable T defined by

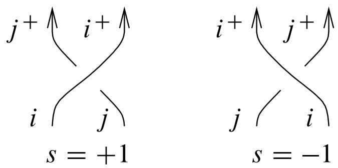
$$A := I - \sum_c \big(T^s E_{i,i^+} + (1-T^s)E_{i,j^+} + E_{j,j^+}\big),$$

where I is the identity matrix and $E_{\alpha\beta}$ denotes the elementary matrix with 1 in row α and column β and zeros elsewhere. The summation is over the crossings c of the diagram D, and once c is chosen, s denotes its sign and i and j denote the labels below the crossing where the label i belongs to the over-strand and j to the under-strand.

Alternatively,

$$A = I + \sum_c A_c,$$

where A_c is a matrix of zeros except for the blocks as follows:



$\longrightarrow$

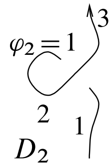
A_c	column i^+	column j^+
row i	$-T^s$	T^s-1
row j	0	-1

(1.1)

For example, if

$$D = D_1 = \uparrow$$

is the diagram with no crossings, the resulting matrix A is the 1×1 identity matrix (1). If $D = D_2$ is the diagram



(here $s = +1$, $(i, j) = (2, 1)$, and $(i^+, j^+) = (3, 2)$), then

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & T-1 & -T \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & T & -T \\ 0 & 0 & 1 \end{pmatrix},$$

and for D_3 as on the first page, we have

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We note without supplying details that the matrix A comes in a straightforward way from Fox calculus as it is applied to the Wirtinger presentation of the fundamental group of the complement of K (using the diagram D). Hence, the determinant of A is equal up to a unit to the normalised Alexander polynomial Δ of K (which satisfies $\Delta(T) = \Delta(T^{-1})$ and $\Delta(1) = 1$). In fact, we have that

$$\Delta = T^{(-\varphi(D) - w(D))/2} \det(A), \tag{1.2}$$

where $\varphi(D) := \sum_k \varphi_k$ is the total rotation number of D and where $w(D) = \sum_c s_c$ is the writhe of D, namely, the sum of the signs s_c of all the crossings c in D.

For our example D_2, $\det(A) = T$, $\varphi(D) = 1$, and $w(D) = 1$, so $\Delta = T^{(-1-1)/2} \cdot T = 1$, as expected for a diagram of the unknot. For D_3, $\det(A) = 1 - T + T^2$, $\varphi(D) = -1$, and $w(D) = 3$, so $\Delta = T^{(1-3)/2}(1 - T + T^2) = T - 1 + T^{-1}$, as expected for the trefoil knot.

We set[1] $G = (g_{\alpha\beta}) = A^{-1}$, and taking our inspiration from physics, we name $g_{\alpha\beta}$ the *Green function* for the diagram D. For our three examples D_1, D_2, and D_3, the

[1]At $T = 1$, the matrix A has 1's on the main diagonal, (-1)'s on the diagonal above it, and 0's everywhere else. Hence, A is invertible at $T = 1$ and hence over the field of rational functions.

Green function G is, respectively,

$$(1), \quad \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \frac{T^3-T^2+T}{T^2-T+1} & 1 & \frac{T^3-T^2+T}{T^2-T+1} & 1 & \frac{T^3-T^2+T}{T^2-T+1} & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{T-T^2}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.3}$$

We can now define our invariant ρ_1. It is the sum of two sums. The first is a sum of a term $R_1(c)$ over all crossings c in D, where for such a crossing we let s denote its sign and we let i and j denote the edge labels of the incoming over- and under-strands, respectively, and where

$$R_1(c) := s(g_{ji}(g_{j^+,j} + g_{j,j^+} - g_{ij}) - g_{ii}(g_{j,j^+} - 1) - 1/2). \tag{1.4}$$

The second sum is a sum over the edges k of D of a correction term dependent on the rotation number φ_k. We multiply the result by Δ^2 to "clear the denominators"[2]:

$$\rho_1 := \Delta^2\Big(\sum_c R_1(c) - \sum_k \varphi_k(g_{kk} - 1/2)\Big). \tag{1.5}$$

Direct calculations show that $\rho_1(D_1) = 0$ (as the sums are empty), $\rho_1(D_2) = 0$, and $\rho_1(D_3) = -T^2 + 2T - 2 + 2T^{-1} - T^{-2}$.

Theorem 1 ("Invariance", proofs in Section 3). *The quantity ρ_1 is a knot invariant.*

As we will see in the next section, ρ_1 has more separation power than the Jones polynomial, yet it is closer to the more topologically meaningful Alexander polynomial Δ: it is cooked up from the same matrix A, and in terms of computational complexity, computing ρ_1 is not very different from computing Δ. In order to compute Δ, we need to compute the determinant of A, while to compute ρ_1, we need to invert A and then compute a sum of $O(n)$ terms that are quadratic in the entries

[2] $R_1(s)$ is quadratic in the entries of G, and hence, it has denominators proportional to Δ^2.

of A^{-1}.[3] We have computed ρ_1 for knots with over 200 crossings using the unsophisticated implementation presented in Section 2.

Topologists should be intrigued! ρ_1 is derived from the same matrix as the Alexander polynomial Δ, yet we have no topological interpretation for ρ_1.

2. Implementation and power

Two of the main reasons we like ρ_1 is that it is very easy to implement and even an unsophisticated implementation runs very fast. To highlight these points, we include a full implementation here, a step-by-step run-through, and a demo run. We write in Mathematica [39], and you can find the notebook displayed here at [16, APAI.nb].

We start by loading the library `KnotTheory‘` [17]. (It is used here only for the list of knots that it contains and to compute other invariants for comparisons.) We also load a minor conversion routine [16, Rot.nb / Rot.m] whose internal workings are irrelevant here.

```
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from http://drorbn.net/APAI to compute rotation numbers.
```

2.1. The program

This done, here is the full ρ_1 program.

```
R₁[s_, i_, j_] := s (g_{ji} (g_{j⁺,j} + g_{j,j⁺} - g_{ij}) - g_{ii} (g_{j,j⁺} - 1) - 1/2);
ρ[K_] := ρ[K] = Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :> (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1 ; 0  -1 ))];
   Δ = T^((-Total[φ] - Total[Cs⟦All,1⟧])/2) Det[A];
   G = Inverse[A];
   ρ1 = Sum_{k=1}^{n} R₁ @@ Cs⟦k⟧ - Sum_{k=1}^{2 n} φ⟦k⟧ (g_{kk} - 1/2);
   Factor@{Δ, Δ^2 ρ1 /. α_⁺ :> α + 1 /. g_{α_,β_} :> G⟦α, β⟧}];
```

[3]We prefer not to be more specific about the complexity of computing ρ_1. It is the same as the complexity of inverting A, and matrix inversion is poly-time, with a rather small exponent, even for matrices with entries in a ring of polynomials (e.g., [38]). We have not explored how much one can further gain by exploiting the fact that A is very sparse.

The program uses mostly the same symbols as the text, so even without any knowledge of Mathematica, the reader should be able to recognise at least formulas (1.1), (1.2), and (1.5) within it. As a further hint, we add that the variable `Cs` ends up storing the list of crossings in a knot K, where each crossing is stored as a triple (s, i, j), where s, i, and j have the same meaning as in (1.1). The conversion routine `Rot` automatically produces `Cs`, as well as a list φ of rotation numbers, given any other knot presentation known to the package ``KnotTheory` ``.

Note that the program outputs the ordered pair (Δ, ρ_1). The Alexander polynomial Δ is anyway computed internally, and we consider the aggregate (Δ, ρ_1) as more interesting than any of its pieces by itself.

2.2. A step-by-step run-through

We start by setting K to be the knot diagram on page 1 using the `PD` notation of ``KnotTheory` `` [17]. We then print `Rot[K]`, which is a list of crossings followed by a list of rotation numbers:

```
K = PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]];
Rot[K]
```

```
{{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}, {0, 0, 0, -1, 0, 0}}
```

Next, we set `Cs` and φ to be the list of crossings and the list of rotation numbers, respectively.

```
{Cs, φ} = Rot[K]
```

```
{{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}, {0, 0, 0, -1, 0, 0}}
```

We set `n` to be the number of crossings, `A` to be the $(2n + 1)$-dimensional identity matrix, and then we iterate over `c` in `Cs`, adding a block as in (1.1) for each crossing:

```
n = Length[Cs];
A = IdentityMatrix[2 n + 1];
Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1 ))];
                                                          (  0      -1   )
```

Here is what `A` comes out to be:

```
A // MatrixForm
```

$$\begin{pmatrix} 1 & -T & 0 & 0 & -1+T & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & -1+T \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1+T & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We set Δ to be the determinant of `A`, with a correction as in (1.2). So, Δ is the Alexander polynomial of K.

```
Δ = T^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A]
```

$$\frac{1-T+T^2}{T}$$

`G` is now the `Inverse` of `A`:

```
G = Inverse[A];
G // MatrixForm
```

$$\begin{pmatrix} 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 \\ 0 & 1 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{T-T^2}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It remains to blindly follow the two parts of equation (1.5):

```
ρ1 = Sum[R1 @@ Cs⟦k⟧, {k, 1, n}] - Sum[φ⟦k⟧ (g_kk - 1/2), {k, 1, 2n}]
```

$$-2+g_{4,4}-g_{1,1}\left(-1+g_{4,4^+}\right)-\left(-1+g_{2,2^+}\right)g_{5,5}-g_{3,3}\left(-1+g_{6,6^+}\right)+g_{2,5}\left(g_{2,2^+}-g_{5,2}+g_{2^+,2}\right)+g_{4,1}\left(-g_{1,4}+g_{4,4^+}+g_{4^+,4}\right)+g_{6,3}\left(-g_{3,6}+g_{6,6^+}+g_{6^+,6}\right)$$

We replace each $g_{\alpha\beta}$ with the appropriate entry of `G`:

```
Δ^2 ρ1 /. α_⁺ :→ α + 1 /. g_{α_,β_} :→ G⟦α, β⟧
```

$$\frac{\left(1-T+T^2\right)^2\left(-1+\frac{T}{\left(1-T+T^2\right)^2}-\frac{-1+\frac{1}{1-T+T^2}}{1-T+T^2}\right)}{T^2}$$

Finally, we output both Δ and ρ_1. We factor them just to put them in a nicer form

```
Factor@{Δ, Δ^2 ρ1 /. α_⁺ :→ α + 1 /. g_{α_,β_} :→ G⟦α, β⟧}
```

$$\left\{\frac{1-T+T^2}{T}, -\frac{(-1+T)^2\left(1+T^2\right)}{T^2}\right\}$$

2.3. A demo run

Here are Δ and ρ_1 of all the knots with up to 6 crossings (a table up to 10 crossings is printed in [15]):

```
TableForm[Table[Join[{K}, ρ[K]], {K, AllKnots[{3, 6}]}], TableAlignments → Center]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Knot[3, 1]	$\frac{1-T+T^2}{T}$	$\frac{(-1+T)^2\,(1+T^2)}{T^2}$
Knot[4, 1]	$-\frac{1-3T+T^2}{T}$	0
Knot[5, 1]	$\frac{1-T+T^2-T^3+T^4}{T^2}$	$\frac{(-1+T)^2\,(1+T^2)\,(2+T^2+2T^4)}{T^4}$
Knot[5, 2]	$\frac{2-3T+2T^2}{T}$	$\frac{(-1+T)^2\,(5-4T+5T^2)}{T^2}$
Knot[6, 1]	$-\frac{(-2+T)\,(-1+2T)}{T}$	$\frac{(-1+T)^2\,(1-4T+T^2)}{T^2}$
Knot[6, 2]	$-\frac{1-3T+3T^2-3T^3+T^4}{T^2}$	$\frac{(-1+T)^2\,(1-4T+4T^2-4T^3+4T^4-4T^5+T^6)}{T^4}$
Knot[6, 3]	$\frac{1-3T+5T^2-3T^3+T^4}{T^2}$	0

Some comments are in order.

- If $\bar{K}$ is the mirror of a knot K, then $\rho_1(\bar{K})(T) = -\rho_1(K)(T^{-1})$. Indeed, in (1.5), both $R_1(c)$ and φ_k flip sign under reflection in a plane perpendicular to the plane of the knot diagram and the matrix A, and hence, also all the $g_{\alpha\beta}$'s are the same except for the substitution $T \to T^{-1}$.
- ρ_1 seems to always be divisible by $(T-1)^2$ and seems to always be palindromic ($\rho_1(T) = \rho_1(T^{-1})$). We are not sure why this is so.
- The last properties taken together would imply that ρ_1 vanishes on amphicheiral knots, such as 4_1 and 6_3 above.

Next is one of our favourites, a knot from [24] (see Figure 1), which is a potential counterexample to the ribbon=slice conjecture [21]. It takes about two minutes to compute ρ_1 for this 48 crossing knot (note that Mathematica prints `Timing` information is seconds, and that this information is highly dependent on the CPU used, how loaded it is, and even on its temperature at the time of the computation):

$$\texttt{Timing@}\rho[\mathrm{EPD}[X_{14,1}, \bar{X}_{2,29}, X_{3,40}, X_{43,4}, \bar{X}_{26,5}, X_{6,95}, X_{96,7}, X_{13,8}, \bar{X}_{9,28}, X_{10,41}, X_{42,11}, \bar{X}_{27,12}, X_{30,15}, \bar{X}_{16,61}, \bar{X}_{17,72}, \bar{X}_{18,83}, X_{19,34}, \bar{X}_{89,20}, \bar{X}_{21,92}, \bar{X}_{79,22}, \bar{X}_{68,23}, \bar{X}_{57,24}, \bar{X}_{25,56}, X_{62,31}, X_{73,32}, X_{84,33}, \bar{X}_{50,35}, X_{36,81}, X_{37,70}, X_{38,59}, \bar{X}_{39,54}, X_{44,55}, X_{58,45}, X_{69,46}, X_{80,47}, X_{48,91}, X_{90,49}, X_{51,82}, X_{52,71}, X_{53,60}, \bar{X}_{63,74}, \bar{X}_{64,85}, \bar{X}_{76,65}, \bar{X}_{87,66}, \bar{X}_{67,94}, \bar{X}_{75,86}, \bar{X}_{88,77}, \bar{X}_{78,93}]]$$

$$\Big\{158.625, \Big\{-\frac{(-1+2T-T^2-T^3+2T^4-T^5+T^8)\,(-1+T^3-2T^4+T^5+T^6-2T^7+T^8)}{T^8},$$
$$\frac{1}{T^{16}}(-1+T)^2\,(5-18T+33T^2-32T^3+2T^4+42T^5-62T^6-8T^7+166T^8-242T^9+108T^{10}+132T^{11}-226T^{12}+148T^{13}-11T^{14}-36T^{15}-11T^{16}+148T^{17}-226T^{18}+132T^{19}+108T^{20}-242T^{21}+166T^{22}-8T^{23}-62T^{24}+42T^{25}+2T^{26}-32T^{27}+33T^{28}-18T^{29}+5T^{30})\Big\}\Big\}$$

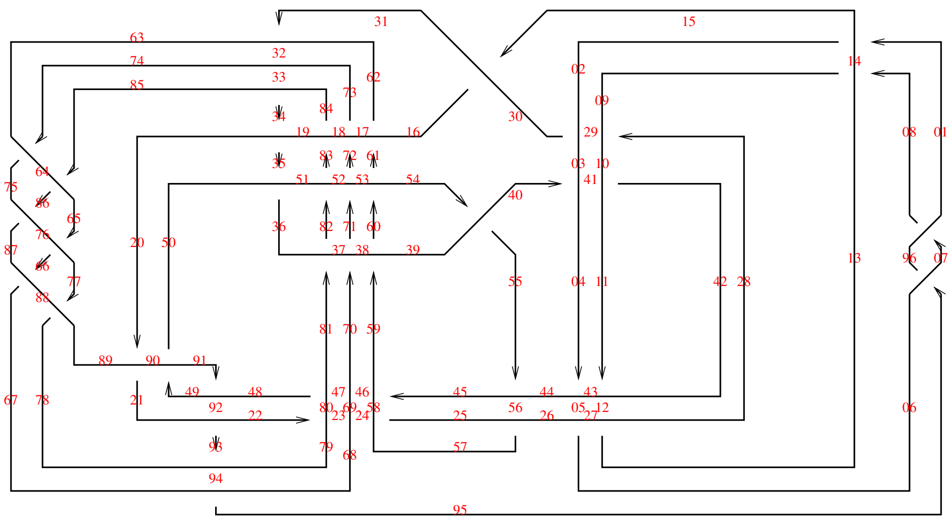


Figure 1. A 48-crossing knot from [24].

2.4. The separation power of ρ_1

Let us check how powerful ρ_1 is on knots with up to 12 crossings:

```
{NumberOfKnots[{3, 12}],
 Length@Union@Table[ρ[K], {K, AllKnots[{3, 12}]}],
 Length@Union@Table[{HOMFLYPT[K], Kh[K]}, {K, AllKnots[{3, 12}]}]}
```

```
{2977, 2882, 2785}
```

So, the pair (Δ, ρ_1) attains 2,882 distinct values on the 2,977 prime knots with up to 12 crossings (a deficit of 95), whereas the pair $(H, Kh) =$ (HOMFLYPT polynomial, Khovanov homology) attains only 2,785 distinct values on the same knots (a deficit of 192).

In our spare time, we computed all of these invariants on all the prime knots with up to 14 crossings. On these 59,937 knots, the pair (Δ, ρ_1) attains 53,684 distinct values (a deficit of 6,253) whereas the pair (H, Kh) attains only 49,149 distinct values on the same knots (a deficit of 10,788).

Hence, the pair (Δ, ρ_1), computable in polynomial time by simple programs, seems stronger than the pair (H, Kh), which is more difficult to program and (for all we know) cannot be computed in polynomial time. We are not aware of another poly-time invariant as strong as the pair (Δ, ρ_1).

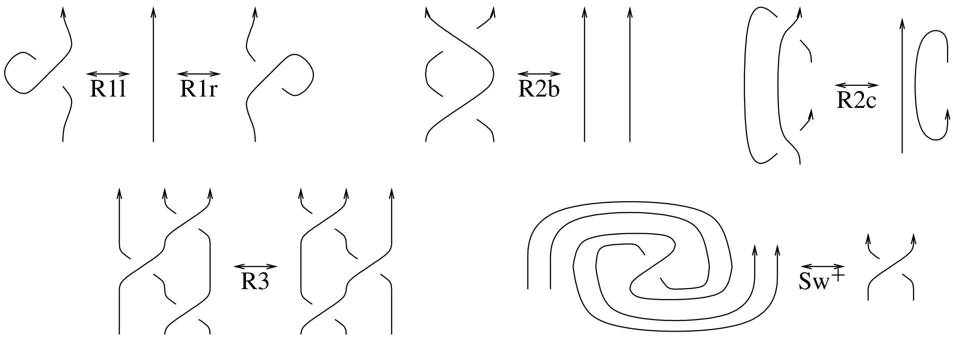


Figure 2. The upright Reidemeister moves: Reidemeister 1 left and right, Reidemeister 2 braid-like and cyclic, Reidemeister 3, and (the $+$) Swirl.

3. Proofs of Theorem 1, the invariance theorem

We tell the proof of the invariance theorem (Theorem 1) in two ways: an elegant and intuitive though slightly lacking telling in Section 3.2, and a complete though slightly dull telling in Section 3.3. But first, a few common elements are given.

3.1. Common elements

Two upright knot diagrams are considered the same (as diagrams) if their underlying knot diagrams are the same and if the respective rotation numbers of their edges are all the same. It is clear that ρ_1 is well defined on upright knot diagrams. To prove Theorem 1 we need to know what to prove. Namely, when do two upright knot diagrams represent the same knot? This is answered in the spirit of the classical Reidemeister theorem by the following:

Theorem 2 ("Upright Reidemeister"). *Two upright knot diagrams represent the same knot if and only if they differ by a sequence of R1l, R2r, R2b, R2c, R3, and* Sw^+ *moves as in Figure 2.*

Sketch of the proof. In the case of round knots (i.e., not "long"), knot diagrams can be turned upright by rotating individual crossings. The only ambiguity here is by powers of the full rotation, the *swirls* Sw^+ and Sw^- (where Sw^- is the same as Sw^+ except with a negative crossing, and we do not need to impose it separately as it follows from Sw^+ and R2). Hence, we have a well-defined map from {knot diagrams} to {upright knot diagrams}$/\mathrm{Sw}^\pm$. It remains to write the usual Reidemeister moves between knot diagrams as moves between upright knot diagrams. The result is the moves R1l, R2r, R2b, R2c, and R3. Note that unoriented knot theory is presented with just three Reidemeister moves, but these split into several versions in the oriented case. The sufficiency of the versions we picked can be found in [33]. In the case of long knots, a minor further complication arises, regarding the rotation numbers of the initial and final edges. We leave the details of the problem and its resolution to the reader. ■

Our key formulas, (1.4) and (1.5) involve the Green function $g_{\alpha\beta}$. We need to know that it is subject to some relations, the g-*rules* of Lemma 3 below, whose proof is so easy that it comes first.

Proof of Lemma 3. The first set of g-rules reads out column β of the equality

$$AG = I,$$

and the second set of g-rules reads out row α of the equality $GA = I$. ■

Lemma 3 (“g-rules”). *Given a fixed upright knot diagram D, its corresponding matrix A, and its inverse $G = (g_{\alpha\beta})$, and given a crossing $c = (s, i, j)$ in D (with s, i, and j as before), the following two sets of relations (the g-rules) hold (with δ denoting the Kronecker delta):*

$$g_{i\beta} = \delta_{i\beta} + T^s g_{i+,\beta} + (1 - T^s) g_{j+,\beta}, \quad g_{j\beta} = \delta_{j\beta} + g_{j+,\beta}, \quad g_{2n+1,\beta} = \delta_{2n+1,\beta} \tag{3.1}$$

and

$$g_{\alpha i} = T^{-s}(g_{\alpha,i+} - \delta_{\alpha,i+}), \quad g_{\alpha j} = g_{\alpha,j+} - (1 - T^s) g_{\alpha i} - \delta_{\alpha,j+}, \quad g_{\alpha,1} = \delta_{\alpha,1}. \tag{3.2}$$

Furthermore, for each fixed β, there are $2n + 1$ g-rules of type (3.1). (The first two depend on a choice of one of n crossings, and the third is fixed, to a total of $2n + 1$ rules.) These fully determine the $2n + 1$ scalars $g_{\alpha\beta}$ corresponding to varying α. Similarly, for each fixed α, there are $2n + 1$ g-rules of type (3.2). These fully determine the $2n + 1$ scalars $g_{\alpha\beta}$ corresponding to varying β.

For later use, we teach our computer about g-rules:

```
δi_,j_ := If[i === j, 1, 0];
gRuless_,i_,j_ := {giβ_ :> δiβ + T^s gi+,β + (1 - T^s) gj+,β, gjβ_ :> δjβ + gj+,β,
   gα_,i :> T^-s (gα,i+ - δα,i+), gα_j :> gα,j+ - (1 - T^s) gαi - δα,j+}
(α_+)+ := α"++"; (* this is for cosmetic reasons only *)
```

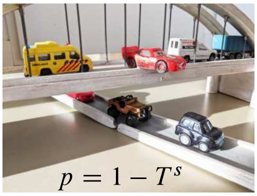
3.2. Cars, traffic counters, and interchanges

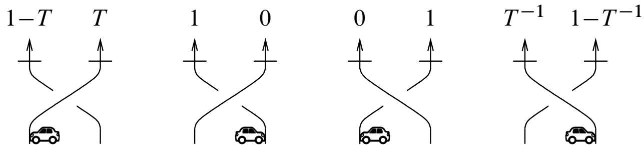
Our first proof of Theorem 1 is slightly informal as it uses the language and intuition of probability theory even though our “probabilities” are merely algebraic formulae and not numbers between 0 and 1. Seasoned mathematicians should see that there is no real problem here. Yet, just to be safe, we also include a fully formal proof in Section 3.3.

Cars (🚙, 🚙) travel on knot diagrams subject to the following three rules, inspired by Jones' "bowling balls" [25] and by Lin, Tian, and Wang's "random walks" [29] (within the proof of Proposition 5 below we will see that these rules are equivalent to the g-rules of equation (3.1) above).

- On plain roads (edges), they travel following the orientation of the edge.
- When reaching an underpass (the lower strand of a crossing), cars drive through (remaining on the lower strand) with probability 1.
- When reaching an overpass, cars pass through with probability T^s (where $s = \pm 1$ is the sign of the crossing), yet drop over to the lower strand with the complementary probability of $1 - T^s$.

These rules can be summarised in the following pictures:





In these pictures, the horizontal struts represent "traffic counters" which measure the amount of traffic that passes through their respective roads, and the output reading of these counters is printed above them. Thus, for example, the last interchange picture indicates that if a unit stream of cars is injected into the diagram on the bottom right and two traffic counters are placed at the top, then the first of these will read a car intensity of T^{-1} and the second $(1 - T^{-1})$.

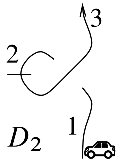
Note that our probabilities are not really probabilities, if only because T and T^{-1} cannot both be between 0 and 1 simultaneously. Yet we will manipulate them algebraically as if they are probabilities, restricting ourselves to equalities and avoiding inequalities. With this restriction, we can use intuition from probability theory. We will pretend that $T^s \sim 1$, or, equivalently, that

$$1 - T^s \sim 0.$$

This has an algebraic meaning that does not refer to inequalities. Namely, certain series can be deemed summable. For example,

$$\sum_{r \geq 0} (1 - T^s)^r = \frac{1}{1 - (1 - T^s)} = T^{-s}.$$

Example 4. Cars are injected on edge #1 of the diagram D_2 of Section 1 as indicated below.



What does the indicated traffic counter on edge #2 measure?
Solution. Every car coming through the interchange from #1 passes through the underpass and comes to #2, so the counter reads "1" just for this traffic. But then, these cars continue and pass on the overpass, and $(1-T)$ of them fall down and continue through edge #2 and get counted again. But then, these fallen cars continue and pass on the overpass once again, and $(1-T)$ of them, meaning $(1-T)^2$ of the original traffic, fall once more and contribute a further reading of $(1-T)^2$. This process continues, and the overall counter reading is

$$1+(1-T)+(1-T)^2+(1-T)^3+\cdots=\frac{1}{1-(1-T)}=T^{-1}.$$

Note that this is exactly the row 1 column 2 entry of the matrix G computed for this tangle in (1.3).

We claim that this is general.

Proposition 5. *For a general knot diagram D, the entry $g_{\alpha\beta}$ of its Green function is equal to the reading of a traffic counter placed at β given that traffic is injected into D at α. (In the case where $\alpha=\beta$, the counter is placed after where the traffic is injected, not before.)*

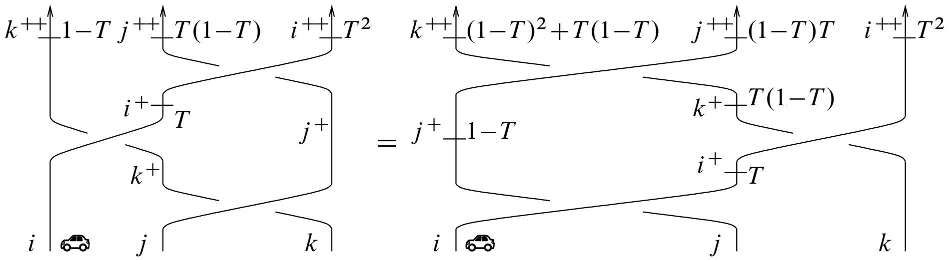
Proof. Consider the g-rules of type (3.1). The third, $g_{2n+1,\beta}=\delta_{2n+1,\beta}$, is the statement that if traffic is injected on the outgoing edge of D, it can only be measured on the outgoing edge of D (so traffic never flows backwards). The second,

$$g_{j\beta}=\delta_{j\beta}+g_{j^+,\beta},$$

is the statement that traffic goes through underpasses undisturbed, so $g_{j\beta}=g_{j^+,\beta}$ unless the traffic counter β is placed between j and j^+, in which case it measures one unit more if the cars are injected before it, at j, rather than after it, at j^+. Similarly, the first of these g-rules, $g_{i\beta}=\delta_{i\beta}+T^s g_{i^+,\beta}+(1-T^s)g_{j^+,\beta}$, is the statement of the behaviour of traffic at overpasses. Thus, the rules in (3.1) are obeyed by cars and traffic counters, and as the rules in (3.1) determine $g_{\alpha\beta}$, the proposition follows. ■

Proposition 6. *The quantity ρ_1 is invariant under R3.*

Proof. We first show that cars entering a multiple interchange styled as the left-hand side of the R3 move exit it with the exact same distribution as cars entering the multiple interchange styled as the right-hand side. The hardest part of that computation is when cars enter at the bottom left (at i), and it boils down to the equality $1-T = (1-T)^2+T(1-T)$:



If cars enter in the middle or at the bottom right (at j or at k), the computation is even easier.[4]

The conclusion is that performing the R3 move does not affect traffic patterns outside the area of the move itself; namely, the Green function $g_{\alpha\beta}$ is unchanged if both α and β are outside the area of the move.

Thus, the only contribution to ρ_1 that may change (see (1.5)) is the contribution coming from the three R_1 terms corresponding to the crossings that move, and we need to know if the following equality holds:

$$\begin{aligned} &R_1(+1,j,k) + R_1(+1,i,k^+) + R_1(+1,i^+,j^+) \\ &\stackrel{?}{=} R_1(+1,i,j) + R_1(+1,i^+,k) + R_1(+1,j^+,k^+). \end{aligned}$$

Both sides here are messy quadratics involving the $g_{\alpha\beta}$'s of both sides, evaluated at $\alpha,\beta \in \{i,j,k,i^+,j^+,k^+,i^{++},j^{++},k^{++}\}$. But we can use the traffic rules (aka the g-rules) to rewrite these quadratics in terms of the $g_{\alpha\beta}$'s with $\alpha,\beta \in \{i^{++},j^{++},k^{++}\}$, and these are unchanged between the sides. So, we simply need to know whether the above equality holds *after* the relevant g-rules have been applied to both sides. We could do that by hand, but it is simpler to appeal to a higher wisdom:

```
lhs = R1[1, j, k] + R1[1, i, k+] + R1[1, i+, j+] //. gRules1,j,k ⋃ gRules1,i,k+ ⋃ gRules1,i+,j+;
rhs = R1[1, i, j] + R1[1, i+, k] + R1[1, j+, k+] //. gRules1,i,j ⋃ gRules1,i+,k ⋃ gRules1,j+,k+;
Simplify[lhs == rhs]
```

```
True
```

∎

[4] Note that this computation is exactly the one that proves that the Burau representation [18] respects R3.

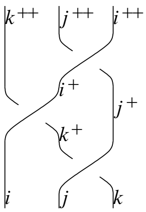
First proof of Theorem 1, "Invariance". We have shown invariance under R3. Invariance under the other moves is shown in a similar way: first one shows that overall traffic patterns are unchanged by each of the moves, and then one verifies that the local contributions to ρ_1 coming from the area changed by each move are equal once the g-rules are used to rewrite them in terms of $g_{\alpha\beta}$'s that are unaffected by the moves. This is shown in greater detail in the following section. ∎

3.3. A more formal version of the proof

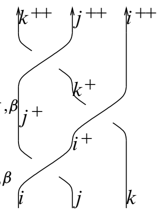
Again, we start with the hardest, R3.

Proposition 7. *The quantity ρ_1 is invariant under R3.*

Proof. We need to know how the Green function $g_{\alpha\beta}$ changes under R3. Here are the two sides of the move, along with the g-rules of type (3.1) corresponding to the crossings within, written with the assumption that β is not in $\{i^+, j^+, k^+\}$, so several of the Kronecker deltas can be ignored. We use g for the Green function at the left-hand side of R3 and g' for the right-hand side:



$$\begin{aligned} g_{i^+,\beta} &= Tg_{i^{++},\beta} + (1-T)g_{j^{++},\beta} \\ g_{j^+,\beta} &= g_{j^{++},\beta} \\ g_{i,\beta} &= \delta_{i\beta} + Tg_{i^+,\beta} + (1-T)g_{k^{++},\beta} \\ g_{k^+,\beta} &= g_{k^{++},\beta} \\ g_{j,\beta} &= \delta_{j\beta} + Tg_{j^+,\beta} + (1-T)g_{k^+,\beta} \\ g_{k,\beta} &= \delta_{k\beta} + g_{k^+,\beta} \end{aligned}$$



$$\begin{aligned} g'_{j^+,\beta} &= Tg'_{j^{++},\beta} + (1-T)g'_{k^{++},\beta} \\ g'_{k^+,\beta} &= g'_{k^{++},\beta} \\ g'_{i^+,\beta} &= Tg'_{i^{++},\beta} + (1-T)g'_{k^+,\beta} \\ g'_{k,\beta} &= \delta_{k\beta} + g'_{k^+,\beta} \\ g'_{i,\beta} &= \delta_{i\beta} + Tg'_{i^+,\beta} + (1-T)g'_{j^+,\beta} \\ g'_{j,\beta} &= \delta_{j\beta} + g'_{j^+,\beta} \end{aligned}$$

A routine computation (eliminating $g_{i^+,\beta}$, $g_{j^+,\beta}$, and $g_{k^+,\beta}$) shows that the first system of 6 equations is equivalent to the following 3 equations:

$$\begin{aligned} g_{i,\beta} &= \delta_{i\beta} + T^2 g_{i^{++},\beta} + T(1-T)g_{j^{++},\beta} + (1-T)g_{k^{++},\beta}, \\ g_{j,\beta} &= \delta_{j\beta} + Tg_{j^{++},\beta} + (1-T)g_{k^{++},\beta}, \quad \text{and} \quad g_{k,\beta} = \delta_{k\beta} + g_{k^{++},\beta}. \end{aligned}$$

Similarly, eliminating $g'_{i^+,\beta}$, $g'_{j^+,\beta}$, and $g'_{k^+,\beta}$ from the second set of equations, we find that it is equivalent to

$$\begin{aligned} g'_{i,\beta} &= \delta_{i\beta} + T^2 g'_{i^{++},\beta} + T(1-T)g'_{j^{++},\beta} + (1-T)g'_{k^{++},\beta}, \\ g'_{j,\beta} &= \delta_{j\beta} + Tg'_{j^{++},\beta} + (1-T)g'_{k^{++},\beta}, \quad \text{and} \quad g'_{k,\beta} = \delta_{k\beta} + g'_{k^{++},\beta}. \end{aligned}$$

But these two sets of equations are the same, and as stated in the g-rules lemma (Lemma 3), along with the g-rules corresponding to the other crossings in D (which are also the same between g and g'), these equations determine $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ for $\alpha, \beta \notin \{i^+, j^+, k^+\}$. So, with this exclusion on α and β, we have that $g_{\alpha\beta} = g'_{\alpha\beta}$.

But this means that the summations (1.5) in the definitions of ρ_1 are equal for the two sides of R3, except perhaps for the three summands on each side that come from the crossings that touch $\{i^+, j^+, k^+\}$.

What remains is completely mechanical. We just need to compute the sum of those three summands for both sides of R3 and apply to it the g-rules of types (3.1) and (3.2) that eliminate the indices $\{i^+, j^+, k^+\}$. The computation is easy enough to be done by hand, yet why bother? Here is the machine version (it takes less typing to apply all relevant g-rules and also eliminate the indices $\{i, j, k\}$):

$$\texttt{lhs = Simplify[}\,R_1[1, j, k] + R_1[1, i, k^+] + R_1[1, i^+, j^+]\ \texttt{//.}\ \texttt{gRules}_{1,j,k} \cup \texttt{gRules}_{1,i,k^+} \cup \texttt{gRules}_{1,i^+,j^+}\texttt{]}$$

$$-\frac{1}{2T^2}\Big(-2(-1+T)\,T\,g_{j^{++},i^{++}}^2 + 2g_{j^{++},i^{++}}\big(T^2 + T^2 g_{i^{++},j^{++}} - 2T^2 g_{j^{++},j^{++}} + g_{k^{++},i^{++}} - 2T g_{k^{++},i^{++}} + T^2 g_{k^{++},i^{++}} - T g_{k^{++},j^{++}} + T^2 g_{k^{++},j^{++}}\big) + 2g_{i^{++},i^{++}}\big(-2T^2 + (-1+T)\,T\,g_{j^{++},i^{++}} + T^2 g_{j^{++},j^{++}} - g_{k^{++},i^{++}} + T g_{k^{++},i^{++}} + T^2 g_{k^{++},k^{++}}\big) + T\Big(3T - 2(-1+T)\,g_{k^{++},i^{++}}^2 + 2T g_{k^{++},j^{++}} + 2T g_{j^{++},k^{++}} g_{k^{++},j^{++}} + 2g_{k^{++},j^{++}}^2 - 2T g_{k^{++},j^{++}}^2 + 2g_{j^{++},j^{++}}\big((-1+T)\,g_{k^{++},i^{++}} + (-1+T)\,g_{k^{++},j^{++}} + T\,(-1+g_{k^{++},k^{++}})\big) - 4T g_{k^{++},j^{++}} g_{k^{++},k^{++}} + 2g_{k^{++},i^{++}}\big(T + T g_{i^{++},k^{++}} - 2(-1+T)\,g_{k^{++},j^{++}} - 2T g_{k^{++},k^{++}}\big)\Big)\Big)$$

$$\texttt{rhs = Simplify[}\,R_1[1, i, j] + R_1[1, i^+, k] + R_1[1, j^+, k^+]\ \texttt{//.}\ \texttt{gRules}_{1,i,j} \cup \texttt{gRules}_{1,i^+,k} \cup \texttt{gRules}_{1,j^+,k^+}\texttt{]}$$

$$-\frac{1}{2T^2}\Big(-2(-1+T)\,T\,g_{j^{++},i^{++}}^2 + 2g_{j^{++},i^{++}}\big(T^2 + T^2 g_{i^{++},j^{++}} - 2T^2 g_{j^{++},j^{++}} + g_{k^{++},i^{++}} - 2T g_{k^{++},i^{++}} + T^2 g_{k^{++},i^{++}} - T g_{k^{++},j^{++}} + T^2 g_{k^{++},j^{++}}\big) + 2g_{i^{++},i^{++}}\big(-2T^2 + (-1+T)\,T\,g_{j^{++},i^{++}} + T^2 g_{j^{++},j^{++}} - g_{k^{++},i^{++}} + T g_{k^{++},i^{++}} + T^2 g_{k^{++},k^{++}}\big) + T\Big(3T - 2(-1+T)\,g_{k^{++},i^{++}}^2 + 2T g_{k^{++},j^{++}} + 2T g_{j^{++},k^{++}} g_{k^{++},j^{++}} + 2g_{k^{++},j^{++}}^2 - 2T g_{k^{++},j^{++}}^2 + 2g_{j^{++},j^{++}}\big((-1+T)\,g_{k^{++},i^{++}} + (-1+T)\,g_{k^{++},j^{++}} + T\,(-1+g_{k^{++},k^{++}})\big) - 4T g_{k^{++},j^{++}} g_{k^{++},k^{++}} + 2g_{k^{++},i^{++}}\big(T + T g_{i^{++},k^{++}} - 2(-1+T)\,g_{k^{++},j^{++}} - 2T g_{k^{++},k^{++}}\big)\Big)\Big)$$

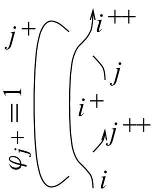
```
lhs == rhs
```

```
True
```

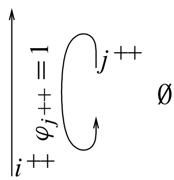
■

Proposition 8. *The quantity ρ_1 is invariant under R2c.*

Proof. We follow the exact same steps as in the case of $R3$. First, we write the g-rules, assuming that β is not in $\{i, j, i^+, j^+\}$:



$$\begin{aligned} g_{i^+,\beta} &= Tg_{i^{++},\beta} + (1-T)g_{j^+,\beta} \\ g_{j,\beta} &= g_{j^+,\beta} \\ g_{i,\beta} &= T^{-1}g_{i^+,\beta} + (1-T^{-1})g_{j^{++},\beta} \\ g_{j^+,\beta} &= g_{j^{++},\beta} \end{aligned}$$



Note that for the right-hand side we allowed ourselves to label the edges i^{++} and j^{++} as the computation is independent of the labelling and the labelling need not be by contiguous integers. (Outside of the move area, we assume that the left-hand side and the right-hand side are labelled in the same way.) Note also that for the right-hand side, there are no relevant g-rules. Now, as in the case of R3, for the left-hand side, we eliminate $g_{i^+,\beta}$ and $g_{j^+,\beta}$ and we are left with the relations

$$g_{i,\beta} = g_{i^{++},\beta} \quad \text{and} \quad g_{j,\beta} = g_{j^{++},\beta}.$$

Otherwise, the g-rules for the left and for the right are the same, and so, their Green functions are the same except if the indices are in $\{i, j, i^+, j^+\}$. (These indices do not even appear in the right-hand side.) Thus, the contribution to ρ_1 from outside the area of the move is the same for both sides.

Next, we write the contribution to ρ_1 coming from the two crossings and one rotation that appear on the left and use the g-rules to push all the indices in $\{i, j, i^+, j^+\}$ up to i^{++} and j^{++}. This can be done by hand, but seeing that we have tools, we use them as follows:

```
Simplify[R1[-1, i, j+] + R1[1, i+, j] - (g_{j+,j+} - 1/2)]
lhs = Simplify[R1[-1, i, j+] + R1[1, i+, j] - (g_{j+,j+} - 1/2) //. gRules_{-1,i,j+} ∪ gRules_{1,i+,j}]
```

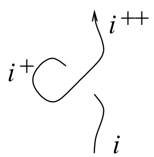
$$\frac{1}{2} - (-1 + g_{j,j^+})\, g_{i^+,i^+} + g_{j,i^+}\,(g_{j,j^+} - g_{i^+,j} + g_{j^+,j}) + g_{i,i}\,(-1 + g_{j^+,j^{++}}) - g_{j^+,i}\,(-g_{i,j^+} + g_{j^{++},j^+} + g_{j^+,j^{++}}) - g_{j^+,j^+}$$

$$\frac{1}{2} - g_{j^{++},j^{++}}$$

This result is clearly equal to the single rotation contribution to ρ_1 that comes from the right-hand side. ■

Proposition 9. *The quantity ρ_1 is invariant under R1l.*

Proof. We start with the relevant g-rules



$$g_{i^+,\beta} = \delta_{i^+,\beta} + T g_{i^{++},\beta} + (1-T) g_{i^+,\beta}$$
$$g_{i,\beta} = \delta_{i,\beta} + g_{i^+,\beta}$$

$\emptyset$

The first of these rules is equivalent to $g_{i^+,\beta} = T^{-1}\delta_{i^+,\beta} + g_{i^{++},\beta}$. For $\beta \neq i, i^+$, we find as before that $g_{i,\beta} = g_{i^{++},\beta}$, and we can ignore the contributions to ρ_1 coming from outside the area of the move. The contribution to ρ_1 coming from the single crossing and single rotation on the left-hand side is computed below and is equal to the empty contribution coming from the right-hand side:

$$\texttt{lhs1} = R_1[1, i^+, i] - (g_{i^+,i^+} - 1/2)$$
$$\texttt{lhs2} = \texttt{lhs1} \;\texttt{//.}\; \{g_{i^+,\beta\_} \mapsto T^{-1}\,\delta_{i^+,\beta} + g_{i^{++},\beta},\; g_{i,\beta\_} \mapsto \delta_{i,\beta} + g_{i^+,\beta}\}$$
$$\texttt{Simplify[lhs2]}$$

$$g^2_{i,i^+} - g_{i^+,i^+} - (-1 + g_{i,i^+})\, g_{i^+,i^+}$$

$$-\frac{1}{T} - g_{i^{++},i^+} - \left(-1 + \frac{1}{T} + g_{i^{++},i^+}\right)\left(\frac{1}{T} + g_{i^{++},i^+}\right) + \left(\frac{1}{T} + g_{i^{++},i^+}\right)^2$$

$$0$$

■

Second proof of Theorem 1, "Invariance". After the upright Reidemeister theorem (Theorem 2) which sets out what we need to do and Propositions 7, 8, and 9 which prove invariance under R3, R2c, and R1l, it remains to show the invariance of ρ_1 under R1r, R2b, and Sw$^+$. This is done exactly as in the examples already shown, so in each case, we show only the punch line:

$$\texttt{Simplify}\big[R_1[1, i, i^+] + (g_{i^+,i^+} - 1/2) \;\texttt{//.}\; \{ \quad \texttt{(* R1r *)}$$
$$g_{i\beta\_} \mapsto \delta_{i\beta} + T\, g_{i^+,\beta} + (1 - T)\, g_{i^{++},\beta},\; g_{i^+,\beta\_} \mapsto \delta_{i^+,\beta} + g_{i^{++},\beta},$$
$$g_{\alpha\_,i} \mapsto T^{-1}\,(g_{\alpha,i^+} - \delta_{\alpha,i^+}),\; g_{\alpha\_,i^+} \mapsto T\, g_{\alpha,i^{++}} - (1 - T)\,\delta_{\alpha,i^+} - T\,\delta_{\alpha,i^{++}}\}\big]$$

$$0$$

(Note that the version of the g-rules we used above easily follows from (3.2)).

$$\texttt{Simplify}\big[R_1[1, i, j] + R_1[-1, i^+, j^+] \;\texttt{//.}\; \texttt{gRules}_{1,i,j} \cup \texttt{gRules}_{-1,i^+,j^+}\big] \quad \texttt{(* R2b *)}$$

$$0$$

$$(g_{i,i} - 1/2) + (g_{j,j} - 1/2) - (g_{i^+,i^+} - 1/2) - (g_{j^+,j^+} - 1/2) \;\texttt{//.}\; \texttt{gRules}_{1,i,j} \quad \texttt{(* Sw}^+\texttt{ *)}$$

$$0$$

■

4. Some context and some morals

We would like to emphasise again that ρ_1 seems very close to the Alexander polynomial, yet we have no topological interpretation for it. Until that changes, where is ρ_1 coming from?

It comes via a lengthy path, which we will only sketch here. For a while now [2–5,7,8,13–15], we have been studying quantum invariants related to the Lie algebra sl^{ε}_{2+}, the 4-dimensional Lie algebra with generators y, b, a, x and brackets

$$[b, x] = \varepsilon x, \quad [b, y] = -\varepsilon y, \quad [b, a] = 0,$$
$$[a, x] = x, \quad [a, y] = -y, \quad [x, y] = b + \varepsilon a,$$

where ε is a scalar. The beauty of this algebra stems from the following.

- It is a "classical double" of a two-dimensional Lie bialgebra $\langle a, x\rangle$, with
$$[a,x]=x,\quad \delta(a)=0,\quad \delta(x)=\varepsilon x\wedge a,$$
and hence, quantisation tools are available and are used below (e.g., [20]).
- At invertible, ε it is isomorphic to $sl_2 \oplus \langle t\rangle$, where t is a central element[5]. Quantum topology tells us that the algebra sl_2 is related to the Jones polynomial. In fact, the universal quantum invariant (see [27, 28, 31]) for the Lie algebra sl_2 is equivalent to the coloured Jones polynomial of [25].
- At $\varepsilon = 0$, it becomes the diamond Lie algebra $\diamond$, a solvable algebra in which computations are easier. The algebra $\diamond$ is the semi-direct product of the unique non-commutative 2D Lie algebra $\mathfrak{a}$ with its dual, and quantum topology tells us that it is related to the Alexander polynomial [1, 11].

The last two facts taken together tell us that the Alexander polynomial is some limit of the coloured Jones polynomial (originally conjectured [30, 34] and proven by other means [12]).

We can make this a bit more explicit. By using the Drinfel'd quantum double construction [19], we find that the universal enveloping algebra $\mathcal{U}(sl^{\varepsilon}_{2+})$ has a quantisation QU, which has an R-matrix solving the Yang–Baxter equation (meaning, satisfying the R3 move, in the appropriate sense). These are given by

$$QU = A\langle y,b,a,x\rangle \Big/ \begin{pmatrix} [b,a]=0,\quad [b,x]=\varepsilon x,\quad [b,y]=-\varepsilon y, \\ [a,x]=x,\quad [a,y]=-y,\quad xy-qyx=\frac{1-\mathbb{e}^{-\hbar(b+\varepsilon a)}}{\hbar} \end{pmatrix},$$

where $A\langle gens\rangle$ is the free associative algebra with generators $gens$, and $q=\mathbb{e}^{\hbar\varepsilon}$, and

$$R=\sum_{m,n\geq 0}\frac{y^n b^m \otimes (\hbar a)^m(\hbar x)^n}{m![n]_q!}$$
$$\left(\text{where } [n]_q! = \prod_{k=1}^{n}\frac{1-q^k}{1-q} \text{ is a "quantum factorial"}\right).$$

Thus, there is an associated universal quantum invariant of knots $Z_\varepsilon(K)\in QU$ (which, as stated, is equivalent to the coloured Jones polynomial). In our talks and papers, we show that Z_ε can be expanded as a power series in ε; that is, at $\varepsilon=0$, it is equivalent to the Alexander polynomial and that, in general, the coefficient $Z_{(k)}$ of ε^k in Z_ε can be computed in polynomial time and is homomorphic, meaning that it leads

[5] Via the isomorphism $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\leftrightarrow \varepsilon^{-1}b+a$, $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\leftrightarrow x$, $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\leftrightarrow \varepsilon^{-1}y$, and $t\leftrightarrow b-\varepsilon a$.

to an "algebraic knot theory" in the sense of (say) [6]. We also know that the excess information in $Z_{(k)}$ (beyond the information in $\{Z_{(0)}, \dots, Z_{(k-1)}\}$) is contained in a single polynomial, ρ_k. The first of these polynomials is ρ_1 of this paper.

But how did we arrive at the specific formulas of this paper? As often seen with quantisations, QU is isomorphic (though only as an algebra, not as a Hopf algebra) with $\mathcal{U}(sl_{2+}^{\varepsilon})$, and the latter can be represented into the Heisenberg algebra

$$\mathbb{H} = A\langle p, x\rangle/([p,x] = 1)$$

via

$$y \to -tp - \varepsilon \cdot xp^2, \quad b \to t + \varepsilon \cdot xp, \quad a \to xp, \quad x \to x,$$

(abstractly, sl_{2+}^{ε} acts on its Verma module

$$\mathcal{U}(sl_{2+}^{\varepsilon})/(\mathcal{U}(sl_{2+}^{\varepsilon})\langle y, a, b - \varepsilon a - t\rangle) \cong \mathbb{Q}[x]$$

by differential operators, namely, via $\mathbb{H}$). So, QU's R-matrix can be expanded in powers of ε and pushed to $\mathcal{U}(sl_{2+}^{\varepsilon})$ and on to $\mathbb{H}$, resulting in $\mathcal{R} = \mathcal{R}_0(1 + \varepsilon\mathcal{R}_1 + \cdots)$, with $\mathcal{R}_0 = \mathbb{e}^{t(xp\otimes 1 - x\otimes p)}$ and $\mathcal{R}_1$ a quartic polynomial in p and x. Now, all the computations for ρ_1 can be carried out by pushing around a rather small number of p's and x's (at most 4), and this can be done using the rules

$$\begin{aligned}(p \otimes 1)\mathcal{R}_0 &= \mathcal{R}_0(\mathbb{e}^t(p \otimes 1) + (1 - \mathbb{e}^t)(1 \otimes p)),\\ (1 \otimes p)\mathcal{R}_0 &= \mathcal{R}_0(1 \otimes p),\end{aligned}$$

which, after setting $T = \mathbb{e}^t$, must remind the reader of equation (1.1). When all the dust settles, the resulting formulas are similar to the ones in equations (1.4) and (1.5) (but only similar, because we applied some ad hoc cosmetics to make the formulas appear nicer).

There are some morals to this story.

(1) The definition of ρ_1 in Section 1 and the proofs of its invariance in Section 3 are clearly much simpler than the origin story, as outlined above. So, quite clearly, we still do not understand ρ_1. There ought to be a room for it directly within topology, which does not require that one would know anything about quantum algebra (and better if that room is large enough to accommodate morals (2) and (6) below).

(2) Like there is ρ_1, there are ρ_k. The origin story tells us that ρ_k should have a formula as a summation over choices of k-tuples of features of the knot (crossings and rotations), just as the formula for ρ_1 is a single summation over these features. The summand for ρ_k will be a degree $2k$ polynomial in the Green function $g_{\alpha\beta}$ (compare with (1.4), which is quadratic). As a k-fold

summation, after inverting A, ρ_k should be computable in $O(n^k)$ additions and multiplications of polynomials in T, where n is the crossing number.

(3) These ρ_k should be equivalent to the invariants in our earlier works [2–5, 7, 8, 13–15].

(4) These ρ_k should be equivalent to the invariants studied earlier by Rozansky and Overbay [32, 34–36], as their quantum origin is essentially the same (though strictly speaking, we have not written proofs of that, and normalizations may differ). Our formulas are significantly simpler and faster to compute than the Rozansky–Overbay formulas, and in our language, it is easier to see the behaviour of ρ_1 under mirror reflection (see Section 2.3).

(5) Like the Rozansky–Overbay invariants, ρ_k should be equivalent to the "higher diagonals" for the Melvin–Morton expansion (e.g., [36]) and should be dominated by the "loop expansion" of the Kontsevich integral [23, 26].

(6) The quantum algebra story extends to other Lie algebras beyond sl_2. So, there should be variants $\rho_k^{\mathfrak{g}}$ of ρ_k at least for every semisimple Lie algebra $\mathfrak{g}$, given by more or less similar formulas. Quantum algebra suggests that $\rho_k^{\mathfrak{g}}$ should be a polynomial in as many variables as the rank of $\mathfrak{g}$ and should in general be stronger than the "base" ρ_k. We have not seriously explored $\rho_k^{\mathfrak{g}}$ yet, though some preliminary work was done by Schaveling [37].

(7) It appears that QU has interesting traces and therefore that there should be a link version of ρ_1. We have not pursued this formally.

(8) QU has a co-product and an antipode, and so, the universal tangle invariant associated with QU has formulas for strand reversal and strand doubling (e.g., [7, 15]). This implies (e.g., by following the ideas of [6]) that there should be formulas for ρ_1 that start with a Seifert surface for the knot. We are pursuing such formulas now; we already know that the degree of ρ_1 is bounded by $2g$, where g is the genus of a knot [15].

(9) For the same reasons, for ribbon knots, ρ_1 should have a formula computable from a ribbon presentation, and its values might be restricted in a manner similar to the Fox–Milnor condition [22]. We are pursuing this now.

(10) The coloured Jones polynomial is invariant under mutation, so we expect ρ_1 to likewise be invariant under mutation (and indeed, also ρ_k), yet we do not have a direct proof of that yet. Note that we can expect $\rho_k^{\mathfrak{g}}$ for higher-rank $\mathfrak{g}$ to no longer be invariant under mutation.

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