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Group-subgroup subfactors revisited

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Abstract. For all Frobenius groups and a large class of finite multiply transitive permutation groups, we show that the corresponding group-subgroup subfactors are completely characterized by their principal graphs. The class includes all the sharply k-transitive permutation groups for $k = 2, 3, 4$, and in particular the Mathieu group M_{11} of degree 11.

In memory of Vaughan Jones

1. Introduction

The classical Goldman's theorem [6] says, in modern term, that every index 2 inclusion $M \supset N$ of type II_1 factors is given by the crossed product $M = N \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the cyclic group of order 2. It is a famous story that this fact is one of the motivating examples when Vaughan Jones introduced his cerebrated notion of index for subfactors [20]. In the case of index 3, there are two different cases: their principal graphs are either the Coxeter graph D_4 or A_5 (see [4, 7] for example). In the D_4 case, the subfactor is given by the crossed product $M = N \rtimes \mathbb{Z}_3$. In the A_5 case, we showed in [11] that there exists a unique subfactor $R \subset N$, up to inner conjugacy, such that

$$M = R \rtimes \mathfrak{S}_3 \supset N = R \rtimes \mathfrak{S}_2$$

holds where $\mathfrak{S}_n$ denotes the symmetric group of degree n. We call such a result *Goldman-type theorem*, uniquely recovering the subfactor R and a group action on it solely from one of the principal graphs of $M \supset N$. More Goldman-type theorems were obtained in [8, 9, 12], but here we should emphasize that only Frobenius groups had been treated until we recently showed a Goldman-type theorem for the alternating groups $\mathfrak{A}_5 > \mathfrak{A}_4$ [19, Theorem A1].

Let G be a finite group, let H be a subgroup of it, and let α be an outer action of G on a factor R. Then, the inclusion

$$M = R \rtimes_\alpha G \supset N = R \rtimes_\alpha H$$

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is called a *group-subgroup subfactor*. Let L be the kernel of the permutation representation of G acting on G/H, which is the largest normal subgroup of G contained in H. Then, the inclusion $M \supset N$ remembers at most the information of $G/L > H/L$, and so, whenever we discuss group-subgroup subfactors, we always assume that L is trivial, or more naturally, we treat G as a transitive permutation group acting on a finite set and H as a point stabilizer. A Frobenius group G is a semi-direct product $K \rtimes H$ with a free H action on $K \setminus \{e\}$. In this paper, we show Goldman-type theorems for all Frobenius groups and for a large class of multiply transitive permutation groups.

One might suspect that every question about group-subgroup subfactors should be reduced to an easy exercise in either permutation group theory or representation theory, which turned out to be not always the case. Indeed, Kodiyalam–Sunder [23] showed that two pairs of groups $\mathfrak{S}_4 > \mathbb{Z}_4$ and $\mathfrak{S}_4 > \mathbb{Z}_2 \times \mathbb{Z}_2$ give isomorphic group-subgroup subfactors, which cannot be understood either in permutation group theory or representation theory. In [14], we gave a complete characterization of two isomorphic group-subgroup subfactors coming from two different permutation groups in terms of fusion categories and group cohomology. To understand this kind of phenomenon, the representation category of a group should be treated as an abstract fusion category, and ordinary representation theory is not strong enough.

When I discussed the above result [14] with Vaughan more than 10 years ago, he asked me whether the Kodiyalam–Sunder-type phenomena occur for primitive permutation groups, or in other words, when H is a maximal subgroup in G. Theorem 2.3 of [14] shows that the answer is ‘no’, and when I told it to him, somehow he looked content. I guess Vaughan believed that one should assume the primitivity of the permutation group G to obtain reasonable results in group-subgroup subfactors. Probably, he was right because the primitivity of G is equivalent to the condition that the corresponding group-subgroup subfactor has no non-trivial intermediate subfactor, and such a subfactor is known to be very rigid. This assumption also rules out the following puzzling example: while the principal graph of the group-subgroup subfactor for $\mathfrak{D}_8 = \mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_2 > \mathbb{Z}_2$ is the Coxeter graph $D_6^{(1)}$, there are 3 other subfactors sharing the same principal graph but they are not group-subgroup subfactors [16, Theorem 3.4]. This means that a Goldman-type theorem never holds for $\mathfrak{D}_8 > \mathbb{Z}_2$. Note that $\mathbb{Z}_2$ is not a maximal subgroup of $\mathfrak{D}_8$, and hence, the $\mathfrak{D}_8$-action on $\mathfrak{D}_8/\mathbb{Z}_2$ is not primitive.

Typical examples of primitive permutation groups are multiply transitive permutation groups, and we mainly work on Goldman-type theorems for them in this paper. We briefly recall the basic definitions related to them here. Let G be a permutation group on a finite set X. For $k \in \mathbb{N}$, we denote by $X^{[k]}$ the set of all ordered tuples $(a_1, a_2, \dots, a_k)$ consisting of distinct elements in X. The group G acts on $X^{[k]}$ by $g \cdot (a_1, a_2, \dots, a_k) = (ga_1, ga_2, \dots, ga_k)$, and we always consider this action. For $x \in X$, we denote by G_x the stabilizer of x in G, and for $(x_1, x_2, \dots, x_k) \in X^{[k]}$, we denote

$$G_{x_1,x_2,\dots,x_k} = \bigcap_{i=1}^{k} G_{x_i}.$$

We say that G is k-transitive if the G-action on $X^{[k]}$ is transitive. This is equivalent to the condition that the $G_{x_1,x_2,\dots,x_{k-1}}$-action on $X \setminus \{x_1, x_2, \dots, x_{k-1}\}$ is transitive. We say that G is regular if G is free and transitive. A Goldman-type theorem for a regular permutation group is nothing but the characterization of crossed products (see [24, 26]).

As will be explained in Section 2.5 in detail, our strategy for proving a Goldman-type theorem for $G > G_{x_1}$ is an induction argument reducing it to that of $G_{x_1} > G_{x_1,x_2}$. Assume that G is k-transitive but not $k+1$-transitive. Then, the first step of the induction is a Goldman-type theorem for $G_{x_1,x_2,\dots,x_{k-1}} > G_{x_1,x_2,\dots,x_k}$, and we need a good assumption on the $G_{x_1,x_2,\dots,x_{k-1}}$-action on $X \setminus \{x_1, x_2, \dots, x_{k-1}\}$ to assure it. Therefore, we will treat the following two cases in this paper.

(i) $G_{x_1,x_2,\dots,x_{k-1}}$ is regular.

(ii) $G_{x_1,x_2,\dots,x_{k-1}}$ is a primitive Frobenius group.

Permutation groups satisfying (i) are called sharply k-transitive, and their complete classification is known. Other than symmetric groups and alternating groups, the following list exhausts all of them (see [10, Chapter XII]).

(1) We denote by $\mathbb{F}_q$ the finite field with q elements. Every sharply 2-transitive group is either a group of transformations of the form $x \mapsto ax^\sigma + b$ of $\mathbb{F}_q$, where $a \in \mathbb{F}_q^\times$, $b \in \mathbb{F}_q$, and $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$, or one of the 7 exceptions. They are all Frobenius groups.

(2) There exist exactly 2 infinite families $L(q)$ and $M(q)$ of sharply 3-transitive permutation groups:

$$L(q) = \mathrm{PGL}_2(q)$$

acting on the projective geometry $\mathrm{PG}_1(q) = (\mathbb{F}_q^2 \setminus \{0\})/\mathbb{F}_q^\times$ over the finite field $\mathbb{F}_q$, and its variant $M(q)$ acting on $\mathrm{PG}_1(q)$ with an involution of $\mathbb{F}_q$ when q is an even power of an odd prime. When q is odd, both of them contain $\mathrm{PSL}_2(q)$ as an index 2 subgroup.

(3) The Mathieu group M_{11} of degree 11 is a sharply 4-transitive group, and the Mathieu group M_{12} of degree 12 is a sharply 5-transitive permutation group.

Conjecture 1.1. *A Goldman-type theorem holds for every sharply k-transitive permutation group.*

In Section 3, we show Goldman-type theorems for all Frobenius groups and verify the conjecture for $k = 2$ as a special case (Theorem 3.1). We also classify related fusion categories generalizing Etingof–Gelaki–Ostrik's result [3, Corollary 7.4] (Theorem 3.5). We verify the conjecture for $k = 3$ in Section 4 (Theorem 4.1) and for $k = 4$ in Section 6 (Theorems 6.1, 6.2, and 6.4). When q is odd, the action of $\mathrm{PSL}_2(q)$ on $\mathrm{PG}_1(q)$ is 2-transitive and it satisfies the condition (ii) above. We will show a Goldman-type theorem for $\mathrm{PSL}_2(q)$ acting on $\mathrm{PG}_1(q)$ in Section 5 (Theorem 5.1).

2-transitive extensions of Frobenius groups (satisfying a certain condition) are called Zassenhaus groups (see [10, Chapter XI] for the precise definition), and there are exactly 4 infinite families of them: $L(q)$, $M(q)$, $\mathrm{PSL}_2(q)$ as above, and the Suzuki groups $Sz(2^{2n+1})$ of degree 2^{4n+2} for $n \geq 1$. One might hope that a Goldman-type theorem would hold for the Suzuki groups too. However, it is difficult to prove it with our technique now because the point stabilizers of the Suzuki groups are non-primitive Frobenius groups and the Frobenius kernels are non-commutative.

2. Preliminaries

2.1. Frobenius groups

A transitive permutation group G on a finite set X is said to be a Frobenius group if it is not regular and every $g \in G \setminus \{e\}$ has at most one fixed point. Let $H = G_{x_1}$ be a point stabilizer. Then, G being Frobenius is equivalent to the condition that the H-action on $X \setminus \{x_1\}$ is free and is further equivalent to the condition that $H \cap gHg^{-1} = \{e\}$ for all $g \in G \setminus H$.

For a Frobenius group G,

$$K = G \setminus \bigcup_{x \in X} G_x$$

is a normal subgroup of G, called the Frobenius kernel, and G is a semi-direct product $K \rtimes H$ (see [27, 8.5.5]). The point stabilizer H is called a Frobenius complement. Now, the set X is identified with K, and the H-action on $X \setminus \{x_1\}$ is identified with that on $K \setminus \{e\}$. It is known that K is nilpotent (Thompson), and H has periodic cohomology (Burnside) in the sense that the Sylow p-subgroups of H are cyclic for odd p and are either cyclic or generalized quaternion for $p = 2$ [27, 10.5.6]. We collect the following properties of Frobenius groups which we will use later.

Recall that a transitive permutation group is primitive if and only if its point stabilizer is maximal in G.

Lemma 2.1. *Let G be a Frobenius group with the kernel K and a complement H. Then, the following hold:*

(1) *G is primitive if and only if K is an elementary abelian p-group $\mathbb{Z}_p^l$ with a prime p and there is no non-trivial H-invariant subgroup of K.*

(2) *The Schur multiplier* $H^2(H, \mathbb{T})$ *is trivial.*

(3) *Every abelian subgroup of* H *is cyclic.*

Proof. (1) Note that G is primitive if and only if there is no non-trivial H-invariant subgroup of K. Assume that G is primitive. Since K is nilpotent, its center $Z(K)$ is not equal to $\{e\}$ and H-invariant, and so, $K = Z(K)$. Let p be a prime so that the p-component K_p of K is not $\{e\}$. Since K_p is H-invariant, we get $K = K_p$. The same argument applied to

$$L = \{x \in K;\ x^p = 0\}$$

shows that K is an elementary abelian p-group.

(2) Since the Schur multiplier is trivial for every cyclic group and generalized quaternion (see, for example, [22, Proposition 2.1.1, Example 2.4.8]), the statement follows from [1, Theorem 10.3].

(3) The statement follows from the fact that every abelian subgroup of a generalized quaternion group is cyclic. ∎

2.2. Sharply k-transitive permutation groups

The reader is refered to [2] for the basics of permutation groups. A transitive permutation group G on a finite set X is said to be sharply k-transitive permutation group if the G-action on $X^{[k]}$ is regular. If the degree of G is n, a sharply k-permutation group has order $n(n-1)\cdots(n-k+1)$.

For $n \in \mathbb{N}$, let $X_n = \{1, 2, \dots, n\}$. Since $X_n^{[n-1]}$ and $X_n^{[n]}$ are naturally identified, the defining action of $\mathfrak{S}_n$ on X_n is both sharply $n-1$ and n-transitive. As this fact might cause confusion, we treat $\mathfrak{S}_n$ as a sharply $n-1$-transitive group in this paper. The natural action of $\mathfrak{A}_n$ on X_n is sharply $n-2$-transitive.

Every sharply 2-transitive permutation group G is known to be a Frobenius group and hence of the form $G = \mathbb{Z}_p^k \rtimes H$ with a prime p and with a Frobenius complement H acting on $\mathbb{Z}_p^k \setminus \{0\}$ regularly. Let $q = p^k$, and let $T(q) = \mathbb{F}_q^\times \rtimes \mathrm{Aut}(\mathbb{F}_q)$, which acts on $\mathbb{F}_q$ as an additive group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^k$. Then, the Zassenhaus theorem says that H is either identified with a subgroup of $T(q)$ or one of the following exceptions: $\mathrm{SL}_2(3)$ acting on $\mathbb{Z}_5^2$, $GL_2(3)$ acting on $\mathbb{Z}_7^2$, $\mathrm{SL}_2(3) \times \mathbb{Z}_5$ acting on $\mathbb{Z}_{11}^2$, $\mathrm{SL}_2(5)$ acting on $\mathbb{Z}_{11}^2$, $GL_2(3) \times \mathbb{Z}_{11}$ acting on $\mathbb{Z}_{23}^2$, $\mathrm{SL}_2(5) \times \mathbb{Z}_7$ acting on $\mathbb{Z}_{29}^2$, and $\mathrm{SL}_2(5) \times \mathbb{Z}_{29}$ acting on $\mathbb{Z}_{59}^2$. The reader is referred to [10, Chapter XII, Section 10] for this fact.

There are two important families $H(q)$ and $S(q)$ of sharply 2-transitive permutation groups. If $G = \mathbb{Z}_p^k \rtimes H$ is a sharply 2-transitive group with an abelian Frobenius complement, it is necessarily of the form $G = \mathbb{F}_q \rtimes \mathbb{F}_q^\times$, which is denoted by $H(q)$. Assume now that p is an odd prime and $q = p^{2l}$. Then, the field $\mathbb{F}_q$ has an involution

$x^\sigma = x^{p^l}$. The group $S(q)$ has a Frobenius complement $\mathbb{F}_q^\times$ as a set, but its action on $\mathbb{F}_q$ is given as follows:

$$a \cdot x = \begin{cases} ax & \text{if } a \text{ is a square in } \mathbb{F}_q^\times, \\ ax^\sigma & \text{if } a \text{ is not a square in } \mathbb{F}_q^\times. \end{cases}$$

For example, the group $S(3^2)$ is isomorphic $\mathbb{Z}_3^2 \rtimes Q_8$. We have small-order coincidences $\mathfrak{S}_3 = H(3)$ and $\mathfrak{A}_4 = H(2^2)$.

There are exactly two families of sharply 3-transitive permutation groups $L(q)$ and $M(q)$, and they are transitive extensions of $H(q)$ and $S(q)$, respectively (see [10, Chapter XI, Section 2]). To describe their actions, it is convenient to identify the projective geometry $\mathrm{PG}_1(q)$ with $\mathbb{F}_q \sqcup \{\infty\}$. The 3-transitive action of $L(q) = \mathrm{PGL}_2(q)$ is given as follows:

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \cdot x = \frac{ax+b}{cx+d}.$$

The group $M(q)$ is $\mathrm{PGL}_2(q)$ as a set, but its action on $\mathrm{PG}_1(q)$ is given by

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \cdot x = \begin{cases} \frac{ax+b}{cx+d} & \text{if } ad-bc \text{ is a square in } \mathbb{F}_q^\times, \\ \frac{ax^\sigma+b}{cx^\sigma+d} & \text{if } ad-bc \text{ is not a square in } \mathbb{F}_q^\times. \end{cases}$$

We have small-order coincidences $\mathfrak{S}_4 = L(3)$ and $\mathfrak{A}_5 = L(2^2)$.

When q is odd, the restriction of the $L(q)$-action on $\mathrm{PG}_1(q)$ to $\mathrm{PSL}_2(q)$ is 2-transitive, and its point stabilizer is isomorphic to $\mathbb{Z}_p^k \rtimes \mathbb{Z}_{(p^k-1)/2}$.

Other than symmetric groups and alternating groups, the Mathieu groups M_{11} and M_{12} are the only sharply 4- and 5-transitive permutation groups, and their degrees are 11 and 12, respectively (see [10, Chapter XII, Section 3]). To show a Goldman-type theorem for the permutation group M_{11} of degree 11, we do not really need its construction. Instead, we only need the fact that this action is a transitive extension of the sharply 3-transitive permutation group $M(3^2)$ on $\mathrm{PG}_1(3^2)$ (see [10, Chapter XII, Theorem 1.3]).

2.3. Group-subgroup subfactors

For a finite index inclusion $M \supset N$ of factors, we need to distinguish the two principal graphs of it and symbols for them. Thus, we mean by the principal graph of $M \supset N$ the induction-reduction graph between N-N bimodules and M-N bimodules arising from the inclusion and denote it by $\mathcal{G}_{M \supset N}$, while we mean by the dual principal graph the induction-reduction graph between M-M bimodules and M-N bimodules and denote it by $\mathcal{G}^d_{M \supset N}$.

Let G be a transitive permutation group on a finite set X, and let $H = G_{x_1}$ with $x_1 \in X$. Let

$$M = R \rtimes_\alpha G \supset N = R \rtimes_\alpha H$$

be a group subgroup subfactor with an outer G-action on a factor R. The reader is referred to [25] for the tensor category structure of the M-M, M-N, N-M, and N-N bimodules arising from the group-subgroup subfactor $M \supset N$. The category of M-M bimodules is equivalent to the representation category $\mathrm{Rep}(G)$ of G, and we use the symbol $\widehat{G}$ to parameterize the equivalence classes of irreducible M-M bimodules. The set of equivalence classes of irreducible M-N bimodules are parameterized by $\widehat{H}$, and $\mathscr{G}^d_{G>H}$ is the induction-reduction graph between $\widehat{G}$ and $\widehat{H}$. For this reason, we denote by $\mathscr{G}^G_H$ the dual principal graph $\mathscr{G}^d_{M\supset N}$.

The description of the category of N-N bimodules is much more involved. We choose one point from each G_{x_1}-orbit in $X \setminus \{x_1\}$ and enumerate them as $x_2, x_3, \ldots,$ x_k. Then, the set of the equivalence classes of irreducible N-N bimodules arising from $M \supset N$ is parameterized by the disjoint union

$$\widehat{G_{x_1}} \sqcup \widehat{G_{x_1,x_2}} \sqcup \cdots \sqcup \widehat{G_{x_1,x_k}},$$

and the graph $\mathscr{G}_{M\supset N}$ is the union of the induction-reduction graph between $\widehat{G_{x_1}}$ and $\widehat{G_{x_1,x_i}}$ over $1 \le i \le k$ with convention $G_{x_1,x_1} = G_{x_1}$. The dimension of the irreducible object corresponding to $\pi \in \widehat{G_{x_1,x_2}}$ is $|G_{x_1}/G_{x_1,x_2}| \dim \pi$. We denote by $\mathscr{G}_{(G,X)}$ or $\mathscr{G}_{G>G_{x_1}}$ the principal graph $\mathscr{G}_{M\supset N}$ depending on the situation.

The category of N-N bimodules for the inclusion $N \supset R$ is equivalent to $\mathrm{Rep}(H)$, and we denote the equivalence classes of irreducible objects of it by $\{[\beta_\pi]\}_{\pi\in\widehat{H}}$. Then, the set $\{[\beta_\pi]\}_{\pi\in\widehat{H}}$ actually coincides with $\widehat{H}$ in $\mathscr{G}_{G>H}$ as equivalence classes of N-N bimodules. (This fact is not usually emphasized, but one can see it from [25].) Let $\iota = {}_M M_N$ be the basic bimodule. Then, the set of equivalence classes of irreducible M-N bimodules arising from $M \supset N$ is given by $\{[\iota \otimes_N \beta_\pi]\}_{\pi\in\widehat{H}}$.

If G is 2-transitive, we have $k = 2$, and the graph $\mathscr{G}_{(G,X)}$ can be obtained from $\mathscr{G}^{G_{x_1}}_{G_{x_1,x_2}}$ by putting an edge of length one to each even vertex of $\mathscr{G}^{G_{x_1}}_{G_{x_1,x_2}}$. More generally, for a bipartite graph $\mathscr{G}$, we denote by $\widetilde{\mathscr{G}}$ the graph obtained by putting an edge of length one to each even vertex of $\mathscr{G}$. Then, we have $\mathscr{G}_{(G,X)} = \widetilde{\mathscr{G}^{G_{x_1}}_{G_{x_1,x_2}}}$.

Let $\mathscr{G}_n$ be a depth 2 graph without multi-edges and with n even vertices (see Figure 1). Assume that $\mathscr{G}_n$ is the principal graph $\mathscr{G}_{M\supset N}$ of a finite-index inclusion $M \supset N$ of factors. Then, the characterization of crossed products shows that $M = N \rtimes_\alpha G$, and the G-action is unique up to inner conjugacy. Thus, a Goldman-type theorem holds for regular permutation groups, but in a weak sense because the graph $\mathscr{G}_n$ determines only the order n of G, and not the group structure unless n is a prime. Even when we specify the dual principal graph of $M \supset N$, it does not distinguish the

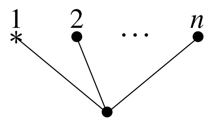

Figure 1. $\mathcal{G}_n$.

dihedral group $\mathfrak{D}_8$ of order 8 and the quaternion group Q_8. As this example suggests, we should clarify what we really mean by a Goldman-type theorem.

Definition 2.2. Let $\mathcal{G}$ be a bipartite graph.

(1) We say that a strong Goldman-type theorem for $\mathcal{G}$ (or for (G, X) if $\mathcal{G} = \mathcal{G}_{(G,X)}$) holds if the following holds: there exists a unique transitive permutation group G on a finite set, up to permutation conjugacy, such that whenever the principal graph of a finite index subfactor $M \supset N$ is $\mathcal{G}$, there exists a unique subfactor R of N, up to inner conjugacy in N, satisfying $M \cap R' = \mathbb{C}$ and
$$M = R \rtimes_\alpha G \supset N = R \rtimes_\alpha H,$$
where H is a point stabilizer of G.

(2) We say that a weak Goldman-type theorem for $\mathcal{G}$ holds if the following holds: whenever the principal graph of a finite index subfactor $M \supset N$ is $\mathcal{G}$, there exists a unique subfactor R of N, up to inner conjugacy in N, satisfying $M \cap R' = \mathbb{C}$ and
$$M = R \rtimes_\alpha G \supset N = R \rtimes_\alpha H,$$
for some transitive permutation group G on a finite set with a point stabilizer of H.

Note that the action α is automatically unique, up to inner conjugacy, thanks to the irreducibility of R in M.

We will show weak Goldman-type theorems for all Frobenius groups (including sharply 2-permutation groups) and strong ones for sharply 3- and 4-permutation groups and for $\mathrm{PSL}_2(q)$ acting on $\mathrm{PG}_1(q)$.

2.4. Intermediate subfactors

In what follows, we use the sector notation for subfactors (see, for example, [13, Section 2] or [15, Section 2.1]), though all results are stated for general factors. The inclusion map $\iota : N \hookrightarrow M$ and its conjugate $\bar{\iota} : N \to M$ in the statements should be read as the basic bimodules $\iota = {}_M M_N$ and $\bar{\iota} = {}_N M_M$, respectively, in the type II_1 case. In the proofs, we always assume that factors involved are either of type II_∞ or type III without mentioning it. In the type II_1 case, this can be justified by either directly working on bimodules instead of sectors or replacing $M \supset N$ with

$$M \otimes B(\ell^2) \supset N \otimes B(\ell^2).$$

For example, assume that a statement insists existence of a subfactor $R \subset N$ satisfying

$$(M \supset N) = (R \rtimes G \supset R \rtimes H).$$

Then, it suffices to prove that there exists a subfactor $P \subset N \otimes B(\ell^2)$ satisfying

$$(M \otimes B(\ell^2) \supset N \otimes B(\ell^2)) = (P \rtimes G \supset P \rtimes H).$$

Indeed, let e be a minimal projection in $B(\ell^2)$, and we choose a projection p in the fixed-point algebra P^G that is equivalent to $1 \otimes e$ in $N \otimes B(\ell^2)$. Then, we get

$$(M \supset N) \cong (p(M \otimes B(\ell^2))p \supset p(N \otimes B(\ell^2))p) = (pPp \rtimes G \supset pPp \rtimes H).$$

For two properly infinite factors A and B and unital homomorphisms ρ, σ from A to B, we say that ρ and σ are equivalent if there exists a unitary $u \in B$ satisfying $\rho = \mathrm{Ad}\, u \circ \sigma$. We denote by $[\rho]$ the equivalence class of ρ, which is called a sector. The statistical dimension $d(\rho)$ of ρ is defined to be the square root of the minimum index $[B : \rho(A)]_0$.

Assume that α is an outer action of a finite group G on a factor M. Let N be the fixed-point algebra M^G, and let ι be the inclusion map $\iota : N \hookrightarrow M$. Then, we have $\alpha_g \iota = \iota$ for all $g \in G$, and the Frobenius reciprocity implies that α_g is contained in $\iota\bar{\iota}$ for all $g \in G$. Since $d(\iota\bar{\iota}) = |G|$, we get

$$[\iota\bar{\iota}] = \bigoplus_{g \in G} [\alpha_g].$$

In fact, the fixed-point subfactor is completely characterized by the fact that $\iota\bar{\iota}$ is decomposed into automorphisms. The other product $\bar{\iota}\iota$ generates a fusion category equivalent to the representation category of G, and $\bar{\iota}\iota$ corresponds to the regular representation.

We collect useful statements for our purpose in the next theorem concerning intermediate subfactors extracted from [18, Corollary 3.10].

Theorem 2.3. *Let $M \supset N$ be an irreducible inclusion of factors with finite index, and let $\iota : N \hookrightarrow M$ be the inclusion map. Let*

$$[\iota\bar{\iota}] = \bigoplus_{\xi \in \Lambda} n_\xi [\xi]$$

be the irreducible decomposition.

(1) *Let P be an intermediate subfactor between M and N, and let $\kappa : P \hookrightarrow M$ be the inclusion map. If $\xi_1, \xi_2 \in \Lambda$ are contained in $\kappa\overline{\kappa}$ and $\xi_3 \in \Lambda$ is contained in $\xi_1\xi_2$, then ξ_3 is contained in $\kappa\overline{\kappa}$.*

(2) *Assume that P and Q are intermediate subfactors between M and N, and the inclusion maps $\kappa : P \hookrightarrow M$ and $\kappa_1 : Q \hookrightarrow M$ satisfy $[\kappa\overline{\kappa}] = [\kappa_1\overline{\kappa_1}]$. If for each $\xi \in \Lambda$ the multiplicity of ξ in $\kappa\overline{\kappa}$ is either 0 or n_ξ, then $P = Q$.*

(3) *Assume that Λ_1 is self-conjugate subset of Λ such that whenever $\xi_3 \in \Lambda$ is contained in $\xi_1\xi_2$ for some $\xi_1, \xi_2 \in \Lambda_1$, we have $\xi_3 \in \Lambda_1$. Then, there exists a unique intermediate subfactor P between M and N such that the inclusion map $\kappa : P \hookrightarrow M$ satisfies*

$$[\kappa\overline{\kappa}] = \bigoplus_{\xi \in \Lambda_1} n_\xi [\xi].$$

2.5. The strategy of the proofs

Let Γ be a doubly transitive permutation group acting on a finite set X, and let $x_1, x_2 \in X$ be distinct points. We further assume that the Γ_{x_1,x_2}-action on $X \setminus \{x_1, x_2\}$ has no orbit of length 1. Our basic strategy to prove a Goldman-type theorem for $\Gamma > \Gamma_{x_1}$ is to reduce it to that of $\Gamma_{x_1} > \Gamma_{x_1,x_2}$. To explain it, we first discuss the relationship between the group-subgroup subfactor of the former and that of the latter. We denote $G = \Gamma_{x_1}$ and $H = \Gamma_{x_1,x_2}$ for simplicity.

Assume that we are given an outer action α of Γ on a factor R. We set $N = R \rtimes_\alpha H$, $M = R \rtimes_\alpha G$, and $L = R \rtimes_\alpha \Gamma$. We denote by $\iota_1 : M \hookrightarrow L$, $\iota_2 : N \hookrightarrow M$, and $\iota_3 : R \hookrightarrow N$ the inclusion maps. Since the Γ-action on X is doubly transitive, there exists $g_0 \in \Gamma$ exchanging x_1 and x_2. Such g_0 normalizes H, and we get $\theta \in \mathrm{Aut}(N)$ extending α_{g_0}, that is, $\theta\iota_3 = \iota_3\alpha_{g_0}$. Let

$$[\iota_3\overline{\iota_3}] = \bigoplus_{\pi \in \widehat{H}} d(\pi)[\beta_\pi]$$

be the irreducible decomposition. The automorphism θ as above is not unique, and there is always a freedom to replace θ with $\theta\beta_\pi$ with $d(\pi) = 1$.

Since

$$[\iota_1\iota_2\theta\iota_3] = [\iota_1\iota_2\iota_3\alpha_{g_0}] = [\iota_1\iota_2\iota_3],$$

we have

$$1 = \dim(\iota_1\iota_2\theta\iota_3, \iota_1\iota_2\iota_3) = (\iota_2\theta\iota_3\overline{\iota_3}\overline{\iota_2}, \overline{\iota_1}\iota_1) = \sum_{\pi \in \widehat{H}} d(\pi) \dim(\iota_2\theta\beta_\pi\overline{\iota_2}, \overline{\iota_1}\iota_1).$$

We claim $(\iota_2\theta\beta_\pi\overline{\iota_2}, \mathrm{id}) = 0$ for all π. Indeed, if it were not the case, we would have π with $d(\beta\pi) = 1$ satisfying $[\iota_2\theta\beta_\pi] = [\iota_2]$ thanks to the Frobenius reciprocity.

However, this implies that $\theta\beta_\pi$ would be contained in $\overline{\iota_2}\iota_2$. Since $d(\theta\beta_\pi) = 1$, this contradicts the assumption that the H-action on $G/H \setminus H$ has no orbit of length 1.

Since Γ is doubly transitive, there exists irreducible τ with $d(\tau) = |X| - 1$ satisfying $[\overline{\iota_1}\iota_1] = [\mathrm{id}] \oplus [\tau]$. On the other hand, we have $d(\iota_2\theta\beta_\pi\overline{\iota_2}) = (|X| - 1)d(\pi)$, which shows that there exists $\pi \in \widehat{H}$ with $d(\pi) = 1$ satisfying $[\tau] = [\iota_2\theta\beta_\pi\overline{\iota_2}]$. This means that by replacing θ with $\theta\beta_\pi$ if necessary, we may always assume

$$[\overline{\iota_1}\iota_1] = [\mathrm{id}] \oplus [\iota_2\theta\overline{\iota_2}].$$

Now, forget about R, α, N, and assume that we are just given an inclusion $L \supset M$ with $\mathcal{G}_{L\supset M} = \mathcal{G}_{\Gamma>G}$. We denote by $\iota_1 : M \hookrightarrow L$ the inclusion map. We assume that a Goldman-type theorem is known for $G > H$. Our task is to recover R and α from the inclusion $L \supset M$. Our strategy is divided into the following steps.

(1) Find a fusion subcategory $\mathcal{C}_1$ in the fusion category $\mathcal{C}$ generated by $\overline{\iota_1}\iota_1$ that looks like the representation category of G.

(2) Show that the object in $\mathcal{C}_1$ corresponding to the induced representation $\mathrm{Ind}_H^G 1$ has a unique Q-system satisfying the following condition: if $N \subset M$ is the subfactor corresponding to the Q-system and $\iota_2 : N \hookrightarrow M$ is the inclusion map, then there exists $\theta \in \mathrm{Aut}(N)$ satisfying

$$[\overline{\iota_1}\iota_1] = [\mathrm{id}] \oplus [\iota_2\theta\overline{\iota_2}].$$

(3) Show $\mathcal{G}_{M\supset N} = \mathcal{G}_{G>H}$.

(4) Apply the Goldman-type theorem for $G > H$ to $M \supset N$, and obtain a subfactor R and an outer action γ of G on $R \subset N$ satisfying $M = R \rtimes_\gamma G$ and $N = R \rtimes_\gamma H$. Show that R is irreducible in L. Let $\iota_3 : R \hookrightarrow N$ be the inclusion map.

(5) Show that $L \supset R$ is a depth 2 inclusion.

(6) Show that there exists $\theta_1 \in \mathrm{Aut}(R)$ satisfying $[\theta\iota_3] = [\iota_3\theta_1]$.

Lemma 2.4. *Assume that the above (1)–(6) are accomplished. Then, there exist a finite group Γ_0 including G as a subgroup of index $|X|$ and an outer action α of Γ_0 on R such that α is an extension of γ and $L = R \rtimes_\alpha \Gamma_0$. Moreover, the action of Γ_0 on Γ_0/G is a doubly transitive extension of the G-action on $X \setminus \{x_0\}$.*

Proof. By (2),

$$[\overline{\iota_3}\overline{\iota_2}\overline{\iota_1}\iota_1\iota_2\iota_3] = [\overline{\iota_3}\overline{\iota_2}(\mathrm{id} \oplus \iota_2\theta\overline{\iota_2})\iota_2\iota_3] = \bigoplus_{g\in G}[\gamma_g] \oplus [\overline{\iota_3}\overline{\iota_2}\iota_2\theta\overline{\iota_2}\iota_2\iota_3],$$

which contains

$$\bigoplus_{g\in G}[\gamma_g] \oplus [\overline{\iota_3}\theta\iota_3] = \bigoplus_{g\in G}[\gamma_g] \oplus [\overline{\iota_3}\iota_3\theta_1] = \bigoplus_{g\in G}[\gamma_g] \oplus \bigoplus_{h\in H}[\gamma_h\theta_1]$$

by (6). Let Γ_0 be the group of 1-dimensional sectors contained in $[\overline{\iota_3}\,\overline{\iota_2}\,\overline{\iota_1}\iota_1\iota_2\iota_3]$. Then, Γ_0 is strictly larger than $[\gamma_G]$, and $R \rtimes \Gamma_0$ is a subfactor of L strictly larger than M. Thanks to Theorem 2.3, there is no non-trivial intermediate subfactor between L and M, and we conclude $L = R \rtimes \Gamma_0$. From the shape of the graph $\mathcal{G}_{\Gamma > G}$, we can see that the Γ_0-action on Γ_0/G is doubly transitive. ■

To identify Γ_0 with Γ, we will use the classification of doubly transitive permutation groups.

In concrete examples treated in this paper, (1) and (3) are purely combinatorial arguments, (2) follows from Theorem 2.3, (4) is an induction hypothesis, and (5) is a simple computation of dimensions. To deal with (6), we give useful criteria now.

Lemma 2.5. *Let G be a transitive permutation group on a finite set with a point stabilizer H, and let α be an outer action of G on a factor R. Let*

$$M = R \rtimes_\alpha G \supset N = R \rtimes_\alpha H.$$

Let L be a factor including M as an irreducible subfactor of index $|G/H| + 1$. We denote by $\iota_1 : M \hookrightarrow L$, $\iota_2 : N \hookrightarrow M$, and $\iota_3 : R \hookrightarrow N$ the inclusion maps. We assume the following two conditions.

(1) *The inclusion $L \supset R$ is irreducible and of depth 2.*

(2) *There exists $\theta \in \mathrm{Aut}(N)$ satisfying $[\overline{\iota_1}\iota_1] = [\mathrm{id}] \oplus [\iota_2\theta\overline{\iota_2}]$.*

Then, we have

$$\dim(\theta\overline{\iota_2}\iota_2\iota_3\overline{\iota_3}\theta^{-1}, \overline{\iota_2}\iota_2\iota_3\overline{\iota_3}) = |H|.$$

Proof. Since $[L : M] = (|G/H| + 1)|G|$, the depth 2 condition implies

$$\begin{aligned}
(|G/H| + 1)|G| &= \dim(\overline{\iota_3}\,\overline{\iota_2}\,\overline{\iota_1}\iota_1\iota_2\iota_3, \overline{\iota_3}\,\overline{\iota_2}\,\overline{\iota_1}\iota_1\iota_2\iota_3) \\
&= \dim(\overline{\iota_3}\,\overline{\iota_2}(\mathrm{id} \oplus \iota_2\theta\overline{\iota_2})\iota_2\iota_3, \overline{\iota_3}\,\overline{\iota_2}(\mathrm{id} \oplus \iota_2\theta\overline{\iota_2})\iota_2\iota_3) \\
&= \dim\Big(\bigoplus_{g \in G} \alpha_g \oplus \overline{\iota_3}\,\overline{\iota_2}\iota_2\theta\overline{\iota_2}\iota_2\iota_3, \bigoplus_{g \in G} \alpha_g \oplus \overline{\iota_3}\,\overline{\iota_2}\iota_2\theta\overline{\iota_2}\iota_2\iota_3\Big) \\
&= |G| + \dim(\overline{\iota_3}\,\overline{\iota_2}\iota_2\theta\overline{\iota_2}\iota_2\iota_3, \overline{\iota_3}\,\overline{\iota_2}\iota_2\theta\overline{\iota_2}\iota_2\iota_3)
\end{aligned}$$

and

$$|G/H||G| = \dim(\overline{\iota_3}\,\overline{\iota_2}\iota_2\theta\overline{\iota_2}\iota_2\iota_3, \overline{\iota_3}\,\overline{\iota_2}\iota_2\theta\overline{\iota_2}\iota_2\iota_3) = \dim(\theta\overline{\iota_2}\iota_2\iota_3\overline{\iota_3}\,\overline{\iota_2}\iota_2\theta^{-1}, \overline{\iota_2}\iota_2\iota_3\overline{\iota_3}\,\overline{\iota_2}\iota_2)$$

by the Frobenius reciprocity. Thus, to prove the statement, it suffices to show

$$[\overline{\iota_2}\iota_2\iota_3\overline{\iota_3}\,\overline{\iota_2}\iota_2] = |G/H|[\overline{\iota_2}\iota_2\iota_3\overline{\iota_3}].$$

Indeed, note that $\iota_2\iota_3\overline{\iota_3}\,\overline{\iota_2}$ is an M-M sector corresponding to the regular representation of G, and hence, $(\iota_2\iota_3\overline{\iota_3}\,\overline{\iota_2})\iota_2$ is an M-N sector corresponding to the restriction of the regular representation of G to H, which is equivalent to $|G/H|$ copies of the regular representation of H. Since $\iota_3\overline{\iota_3}$ is an N-N sector corresponding the regular representation of H, we get

$$[(\iota_2\iota_3\overline{\iota_3}\overline{\iota_2})\iota_2] = |G/H|[\iota_2(\overline{\iota_3}\iota_3)],$$

which finishes the proof. ∎

In concrete cases where Lemma 2.5 is applied, we can further show

$$\dim(\theta\iota_3\overline{\iota_3}\theta^{-1}, \iota_3\overline{\iota_3}) = |H|,$$

resulting in $[\theta\iota_3\overline{\iota_3}\theta^{-1}] = [\iota_3\overline{\iota_3}]$.

From [17, Theorem 3.3 and Lemma 4.1], we can show the following global invariance criterion.

Lemma 2.6. *Let H be a finite group, and let α be an outer action of H on a factor R. Let $N = R \rtimes_\alpha H$, and let $\iota : R \hookrightarrow N$ be the inclusion map. We assume that there is no non-trivial abelian normal subgroup $K \triangleleft H$ with a non-degenerate cohomology class $\omega \in H^2(\widehat{K}, \mathbb{T})$ invariant under the H-action by conjugation. If $\theta \in \mathrm{Aut}(N)$ satisfies $[\theta\iota\overline{\iota}\theta^{-1}] = [\iota\overline{\iota}]$, there exists $\theta_1 \in \mathrm{Aut}(R)$ satisfying*

$$[\theta\iota] = [\iota\theta_1].$$

Even when the cohomological assumption in Lemma 2.6 is not fulfilled, we still have a chance to apply the following criterion. For an inclusion $N \supset R$ of factors, we denote by $\mathrm{Aut}(N, R)$ the set of automorphisms of N globally preserving R.

Lemma 2.7. *Let $N \supset R$ be an irreducible inclusion of factors with finite index, and let P be an intermediate subfactor between N and R. We denote by $\iota : R \hookrightarrow N$ and $\kappa : P \hookrightarrow N$ the inclusion maps. Let*

$$[\iota\overline{\iota}] = \bigoplus_{\xi\in\Lambda} n_\xi[\xi]$$

be the irreducible decomposition. We assume that for each $\xi \in \Lambda$ the multiplicity of ξ in $\kappa\overline{\kappa}$ is either 0 or n_ξ. If $\theta \in \mathrm{Aut}(N, R)$ satisfies $[\theta\kappa\overline{\kappa}\theta^{-1}] = [\kappa\overline{\kappa}]$, then $\theta(P) = P$.

Proof. Let $Q = \theta(P)$, let $\varphi : P \to Q$ be the restriction of θ to P regarded as an isomorphism from P onto Q, and let $\kappa_1 : Q \hookrightarrow N$ be the inclusion map. Then, by definition, we have $\theta \circ \kappa = \kappa_1 \circ \varphi$. Thus,

$$[\kappa_1\overline{\kappa_1}] = [\kappa_1\varphi\overline{\varphi}\overline{\kappa_1}] = [\theta][\kappa\overline{\kappa}][\theta^{-1}] = [\kappa\overline{\kappa}],$$

and the statement follows from Theorem 2.3. ∎

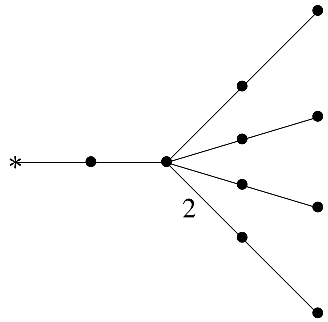

Figure 2. $\mathcal{G}_{(1^4,2),1} = \mathcal{G}_{S(3^2)>Q_8}$.

3. Goldman-type theorems for Frobenius groups

In this section, we establish weak Goldman-type theorems for all Frobenius groups, generalizing results obtained in [12].

For a tuple of natural numbers $\mathbf{m} = (m_0, m_1, \dots, m_l)$ with $m_0 = 1$ and $l \geq 1$ and a natural number n, we assign a bipartite graph $\mathcal{G}_{\mathbf{m},n}$ as follows. Let $I = \{0, 1, \dots, l\}$, and let J be an index set with $|J| = n$. The set of even vertices is $\{v_i^0\}_{i\in I} \sqcup \{v_j^2\}_{j\in J}$ and the set of odd vertices is $\{v_i^1\}_{i\in I}$. The only non-zero entries of the adjacency matrix Δ of $\mathcal{G}_{\mathbf{m},n}$ are

$$\Delta(v_i^0, v_i^1) = \Delta(v_i^1, v_i^0) = 1 \quad \forall i \in I,$$
$$\Delta(v_i^1, v_j^2) = \Delta(v_j^2, v_i^1) = m_i \quad \forall i \in I,\ \forall j \in J.$$

The vertex v_0^0 is treated as a distinguished vertex $*$.

We use notation $k^a = \overbrace{k, k, \dots, k}^{a}$ for short. With this convention, the graph $\mathcal{G}_{m,n}$ considered in [12] is $\mathcal{G}_{(1^m),n}$. An edge with a number b means a multi-edge with multiplicity b.

Let

$$m := \|\mathbf{m}\|^2 = \sum_{i=0}^{l} m_i^2.$$

Then, the Perron–Frobenius eigenvalue of Δ is $\sqrt{1+mn}$. The Perron–Frobenius eigenvector d with normalization $d(v_0^0) = 1$ is

$$d(v_i^0) = m_i, \quad d(v_i^1) = m_i\sqrt{1+mn}, \quad d(v_j^2) = m.$$

Let $G = K \rtimes H$ be a Frobenius group with the Frobenius kernel K and a Frobenius complement H. Then, we have $\mathcal{G}_{G>H} = \mathcal{G}_{\mathbf{m},n}$, where n is the number of H-orbits in $K \setminus \{e\}$ and $\mathbf{m}$ is the ranks of the irreducible representations of H. Therefore, we have $|H| = m$ and $|K| = 1 + mn$ (see Figure 2). If moreover K is abelian, the graph $\mathcal{G}_H^G$ is also $\mathcal{G}_{\mathbf{m},n}$.

Conversely, we can show the following theorem.

Theorem 3.1. *Let $N \supset P$ be a finite index inclusion of factors with $\mathcal{G}_{N\supset P} = \mathcal{G}_{\mathbf{m},n}$. Then, there exists a unique subfactor $R \subset P$, up to inner conjugacy, such that $N \cap R' = \mathbb{C}$, and there exists a Frobenius group $G = K \rtimes H$ with the Frobenius kernel K and a Frobenius complement H satisfying $|K| = 1 + mn$, $|H| = m$, the tuple $(m_0, m_1, \dots, m_l)$ being the ranks of the irreducible representations of H, and*

$$N = R \rtimes G \supset P = R \rtimes H.$$

Moreover,

(1) *If $n = 1$, then $1 + m$ is a prime power p^k with a prime p and $K = \mathbb{Z}_p^k$. The G-action on G/H is sharply 2-transitive. The dual principal graph is also $\mathcal{G}_{\mathbf{m},1}$ in this case.*

(2) *If $n = 2$ or $n = 3$, then $1 + mn$ is a prime power p^k with a prime p, and G is a primitive Frobenius group with $K = \mathbb{Z}_p^k$. The dual principal graph is also $\mathcal{G}_{\mathbf{m},n}$ in this case.*

We prove the theorem in several steps. Let $\iota : P \hookrightarrow N$ be the inclusion map. We denote by α_i the irreducible endomorphism of N corresponding to v_i^0 and by ρ_j the ones corresponding to v_j^2. Then, $\iota \circ \alpha_i$ corresponds to v_i^1 (see Figure 3). From the graph $\mathcal{G}_{\mathbf{m},n}$, we get the following fusion rules:

$$\begin{aligned} [\bar{\iota}][\iota] &= [\mathrm{id}] \oplus \bigoplus_{j\in J} [\rho_j], \\ [\iota][\rho_j] &= \bigoplus_{i\in I} m_i [\iota\alpha_i], \\ [\bar{\iota}][\iota\alpha_i] &= [\alpha_i] + m_i \bigoplus_{j\in J} [\rho_j], \\ d(\alpha_i) &= m_i, \quad d(\iota) = \sqrt{1 + mn}, \quad d(\rho_j) = m. \end{aligned}$$

Let $\mathcal{C}$ be the fusion category generated by $\bar{\iota}\iota$. Then, since $d(\alpha_{i_1}\alpha_{i_2})$ is smaller than $m = d(\rho_j)$, we have a fusion subcategory $\mathcal{C}_0$ with the set (of equivalence classes) of simple objects $\mathrm{Irr}(\mathcal{C}_0) = \{\alpha_i\}_{i\in I}$.

We introduce involutions of I and J by $[\overline{\alpha_i}] = [\alpha_{\bar{i}}]$ and $[\overline{\rho_j}] = [\rho_{\bar{j}}]$. Note that $\rho_j \rho_{\bar{j}}$ contains α_i at most $d(\alpha_i) = m_i$ times (see [18, p. 39]). Since it contains id, dimension counting shows that it contains α_i with full multiplicity m_i. Thus, the Frobenius reciprocity implies

$$[\alpha_i \rho_j] = m_i [\rho_j].$$

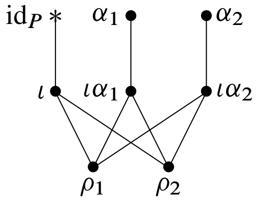

Figure 3. $\mathcal{G}_{(1^3),2} = \mathcal{G}_{\mathbb{Z}_7 \rtimes \mathbb{Z}_3 > \mathbb{Z}_3}$.

Lemma 3.2. *Let the notation be as above. There exist a unique intermediate subfactor $P \supset R_j \supset \rho_j(P)$ and an isomorphism $\theta_j : R_{\bar{j}} \to R_j$ for each $j \in J$ such that if $\kappa_j : R_j \hookrightarrow P$ is the inclusion map,*

$$[\rho_j] = [\kappa_j \theta_j \overline{\kappa_{\bar{j}}}],$$
$$[\kappa_j \overline{\kappa_j}] = \bigoplus_{i \in I} m_i [\alpha_i].$$

Moreover, $P \supset R_j$ is a depth 2 inclusion of index m.

Proof. Theorem 2.3 shows that there exists a unique intermediate subfactor $P \supset R_j \supset \rho_j(P)$ such that if $\kappa_j : R_j \hookrightarrow P$ is the inclusion map, we have

$$[\kappa_j \overline{\kappa_j}] = \bigoplus_{i \in I} m_i [\alpha_i].$$

Since $m_i = d(\alpha_i)$, Frobenius reciprocity implies

$$[\alpha_i][\kappa_i] = m_i [\kappa_i],$$

and $P \supset R_j$ is a depth 2 inclusion of index m.

Let σ_j be ρ_j regarded as a map from P to R_j. By definition, we have $\rho_j = \kappa_j \circ \sigma_j$, and since $d(\rho_j) = m$ and $d(\kappa_j) = \sqrt{m}$, we get $d(\sigma_j) = \sqrt{m}$. Taking conjugate, we get

$$[\rho_{\bar{j}}] = [\overline{\sigma_j}][\overline{\kappa_j}].$$

Perturbing $\overline{\sigma_j}$ by an inner automorphism if necessary, we may and do assume $\rho_{\bar{j}} = \overline{\sigma_j} \circ \overline{\kappa_j}$. Since $[\overline{\sigma_j} \sigma_j]$ contains id and is contained in $\rho_{\bar{j}} \rho_j$, dimension counting shows

$$[\overline{\sigma_j} \sigma_j] = \bigoplus_{i \in I} m_i [\alpha_i],$$

and Theorem 2.3 implies $\overline{\sigma_j}(R_j) = R_{\bar{j}}$. Let θ_j be the inverse of $\overline{\sigma_j}$, which is an isomorphism from $R_{\bar{j}}$ onto R_j. Then, we get $\rho_{\bar{j}} = \kappa_{\bar{j}} \circ \theta_j^{-1} \circ \overline{\kappa_j}$, and

$$[\rho_j] = [\kappa_j \theta_j \overline{\kappa_{\bar{j}}}].$$ ■

Lemma 3.3. *With the above notation, $\overline{\kappa}_j\kappa_k$ is decomposed into 1-dimensional sectors for all $j,k\in J$.*

Proof. Let

$$[\overline{\kappa_j}\kappa_k]=\bigoplus_{a\in\Lambda_{j,k}} n^a_{jk}[\xi^a_{jk}]$$

be the irreducible decomposition. Since

$$[\overline{\kappa_j}\kappa_k\overline{\kappa_k}\kappa_l]=\bigoplus_{i\in I} m_i[\overline{\kappa_j}\alpha_i\kappa_l]=\bigoplus_{i\in I} m_i^2[\overline{\kappa_j}\kappa_l]=m[\overline{\kappa_j}\kappa_l],$$

the product $\xi^a_{jk}\xi^b_{kl}$ is a direct sum of irreducibles from $\{\xi^c_{jl}\}_{c\in\Lambda_{j,l}}$. Since $[\overline{\overline{\kappa_k}\kappa_j}]=[\overline{\kappa_j}\kappa_k]$, we can arrange the index sets so that for any $a\in\Lambda_{j,k}$ there exists $\bar{a}\in\Lambda_{k,j}$ satisfying $[\overline{\xi^a_{jk}}]=[\xi^{\bar{a}}_{kj}]$.

Since

$$\delta_{j,k}=\dim(\rho_j,\rho_k)=\dim(\overline{\kappa_{\bar{j}}}\kappa_{\bar{k}},\theta_j^{-1}\overline{\kappa_j}\kappa_k\theta_k),$$

we have

$$\{[\theta_j^{-1}][\xi^a_{jj}][\theta_j]\}_{a\in\Lambda_{j,j}}\cap\{[\xi^b_{\bar{j}\bar{j}}]\}_{b\in\Lambda_{\bar{j},\bar{j}}}=[\mathrm{id}],\tag{3.1}$$

and for $j\neq k$,

$$\{[\theta_j^{-1}][\xi^a_{jk}][\theta_k]\}_{a\in\Lambda_{j,k}}\cap\{[\xi^b_{\bar{j}\bar{k}}]\}_{b\in\Lambda_{\bar{j},\bar{k}}}=\emptyset.$$

Assume we have ξ^a_{jk} with $d(\xi^a_{jk})>1$. Since $\kappa_{\bar{j}}\theta_j^{-1}\xi^a_{jk}\theta_k\overline{\kappa_{\bar{k}}}$ is contained in $\rho_{\bar{j}}\rho_k$, the former contains either α_i with $i\in I$ or ρ_l with $l\in J$. The first case never occurs because

$$\dim(\kappa_{\bar{j}}\theta_j^{-1}\xi^a_{jk}\theta_k\overline{\kappa_{\bar{k}}},\alpha_i)=\dim(\theta_j^{-1}\xi^a_{jk}\theta_k,\overline{\kappa_{\bar{j}}}\alpha_i\kappa_{\bar{k}})=m_i\dim(\theta_j^{-1}\xi^a_{jk}\theta_k,\overline{\kappa_{\bar{j}}}\kappa_{\bar{k}})=0.$$

Thus,

$$0\neq\dim(\kappa_{\bar{j}}\theta_j^{-1}\xi^a_{jk}\theta_k\overline{\kappa_{\bar{k}}},\rho_l)=\dim(\theta_j^{-1}\xi^a_{jk}\theta_k,\overline{\kappa_{\bar{j}}}\kappa_l\theta_l\overline{\kappa_{\bar{l}}}\kappa_{\bar{k}}),$$

and there exist $\xi^b_{\bar{j}l}$ and $\xi^c_{\bar{l}\bar{k}}$ such that $\theta_j^{-1}\xi^a_{jk}\theta_k$ is contained in $\xi^b_{\bar{j}l}\theta_l\xi^c_{\bar{l}\bar{k}}$. In fact, the latter is irreducible because of

$$\dim(\xi^b_{\bar{j}l}\theta_l\xi^c_{\bar{l}\,\bar{k}},\xi^b_{\bar{j}l}\theta_l\xi^c_{\bar{l}\,\bar{k}})=(\theta_l^{-1}\xi^{\bar{b}}_{l\bar{j}}\xi^b_{\bar{j}l}\theta_l,\xi^c_{\bar{l}\,\bar{k}}\xi^{\bar{c}}_{\bar{k}\bar{l}})$$

and equation (3.1). Therefore, we get

$$[\theta_j^{-1}\xi^a_{jk}\theta_k]=[\xi^b_{\bar{j}l}\theta_l\xi^c_{\bar{l}\,\bar{k}}].\tag{3.2}$$

Since $d(\xi^a_{jk})>1$, we have either $d(\xi^b_{\bar{j}l})>1$ or $d(\xi^c_{\bar{l}\,\bar{k}})>1$. We first assume that $d(\xi^c_{\bar{l}\,\bar{k}})>1$. We have $[\xi^a_{jk}\theta_k]=[\theta_j\xi^b_{\bar{j}l}\theta_l\xi^c_{\bar{l}\,\bar{k}}]$. Since $\kappa_j\theta_j\xi^b_{\bar{j}l}\theta_l\overline{\kappa_{\bar{l}}}$ is contained in $\rho_j\rho_l$, the former contains either α_i with $i\in I$ or ρ_r with $r\in J$. In the first case, we have

$$0\neq\dim(\kappa_j\theta_j\xi^b_{\bar{j}l}\theta_l\overline{\kappa_{\bar{l}}},\alpha_i)=\dim(\theta_j\xi^b_{\bar{j}l}\theta_l,\overline{\kappa_j}\alpha_i\kappa_{\bar{l}})=m_i\dim(\theta_j\xi^b_{\bar{j}l}\theta_l,\overline{\kappa_j}\kappa_{\bar{l}}),$$

and there exists $\xi^d_{j\bar{l}}$ satisfying $[\theta_j\xi^b_{\bar{j}l}\theta_l] = [\xi^d_{j\bar{l}}]$ and $[\xi^a_{jk}\theta_k] = [\xi^d_{j\bar{l}}\xi^c_{\bar{l}k}]$. By the Frobenius reciprocity, there exists $\xi^e_{k\bar{k}}$ satisfying $[\theta_k] = [\xi^e_{k\bar{k}}]$. Since

$$[\kappa_k\overline{\kappa_k}\kappa_{\bar{k}}] = \bigoplus_{i_1\in I} m_{i_1}[\alpha_g\kappa_{\bar{k}}] = m[\kappa_{\bar{k}}],$$

we get $[\kappa_k\xi^e_{k\bar{k}}] = [\kappa_{\bar{k}}]$, and

$$[\rho_k] = [\kappa_k\theta_k\overline{\kappa_{\bar{k}}}] = [\kappa_k\xi^e_{k\bar{k}}\overline{\kappa_{\bar{k}}}] = [\kappa_{\bar{k}}\overline{\kappa_{\bar{k}}}] = \bigoplus_{i_1\in I}[\alpha_{i_1}],$$

which is a contradiction. Thus, we are left with

$$0 \neq \dim(\kappa_j\theta_j\xi^b_{\bar{j}l}\theta_l\overline{\kappa_{\bar{l}}}, \rho_r) = \dim(\theta_j\xi^b_{\bar{j}l}\theta_l, \overline{\kappa_j}\kappa_r\theta_r\overline{\kappa_{\bar{r}}}\kappa_{\bar{l}}),$$

which shows that there exist ξ^e_{jr} and $\xi^f_{\bar{r}\bar{l}}$ satisfying

$$\dim(\theta_j\xi^b_{\bar{j}l}\theta_l, \xi^e_{jr}\theta_r\xi^f_{\bar{r}\bar{l}}) \neq 0.$$

As before, the right-hand side is irreducible, and we get $[\theta_j\xi^b_{\bar{j}l}\theta_l] = [\xi^e_{jr}\theta_r\xi^f_{\bar{r}\bar{l}}]$ and $[\xi^a_{jk}\theta_k] = [\xi^e_{jr}\theta_r\xi^f_{\bar{r}\bar{l}}\xi^c_{\bar{l}k}]$. Since the left-hand side is irreducible, so is $\xi^f_{\bar{r}\bar{l}}\xi^c_{\bar{l}k}$, and there exists $\xi^s_{\bar{r}k}$ satisfying $[\xi^f_{\bar{r}\bar{l}}\xi^c_{\bar{l}\,k}] = [\xi^s_{\bar{r}k}]$, and $[\xi^a_{jk}\theta_k] = [\xi^e_{jr}\theta_r\xi^s_{\bar{r}\,k}]$. Note that we have $d(\xi^s_{\bar{r}k}) > 1$. By the Frobenius reciprocity,

$$1 = \dim(\xi^a_{jk}\theta_k, \xi^e_{jr}\theta_r\xi^s_{\bar{r}k}) = \dim(\theta_r^{-1}\xi^{\bar{e}}_{rj}\xi^a_{jk}\theta_k, \xi^s_{\bar{r}k}),$$

and there exists ξ^t_{rk} satisfying $[\theta_r^{-1}\xi^t_{rk}\theta_k] = [\xi^s_{\bar{r}k}]$, which contradicts equation (3.1).

Now, the only possibility is $d(\xi^b_{\bar{j}l}) > 1$. Taking conjugate of equation (3.2), we get

$$[\theta_k^{-1}\xi^{\bar{a}}_{kj}\theta_j] = [\xi^{\bar{c}}_{k\bar{l}}\theta_l^{-1}\xi^{\bar{b}}_{l\bar{j}}],$$

and $[\xi^{\bar{a}}_{kj}\theta_j] = [\theta_k\xi^{\bar{c}}_{k\bar{l}}\theta_l^{-1}\xi^{\bar{b}}_{l\bar{j}}]$. Since $\kappa_k\theta_k\xi^{\bar{c}}_{k\bar{l}}\theta_l^{-1}\overline{\kappa_l}$ is contained in $\rho_k\rho_{\bar{l}}$, a similar argument as above works, and we get a contradiction again. Therefore, $d(\xi^a_{jk}) = 1$ for all j, k, a. ∎

Proof of Theorem 3.1. We fix $j_0 \in J$. Since $\overline{\kappa_{j_0}}\kappa_k$ contains an isomorphism

$$\varphi_j : R_j \to R_{j_0},$$

by the Frobenius reciprocity, we get $[\kappa_j] = [\kappa_{j_0}\varphi_j]$. Thus, there exists a unitary $u_j \in P$ satisfying $\mathrm{Ad}\, u_j \circ \kappa_j = \kappa_{j_0} \circ \varphi_j$, which means that for every $x \in R_j$,

$$u_j x u_j^* = \varphi_j(x).$$

This implies $u_j R_j u_j^* = R_{j_0}$. By replacing ρ_j with $\mathrm{Ad}\, u_j \circ \rho_j$ if necessary, we may assume $R_j = R_{j_0}$ for all $j \in J$. We denote $R = R_{j_0}$ and $\kappa = \kappa_{j_0}$ for simplicity. Now, we have $\theta_j \in \mathrm{Aut}(R)$ and $[\rho_j] = [\kappa\theta_j\bar{\kappa}]$.

Since $\bar{\kappa}\kappa$ is decomposed into 1-dimensional sectors, the inclusion $P \supset R$ is a crossed product by a finite group of order m, say, H, and there exists an outer action β of H on R such that $P = R \rtimes_\beta H$, and

$$[\bar{\kappa}\kappa] = \bigoplus_{h \in H} [\beta_h].$$

Note that $N \supset R$ is irreducible because

$$\dim(\iota\kappa, \iota\kappa) = \dim(\bar{\iota}\iota, \kappa\bar{\kappa}) = 1.$$

Now, we have

$$[\overline{(\iota\kappa)}\iota\kappa] = [\bar{\kappa}\,\bar{\iota}\iota\kappa] = [\bar{\kappa}\kappa] \oplus \bigoplus_{j \in J} [\bar{\kappa}\rho_j\kappa] = \bigoplus_{h \in H} [\beta_h] \oplus \bigoplus_{j \in J,\, h_1, h_2 \in H} [\beta_{h_1}\theta_j\beta_{h_2}].$$

This shows that there exists a finite group G including H, and its outer action γ on R extending β satisfying $N = R \rtimes_\gamma G$. Moreover,

$$\bigoplus_{g \in G} [\gamma_g] = \bigoplus_{h \in H} [\beta_h] \oplus \bigoplus_{j \in J,\, h_1, h_2 \in H} [\beta_{h_1}\theta_j\beta_{h_2}]$$

holds, which shows that every (H, H)-double coset except for H has size $|H|^2$. Therefore, G is a Frobenius group with a Frobenius complement H, and it is of the form $K \rtimes H$ with the Frobenius kernel K. Since $|K| = [N : P]$, we get $|K| = 1 + mn$.

When $n = 1$, we have $|K| = |H| + 1$, and G acting on G/H is a sharply 2-transitive permutation group.

For (2), it suffices to show that H is maximal in G. For this, it suffices to show that there is no non-trivial intermediate subfactor between N and P. Assume $n = 2$ first. Suppose that Q is a non-trivial intermediate subfactor, and let $\iota_1 : P \hookrightarrow Q$ be the inclusion map. Since $[\bar{\iota}\iota] = [\mathrm{id}] \oplus [\rho_1] \oplus [\rho_2]$, we have either $[\bar{\iota}_1\iota_1] = [\mathrm{id}] \oplus [\rho_1]$ or $[\bar{\iota}_1\iota_1] = [\mathrm{id}] \oplus [\rho_2]$. In any case, we get $[Q : P] = 1 + m$, and

$$[N : Q] = \frac{[N : P]}{[Q : P]} = \frac{1 + 2m}{1 + m} = 2 - \frac{1}{1 + m},$$

which is forbidden by the Jones theorem.

The case $n = 3$ can be treated in a similar way. ∎

Remark 3.4. The above theorem together with the classification of sharply 2-transitive permutation groups with abelian point stabilizers shows that the graph $\mathcal{G}_{(1^m),1}$

uniquely characterizes the group-subgroup subfactor for $H(q) = \mathbb{F}_q \rtimes \mathbb{F}_q^\times > \mathbb{F}_q^\times$ with $q = m + 1$. In the case of non-commutative H, probably the graph $\mathcal{G}_{\mathbf{m},1}$ does not uniquely determine the group $K \rtimes H$ in general. However, [10, Chapter XII, Theorem 9.7] shows that possibilities of $K \rtimes H$ for a given $q = m + 1$ are very much restricted. For example, the graph $\mathcal{G}_{(1^4,2),1}$ uniquely characterizes $S(3^2) > Q_8$.

In the rest of this section, we classify related fusion categories, which is a generalization of [3, (7.1)].

Let $\mathcal{C}_0$ be a C^*-fusion category with the set of (equivalence classes of) simple objects $\mathrm{Irr}(\mathcal{C}) = \{\alpha_i\}_{i\in I}$. We may assume $0 \in I$ and $\alpha_0 = 1$. Let $\mathcal{C}$ be a fusion category containing $\mathcal{C}_0$ with $\mathrm{Irr}(\mathcal{C}) = \{\alpha_i\}_{i\in I} \cup \{\rho\}$. Then, we have

$$\alpha_i \otimes \rho \cong \rho \otimes \alpha_i = d(\alpha_i)\rho.$$

Indeed, if $\alpha_i \otimes \rho$ contained α_j, the Frobenius reciprocity implies that $\alpha_{\bar{i}} \otimes \alpha_j$ would contain ρ, which is impossible, and the claim holds. In particular, $m_i = d(\alpha_i)$ is an integer. By the Frobenius reciprocity again, we get

$$\rho \otimes \rho \cong \bigoplus_{i\in I} m_i\alpha_i \oplus k\rho,$$

where k is a non-negative integer. We now consider the case with $k = m - 1$, where

$$m = \sum_{i\in I} m_i^2.$$

Then, $d(\rho) = m$.

Theorem 3.5. *Let $\mathcal{C}$ be as above. Then, there exists a sharply 2-transitive permutation group $G = K \rtimes H$ with the Frobenius kernel K and a Frobenius complement H such that $\mathcal{C}_0$ is equivalent to the representation category of H. In particular, the number $m + 1$ is a prime power p^k. The category $\mathcal{C}$ is classified by*

$$\{\omega \in H^3(K \rtimes H, \mathbb{T}) \mid \omega|_H = 0\}/\,\mathrm{Aut}(K \rtimes H, H),$$

(or equivalently by $H^3(K, \mathbb{T})^H / N_{\mathrm{Aut}(K)}(H)$).

Proof. For the proof of Theorem 3.5, we may assume that the category $\mathcal{C}$ is embedded in $\mathrm{End}(P)$ for a type III factor P.

In the same way as in the proofs of Theorem 3.1, there exist a unique subfactor $R \subset P$, up to inner conjugacy, a unique finite group H of order m, $\theta \in \mathrm{Aut}(R)$, and an outer action β of H on R such that

$$P = R \rtimes_\beta H,$$

and if $\kappa :\hookrightarrow P$ is the inclusion map,

$$[\kappa\overline{\kappa}] = \bigoplus_{i\in I} m_i[\alpha_i],$$
$$[\overline{\kappa}\kappa] = \bigoplus_{h\in H} [\beta_h],$$
$$[\rho] = [\kappa\theta\overline{\kappa}].$$

Let G be the group generated by $[\beta_H] = \{[\beta_h]\}_{h\in H}$ and $[\theta]$ in $\mathrm{Out}(R)$. We will show

$$G = [\beta_H] \sqcup [\beta_H][\theta][\beta_H],$$

whose order is $m(m+1)$, and it is a Frobenius group with a Frobenius complement $[\beta_H]$.

The proof of Lemma 3.3 shows $[\theta] \notin [\beta_H]$,

$$[\theta][\beta_H][\theta^{-1}] \cap [\beta_H] = [\mathrm{id}],$$

and $|[\beta_H][\theta][\beta_H]| = m^2$. Let $G_0 = [\beta_H] \cup [\beta_H][\theta][\beta_H]$, which is a subset of G with

$$|G_0| = m(m+1).$$

To prove that G_0 coincides with G, it suffices to show $[\theta][\beta_H][\theta] \subset G_0$ and $[\theta^{-1}] \in [\beta_H][\theta][\beta_H]$.

Let $h \in H$. Since $\kappa\theta\beta_h\theta\overline{\kappa}$ is contained in ρ^2, it contains either α_i with $i \in I$ or ρ. If it contains α_i, we have

$$0 \neq \dim(\kappa\theta\beta_h\theta\overline{\kappa}, \alpha_i) = \dim(\theta\beta_h\theta, \overline{\kappa}\alpha_i\kappa) = m_i \dim(\theta\beta_h\theta, \overline{\kappa}\kappa)$$
$$= m_i \sum_{k\in H} \dim(\theta\beta_h\theta, \beta_k),$$

which shows $[\theta\beta_h\theta] \in [\beta_H]$. If it contains ρ,

$$0 \neq \dim(\kappa\theta\beta_h\theta\overline{\kappa}, \kappa\theta\overline{\kappa}) = \dim(\theta\beta_h\theta, \overline{\kappa}\kappa\theta\overline{\kappa}\kappa) = \sum_{k,l} \dim(\theta\beta_h\theta, \beta_k\theta\beta_l),$$

which shows $[\theta\beta_h\theta] \in [\beta_H][\theta][\beta_H]$. Therefore, we get $[\theta][\beta_H][\theta] \subset G_0$.

Since ρ is self-conjugate, we have

$$1 = \dim(\overline{\rho}, \rho) = \dim(\kappa\theta^{-1}\overline{\kappa}, \kappa\theta\overline{\kappa}) = \dim(\theta^{-1}, \overline{\kappa}\kappa\theta\overline{\kappa}\kappa) = \sum_{h,k\in H} \dim(\theta^{-1}, \beta_h\theta\beta_k),$$

which shows $[\theta^{-1}] \in [\beta_H][\theta][\beta_H]$. Therefore, we get $G = G_0$.

Since G has only two (H,H)-double cosets and the size of $[\beta_H][\theta][\beta_H]$ is $|H|^2$, the group G is a Frobenius group with a Frobenius complement $[\beta_H]$. Moreover, the G action on G/H is sharply 2-transitive.

For the classification of the category $\mathcal{C}$, we may assume that R is the injective type III_1 factor. Then, the conjugacy class of G in $\mathrm{Out}(R)$ is completely determined by its obstruction class $\omega \in H^3(G,\mathbb{T})$. Since H has a lifting $\beta_H \subset \mathrm{Aut}(R)$, the restriction of ω to H is trivial. Since H is a Frobenius complement, the Schur multiplier $H^2(H,\mathbb{T})$ is trivial, and the lifting is unique, up to cocycle conjugacy, and one can uniquely recover P from R and $[\beta_H]$. This means that the generator ρ of the category $\widetilde{\mathcal{C}}$ is uniquely determined by ω. On the other hand, there always exists a G-kernel in $\mathrm{Out}(R)$ for a given $\omega \in H^3(G,\mathbb{T})$, which shows the existence part of the statement.

Finally, since $|K|$ and $|H|$ are relatively prime, we have

$$E_2^{p,q} = H^p(H, H^q(K,\mathbb{T})) = 0$$

for $p,q \geq 1$ in the Lindon/Hochschild–Serre spectral sequence for $G = K \rtimes H$. Thus, the group

$$\{\omega \in H^3(G,\mathbb{T}) \mid \omega|_H = 0\}$$

is isomorphic to $H^3(K,\mathbb{T})^H$. ■

When H is abelian (in fact cyclic in this case), the group $H^3(K,\mathbb{T})^H$ is explicitly computed in [3, Corollary 7.4].

4. Goldman-type theorems for sharply 3-transitive permutation groups

Let $\mathbf{m}$, n, I, and m be as in the previous section. Now, we consider the graph $\widetilde{\mathcal{G}_{\mathbf{m},1}}$ (see Section 2.3 for the definition of $\widetilde{\mathcal{G}}$ for a given $\mathcal{G}$), which is described as follows. The set of even vertices of $\widetilde{\mathcal{G}_{\mathbf{m},1}}$ is

$$\{v_i^0\}_{i\in I} \sqcup \{v_i^2\}_{\in I} \sqcup \{v^4\};$$

the set of odd vertices is

$$\{v_i^1\}_{i\in I} \sqcup \{v^3\}.$$

The only non-zero entries of the adjacency matrix Δ of $\widetilde{\mathcal{G}_{\mathbf{m},1}}$ are

$$\begin{aligned}
\Delta(v_i^0, v_i^1) &= \Delta(v_i^1, v_i^0) = 1 \qquad \forall i \in I,\\
\Delta(v_i^1, v_i^2) &= \Delta(v_i^2, v_i^1) = 1 \qquad \forall i \in I,\\
\Delta(v_i^2, v^3) &= \Delta(v^3, v_i^2) = m_i \quad\ \forall i \in I,\\
\Delta(v^3, v^4) &= \Delta(v^4, v^3) = 1.
\end{aligned}$$

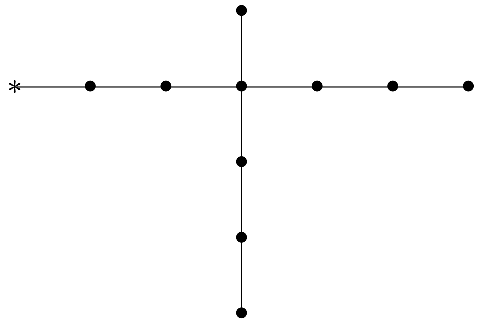

Figure 4. $\widetilde{\mathscr{G}_{(1^3),1}} = \mathscr{G}_{(L(2^2),\mathrm{PG}_1(2^2))} = \mathscr{G}_{(\mathfrak{A}_5, X_5)}$.

The vertex v_0^0 is treated as a distinguished vertex $*$. The Perron–Frobenius eigenvalue of Δ is $\sqrt{2+m}$ (see Figure 4). The Perron–Frobenius eigenvector d normalized as $d(v_0^0) = 1$ is

$$d(v_i^0) = m_i, \quad d(v_i^2) = m_i(1+mn), \quad d(v_4) = m.$$

$$d(v_i^1) = m_i\sqrt{2+mn}, \quad d(v_3) = m\sqrt{2+mn}.$$

In [19], we showed that a strong Goldman-type theorem holds for $\widetilde{\mathscr{G}_{(1^3),1}}$. Now, we show it for general sharply 3-transitive permutation groups.

Although we excluded the case $\mathbf{m} = (1)$ in the definition of $\mathscr{G}_{\mathbf{m},1}$ in Section 3, the graph itself makes sense for $\mathbf{m} = (1)$, and we include this case in the next theorem.

Theorem 4.1. *Let $M \supset N$ be a finite index subfactor with $\mathscr{G}_{M\supset N} = \widetilde{\mathscr{G}_{\mathbf{m},1}}$. Then,*

$$q = 1 + m$$

is a prime power, and there exists a unique subfactor $R \subset N$ that is irreducible in M such that if $\mathbf{m} = 1^m$,

$$M = R \rtimes L(q) \supset N = R \rtimes H(q),$$

and otherwise,

$$M = R \rtimes M(q) \supset N = R \rtimes S(q).$$

Proof. If $\mathbf{m} = (1)$, the graph $\widetilde{\mathscr{G}_{(1),1}}$ is nothing but the Coxeter graph A_5, and the statement follows from [11] as $(\mathfrak{S}_3, X_3) \cong (\mathrm{PGL}_2(2), \mathrm{PG}_1(2))$. We assume $\mathbf{m} \neq (1)$ in what follows.

We follow the strategy described in Section 2.5 taking the 6 steps.

(1) Let $\varepsilon : N \hookrightarrow M$ be the inclusion map, and let $\mathscr{C}$ be the fusion category generated by $\bar{\varepsilon}\varepsilon$. We first parameterize $\mathrm{Irr}(\mathscr{C})$. Let $[\bar{\varepsilon}\varepsilon] = [\mathrm{id}] \oplus [\sigma]$ be the irreducible

decomposition, which means that σ corresponds to the vertex v_0^2. We denote by α_i' and ρ' the endomorphisms of N corresponding to v_i^0 and v^4, respectively. Then, $\varepsilon\alpha_i'$, $\sigma\alpha_i'$, and $\varepsilon\rho'$ are irreducible, and they correspond to v_i^1, v_i^2, v^3, respectively. Thus,

$$\mathrm{Irr}(\mathcal{C}) = \{\alpha_i'\}_{i\in I} \sqcup \{\sigma\alpha_i'\}_{i\in I} \sqcup \{\rho'\}.$$

We have

$$d(\alpha_i) = m_i, \quad d(\varepsilon) = \sqrt{2+m}, \quad d(\sigma) = 1+m, \quad d(\rho') = m.$$

Two endomorphisms σ and ρ' are self-conjugate. We introduce two involutions of I by $\overline{[\alpha_i']} = [\alpha_{\bar{i}}']$ and $\overline{[\sigma\alpha_i']} = [\sigma\alpha_{i^*}']$. Then, they are related by

$$[\sigma\alpha_{i^*}'] = [\alpha_{\bar{i}}'\sigma].$$

By dimension counting, we see that there exists a fusion subcategory $\mathcal{C}_0$ of $\mathcal{C}$ with

$$\mathrm{Irr}(\mathcal{C}_0) = \{\alpha_i'\}_{i\in I}.$$

We claim that there exists another fusion subcategory $\mathcal{C}_1$ of $\mathcal{C}$ with

$$\mathrm{Irr}(\mathcal{C}_1) = \{\alpha_i'\}_{i\in I} \sqcup \{\rho'\}.$$

Indeed, if $\rho'\alpha_i$ contained $\sigma\alpha_{i_1}'$, the Frobenius reciprocity implies that $\sigma\alpha_{i_1}'\alpha_{\bar{i}}'$ would contain ρ', and hence, $\sigma\alpha_{i_2}'$ would contain ρ' for some i_2, which is a contradiction. Thus, $\rho\alpha_i$ is decomposed into a direct sum of sectors in $\{\alpha_{i_1}'\}_{i_1\in I} \cup \{\rho'\}$, and dimension counting shows

$$[\rho'\alpha_i'] = m_i[\rho'], \quad [\alpha_i'\rho] = m_i[\rho],$$

where the second equality follows from the first one by conjugation.

From the shape of the graph $\widetilde{\mathcal{G}_{\mathbf{m},1}}$, we can see

$$[\sigma^2] = [\mathrm{id}] \oplus [\rho'] \oplus \bigoplus_{i\in I} m_i[\sigma\alpha_i'], \tag{4.1}$$

$$[\sigma\rho'] = \bigoplus_{i\in I} m_i[\sigma\alpha_i']. \tag{4.2}$$

Using these and associativity, we have

$$\begin{aligned}[\sigma][\sigma\rho'] &= \bigoplus_{i_1\in I} m_i[\sigma][\sigma\alpha_i'] \\ &= \bigoplus_{i\in I} m_i\Big([\mathrm{id}] \oplus [\rho'] \oplus \bigoplus_{i'\in I} m_{i'}[\sigma\alpha_{i'}']\Big)[\alpha_i'] \\ &= \bigoplus_{i\in I} m_i[\alpha_i'] \oplus m[\rho'] \oplus \bigoplus_{i,i'} m_{i'}[\sigma\alpha_{i'}'\alpha_i'].\end{aligned}$$

On the other hand,

$$[\sigma][\sigma\rho'] = [\sigma^2][\rho'] = \Big([\mathrm{id}] \oplus [\rho'] \oplus \bigoplus_{i\in I} m_i[\sigma\alpha'_i]\Big)[\rho']$$
$$= [\rho'] \oplus [\rho'^2] \oplus \bigoplus_{i\in I} m_i^2[\sigma\rho'] = [\rho'] \oplus [\rho'^2] \oplus m \bigoplus_{i\in I} m_i[\sigma\alpha'_i].$$

Since $\sigma\alpha'_{i'}\alpha'_i$ is a direct sum of irreducibles of the form $\sigma\alpha'_{i''}$, the endomorphism ρ^2 contains

$$\bigoplus_{i\in I} m_i[\alpha'_i] \oplus (m-1)[\rho'],$$

and comparing dimensions, we get

$$[\rho'^2] = \bigoplus_{i\in I} m_i[\alpha'_i] \oplus (m-1)[\rho'].$$

Therefore, the claim is shown.

(2) Form equation (4.1) and Theorem 2.3, there exists a unique intermediate subfactor $N \supset P \supset \sigma(N)$ such that if $\iota : P \hookrightarrow N$ is the inclusion map, we have

$$[\iota\bar{\iota}] = [\mathrm{id}] \oplus [\rho'].$$

Let $\mathcal{C}_2$ be the fusion category generated by $\bar{\iota}\iota$. As in the proof of Lemma 3.2, there exists $\tau \in \mathrm{Aut}(P)$ satisfying

$$[\sigma] = [\iota\tau\bar{\iota}].$$

(3) From the fusion rules of $\mathcal{C}_1$, we can see that the dual principal graph $\mathcal{G}^d_{N\supset P}$ is $\mathcal{G}_{\mathbf{m},1}$, and Theorem 3.1 (1) shows that so is the principal graph $\mathcal{G}_{N>P}$ too. Therefore, we can arrange the labeling of irreducibles of $\mathcal{C}_2$ so that

$$\mathrm{Irr}(\mathcal{C}_2) = \{\alpha_i\} \sqcup \{\rho_i\},$$

and $[\alpha'_i\iota] = [\iota\alpha_i]$ and $[\bar{\iota}\iota] = [\mathrm{id}] \oplus [\rho]$.

(4) Now, we apply Theorem 3.1, and we get a unique subfactor $R \subset P$, up to inner conjugacy such that $R' \cap P = \mathbb{C}$, and there exists an outer action β of a Frobenius group $K \rtimes H$ satisfying

$$N = R \rtimes_\beta (H \ltimes K) \supset P = R \rtimes_\beta H.$$

Moreover, the $K \rtimes H$-action on $(K \rtimes H)/H$ is sharply 2-transitive. We denote by $\kappa : R \hookrightarrow P$ the inclusion map. Then, we have

$$[\iota\kappa\bar{\kappa}\bar{\iota}] = \bigoplus_{i\in I} m_i[\iota\alpha_i\bar{\iota}] = \bigoplus_{i\in I} m_i[\alpha'_i\iota\bar{\iota}] = \bigoplus_{i\in I} m_i[\alpha'_i]([\mathrm{id}] \oplus \rho') = \bigoplus_{i\in I} m_i[\alpha'_i] \oplus m[\rho'],$$

which shows

$$\dim(\varepsilon\iota\kappa, \varepsilon\iota\kappa) = \dim(\bar{\varepsilon}\varepsilon, \iota\kappa\bar{\kappa}\,\bar{\iota}) = 1,$$

and R is irreducible in M.

(5) Since

$$[M : P] = [M : N][N : P][P : R] = (m+2)(m+1)m,$$

to prove that the inclusion $L \supset R$ is of depth 2, it suffices to show that the number $(m+2)(m+1)m$ coincides with the following dimension:

$$\begin{aligned}\dim(\varepsilon\iota\kappa\overline{(\varepsilon\iota\kappa)}, \varepsilon\iota\kappa\overline{(\varepsilon\iota\kappa)}) &= \dim(\bar{\varepsilon}\varepsilon\iota\kappa\bar{\kappa}\,\bar{\iota}\iota\kappa\bar{\kappa}\,\bar{\iota}\,\bar{\varepsilon}\varepsilon) \\ &= \dim((\mathrm{id}\oplus\sigma)\iota\kappa\bar{\kappa}\,\bar{\iota}, \iota\kappa\bar{\kappa}\,\bar{\iota}(\mathrm{id}\oplus\sigma)).\end{aligned}$$

Note that $[\sigma]$ commutes with $[\rho']$ and

$$\bigoplus_{i\in I} m_i[\alpha_i'],$$

and hence with $[\iota\kappa\bar{\kappa}\,\bar{\iota}]$. Thus, this number is equal to

$$= \dim((\mathrm{id}\oplus\sigma)\iota\kappa\bar{\kappa}\,\bar{\iota}, (\mathrm{id}\oplus\sigma)\iota\kappa\bar{\kappa}\,\bar{\iota}) = \dim((\mathrm{id}\oplus\sigma)^2, (\iota\kappa\bar{\kappa}\,\bar{\iota})^2).$$

Since the fusion category generated by $\iota\kappa\bar{\kappa}\,\bar{\iota}$ is equivalent to the representation category $\mathrm{Rep}(K \rtimes H)$ and $\iota\kappa\bar{\kappa}\,\bar{\iota}$ corresponds to the regular representation of $K \rtimes H$, we get

$$[(\iota\kappa\bar{\kappa}\,\bar{\iota})^2] = m(m+1)[\iota\kappa\bar{\kappa}\,\bar{\iota}].$$

Thus,

$$\begin{aligned}&\dim((\mathrm{id}\oplus\sigma)^2, (\iota\kappa\bar{\kappa}\,\bar{\iota})^2) \\ &\quad = m(m+1)(\mathrm{id}\oplus 2\sigma\oplus\sigma^2, \iota\kappa\bar{\kappa}\,\bar{\iota}) \\ &\quad = m(m+1)\dim\Big(2\mathrm{id}\oplus\rho\oplus 2\sigma\oplus\bigoplus_{i\in I} m_i\sigma\alpha_i', \bigoplus_{i\in I} m_i\alpha_i'\oplus m\rho'\Big) \\ &\quad = m(m+1)(m+2),\end{aligned}$$

and the inclusion $M \supset R$ is of depth 2.

(6) Now, Lemma 2.5 shows that we have

$$m = \dim(\tau\bar{\iota}\iota\kappa\bar{\kappa}\tau^{-1}, \bar{\iota}\iota\kappa\bar{\kappa})$$

and

$$[\bar{\iota}\iota\kappa\bar{\kappa}] = \Big[(\mathrm{id}\oplus\rho)\bigoplus_{i\in I} m_i\alpha_i\Big] = \bigoplus_{i\in I} m_i[\alpha_i]\oplus m[\rho].$$

Dimension counting implies

$$m = \dim\Big(\bigoplus_{i\in I} m_i \tau\alpha_i\tau^{-1}, \bigoplus_{i\in I} m_i\alpha_i\Big),$$

and this is possible only if $[\tau\kappa\bar{\kappa}\tau^{-1}] = [\kappa\bar{\kappa}]$. Since H is a Frobenius complement, every abelian subgroup of H is cyclic, and Lemma 2.6 implies there exists $\tau_1 \in \mathrm{Aut}(R)$ satisfying $[\tau\kappa] = [\kappa][\tau_1]$.

Now, Lemma 2.4 shows that there exists a group G including $K \rtimes H$, and outer G-action on R extending β satisfying $M = R \rtimes_\gamma G$. The principal graph $\mathcal{G}_{M\supset N}$ shows that the G-action on $G/(K \rtimes H)$ is 3-transitive. Since

$$|G/(K \rtimes H)| = m+2 \quad \text{and} \quad |G| = m(m+1)(m+2),$$

the permutation group G is sharply 3-transitive. Now, the statement follows from the classification of sharply 3-transitive permutation groups. ∎

We devote the rest of this section to a preparation of the Goldman-type theorem for the Mathieu groups M_{11}. Since $M(3^2)$ and $S(3^2)$ are a point stabilizer and a two-point stabilizer of the sharply 4-transitive action of M_{11}, we denote $M(3^2) = M_{10}$ and $S(3^2) = M_9$. We first determine the dual principal graph $\mathcal{G}_N^M$ in the case of $M_{10} > M_9$. Since this graph is the induction-reduction graph $\mathcal{G}_{M_9}^{M_{10}}$, the irreducible M-M sectors are parameterized by the irreducible representations of M_{10}, whose ranks are 1, 1, 9, 9, 10, 10, 10, 16 (see [5, Table 8]).

We parameterize the irreducible N-N and M-N sectors as in the above proof and Figure 5. Theorem 3.1 (1) shows that $N \supset P$ and its dual inclusion are isomorphic subfactors associated with $(S(3^2), \mathbb{F}_{3^2})$ (see Remark 3.4), and the two fusion categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent. On the other hand, the fusion subcategory generated by $\kappa\bar{\kappa}$ in $\mathcal{C}_2$ is equivalent to $\mathrm{Rep}(Q_8)$. Thus, the fusion category $\mathcal{C}_0$ is equivalent to $\mathrm{Rep}(Q_8)$. In particular, we have $\bar{i} = i$ for all i. Since at least one of $\{1,2,3\}$ is fixed by the other involution $i \mapsto i^*$, we may and do assume $1^* = 1$, and $\sigma\alpha_1$ is self-conjugate. Since $d(\alpha_4') = 2$, the two sectors α_4' and $\sigma\alpha_4'$ are self-conjugate.

Let $[\varepsilon\bar{\varepsilon}] = [\mathrm{id}] \oplus [\pi]$ be the irreducible decomposition. Then, $d(\pi) = 9$. Since

$$\begin{aligned}\dim(\varepsilon\alpha_i'\bar{\varepsilon}, \varepsilon\alpha_i'\bar{\varepsilon}) &= \dim(\bar{\varepsilon}\varepsilon\alpha_i', \alpha_i'\bar{\varepsilon}\varepsilon) = \dim((\mathrm{id}\oplus\sigma)\alpha_i', \alpha_i'(\mathrm{id}\oplus\sigma)) \\ &= 1 + \dim(\sigma\alpha_i', \sigma\alpha_{i^*}'),\end{aligned}$$

if $i^* = i$, the endomorphism $\varepsilon\alpha_i'\bar{\varepsilon}$ is decomposed into two irreducibles, and otherwise, it is irreducible. Thus, $\varepsilon\alpha_1'\bar{\varepsilon}$ is decomposed into two irreducibles. Since $d(\varepsilon\alpha_1'\bar{\varepsilon}) = 10$, it is a direct sum of a 1-dimensional representation and a 9-dimensional representation, and we denote the former by χ. Then, the Frobenius reciprocity implies $[\chi\varepsilon] = [\varepsilon\alpha_1']$, and

$$[\varepsilon\alpha_1'\bar{\varepsilon}] = [\chi\varepsilon\bar{\varepsilon}] = [\chi] \oplus [\chi\pi].$$

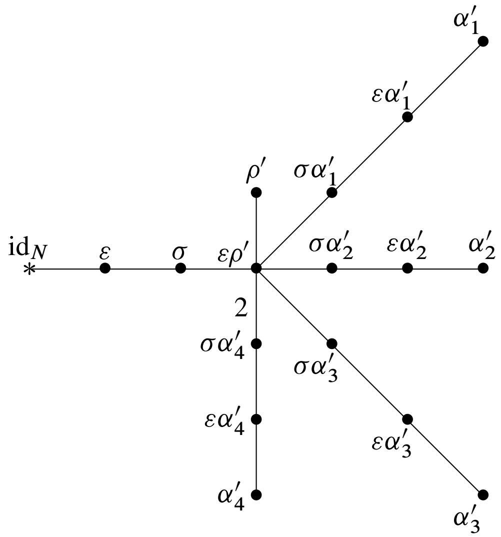

Figure 5. $\widetilde{\mathcal{G}_{(1^4,2),1}} = \mathcal{G}_{(M(3^2),\mathrm{PG}_1(3^2))}$.

Since $\varepsilon\alpha'_i\overline{\varepsilon}$, for $i = 2, 3$, cannot contain a 1-dimensional representation, we have $2^* = 3$, and $\xi := \varepsilon\alpha'_2\overline{\varepsilon}$ is irreducible. By

$$\begin{aligned}[][\varepsilon\alpha'_i\overline{\varepsilon}][\varepsilon] &= [\varepsilon\alpha'_i(\mathrm{id}\oplus\sigma)] = [\varepsilon\alpha'_i] \oplus [\varepsilon\alpha'_i\sigma] \\ &= [\varepsilon\alpha'_i] \oplus [\varepsilon\sigma\alpha'_{i^*}] = [\varepsilon\alpha'_i] \oplus [\varepsilon\alpha'_{i^*}] \oplus d(\alpha'_i)[\varepsilon\rho'],\end{aligned}$$

and the Frobenius reciprocity, we also have $[\varepsilon\alpha'_2\overline{\varepsilon}] = [\xi]$, and

$$[\xi\varepsilon] = [\varepsilon\alpha'_2] \oplus [\varepsilon\alpha'_3] \oplus [\varepsilon\rho'].$$

Since $d(\varepsilon\alpha'_4\overline{\varepsilon}) = 20$, we have

$$[\varepsilon\alpha'_4\overline{\varepsilon}] = [\eta_1] \oplus [\eta_2],$$

with $d(\eta_1) = d(\eta_2) = 10$, and

$$\begin{aligned}[][\eta_1\varepsilon] &= [\varepsilon\alpha'_4] \oplus [\varepsilon\rho'], \\ [\eta_2\varepsilon] &= [\varepsilon\alpha'_4] \oplus [\varepsilon\rho'].\end{aligned}$$

There is one irreducible representation of M_{10} missing, which we denote by ζ. By the Frobenius reciprocity and $d(\zeta) = 16$, we get

$$[\varepsilon\rho'\overline{\varepsilon}] = [\pi] \oplus [\chi\pi] \oplus [\xi] \oplus [\eta_1] \oplus [\eta_2] \oplus 2[\zeta].$$

Thus, the graph $\mathcal{G}_{M_9}^{M_{10}}$ is as in Figure 6.

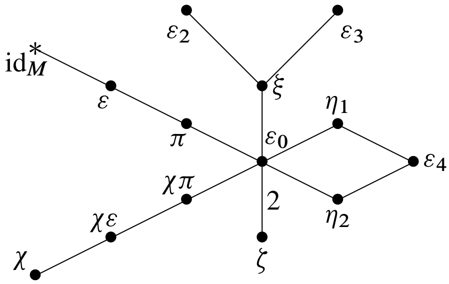

Figure 6. $\mathscr{G}_{M_9}^{M_{10}}$.

Theorem 4.2. *Let $M \supset N$ be a finite-index subfactor with $\mathscr{G}^d_{M\supset N} = \mathscr{G}_{M_9}^{M_{10}}$. Then, we have*

$$\mathscr{G}_{M\supset N} = \mathscr{G}_{M_{10}>M_9}.$$

In consequence, there exists a unique subfactor $R \subset N$ up to inner conjugacy, which is irreducible in M such that

$$M = R \rtimes M_{10} \supset N = R \rtimes M_9.$$

We divide the proof into a few steps. We parameterize the M-M sectors and M-N sectors as in Figure 6. Then,

$$d(\chi) = 1, \quad d(\pi) = 9, \quad d(\xi) = d(\eta_1) = d(\eta_2) = 10, \quad d(\zeta) = 16,$$
$$d(\varepsilon) = d(\varepsilon_2) = d(\varepsilon_3) = \sqrt{10}, \quad d(\varepsilon_4) = 2\sqrt{10}, \quad d(\varepsilon_0) = 8\sqrt{10}.$$

From the graph, we can see that π, $\chi\pi$, χ, ζ are self-conjugate,

$$\{[\overline{\xi}], [\overline{\eta_1}], [\overline{\eta_1}]\} = \{[\xi], [\eta_1], [\eta_2]\},$$

and this with the graph symmetry implies

$$[\chi^2] = [\mathrm{id}], \quad [\chi\pi] = [\pi\chi], \quad [\chi\zeta] = [\zeta\chi] = [\zeta], \quad [\chi\xi] = [\xi],$$
$$\{[\chi\eta_1], [\chi\eta_2]\} = \{[\eta_1], [\eta_2]\}.$$

The basic fusion rules coming from the graph are

$$[\pi\varepsilon] = [\varepsilon] \oplus [\varepsilon_0], \quad [\zeta\varepsilon] = 2[\varepsilon_0], \quad [\xi\varepsilon] = [\varepsilon_2] \oplus [\varepsilon_3] \oplus [\varepsilon_0],$$
$$[\eta_1\varepsilon] = [\eta_2\varepsilon] = [\varepsilon_4] \oplus [\varepsilon_0],$$
$$[\varepsilon\overline{\varepsilon}] = [\mathrm{id}] \oplus [\pi], \quad [\varepsilon_2\overline{\varepsilon}] = [\varepsilon_3\overline{\varepsilon}] = [\xi], \quad [\varepsilon_4\overline{\varepsilon}] = [\eta_1] \oplus [\eta_2],$$
$$[\varepsilon_0\overline{\varepsilon}] = [\pi] \oplus [\chi\pi] \oplus [\xi] \oplus [\eta_1] \oplus [\eta_2] \oplus 2[\zeta].$$

We denote the last sector by Σ for simplicity. Then, we have $\overline{\Sigma} = \Sigma$, and associativity implies

$$[\pi^2] = [\mathrm{id}] \oplus \Sigma, \quad [\xi\pi] = [\xi] \oplus \Sigma, \quad [\eta_1\pi] = [\eta_2] \oplus \Sigma,$$
$$[\eta_2\pi] = [\eta_1] \oplus \Sigma, \quad [\zeta\pi] \oplus [\zeta] = 2\Sigma.$$

The Frobenius reciprocity implies

$$\dim(\bar{\xi}\xi, \pi) = \dim(\bar{\eta}_2\eta_1, \pi) = \dim(\bar{\eta}_1\eta_2, \pi) = 2, \tag{4.3}$$
$$\dim(\bar{\eta}_i\xi, \pi) = \dim(\bar{\xi}\eta_i, \pi) = \dim(\bar{\eta}_i\eta_i, \pi) = 1. \tag{4.4}$$
$$\dim(\bar{\xi}\zeta, \pi) = \dim(\bar{\eta}_i\zeta, \pi) = 2, \tag{4.5}$$
$$\dim(\bar{\zeta}\zeta, \pi) = 3. \tag{4.6}$$

Lemma 4.3. *With the above notation, we have* $[\bar{\xi}] = [\xi]$ *and* $[\chi\eta_1] = [\eta_1\chi] = [\eta_2]$.

Proof. Note that we have $[\chi\xi] = [\xi]$. First, we claim $[\xi\chi] = [\xi]$. Indeed, assume that it is not the case. Then, we may assume $[\xi\chi] = [\eta_1]$, which implies

$$[\chi\eta_1] = [\chi\xi\chi] = [\xi\chi] = [\eta_1],$$

and so, $[\chi\eta_2] = [\eta_2]$. Since $\{[\bar{\xi}], [\bar{\eta}_1], [\bar{\eta}_2]\} = \{[\xi], [\eta_1], [\eta_2]\}$, we get a contradiction, and the claim holds.

Now, to prove the statement, it suffices to show $[\eta_2\chi] = [\eta_3]$. For this, we assume $[\eta_1\chi] = [\eta_1]$ (and consequently $[\eta_2\chi] = [\eta_2]$) and will deduce contradiction. Taking conjugate, we also have $[\chi\eta_1] = [\eta_1]$ and $[\chi\eta_2] = [\eta_2]$ in this case. Then, since $[\bar{\xi}\xi]$ contains π with multiplicity 2 and $[\chi\bar{\xi}] = [\bar{\xi}]$, it contains $[\chi\pi]$ with multiplicity 2, and so, dimension counting shows

$$[\bar{\xi}\xi] = [\mathrm{id}] \oplus [\chi] \oplus 2[\pi] \oplus 2[\chi\pi] \oplus 2[\zeta] \oplus 3 \times 10 \dim, \tag{4.7}$$

where $3 \times 10 \dim$ means a direct sum of 3 elements from $\{\xi, \eta_1, \eta_2\}$. In the same way, we get

$$[\bar{\eta}_1\eta_1] = [\mathrm{id}] \oplus [\chi] \oplus [\pi] \oplus [\chi\pi] \oplus 80 \dim,$$

where the last part is decomposed as either $80 = 5 \times 16$ or $80 = 8 \times 10$. Also, we get

$$[\bar{\eta}_1\eta_2] = [\bar{\eta}_2\eta_1] = 2[\pi] \oplus 2[\chi\pi] \oplus 4[\zeta].$$

This implies

$$\begin{aligned} 0 = \dim(\bar{\eta}_1\eta_2, \xi) &= \dim(\bar{\eta}_1\eta_2, \eta_1) = \dim(\bar{\eta}_1\eta_2, \eta_2) \\ &= \dim(\bar{\eta}_2\eta_1, \xi) = \dim(\bar{\eta}_2\eta_1, \eta_1) = \dim(\bar{\eta}_2\eta_1, \eta_2). \end{aligned} \tag{4.8}$$

Also, the Frobenius reciprocity implies

$$d(\eta_1\zeta, \eta_2) = 4.$$

Since $[\zeta\chi] = [\zeta]$, equation (4.5) shows

$$\dim(\eta_1\zeta, \chi\pi) = \dim(\eta_1\zeta, \pi\chi) = \dim(\eta_1\zeta, \pi) = 2$$

and

$$[\eta_1\zeta] = 2[\pi] \oplus 2[\chi\pi] \oplus 4[\zeta] \oplus 4[\eta_2] \oplus 2 \times 10 \text{ dim}.$$

Since $\dim(\eta_1, \eta_1\zeta) = \dim(\overline{\eta_1}\eta_1, \zeta)$ is either 5 or 0, we get

$$[\eta_1\zeta] = 2[\pi] \oplus 2[\chi\pi] \oplus 4[\zeta] \oplus 4[\eta_2] \oplus 2[\xi]$$

and

$$[\overline{\eta_1}\eta_1] = [\text{id}] \oplus [\chi] \oplus [\pi] \oplus [\chi\pi] \oplus 8 \times 10 \text{ dim}. \tag{4.9}$$

A similar reasoning shows

$$[\overline{\eta_i}\xi] = [\pi] \oplus [\chi\pi] \oplus 2[\zeta] \oplus 5 \times 10 \text{ dim}, \tag{4.10}$$

For the contragredient map, we have the following 3 possibilities up to relabeling η_1 and η_2:

(i) $[\overline{\xi}] = [\xi]$, $[\overline{\eta_1}] = [\eta_1]$, $[\overline{\eta_2}] = [\eta_2]$,

(ii) $[\overline{\xi}] = [\xi]$, $[\overline{\eta_1}] = [\eta_2]$, $[\overline{\eta_2}] = [\eta_1]$,

(iii) $[\overline{\xi}] = [\eta_1]$, $[\overline{\eta_1}] = [\xi]$, $[\overline{\eta_2}] = [\eta_2]$.

However, direct computation shows that there are no fusion rules consistent with equations (4.7), (4.8), (4.9), and (4.10) in each case. ∎

Lemma 4.4. *With the above notation,*

$$\begin{aligned}
&[\chi\varepsilon_2] = [\varepsilon_3],\\
&[\varepsilon_2\overline{\varepsilon_2}] = [\text{id}] \oplus [\pi],\\
&[\pi\varepsilon_2] = [\varepsilon_0] \oplus [\varepsilon_2], \quad [\pi\varepsilon_3] = [\varepsilon_0] \oplus [\varepsilon_3],\\
&[\pi\varepsilon_4] = 2[\varepsilon_0] \oplus [\varepsilon_4],\\
&[\pi\varepsilon_0] = [\varepsilon] \oplus [\chi\varepsilon] \oplus [\varepsilon_2] \oplus [\varepsilon_3] \oplus 2[\varepsilon_4] \oplus 8[\varepsilon_0].
\end{aligned}$$

Proof. Since $d(\varepsilon_2\overline{\varepsilon_2}) = 10$ and $\varepsilon_2\overline{\varepsilon_2}$ contains id, we have only the following two possibilities:

$$\begin{aligned}
&[\varepsilon_2\overline{\varepsilon_2}] = [\text{id}] \oplus [\pi],\\
&[\varepsilon_2\overline{\varepsilon_2}] = [\text{id}] \oplus [\chi\pi].
\end{aligned}$$

Since $\varepsilon_2\overline{\varepsilon_2}$ does not contain χ in any case, we have $[\chi\varepsilon_2] \neq [\varepsilon_2]$, and so, $[\chi\varepsilon_2] = [\varepsilon_3]$.

Assume that $[\varepsilon_2\overline{\varepsilon_2}] = [\mathrm{id}] \oplus [\chi\pi]$ holds. Then,

$$\dim(\eta_1\varepsilon_2, \eta_1\varepsilon_2) = \dim(\eta_1, \eta_1\varepsilon_2\overline{\varepsilon_2}) = \dim(\eta_1, \eta_1(\mathrm{id} \oplus \chi\pi)) = 1 + \dim(\eta_1, \eta_2\pi) = 3.$$

Since $d(\eta_1\varepsilon_2) = 10\sqrt{10}$, we have

$$[\eta_1\varepsilon_2] = [\varepsilon_0] \oplus 2 \times \sqrt{10}\,\dim.$$

However, we have

$$\begin{aligned}
\dim(\eta_1\varepsilon_2, \varepsilon) &= \dim(\eta_1, \varepsilon\overline{\varepsilon_2}) = \dim(\eta_1, \overline{\xi}) = \dim(\eta_1, \xi) = 0,\\
\dim(\eta_1\varepsilon_2, \chi\varepsilon) &= \dim(\eta_2\varepsilon_2, \varepsilon) = \dim(\eta_2, \varepsilon\overline{\varepsilon_2}) = \dim(\eta_2, \overline{\xi}) = \dim(\eta_2, \xi) = 0,\\
\dim(\eta_1\varepsilon_2, \varepsilon_2) &= \dim(\eta_1, \mathrm{id} \oplus \chi\pi) = 0.\\
\dim(\eta_1\varepsilon_2, \varepsilon_3) &= \dim(\eta_1\varepsilon_2, \chi\varepsilon_2) = \dim(\eta_1, \chi \oplus \pi) = 0,
\end{aligned}$$

and we get a contradiction. Therefore, we get $[\varepsilon_2\overline{\varepsilon_2}] = [\mathrm{id}] \oplus [\pi]$.

The Frobenius reciprocity implies $\dim(\pi\varepsilon_2, \varepsilon_2) = 1$. Since $d(\pi\varepsilon_2) = 9\sqrt{10}$, we get $[\pi\varepsilon_2] = [\varepsilon_2] \oplus [\varepsilon_0]$, and $[\pi\varepsilon_3] = [\varepsilon_3] \oplus [\varepsilon_0]$ in the same way.

By associativity,

$$\begin{aligned}
2[\pi\varepsilon_0] &= [\pi\zeta\varepsilon] = [\overline{\zeta\pi}\varepsilon]\\
&= [(2\pi \oplus 2\chi\pi \oplus 2\xi \oplus 2\eta_1 \oplus 2\eta_2 \oplus 3\zeta)\varepsilon]\\
&= 2([\varepsilon] \oplus [\varepsilon_0]) \oplus 2([\chi\varepsilon] \oplus [\varepsilon_0]) \oplus 2([\varepsilon_2] \oplus [\varepsilon_3] \oplus [\varepsilon_0])\\
&\quad \oplus 2([\varepsilon_4] \oplus [\varepsilon_0]) \oplus 2([\varepsilon_4] \oplus [\varepsilon_0]) \oplus 6[\varepsilon_0],
\end{aligned}$$

which shows the last equation. The Frobenius reciprocity together with the equations obtained so far implies the fourth one. ∎

Proof of Theorem 4.2. It suffices to show $\mathscr{G}_{M\supset N} = \mathscr{G}_{M_{10}>M_9}$ (which is $\widetilde{\mathscr{G}_{(1^42),1}}$). Let $[\overline{\varepsilon}\varepsilon] = [\mathrm{id}] \oplus [\sigma]$ be the irreducible decomposition. Since

$$[\varepsilon\overline{\varepsilon}\varepsilon] = [(\mathrm{id} \oplus \pi)\varepsilon] = 2[\varepsilon] \oplus [\varepsilon_0],$$

we get $[\varepsilon\sigma] = [\varepsilon] \oplus [\varepsilon_0]$.

Since

$$\dim(\overline{\varepsilon}\chi\varepsilon, \overline{\varepsilon}\chi\varepsilon) = \dim(\varepsilon\overline{\varepsilon}\chi, \chi\varepsilon\overline{\varepsilon}) = \dim(\chi \oplus \pi\chi, \chi \oplus \chi\pi) = 2,$$

the sector $\overline{\varepsilon}\chi\varepsilon$ is decomposed into two distinct irreducibles. Since $d(\overline{\varepsilon}\chi\varepsilon) = 10$ and

$$[\varepsilon\overline{\varepsilon}\chi\varepsilon] = [(\mathrm{id} \oplus \pi)\chi\varepsilon] = [\chi\varepsilon] \oplus [\chi][\pi\varepsilon] = 2[\chi\varepsilon] \oplus [\mu_0],$$

one of the irreducible components of $\overline{\varepsilon}\chi\varepsilon$ is an automorphism of N, say, α_1, and the Frobenius reciprocity implies $[\chi\varepsilon] = [\varepsilon\alpha_1]$. Thus,

$$[\overline{\varepsilon}\chi\varepsilon] = [\sigma\alpha_1] \oplus [\alpha_1],$$

and $[\varepsilon\sigma\alpha_1] = [\chi\varepsilon] \oplus [\varepsilon_0]$. Since

$$[\varepsilon\sigma\alpha_1] = [(\varepsilon \oplus \varepsilon_0)\alpha_1],$$

we get $[\varepsilon_0][\alpha_1] = [\varepsilon_0]$.

In the same way, Lemma 4.4 implies

$$\dim(\overline{\varepsilon}\varepsilon_2, \overline{\varepsilon}\varepsilon_2) = (\varepsilon\overline{\varepsilon}, \varepsilon_2\overline{\varepsilon_2}) = \dim(\mathrm{id} \oplus \pi, \mathrm{id} \oplus \pi) = 2,$$

and there exists $\alpha_2 \in \mathrm{Aut}(N)$ satisfying $[\varepsilon_2] = [\varepsilon\alpha_2]$, and

$$[\overline{\varepsilon}\varepsilon_2] = [\sigma\alpha_2] \oplus [\alpha_2].$$

Letting $[\alpha_3] = [\alpha_1\alpha_2]$, we get

$$[\varepsilon_3] = [\chi\varepsilon_2] = [\chi\varepsilon\alpha_2] = [\varepsilon\alpha_1\alpha_2] = [\varepsilon\alpha_3]$$

and

$$[\overline{\varepsilon}\varepsilon_3] = [\sigma\varepsilon_3] \oplus [\alpha_3].$$

Since

$$[\varepsilon\overline{\varepsilon}\varepsilon_2] = [(\mathrm{id} \oplus \pi)\varepsilon_2] = 2[\varepsilon_2] \oplus [\varepsilon_0],$$

we get $[\varepsilon\sigma\alpha_2] = [\varepsilon_2] \oplus [\varepsilon_0]$. Since

$$[\varepsilon\sigma\alpha_2] = [(\varepsilon \oplus \varepsilon_0)\alpha_2] = [\varepsilon\alpha_2] \oplus [\varepsilon_0\alpha_2],$$

we get $[\varepsilon_0\alpha_2] = [\varepsilon_0]$, and $[\varepsilon_0\alpha_3] = [\varepsilon_0]$ too.

Lemma 4.4 implies

$$\dim(\overline{\varepsilon}\varepsilon_4, \overline{\varepsilon}\varepsilon_4) = \dim(\varepsilon_4, \varepsilon\overline{\varepsilon}\varepsilon_4) = (\varepsilon_4, (\mathrm{id} \oplus \pi)\varepsilon_4) = 1 + (\varepsilon_4, \pi\varepsilon_4) = 2,$$

and $\overline{\varepsilon}\varepsilon_4$ is decomposed into two distinct irreducibles, say, $\widehat{\eta}_1$ and $\widehat{\eta}_2$. On the other hand, we have

$$[\varepsilon\overline{\varepsilon}\varepsilon_4] = [(\mathrm{id} \oplus \pi)\varepsilon_4] = 2[\varepsilon_4] \oplus 2[\varepsilon_0].$$

Thus, there are the following two possibilities.

(i) $[\varepsilon\widehat{\eta}_1] = [\varepsilon\widehat{\eta}_2] = [\varepsilon_4] \oplus [\varepsilon_0]$.

(ii) $[\varepsilon\widehat{\eta}_1] = [\varepsilon_4] \oplus 2[\varepsilon_0]$ and $[\varepsilon\widehat{\eta}_2] = [\varepsilon_4]$.

Assume that the case (i) occurs. Then, $d(\widehat{\eta}_1) = d(\widehat{\eta}_2) = 10$. Lemma 4.4 implies

$$\dim(\overline{\varepsilon}\varepsilon_0, \overline{\varepsilon}\varepsilon_0) = (\varepsilon_0, \varepsilon\overline{\varepsilon}\varepsilon_0) = 1 + \dim(\varepsilon_0, \pi\varepsilon_0) = 9.$$

Thus, the Frobenius reciprocity together with the fusion rules obtained so far shows that there exists distinct irreducibles ρ_1, ρ_2, ρ_3 with $d(\rho_1) = d(\rho_2) = d(\rho_3) = 8$ satisfying

$$[\overline{\varepsilon}\varepsilon_0] = \bigoplus_{i=0}^{3}[\sigma\alpha_i] \oplus [\widehat{\eta}_1] \oplus [\widehat{\eta}_2] \oplus [\rho_1] \oplus [\rho_2] \oplus [\rho_3],$$
$$[\varepsilon\rho_1] = [\varepsilon\rho_2] = [\varepsilon\rho_3] = [\varepsilon_0],$$

where $\alpha_0 = \mathrm{id}$. For the fusion category $\mathcal{C}$ generated by $\overline{\varepsilon}\varepsilon$, we have

$$\mathrm{Irr}(\mathcal{C}) = \{[\alpha_i]\}_{i=0}^{4} \sqcup \{[\sigma\alpha_i]\}_{i=0}^{3} \sqcup \{[\widehat{\eta}_1], [\widehat{\eta}_2], [\rho_1], [\rho_2], [\rho_3]\}.$$

Let $\Lambda = \{[\alpha_i]\}_{i=0}^{4}$, which forms a group of order 4. Then, the Λ-action on the set $\{[\rho_1], [\rho_2], [\rho_3]\}$ by left multiplication has a fixed point, and we may assume that it is $[\rho_1]$. Thus, there exists an intermediate subfactor of index 4 between $N \supset \rho_1(N)$, and ρ_1 factorizes as $\rho_1 = \mu_1\mu_2$ with $d(\mu_1) = 2$, $d(\mu_2) = 4$. Since $\overline{\mu_2}\mu_2$ is contained in $\overline{\rho_1}\rho_1$, it belongs to $\mathcal{C}$. However, we have $d(\overline{\mu}\mu) = 16$, and $\overline{\mu}\mu$ contains either 1, 2 or 4 automorphisms, which is impossible because $d(\sigma\alpha_i) = 9$, $d(\widehat{\eta}_i) = 10$, and $d(\rho_i) = 8$. Therefore, (i) never occurs.

Now, we are left with the case (ii). In this case, we have $d(\widehat{\eta}_1) = 2$, and

$$[\overline{\varepsilon}\varepsilon_4] = [\overline{\varepsilon}\varepsilon\widehat{\eta}_2] = [(\mathrm{id} \oplus \sigma)\widehat{\eta}_2],$$

which implies $[\widehat{\eta}_1] = [\sigma\widehat{\eta}_2]$. The Frobenius reciprocity and $\dim(\overline{\varepsilon}\varepsilon_0, \overline{\varepsilon}\varepsilon_0) = 9$ imply that there exists an irreducible ρ satisfying

$$[\overline{\varepsilon}\varepsilon_0] = \bigoplus_{i=0}^{3}[\sigma\alpha_i] \oplus 2[\sigma\widehat{\eta}_2] \oplus [\rho],$$
$$[\varepsilon\rho] = [\varepsilon_0],$$

which shows $\mathcal{G}_{M\supset N} = \mathcal{G}_{(1^4 2),1}$. ■

5. Goldman-type theorems for $(\mathrm{PSL}_2(q), \mathrm{PG}_1(q))$

Theorem 5.1. *Let $M \supset N$ be a finite index subfactor with $\mathcal{G}_{M\supset N} = \widetilde{\mathcal{G}_{(1^m),2}}$. Then, $q = 1 + 2m$ is an odd prime power, and there exists a subfactor $R \subset N$ up to inner conjugacy such that R is irreducible in M and*

$$M = R \rtimes \mathrm{PSL}_2(q) \supset N = R \rtimes \Lambda,$$

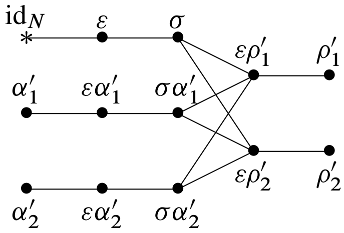

Figure 7. $\widetilde{\mathscr{G}_{(1^3),2}}$.

where

$$\Lambda = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} ;\ a \in \mathbb{F}_q^\times,\ b \in \mathbb{F}_q \right\} / \{\pm 1\}.$$

Proof. Note that if $m = 1$, we have $\widetilde{\mathscr{G}_{(1),2}} = E_6^{(1)} = \mathscr{G}_{(1^3),1}$, and the statement follows from [8] (or Theorem 3.1) as we have $(\mathfrak{A}_4, X_4) \cong (\mathrm{PSL}_2(3), \mathrm{PG}_1(3))$. We assume $m > 1$ in what follows.

Let $\varepsilon : N \hookrightarrow M$ be the inclusion map, and let $[\bar{\varepsilon}\varepsilon] = [\mathrm{id}] \oplus [\sigma]$ be the irreducible decomposition. Let $\mathscr{C}$ be the fusion category generated by $\bar{\varepsilon}\varepsilon$, and let I be the group of (the equivalence classes of) the invertible objects in $\mathscr{C}$. Then, $|I| = m$.

We can make the following parameterization of irreducible N-N and M-N sectors, respectively (see Figure 7):

$$\{\alpha'_i\}_{i \in I} \sqcup \{\sigma\alpha'_i\}_{i \in I} \sqcup \{\rho'_1, \rho'_2\},$$
$$\{\varepsilon\alpha'_i\}_{i \in I} \sqcup \{\varepsilon\rho'_1, \varepsilon\rho'_2\},$$

with properties

$$d(\alpha'_i) = 1, \quad d(\varepsilon) = \sqrt{2 + 2m}, \quad d(\sigma) = 1 + 2m, \quad d(\rho'_1) = d(\rho'_2) = m,$$
$$[\bar{\varepsilon}\varepsilon] = [\mathrm{id}] \oplus [\sigma],$$
$$[\alpha'_{i_1}\alpha'_{i_2}] = [\alpha'_{i_1 i_2}],$$
$$[\varepsilon\sigma] = [\varepsilon] \oplus [\varepsilon\rho_1] \oplus [\varepsilon\rho_2],$$
$$[\sigma\rho'_1] = [\sigma\rho'_2] = \bigoplus_{i \in I} [\sigma\alpha'_i], \tag{5.1}$$
$$[\sigma^2] = [\mathrm{id}] \oplus [\rho'_1] \oplus [\rho'_2] \oplus 2 \bigoplus_{i \in I} [\sigma\alpha'_i]. \tag{5.2}$$

By definition of I, we have $[\overline{\alpha'_i}] = [\alpha'_{i^{-1}}]$. We can introduce another involution in I by $[\overline{(\sigma\alpha'_i)}] = [\sigma\alpha'_{i^*}]$. We also introduce an involution in $\{1, 2\}$ by $[\overline{\rho'_j}] = [\rho'_{\bar{j}}]$. Taking conjugation of equation (5.1), we also have

$$[\rho_1'\sigma] = [\rho_2'\sigma] = \bigoplus_{i\in I}[\sigma\alpha_i'].$$

We claim that there exists a fusion subcategory $\mathcal{C}_1$ of $\mathcal{C}$ satisfying

$$\mathrm{Irr}(\mathcal{C}_1) = \{\alpha_i'\}_{i\in I} \sqcup \{\rho_1, \rho_2\}.$$

Indeed, let

$$I_j = \{i \in I;\ [\alpha_i'][\rho_j'] = [\rho_j']\},$$
$$I_j' = \{i \in I;\ [\rho_j'][\alpha_i'] = [\rho_j']\}.$$

Since the group I acts on the 2-point set $\{[\rho_1'], [\rho_2']\}$ by left (and also right) multiplication, we have the following two cases.

(i) $I_1 = I_2 = I$. In this case, we also have $I_1' = I_2' = I$ as $\{[\overline{\rho_1'}], \{[\overline{\rho_2'}]\}\} = \{[\rho_1'], [\rho_2']\}$.

(ii) $|I_1| = |I_2| = m/2$. In this case, we also have $|I_1'| = |I_2'| = m/2$.

Assume that (i) occurs first. Then, the Frobenius reciprocity implies

$$[\rho_j'\overline{\rho_j'}] = \bigoplus_{i\in I}[\alpha_i'] \oplus a_{j1}[\rho_1'] \oplus a_{j2}[\rho_2'] \oplus \bigoplus_{i\in I} b_{ji}[\sigma\alpha_i'].$$

Let

$$b_j = \sum_{i\in I} b_{ji}.$$

Then,

$$m^2 = m + (a_{j1} + a_{j2})m + b_j(2m+1),$$

and we see that m divides b_j. If $b_j \geq m$, we would have $m \geq 2m+1$, which is contradiction. Thus, $b_{ji} = 0$ for all i, j. The Frobenius reciprocity shows that neither $[\rho_1'\overline{\rho_2'}]$ nor $[\rho_2'\overline{\rho_1'}]$ contain any automorphism, and a similar argument as above shows that $\rho_1'\overline{\rho_2'}$ and $\rho_2'\overline{\rho_1'}$ are also direct sums of sectors in $\{\alpha_i'\} \sqcup \{\rho_1', \rho_2'\}$. This proves the claim in the case (i).

Assume that (ii) occurs now. Then, $l = m/2$ is a natural number. A similar argument as above shows that for

$$a_j = \dim(\rho_j'\overline{\rho_j'}, \rho_1') + \dim(\rho_j'\overline{\rho_j'}, \rho_2'),$$
$$b_j = \sum_{i\in I}\dim(\rho_j'\overline{\rho_j'}, \sigma\alpha_i'),$$

we have

$$4l^2 = l + 2a_jl + b_j(4l+1).$$

This shows that l divides b_j, and so, $b_j = 0$. Note that there exists $i_0 \in I$ satisfying

$$[\rho'_1] = [\alpha'_{i_0}\rho'_2],$$

which implies

$$[\rho'_1\overline{\rho'_2}] = [\rho'_1\overline{\rho'_1}\alpha'_{i_0}], \quad [\rho'_2\overline{\rho'_1}] = [\alpha'_{i_0^{-1}}\rho'_1\overline{\rho'_1}].$$

Therefore, $\rho'_{j_1}\overline{\rho'_{j_2}}$, $1 \leq j_1, j_2 \leq 2$ are direct sums of sectors in $\{\alpha'_i\} \sqcup \{\rho'_1, \rho'_2\}$, which shows the claim in the case (ii).

The rest of the proof is very much similar to that of Theorem 4.1, and we briefly address it except for the last part deciding the group structure of Γ. Theorem 2.3 and equation (5.2) show that there exists a unique intermediate subfactor P between N and $\sigma(N)$ such that if $\iota : P \hookrightarrow N$ denotes the inclusion map, we have

$$[\iota\bar{\iota}] = [\mathrm{id}] \oplus [\rho_1] \oplus [\rho_2].$$

Moreover, there exists $\tau \in \mathrm{Aut}(P)$ satisfying $[\sigma] = [\iota\tau\bar{\iota}]$. The fusion rules of $\mathcal{C}_1$ tell that the dual principal graph $\mathcal{G}^d_{N\supset M}$ is $\mathcal{G}_{(1^m),2}$, and Theorem 3.1 shows that $\mathcal{G}_{M\supset N}$ is also $\mathcal{G}_{(1^m),2}$. The group I is the cyclic group $\mathbb{Z}_m$ now. Let $\mathcal{C}_2$ be the fusion category generated by $\bar{\iota}\iota$. Then, we can parameterize $\mathrm{Irr}(\mathcal{C}_2)$ so that

$$\begin{aligned}\mathrm{Irr}(\mathcal{C}_2) &= \{[\alpha_i]\}_{i\in I} \sqcup \{[\rho_1], [\rho_2]\}, \\ [\iota\alpha_i] &= [\alpha'_i\iota], \\ [\bar{\iota}\iota] &= [\mathrm{id}] \oplus [\rho_1] \oplus [\rho_2].\end{aligned}$$

Applying Theorem 3.1, we see that there exists a unique subfactor $R \subset P$, up to inner conjugacy, that is irreducible in M such that there exists a primitive Frobenius group $K \rtimes H$ with $|H| = m$, $|K| = 1 + 2m$ and an outer action β of it on R satisfying

$$N = R \rtimes_\beta (K \rtimes H) \supset P = R \rtimes_\beta H.$$

Note that the number $q = 1 + 2m$ is an odd prime power p^k and $K = \mathbb{Z}_p^k$, $H = \mathbb{Z}_m$. Moreover, there exists a group Γ including $K \rtimes H$ such that β extends to an outer action γ of Γ satisfying

$$M = R \rtimes_\gamma \Gamma.$$

From the graph $\mathcal{G}_{M\supset N}$, we can see that the Γ-action on $\Gamma/(K \rtimes H)$ is a 2-transitive, but not 3-transitive, extension of the Frobenius group $K \rtimes H$ acting on $(K \rtimes H)/H$. Note that $|\Gamma| = (2m + 2)(2m + 1)m$. Thus, [10, Chapter XI, Theorem 1.1] shows that Γ is a Zassenhaus group. The order of Γ shows that it is not one of the Suzuki groups. Since Γ is not 3-transitive, we conclude from [10, Chapter XI, Theorem 11.16] that $\Gamma = \mathrm{PSL}_2(q)$. ■

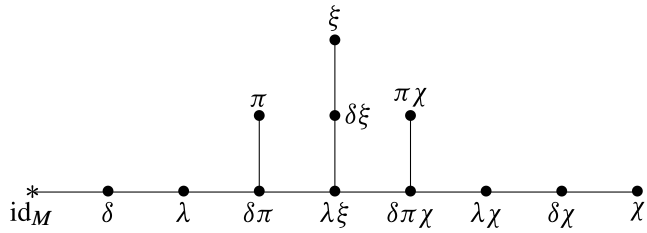

Figure 8. $\mathscr{G}_{(\mathfrak{S}_5,X_5)}$.

6. Goldman-type theorems for sharply 4-transitive permutation groups

Since the finite depth subfactors of index 5 are completely classified in [19], the only point of the following theorem is to see how to find a subfactor R and an $\mathfrak{S}_5$-action on it from the principal graph.

Theorem 6.1. *Let $L \supset M$ be a finite index inclusion of factors with $\mathscr{G}_{L\supset M} = \mathscr{G}_{(\mathfrak{S}_5,X_5)}$. Then, there exists a unique subfactor $R \subset M$, up to inner conjugacy, such that $R' \cap L = \mathbb{C}$ and there exists an outer action γ of $\mathfrak{S}_5$ on R satisfying*

$$L = R \rtimes_\gamma \mathfrak{S}_5 \supset M = R \rtimes_\gamma \mathfrak{S}_4.$$

Proof. We follow the strategy described in Section 2.5.

(1) Let $\delta : M \hookrightarrow L$ be the inclusion map, and let $[\overline{\delta}\delta] = [\mathrm{id}] \oplus [\lambda]$ be the irreducible decomposition. We parameterize the irreducible M-M sectors and the L-M sectors generated by δ as in Figure 8. Then, we have

$$d(\lambda) = 4, \quad d(\pi) = 3, \quad d(\xi) = 2, \quad d(\chi) = 1, \quad d(\delta) = \sqrt{5}.$$

From the graph, we can see that all the M-M sectors are self-conjugate, which implies $[\chi\lambda] = [\lambda\chi]$, $[\chi\pi] = [\pi\chi]$. The graph symmetry implies $[\xi\chi] = [\xi]$, and since ξ is self-conjugate, we get

$$[\xi^2] = [\mathrm{id}] \oplus [\chi] \oplus [\xi]$$

by dimension counting.

The basic fusion rules coming from the graph are

$$\begin{aligned} [\lambda^2] &= [\mathrm{id}] \oplus [\lambda] \oplus [\pi] \oplus [\lambda\xi], \\ [\lambda\pi] &= [\lambda] \oplus [\lambda\xi], \\ [\lambda(\lambda\xi)] &= [\lambda] \oplus [\lambda\chi] \oplus [\pi] \oplus [\pi\chi] \oplus [\xi] \oplus 2[\lambda\xi]. \end{aligned} \tag{6.1}$$

Taking conjugate, we also have

$$[\pi\lambda] = [\lambda] \oplus [\lambda\xi].$$

Now, direct computation using the Frobenius reciprocity and associativity shows the following fusion rules:

$$[\pi^2] = [\mathrm{id}] \oplus [\pi] \oplus [\pi\chi] \oplus [\xi],$$
$$[\pi\xi] = [\xi\pi] = [\pi] \oplus [\pi\chi].$$

Let $\mathcal{C}$ be the fusion category generated by $\overline{\delta}\delta$. Then, the above fusion rules show that there exists a fusion subcategory $\mathcal{C}_1$ of $\mathcal{C}$ with

$$\mathrm{Irr}(\mathcal{C}_1) = \{\mathrm{id}, \chi, \xi, \pi, \pi\chi\}.$$

(2) Theorem 2.3 and equation (6.1) imply that there exists a unique intermediate subfactor N between M and $\lambda(M)$ such that if $\varepsilon : N \hookrightarrow M$ is the inclusion map, we have

$$[\varepsilon\overline{\varepsilon}] = [\mathrm{id}] \oplus [\pi].$$

In the same way as in the proof of Lemma 3.2, there exists $\varphi \in \mathrm{Aut}(N)$ satisfying $[\lambda] = [\varepsilon\varphi\overline{\varepsilon}]$.

(3) Note that we have $[M : N] = 1 + d(\pi) = 4$. Thanks to the classification of subfactors of index 4 (see [21, Section 3.2]) and $\mathrm{Irr}(\mathcal{C}_1)$, we can see that $\mathcal{G}^d_{M\supset N}$ is the Coxeter graph $E_7^{(1)}$, and so is $\mathcal{G}_{M\supset N}$ too. Note that we have $E_7^{(1)} = \widetilde{\mathcal{G}_{(1^2),1}}$, and $(L(3), \mathrm{PG}_1(3)) \cong (\mathfrak{S}_4, X_4)$. Let $\mathcal{C}_2$ be the fusion category generated by $\overline{\varepsilon}\varepsilon$. As in Theorem 4.1, we can parameterize $\mathrm{Irr}(\mathcal{C}_2)$ as

$$\mathrm{Irr}(\mathcal{C}_2) = \{\mathrm{id}, \alpha', \rho', \sigma, \sigma\alpha'\},$$

with the following properties:

$$d(\alpha') = 1, \quad d(\rho') = 2, \quad d(\sigma) = 3,$$
$$[\alpha'^2] = [\mathrm{id}],$$
$$[\alpha'\rho] = [\rho'\alpha'] = [\rho'],$$
$$[\rho'^2] = [\mathrm{id}] \oplus [\alpha'] \oplus [\rho'],$$
$$[\sigma^2] = [\mathrm{id}] \oplus [\sigma] \oplus [\rho'] \oplus [\sigma\alpha'],$$
$$[\alpha'\sigma] = [\sigma\alpha'],$$
$$[\sigma\rho'] = [\rho'\sigma] = [\sigma] \oplus [\sigma\alpha'],$$
$$[\overline{\varepsilon}\varepsilon] = [\mathrm{id}] \oplus [\sigma].$$

(4) Theorem 4.1 shows that there exists a unique subfactor $R \subset N$, up to inner conjugacy such that $R' \cap M = \mathbb{C}$, and there exists an outer action β of $\mathfrak{S}_4$ on R satisfying

$$M = R \rtimes_\beta \mathfrak{S}_4 \supset N = R \rtimes_\beta \mathfrak{S}_3.$$

To use notation consistent with that in Theorems 3.1 and 4.1, we let $P = R \rtimes_\beta \mathfrak{S}_3 \subset N$, and we let $\iota : P \hookrightarrow N$ and $\kappa : R \hookrightarrow P$ be the inclusion maps. Let $\varepsilon_1 = \varepsilon\iota\kappa$. Then, $\varepsilon_1\overline{\varepsilon_1}$ corresponds to the regular representation of $\mathfrak{S}_4$, and

$$[\varepsilon_1\overline{\varepsilon_1}] = [\mathrm{id}] \oplus [\chi] \oplus 2[\xi] \oplus 3[\pi] \oplus 3[\pi\chi].$$

Thus, since $[\overline{\delta}\delta] = [\mathrm{id}] \oplus [\lambda]$,

$$\dim(\delta\varepsilon_1, \delta\varepsilon_1) = \dim(\overline{\delta}\delta, \varepsilon_1\overline{\varepsilon_1}) = 1,$$

and $L \supset R$ is irreducible.

(5) Note that we have $[L : R] = 120$. On the other hand,

$$\dim(\delta\varepsilon_1\overline{(\delta\varepsilon_1)}, \delta\varepsilon_1\overline{(\delta\varepsilon_1)}) = \dim(\overline{\delta}\delta\varepsilon_1\overline{\varepsilon_1}, \varepsilon_1\overline{\varepsilon_1}\overline{\delta}\delta).$$

Note that $[\lambda]$ commutes with $[\varepsilon_1\overline{\varepsilon_1}]$, and $[(\varepsilon_1\overline{\varepsilon_1})^2] = |\mathfrak{S}_4|[\varepsilon_1\overline{\varepsilon_1}]$. Thus,

$$\begin{aligned}\dim(\overline{\delta}\delta\varepsilon_1\overline{\varepsilon_1}, \varepsilon_1\overline{\varepsilon_1}\overline{\delta}\delta) &= \dim(\overline{\delta}\delta\varepsilon_1\overline{\varepsilon_1}, \overline{\delta}\delta\varepsilon_1\overline{\varepsilon_1}) = \dim((\overline{\delta}\delta)^2, (\varepsilon_1\overline{\varepsilon_1})^2)\\ &= 24\dim((\mathrm{id} \oplus \lambda)^2, \varepsilon_1\overline{\varepsilon_1}) = 120.\end{aligned}$$

Thus, the inclusion $L \supset R$ is of depth 2.

(6) We denote $\iota_3 = \iota\kappa$. By Lemma 2.5, we get

$$\dim(\varphi\overline{\varepsilon}\varepsilon\iota_3\overline{\iota_3}\varphi^{-1}, \overline{\varepsilon}\varepsilon\iota_3\overline{\iota_3}) = |\mathfrak{S}_3| = 6.$$

Note that $\iota_3\overline{\iota_3}$ corresponds to the regular representation in $\mathrm{Rep}(\mathfrak{S}_3)$, and

$$[\iota_3\overline{\iota_3}] = [\mathrm{id}] \oplus [\alpha'] \oplus 2[\rho'].$$

Thus,

$$[\overline{\varepsilon}\varepsilon\iota_3\overline{\iota_3}] = [(\mathrm{id} \oplus \sigma)(\mathrm{id} \oplus \alpha' \oplus 2\rho')] = [\mathrm{id}] \oplus [\alpha'] \oplus 2[\rho'] \oplus 3[\sigma] \oplus 3[\sigma\alpha'].$$

Dimension counting implies

$$\dim(\varphi(\mathrm{id} \oplus \alpha' \oplus 2\rho')\varphi^{-1}, \mathrm{id} \oplus \alpha' \oplus 2\rho') = 6,$$

and $[\varphi\iota_3\overline{\iota_3}\varphi^{-1}] = [\iota_3\overline{\iota_3}]$.

Now, we can apply Lemma 2.6 to $\mathfrak{S}_3$, and we obtain $\varphi_1 \in \mathrm{Aut}(R)$ satisfying

$$[\varphi\varepsilon_1] = [\varepsilon_1\varphi_1].$$

Lemma 2.4 implies that there exists a group Γ including $\mathfrak{S}_4$ such that β extends to an outer action γ of Γ satisfying $L = R \rtimes_\gamma \Gamma$. Note that

$$|\Gamma| = [L : R] = 120.$$

Since the graph $\mathcal{G}_{(\mathfrak{S}_5, X_5)}$ shows that the Γ-action on $\Gamma/\mathfrak{S}_4$ is a 3-transitive extension of $(\mathfrak{S}_4, X_4)$, we conclude $\Gamma = \mathfrak{S}_5$. ∎

The remaining two cases are the most subtle because we cannot apply Lemma 2.6 to either $\mathfrak{A}_4 = H(2^2) = \mathbb{Z}_2^2 \rtimes \mathbb{Z}_3$ or $M_9 = S(3^2) = \mathbb{Z}_3^3 \rtimes Q_8$ in step (6).

Since $\mathcal{G}_{(\mathfrak{A}_6, X_6)} = \widetilde{\mathcal{G}_{\mathfrak{A}_4}^{\mathfrak{A}_5}}$, we can obtain it from the induction-reduction graph $\mathcal{G}_{\mathfrak{A}_4}^{\mathfrak{A}_5}$ between $\mathfrak{A}_5$ and $\mathfrak{A}_4$ (see, for example, [19] for the latter).

Theorem 6.2. *Let $L \supset M$ be a finite-index inclusion of factors with $\mathcal{G}_{L \supset M} = \mathcal{G}_{(\mathfrak{A}_6, X_6)}$. Then, there exists a unique subfactor $R \subset M$, up to inner conjugacy, such that*

$$R' \cap L = \mathbb{C},$$

and there exists an outer action γ of $\mathfrak{A}_6$ on R satisfying

$$L = R \rtimes_\gamma \mathfrak{A}_6 \supset M = R \rtimes_\gamma \mathfrak{A}_5.$$

Proof. (1) Let $\delta : M \hookrightarrow L$ be the inclusion map, and let

$$[\bar{\delta}\delta] = [\mathrm{id}] \oplus [\lambda]$$

be the irreducible decomposition. We parameterize the irreducible M-M sectors and the L-M sectors generated by δ as in Figure 9. Then, we have

$$d(\lambda) = d(\xi_1) = d(\xi_2) = d(\xi_3) = 5, \quad d(\pi) = 4,$$
$$d(\mu) = 15, \quad d(\eta_1) = d(\eta_2) = 3, \quad d(\delta) = \sqrt{6}.$$

From the graph, we can see that λ, π, and μ are self-conjugate, and

$$\{[\bar{\xi}_1], [\bar{\xi}_2], [\bar{\xi}_3]\} = \{[\xi_1], [\xi_2], [\xi_3]\}, \quad \{[\bar{\eta}_1], [\bar{\eta}_2]\} = \{[\eta_1], [\eta_2]\}.$$

We use the notation $[\bar{\xi}_i] = [\xi_{\bar{i}}]$ and $[\bar{\eta}_j] = [\eta_{\bar{j}}]$ for simplicity.

The basic fusion rules coming from the graph and their conjugate are

$$[\lambda^2] = [\mathrm{id}] \oplus [\lambda] \oplus [\pi] \oplus [\mu], \tag{6.2}$$
$$[\lambda\pi] = [\pi\lambda] = [\lambda] \oplus [\mu], \tag{6.3}$$
$$[\lambda\mu] = [\mu\lambda] = [\lambda] \oplus [\pi] \oplus [\xi_1] \oplus [\xi_2] \oplus [\xi_3] \oplus [\eta_1] \oplus [\eta_2] \oplus 3[\mu], \tag{6.4}$$
$$[\lambda\xi_i] + [\xi_i] = [\xi_i\lambda] \oplus [\xi_i] = [\xi_1] \oplus [\xi_2] \oplus [\xi_3] \oplus [\mu], \tag{6.5}$$
$$[\lambda\eta_i] = [\eta_i\lambda] = [\mu]. \tag{6.6}$$

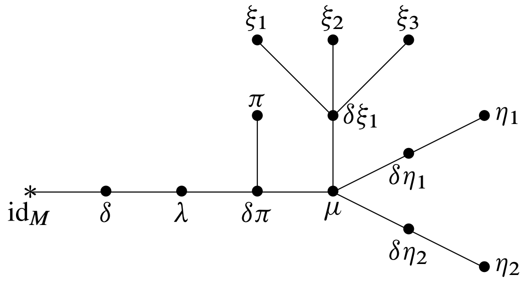

Figure 9. $\mathscr{G}_{(\mathfrak{A}_6, X_6)}$.

By associativity, we get

$$[\pi^2] \oplus [\mu\pi] = [\mathrm{id}] \oplus [\lambda] \oplus [\pi] \oplus [\xi_1] \oplus [\xi_2] \oplus [\xi_3] \oplus [\eta_1] \oplus [\eta_2] \oplus 3[\mu], \tag{6.7}$$

$$[\pi\mu] \oplus [\mu^2] = [\mathrm{id}] \oplus 4[\lambda] \oplus 3[\pi] \oplus 4[\xi_1] \oplus 4[\xi_2] \oplus 4[\xi_3] \oplus 2[\eta_1] \oplus 2[\eta_2] \oplus 12[\mu], \tag{6.8}$$

$$[\pi\xi_i] \oplus [\mu\xi_i] = [\xi_i] \oplus [\lambda] \oplus [\pi] \oplus [\xi_1] \oplus [\xi_2] \oplus [\xi_3] \oplus [\eta_1] \oplus [\eta_2] \oplus 4[\mu], \tag{6.9}$$

$$[\eta_i] \oplus [\pi\eta_i] \oplus [\mu\eta_i] = [\lambda] \oplus [\pi] \oplus [\xi_1] \oplus [\xi_2] \oplus [\xi_3] \oplus [\eta_1] \oplus [\eta_2] \oplus 2[\mu]. \tag{6.10}$$

Equation (6.3) shows

$$1 = \dim(\lambda\pi, \mu) = \dim(\lambda, \mu\pi).$$

Since $d(\eta_i\pi) < d(\mu)$, we have

$$0 = \dim(\eta_i\pi, \mu) = \dim(\eta_i, \mu\pi).$$

Equation (6.7) shows that π^2 contains id, η_1,η_2, and it cannot contain μ by dimension counting, which implies $\dim(\pi, \mu\pi) = 0$ by the Frobenius reciprocity. Equation (6.7) again shows that $\mu\pi$ contains μ with multiplicity 3 and π^2 contains π with multiplicity 1. Thus, we get

$$[\pi^2] = [\mathrm{id}] \oplus [\pi] \oplus [\eta_1] \oplus [\eta_2] \oplus 5 \dim, \quad [\mu\pi] = [3\mu] \oplus [\lambda] \oplus 10 \dim,$$

where the remainder is $\xi_1 \oplus \xi_2 \oplus \xi_3$. Therefore, we may and do assume that π^2 contains ξ_1, and we get

$$[\pi^2] = [\mathrm{id}] \oplus [\pi] \oplus [\xi_1] \oplus [\eta_1] \oplus [\eta_2], \tag{6.11}$$

$$[\mu\pi] = [3\mu] \oplus [\lambda] \oplus [\xi_2] \oplus [\xi_3]. \tag{6.12}$$

In consequence, ξ_1 is self-conjugate. Taking conjugate of equation (6.12), we also get $[\mu\pi] = [\pi\mu]$, and equation (6.8) implies

$$[\mu^2] = [\mathrm{id}] \oplus 3[\lambda] \oplus 3[\pi] \oplus 4[\xi_1] \oplus 3[\xi_2] \oplus 3[\xi_3] \oplus 2[\eta_1] \oplus 2[\eta_2] \oplus 9[\mu]. \quad (6.13)$$

The Frobenius reciprocity implies

$$[\pi\xi_1] = [\pi] \oplus 16 \dim, \quad [\mu\xi_1] = 4[\mu] \oplus [\lambda] \oplus 10 \dim,$$

and equation (6.9) with dimension counting implies

$$[\pi\xi_1] = [\pi] \oplus [\eta_1] \oplus [\eta_2] \oplus 10 \dim, \quad [\mu\xi_1] = 4[\mu] \oplus [\lambda] \oplus 10 \dim,$$

where the remainder is $2[\xi_1] \oplus [\xi_2] \oplus [\xi_3]$.

For $i = 2, 3$, equations (6.9) and (6.13) show that we have

$$3 = \dim(\xi_i, \mu^2) = \dim(\mu\xi_i, \mu),$$

and $\mu\xi_i$ contains μ with multiplicity 3, while it does not contain π as

$$0 = \dim(\pi^2, \xi_i) = \dim(\pi, \pi\xi_i).$$

Thus,

$$[\pi\xi_i] = [\mu] \oplus 5 \dim, \quad [\mu\xi_i] = [\lambda] \oplus [\pi] \oplus 3[\mu] \oplus [\eta_1] \oplus [\eta_2] \oplus 15 \dim,$$

where the remainder is $[\xi_i] \oplus [\xi_1] \oplus [\xi_2] \oplus [\xi_3]$. If $\mu\xi_i$ contained ξ_i with multiplicity 2, the Frobenius reciprocity implies that $\xi_i\overline{\xi_i}$ would contain μ with multiplicity 2, which is impossible. Thus, we get

$$[\pi\xi_i] = [\mu] \oplus [\xi_i], \qquad i = 2, 3, \quad (6.14)$$

$$[\mu\xi_i] = [\lambda] \oplus [\pi] \oplus [\xi_1] \oplus [\xi_2] \oplus [\xi_3] \oplus [\eta_1] \oplus [\eta_2] \oplus 3[\mu], \quad i = 2, 3. \quad (6.15)$$

Equation (6.14) shows

$$0 = \dim(\pi\xi_i, \xi_1) = \dim(\xi_i, \pi\xi_1), \quad i = 2, 3.$$

Thus,

$$[\pi\xi_1] = [\pi] \oplus [\eta_1] \oplus [\eta_2] \oplus 2[\xi_1], \quad (6.16)$$

$$[\mu\xi_1] = 4[\mu] \oplus [\lambda] \oplus [\xi_1] \oplus [\xi_2]. \quad (6.17)$$

The Frobenius reciprocity together with the fusion rules obtained so far implies

$$[\pi\eta_1] = [\pi] \oplus [\xi_1] \oplus [\eta_2],$$

$$[\pi\eta_2] = [\pi] \oplus [\xi_1] \oplus [\eta_1],$$

$$[\mu\eta_i] = 2[\mu] \oplus [\lambda] \oplus [\xi_2] \oplus [\xi_3].$$

Let $\mathcal{C}$ be the fusion category generated by $\bar{\delta}\delta$. Then, the above computation shows that the fusion subcategory $\mathcal{C}_1$ of $\mathcal{C}$ generated by π satisfies

$$\mathrm{Irr}(\mathcal{C}_1) = \{\mathrm{id}, \pi, \xi_1, \eta_1, \eta_2\}.$$

(2) Theorem 2.3 and equation (6.2) imply that there exists a unique intermediate subfactor N between M and $\lambda(M)$ such that if $\varepsilon : N \hookrightarrow M$ is the inclusion map, we have

$$[\varepsilon\bar{\varepsilon}] = [\mathrm{id}] \oplus [\pi].$$

Note that we have $d(\varepsilon) = \sqrt{5}$. In the same way as in the proof of Lemma 3.2, there exists $\varphi \in \mathrm{Aut}(N)$ satisfying $[\lambda] = [\varepsilon\varphi\bar{\varepsilon}]$.

(3) Since

$$\dim(\pi\varepsilon, \pi\varepsilon) = \dim(\pi^2, \varepsilon\bar{\varepsilon}) = \dim(\pi^2, \mathrm{id} \oplus \pi) = 2,$$

there exists an irreducible sector ε' with $[\pi\varepsilon] = [\varepsilon] \oplus [\varepsilon']$ and $d(\varepsilon') = 3\sqrt{5}$. Since

$$[\pi\varepsilon\bar{\varepsilon}] = [\pi(\mathrm{id} \oplus \pi)] = [\mathrm{id}] \oplus 2[\pi] \oplus [\xi_1] \oplus [\eta_1] \oplus [\eta_2],$$

we get

$$[\varepsilon'\bar{\varepsilon}] = [\pi] \oplus [\xi_1] \oplus [\eta_1] \oplus [\eta_2].$$

The Frobenius reciprocity and dimension counting show $[\eta_1\varepsilon] = [\eta_2\varepsilon] = [\varepsilon']$. Since ξ_1 is self-conjugate,

$$\begin{aligned}\dim(\xi_1\varepsilon, \xi_1\varepsilon) &= \dim(\xi_1, \xi_1\varepsilon\bar{\varepsilon}) = \dim(\xi_1, \xi_1(\mathrm{id} \oplus \pi)) \\ &= 1 + \dim(\xi_1, \xi_1\pi) = 1 + \dim(\xi_1, \pi\xi_1),\end{aligned}$$

and equation (6.16) shows $\dim(\xi_1\varepsilon, \xi_1\varepsilon) = 3$. This together with the Frobenius reciprocity imply that there exist irreducible sectors ε'' and ε''' satisfying $d(\varepsilon'') = d(\varepsilon'') = \sqrt{5}$,

$$[\xi_1\varepsilon] = [\varepsilon'] \oplus [\varepsilon''] \oplus [\varepsilon'''],$$

and $[\varepsilon''\bar{\varepsilon}] = [\varepsilon'''\bar{\varepsilon}] = [\xi_1]$. The above computation shows that the dual principal graph $\mathcal{G}^d_{M\supset N}$ is $\mathcal{G}^{\mathfrak{A}_5}_{\mathfrak{A}_4}$, and the classification of finite depth subfactors of index 5 shows that $\mathcal{G}_{M\supset N}$ is $\mathcal{G}_{(\mathfrak{A}_5, X_5)}$ (see [19]). Note that we have

$$(\mathfrak{A}_5, X_5) = (L(2^2), \mathrm{PG}_1(2^2)) \quad \text{and} \quad \mathcal{G}_{M\supset N} = \widetilde{\mathcal{G}_{(1^3),1}}.$$

Let $\mathcal{C}_2$ be the fusion category generated by $\bar{\varepsilon}\varepsilon$. As in the proof of Theorem 4.1, we can parameterize $\mathrm{Irr}(\mathcal{C}_2)$ as

$$\mathrm{Irr}(\mathcal{C}_2) = \{\mathrm{id}, \alpha', \alpha'^2, \rho', \sigma, \sigma\alpha', \sigma\alpha'^2\},$$

with the following properties:

$$
\begin{aligned}
d(\alpha') = 1, \quad d(\rho') &= 3, \quad d(\sigma) = 4,\\
[\alpha'^3] &= [\mathrm{id}],\\
[\alpha'\rho] &= [\rho'\alpha'] = [\rho'],\\
[\rho'^2] &= [\mathrm{id}] \oplus [\alpha'] \oplus [\alpha'^2] + 2[\rho'],\\
[\sigma^2] &= [\mathrm{id}] \oplus [\rho'] \oplus [\sigma] \oplus [\sigma\alpha'] \oplus [\sigma\alpha'^2],\\
[\alpha'\sigma] &= [\sigma\alpha'^2],\\
[\rho'\sigma] &= [\sigma\rho'] = [\sigma] \oplus [\sigma\alpha'] \oplus [\sigma\alpha'^2],\\
[\overline{\varepsilon}\varepsilon] &= [\mathrm{id}] \oplus [\sigma].
\end{aligned}
$$

(4) Theorem 4.1 shows that there exists a unique subfactor $R \subset N$, up to inner conjugacy, such that $R' \cap M = \mathbb{C}$, and there exists an outer action β of $\mathfrak{A}_5$ on R satisfying

$$M = R \rtimes_\beta \mathfrak{A}_5 \supset N = R \rtimes_\beta \mathfrak{A}_4.$$

Let $P = R \rtimes_\beta \mathfrak{A}_3 \subset N$, and let $\iota : P \hookrightarrow N$ and $\kappa : R \hookrightarrow P$ be the inclusion maps. Let $\varepsilon_1 = \varepsilon\iota\kappa$. Then, $\varepsilon_1\overline{\varepsilon_1}$ corresponds to the regular representation of $\mathfrak{A}_5$, and

$$[\varepsilon_1\overline{\varepsilon_1}] = [\mathrm{id}] \oplus 3[\eta_1] \oplus 3[\eta_2] \oplus 4[\pi] \oplus 5[\xi].$$

Thus, since $[\overline{\delta}\delta] = [\mathrm{id}] \oplus [\lambda]$,

$$\dim(\delta\varepsilon_1, \delta\varepsilon_1) = \dim(\overline{\delta}\delta, \varepsilon_1\overline{\varepsilon_1}) = 1,$$

and $L \supset R$ is irreducible.

(5) Note that we have $[L : R] = 6|\mathfrak{A}_5| = 360$. On the other hand, since $[\lambda]$ commutes with $[\varepsilon_1\overline{\varepsilon_1}]$, and $[(\varepsilon_1\overline{\varepsilon_1})^2] = |\mathfrak{A}_5|[\varepsilon_1\overline{\varepsilon_1}]$,

$$
\begin{aligned}
\dim(\delta\varepsilon_1\overline{(\delta\varepsilon_1)}, \delta\varepsilon_1\overline{(\delta\varepsilon_1)}) &= \dim(\overline{\delta}\delta\varepsilon_1\overline{\varepsilon_1}, \varepsilon_1\overline{\varepsilon_1}\overline{\delta}\delta) = \dim(\overline{\delta}\delta\varepsilon_1\overline{\varepsilon_1}, \overline{\delta}\delta\varepsilon_1\overline{\varepsilon_1})\\
&= \dim((\overline{\delta}\delta)^2, (\varepsilon_1\overline{\varepsilon_1})^2) = 60\dim((\mathrm{id} \oplus \lambda)^2, \varepsilon_1\overline{\varepsilon_1})\\
&= 60\dim(2\mathrm{id} \oplus \pi \oplus 3\lambda \oplus \mu, \varepsilon_1\overline{\varepsilon_1}) = 360.
\end{aligned}
$$

Therefore, the inclusion $L \supset R$ is of depth 2.

(6) We denote $\iota_3 = \iota\kappa$. By Lemma 2.5, we get

$$\dim(\varphi\overline{\varepsilon}\varepsilon\iota_3\overline{\iota_3}\varphi^{-1}, \overline{\varepsilon}\varepsilon\iota_3\overline{\iota_3}) = |\mathfrak{A}_4| = 12.$$

Note that $\iota_3\overline{\iota_3}$ corresponds to the regular representation of $\mathfrak{A}_4$, and

$$[\iota_3\overline{\iota_3}] = [\mathrm{id}] \oplus [\alpha'] \oplus [\alpha'^2] \oplus 3[\rho'].$$

Thus,

$$
\begin{aligned}
[\overline{\varepsilon}\varepsilon\iota_3\overline{\iota_3}] &= [(\mathrm{id}\oplus\sigma)(\mathrm{id}\oplus\alpha'\oplus\alpha'^2\oplus 3\rho')] \\
&= [\mathrm{id}]\oplus[\alpha']\oplus[\alpha'^2]\oplus 3[\rho']\oplus 4[\sigma]\oplus 4[\sigma\alpha']\oplus 4[\sigma\alpha'^2].
\end{aligned}
$$

Dimension counting implies

$$\dim(\varphi(\mathrm{id}\oplus\alpha'\oplus\alpha'^2\oplus 3\rho')\varphi^{-1}, \mathrm{id}\oplus\alpha'\oplus\alpha'^2\oplus 3\rho') = 12,$$

and $[\varphi\iota_3\overline{\iota_3}\varphi^{-1}] = [\iota_3\overline{\iota_3}]$. We also have

$$(\varphi(\sigma\oplus\sigma\alpha'\oplus\sigma\alpha'^2)\varphi^{-1}, (\sigma\oplus\sigma\alpha'\oplus\sigma\alpha'^2)) = 0. \tag{6.18}$$

To finish the proof, we cannot apply Lemma 2.6 to $\mathfrak{A}_4$, and we make a little detour. We examine the automorphism $\varphi\in\mathrm{Aut}(N)$ more carefully. We first claim $[\varphi^2] = [\mathrm{id}]$. Indeed, since λ is self-conjugate,

$$1 = \dim(\varepsilon\varphi\overline{\varepsilon}, \varepsilon\varphi^{-1}\overline{\varepsilon}) = \dim(\overline{\varepsilon}\varepsilon\varphi, \varphi^{-1}\overline{\varepsilon}\varepsilon) = \dim(\varphi\oplus\sigma\varphi, \varphi^{-1}\oplus\varphi^{-1}\sigma),$$

and either $[\varphi^2] = [\mathrm{id}]$ or $[\varphi\sigma\varphi] = [\sigma]$ holds. Assume that the latter holds. Then,

$$
\begin{aligned}
[\lambda^2] &= [\varepsilon\varphi\overline{\varepsilon}\varepsilon\varphi\overline{\varepsilon}] = [\varepsilon\varphi(\mathrm{id}\oplus\sigma)\varphi\overline{\varepsilon}] \\
&= [\varepsilon\varphi^2\overline{\varepsilon}]\oplus[\varepsilon\varphi\sigma\varphi\overline{\varepsilon}] = [\varepsilon\varphi^2\overline{\varepsilon}]\oplus[\varepsilon\sigma\overline{\varepsilon}].
\end{aligned}
$$

Since

$$[\varepsilon\overline{\varepsilon}]\oplus[\varepsilon\sigma\overline{\varepsilon}] = [(\varepsilon\overline{\varepsilon})^2] = ([\mathrm{id}]\oplus[\pi])^2 = 2[\mathrm{id}]\oplus 3[\pi]\oplus[\xi_1]\oplus[\eta_1]\oplus[\eta_2],$$

we get

$$[\lambda^2] = [\varepsilon\varphi^2\overline{\varepsilon}]\oplus[\mathrm{id}]\oplus 2[\pi]\oplus[\xi_1]\oplus[\eta_1]\oplus[\eta_2],$$

which is a contradiction. Thus, the claim is shown, and we also have $[\varphi\sigma\varphi]\neq[\sigma]$.

Let $\omega = \sigma\varphi\overline{\varepsilon}$. We show the following 3 properties of ω.

(i) ω is irreducible.

(ii) $\dim(\rho', \omega\overline{\omega}) = 1$.

(iii) $[\varphi\omega] = [\omega]$.

Indeed, thanks to equation (6.18), we get

$$\dim(\omega,\omega) = \dim(\sigma^2, \varphi\overline{\varepsilon}\varepsilon\varphi^{-1}) = \dim(\mathrm{id}\oplus\rho'\oplus\sigma\oplus\sigma\alpha'\oplus\sigma\alpha'^2, \varphi(\mathrm{id}\oplus\sigma)\varphi^{-1}) = 1,$$

and ω is irreducible. (ii) also follows from equation (6.18) as we have

$$\dim(\rho', \omega\overline{\omega}) = \dim(\rho'\omega, \omega) = \dim(\sigma\rho'\sigma, \varphi(\mathrm{id}\oplus\sigma)\varphi^{-1}),$$

and $\sigma\rho'\sigma$ contains id with multiplicity 1. (iii) follows from

$$\begin{aligned}1 &= \dim(\lambda, \lambda^2) = \dim(\varepsilon\varphi\bar{\varepsilon}, \varepsilon\varphi\bar{\varepsilon}\varepsilon\varphi\bar{\varepsilon}) = \dim(\bar{\varepsilon}\varepsilon\varphi\bar{\varepsilon}, \varphi\bar{\varepsilon}\varepsilon\varphi\bar{\varepsilon}) \\ &= \dim((\mathrm{id}\oplus\sigma)\varphi\bar{\varepsilon}, \varphi(\mathrm{id}\oplus\sigma)\varphi\bar{\varepsilon}) = \dim(\varphi\bar{\varepsilon}\oplus\omega, \bar{\varepsilon}\oplus\varphi\omega) \\ &= \dim(\varphi, \bar{\varepsilon}\varepsilon) + \dim(\omega, \varphi\omega) = \dim(\omega, \varphi\omega).\end{aligned}$$

The proof of Theorem 4.1 shows that there exists $\tau \in \mathrm{Aut}(P)$ such that σ factorizes as $\sigma = \iota\tau\bar{\iota}$. Thus, we have $N \supset P \supset \omega(M)$. Since $[\iota\bar{\iota}] = [\mathrm{id}] \oplus [\rho']$, Lemma 2.7 shows that there exists a unitary $u \in N$ satisfying $\mathrm{Ad}\, u \circ \varphi(P) = P$, which means that there exists $\psi \in \mathrm{Aut}(P)$ satisfying $[\varphi\iota] = [\iota\psi]$. Now, we have

$$12 = \dim(\iota\psi\kappa\bar{\kappa}\psi^{-1}\bar{\iota}, \iota\kappa\bar{\kappa}\,\bar{\iota}) = \dim(\psi\kappa\bar{\kappa}\psi^{-1}, \bar{\iota}\iota\kappa\bar{\kappa}\,\bar{\iota}\iota).$$

We parameterize P-P sectors generated by $\bar{\iota}\iota$ as in the proof of Theorem 3.1. Then, $[\bar{\iota}\iota] = [\mathrm{id}] \oplus [\rho]$, $[\kappa\bar{\kappa}] = [\mathrm{id}] \oplus [\alpha] \oplus [\alpha^2]$, $d(\rho) = 3$, $d(\alpha) = 1$, $\alpha^3 = \mathrm{id}$, and they satisfy the following fusion rules:

$$\begin{aligned}[][\alpha\rho] &= [\rho\alpha] = [\rho], \\ [\rho^2] &= [\mathrm{id}] \oplus [\alpha] \oplus [\alpha^2] \oplus 2[\rho].\end{aligned}$$

Now, we have

$$[\bar{\iota}\iota\kappa\bar{\kappa}\,\bar{\iota}\iota] = 4([\mathrm{id}] \oplus [\alpha] \oplus [\alpha^2] \oplus 3[\rho]),$$

and we get

$$3 = \dim(\psi(\mathrm{id}\oplus\alpha\oplus\alpha^2)\psi^{-1}, \mathrm{id}\oplus\alpha\oplus\alpha^2\oplus 3\rho).$$

Thus, $[\psi\kappa\bar{\kappa}\psi^{-1}] = [\kappa\bar{\kappa}]$. Lemma 2.6 shows that there exists $\varphi_1 \in \mathrm{Aut}(R)$ satisfying $[\psi\kappa] = [\kappa\varphi_1]$, and so, $[\varphi\iota\kappa] = [\iota\kappa\varphi_1]$. Lemma 2.4 shows that there exists a group Γ containing $\mathfrak{A}_5$ such that γ extends to an outer action of Γ on R such that

$$L = R \rtimes \Gamma.$$

The shape of graph $\mathcal{G}_{L\supset M}$ shows that the Γ-action on the set $\Gamma/\mathfrak{A}_5$ is 3-transitive extension of $(\mathfrak{A}_5, X_5)$, and we conclude that $\Gamma = \mathfrak{A}_6$. ■

Remark 6.3. A similar argument works for $(\mathfrak{S}_6, X_6)$. In this case, we can apply Lemma 2.6 to $\mathfrak{S}_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ instead of $\mathfrak{A}_3 = \mathbb{Z}_3$ at the last step.

Note that we computed the graph $\mathcal{G}_{M_9}^{M_{10}}$ in Section 4, and the graph $\mathcal{G}_{M_{11}>M_{10}}$ for the Mathieu group M_{11} can be obtained by $\mathcal{G}_{M_{11}>M_{10}} = \widetilde{\mathcal{G}_{M_9}^{M_{10}}}$.

Theorem 6.4. *Let $L \supset M$ be a finite-index inclusion of factors with*

$$\mathcal{G}_{L\supset M} = \mathcal{G}_{M_{11}>M_{10}}.$$

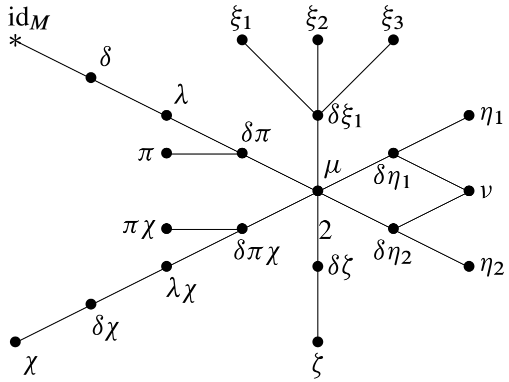

Figure 10. $\mathcal{G}_{M_{11}>M_{10}}$.

Then, there exists a unique subfactor $R \subset M$, up to inner conjugacy, such that

$$R' \cap L = \mathbb{C},$$

and there exists an outer action γ of M_{11} on R satisfying

$$L = R \rtimes_\gamma M_{11} \supset M = R \rtimes_\gamma M_{10}.$$

Proof. (1) Let $\delta : M \hookrightarrow L$ be the inclusion map, and let $[\bar{\delta}\delta] = [\mathrm{id}] \oplus [\lambda]$ be the irreducible decomposition. We parameterize the irreducible M-M sectors and the L-M sectors generated by δ as in Figure 10. Then, we have

$$d(\chi) = 1, \quad d(\pi) = 9, \quad d(\lambda) = d(\xi_1) = d(\xi_2) = d(\xi_3) = d(\eta_1) = d(\eta_2) = 10,$$
$$d(\zeta) = 16, \quad d(\nu) = 20, \quad d(\mu) = 80, \quad d(\delta) = \sqrt{11}.$$

From the graph, we can see that λ, π, $\pi\chi$, μ, ν, and χ are self-conjugate, and

$$\{[\chi\lambda], [\bar{\xi}_1], [\bar{\xi}_2], [\bar{\xi}_3], [\bar{\eta}_1], [\bar{\eta}_2]\} = \{[\lambda\chi], [\xi_1], [\xi_2], [\xi_3], [\eta_1], [\eta_2]\}.$$

Since $\pi\chi$ is self-conjugate, we have $[\pi\chi] = [\chi\pi]$. By the graph symmetry, we have $[\zeta\chi] = [\zeta]$, $[\mu\chi] = [\mu]$, $[\nu\chi] = [\nu]$, and

$$\{[\xi_1\chi], [\xi_2\chi], [\xi_3\chi]\} = \{[\xi_1], [\xi_2], [\xi_3]\},$$
$$\{[\eta_1\chi], [\eta_2\chi]\} = \{[\eta_1], [\eta_2]\}.$$

The basic fusion rules coming from the graph are

$$[\lambda^2] = [\mathrm{id}] \oplus [\lambda] \oplus [\pi] \oplus [\mu], \tag{6.19}$$
$$[\lambda\pi] = [\lambda] \oplus [\mu], \tag{6.20}$$

$$\begin{aligned}[\lambda\mu] = [\lambda] \oplus [\lambda\chi] \oplus [\pi] \oplus [\pi\chi] \oplus [\xi_1] \oplus [\xi_2] \oplus [\xi_3] \\ \oplus [\eta_1] \oplus [\eta_2] \oplus 2[\zeta] \oplus 2[\nu] \oplus 8[\mu], \end{aligned} \tag{6.21}$$

$$[\lambda\xi_i] + [\xi_i] = [\xi_1] \oplus [\xi_2] \oplus [\xi_3] \oplus [\mu], \tag{6.22}$$

$$[\lambda\eta_i] = [\nu] \oplus [\mu], \tag{6.23}$$

$$[\lambda\zeta] = 2[\mu], \tag{6.24}$$

$$[\lambda\nu] = [\eta_1] \oplus [\eta_2] \oplus [\nu] \oplus 2[\mu]. \tag{6.25}$$

Since the right-hand sides of equations (6.20), (6.21), (6.23), and (6.24) are self-conjugate, we have $[\lambda\pi] = [\pi\lambda]$, $[\lambda\mu] = [\mu\lambda]$, $[\lambda\eta_i] = [\overline{\eta}_i\lambda]$, and $[\lambda\zeta] = [\zeta\lambda]$. Since $[\lambda^2\pi] = [\pi\lambda^2]$, we get $[\mu\pi] = [\pi\mu]$.

By associativity, we get

$$\begin{aligned}[\pi^2] \oplus [\mu\pi] = [\mathrm{id}] \oplus [\lambda] \oplus [\lambda\chi] \oplus [\pi] \oplus [\pi\chi] \oplus [\xi_1] \oplus [\xi_2] \\ \oplus [\xi_3] \oplus [\eta_1] \oplus [\eta_2] \oplus 2[\zeta] \oplus 2[\nu] \oplus 8[\mu], \end{aligned} \tag{6.26}$$

$$\begin{aligned}[\pi\mu] \oplus [\mu^2] = [\mathrm{id}] \oplus [\chi] \oplus 9[\lambda] \oplus 9[\lambda\chi] \oplus 8[\pi] \oplus 8[\pi\chi] \oplus 9[\xi_1] \oplus 9[\xi_2] \\ \oplus 9[\xi_3] \oplus 9[\eta_1] \oplus 9[\eta_2] \oplus 14[\zeta] \oplus 18[\nu] \oplus 72[\mu], \end{aligned} \tag{6.27}$$

$$\begin{aligned}[\pi\xi_i] \oplus [\mu\xi_i] = [\lambda] \oplus [\lambda\chi] \oplus [\pi] \oplus [\pi\chi] \oplus [\xi_i] \oplus [\xi_1] \oplus [\xi_2] \oplus [\xi_3] \\ \oplus [\eta_1] \oplus [\eta_2] \oplus 2[\zeta] \oplus 2[\nu] \oplus 9[\mu], \end{aligned} \tag{6.28}$$

$$\begin{aligned}[\eta_i] \oplus [\pi\eta_i] \oplus [\mu\eta_i] = [\lambda] \oplus [\lambda\chi] \oplus [\pi] \oplus [\pi\chi] \oplus [\xi_1] \oplus [\xi_2] \oplus [\xi_3] \oplus 2[\eta_1] \\ \oplus 2[\eta_2] \oplus 2[\zeta] \oplus 2[\nu] \oplus 9[\mu], \end{aligned} \tag{6.29}$$

$$\begin{aligned}[\pi\zeta] \oplus [\mu\zeta] = 2[\lambda] \oplus 2[\lambda\chi] \oplus 2[\pi] \oplus 2[\pi\chi] \oplus 2[\xi_1] \oplus 2[\xi_2] \oplus 2[\xi_3] \\ \oplus 2[\eta_1] \oplus 2[\eta_2] \oplus 3[\zeta] \oplus 4[\nu] \oplus 14[\mu], \end{aligned} \tag{6.30}$$

$$\begin{aligned}[\pi\nu] \oplus [\mu\nu] = 2[\lambda] \oplus 2[\lambda\chi] \oplus 2[\pi] \oplus 2[\pi\chi] \oplus 2[\xi_1] \oplus 2[\xi_2] \oplus 2[\xi_3] \\ \oplus 2[\eta_1] \oplus 2[\eta_2] \oplus 4[\zeta] \oplus 5[\nu] \oplus 18[\mu]. \end{aligned} \tag{6.31}$$

We give a criterion to separate the summations of the left-hand sides. For irreducible X and Y, we have

$$\dim(\lambda\pi X, Y) = \dim(\pi X, \lambda Y),$$

and on the other hand,

$$\dim(\lambda\pi X, Y) = \dim((\lambda \oplus \mu)X, Y) = \dim(\lambda X, Y) + \dim(\mu X, Y).$$

Thus,

$$\dim(\pi X \oplus \mu X, Y) = \dim(\pi X, Y \oplus \lambda Y) - \dim(\lambda X, Y). \tag{6.32}$$

The Frobenius reciprocity implies $\dim(\pi^2, \lambda) = 0$ and $\dim(\mu\pi, \lambda) = 1$. We claim that π^2 does not contain μ. Assume on the contrary that π^2 contains μ. Then, equation (6.26) implies $\dim(\mu\pi, \mu) = 7$. Since $[\lambda]$ commutes with $[\zeta]$, equation (6.24) shows that $2[\mu] = [\zeta\lambda]$, and

$$14 = \dim(2\mu\pi, \mu) = \dim(\zeta\lambda\pi, \mu) = \dim(\zeta(\lambda \oplus \mu), \mu) = \dim(2\mu \oplus \zeta\mu, \mu).$$

Since μ and ζ are self-conjugate, we get $\dim(\mu\zeta, \mu) = 12$. However, this and equation (6.30) show $\dim(\pi\zeta, \mu) = 2$, which is impossible because $d(\pi\zeta) < 2d(\mu)$. Therefore, the claim is shown. The Frobenius reciprocity implies that we have

$$\dim(\mu\pi, \pi) = 0.$$

Since $[\mu\pi\chi] = [\mu\chi\pi] = [\mu\pi]$, we get $\dim(\mu\pi, \pi\chi) = 0$ too. Thus, dimension counting shows that we may put

$$[\pi^2] = [\mathrm{id}] \oplus [\pi] \oplus [\pi\chi] \oplus 2[\zeta] \oplus \bigoplus_{i=1}^{3} a_i[\xi_i] \oplus \bigoplus_{i=1}^{2} b_i[\eta_i] \oplus c[\nu],$$

where a_i, b_i, and c are non-negative integers satisfying

$$\sum_{i=1}^{3} a_i + \sum_{i=1}^{2} b_i + 2c = 3.$$

Applying equation (6.32) to this, we obtain $a_1 + a_2 + a_3 = 1$ and $b_i = 1 - c$. We may and do assume $a_1 = 1$, $a_2 = a_3 = 0$, and

$$[\pi^2] = [\mathrm{id}] \oplus [\pi] \oplus [\pi\chi] \oplus 2[\zeta] \oplus [\xi_1] \oplus (1-c)[\eta_1] \oplus (1-c)[\eta_2] \oplus c[\nu],$$
$$[\mu\pi] = 8[\mu] \oplus [\lambda] \oplus [\lambda\chi] \oplus [\xi_2] \oplus [\xi_3] \oplus c[\eta_1] \oplus c[\eta_2] \oplus (2-c)[\nu].$$

Since $[\mu\pi] = [\pi\mu]$, equation (6.27) shows $\dim(\mu^2, \xi_i) = 8$ for $i = 2, 3$, and the Frobenius reciprocity and equation (6.28) show that $\pi\xi_i$ contains μ. Thus,

$$[\pi\xi_i] = [\mu] \oplus 10 \dim, \quad i = 2, 3.$$

If ξ_i were not contained in $\pi\xi_i$, equation (6.28) implies that $\mu\xi_i$ would contain ξ_i with multiplicity 2, and consequently, $\xi_i\overline{\xi_i}$ would contain μ with multiplicity 2, which is a contradiction because $d(\xi_i)^2 < 2d(\mu)$. Thus, we have

$$[\pi\xi_i] = [\mu] \oplus [\xi_i], \qquad i = 2, 3,$$
$$[\mu\xi_i] = [\lambda] \oplus [\lambda\chi] \oplus [\pi] \oplus [\pi\chi] \oplus [\xi_1] \oplus [\xi_2] \oplus [\xi_3] \oplus [\eta_1] \oplus [\eta_2] \oplus 2[\zeta] \oplus 2[\nu] \oplus 8[\mu], \qquad i = 2, 3.$$

Since our argument is already long, we state the next claim as a separate lemma. ∎

Lemma 6.5. *With the above notation, we have* $c = 0$.

Proof. Assume on the contrary that $c = 1$. Since

$$2[\pi\mu] = [\pi\lambda\zeta] = [(\mu \oplus \lambda)\zeta] = [\mu\zeta] \oplus 2[\mu],$$

we can obtain the irreducible decomposition of $\mu\zeta$ and $\pi\zeta$.

Now, equation (6.32), the Frobenius reciprocity, and dimension counting show the following:

$$[\pi^2] = [\mathrm{id}] \oplus [\pi] \oplus [\pi\chi] \oplus 2[\zeta] \oplus [\xi_1] \oplus [\nu], \tag{W1}$$

$$[\mu\pi] = [\lambda] \oplus [\lambda\chi] \oplus [\xi_2] \oplus [\xi_3] \oplus [\eta_1] \oplus [\eta_2] \oplus [\nu] \oplus 8[\mu], \tag{W2}$$

$$\begin{aligned}[\mu^2] = {} & [\mathrm{id}] \oplus [\chi] \oplus 8[\lambda] \oplus 8[\lambda\chi] \oplus 8[\pi] \oplus 8[\pi\chi] \\ & \oplus 9[\xi_1] \oplus 8[\xi_2] \oplus 8[\xi_3] \oplus 8[\eta_1] \oplus 8[\eta_2] \oplus 14[\zeta] \oplus 17[\nu] \oplus 64[\mu],\end{aligned} \tag{W3}$$

$$[\pi\zeta] = 2[\pi] \oplus 2[\pi\chi] \oplus 2[\xi_1] \oplus 3[\zeta] \oplus 2[\nu], \tag{W4}$$

$$[\mu\zeta] = 2[\lambda] \oplus 2[\lambda\chi] \oplus 2[\xi_2] \oplus 2[\xi_3] \oplus 2[\eta_1] \oplus 2[\eta_2] \oplus 2[\nu] \oplus 14[\mu], \tag{W5}$$

$$[\pi\xi_1] = [\pi] \oplus [\pi\chi] \oplus 2[\zeta] \oplus 2[\xi_1] \oplus [\nu], \tag{W6}$$

$$[\mu\xi_1] = [\lambda] \oplus [\lambda\chi] \oplus [\xi_2] \oplus [\xi_3] \oplus [\eta_1] \oplus [\eta_2] \oplus [\nu] \oplus 9[\mu], \tag{W7}$$

$$[\pi\nu] = [\pi] \oplus [\pi\chi] \oplus 2[\zeta] \oplus [\xi_1] \oplus 2[\nu] \oplus [\mu], \tag{W8}$$

$$\begin{aligned}[\mu\nu] = {} & 2[\lambda] \oplus 2[\lambda\chi] \oplus [\pi] \oplus [\pi\chi] \\ & \oplus [\xi_1] \oplus 2[\xi_2] \oplus 2[\xi_3] \oplus 2[\eta_1] \oplus 2[\eta_2] \oplus 2[\zeta] \oplus 3[\nu] \oplus 17[\mu].\end{aligned} \tag{W9}$$

Here, the letter ‘W’ stands for wrong equations. Since the right-hand sides are self-conjugate, we see that $[\mu]$ commutes with $[\pi]$, $[\xi_1]$, $[\zeta]$, and $[\nu]$.

An argument similar to the case of $\pi\xi_i$ with $i = 2, 3$ shows $[\pi\eta_1] = [\mu] \oplus [\eta_2]$ and $[\pi\eta_2] = [\mu] \oplus [\eta_1]$. Equation (6.28) shows

$$2 = \dim(\mu\eta_i, \zeta) = \dim(\eta_i\zeta, \mu),$$

and consequently, $[\eta_i\zeta] = 2[\mu]$. In the same way, we have $[\xi_2\zeta] = [\xi_3\zeta] = 2[\mu]$, and taking conjugate, we also get

$$[\xi_2\zeta] = [\xi_3\zeta] = [\eta_1\zeta] = [\eta_2\zeta] = [\zeta\xi_2] = [\zeta\xi_3] = [\zeta\eta_1] = [\zeta\eta_2] = 2[\mu], \tag{W10}$$

$$[\pi\eta_1] = [\mu] \oplus [\eta_2], \quad [\pi\eta_1] = [\mu] \oplus [\eta_1], \tag{W11}$$

$$\begin{aligned}[\mu\xi_2] = [\mu\xi_3] = [\mu\eta_1] = [\mu\eta_2] & \\ = [\lambda] \oplus [\lambda\chi] \oplus [\pi] \oplus [\pi\chi] \oplus [\xi_1] \oplus [\xi_2] \oplus [\xi_3] \oplus [\eta_1] \oplus [\eta_2] & \\ \oplus 2[\zeta] \oplus 2[\nu] \oplus 8[\mu]. &\end{aligned} \tag{W12}$$

From

$$2[\mu\xi_2] = [\zeta\lambda\xi_1] = [\zeta(\mu \oplus \xi_2 \oplus \xi_3)] = [\zeta\mu] \oplus [\zeta(\xi_1 \oplus \xi_3)] = [\zeta\mu] \oplus 2[\mu] \oplus [\zeta\xi_1],$$

we get the irreducible decomposition of $\zeta\xi_1$.

From

$$2[\mu\eta_i] = [\zeta\lambda\eta_i] = [\zeta(\mu \oplus \nu)] = [\zeta\mu] \oplus [\zeta\nu],$$

we get the irreducible decomposition of $[\zeta\nu]$. The Frobenius reciprocity and dimension counting show

$$[\zeta\xi_1] = 2[\pi] \oplus 2[\pi\chi] \oplus 4[\zeta] \oplus 2[\xi_1] \oplus 2[\nu], \tag{W13}$$

$$[\zeta\nu] = 2[\pi] \oplus 2[\pi\chi] \oplus 4[\zeta] \oplus 2[\xi_1] \oplus 2[\nu] \oplus 2[\mu], \tag{W14}$$

$$[\zeta^2] = [\mathrm{id}] \oplus [\chi] \oplus 3[\pi] \oplus 3[\pi\chi] \oplus 5[\zeta] \oplus 4[\xi_1] \oplus 4[\nu]. \tag{W15}$$

Next, we determine the left multiplications of $[\xi_1]$ and $[\nu]$ by applying associativity to $[\pi^2 X]$. The two equations

$$[\pi(\pi\xi_1)] = [\pi(\pi \oplus \pi\chi \oplus 2\zeta \oplus 2\xi_1 \oplus \nu)],$$
$$[\pi^2\xi_1] = [(\mathrm{id} \oplus \pi \oplus \pi\chi \oplus 2\zeta \oplus \xi_1 \oplus \nu)\xi_1]$$

show

$$[\xi_1^2] \oplus [\nu\xi_1] = [\mathrm{id}] \oplus [\chi] \oplus 3[\pi] \oplus 3[\pi\chi] \oplus 4[\zeta] \oplus 2[\xi_1] \oplus 4[\nu] \oplus [\mu].$$

By the Frobenius reciprocity and dimension computing, we get

$$[\xi_1^2] = [\mathrm{id}] \oplus [\chi] \oplus 2[\pi] \oplus 2[\pi\chi] \oplus 2[\zeta] \oplus [\xi_1] \oplus [\nu], \tag{W16}$$

$$[\nu\xi_1] = [\pi] \oplus [\pi\chi] \oplus 2[\zeta] \oplus [\mu] \oplus [\xi_1] \oplus 3[\nu]. \tag{W17}$$

The two equations

$$[\pi(\pi\nu)] = [\pi(\pi \oplus \pi\chi \oplus 2\zeta \oplus \xi_1 \oplus 2\nu \oplus \mu)],$$
$$[\pi^2\nu] = [(\mathrm{id} \oplus \pi \oplus \pi\chi \oplus 2\zeta \oplus \xi_1 \oplus \nu)\nu]$$

show

$$[\xi_1\nu] \oplus [\nu^2] = [\mathrm{id}] \oplus [\chi] \oplus [\lambda] \oplus [\lambda\chi] \oplus 3[\pi] \oplus 3[\pi\chi]$$
$$\oplus 4[\zeta] \oplus 4[\xi_1] \oplus 3[\nu] \oplus 4[\mu] \oplus [\xi_2] \oplus [\xi_3] \oplus [\eta_1] \oplus [\eta_2],$$

and

$$[\nu^2] = [\mathrm{id}] \oplus [\chi] \oplus [\lambda] \oplus [\lambda\chi] \oplus 2[\pi] \oplus 2[\pi\chi]$$
$$\oplus 2[\zeta] \oplus 3[\mu] \oplus 3[\xi_1] \oplus [\xi_2] \oplus [\xi_3] \oplus [\eta_1] \oplus [\eta_2]. \tag{W18}$$

The two equations

$$[\pi(\pi\eta_1)] = [\pi\mu] \oplus [\pi\eta_2] = [\pi\mu] \oplus [\mu] \oplus [\eta_1],$$
$$[\pi^2\eta_1] = [(\mathrm{id} \oplus \pi \oplus \pi\chi \oplus 2\zeta \oplus \xi_1 \oplus \nu)\eta_1]$$
$$= [\eta_1] \oplus [\eta_2] \oplus [\chi\eta_2] \oplus 6[\mu] \oplus [\xi_1\eta_1] \oplus [\nu\eta_1]$$

show that $[\chi\eta_2] = [\eta_1]$, and

$$[\xi_1\eta_1] \oplus [\nu\eta_1] = [\lambda] \oplus [\lambda\chi] \oplus [\xi_2] \oplus [\xi_3] \oplus [\nu] \oplus 3[\mu].$$

In a similar way, we have $[\chi\xi_2] = [\xi_3]$ and

$$[\xi_1\xi_2] \oplus [\nu\xi_2] = [\lambda] \oplus [\lambda\chi] \oplus [\eta_1] \oplus [\eta_2] \oplus [\nu] \oplus 3[\mu].$$

We claim

$$\{[\overline{\xi_2}], [\overline{\xi_3}], [\overline{\eta_1}], [\overline{\eta_2}]\} = \{[\xi_2], [\xi_3], [\eta_1], [\eta_2]\}.$$

Indeed, since $[\lambda\chi\xi_2] = [\lambda\xi_3]$ does not contain id, we see that $\lambda\chi$ is not the conjugate sector of ξ_2. A similar argument applied to ξ_3, η_1, and η_2 shows the claim.

Assume first that $[\overline{\xi_2}]$ is either $[\eta_1]$ or $[\eta_2]$. Note that in this case $[\overline{\xi_3}] = [\overline{\xi_2}\chi]$ is also either η_1 or η_2. Then,

$$\dim(\xi_1\eta_2, \lambda) = \dim(\lambda\xi_1, \overline{\eta_2}) = 1,$$

and $\dim(\xi_1\eta_2, \lambda\chi) = 1$ in the same way. We have

$$\dim(\nu\xi_2, \lambda) = \dim(\lambda\nu, \overline{\xi_2}) = 1,$$

and $\dim(\nu\xi_2, \lambda\chi) = 1$ in the same way. Thus,

$$[\xi_1\eta_1] = [\xi_1\eta_2] = [\lambda] \oplus [\lambda\chi] \oplus [\mu], \tag{W19}$$

$$[\nu\eta_1] = [\nu\eta_2] = [\xi_2] \oplus [\xi_3] \oplus [\nu] \oplus 2[\mu], \tag{W20}$$

$$[\xi_1\xi_2] = [\xi_1\xi_3] = [\eta_1] \oplus [\eta_2] \oplus [\mu], \tag{W21}$$

$$[\nu\xi_2] = [\nu\xi_3] = [\lambda] \oplus [\lambda\chi] \oplus [\nu] \oplus 2[\mu]. \tag{W22}$$

Multiplying the both sides of equations (W20) and (W21) by $[\lambda]$ from the left, we get

$$[\eta_1^2] \oplus [\eta_2\eta_1] = [\eta_1\eta_2] \oplus [\eta_2^2] = 2[\xi_1] \oplus [\eta_1] \oplus [\eta_2] \oplus 2[\mu],$$
$$[\xi_2^2] \oplus [\xi_3\xi_2] = [\xi_2\xi_3] \oplus [\xi_3^2] = 2[\mu] \oplus 2[\nu].$$

Taking conjugate, we get a contradiction.

Assume now that $[\overline{\xi_2}]$ is either $[\xi_2]$ or $[\xi_3]$. In this case, $[\overline{\xi_3}]$ is either $[\xi_2]$ or $[\xi_3]$ too. The Frobenius reciprocity and dimension counting show

$$[\xi_1\eta_1] = [\xi_1\eta_2] = [\mu] \oplus [\xi_2] \oplus [\xi_3], \tag{W23}$$

$$[\nu\eta_1] = [\nu\eta_2] = [\lambda] \oplus [\lambda\chi] \oplus [\nu] \oplus 2[\mu], \tag{W24}$$

$$[\xi_1\xi_2] = [\xi_1\xi_3] = [\lambda] \oplus [\lambda\chi] \oplus [\mu], \tag{W25}$$

$$[\nu\xi_2] = [\nu\xi_3] = [\eta_1] \oplus [\eta_2] \oplus [\nu] \oplus 2[\mu]. \tag{W26}$$

Multiplying both sides of equations (W23) and (W26) by $[\lambda]$ from the left, we get

$$[\xi_2\eta_1] \oplus [\xi_3\eta_1] = [\xi_2\eta_2] \oplus [\xi_3\eta_2] = 2[\xi_1] \oplus [\xi_2] \oplus [\xi_3] \oplus 2[\mu],$$
$$[\eta_1\xi_2] \oplus [\eta_2\xi_2] = [\eta_1\xi_3] \oplus [\eta_2\xi_3] = 2[\nu] \oplus 2[\mu],$$

which is a contradiction again. Finally, we conclude that $c = 0$. ■

Continuation of the proof of Theorem 6.4. The above lemma and equation (6.26) show

$$[\pi^2] = [\mathrm{id}] \oplus [\pi] \oplus [\pi\chi] \oplus 2[\zeta] \oplus [\xi_1] \oplus [\eta_1] \oplus [\eta_2], \tag{6.33}$$
$$[\mu\pi] = 8[\mu] \oplus [\lambda] \oplus [\lambda\chi] \oplus [\xi_2] \oplus [\xi_3] \oplus 2[\nu]. \tag{6.34}$$

From equation (6.33), we can see

$$\{[\bar{\xi_1}], [\bar{\eta_1}], [\bar{\eta_2}]\} = \{[\xi_1], [\eta_1], [\eta_2]\}, \tag{6.35}$$

and in consequence,

$$\{[\chi\lambda], [\bar{\xi_2}], [\bar{\xi_3}]\} = \{[\lambda\chi], [\xi_2], [\xi_3]\}.$$

Since

$$2[\pi\mu] = [\pi\lambda\zeta] = [(\mu \oplus \lambda)\zeta] = [\mu\zeta] \oplus 2[\mu],$$

we get

$$[\mu\zeta] = 2[\lambda] \oplus 2[\lambda\chi] \oplus 2[\xi_2] \oplus 2[\xi_3] \oplus 4[\nu] \oplus 14[\mu],$$

and from equation (6.30),

$$[\pi\zeta] = 2[\pi] \oplus 2[\pi\chi] \oplus 2[\xi_1] \oplus 2[\eta_1] \oplus 2[\eta_2] \oplus 3[\zeta].$$

Equation (6.34) shows that $\pi\nu$ contains μ with multiplicity 2. If $\pi\nu$ contained ν with multiplicity at most 1, equation (6.31) shows that ν^2 would contain μ with multiplicity 4, which is impossible because $d(\nu^2) = 4d(\mu)$ and ν^2 contains id. Thus, we get

$$[\pi\nu] = 2[\mu] \oplus 2[\nu].$$

Now, the Frobenius reciprocity implies that neither $\pi\xi_1$, $\pi\eta_1$, nor $\pi\eta_2$ contains λ, $\lambda\chi$, ξ_2, ξ_3, ν, and we get

$$[\pi\xi_1] = [\pi] \oplus [\pi\chi] \oplus 2[\xi_1] \oplus [\eta_1] \oplus [\eta_2] \oplus 2[\zeta],$$
$$[\pi\eta_1] = [\pi] \oplus [\pi\chi] \oplus [\xi_1] \oplus [\eta_1] \oplus 2[\eta_2] \oplus 2[\zeta],$$
$$[\pi\eta_2] = [\pi] \oplus [\pi\chi] \oplus [\xi_1] \oplus 2[\eta_1] \oplus [\eta_2] \oplus 2[\zeta].$$

The above fusion rules show that the fusion category $\mathcal{C}_1$ generated by π satisfies

$$\mathrm{Irr}(\mathcal{C}_1) = \{\mathrm{id}, \chi, \pi, \pi\chi, \xi_1, \eta_1, \eta_2, \zeta\}.$$

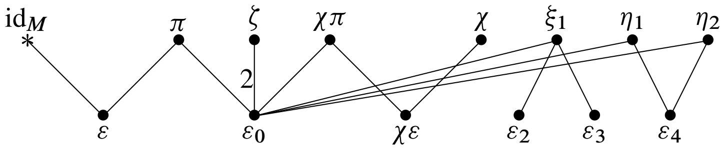

Figure 11. $\mathcal{G}^d_{M\supset N}$.

(2) Theorem 2.3 and equation (6.19) imply that there exists a unique intermediate subfactor N between M and $\lambda(M)$ such that if $\varepsilon : N \hookrightarrow M$ is the inclusion map, we have

$$[\varepsilon\bar{\varepsilon}] = [\mathrm{id}] \oplus [\pi].$$

Note that we have $d(\varepsilon) = \sqrt{10}$. In the same way as in the proof of Lemma 3.2, there exists $\varphi \in \mathrm{Aut}(N)$ satisfying $[\lambda] = [\varepsilon\varphi\bar{\varepsilon}]$.

(3) We show the dual principal graph $\mathcal{G}^d_{M\supset N}$ is $\mathcal{G}^{M_{10}}_{M_9}$ computed in Section 4 (see Figure 11). Since

$$\dim(\pi\varepsilon, \pi\varepsilon) = \dim(\pi, \pi\varepsilon\bar{\varepsilon}) = \dim(\pi, \pi(1 \oplus \pi)) = 2,$$

there exists an irreducible ε_0 satisfying $[\pi\varepsilon] = [\varepsilon] \oplus [\varepsilon_0]$ and $d(\varepsilon_0) = 8\sqrt{10}$. Since equation (6.35) and

$$[\pi\varepsilon\bar{\varepsilon}] = [\pi] \oplus [\pi^2] = [\mathrm{id}] \oplus 2[\pi] \oplus [\chi\pi] \oplus [\xi_1] \oplus [\eta_1] \oplus [\eta_2] \oplus 2[\zeta],$$

we get

$$[\varepsilon_0\bar{\varepsilon}] = [\pi] \oplus [\chi\pi] \oplus [\bar{\xi}_1] \oplus [\bar{\eta}_1] \oplus [\bar{\eta}_2] \oplus 2[\zeta].$$

By the Frobenius reciprocity,

$$[\zeta\varepsilon] = 2[\varepsilon_0].$$

Since

$$\dim(\bar{\xi}_1\varepsilon, \bar{\xi}_1\varepsilon) = \dim(\bar{\xi}_1, \bar{\xi}_1(\mathrm{id} \oplus \pi)) = 3,$$

there exist two irreducibles ε_2 and ε_3 satisfying

$$[\bar{\xi}_1\varepsilon] = [\varepsilon_0] \oplus [\varepsilon_2] \oplus [\varepsilon_3],$$

and $d(\varepsilon_2) + d(\varepsilon_3) = 2\sqrt{10}$. By the Frobenius reciprocity, we get

$$d(\varepsilon_2) = d(\varepsilon_3) = \sqrt{10}$$

and

$$[\varepsilon_2\bar{\varepsilon}] = [\varepsilon_2\bar{\varepsilon}] = [\bar{\xi}_1].$$

In a similar way, we can show

$$\dim(\overline{\eta_1}\varepsilon, \overline{\eta_1}\varepsilon) = \dim(\overline{\eta_2}\varepsilon, \overline{\eta_2}\varepsilon) = \dim(\overline{\eta_1}\varepsilon, \overline{\eta_2}\varepsilon) = 2,$$

and there exists irreducible ε_4 satisfying

$$[\eta_1\varepsilon] = [\eta_2\varepsilon] = [\varepsilon_4],$$

and $d(\varepsilon_4) = 2\sqrt{10}$. The Frobenius reciprocity shows

$$[\varepsilon_4\bar{\varepsilon}] = [\overline{\eta_1}] \oplus [\overline{\eta_2}].$$

Note that ξ_1 is self-conjugate and $\{[\overline{\eta_1}], [\overline{\eta_2}]\} = \{[\eta_1], [\eta_2]\}$. Thus, we get

$$\mathcal{G}^d_{M\supset N} = \mathcal{G}^{M_{10}}_{M_9}.$$

Now, Theorem 4.2 implies that $\mathcal{G}_{M\supset N} = \mathcal{G}_{M_{10}>M_9}$.

The rest of the proof is very much similar to that of Theorem 6.2, and we make only points different from it.

(4) Theorem 4.1 shows that there exists a unique subfactor $R \subset N$, up to inner conjugacy, such that $R' \cap M = \mathbb{C}$ and there exists an outer action β of M_{10} on R satisfying

$$M = R \rtimes_\beta M_{10} \supset N = R \rtimes_\beta M_9.$$

The inclusion $L \supset R$ is irreducible.

(5) To prove that $L \supset R$ is of depth 2, it suffices to show that $[\lambda]$ commutes with

$$[\mathrm{id}] \oplus [\chi] \oplus 9[\pi] \oplus 9[\pi\chi] \oplus 10[\xi_1] \oplus 10[\eta_1] \oplus 10[\eta_2] \oplus 16[\zeta],$$

which corresponds to the regular representation of M_{11}. Indeed, it follows from

$$\begin{aligned}
&[\lambda]([\mathrm{id}] \oplus [\chi] \oplus 9[\pi] \oplus 9[\pi\chi] \oplus 10[\xi_1] \oplus 10[\eta_1] \oplus 10[\eta_2] \oplus 16[\zeta]) \\
&\quad = [\lambda] \oplus [\lambda\chi] \oplus 9([\lambda] \oplus [\mu]) \oplus 9([\lambda\chi] \oplus [\mu]) \oplus 10([\mu] \oplus [\xi_2] \oplus [\xi_3]) \\
&\qquad \oplus 10([\mu] \oplus [\nu]) \oplus 10([\mu] \oplus [\nu]) \oplus 32[\mu] \\
&\quad = 10([\lambda] \oplus [\lambda\chi] \oplus [\xi_2] \oplus [\xi_3] \oplus 2[\nu] \oplus 8[\mu]),
\end{aligned}$$

which is self-conjugate as we can take the conjugate of the both sides.

(6) We can apply Lemma 2.6 to Q_8 to finish the proof. ∎

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