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Abelian TQFTS and Schrödinger local systems

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Abstract. In this paper, we construct an action of 3-cobordisms on the finite dimensional Schrödinger representations of the Heisenberg group by Lagrangian correspondences. In addition, we review the construction of the abelian Topological Quantum Field Theory (TQFT) associated with a q-deformation of $U(1)$ for any root of unity q. We prove that, for 3-cobordisms compatible with Lagrangian correspondences, there is a normalization of the associated Schrödinger bimodule action that reproduces the abelian TQFT.

The full abelian TQFT provides a projective representation of the mapping class group $\mathrm{Mod}(\Sigma)$ on the Schrödinger representation, which is linearizable at odd root of 1. Motivated by homology of surface configurations with Schrödinger representation as local coefficients, we define another projective action of $\mathrm{Mod}(\Sigma)$ on Schrödinger representations. We show that the latter is not linearizable by identifying the associated 2-cocycle.

To the memory of Vaughan Jones
The founder of quantum topology

1. Introduction

The discovery of the Jones polynomial revolutionized low-dimensional topology. The new link invariants constructed by Jones, Kauffman, HOMFLY-PT, and Reshetikhin–Turaev, etc. were extended to mapping class group representations, later shown to be asymptotically faithful, and to 3-manifold invariants. These developments have reached their peak in constructions of Topological Quantum Field Theories (TQFTs) [11, 36]. The scope of ideas initiated by Vaughan Jones built the foundations for the new domain of mathematics—the quantum topology. One of the main open problems in quantum topology is to understand the topological nature of quantum invariants.

In the 1990s, Lawrence [28] initiated a program aimed at homological interpretation of quantum invariants. In 2001, Bigelow [8] was able to read the Jones polynomial from the intersection pairing on the twisted homology of the configuration space $\mathrm{Conf}_n(\mathbb{D}^2_m)$ of n points in m-punctured disc $\mathbb{D}^2_m$. This construction led

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to a family of representations (indexed by n) of the braid group B_m, which recovers for $n = 1$ the Burau representation. A spectacular achievement was the proof by Bigelow [10] and Krammer [27] that this braid group representation for $n = 2$ is faithful, showing the linearity of the braid group. Bigelow's construction was extended later to other quantum link invariants (see [1, 2, 9, 30] and references therein).

Recently, homological mapping class group representations were constructed by the third author together with Palmer and Shaukat [13]. The idea here was to use a *Heisenberg* cover of the space $\mathrm{Conf}_n(\Sigma)$ of unordered n configurations in a surface Σ with one boundary component, whose group of deck transformations is the *Heisenberg group* $\mathcal{H}(\Sigma)$. Recall that $\mathcal{H}(\Sigma) = \mathbb{Z} \times H_1(\Sigma, \mathbb{Z})$ has the group law

$$(k, x)(l, y) = (k + l + x \cdot y, x + y),$$

where $x \cdot y$ is the intersection pairing. In detail, the surface braid group $B_n(\Sigma) := \pi_1(\mathrm{Conf}_n(\Sigma))$ surjects onto $\mathcal{H}(\Sigma)$ (see Section 4.1). The kernel of this map is a characteristic normal subgroup of the surface braid group by [13, Proposition 8], which determines the Heisenberg cover $\widetilde{\mathrm{Conf}}_n(\Sigma)$.

Since the group of deck transformations $\mathcal{H}(\Sigma)$ acts on the chain groups of the Heisenberg cover, any $\mathcal{H}(\Sigma)$-module M can be used to construct a twisted homology as follows. We first extend the action of the group to its group algebra $\mathbb{C}[\mathcal{H}] \to \mathrm{End}(M)$ by linearity and then construct a complex

$$C_\bullet\big(\widetilde{\mathrm{Conf}}_n(\Sigma) \otimes_{\mathbb{C}[\mathcal{H}(\Sigma)]} M\big).$$

Its homology, known as twisted homology of $\mathrm{Conf}_n(\Sigma)$ with coefficients in M, is denoted by $H_\bullet(\mathrm{Conf}_n(\Sigma), M)$ (compare [18, Chapter 5], [25, Chapter 3.H]). An interesting choice of M provides a finite dimensional Schrödinger representation $W_q(L)$ of a finite quotient of $\mathcal{H}(\Sigma)$, which depends on a choice of a Lagrangian $L \subset H_1(\Sigma, \mathbb{Z})$ and a root of unity q. If the order of q is odd, the resulting mapping class group representations were recently shown to contain the quantum representations arising from the non-semisimple TQFT for the small quantum $\mathfrak{sl}_2$ by De Renzi and Martel [35]. In particular, they defined the action of the quantum $\mathfrak{sl}_2$ on the Schrödinger homology explicitly and showed that it commutes with the action of the mapping class group.

To complete Lawrence–Bigelow program we are lacking homological interpretation of quantum 3-manifold invariants and of the action of 3-cobordisms on Schrödinger homologies. This paper is a first step in this direction. Here, we construct an action of 3-cobordisms on Schrödinger representations by Lagrangian correspondences. In addition, we show that on a certain subcategory of extended 3-cobordisms and after a suitable normalization this action recovers the abelian TQFT.

Abelian TQFTs are functorial extensions of 3-manifold invariants constructed by Murakami–Ohtsuki–Okada from linking matrices [34]. Their connections with theta functions and Schrödinger representations, in the case when the quantum parameter (called t in these papers) is a root of unity of order divisible by 4, were extensively studied by Gelca and collaborators [19–22]. Here, we work with an arbitrary root of unity. We show that interesting cases are if the order is either odd or divisible by 4. In the latter case, we complete the work of Gelca et al. by constructing TQFTs via *modularization functor*. In addition, we discuss refined TQFTs corresponding to the choice of a spin structure or a first cohomology class on 3-manifolds.

Our preferred cobordism category is the Crane–Yetter category 3Cob of connected oriented 3-cobordisms between connected surfaces with one boundary component and with the boundary connected sum as the monoidal structure. This category has a beautiful algebraic presentation: it is monoidally generated by the Hopf algebra object—the torus with one boundary component [5, 14]. By the result of [7], for any finite unimodular ribbon category $\mathcal{C}$, there exists a monoidal TQFT functor from $3\mathrm{Cob}^\sigma$ to $\mathcal{C}$ determined by sending the torus with one boundary component to the terminal object of $\mathcal{C}$, called the *end*. Here, $3\mathrm{Cob}^\sigma$ is the category of *extended* 3-cobordisms, whose objects are connected surfaces with one boundary component equipped with a choice of Lagrangian, and morphisms are 3-cobordisms equipped with natural numbers called weights. The composition includes a correction term given by a Maslov index. Note that if $\mathcal{C}$ is the category of modules over a unimodular ribbon Hopf algebra H, then the *end* of $\mathcal{C}$ is the adjoint representation $(H, \triangleright)$, where $\triangleright$ denotes the adjoint action.

Let us define a subcategory $3\mathrm{Cob}^{\mathrm{LC}}$ of $3\mathrm{Cob}^\sigma$ having the same objects, but a smaller set of morphisms. A cobordism $C = (C, 0)$ belongs to $3\mathrm{Cob}^{\mathrm{LC}}((\Sigma_-, L_-), (\Sigma_+, L_+))$ if and only if $L_+ = L_C.L_-$, where

$$L_C \cdot L_- = \{y \in H_1(\Sigma_+) \mid \exists\, x \in L_-,\ (x, y) \in L_C\},$$
$$L_C = \operatorname{Ker}\big(i_* : H_1(\partial C, \mathbb{Z}) = H_1(-\Sigma_-, \mathbb{Z}) \oplus H_1(\Sigma_+, \mathbb{Z}) \to H_1(C, \mathbb{Z})\big).$$

We say that L_+ is determined by the *Lagrangian correspondence* given by C.

In subcategory $3\mathrm{Cob}^{\mathrm{LC}}$, all anomalies vanish. We get a linear representation of the subgroup of the mapping class group fixing a Lagrangian. The full mapping class group is replaced by a groupoid whose objects are Lagrangians and morphisms are compatible mapping classes. This *action groupoid* is a subcategory in $3\mathrm{Cob}^{\mathrm{LC}}$.

Assume $q \in \mathbb{C}$ is a primitive pth root of unity of order $p \geq 3$ and $p \not\equiv 2 \pmod 4$. Let $p' = p$ if p is odd, and $p' = p/2$ otherwise. We define the finite Heisenberg group $\mathcal{H}_p(\Sigma)$ as a quotient of $\mathcal{H}(\Sigma)$ by the normal subgroup

$$I_p := \{(pk, p'x) \mid k \in \mathbb{Z}, x \in H_1(\Sigma, \mathbb{Z})\}.$$

The group $\mathcal{H}_p(\Sigma)$ is isomorphic to a $\mathbb{Z}_p$-extension of $H_1(\Sigma, \mathbb{Z}_{p'})$, where we use the shorthand $\mathbb{Z}_p$ for $\mathbb{Z}/p\mathbb{Z}$ (see Section 4.2 for more details).

For a given Lagrangian submodule $L \subset H_1(\Sigma, \mathbb{Z})$, let $L_p = L \otimes \mathbb{Z}_{p'} \subset H_1(\Sigma, \mathbb{Z}_{p'})$ and $\widetilde{L}_p = \mathbb{Z}_p \times L_p \subset \mathcal{H}_p(\Sigma)$ be a maximal abelian subgroup.

Denote by $\mathbb{C}_q$ a 1-dimensional representation of $\widetilde{L}_p$, where (k, x) acts by q^k. Then, inducing from $\mathbb{C}_q$, we obtain

$$W_q(L) = \mathbb{C}[\mathcal{H}_p(\Sigma)] \otimes_{\mathbb{C}[\widetilde{L}_p]} \mathbb{C}_q$$

a p'^g-dimensional *Schrödinger* representation of $\mathcal{H}_p(\Sigma)$. Note that as a $\mathbb{C}[\mathcal{H}_p(\Sigma)]$-module $W_q(L)$ is generated by $\mathbf{1} \in \mathbb{C}_q$. Given a cobordism in the category $3\mathrm{Cob}^{\mathrm{LC}}$, $C : (\Sigma_-, L_-) \to (\Sigma_+, L_+)$, with $L_+ = L_C . L_-$, we have a Schrödinger representation $W(L_C)$ of the Heisenberg group $\mathcal{H}(\partial C)$ which can be considered as a $(\mathbb{C}[\mathcal{H}(\Sigma_+)]$, $\mathbb{C}[\mathcal{H}(\Sigma_-)])$-bimodule, after identifying the subgroup $\mathcal{H}(-\Sigma_-) \subset \mathcal{H}(\partial C)$ with the group $\mathcal{H}(\Sigma_-)^{\mathrm{op}}$, and defining a right action of $\mathcal{H}(\Sigma_-)$ on $W_q(L_C)$ as the left action of the same element of $\mathcal{H}(-\Sigma_-)$.

The main results of this paper can be formulated as follows.

Theorem 1. *Assume that* $p \not\equiv 2 \pmod 4$. *For any cobordism* C *from* (Σ_-, L_-) *to* (Σ_+, L_+) *in* $3\mathrm{Cob}^{\mathrm{LC}}$*, there exists an isomorphism of* $\mathbb{Z}[\mathcal{H}(\Sigma_+)]$*-modules*

$$\psi_C : W_q(L_C) \otimes_{\mathbb{C}[\mathcal{H}_p(\Sigma_-)]} W_q(L_-) \xrightarrow{\sim} W_q(L_+)$$

sending $\mathbf{1} \otimes \mathbf{1}$ *to* $\mathbf{1}$. *In addition, the map*

$$\begin{aligned} F_C : W_q(L_-) &\to W_q(L_+), \\ w &\mapsto \psi_C(\mathbf{1} \otimes w), \end{aligned}$$

defines a monoidal functor $F : 3\mathrm{Cob}^{\mathrm{LC}} \to \mathrm{Vect}_{\mathbb{C}}$ *which associates with an object* (Σ, L) *the finite dimensional Schrödinger representation* $W_q(L)$.

The proof uses a modification of the Juhasz's presentation of cobordisms categories, which works for $3\mathrm{Cob}^{\mathrm{LC}}$ and is presented in appendix. Throughout this paper, we refer to the functor F as *Schrödinger* TQFT.

Our second result compares the abelian and Schrödinger TQFTs. In particular, we will show that TQFT maps for a given cobordism C coincide up to a normalization. The normalising coefficient, which we denote by $Z(\check{C})$, is actually the Murakami–Ohtsuki–Okada invariant of a closed 3-manifold $\check{C}$ obtained from C by gluing of two standard handlebodies $(H_\pm, L_\pm)$ with $\partial H_\pm = \Sigma_\pm$, $L_\pm$ generated by meridians, along diffeomorphisms identifying the Lagrangians. This leads to the following theorem.

Theorem 2. *The monoidal functor* $\check{F} : 3\mathrm{Cob}^{\mathrm{LC}} \to \mathrm{Vect}_{\mathbb{C}}$ *sending a cobordism* C *to*

$$\begin{aligned} \check{F}_C : W_q(L_-) &\to W_q(L_+), \\ w &\mapsto Z(\check{C}) \psi_C(\mathbf{1} \otimes w) \end{aligned}$$

coincides with the abelian TQFT at q *restricted to* $3\mathrm{Cob}^{\mathrm{LC}}$.

Observe that the normalization coefficient $Z(\check{C})$ is equal to zero if and only if there exists $\alpha \in H^1(\check{C}, \mathbb{Z}_{p'})$ with non zero triple product $\alpha \cup \alpha \cup \alpha$ [34, Theorem 3.2], however the Schrödinger action is always non vanishing.

Finally, in Section 4, we study the projective action of the full mapping class group $\mathrm{Mod}(\Sigma)$ on the Schrödinger representations. We use here the Stone–von Neumann theorem to identify Schrödinger representations for different Lagrangians. The symplectic action sends $f \in \mathrm{Mod}(\Sigma)$ to the automorphim $(k, x) \mapsto (k, f_*(x))$ of the Heisenberg group. We use this automorphism to build a projective action of $\mathrm{Mod}(\Sigma)$ on the Schrödinger representation $W_q(L)$ known as Weil representation. However, if we ask the natural $\mathrm{Mod}(\Sigma)$ action on the surface braid group $B_n(\Sigma)$ to commute with the projection to $\mathcal{H}(\Sigma)$ we get a different automorphism

$$f_{\mathcal{H}}(k, x) = (k + \theta_f(x), f_*(x))$$

with $f \mapsto \theta_f \in \mathrm{Hom}(H_1(\Sigma), \mathbb{Z})$ a crossed homomorphism. For odd p, $f_{\mathcal{H}}$ can be used to construct another projective action on the Schrödinger representations. This action is actually compatible with the corresponding local systems on $\mathrm{Conf}_n(\Sigma)$. Our analysis shows that in the odd case the symplectic action on Schrödinger representations is linearizable; however, the latter action intertwining $f_{\mathcal{H}}$ does not.

We plan to use these results to construct an action of cobordisms on the homology of $\mathrm{Conf}_n(\Sigma)$ twisted by Schrödinger representations and provide a homological interpretation of the Kerler–Lyubashenko TQFTs. Our long term goal will be to use infinite dimensional Schrödinger representations to construct TQFTs with generic quantum parameter q, rather than at a root of unity. An existence of such TQFTs was predicted by physicists. They are expected to play a crucial role in the categorification of quantum 3-manifold invariants [24]. Lagrangian Floer homology may serve as an inspiration for this purpose.

The paper is organized as follows. In Section 2, we review representation theoretical and skein constructions of abelian TQFTs, we discuss modularization functors, refinements as well as the action of the mapping class group and its extensions. In Section 3, we prove the two main theorems. In Section 4, we define the two projective mapping class group actions on the Schrödinger representations and study the associated 2-cocycles. Juhasz construction for Lagrangian cobordisms is recalled in the appendix.

2. Abelian TQFTs

In this section, we review representation theoretical and skein constructions of abelian TQFTs, we discuss modularization functors, refinements as well as the action of the mapping class group and its extensions.

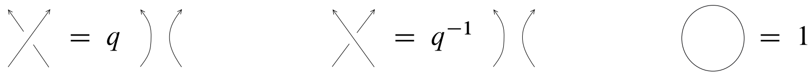

Figure 1. $U(1)$ skein relations.

2.1. Algebraic approach

Let $q \in \mathbb{S}^1 \subset \mathbb{C}$ be a primitive pth root of 1 and $p \geq 3$ is an integer. Let $p' = p$ if p is odd and $p' = p/2$ if p even. Consider the group algebra $H = \mathbb{C}[K]/(K^p - 1)$ of the cyclic group. This algebra can be identified with the Cartan part of the quantum $\mathfrak{sl}_2$ at q by extending the group monomorphism

$$U(1) \to SL(2, \mathbb{C}),$$
$$z \to \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}.$$

For this reason, abelian TQFTs are also called $U(1)$ TQFTs.

The algebra H has a natural Hopf algebra structure with a grouplike generator, i.e., $\Delta(K) = K \otimes K$, $S(K) = K^{-1}$. Moreover, H is a ribbon Hopf algebra with R-matrix, and its inverse is given by

$$R = \frac{1}{p} \sum_{0 \leq i,j \leq p-1} q^{-ij} K^i \otimes K^j, \quad R^{-1} = \frac{1}{p} \sum_{0 \leq i,j \leq p-1} q^{ij} K^{-i} \otimes K^{-j},$$

the ribbon elements

$$v = \frac{1}{p} \sum_{0 \leq i,j \leq p-1} q^{i(j-i)} K^j, \quad v^{-1} = \frac{1}{p} \sum_{0 \leq i,j \leq p-1} q^{i(i-j)} K^{-j},$$

and the trivial pivotal structure.

Similarly to the $U_q(\mathfrak{sl}_2)$ case, the representation category H−mod has p simple modules V_k for $0 \leq k \leq p-1$. However, here, V_k is the 1-dimensional representation determined by its character $K \mapsto q^k$. Also, in our case, the fusion rules are very simple: $V_i \otimes V_j = V_{i+j}$, where the index $i + j$ is taken modulo p. Hence, all objects V_j are *invertible*, meaning that for each j there exists $k = p - j$ such that $V_j \otimes V_k = V_0$, where V_0 is the tensor unit of H−mod. The R-matrix is acting by q^{kl} on $V_k \otimes V_l$.

2.2. Skein approach

For explicit computations, it is more convenient to work with a skein theoretic construction.

Consider the skein relations depicted in Figure 1. Given a 3-manifold M, a skein module $S(M)$ is a $\mathbb{C}$-vector space generated by framed links in M modulo the skein relations. For a surface F it is customary to denote by $S(F)$ the skein algebra $S(F \times [0,1])$. We will usually identify a coloring of a component K of a framed link with an element of the skein algebra $S(A)$, where the annulus A is embedded along K by using the framing. For example, V_j-coloring is represented by an element y^j, where y is the core of A. Here, we use the usual algebra structure on $S(A)$ to identify y^j with j parallel copies of y. The Kirby color is

$$\Omega \equiv \sum_{k=0}^{p-1} y^k \in S(A).$$

The Ω-colored $(+1)$-framed unknot gets value

$$G \equiv \sum_{k=0}^{p-1} q^{k^2} = \begin{cases} \varepsilon\sqrt{p}(1+\sqrt{-1}) & \text{if } p \equiv 0 \pmod 4, \\ \pm\sqrt{p} & \text{if } p \equiv 1 \pmod 4, \\ 0 & \text{if } p \equiv 2 \pmod 4, \\ \pm\sqrt{-p} & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{1}$$

by the well-known Gauss formula, where ε is a 4th root of 1. For $p = 0 \pmod 4$, we explain in the next section, why the sums for G and Ω should be taken till $p'-1$, and we denote them by

$$g \equiv \sum_{k=0}^{p'-1} q^{k^2} \quad \text{and} \quad \omega \equiv \sum_{k=0}^{p'-1} y^k. \tag{2}$$

Recall that for p odd, $p' = p$ and $g = G$, $\omega = \Omega$. Define η and κ by $|g| = \eta^{-1}$ and $\kappa = \eta g$. Then, $\eta^{-1} = \sqrt{p'}$ while κ is an eighth root of unity for $p \equiv 0 \pmod 4$ and $\kappa \in \{1, \sqrt{-1}\}$ for p odd. Hence, for all p except $p \equiv 2 \pmod 4$, we can define the invariant of a closed 3-manifold M obtained by surgery on S^3 along a framed n component link L as follows:

$$Z(M) = \kappa^{-\mathrm{sign}(L)}\eta\langle \eta\omega, \ldots, \eta\omega\rangle_L,$$

where $\mathrm{sign}(L)$ is the signature of the linking matrix and $\langle x_1, \ldots, x_n\rangle_L$ denotes the evaluation of L whose ith component is colored by x_i in the skein algebra $S(\mathbb{R}^3)$.

The normalization is chosen in such a way that

$$Z(S^2 \times S^1) = \eta\langle \eta\omega\rangle_{\text{0-framed unknot}} = \eta^2 \sum_{k=0}^{p'-1} \langle y^k\rangle_{\text{unknot}} = \eta^2 p' = 1,$$

$$Z(S^3) = \eta = 1/\sqrt{p'}.$$

The right Dehn twist along a curve γ is represented by coloring the curve γ with

$$\eta\omega_- = \eta \sum_{k=0}^{p'-1} q^{-k^2} y^k,$$

where y is the core of γ.

If $p \equiv 0 \pmod 4$, then we split $\omega = \omega_0 + \omega_1$ into even and odd colors. Then, we define an additional topological structure on M that determines a $\mathbb{Z}/2\mathbb{Z}$-grading on the components of L, and thus a $\mathbb{Z}/2\mathbb{Z}$-grading on their colorings. In particular, for $p \equiv 4 \pmod 8$, we construct an invariant of the pair (M, s):

$$Z(M,s) = \kappa^{-\operatorname{sign}(L)} \eta \langle \eta\omega_{s_1}, \dots, \eta\omega_{s_n} \rangle_L,$$

where $(s_1, \dots, s_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ satisfying

$$\sum_{j=1}^{n} L_{ij} s_j = L_{ii} \pmod 2$$

determines a characteristic sublink of L corresponding to the spin structure s on M. Analogously, for $p \equiv 0 \pmod 8$, we construct invariants of a pair (M, h):

$$Z(M,h) = \kappa^{-\operatorname{sign}(L)} \eta \langle \eta\omega_{h_1}, \dots, \eta\omega_{h_n} \rangle_L,$$

where $(h_1, \dots, h_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ satisfying

$$\sum_{j=1}^{n} L_{ij} h_j = 0 \pmod 2$$

determines the first cohomology class $h \in H^1(M, \mathbb{Z}/2\mathbb{Z})$. In both cases, $Z(M)$ is the sum of the refined invariants over all choices of the additional structure.

2.3. Modularization and refinements

A $\mathbb{C}$-linear ribbon category with a finite number of dominating simple objects is called *premodular*. If, in addition, the monodromy S-matrix is invertible, then the category is modular. In our case, the S-matrix, whose (i, j) component is the invariant q^{2ij} of the (i, j)-colored Hopf link, is invertible only for odd p, and in this case, $H-$mod is *modular*, providing an abelian TQFT by standard constructions [36] or [11].

We call a premodular category $\mathcal{C}$ *modularizable*, if there exists a braided monoidal essentially surjective functor from $\mathcal{C}$ to a modular category, sending the subcategory of transparent objects to the tensor unit. In [16, Proposition 4.2], Bruguières gave a simple criterion for a premodular category to be modularizable, see also [33]. In particular, such category cannot contain *transparent* objects with twist coefficient -1. Recall that an object is called transparent if it has trivial braiding with any other object. Observe that the row in the S-matrix corresponding to the transparent object is colinear with the one for the tensor unit.

If p is even, H-mod is a premodular category. The object $V_{p'}$ is transparent and has twist coefficient $q^{p'^2}$, which is 1 if p' is even and -1 if p' is odd. Using results of [16], we deduce that in the case when $p \equiv 0 \pmod 4$, H-mod is modularizable. The resulting modular category has p' simple objects, that are all invertible. The new Kirby color is given in (2). Hence, we have $\eta = |g|^{-1} = (\sqrt{p'})^{-1}$ in all cases when invariant is defined.

Furthermore, if $p \equiv 4 \pmod 8$, the object $V_{p'/2}$ has twist coefficient -1. From [4], we deduce that our category in this case is actually *spin modular*, hence providing an abelian spin TQFT for 3-cobordisms equipped with a spin structure. Analogously, if $p \equiv 0 \pmod 8$, we can construct a refined TQFT that gives rise to invariants of 3-cobordisms equipped with first cohomology classes over $\mathbb{Z}/2\mathbb{Z}$. We refer to [4] for details about the construction of the refined invariants and their properties.

In the case $p \equiv 2 \pmod 4$, $H-$mod is not modularizable. The best we can do in this case to obtain 3-manifold invariants is to consider the degree 0 subcategory with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading given by the action of $K^{p'}$. The corresponding invariants will coincide with those obtained with the quantum parameter of odd order equals p'.

To construct a map associated by an abelian TQFT with a 3-cobordism $C : \Sigma_- \to \Sigma_+$, we first need to choose parametrizations of surfaces $\Sigma_\pm$, i.e., diffeomorphisms $\phi_\pm : \Sigma_{g_\pm} \to \Sigma_\pm$, where Σ_g is the standard genus g surface. If $p \not\equiv 2 \pmod 4$, the TQFT vector space associated with Σ_g has dimension p'^g. A basis $\{y^{\mathbf{i}}, \mathbf{i} = (i_1, \dots, i_g), 0 \le i_j \le p' - 1\}$ is given by p' colorings of g cores of the 1-handles of a bounding handlebody H_g. The $(\mathbf{i}, \mathbf{j})$-matrix element of the TQFT map is constructed as follows: We glue the standard handlebodies $H_{g_\pm}$ to C along the parametrizations. Inside H_{g_-} we put the link $y^{\mathbf{j}}$ and inside H_{g_+} the link $y^{\mathbf{i}}$. The result is a closed 3-manifold $\check{M} = S^3(L)$ with a collection of circles $c^+ \cup c^-$ inside; then,

$$Z(C)^{\mathbf{i}}_{\mathbf{j}} := \kappa^{-\mathrm{sign}(L)} \eta^{g_+} \langle \eta\omega, \dots, \eta\omega, y^{\mathbf{i}}, y^{\mathbf{j}} \rangle_{L \cup c_+ \cup c_-}.$$

By using the universal construction [11], this map can also be computed by gluing just one handlebody $(H_{g_-}, y^{\mathbf{j}})$ to C and by evaluating the result in the skein of $C \cup H_{g_-}$. The parametrization reduces in this approach to the choice of Lagrangian $L \subset H_1(\Sigma, \mathbb{Z})$, which is equal to ker : $H_1(\Sigma, \mathbb{Z}) \to H_1(H_g, \mathbb{Z})$, and its complement $L^\vee$. Since all curves representing elements of L are trivial in the skein of H_g, the basis curves $y^{\mathbf{j}}$ of the TQFT vector space are parametrized by a basis of $L^\vee$.

In this paper, we will be particularly interested in the Crane–Yetter category 3Cob of connected 3-cobordisms between connected surfaces with one boundary component. In this category, the monoidal product is given by the boundary connected sum rather than by the disjoint union, thus leading to a rich algebraic structure [14]. By the result of [7], the category $3\mathrm{Cob}^\sigma$ of extended 3-cobordisms is universal in the sense that for any finite unimodular ribbon category $\mathcal{C}$, there exists a TQFT functor $F : 3\mathrm{Cob}^\sigma \to \mathcal{C}$ defined by sending the torus with one boundary component to the *end* of $\mathcal{C}$. In our case, for odd p,

$$\mathrm{end}(H\text{−mod}) = \bigoplus_{j=0}^{p-1} V_j .$$

Modularization creates an isomorphism $V_k \cong V_k \otimes V_{p'}$; hence, for $p \equiv 0 \pmod 4$, we have

$$\mathrm{end}(H\text{−mod}) = \bigoplus_{j=0}^{p'-1} V_j .$$

In both cases, the vector space associated by F to a genus g surface with one boundary component has dimension p'^g.

For even p', refined TQFTs on $3\mathrm{Cob}^\sigma$ can be constructed along the lines of [6]. On the standard cobordism category, this was done in [3, 12].

2.4. Extended cobordisms and Lagrangian correspondence

Let us recall that the skein or Reshetikhin–Turaev TQFT constructions give rise to projective representations of the mapping class group and the gluing formula has a so-called *framing anomaly* which can be resolved by using *extended cobordisms*. The latter are given by a pair: a 3-cobordism between surfaces equipped with Lagrangian subspaces in the first homology group and a natural number. This approach leads to a representation of a certain central extension of the mapping class group.

If p is odd, then the framing anomaly κ, defined as $\frac{g}{|g|}$, where the Gauss sum g in (1), is a 4th root of 1. From [23, Remark 6.9], we can deduce that the corresponding TQFT contains a native representation of the mapping class group. This is because, the central generator of the extension acts by $\kappa^4 = 1$; hence, the index 4 subgroup described in [23] is the trivial extension. Recall that the metaplectic group Mp_{2g} is the non trivial double cover of the symplectic group Sp_{2g}. The metaplectic mapping class group is the pull back of this double cover using the symplectic action. In the case $p \equiv 0 \pmod 4$, the framing anomaly κ is a primitive 8th root of unity and the above argument shows that the TQFT contains a native representation of the metaplectic mapping class group.

To avoid anomaly issues in general, we will work with a subcategory $3\mathrm{Cob}^{\mathrm{LC}}$ of the category of connected *extended* 3-cobordisms between connected surfaces with one boundary component equipped with Lagrangians. Objects of $3\mathrm{Cob}^{\mathrm{LC}}$ are as in

$3\mathrm{Cob}^\sigma$, namely, pairs: a connected surface with one boundary component Σ and a Lagrangian subspace $L \subset H_1(\Sigma, \mathbb{Z})$. Recall that a Lagrangian is a maximal submodule with vanishing intersection pairing. For any 3-cobordism $C : \Sigma_- \to \Sigma_+$, we define a Lagrangian correspondence

$$L_C = \mathrm{Ker}\big(i_* : H_1(\partial C, \mathbb{Z}) = H_1(-\Sigma_-, \mathbb{Z}) \oplus H_1(\Sigma_+, \mathbb{Z}) \to H_1(C, \mathbb{Z})\big).$$

Now, given Lagrangians $L_\pm \subset H_1(\Sigma_\pm, \mathbb{Z})$ the action of L_C on L_- is defined as follows:

$$L_C \cdot L_- = \big\{y \in H_1(\Sigma_+) \mid \exists\, x \in L_-, (x, y) \in L_C\big\}.$$

The pair $(C, 0)$ belongs to $3\mathrm{Cob}^{\mathrm{LC}}((\Sigma_-, L_-), (\Sigma_+, L_+))$ if and only if $L_C \cdot L_- = L_+$. If we restrict to mapping cylinders, we obtain the so-called *action groupoid* of the mapping class group action on Lagrangian subspaces.

Restriction of the TQFT functor to $3\mathrm{Cob}^{\mathrm{LC}}$ kills all Maslov indices needed to compute framing anomalies in gluing formulas (compare [23, Section 2]).

3. Proofs

In this section, we prove our two main results. We will always assume that $p \geq 3$, $p \not\equiv 2 \pmod 4$ and $p' = p$ if p is odd and $p' = p/2$ if p is even.

For a Lagrangian submodule $L \subset H_1(\Sigma, \mathbb{Z})$ let $L^\vee$ be a complement of L. Then, we set $L_p := L \otimes \mathbb{Z}_{p'}$, and $L_p^\vee := L^\vee \otimes \mathbb{Z}_{p'}$. The finite quotient $\mathcal{H}_p(\Sigma) := \mathcal{H}(\Sigma)/I_p$ of the Heisenberg group defined in introduction, coincides with the semidirect product

$$\mathcal{H}_p(\Sigma) \cong (\mathbb{Z}_p \times L_p) \ltimes L_p^\vee,$$

where the multiplication on the right-hand side is given by $(k, a, b)(k', a', b') = (k + k' + 2a.b', a + a', b + b')$. The isomorphism between two models is induced by the homomorphism

$$\mathcal{H}(\Sigma) \to (\mathbb{Z}_p \times L_p) \ltimes L_p^\vee : (k, a + b) \mapsto (k + a \cdot b \pmod p, a \pmod{p'}, b \pmod{p'}),$$

where $a \in L$, $b \in L^\vee$ (see [22, Proposition 2.3] for more details).

Using this isomorphism it is easy to check that $\widetilde{L}_p = \mathbb{Z}_p \times L_p \subset \mathcal{H}_p(\Sigma)$ is a maximal abelian subgroup. Let q be a primitive pth root of unity. Denote by $\mathbb{C}_q$ a 1-dimensional representation of $\widetilde{L}_p$, where (k, x) acts by q^k. Then, inducing from $\mathbb{C}_q$ we define

$$W_q(L) = \mathbb{C}[\mathcal{H}_p(\Sigma)] \otimes_{\mathbb{C}[\widetilde{L}_p]} \mathbb{C}_q,$$

a p'^g-dimensional *Schrödinger representation* of the finite Heisenberg group $\mathcal{H}_p(\Sigma)$. Let us denote by $\mathbf{1}$ the canonical generator of $W_q(L)$ as $\mathbb{C}[\mathcal{H}_p(\Sigma)]$-module. Moreover, throughout this section to *simplify notation*, we denote $L \otimes \mathbb{Z}_{p'}$ by L.

Any Lagrangian $L^\vee \subset H_1(\Sigma, \mathbb{Z})$ complementary to L provides a basis for $W_q(L)$ indexed by $\mathsf{L}^\vee$. Given $b \in \mathsf{L}^\vee$, we denote by v_b the corresponding basis vector. In this basis the left action of the Heisenberg group is as follows.

- The central generator $u = (1, 0)$ acts by $v_b \mapsto q v_b$.
- For $y \in L^\vee$, $(0, y)$ acts by translation: $v_b \mapsto v_{b+y}$, where the index is modulo p'.
- For $x \in L$, $(0, x)$ acts by $v_b \mapsto q^{2x \cdot b}\, v_b$, where $x \cdot b$ is computed by lifting b to any preimage in $H_1(\Sigma)$ and is well defined since q is a pth root of unity.

In the last step, we used the rule $(0, x)(0, b) = (x \cdot b, x + b) = (0, b)(2x \cdot b, x)$.

Proof of Theorem 1. Let $C \in 3\mathrm{Cob}^{\mathrm{LC}}((\Sigma_-, L_-), (\Sigma_+, L_+))$. Then, we have three Heisenberg groups $\mathcal{H}(\Sigma_-)$, $\mathcal{H}(\Sigma_+)$ and $\mathcal{H}(\partial C)$, and respective Schrödinger representations $W_q(L_-)$, $W_q(L_+)$ and $W_q(L_C)$.

Using that $\partial C = -\Sigma_- \bigcup_{S^1} \Sigma_+$ and the inclusions $H_1(-\Sigma_-, \mathbb{Z}) \to H_1(\partial C, \mathbb{Z})$, $H_1(\Sigma_+, \mathbb{Z}) \to H_1(\partial C, \mathbb{Z})$ we have commuting actions of $\mathcal{H}(-\Sigma_-)$ and $\mathcal{H}(\Sigma_+)$ on $W_q(L_C)$. Actually, $W_q(L_C)$ can be viewed as a $(\mathbb{C}[\mathcal{H}(\Sigma_+)], \mathbb{C}[\mathcal{H}(\Sigma_-)])$-bimodule, after identifying the group $\mathcal{H}(-\Sigma_-)$ with $\mathcal{H}(\Sigma_-)^{\mathrm{op}}$, and defining a right action of $\mathcal{H}(\Sigma_-)$ on $W_q(L_C)$ as the left action of the same element of $\mathcal{H}(-\Sigma_-)$. Then, we can form the tensor product $W_q(L_C) \otimes_{\mathbb{C}[\mathcal{H}(\Sigma_-)]} W_q(L_-)$ and compare it with $W_q(L_+)$.

Any morphism in $3\mathrm{Cob}^{\mathrm{LC}}$ can be decomposed into simple ones, which are mapping cylinders and index 1 or 2 surgeries (see Section 5 for details). We will first prove the isomorphism $W_q(L_C) \otimes_{\mathbb{C}[\mathcal{H}(\Sigma_-)]} W_q(L_-) \cong W_q(L_+)$ and compute the induced maps $F_C : W_q(L_-) \to W_q(L_+)$ for simple cobordisms. Then, we use Corollary 16 to argue that the bimodule associated with a composition of simple cobordisms does not depend on the choice of the decomposition.

For a mapping cylinder $C_f : (\Sigma_-, L_-) \to (\Sigma_+, L_+)$, where the diffeomorphism $f : \Sigma_- \to \Sigma_+$ sends L_- to $f_*(L_-) = L_+$, we have

$$L_{C_f} = \{(-x, f_*(x)),\ x \in H_1(\Sigma_-, \mathbb{Z})\}.$$

We choose a Lagrangian $L_-^\vee$ complementary to L_-. Then, $L_+^\vee = f_*(L_-^\vee)$ is complementary to L_+. The submodule

$$L_{C_f}^\vee = L_- \oplus L_+^\vee \subset H_1(\partial C_f, \mathbb{Z})$$

is Lagrangian and complementary to L_{C_f}. Indeed, if $(-x, f_*(x))$ belongs to $L_{C_f}^\vee$, then $x \in L_-$ and $f_*(x) \in L_+^\vee \cap L_+ = \{0\}$, showing that $L_{C_f}^\vee \cap L_{C_f} = \{0\}$. Recall that for all kinds of Lagrangians L, the notation L means $L \otimes \mathbb{Z}_{p'}$. A $\mathbb{C}$-basis b_y^+ for $W_q(L_+)$ is labeled by elements $y \in \mathsf{L}_+^\vee$. In all computations below, we use the same notation for elements of $L^\vee$ as elements of $\mathcal{H}(\Sigma)$ acting on a module and $\mathsf{L}^\vee$ as the indexing set of a basis of $W_q(L)$. It should not be confusing since in the second case they are just considered modulo p'. Likewise for any element $(k, x) \in \widetilde{L}$ its image in $\widetilde{L}_p$ is just $(k \pmod p, x \pmod{p'})$; hence, it acts again by q^k on $\mathbb{C}_q$.

We have bases $\{B_z, z \in \mathsf{L}_{C_f}^{\vee}\}$ for $W_q(L_{C_f})$, and $\{b_x^-, x \in \mathsf{L}_-^{\vee}\}$ for $W_q(L_-)$. As a vector space the tensor product is generated by

$$\left\{B_z \otimes b_x^-, z \in \mathsf{L}_{C_f}^{\vee}, x \in \mathsf{L}_-^{\vee}\right\}$$

with relations coming from the action by elements in $\mathcal{H}(\Sigma_-)$. We write $z \in \mathsf{L}_{C_f}^{\vee}$ as $z = (z_-, z_+)$, $z_- \in \mathsf{L}_-$, $z_+ \in \mathsf{L}_+^{\vee} = f_*(\mathsf{L}_-^{\vee})$.

For an element $y \in L_-$, we get the relation:

$$\begin{aligned} q^{2y.x} B_{(z_-,z_+)} \otimes b_x^- &= B_{(z_-,z_+)}(0, y) \otimes b_x^- = (0, (y, 0)) B_{(z_-,z_+)} \otimes b_x^- \\ &= B_{(z_-+y,z_+)} \otimes b_x^-. \end{aligned}$$

This reduces the set of generators to $\{B_{(0,z_+)} \otimes b_x^-, x \in \mathsf{L}_-^{\vee}, z_+ \in \mathsf{L}_+^{\vee}\}$.

For an element $x \in L_-^{\vee}$, we get another relation

$$\begin{aligned} B_{(0,z_+)} \otimes b_x^- &= B_{(0,z_+)}(0, x) \otimes \mathbf{1} = (0, (x, 0)) B_{(0,z_+)} \otimes \mathbf{1} \\ &= (0, (0, f_*(x))(0, (x, -f_*(x)))) B_{(0,z_+)} \otimes \mathbf{1} \\ &= q^{-2 f_*(x) \cdot z_+} B_{(0,z_+ + f_*(x))} \otimes \mathbf{1}, \end{aligned} \tag{3}$$

where the intersection is written on the positively oriented Σ_+. This further reduces the generators to $\{B_{(0,z_+)} \otimes \mathbf{1}, z_+ \in \mathsf{L}_+^{\vee}\}$. Since any relation coming from any element in $\mathcal{H}(\Sigma_-)$ can be deduced from the previously written ones, we get that $\{B_{(0,y)} \otimes \mathbf{1}, y \in \mathsf{L}_+^{\vee}\}$ represents a $\mathbb{C}$-basis for the tensor product $W_q(L_C) \otimes_{\mathbb{C}[\mathcal{H}(\Sigma_-)]} W_q(L_-)$. It follows that the $\mathbb{C}[\mathcal{H}(\Sigma_+)]$-module map

$$\psi_C : W_q(L_C) \otimes_{\mathbb{C}[\mathcal{H}(\Sigma_-)]} W_q(L_-) \to W_q(L_+),$$

which sends $\mathbf{1} \otimes \mathbf{1}$ to $\mathbf{1}$ is an isomorphism. Moreover, the map

$$\begin{aligned} F_C : W_q(L_-) &\to W_q(L_C) \otimes_{\mathbb{C}[\mathcal{H}(\Sigma_-)]} W_q(L_-) \cong W_q(L_+), \\ b_x^- &\mapsto \psi_C(\mathbf{1} \otimes b_x^-) \end{aligned}$$

sends a basis vector b_x^- to $b_{f_*(x)}^+$, for any $x \in \mathsf{L}_-^{\vee}$, by using (3) with $z_+ = 0$.

In the case of a simple cobordism $C : (\Sigma_-, L_-) \to (\Sigma_+, L_+)$ corresponding to an index 1 surgery, the genus increases by 1. We have

$$L_C = \left\{(-x_-, x_+), x_- \in H_1(\Sigma_-, \mathbb{Z})\right\} \oplus \mathbb{Z}(0, \mu),$$

where μ is a meridian of the new handle and x_+ is the class x_- pushed in Σ_+. Let λ be a longitude for the new handle. We choose a Lagrangian $L_-^{\vee}$ complementary to L_-. By pushing through the cobordism, we may also consider $L_-^\vee$ as a subspace in $H_1(\Sigma_+, \mathbb{Z})$. The span of $L_-^\vee$ and λ gives a Lagrangian $L_+^\vee$ complementary to L_+. Then, $L_C^\vee = L_-^\vee \oplus L_+^\vee$ is complementary to L_C and the previous argument constructs the isomorphism. Here, the map

$$F_C : W_q(L_-) \to W_q(L_C) \otimes_{\mathbb{C}[\mathcal{H}(\Sigma_-)]} W_q(L_-) \cong W_q(L_+)$$

sends a basis vector b_x^- to b_x^+, where $x \in \mathsf{L}_-^\vee = L_-^\vee \otimes \mathbb{Z}_{p'}$.

Let us consider a simple cobordism $C : (\Sigma_-, L_-) \to (\Sigma_+, L_+)$ corresponding to an index 2 surgery on a curve γ. Let δ be a curve in Σ_- such that $\gamma.\delta = 1$. The curves γ and δ determine a genus one subsurface Σ_1. Outside Σ_1 the cobordism is trivial. Denote by $\Sigma \subset \Sigma_-$ the complement of Σ_1 which we consider also as a subsurface of Σ_+. We arrange the splitting so that $\Sigma_- = \Sigma \natural \Sigma_1$ is a boundary connected sum. Then, all Lagrangian subspaces and Schrödinger modules split. Over Σ the cobordism is trivial and the expected result is clear, so that it is enough to compute in the genus 1 case, $\Sigma_- = \Sigma_1$ and $\Sigma_+ = D^2$. The Lagrangian L_- is generated by a simple curve m. A complementary Lagrangian $L_-^\vee$ is generated by l with $m.l = 1$. We have $\gamma = \alpha m + \beta l$, $\gcd(\alpha, \beta) = 1$. The Lagrangian correspondence is

$$L_C = \mathbb{Z}(\gamma, 0) \text{ with complement } L_C^\vee = \mathbb{Z}(\delta, 0),$$

where $\delta = um + vl$, $\alpha v - \beta u = 1$. Then, $m = v\gamma - \beta\delta$, $l = -u\gamma + \alpha\delta$. We have bases $B_{k\delta}$ and $b_{\nu l}$, $0 \le k, \nu < p'$ for $W_q(L_C)$ and $W_q(L_-)$, respectively. Using l, we get the relation

$$\begin{aligned} B_{k\delta} \otimes b_{(\nu+1)l} &= B_{k\delta}(0, -u\gamma + \alpha\delta) \otimes b_{\nu l} = (0, \alpha\delta)(u\alpha, -u\gamma) B_{k\delta} \otimes b_{\nu l} \\ &= q^{u\alpha + 2ku} B_{(\alpha+k)\delta} \otimes b_{\nu l}, \end{aligned}$$

where we used intersection on $-\Sigma_-$. This reduces the set of generators to $B_{k\delta} \otimes \mathbf{1}$, $0 \le k < p'$. The relation coming from m then gives

$$B_{k\delta} \otimes \mathbf{1} = B_{k\delta}(0, v\gamma - \beta\delta) \otimes \mathbf{1} = (0, -\beta\delta)(\beta v, v\gamma) B_{k\delta} \otimes \mathbf{1} = q^{\beta v - 2kv} B_{(k-\beta)\delta} \otimes \mathbf{1}. \tag{4}$$

If the surgery curve γ is in L_-, we can choose $m = \gamma$, $l = \delta$. The last relation gives in this case $B_{k\delta} \otimes \mathbf{1} = q^{-2k} B_{k\delta} \otimes \mathbf{1}$. Hence, we have $B_{k\delta} \otimes \mathbf{1} = 0$ for $0 < k < p'$ and the tensor product is $\mathbb{C}$-generated by $\mathbf{1} \otimes \mathbf{1}$. The equalities

$$B_{k\delta} \otimes \mathbf{1} = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{p'}, \\ 0 & \text{else} \end{cases}$$

define an isomorphism $W_q(L_C) \otimes_{\mathbb{C}[\mathcal{H}(\Sigma_-)]} W_q(L_-) \cong \mathbb{C}_q$. In particular,

$$F_C(b_{kl}) = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{p'}, \\ 0 & \text{else.} \end{cases}$$

If the surgery curve γ is not in L_-, then $\beta \neq 0$. Let $d = \gcd(\beta, p')$; then, the order of β modulo p' is $a = \frac{p'}{d}$. Hence, relation (4) reduces the generators to $\{B_{k\delta} \otimes \mathbf{1}, 0 \leq k < d\}$. Finally, the action of $(0, am)_{-\Sigma_-}$ gives the following relation:

$$B_{k\delta} \otimes \mathbf{1} = (0, av\gamma - a\beta\delta) B_{k\delta} \otimes \mathbf{1} = (0, -a\beta\delta)(a^2\beta v, av\gamma) B_{k\delta} \otimes \mathbf{1} = q^{-2kav} B_{k\delta} \otimes \mathbf{1},$$

since $q^{a^2\beta} = 1$ and the intersection pairing is taken on $-\Sigma_-$. It follows $B_{k\delta} \otimes \mathbf{1} = 0$ unless k is divisible by d; hence, $\mathbf{1} \otimes \mathbf{1}$ generates $W_q(L_C) \otimes_{\mathbb{C}[\mathcal{H}_p(\Sigma_-)]} W(L_-) \cong \mathbb{C}_q$, as expected.

We are left with computing F_C. The action of $(0, a\gamma) = (0, a\alpha m + a\beta l)$ gives

$$\mathbf{1} \otimes b_{kl} = \mathbf{1} \otimes (0, a\beta l)(a^2\alpha\beta, a\alpha m) b_{kl} = q^{-2ka\alpha} \mathbf{1} \otimes b_{kl}$$

implying that $\mathbf{1} \otimes b_{kl} = 0$ if $d \nmid k$. If $d \mid k$, we set $k\alpha = k'\beta$ and compute

$$\begin{aligned} \mathbf{1} \otimes b_{kl} &= \mathbf{1}(0, -ku\gamma + k\alpha\delta) \otimes \mathbf{1} = (0, k\alpha\delta)(k^2 u\alpha, -ku\gamma)\mathbf{1} \otimes \mathbf{1} \\ &= q^{k^2 u\alpha} B_{k\alpha\delta} \otimes \mathbf{1} = q^{kk'\beta u} B_{k'\beta\delta} \otimes \mathbf{1}. \end{aligned}$$

From the action of $(0, k'm)$, we get

$$\begin{aligned} \mathbf{1} \otimes b_{kl} &= q^{kk'\beta u}(0, k'v\gamma - k'\beta\delta) B_{k'\beta\delta} \otimes \mathbf{1} \\ &= q^{kk'\beta u}(-(k')^2 v\beta, k'v\gamma)(0, -k'\beta\delta) B_{k'\beta\delta} \otimes \mathbf{1} \\ &= q^{kk'\beta u - kk'\alpha v} \mathbf{1} \otimes \mathbf{1} = q^{-kk'} \mathbf{1} \otimes \mathbf{1} = q^{-\alpha k^2/\beta} \mathbf{1} \otimes \mathbf{1}. \end{aligned}$$

We deduce that

$$F_C(b_{kl}) = \begin{cases} 0 & \text{if } d \nmid k, \\ q^{-\alpha k^2/\beta} & \text{else.} \end{cases}$$

In order to complete the proof and construct Schrödinger TQFT, we need to check the conditions in Corollary 16 for F.

The first relation holds trivially since any $d \in \mathrm{Diff}(\Sigma)$, which is isotopic to identity, induces identity automorphism of $H_1(\Sigma)$ and as follows, identity automorphism of the corresponding Schrödinger representation.

The second relation also holds since d and $d^{\mathbb{S}}$ can be extended to a diffeomorphism $M(\mathbb{S}) \to M(\mathbb{S}')$ which induces the relation.

If attaching and belt spheres $\mathbb{S}$ and $\mathbb{S}'$ of two index 1 or 2 surgeries do not intersect on a surface $\Sigma \in \mathbf{Surf}^{\mathrm{LC}}$, then it can be cut into two pieces $\Sigma = \Sigma_1 \natural \Sigma_2$, such that $\mathbb{S} \subset \Sigma_1$ and $\mathbb{S}' \subset \Sigma_2$. It can be naturally extended to $M(\mathbb{S})$ and $M(\mathbb{S}')$ so that all Lagrangians splits into direct sums, and Schrödinger representations into tensor product (see discussion of monoidality below). It follows that $F_{M(\mathbb{S})}$ and $F_{M(\mathbb{S}')}$ act independently on the corresponding factors, hence commute. It proves that relation 3 holds.

Since definition of F does not depend on the choice of the orientation on $\mathbb{S}$, relation (5) holds.

The rest of the proof deals with checking that F preserves relation (4). For that, we compute Lagrangian correspondences and the composition of two F maps using some choice of bases.

Consider a composition of two cobordisms

$$M(\mathbb{S}_2) \circ M(\mathbb{S}_1) : (\Sigma_-, L_-) \to (\Sigma, L) \to (\Sigma_+, L_+) \tag{5}$$

of indices 1 and 2, such that the belt sphere $b(\mathbb{S}_1)$ and the attaching $a(\mathbb{S}_2)$ intersect at a single point. Denote by $L_{M(\mathbb{S}_1)}$ and $L_{M(\mathbb{S}_2)}$ the Lagrangian correspondences induced by these cobordisms. Let μ be the meridian of the handle attached in the first surgery (i.e., $\mu = b(\mathbb{S}_1)$). Let $\gamma = a(\mathbb{S}_2)$ be the attaching sphere of the second surgery. The composition (5) defines a diffeomorphism $\varphi : \Sigma_- \to \Sigma_+$ uniquely up to isotopy by the property $\varphi|_{\Sigma_- \cap \Sigma_+} = \mathrm{id}$ [26]. Then, it induces the isomorphism $\varphi_* : H_1(\Sigma_-) \to H_1(\Sigma_+)$.

Let us choose some orientation on γ and μ, such that $\mu.\gamma = 1$.

The first homology group of Σ can be represented as $H_1(\Sigma) \simeq H_1(\Sigma_-) \oplus \mathbb{Z}(\mu, \gamma)$ since $\mu \cdot \gamma = 1$. Let $i_0 : H_1(\Sigma_-) \to H_1(\Sigma)$ be the corresponding embedding.

Since $L_{M(\mathbb{S}_1)} = \{(x, -i_0(x)) \mid x \in H_1(\Sigma_-)\} \oplus \mathbb{Z}(0, \mu) \subset H_1(-\Sigma_-) \oplus H_1(\Sigma)$ the image of L_- in $H_1(\Sigma)$ equals

$$L = L_{M(\mathbb{S}_1)} \cdot L_- = i_0(L_-) \oplus \mathbb{Z}(\mu).$$

The manifold $M(\mathbb{S}_2)$ is homotopy equivalent to $\Sigma \bigcup_{\partial D^2} D^2$, where $\partial D^2 = S^1 \to \Sigma$ is the inclusion of γ. Hence, $H_1(M(\mathbb{S}_2)) \simeq H_1(\Sigma)/[\gamma]$.

Let $h_1 \simeq S^1 \times D^1$ be the handle glued in the first surgery and $h_2 \simeq D^2 \times S^0$ in the second. Then, φ maps $(\Sigma_- \cap \mathrm{Im}(\mathbb{S}_2)) \cup \mathrm{Im}(\mathbb{S}_1) \simeq D^2$ to $(h_1 \backslash \mathrm{Im}(\mathbb{S}_2)) \cup h_2 \simeq D^2$ and is identical on $\Sigma \cap \Sigma_-$.

Consider a cycle x representing a class in $H_1(\Sigma_-)$. Let us first consider the case when $[x].\gamma = 1$ and $x \cap \mathrm{Im}(\mathbb{S}_2) = I_x \simeq [0, 1]$. Then, $\varphi|_{x \backslash I_x} = \mathrm{id}$ and φ maps I_x to an interval on $(h_1 \backslash \mathrm{Im}(\mathbb{S}_2)) \cup h_2$ connecting two points on its boundary. It can be represented (up to homotopy) as $(\mu \backslash \mathrm{Im}\mathbb{S}_2) \cup y \cup y'$, where y and y' are two chords—one on each of two discs of h_2. This means that the class $\varphi_*(x)$ can be represented as $[x] - [\mu]$ in $M(\mathbb{S}_2)$. If we consider now a class $[x] \in H_1(\Sigma_-)$ with an arbitrary intersection number $[x] \cdot \gamma$, an analogous construction gives $\varphi_*(x) = [x] - ([x] \cdot \gamma)\mu$ in $H_1(M(\mathbb{S}_2))$. Hence, the image of $(i_0(x) + \gamma.x, -\varphi_*(x)) \in H_1(\Sigma) \oplus H_1(\Sigma_+)$ is equal to zero under the homomorphism $H_1(\Sigma) \oplus H_1(\Sigma_+) \to H_1(M(\mathbb{S}_2))$. Therefore,

$$L_{M(\mathbb{S}_2)} = \left\{(i_0(x) + (\gamma.x)\mu, -\varphi_*(x)) | x \in H_1(\Sigma_-)\right\} \oplus \mathbb{Z}(\gamma, 0).$$

Let $L_-^\vee$ be a complement to L_-. Then, $L^\vee = i_0(L_-^\vee) \oplus \mathbb{Z}(\lambda)$ is a complement to L and $L^\vee_{M(\mathbb{S}_2)} = \{(i_0(x), \varphi_*(y)) \mid x \in L_-, y \in L_-^\vee\} \oplus \mathbb{Z}(\mu, 0)$ is a complementary Lagrangian to $L_{M(\mathbb{S}_2)}$. Indeed, if $(x, y) \in L_- \oplus L_-^\vee$ and $(i_0(x) + k\mu, \varphi_*(y)) \in L_{M(\mathbb{S}_2)}$ then $x = -y = 0, k = 0$ since $L_- \cap L_-^\vee = \{0\}$, so $L_{M(\mathbb{S}_2)} \cap L^\vee_{M(\mathbb{S}_2)} = \{0\}$. Restriction of the intersection pairing on $L^\vee_{M(\mathbb{S}_2)}$ is equal to zero since it is a direct sum of two Lagrangians, and $L_{M(\mathbb{S}_2)} + L^\vee_{M(\mathbb{S}_2)} = H_1(\Sigma) \oplus H_1(\Sigma_+)$ since any element $(i_0(u) + n\mu + k\lambda, \varphi_*(v)), u, v \in H_1(\Sigma_-)$ can be decomposed as

$$\begin{aligned}
&(i_0(u) + n\mu + k\gamma, \varphi_*(v)) \\
&\quad = \big(i_0(u_{L_-^\vee} - v_{L_-}) + (\gamma \cdot i_0(u_{L_-^\vee} - v_{L_-}))\mu + k\gamma, -\varphi_*(u_{L_-^\vee} - v_{L_-})\big) \\
&\qquad + \big(i_0(u_{L_-} + v_{L_-}) + (n - (\gamma \cdot i_0(u_{L_-^\vee} - v_{L_-})))\mu, \varphi_*(u_{L_-^\vee} + v_{L_-^\vee})\big), \qquad (6)
\end{aligned}$$

where $u = u_{L_-} + u_{L_-^\vee}$, $v = v_{L_-} + v_{L_-^\vee}$ and $u_{L_-}, v_{L_-} \in L_-, u_{L_-^\vee}, v_{L_-^\vee} \in L_-^\vee$.

Choose $L_+^\vee = \varphi_*(L_-^\vee)$. Then, we can choose bases of Schrödinger representations as follows:

$$\begin{aligned}
W_q(L_-) &: \{b_x^- | x \in \mathsf{L}_-^\vee\}; \\
W_q(L) &: \{b_{x+n\lambda} | x \in \mathsf{L}_-^\vee, n \in \mathbb{Z}\}; \\
W_q(L_+) &: \{b_x^+ | x \in \mathsf{L}_-^\vee\}; \\
W_q(L_{M(\mathbb{S}_2)}) &: \{B_{(x+a\mu,y)} | (x, y) \in \mathsf{L}_- \oplus \mathsf{L}_-^\vee, a \in \mathbb{Z}\}.
\end{aligned}$$

The map $F_{M(\mathbb{S}_1)}$ sends b_x^- to b_x then. Consider $\mathbf{1} \otimes b_x \in W_q(L_{M(\mathbb{S}_2)}) \otimes W_q(L)$ for $x \in \mathsf{L}_-^\vee$. It can be rewritten using (6) as follows:

$$\begin{aligned}
\mathbf{1} \otimes b_x &= \mathbf{1} \otimes (0, i_0(x))\mathbf{1} = (0, (i_0(x), 0))\mathbf{1} \otimes \mathbf{1} \\
&= (0, (-(\gamma \cdot i_0(x))\mu, \varphi_*(x)))(0, (i_0(x) + (\gamma \cdot i_0(x))\mu), -\varphi_*(x))\mathbf{1} \otimes \mathbf{1} \\
&= (0, (0, \varphi_*(x)))\mathbf{1} \otimes (0, -(\gamma \cdot i_0(x))\mu)\mathbf{1} = (0, (0, \varphi_*(x)))\mathbf{1} \otimes \mathbf{1},
\end{aligned}$$

which means that the image of b_x is $b_{\varphi_*(x)}$ and it coincides with the action of the mapping cylinder C_φ associated to φ.

Finally, we have to prove that F preserves the monoidal structure. The monoidal product $\Sigma_1 \natural \Sigma_2$ of two surfaces $\Sigma_1, \Sigma_2 \in 3\mathrm{Cob}^{\mathrm{LC}}$ with one boundary component is induced by boundary connected sum $\partial\Sigma_1 \# \partial\Sigma_2 \simeq S^1$. Since $H_1(\Sigma_1 \natural \Sigma_2) = H_1(\Sigma_1) \oplus H_1(\Sigma_2)$, the new Lagrangian is $L = L_1 \oplus L_2$. It means that the Heisenberg group is the direct product over the center of the two Heisenberg groups associated to (Σ_1, L_1) and (Σ_2, L_2). It induces the isomorphsm $W_q(L) \simeq W_q(L_1) \otimes W_q(L_2)$. Similarly, one can check that F preserves the monoidal structure on morphisms. ■

It remains to compare F with the abelian TQFT.

Proof of Theorem 2. We will use the skein model from Section 2. Following [22, Theorem 4.5] the Heisenberg group algebra $\mathbb{C}[\mathcal{H}(\Sigma)]$ can be identified with the $U(1)$-skein algebra $S(\Sigma)$. This makes the TQFT vector space $V(\Sigma, L)$ to a module over $\mathbb{C}[\mathcal{H}(\Sigma)]$. Actually, it is isomorphic to the Schrödinger representation (see [22, Theorem 4.7] for p even).

Here, we construct the isomorphism explicitly. Let us denote by $S_p(\Sigma)$ the reduced $U(1)$ skein algebra, where q is specified to the pth root of unity and a p' copies of any curve are removed [22, def 4.3]. Then, $S_p(\Sigma)$ is identified with $\mathbb{C}[\mathcal{H}_p(\Sigma)]$ by sending a simple closed curve γ with blackboard framing to the class of the image of

$$(0, [\gamma]) \in \mathbb{Z} \times H_1(\Sigma, \mathbb{Z}) = \mathcal{H}(\Sigma)$$

in $\mathcal{H}_p(\Sigma)$. Let H be a handlebody with boundary Σ such that L is the kernel of the inclusion $H_1(\Sigma, \mathbb{Z}) \hookrightarrow H_1(H, \mathbb{Z})$. Then, the TQFT vector space $V(\Sigma, L)$ is the quotient of $S_p(\Sigma)$ by the subspace generated by $\gamma - 1$, where γ is a simple curve that bounds in H or equivalently such that $[\gamma] \in L$. Using the isomorphism $S(\Sigma) \cong \mathbb{C}[\mathcal{H}(\Sigma)]$, we deduce that the quotients $V(\Sigma, L)$ and $W_q(L)$ are isomorphic.

A basis $\{b_x, x \in \mathsf{L}^\vee\}$ for $W_q(L)$ can be represented by skein elements $\{y_x, x \in \mathsf{L}^\vee\}$ in H providing a basis for $V(\Sigma, L)$. Here, for an embedded curve x in Σ, the element y_x is obtained by pushing x in H with blackboard framing and then by taking its skein class. For example, the element y_{3x} corresponds to the three parallel copies of y_x obtained by using the blackboard framing. We are now able to compare $\check{F}_C = Z(\check{C})F_C$ with the TQFT map on simple cobordisms.

Let us consider a mapping cylinder $C_f : (\Sigma_-, L_-) \to (\Sigma_+, L_+)$ with $g_- = g_+ = g$. A basis for the TQFT vector space (identified with the Schrödinger representation $W_q(L_-)$) is represented by a handlebody H_-, with $\partial H_- = \Sigma_-$ and with the cores l_i, $1 \le i \le g$, of its handles colored by y^k, $0 \le k \le p'$. The TQFT map is represented by gluing the mapping cylinder C_f to the handlebody H_-. This results in a handlebody H_+ with boundary Σ_+. Moreover, when pushing the colored curve l across the cylinder, we get a curve parallel to $f(l)$. Hence, the TQFT map sends y^k in H_- to $f(y^k)$ in H_+, matching F_{C_f}. Note that $\check{C}_f$ is an integral homology connected sum of g copies of $S^2 \times S^1$, since f preserves the Lagrangians. Hence, in our normalization, $Z(\check{C}_f) = 1$.

In the case of an index 1 surgery $C : (\Sigma_-, L_-) \to (\Sigma_+, L_+)$, the TQFT map is represented by the inclusion of a handlebody $H_- \hookrightarrow H_+ = H_- \bigcup_{\Sigma_-} C$, where $\partial H_- = \Sigma_-$ and

$$\ker(H_1(\Sigma, \mathbb{Z}) \hookrightarrow H_1(H_-, \mathbb{Z})) = L_-.$$

This inclusion map matches again F_C with $Z(\check{C}) = 1$.

In the case of an index 2 surgery on a curve γ, we only need to consider the case where Σ_- is a genus 1 surface. Then, the TQFT map $Z(C) : V(\Sigma_-, L_-) \to \mathbb{C}_q$ is

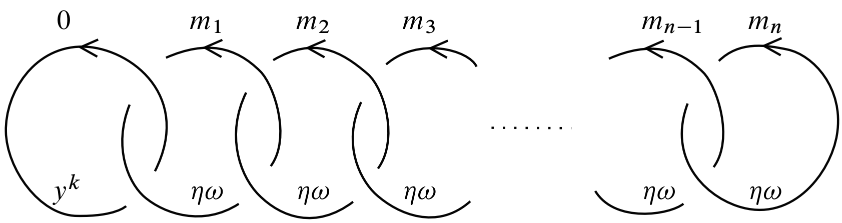

Figure 2. Surgery link for the lens space where the upper indices correspond to the framings and the lower ones to the colors.

given by the evaluation of the skein (H_-, x) inside $M_\gamma = (H_- \bigcup_{\Sigma_-} C) \bigcup_{S^2} D^3$. If $\gamma = m$, $M_\gamma = S^1 \times S^2$ and the evaluation reduces to a Hopf link with one Kirby-colored component, which is zero unless $x = 0$, when it is 1.

If $\gamma = l$, $M_\gamma = S^3$ and the evaluation is 1 for all $x = y^k$. Hence, in both cases, we recover F_C.

More generally, for $\gamma = \alpha m + \beta l$ with $\beta \neq 0$, the manifold M_γ is the lens space $L(\beta, \alpha)$. Let us choose a continued fraction decomposition $\beta/\alpha = [m_1, \ldots, m_n]$ as in [29]. Then, a surgery link L for M_γ is the length n Hopf chain with framings m_i (see Figure 2). Hence, the TQFT map sends y^k to the following number:

$$Z(C)(y^k) = \kappa^{-\operatorname{sign}(L)} \eta^n \sum_{j_1, \ldots, j_n = 1}^{p'} q^{\sum_{i=1}^n m_i j_i^2} q^{2kj_1} q^{2\sum_{i=1}^{n-1} j_i j_{i+1}}.$$

Since a recursive computation of this sum was done in [29], we present here just the result:

$$Z(C)(y^k) = \begin{cases} 0 & d \nmid k, \\ q^{-\frac{\alpha k^2}{\beta}} Z(L(\beta, \alpha)) & \text{else,} \end{cases}$$

where $d = \gcd(\beta, p')$. This coincides with F_C on this cobordism with

$$Z(\check{C}) = Z(L(\beta, \alpha)).$$

Since any cobordism is a composition of simple ones and F_C is functorial, for any cobordism C, we have the TQFT map

$$Z(C) : V(\Sigma_-) \cong W_q(L_-) \to V(\Sigma_+) \cong W_q(L_+)$$

which equals, up to a coefficient, the inclusion map $W_q(L_-) \to W_q(L_C) \otimes_{\mathbb{C}[\mathcal{H}_p(\Sigma)]} W_q(L_-)$ composed with the isomorphism from Theorem 1. Closing with handlebodies compatible with the Lagrangians, we get that the coefficient is $Z(\check{C})$ which completes the proof. ■

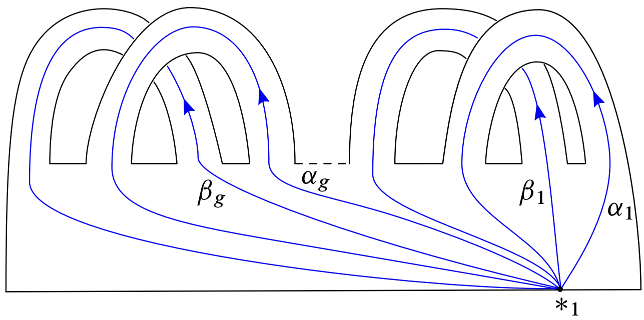

Figure 3. Model for Σ.

4. Schrödinger local systems on surface configurations

In this section, we construct and study two projective representations of $\mathrm{Mod}(\Sigma)$ on Schrödinger representations. Note that the results of this section are independent from the rest of the paper.

4.1. Heisenberg group as a quotient of the surface braid group

Let Σ be an oriented surface of genus g with one boundary component. For $n \geq 2$, the unordered configuration space of n points in Σ is

$$\mathrm{Conf}_n(\Sigma) = \big\{\{c_1, \ldots, c_n\} \subset \Sigma \mid c_i \neq c_j \text{ for } i \neq j\big\}.$$

The surface braid group is then defined as $B_n(\Sigma) = \pi_1(\mathrm{Conf}_n(\Sigma), *)$. To construct a presentation, we fix based loops, $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ on Σ, as depicted in Figure 3. The base point $*_1$ on Σ belongs to the base configuration $*$ in $\mathrm{Conf}_n(\Sigma)$. By abuse of notation, we use α_r, β_s also for the loops in $\mathrm{Conf}_n(\Sigma)$ where only the point at $*_1$ is moving along the corresponding curve. We write composition of loops from right to left. The braid group $B_n(\Sigma)$ has generators $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ together with the classical braid generators $\sigma_1, \ldots, \sigma_{n-1}$, and relations

$$\begin{cases} [\sigma_i, \sigma_j] = 1 & \text{for } |i-j| \geq 2, \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1, \\ [\zeta, \sigma_i] = 1 & \text{for } i > 1 \text{ and all } \zeta \text{ among the } \alpha_r, \beta_s, \\ [\zeta, \sigma_1\zeta\sigma_1] = 1 & \text{for all } \zeta \text{ among the } \alpha_r, \beta_s, \\ [\zeta, \sigma_1^{-1}\eta\sigma_1] = 1 & \text{for all } \zeta \neq \eta \text{ among the } \alpha_r, \beta_s \text{ with } \{\zeta,\eta\} \neq \{\alpha_r, \beta_r\}, \\ \sigma_1\beta_r\sigma_1\alpha_r\sigma_1 = \alpha_r\sigma_1\beta_r & \text{for all } r. \end{cases} \tag{7}$$

We denote by $x.y$ the standard intersection form on $H_1(\Sigma,\mathbb{Z})$. The Heisenberg group $\mathcal{H}(\Sigma)$ is the central extension of the homology group $H_1(\Sigma,\mathbb{Z})$ by the intersection 2-cocycle $(x,y)\mapsto x\cdot y$. As a set, $\mathcal{H}(\Sigma)$ is equal to $\mathbb{Z}\times H_1(\Sigma,\mathbb{Z})$, with the group structure

$$(k,x)(l,y) = (k+l+x\cdot y, x+y). \tag{8}$$

We use the notation a_r, b_s for the homology classes of α_r, β_s, respectively. Let us denote by $[\sigma_1, B_n(\Sigma)]$ the normal subgroup of the surface braid group $B_n(\Sigma)$ generated by the commutators $\{[\sigma_1, x],\ x\in B_n(\Sigma)\}$. From the presentation above, we obtain the following (see [13] for more details).

Proposition 3. *For each $g\geq 0$ and $n\geq 2$, the quotient*

$$B_n(\Sigma)/[\sigma_1, B_n(\Sigma)] \xrightarrow{\sim} \mathcal{H}(\Sigma)$$

is isomorphic to the Heisenberg group. An isomorphism is induced by the surjective homomorphism

$$\phi\colon B_n(\Sigma)\to\mathcal{H}(\Sigma)$$

sending each σ_i to $u=(1,0)$, α_r to $\tilde{a}_r=(0,a_r)$, β_s to $\tilde{b}_s=(0,b_s)$.

It follows that any representation of the Heisenberg group $\mathcal{H}(\Sigma)$ is also a representation of the surface braid group $B_n(\Sigma)=\pi_1(\mathrm{Conf}_n(\Sigma),*)$ and hence provides a local system on the configuration space $\mathrm{Conf}_n(\Sigma)$.

Let us denote by $\mathrm{Aut}^+(\mathcal{H}(\Sigma))$ the group of automorphisms of $\mathcal{H}(\Sigma)$ acting by identity on the center $\langle u\rangle=\mathbb{Z}\times 0$; namely, an element of this group sends (n,x) to $(c(x)+n, l(x))$ for some $l\in \mathrm{Sp}(H_1(\Sigma))$ and $c\in H^1(\Sigma)$. By [13, Lemma 15], we have the following split short exact sequence:

$$1\to H^1(\Sigma,\mathbb{Z})\xrightarrow{j}\mathrm{Aut}^+(\mathcal{H}(\Sigma))\xrightarrow{l}\mathrm{Sp}(H_1(\Sigma))\to 1,$$

where $j(c)=[(k,x)\to(k+c(x),x)]$ and $\mathrm{Sp}(H_1(\Sigma))$ is the symplectic group preserving the intersection pairing. The homomorphism l has a section

$$s: g\mapsto [(k,x)\mapsto (k,g(x))] \tag{9}$$

providing a semi-direct decomposition $\mathrm{Aut}^+(\mathcal{H}(\Sigma))\cong \mathrm{Sp}(H_1(\Sigma))\ltimes H^1(\Sigma;\mathbb{Z})$.

Let us denote by $\mathrm{Mod}(\Sigma)$ the mapping class group. Its action on $H_1(\Sigma,\mathbb{Z})$ preserves the symplectic form, and hence, using the section s from (9), we get a *symplectic action* of the mapping class group on the Heisenberg group, where $f\in\mathrm{Mod}(\Sigma)$ acts by

$$(k,x)\mapsto (k, f_*(x)). \tag{10}$$

On the other hand, the quotient map $\phi: B_n(\Sigma)\to\mathcal{H}(\Sigma)$ induces a different action of $\mathrm{Mod}(\Sigma)$ on $\mathcal{H}(\Sigma)$. The following proposition is proved in [13, Section 3].

Proposition 4. *For $f \in \mathrm{Mod}(\Sigma)$, there exists a unique homomorphism $f_{\mathcal{H}}\colon \mathcal{H}(\Sigma) \to \mathcal{H}(\Sigma)$ such that the following square commutes:*

$$\begin{array}{ccc} B_n(\Sigma) & \xrightarrow{f_{B_n(\Sigma)}} & B_n(\Sigma) \\ \phi \downarrow & & \downarrow \phi \\ \mathcal{H}(\Sigma) & \xrightarrow{f_{\mathcal{H}}} & \mathcal{H}(\Sigma) \end{array} \tag{11}$$

We obtain an action of $\mathrm{Mod}(\Sigma)$ on the Heisenberg group $\mathcal{H}(\Sigma)$ given by

$$\begin{aligned} \mathrm{Mod}(\Sigma) &\to \mathrm{Aut}^+(\mathcal{H}(\Sigma)), \\ f &\mapsto f_{\mathcal{H}} : (k, x) \mapsto (k + \theta_f(x), f_*(x)), \end{aligned} \tag{12}$$

where the map $\theta : \mathrm{Mod}(\Sigma) \to H^1(\Sigma, \mathbb{Z})$ sending f to $\theta_f \in \mathrm{Hom}(H_1(\Sigma), \mathbb{Z})$ satisfies the *crossed homomorphism* property $\theta_{g \circ f}(x) = \theta_f(x) + \theta_g(f_*(x))$. Clearly, both actions coincide on $\mathrm{Sp}(H_1(\Sigma))$, i.e., $l(f_{\mathcal{H}}) = f_*$. However, f_* and $f_{\mathcal{H}}$ are different on $\mathcal{H}(\Sigma)$.

It is shown in [13, Proposition 19] that the crossed homomorphism θ is given by a formula due to Morita [31, Section 6], which is stated as follows. For $\gamma \in \pi_1(\Sigma)$, let us denote by γ_i the element in the free group generated by α_i, β_i that is the image of γ under the homomorphism that maps the other generators to 1. Then, we have a decomposition

$$\gamma_i = \alpha_i^{\nu_1} \beta_i^{\mu_1} \cdots \alpha_i^{\nu_m} \beta_i^{\mu_m},$$

where ν_j and μ_j are 0, -1 or 1. The integer $d_i(\gamma)$ is then defined by

$$\begin{aligned} d_i(\gamma) &= \sum_{j=1}^{m} \nu_j \sum_{k=j}^{m} \mu_k - \sum_{j=1}^{m} \mu_j \sum_{k=j+1}^{m} \nu_k \\ &= \sum_{j=1}^{m} \sum_{k=1}^{m} \iota_{jk} \nu_j \mu_k, \end{aligned}$$

where $\iota_{jk} = +1$ when $j \leq k$ and $\iota_{jk} = -1$ when $j > k$. The formula for the crossed homomorphism is

$$\theta(f)([\gamma]) = \sum_{i=1}^{g} d_i(f_\sharp(\gamma)) - d_i(\gamma).$$

4.2. A variant of Weil representation

Recall that for a Lagrangian subspace L and q a pth root of unity, we defined the Schrödinger representation $W_q(L)$ of the finite quotient $\mathcal{H}_p(\Sigma)$ of the Heisenberg group. The following finite dimensional version of the famous Stone–von Neumann theorem holds.

Theorem 5 (Stone–von Neumann). *For q a root of unity of order p, $W_q(L)$ is the unique irreducible unitary representation of $\mathcal{H}_p(\Sigma)$, up to unitary isomorphism, where the central generator $u = (1, 0)$ acts by q.*

A proof for even p can be found in [22, Theorem 2.4]. The odd case works similarly.

For the rest of this section, we fix an *odd* integer $p \geq 3$. Denote by $\rho : \mathcal{H}_p \to GL(W_q(L))$ the Schrödinger representation. For any automorphism $\tau \in \mathrm{Aut}^+(\mathcal{H}_p)$, we have another representation $\rho \circ \tau : \mathcal{H}_p \to GL(W_q(L))$ called *twisted* representation of $\mathcal{H}_p$ and denoted by ${}_\tau W_q(L)$. From the Stone–von Neumann theorem, we get an isomorphism of representation $W_q(L) \approx {}_\tau W_q(L)$ which is well defined up to a scalar in S^1.

For odd p, the action of $\mathrm{Mod}(\Sigma)$ on the Heisenberg group from Proposition 4 passes to the finite quotient, and we denote by $f_{\mathcal{H}_p}$ the resulting automorphism for every $f \in \mathrm{Mod}(\Sigma)$.

Hence, for a mapping class f, we obtain a unitary isomorphism $\mathcal{S}_{\mathcal{H}}(f) : W_q(L) \xrightarrow{\sim} {}_{f_{\mathcal{H}_p}} W_q(L)$ defined by the following commutative diagram:

$$\begin{array}{ccc} W_q(L) & \xrightarrow{\mathcal{S}_{\mathcal{H}}(f)} & {}_{f_{\mathcal{H}_p}} W_q(L) \\ {\scriptstyle \rho(k,x)} \downarrow & & \downarrow {\scriptstyle \rho(f_{\mathcal{H}_p}(k,x))} \\ W_q(L) & \xrightarrow{\mathcal{S}_{\mathcal{H}}(f)} & {}_{f_{\mathcal{H}_p}} W_q(L) \end{array}$$

Applying the Stone–von Neumann theorem, this provides a homomorphism

$$\mathcal{S}_{\mathcal{H}} : \mathrm{Mod}(\Sigma) \to \mathrm{PU}(W_q), \quad \text{where } \mathrm{PU}(W_q) = \mathrm{U}(W_q(L))/\mathbb{S}^1$$

is the projective unitary group which does not depend on the choice of L.

Denote ${}_{f_*} W_q(L)$ the Schrödinger representation twisted by the symplectic action, we also have an isomorphism $\mathcal{S}(f) : W_q(L) \xrightarrow{\sim} {}_{f_*} W_q(L)$ defined, up to a scalar in $\mathbb{S}^1 \subset \mathbb{C}$, by the condition

$$\rho(k, f_*(x)) \circ \mathcal{S}(f) = \mathcal{S}(f) \circ \rho(k, x) \quad \text{for any } (k, x) \in \mathcal{H}_p(\Sigma). \tag{13}$$

This provides another homomorphism:

$$\mathcal{S} : \mathrm{Mod}(\Sigma) \to \mathrm{PU}(W_q).$$

The next result was proven independently by Gelca with collaborators [22, Theorem 8.1], [21], [19, Chapter 7].

Corollary 6. *The homomorphism $\mathcal{S} : \mathrm{Mod}(\Sigma) \to \mathrm{PU}(W_q)$ given by the symplectic action is isomorphic to the one resulting from the abelian TQFT on* 3Cob *described in Section 2.*

In general, any projective representation of a group G can be linearized on an appropriate central extension. Given an homomorphism $R : G \to PGL(V)$, where V is a complex vector space, a choice of lift (as a set map) $\widetilde{R} : G \to GL(V)$ defines a defect map $c : G \times G \to \mathbb{C}^*$, by $\widetilde{R}(gg') = c(g, g')\widetilde{R}(g)\widetilde{R}(g')$. In the case of a projective unitary representation, the map c takes values in $\mathbb{S}^1$. It is well known from basic group cohomology theory that c is a 2-cocycle defining a central extension of G on which R can be linearized. This central extension is classified by the class $[c] \in H^2(G, \mathbb{C}^*)$. If this class can be reduced to a subgroup, then the linearization already arises on a smaller extension. If $[c] = 0$, the minimal extension is G itself.

Projective actions of $\mathrm{Mod}(\Sigma)$ on the Schrödinger representations are naturally equipped with such cohomology classes, determined by the Stone–von Neumann isomorphisms. As explained in Section 2.4, the homomorphism $\mathcal{S}$ can be linearized and we use the same notation for a linearization $\mathcal{S} : \mathrm{Mod}(\Sigma) \to U(W_q(L))$. We will show that the extension which linearizes the projective representation $\mathcal{S}_{\mathcal{H}}$ is non trivial by computing its classifying 2-cocycle.

A key observation is that, for a mapping class f, the automorphism $f_{\mathcal{H}_p} : \mathcal{H}_p(\Sigma) \to \mathcal{H}_p(\Sigma)$ is equal to the symplectic one composed with an inner automorphism

$$f_{\mathcal{H}_p}(k, x) = (k + \theta_f(x), f_*(x)) = (0, f_*(t_f))(k, f_*(x))(0, -f_*(t_f)),$$

where $2t_f \in H_1(\Sigma, \mathbb{Z}_p)$ is the Poincaré dual of θ_f, i.e., $\theta_f(x) = 2t_f \cdot x$. Here, we use that 2 is invertible modulo p and that $f_*(t_f) \cdot f_*(x) = t_f \cdot x$. Acting on $W_q(L)$, we get the following commutative diagram:

$$\begin{array}{ccccc}
W_q(L) & \xrightarrow{\mathcal{S}(f)} & {}_{f_*}W_q(L) & \xrightarrow{\rho(0,f_*(t_f))} & {}_{f_{\mathcal{H}_p}}W_q(L) \\
\Big\downarrow{\scriptstyle \rho(k,x)} & & \Big\downarrow{\scriptstyle \rho(k,f_*(x))} & & \Big\downarrow{\scriptstyle \rho(f_{\mathcal{H}_p}(k,x))} \\
W_q(L) & \xrightarrow{\mathcal{S}(f)} & {}_{f_*}W_q(L) & \xrightarrow{\rho(0,f_*(t_f))} & {}_{f_{\mathcal{H}_p}}W_q(L)
\end{array}$$

Hence, the two projective actions are related as follows:

$$\mathcal{S}_{\mathcal{H}}(f) = \rho(0, f_*(t_f)) \circ \mathcal{S}(f) = \mathcal{S}(f) \circ \rho(0, t_f).$$

We can now compute the cocycle from the intertwining isomorphism

$$\begin{aligned}
W_q(L) &\cong {}_{(fg)_{\mathcal{H}}}W_q(L) = {}_{g_{\mathcal{H}}}({}_{f_{\mathcal{H}}}W_q(L)), \\
\mathcal{S}_{\mathcal{H}}(f) \circ \mathcal{S}_{\mathcal{H}}(g) &= \mathcal{S}(f) \circ \rho(0, t_f) \circ \mathcal{S}(g) \circ \rho(0, t_g) \\
&= \mathcal{S}(f) \circ \mathcal{S}(g) \circ \rho(0, g_*^{-1}(t_f)) \circ \rho(0, t_g) \\
&= \mathcal{S}(f) \circ \mathcal{S}(g) \circ \rho(g_*^{-1}(t_f) \cdot t_g, g_*^{-1}(t_f) + t_g) \\
&= q^{g_*^{-1}(t_f) \cdot t_g} \mathcal{S}_{\mathcal{H}}(fg).
\end{aligned}$$

Here, we used that the crossed homomorphism property $\theta_{fg} = \theta_g + g^*(\theta_f)$ implies for the Poincaré dual $t_{fg} = t_g + g_*^{-1}(t_f)$. Using $t_{gg^{-1}} = t_{g^{-1}} + g_*(t_g) = 0$, we get that the cocycle is equal to $q^{c(f,g)}$ where $c(f,g) = g_*^{-1}(t_f) \cdot t_g = t_f \cdot g_*(t_g) = -t_f \cdot t_{g^{-1}}$.

Morita studied in [32] the intersection cocycle $(f,g) \mapsto c_{\mathrm{Mor}}(f,g) = t_{f^{-1}} \cdot t_g = c(g,f)$ which represents $12c_1$, where c_1 is the Chern class generating $H^2(\mathrm{Mod}(\Sigma), \mathbb{Z}) = \mathbb{Z}$ for surfaces of genus at least 3. The Meyer cocycle $\tau(f,g)$ is the signature of the oriented 4-dimensional manifold defined as the surface bundle over the pair of pants with monodromy f and g on 2 boundary components. This definition is symmetric in f and g so that we have $\tau(f,g) = \tau(g,f)$. From Morita work we have that $[c_{\mathrm{Mor}}] = 3[\tau] = 12c_1$. By switching the variable, we get $[c] = 3[\tau] = 12c_1$. It follows that for odd p the projective action $\mathcal{S}_{\mathcal{H}} : \mathrm{Mod}(\Sigma) \to PU(W_q)$ cannot be linearized on the mapping class group while the symplectic action does.

By Corollary 6, abelian, and hence, the Schrödinger TQFT, reproduces symplectic action

$$\mathcal{S} : \mathrm{Mod}(\Sigma) \to \mathrm{PU}(W_q)$$

after restricting to mapping classes preserving Lagrangians. We just argued that $\mathcal{S}_{\mathcal{H}}$ is essentially different from $\mathcal{S}$. This leads us to the following problem.

Problem 7. *Construct a TQFT on* $3\mathrm{Cob}^{\mathrm{LC}}$ *using Schrödinger representations that recovers the action* $\mathcal{S}_{\mathcal{H}}$ *on mapping cylinders.*

5. Appendix

For the reader's convenience, we adapt here the Juhász presentation of cobordism categories from [26] to $3\mathrm{Cob}^{\mathrm{LC}}$.

Let us define the category $\mathbf{Surf}^{\mathrm{LC}}$, which is an analog of $\mathbf{Man}_2$ in [26], as follows. An object of $\mathbf{Surf}^{\mathrm{LC}}$ is an oriented compact surface Σ with S^1-parametrized boundary $S^1 \xrightarrow{\simeq} \partial\Sigma$ equipped with a Lagrangian $L \subset H_1(\Sigma;\mathbb{Z})$. A morphism from (Σ_-, L_-) to (Σ_+, L_+) is a diffeomorphism $d : \Sigma_- \to \Sigma_+$ preserving the boundary and Lagrangians so that $d_*(L_-) = L_+$.

A framed sphere in $(\Sigma, L) \in \mathbf{Surf}^{\mathrm{LC}}$ is a smooth orientation-preserving embedding $\mathbb{S} : S^k \times D^{2-k} \hookrightarrow \Sigma$ for $k = 0, 1$ such that $\mathrm{Im}(\mathbb{S}) \cap \partial\Sigma = \emptyset$. By performing surgery on Σ along $\mathbb{S}$, we construct

$$\Sigma(\mathbb{S}) = (\Sigma \backslash \mathrm{Im}(\mathbb{S})) \bigcup_{S^k \times S^{1-k}} D^{k+1} \times S^{1-k}$$

for $k = 0, 1$. The trace of the surgery

$$M(\mathbb{S}) = (\Sigma \times I) \bigcup_{\mathrm{Im}(\mathbb{S}) \times \{1\}} D^{k+1} \times D^{2-k}$$

is a 3-cobordism from Σ to $\Sigma(\mathbb{S})$ obtained by attaching the handle $D^{k+1} \times D^{2-k}$ to $\Sigma \times I$ with the Lagrangian

$$L_{M(\mathbb{S})} := \operatorname{Ker}\big(i_* : H_1(\Sigma) \oplus H_1(\Sigma(\mathbb{S})) \to H_1(M(\mathbb{S}))\big).$$

Let us now define the induced Lagrangian in $H_1(\Sigma(\mathbb{S}); \mathbb{Z})$ as $L_{M(\mathbb{S})} \cdot L$. We conclude that the surgery along $\mathbb{S}$ induces a morphism in $3\mathrm{Cob}^{\mathrm{LC}}$ between (Σ, L) and $(\Sigma(\mathbb{S}), L_{M(\mathbb{S})} \cdot L)$ which we denote by $(M(\mathbb{S}), L_{M(\mathbb{S})})$.

Let $\mathcal{G}^{\mathrm{LC}}$ be the directed graph obtained from the category $\mathbf{Surf}^{\mathrm{LC}}$ by adding an edge $e_{\Sigma,\mathbb{S}}$ from (Σ, L) to $(\Sigma(\mathbb{S}), L_{M(\mathbb{S})} \cdot L)$ for every (Σ, L) and $\mathbb{S}$ in Σ.

Let $\mathcal{F}(\mathcal{G}^{\mathrm{LC}})$ be the free category generated by $\mathcal{G}^{\mathrm{LC}}$.

Definition 8 ([26, Definition 1.4]). The set of relations $\mathcal{R}$ in $\mathcal{F}(\mathcal{G}^{\mathrm{LC}})$ is defined as follows:

(1) $e_d \circ e_{d'} = e_{d \circ d'}$, where $e_d, e_{d'}$ and $e_{d \circ d'}$ are edges corresponding to diffeomorphisms. If d is isotopic to the identity, then $e_d = \mathrm{id}_\Sigma$. Also, $e_{\Sigma,\varnothing} = \mathrm{id}$;

(2) for an orientation-preserving diffeomorphism $d : \Sigma \to \Sigma'$ sending the Lagrangian L to L' and for a framed sphere $\mathbb{S} \subset \Sigma$, let $\mathbb{S}' = d \circ \mathbb{S}$ and $d^{\mathbb{S}}$ be the induced diffeomorphism, then $e_{\Sigma,\mathbb{S}'} \circ e_d = e_{d^{\mathbb{S}}} \circ e_{\Sigma,\mathbb{S}}$;

(3) if $\mathbb{S}$ and $\mathbb{S}'$ are two disjoint framed spheres in Σ, $(\Sigma, L) \in 3\mathrm{Cob}^{\mathrm{LC}}$, then $e_{\Sigma,\mathbb{S}}$ and $e_{\Sigma,\mathbb{S}'}$ commute;

(4) if $\mathbb{S}' \subset \Sigma(\mathbb{S})$ is a framed sphere of index 1, $\mathbb{S}$ is a framed sphere of index 2 and the attaching sphere $a(\mathbb{S}')$ intersects the belt sphere $b(\mathbb{S})$ transversely at one point. Then, $e_{\Sigma(\mathbb{S}),\mathbb{S}'} \circ e_{\Sigma,\mathbb{S}} = e_\varphi$, where $\varphi : \Sigma \to \Sigma(\mathbb{S})(\mathbb{S}')$ is a diffeomorphism which is identical on $\Sigma \cap \Sigma(\mathbb{S})(\mathbb{S}')$ and unique up to isotopy (see details in [26, Definition 2.17]);

(5) $e_{\Sigma,\mathbb{S}} = e_{\Sigma,\overline{\mathbb{S}}}$, where $\overline{\mathbb{S}}$ is the same sphere with the opposite orientation.

Define the functor $P : \mathcal{F}(\mathcal{G}^{\mathrm{LC}}) \to 3\mathrm{Cob}^{\mathrm{LC}}$ by sending vertices to itself, diffeomorphisms to mapping cylinders and the edges $e_{M,\mathbb{S}}$ to cobordisms $M(\mathbb{S})$. The aim of this section is to prove that P descends to a functor on $\mathcal{F}(\mathcal{G}^{\mathrm{LC}})/\mathcal{R}$, which is an isomorphism of categories.

5.1. Morse data and Cerf decompositions

Given two objects (Σ_-, L_-) and (Σ_+, L_+) in $3\mathrm{Cob}^{\mathrm{LC}}$ and $[M] \in 3\mathrm{Cob}^{\mathrm{LC}}((\Sigma_-, L_-), (\Sigma_+, L_+))$, where $L_- \subset H_1(\Sigma_-)$, $L_+ \subset H_1(\Sigma_+)$ and $L_M \subset H_1(\Sigma_-) \oplus H_1(\Sigma_+)$ are the corresponding Lagrangians with $L_+ = L_M.L_-$. Here, we write $[M]$ to emphasize that we refer to an equivalence class of 3-cobordisms. Recall that two 3-cobordisms are equivalent if they are diffeomorphic and the restriction of the diffeomorphism to the boundary respects the parametrizations (compare [26, Definition 2.2]).

Let us choose a representative M of the equivalence class $[M]$. The boundary ∂M is decomposed as

$$\partial M = \partial_-(M) \cup \partial_0(M) \cup \partial_+(M),$$

where $\partial_\pm(M)$ is identified with $\Sigma_\pm$ and $\partial_0(M)$ is parametrized by $I \times S^1$.

Definition 9. A Morse datum for M consists of a pair $(f, \mathbf{b}, v)$ of a Morse function on M, an ordered tuple

$$\mathbf{b} = (0 = b_0 < b_1 < \cdots < b_m = 1) \subset \mathbb{R},$$

and a gradient-like field v for f such that the following statements hold:

- on $I \times S^1$, f coincides with the first coordinate map $(t, x) \mapsto t$ and $v = \frac{\partial}{\partial t}$;
- $\Sigma_- = f^{-1}(b_0)$ and $\Sigma_+ = f^{-1}(b_m)$ are the sets of minima and maxima of f;
- each $f^{-1}(b)$ is connected, and f has no critical points of indices 0 and 3;
- f has different values at critical points;
- all critical points belong to the interior of M;
- $b_1, \ldots, b_{m-1}$ are regular values of f, such that each (b_i, b_{i+1}) contains at most one critical point.

Lemma 2-1 in [15] can be applied here with $Y = \partial_0(M)$ proving that Morse functions with the required boundary conditions exist on any morphism M in $3\mathrm{Cob}^{\mathrm{LC}}$. Using that Morse functions with different values at critical points are generic, in particular, dense in the space of functions the proof given there show existence of a Morse function with the required boundary conditions and distinct values at critical points. We obtain a gradient-like vector field by using a Riemannian metric (cf. Lemma 1-6 in [15]). We first choose a collar of $\partial_0(M)$, $]-1, 0] \times I \times S^1 \subset M$ on which we put the product Riemannian metric g_1. Then, we take any Riemannian metric g_2 on $M \backslash]-\frac{1}{2}, 0] \times I \times S^1$. Using a partition of unity we build a metric g on M which coincides with g_1 on $[-\frac{1}{2}, 0]$ and g_2 on $M \backslash]-1, 0] \times I \times S^1$. Using g, we get a gradient vector field ∇f for the Morse function f and obtain a Morse datum (compare with Lemma 1.7 in [15]).

We call a 3-cobordism M *simple* if it admits a Morse function with at most one critical point. A Cerf decomposition of M is a presentation of M as a composition of simple cobordisms. We will need here a refined notion.

Definition 10. A *parametrized Cerf decomposition* of (M, L_M) consists of the following data:

- a Cerf decomposition

$$(M, L_M) = (M_m, L_{M_m}) \circ (M_{m-1}, L_{M_{m-1}}) \circ \cdots \circ (M_1, L_{M_1}),$$

where

$$(M_i, L_{M_i}) : (\Sigma_{i-1}, L_{i-1}) \to (\Sigma_i, L_i)$$

are simple cobordisms, $L_0 = L_-$ and the other L_i are defined inductively by $L_i := L_{M_i}.L_{i-1}$ with a $L_{M_i} = \mathrm{Ker}(H_1(\Sigma_{i-1}) \oplus H_1(\Sigma_i) \to H_1(M_i))$;

- a framed attaching sphere $\mathbb{S}_i \in \Sigma_i$ and a diffeomorphism $D_i : M(\mathbb{S}_i) \to M_i$ such that $D_i|_{\Sigma_i(\mathbb{S}_i)} : \Sigma_i(\mathbb{S}_i) \to \Sigma_{i+1}$ is an orientation preserving diffeomorphism, well defined up to isotopy, and $D_i|_{\Sigma_{i-1}} = \mathrm{id}$.

Proposition 11. *Each simple cobordism M is either diffeomorphic to a mapping cylinder or to a 3-cobordism $M(\mathbb{S})$ obtained by index 1 or 2 surgery.*

Proof. A simple cobordism M admits a Morse datum (f, v) with at most one critical point of index $k = 1, 2$. In the case with no critical point, the flow defines a trivialization $M \cong I \times \Sigma_-$ which is a mapping cylinder. In the case of a critical point p, the stable and unstable manifolds $M^s(p)$, $M^u(p)$ of the critical point p do not intersect $\partial_0(M)$, then the intersection of $M^s(p)$ with Σ_- defines a $(k-1)$-sphere (k is the index) which can be framed, giving a framed sphere $\mathbb{S}$ and a diffeomorphism $\Sigma_-(\mathbb{S}) \cong M$ well defined up to isotopy [26, Section 2.2]. ■

More generally, for any cobordism M in $3\mathrm{Cob}^{\mathrm{LC}}$, a Morse datum $(f, \mathbf{b}, v)$ induces a parametrized Cerf decomposition. Indeed, simple cobordisms between level sets $\Sigma_i = f^{-1}(b_i) = M_i \cap M_{i+1}$ are $M_i = f^{-1}([b_{i-1}, b_i])$. Furthermore, the flow of v defines the induced attaching sphere on each level set as the intersection with the stable manifold $M^s(p)$ of the corresponding critical point if it exists. If not, this flow defines a diffeomorphism between Σ_i and Σ_{i+1} which is identity on the boundary.

For the reader convenience, let us show that Cerf decompositions preserve Lagrangian correspondences.

Lemma 12. *Assume (Σ_i, L_i) are objects of $3\mathrm{Cob}^{\mathrm{LC}}$ for $0 \le i \le m$. For a Cerf decomposition*

$$(M, L_M) = (M_m, L_{M_m}) \circ (M_{m-1}, L_{M_{m-1}}) \circ \cdots \circ (M_1, L_{M_1}),$$

we have $L_m = L_M \cdot L_0$.

Proof. It suffices to show that for two morphisms $M_1 : (\Sigma_-, L_-) \to (\Sigma, L)$ and $M_2 : (\Sigma, L) \to (\Sigma_+, L_+)$ from $3\mathrm{Cob}^{\mathrm{LC}}$, their composition is also a morphism from $3\mathrm{Cob}^{\mathrm{LC}}$, i.e.,

$$L_{M_2} \cdot (L_{M_1} \cdot L_-) = L_{M_2 \circ M_1} \cdot L_-. \tag{14}$$

Since both sides of (14) are Lagrangians, it is enough to show that any $x \in L_{M_2} \cdot (L_{M_1} \cdot L_-)$ also belongs to $L_{M_2 \circ M_1} \cdot L_-$. By the definition of Lagrangian correspondence $\exists y \in H_1(\Sigma)$ and $\exists z \in L_{\Sigma_-}$, such that $(x, y) \in L_{M_2}$ and $(y, z) \in L_{M_1}$. Hence,

$(x, z) \in L_{M_2 \circ M_1}$, since the image of (x, z) in $H_1(M_2 \circ M_1)$ can be rewritten as $x + z = y - y = 0$. ■

Definition 13. A *diffeomorphism equivalence* of parametrized Cerf decompositions

$$(M_m, L_{M_m}) \circ \cdots \circ (M_1, L_{M_1}) \quad \text{and} \quad (M'_m, L_{M'_m}) \circ \cdots \circ (M'_1, L_{M'_1})$$

of the same length is a collection of diffeomorphisms $\phi_i : M_i \to M'_i$ such that

- they are identities on the ends: $\phi_1|_{\Sigma_-} = \mathrm{id}_{\Sigma_-}$ and $\phi_m|_{\Sigma_+} = \mathrm{id}_{\Sigma_+}$;
- they are compatible: $\phi_i|_{\Sigma_i} = \phi_{i+1}|_{\Sigma_i}$;
- they preserve Lagrangians: $(\phi_i|_{\Sigma_i})_*(L_i) = L'_i$.

As explained in [26] (before Theorem 2.24) diffeomorphism equivalences are induced by left-right equivalence of Morse data. Let us recall here the definition of the latter for completeness. Two Morse data $(f, \mathbf{b}, v)$ and $(f', \mathbf{b}', v')$ are related by a *left-right equivalence* if there are diffeomorphisms $\Phi : M \to M$ and $\phi : \mathbb{R} \to \mathbb{R}$, such that $f' = \phi \circ f \circ \Phi^{-1}$, $\mathbf{b}' = \phi(\mathbf{b})$, $v' = \Phi_*(v)$, $\Phi|_{\Sigma_\pm}$ are isotopic to id. Then, for a given Cerf decomposition, we can obtain a diffeomorphism equivalent one by setting $M'_i := \Phi(M_i)$.

Theorem 14. *Any two parametrized Cerf decompositions of (M, L_M) are connected by sequence of the following moves (and their inverse):*

1. *critical point cancellation: compositions of two simple morphisms (M_i, L_{M_i}) and $(M_{i+1}, L_{M_{i+1}})$ of indices 1 and 2, whose belt and attaching spheres intersect transversally in one point, are replaced by the cylindrical morphism $(\Sigma_{i-1} \times [0, 1], L_{\Sigma_{i-1}\times[0,1]})$;*
2. *critical point crossing: two simple morphisms (M_i, L_{M_i}) and $(M_{i+1}, L_{M_{i+1}})$ of indices l and k, whose belt and attaching spheres do not intersect (for some choice of metric), are replaced by the pair of morphisms $(M'_i, L_{M'_i})$ and $(M'_{i+1}, L_{M'_{i+1}})$ of indices k and l, such that $(M_{i+1} \circ M_i, L_{M_{i+1}\circ M_i}) \simeq (M'_{i+1} \circ M'_i, L_{M'_{i+1}\circ M'_i})$;*
3. *cylinder gluing: two simple cobordisms (M_i, L_{M_i}) and $(M_{i+1}, L_{M_{i+1}})$, one of which is a mapping cylinder, are replaced by the composition $(M_{i+1} \circ M_i, L_{M_{i+1}\circ M_i})$;*
4. *diffeomorphism equivalences.*

Proof. For each cobordism M in $3\mathrm{Cob}^{LC}$, we construct an associated cobordism $\widetilde{M}$ between closed surfaces by gluing a cylinder $D^2 \times I$ along the parameterised boundary $\partial_0(M)$. Then, we extend both Cerf decompositions to it and apply [26, Theorem 1.7] within its usual setting. In order to restrict back to surfaces with boundaries, we have to make sure that all the moves can be performed in such way that it does not affect the glued cylinder.

(1,2) The critical point cancellation and critical point crossing moves can be obtained by an isotopy supported in a tubular neighborhood of stable and unstable manifolds [17]. Since all belt and attaching spheres belong to the interior of M, we can always choose this neighborhood to be inside $\mathrm{Int}(M)$. Thus, this isotopy can be restricted to M.

(3) Let $\widetilde{M}_i$ be a mapping cylinder associated with a diffeomorphism $d : \widetilde{\Sigma}_{i-1} \to \widetilde{\Sigma}_i$, then the composition of this cobordism with $\widetilde{M}(\mathbb{S})$ is diffeomorphic to $\widetilde{M}(\mathbb{S}') : \widetilde{\Sigma}_i \to \widetilde{\Sigma}_i(\mathbb{S}')$, where $\mathbb{S}' = d \circ \mathbb{S}$. Since diffeomorphism d is identical outside Σ_i in $\widetilde{\Sigma}_i$, the induced diffeomorphism of cobordisms is identical on $D^2 \times I$ and can be restricted.

(4,5) Moves 4 and 5 are induced by isotopies which are identical on $D^2 \times I$, hence, can be restricted to M.

Since each pair of Cerf decompositions is connected by a move relating diffeomorphic cobordisms, it preserves Lagrangian correspondences. ∎

Theorem 15. *The functor $P : \mathcal{F}(\mathcal{G}^{\mathrm{LC}}) \to 3\mathrm{Cob}^{\mathrm{LC}}$ descends to a functor $\mathcal{F}(\mathcal{G}^{\mathrm{LC}})/\mathcal{R} \xrightarrow{\simeq} 3\mathrm{Cob}^{\mathrm{LC}}$ which is an isomorphism of categories.*

Proof. First, we should check that relations $\mathcal{R}$ hold in $3\mathrm{Cob}^{\mathrm{LC}}$. Relation (1) in $\mathcal{R}$ can be realized by moves (3) and (4) in Theorem 14. Relation (2) follows from move (4). Relation (3) follows from the critical point crossing move. Relation (4) also follows from the critical point cancellation move. The last relation follows from the fact that $M(\mathbb{S}) = M(\overline{\mathbb{S}})$.

Now, we prove that P is a bijection on the hom-sets. If we have a morphism $(M, L) \in 3\mathrm{Cob}^{\mathrm{LC}}$, then it admits a Morse datum, and hence, as discussed above, the induced parametrized Cerf decomposition

$$(M, L) = (M_m, L_{M_m}) \circ (M_{m-1}, L_{M_{m-1}}) \circ \cdots \circ (M_1, L_{M_1})$$

with M_i in $3\mathrm{Cob}^{\mathrm{LC}}$ for $1 \le i \le m$ by Lemma 12. Then, assigning to each simple morphism (M_i, L_{M_i}) an edge in $\mathcal{G}^{\mathrm{LC}}$ ($e_{\Sigma,\mathbb{S}}$ to surgeries and e_d to mapping cylinders), we obtain a preimage of (M, L) in $\mathcal{G}^{\mathrm{LC}}$. This proves that P is surjective onto morphisms of $3\mathrm{Cob}^{\mathrm{LC}}$.

Assume we have two morphisms f, f' in $\mathcal{F}(\mathcal{G})/\mathcal{R}$ which give the same morphism in $3\mathrm{Cob}^{\mathrm{LC}}$. Each of them can be represented by some composition of edges in $\mathcal{G}$, inducing natural Cerf decompositions on $P(f)$ and $P(f')$. Then, these decompositions are connected by a sequence of moves listed in Theorem 14. Since each move corresponds to a relation in $\mathcal{R}$ the morphisms f and f' belong to the same equivalence class in $\mathcal{F}(\mathcal{G}^{\mathrm{LC}})/\mathcal{R}$. This proves that P is injective. ∎

As a corollary, we have the following construction of TQFTs on $3\mathrm{Cob}^{\mathrm{LC}}$.

Corollary 16. *Assume a map* $F : \mathcal{G}^{\mathrm{LC}} \to \mathrm{Vect}_{\mathbb{C}}$ *(sending vertices to vector spaces and arrows to linear maps) satisfies the following properties:*

(1) $F(e_{\Sigma,\varnothing}) = \mathrm{id}$ *and if* d *is isotopic to the identity, then* $F(d) = \mathrm{id}$*,*

(2) *for an orientation preserving diffeomorphism* $d : \Sigma \to \Sigma'$ *sending the Lagrangian* L *to* L' *and a framed sphere* $\mathbb{S} \subset \Sigma$*, let* $\mathbb{S}' = d \circ \mathbb{S}$ *and* $d^{\mathbb{S}}$ *be the induced diffeomorphism, then* $F(e_{\Sigma,\mathbb{S}'}) \circ F(e_d) = F(e_{d^{\mathbb{S}}}) \circ F(e_{\Sigma,\mathbb{S}})$*,*

(3) *if* $\mathbb{S}$ *and* $\mathbb{S}'$ *are two disjoint framed spheres in* Σ*,* $(\Sigma, L) \in 3\mathrm{Cob}^{\mathrm{LC}}$*, then* $F(e_{\Sigma,\mathbb{S}})$ *and* $F(e_{\Sigma,\mathbb{S}'})$ *commute,*

(4) *if* $\mathbb{S}' \subset \Sigma(\mathbb{S})$ *is a framed sphere of index 1,* $\mathbb{S}$ *is a framed sphere of index 0 and the attaching sphere* $a(\mathbb{S}')$ *intersects the belt sphere* $b(\mathbb{S})$ *transversely in one point. Then, there is a diffeomorphism* $\varphi : \Sigma \to \Sigma(\mathbb{S})(\mathbb{S}')$ *for which* $F(e_{\Sigma(\mathbb{S}),\mathbb{S}'}) \circ F(e_{\Sigma,\mathbb{S}}) = F(e_\varphi)$*,*

(5) $F(e_{\Sigma,\mathbb{S}}) = F(e_{\Sigma,\overline{\mathbb{S}}})$*, where* $\overline{\mathbb{S}}$ *is the same sphere with the opposite orientation.*

Then, F *descends to the unique functor* $F : 3\mathrm{Cob}^{\mathrm{LC}} \to \mathrm{Vect}_{\mathbb{C}}$.

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