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Upper bounds for the relaxed area of $\mathbb{S}^1$-valued Sobolev maps and its countably subadditive interior envelope

Giovanni Bellettini, Riccardo Scala and Giuseppe Scianna

Abstract. Given a connected bounded open Lipschitz set $\Omega \subset \mathbb{R}^2$, we show that the relaxed Cartesian area functional $\overline{\mathcal{A}}(u, \Omega)$ of a map $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ is finite, and we provide a useful upper bound for its value. Using this estimate, we prove a modified version of a De Giorgi conjecture adapted to $W^{1,1}(\Omega; \mathbb{S}^1)$, on the largest countably subadditive set function $\overline{\overline{\mathcal{A}}}(u, \cdot)$ smaller than or equal to $\overline{\mathcal{A}}(u, \cdot)$.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. For a given $v \in C^1(\Omega; \mathbb{R}^2)$, we indicate by

$$\mathcal{A}(v, \Omega) := \int_\Omega \sqrt{1 + |\nabla v|^2 + |Jv|^2}\, dx$$

the classical 2-dimensional area of the graph $G_v = \{(x, y) \in \Omega \times \mathbb{R}^2 : y = v(x)\}$ of v, where

$$Jv = \frac{\partial v_1}{\partial x_1}\frac{\partial v_2}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\frac{\partial v_1}{\partial x_2}$$

denotes the Jacobian determinant of v. For any $u \in L^1(\Omega; \mathbb{R}^2)$, we consider the L^1-relaxed area of the graph of u, namely

$$\overline{\mathcal{A}}(u, \Omega) := \inf\Big\{\liminf_{k \to +\infty} \mathcal{A}(v_k, \Omega),\ v_k \in C^1(\Omega; \mathbb{R}^2),\ v_k \to u \text{ in } L^1(\Omega; \mathbb{R}^2)\Big\}. \tag{1.1}$$

It is well known that, when v is scalar valued, the study of the relaxed area is crucial in the analysis of the Cartesian Plateau problem [21]. In higher codimension, the characterization of the domain $\mathrm{Dom}(\overline{\mathcal{A}}(\cdot, \Omega))$ of $\overline{\mathcal{A}}(\cdot, \Omega)$, and the computation of its corresponding values, seem at the moment out of reach, due to the presence of highly nonlocal phenomena. More specifically, for a given $u \in L^1(\Omega; \mathbb{R}^2)$, the set function $\Omega \supseteq A \to \overline{\mathcal{A}}(u, A)$ turns out to be not subadditive when restricted to open sets (This is true for general maps, apart

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from some specific cases which trivialize the functional, see [1] for details.) In particular, $\overline{\mathcal{A}}(u,\cdot)$ is not a measure, and thus it cannot be represented in integral form; for this reason, only a few partial results are available (see, e.g., [5,9,10]). In these references, it is shown that nonlocality is due to at least two reasons: one is the presence of singularities in the map u; the other one is the possible interaction of such singularities with $\partial\Omega$. In both cases, it appears that, in general, interesting and rather involved Plateau-type problems must be solved, in order to get the exact value of $\overline{\mathcal{A}}(u,\Omega)$ (see the discussion below on the maps u_V and u_T; see also [7]). So, the computation of $\overline{\mathcal{A}}(u,\Omega)$ is, in general, quite difficult; on the other hand, looking for upper bounds that do not take into account the above mentioned Plateau problems in full generality, seems realistic[1].

In this paper, we are concerned with maps in

$$W^{1,1}(\Omega;\mathbb{S}^1) := \{u \in W^{1,1}(\Omega;\mathbb{R}^2) : |u| = 1 \text{ a.e. in } \Omega\},$$

where $\mathbb{S}^1 = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$. Given a distribution $\Lambda \in \mathcal{D}'(\Omega)$, let us introduce the quantity

$$\|\Lambda\|_{\text{flat},\alpha} := \sup\left\{\langle\Lambda,\varphi\rangle : \varphi \in \text{Lip}_0(\Omega),\, \|\varphi\|_{L^\infty(\Omega)} \le 1,\ \alpha\|\nabla\varphi\|_{L^\infty(\Omega)} \le 1\right\}, \tag{1.2}$$

where

$$\alpha := \frac{|B_1|}{|\partial B_1|} = \frac{1}{2} \tag{1.3}$$

and $\text{Lip}_0(\Omega)$ are the Lipschitz functions on Ω vanishing on $\partial\Omega$. Our first result (Section 6) reads as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a connected bounded open set with Lipschitz boundary, and let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$. Then*

$$\overline{\mathcal{A}}(u,\Omega) \le \int_\Omega \sqrt{1+|\nabla u|^2}\,dx + \|\text{Det}(\nabla u)\|_{\text{flat},\alpha} < +\infty. \tag{1.4}$$

In particular,

$$W^{1,1}(\Omega;\mathbb{S}^1) \subset \text{Dom}(\overline{\mathcal{A}}(\cdot,\Omega)).$$

Estimate (1.4) in general is not sharp. Indeed, consider the map $u_V(x) := x/|x|$ defined on the open pointed disc $B_r(0)\setminus\{0\}$ of radius $r>0$, whose distributional Jacobian determinant is $\text{Det}(\nabla u_V) = \pi\delta_0$. Theorem 1.1 implies that

$$\overline{\mathcal{A}}(u_V, B_r(0)) \le \int_{B_r(0)} \sqrt{1+|\nabla u|^2}\,dx + \min\{2\pi r, \pi\}. \tag{1.5}$$

On the other hand, according to Theorem 1.1 in [6], one has

$$\overline{\mathcal{A}}(u_V, B_r(0)) = \int_{B_r(0)} \sqrt{1+|\nabla u|^2}\,dx + F(r),$$

[1] Notice that, if one replaces in (1.1) the L^1 convergence with stronger topologies, some sharp estimates can be given (see for instance [3,16,23], where the strict convergence in BV has been investigated).

where the singular contribution $F(r) \in (0, \pi]$ has the meaning of the area of a minimal surface solving a suitable non-parametric Plateau problem with partial free boundary. Specifically, $F(r)$ coincides with half of the area of a sort of catenoid $S \subset \mathbb{R}^3 = \mathbb{R}^2_{\text{target}} \times \mathbb{R}$ with boundary $(\mathbb{S}^1 \times \{0\}) \cup (\mathbb{S}^1 \times \{2r\})$ and constrained to contain the segment $\{0\} \times [0, 2r]$. In particular, it can be seen that there exists a number $\bar{r} \in (0, 1/2)$ such that for $r \geq \bar{r}$ this catenoid reduces to two discs, and in this case $F(r) = \pi$, whereas for $r \in (0, \bar{r})$ there exists a non-trivial catenoid whose area is strictly smaller than the lateral area of the solid portion of the (smallest) cylinder containing it, namely $F(r) < 2\pi r$. This shows that for $r \geq \bar{r}$ estimate in (1.5) is an equality, and that for $r < \bar{r}$ is not sharp. We emphasize that a more precise estimate than (1.4), and hopefully the sharp value of the left-hand side, seems quite difficult to obtain. On the one hand we expect that, when the singularities of a map u are far from each other, (1.4) becomes sharp[2]. However, in the opposite case, a characterization as in (1.5) needs some strong improvements of the techniques used in [6]. Indeed, in [6] the rotational invariance of the domain and of the map u_V itself are strongly exploited to prove the lower bound, which is based on a cylindrical Steiner-type symmetrization for integral currents. A similar technique has been employed in [25] (see also [8]), yielding the value of $\overline{\mathcal{A}}(u_T, B_r(0))$, where u_T is the symmetric triple junction map, a piecewise constant map taking three values in $\mathbb{S}^1$, each value on a 120° sector. Also, in that case the symmetries of the source and the target spaces allow to use such symmetrization techniques. Without these symmetries, at the moment little can be said about the exact expression of $\overline{\mathcal{A}}(\cdot, \Omega)$.

So, the nonlocality of the L^1-relaxed functional seems not removable. Thus, following De Giorgi [18], it seems interesting to consider a further "relaxation", this time looking at the functional $\overline{\mathcal{A}}(u, \cdot)$, i.e., looking at it as a function of the open set: for every $V \subseteq \Omega$, we set

$$\overline{\overline{\mathcal{A}}}(u, V) := \inf\left\{ \sum_{k=1}^{\infty} \overline{\mathcal{A}}(u, A_k) : A_k \subseteq \Omega \text{ open}, \bigcup_{k=1}^{\infty} A_k \supseteq V \right\}. \tag{1.6}$$

Actually, notice that, for all $u \in L^1(\Omega; \mathbb{R}^2)$ with $\overline{\mathcal{A}}(u, \Omega) < +\infty$, $\overline{\overline{\mathcal{A}}}(u, \cdot)$ is the trace of a regular Borel measure restricted to open sets (in Ω).

The estimate provided by Theorem 1.1 allows us to prove (Proposition 7.1) that

$$\overline{\overline{\mathcal{A}}}(u, A) = \int_A \sqrt{1 + |\nabla u|^2}\, dx, \quad \forall u \in W^{1,1}(\Omega; \mathbb{S}^1), \text{ for every open set } A \subseteq \Omega.$$

Using this, we are able to show our next main result (Corollary 7.6):

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a connected bounded open set with Lipschitz boundary, and let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. Then*

$$\overline{\overline{\mathcal{A}}}(u, \Omega) = \inf\left\{ \overline{\mathcal{A}}(u, \Omega \setminus C) : C \subset \Omega, \mathcal{H}^0(C) < +\infty \right\}.$$

This theorem positively answers to an adaptation of a De Giorgi conjecture, see Conjecture 3 in [18], provided one restricts the analysis to the space $W^{1,1}(\Omega; \mathbb{S}^1)$.

[2] For instance, under the further assumption that the 1-current $S_{\min}$ given by Lemma 3.6 below vanishes.

Before concluding the introduction, it is worth recalling that in several works (see [12, 14, 15] and references therein; a general survey can also be found in [13]), the authors studied the analogue of our relaxation problem, with the area functional replaced by the total variation, for $W^{1,1}$ maps defined on a closed simply connected surface taking values in $\mathbb{S}^1$. They were able to characterize the corresponding relaxed functional, and showed that the singular contribution is given by

$$L(\Lambda) := \sup_{\varphi \in \mathrm{Lip}_0(\Omega),\, \mathrm{lip}(\varphi)\leq 1} \langle \Lambda, \varphi \rangle = \inf\{|S|_\Omega : S \in \mathcal{D}_1(\Omega), T = \partial S\},$$

which has the geometric meaning of the (geodesic) length of a minimal connection between the poles of Λ. The case considered in the present paper seems much more involved, due to the presence of the minimal surfaces briefly discussed above.

The plan of the paper is the following. In Section 2, we fix the setting and notation needed in the sequel. In Section 3, we investigate the minimization problem dual to (1.2) (see (2.10) and (2.11) below), and we prove some regularity result for the minimizing currents (see also Remark A.3 in Appendix A). In Section 4, we collect some results on the distributional Jacobian for Sobolev maps taking values in the circle. We briefly add some details to extend the well-known results for simply connected domains [14,15] to the case of non-simply connected domains, for the reader convenience. Notice that many of these results were stated in the aforementioned references and also summarized in [13]. In Section 5, we prove a density result for circle valued Sobolev maps, see Proposition 5.1, which needs some preparatory lemmata. Finally, in Section 6, we prove Theorem 1.1, whereas in the last section we investigate the countably subadditive interior envelope of the relaxed area functional and we prove Theorem 1.2.

To conclude, we mention that it would be interesting to extend Theorem 1.1 to maps $u \in \mathrm{BV}(\Omega; \mathbb{S}^1)$. We leave this effort for future investigations; we only mention that in [4], some estimates are given for specific piecewise constant maps.

2. Notation and preliminaries

In what follows, $\Omega \subset \mathbb{R}^2$ is a fixed connected (but not necessarily simply connected) bounded open set with Lipschitz boundary. We denote by $d(\cdot, \partial\Omega)$ the distance from $\partial\Omega$, and following [14], p. 96, we denote by $d_\Omega : \overline{\Omega} \times \overline{\Omega} \to [0, +\infty)$ the function

$$d_\Omega(x, y) := \min\big\{|x - y|, d(x, \partial\Omega) + d(y, \partial\Omega)\big\}.$$

Hence, if $d_\Omega(x, y) = |x - y|$, then the closed segment $\overline{xy}$ joining x and y is contained in $\overline{\Omega}$.

Given a vector $V = (V_1, V_2) \in \mathbb{R}^2$, we set $V^\perp := (-V_2, V_1)$ its $\pi/2$-counterclockwise rotation. If $V = \nabla u$, then $\nabla^\perp u$ stands for $(\nabla u)^\perp = (-\partial u/\partial x_2, \partial u/\partial x_1)$. The distributional divergence of a vector field $V = (V_1, V_2) \in L^1(\Omega; \mathbb{R}^2)$ is the distribution

$$\langle \mathrm{Div}\, V, \varphi\rangle := -\int_\Omega V \cdot \nabla\varphi \, dx, \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega). \tag{2.1}$$

If the vector field V is sufficiently smooth, $\mathrm{Div}\, V$ equals the pointwise divergence $\mathrm{div}\, V = \partial V_1/\partial x_1 + \partial V_2/\partial x_2$.

The distributional curl of $V \in L^1(\Omega;\mathbb{R}^2)$ is the distribution

$$\langle \mathrm{Curl}\, V, \varphi\rangle := \int_\Omega V \cdot \nabla^\perp \varphi \, dx, \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega). \tag{2.2}$$

If V is sufficiently smooth, then $\mathrm{Curl}\, V = \mathrm{div}(V^\perp) = -\partial V_2/\partial x_1 + \partial V_1/\partial x_2$.

The symbol $\mathrm{BV}(A)$ (respectively, $\mathrm{SBV}(A)$) denotes the space of functions of bounded variation (respectively, special functions of bounded variation) in the open set $A \subseteq \mathbb{R}^2$; if $u \in \mathrm{BV}(A)$, ∇u stands for the absolutely continuous part of the gradient measure Du. Further, $[\![u]\!]$ stands for the difference $u^+ - u^-$ of the two traces of u on its jump set J_u, provided a unit normal vector field to J_u is assigned. We denote by $\mathrm{BV}(A;\mathbb{R}^2)$ the space of functions of bounded variation in A taking values in $\mathbb{R}^2$; if $u \in \mathrm{BV}(A;\mathbb{R}^2)$, $|\nabla u|$ stands for the Frobenius norm of ∇u; see [2].

Definition 2.1 (Dipole map). Let $p, n \in \mathbb{R}^2$ be distinct, and consider two polar coordinate systems (ρ_p, θ_p) and (ρ_n, θ_n) centered at p and n, respectively, chosen[3] so that both θ_p and θ_n have a jump of size 2π on $\ell_n \subset \ell$, where ℓ is the line containing $\overline{pn}$, and ℓ_n is the halfline with endpoint n and not containing p. We let $w_{p,n} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^2)$ be the dipole map, defined as

$$w_{p,n} := \theta_p - \theta_n. \tag{2.3}$$

Notice that $w_{p,n}$ does not jump on ℓ_n, while it jumps (of size 2π) on the relative interior of $\overline{pn}$. Notice also that there exists a constant $C > 0$ such that

$$|\nabla w_{p,n}(x)| \le |\nabla \theta_p(x)| + |\nabla \theta_n(x)| \le C\Big(\frac{1}{|x-p|} + \frac{1}{|x-n|}\Big), \quad \forall x \in \mathbb{R}^2 \setminus \ell. \tag{2.4}$$

For any open set $A \subset \mathbb{R}^2$, $\mathcal{D}_0(A)$ and $\mathcal{D}_1(A)$ denote the 0-dimensional and 1-dimensional currents in A, respectively. The symbol $|\Lambda|_A$ stands for the mass of a current Λ in A, while $\mathrm{supp}(\Lambda)$ denotes the support of Λ [19].

2.1. Lipschitz maps; the flat norm

For any bounded open set $A \subset \mathbb{R}^2$, we let $\mathrm{Lip}_0(A)$ be the space of Lipschitz functions on A vanishing on ∂A, endowed with the norm

$$\|\varphi\|_{\mathrm{Lip}_0(A)} := \max\big\{\|\varphi\|_{L^\infty(A)}, \mathrm{lip}(\varphi, A)\big\}, \tag{2.5}$$

where

$$\mathrm{lip}(\varphi, A) := \sup_{\substack{x,y\in A\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|}.$$

In the Banach space $\mathrm{Lip}_0(A)$, the norms $\mathrm{lip}(\cdot, A)$ and $\|\cdot\|_{\mathrm{Lip}_0(A)}$ are equivalent.

In what follows, it is also convenient to introduce the equivalent norm

$$\|\varphi\|_{\mathrm{Lip}_0(A),\alpha} = \max\big\{\|\varphi\|_{L^\infty(A)}, \alpha\, \mathrm{lip}(\varphi, A)\big\}, \quad \forall \varphi \in \mathrm{Lip}_0(A), \tag{2.6}$$

with α as in (1.3). For all these norms, we drop in the sequel the symbol A when $A = \Omega$.

[3] The orientation of these systems is always counterclockwise.

We denote by $\mathrm{Lip}_0(A)'$ the dual space of $\mathrm{Lip}_0(A)$ (endowed with one of these norms). The (equivalent to each other) dual norms to (2.5) and (2.6) on $\mathrm{Lip}_0(A)'$ are, respectively,

$$\|\Lambda\|_{\mathrm{flat},A} := \sup_{\substack{\varphi\in\mathrm{Lip}_0(A)\\ \|\varphi\|_{\mathrm{Lip}_0(A)}\le 1}} \langle\Lambda,\varphi\rangle \quad \text{and} \quad \|\Lambda\|_{\mathrm{flat},\alpha,A} := \sup_{\substack{\varphi\in\mathrm{Lip}_0(A)\\ \|\varphi\|_{\mathrm{Lip}_0(A),\alpha}\le 1}} \langle\Lambda,\varphi\rangle, \tag{2.7}$$

for all $\Lambda \in \mathrm{Lip}_0(A)'$, see (1.2). Again, for these dual norms we usually drop the symbol A when $A = \Omega$. The reason of the notation $\|\cdot\|_{\mathrm{flat}}$ is explained by formula (2.10) below.

2.2. The classes $X(\Omega)$ and $X_f(\Omega)$

Let $((x_i, y_i))_{i\in\mathbb{N}} \subset \overline{\Omega}\times\overline{\Omega}$ be a sequence of pairs of points of $\overline{\Omega}$ for which $\sum_{i=1}^{\infty} d_\Omega(x_i, y_i) < +\infty$. We shall always suppose that $x_i \neq y_i$, while we do not exclude that $x_i = x_j$ and/or $y_h = y_k$ for some $i \neq j$, $h \neq k$. Namely, $((x_i, y_i))_{i\in\mathbb{N}} \subset \overline{\Omega}\times\overline{\Omega} \setminus \mathrm{Diag}_\Omega$, where Diag_Ω is the diagonal of $\overline{\Omega}\times\overline{\Omega}$.

The measures

$$\Lambda_n := \sum_{i=1}^{n} (\delta_{x_i} - \delta_{y_i}), \quad n \in \mathbb{N},$$

converge in $\mathrm{Lip}_0(\Omega)'$ to $\sum_{i=1}^{\infty}(\delta_{x_i} - \delta_{y_i})$. Indeed, for any $\varphi \in \mathrm{Lip}_0(\Omega)$ with $\|\varphi\|_{\mathrm{Lip}_0} \le 1$ and any $n \in \mathbb{N}$, setting

$$\begin{aligned} I_n &:= \{i \ge n+1 : d_\Omega(x_i, y_i) = |x_i - y_i|\},\\ B_n &:= \{i \ge n+1 : d_\Omega(x_i, y_i) = d(x_i, \partial\Omega) + d(y_i, \partial\Omega)\}, \end{aligned}$$

we have

$$\begin{aligned} \Big|\Big\langle \sum_{i=n+1}^{\infty} \delta_{x_i} - \delta_{y_i}, \varphi\Big\rangle\Big| &= \Big|\sum_{i=n+1}^{\infty} (\varphi(x_i) - \varphi(y_i))\Big| \\ &\le \Big|\sum_{i\in I_n}(\varphi(x_i) - \varphi(y_i))\Big| + \sum_{i\in B_n}(|\varphi(x_i)| + |\varphi(y_i)|) \\ &\le \sum_{i\in I_n}|x_i - y_i| + \sum_{i\in B_n}(d(x_i,\partial\Omega) + d(y_i,\partial\Omega)) = \sum_{i=n+1}^{\infty} d_\Omega(x_i, y_i) \to 0 \end{aligned}$$

as $n \to +\infty$, where in the last inequality we have used $\|\nabla\varphi\|_\infty \le 1$ and $\varphi = 0$ on $\partial\Omega$.

Remark 2.2 (On the non-uniqueness of the representation). The representation $\Lambda = \sum_{i=1}^{\infty}(\delta_{x_i} - \delta_{y_i})$ is not unique, since two sequences $((x_i, y_i))_{i\in\mathbb{N}} \subset \overline{\Omega}\times\overline{\Omega}\setminus\mathrm{Diag}_\Omega$ and $((\widehat{x}_i, \widehat{y}_i))_{i\in\mathbb{N}} \subset \overline{\Omega}\times\overline{\Omega}\setminus\mathrm{Diag}_\Omega$ with $\sum_{i\in\mathbb{N}} d_\Omega(x_i, y_i) < +\infty$, $\sum_{i\in\mathbb{N}} d_\Omega(\widehat{x}_i, \widehat{y}_i) < +\infty$, define the same linear functional on $\mathrm{Lip}_0(\Omega)$ if

$$\Big\langle \sum_{i=1}^{\infty}(\delta_{x_i} - \delta_{y_i}), \varphi\Big\rangle = \Big\langle \sum_{i=1}^{\infty}(\delta_{\widehat{x}_i} - \delta_{\widehat{y}_i}), \varphi\Big\rangle, \quad \forall\varphi \in \mathrm{Lip}_0(\Omega).$$

We emphasize that the hypothesis that $((x_i, y_i))_{i\in\mathbb{N}} \subset \overline{\Omega}\times\overline{\Omega}\setminus\mathrm{Diag}_\Omega$ (instead that $((x_i, y_i))_{i\in\mathbb{N}} \subset \Omega\times\Omega\setminus\mathrm{Diag}_\Omega$) is done for convenience, and it may happen that for some $i \in \mathbb{N}$, either $x_i \in \partial\Omega$ or $y_i \in \partial\Omega$ (or both). Of course, if $x_i \in \partial\Omega$, then $\delta_{x_i} = 0$ in $\mathrm{Lip}_0(\Omega)'$; the presence of x_i affects the representation of Λ, but not its action on $\mathrm{Lip}_0(\Omega)$. Nevertheless, we can always assume that for all $i \in \mathbb{N}$, at least one among x_i and y_i belongs to Ω. To indicate such a property, we briefly write

$$(x_i, y_i) \in \overline{\Omega} \times \overline{\Omega} \setminus \mathrm{Diag}^\star_\Omega.$$

Preferred representations will be discussed at the beginning of Section 3.

Definition 2.3. We set[4]

$$\begin{aligned} X(\Omega) :=\Big\{\Lambda \in \mathrm{Lip}_0(\Omega)' : \exists((x_i, y_i))_{i\in\mathbb{N}} \subset \overline{\Omega} \times \overline{\Omega} \setminus \mathrm{Diag}^\star_\Omega, \\ \sum_{i=1}^{\infty} d_\Omega(x_i, y_i) < +\infty, \Lambda = \sum_{i=1}^{\infty}(\delta_{x_i} - \delta_{y_i})\Big\} \cup \{0\}. \end{aligned} \tag{2.8}$$

We have seen that

$$\forall \Lambda \in X(\Omega), \quad \langle \Lambda, \varphi\rangle = \sum_{i=1}^{\infty}(\varphi(x_i) - \varphi(y_i)), \quad \forall \varphi \in \mathrm{Lip}_0(\Omega), \tag{2.9}$$

the series in (2.9) being convergent.

Definition 2.4. We set

$$\begin{aligned} X_f(\Omega) :=\Big\{ T \in \mathrm{Lip}_0(\Omega)' : \exists m \in \mathbb{N},\ \exists (x_i, y_i) \in \overline{\Omega} \times \overline{\Omega} \setminus \mathrm{Diag}^\star_\Omega \text{ for } i = 1, \dots, m, \\ T = \sum_{i=1}^{m}(\delta_{x_i} - \delta_{y_i})\Big\} \cup \{0\}. \end{aligned}$$

Every $T \in X_f(\Omega)$ is a Radon measure, and can be identified with an integral 0-current in $\mathcal{D}_0(\Omega)$.

Remark 2.5. If $\Lambda \in X(\Omega)$, then, adapting the arguments of Proposition 18 in [24], it readily follows that the suprema in (2.7) are attained (taking into account that we have Lipschitz maps which are null on $\partial\Omega$).

2.3. The classes $\mathcal{R}_f$ and $\mathcal{S}$

In the sequel, we need to consider the following classes of rectifiable currents in $\mathbb{R}^2$:

$$\mathcal{R}_f := \Big\{ R \in \mathcal{D}_0(\mathbb{R}^2) : R = \sum_{i=1}^{n} \sigma_i \delta_{z_i} \text{ for some } n \geq 0,\ z_i \in \mathbb{R}^2, \sigma_i \in \{-1, +1\}\Big\}$$

[4] We take the union with $\{0\}$ since Λ could be the Jacobian determinant of a suitable map (see (4.10)), and we want to include the case in which the map is constant.

and

$$\mathcal{S} := \Big\{ S \in \mathcal{D}_1(\mathbb{R}^2) : S = \sum_{k=1}^{\infty} [\![\overline{x_k y_k}]\!] \text{ for some sequence } ((x_k, y_k))_k \subset \mathbb{R}^2, \\ \sum_{k=1}^{\infty} |y_k - x_k| < +\infty \Big\},$$

and denote by $\mathcal{R}_f(A)$ and $\mathcal{S}(A)$ the classes written above when the currents are restricted to an open set $A \subset \mathbb{R}^2$.

By [19] (see p. 367) and Lemma A.1 in Appendix A, for all $\Lambda \in X(\Omega)$,

$$\|\Lambda\|_{\text{flat}} = \inf\{|R|_\Omega + |S|_\Omega : (R, S) \in \mathcal{D}_0(\Omega) \times \mathcal{D}_1(\Omega),\ \Lambda = R + \partial S\}, \tag{2.10}$$

and similarly, for all $\Lambda \in X(\Omega)$,

$$\|\Lambda\|_{\text{flat},\alpha} = \inf\big\{|R|_\Omega + \alpha^{-1}|S|_\Omega : (R, S) \in \mathcal{D}_0(\Omega) \times \mathcal{D}_1(\Omega),\ \Lambda = R + \partial S\big\}, \tag{2.11}$$

where we recall that α is defined in (1.3). We shall prove that the infimum in (2.11) is attained and that, if $\Lambda \in X_f(\Omega)$, the minimizers $R_{\min}$ and $S_{\min}$ satisfy $R_{\min} \in \mathcal{R}_f$ and $S_{\min} \in \mathcal{S}$ (similar properties hold for (2.10)).

3. A minimization problem for atomic distributions

Our aim in this section is to show that, for all $\Lambda \in X(\Omega)$, the infimum on the right-hand side of (2.11) is a minimum, and to analyze the regularity of its minimizers (Proposition 3.5); this will be done supposing first that, in place of Λ, we consider $T \in X_f(\Omega)$.

3.1. Properties (P) and (P$_\mathrm{f}$)

Given a distribution $\Lambda = \sum_{i=1}^{\infty}(\delta_{x_i} - \delta_{y_i}) \in X(\Omega)$, we can modify the set of points x_i, y_i in the following way. Take $i \in \mathbb{N}$. Then,

- if $d_\Omega(x_i, y_i) = |x_i - y_i|$, we introduce two (coinciding) points $\hat{x}_i = \hat{y}_i$ at the center of the segment $\overline{x_i y_i}$;
- if $d_\Omega(x_i, y_i) = d(x_i, \partial\Omega) + d(y_i, \partial\Omega)$, we choose two points $\hat{x}_i, \hat{y}_i \in \partial\Omega$ so that

$$d(x_i, \partial\Omega) = |x_i - \hat{y}_i| \quad \text{and} \quad d(y_i, \partial\Omega) = |\hat{x}_i - y_i|.$$

In this way,

$$\sum_{i=1}^{\infty}(|\hat{x}_i - y_i| + |x_i - \hat{y}_i|) = \sum_{i=1}^{\infty} d_\Omega(x_i, y_i) < +\infty,$$

and we can write

$$\Lambda = \sum_{i=1}^{\infty}(\delta_{x_i} - \delta_{y_i}) = \sum_{i=1}^{\infty}(\delta_{\hat{x}_i} - \delta_{y_i}) + \sum_{i=1}^{\infty}(\delta_{x_i} - \delta_{\hat{y}_i}) \quad \text{in } \mathcal{D}'(\Omega).$$

In particular, we may assume, after relabelling and renaming the points, that:

(P) There are sequences $((x_i, y_i)) \subset \overline{\Omega} \times \overline{\Omega} \setminus \mathrm{Diag}_{\Omega}$ such that

$$\Lambda = \sum_{i=1}^{\infty} (\delta_{x_i} - \delta_{y_i}) \quad \text{and} \quad \sum_{i=1}^{\infty} |x_i - y_i| = \sum_{i=1}^{\infty} d_{\Omega}(x_i, y_i) < +\infty. \tag{3.1}$$

Using that $\Lambda = \sum_{i=1}^{\infty} (\delta_{x_i} - \delta_{y_i})$ admits a representation as in (3.1), in (2.11) we can choose as a competitor the pair (R, S), with $R = 0$ and $S = \sum_{i=1}^{\infty} [\![\overline{y_i x_i}]\!]$, and we obtain

$$\|\Lambda\|_{\mathrm{flat},\alpha} \leq 2 \sum_{i=1}^{\infty} |x_i - y_i|. \tag{3.2}$$

Recall that there can be repetitions between the x_i, as well as between the y_i.

Now, let $T = \sum_{i=1}^{n} (\delta_{x_i} - \delta_{y_i}) \in X_f(\Omega)$. After relabelling (and keeping the same symbols, for simplicity), T admits the representation

$$T = \sum_{k \in J^+} \delta_{x_k} - \sum_{k \in J^-} \delta_{y_k}, \quad \text{for } x_k, y_k \in \Omega,\ x_k \neq y_k,$$

in $\mathcal{D}'(\Omega)$, where J^+ and J^- are finite (possibly empty) subsets of $\mathbb{N}$, and $J^+ \cap J^- = \emptyset$. It is convenient to add some atoms to T as follows: for any $k \in J^+$, we consider a point $\widehat{y}_k \in \partial\Omega$ so that $|x_k - \widehat{y}_k| = d(x_k, \partial\Omega)$, and similarly, for any $k \in J^-$, we consider a point $\widehat{x}_k \in \partial\Omega$, so that $|\widehat{x}_k - y_k| = d(y_k, \partial\Omega)$. In this way, again without changing the notation and calling once more $\widehat{y}_k$ by y_k and $\widehat{x}_k$ by x_k for simplicity, setting $I = J^+ \cup J^-$, we can always write T as

$$T = \sum_{k \in I} (\delta_{x_k} - \delta_{y_k}), \tag{3.3}$$

with the following additional property:

($\mathrm{P_f}$) for every $k \in I$, one and only one of the x_k and y_k belongs to Ω, $x_i \neq y_j$ for any $x_i, y_j \in \{x_k, y_h : x_k \in \Omega, y_h \in \Omega\}$, and $|x_k - y_k| = d_{\Omega}(x_k, y_k)$.

This implies that in (3.3) there are no cancellations in Ω. Recall that there can be repetitions between the x_i, as well as between the y_i.

3.2. Analysis of the minimum problem (2.11)

Let $T \in X_f(\Omega)$ be represented as in (3.3) and satisfying ($\mathrm{P_f}$). We consider a disjoint partition $\{I_P, I_D\}$ of I (i.e., $I = I_P \cup I_D$, $I_P \cap I_D = \emptyset$, where we allow I_P or I_D to be empty) and, provided $I_D \neq \emptyset$, an injective map $\tau\colon I_D \to I$. Along with this, we define the currents

$$\begin{cases} R_\tau := \displaystyle\sum_{k \in I_P} \delta_{x_k} - \sum_{j \in I \setminus \tau(I_D)} \delta_{y_j}, \quad S_\tau := \sum_{k \in I_D} [\![\overline{y_{\tau(k)} x_k}]\!] & \text{if } I_D \neq \emptyset, \\ R_\tau := T, \quad S_\tau := 0 & \text{if } I_D = \emptyset \end{cases} \tag{3.4}$$

(clearly $R_\tau \in \mathcal{R}_f$, and S_τ, being a finite sum, belongs to $\mathcal{S}$). Namely, we split the set I as the union of $\tau(I_D)$ and $I \setminus \tau(I_D)$; a point labelled by an index $h = \tau(k) \in \tau(I_D)$ is coupled with x_k, while a point labelled by an index $k \in I \setminus \tau(I_D)$ is uncoupled.

Notice that

$$\begin{aligned} R_\tau + \partial S_\tau &= \sum_{k\in I_P} \delta_{x_k} + \sum_{k\in I_D} \delta_{x_k} - \Big(\sum_{j\in I\setminus \tau(I_D)} \delta_{y_j} + \sum_{k\in I_D} \delta_{y_{\tau(k)}}\Big) \\ &= \sum_{k\in I} \delta_{x_k} - \sum_{k\in I} \delta_{y_k} = T, \quad \text{in } \mathcal{D}_0(\Omega). \end{aligned}$$

Lemma 3.1. *For any $T \in X_f(\Omega)$, we have*

$$\begin{aligned} &\min\big\{|R_\tau|_\Omega + \alpha^{-1}|S_\tau|_\Omega : (R_\tau, S_\tau) \textit{ as in } (3.4)\big\} \\ &\qquad = \min\big\{|R|_\Omega + \alpha^{-1}|S|_\Omega : (R,S) \in \mathcal{R}_f \times \mathcal{S}, T = R + \partial S \textit{ in } \mathcal{D}_0(\mathbb{R}^2)\big\}, \end{aligned} \tag{3.5}$$

where on the left-hand side, the minimum[5] *is taken over all disjoint partitions $\{I_D, I_P\}$ of I, and all injective maps $\tau: I_D \to I$, as above. In particular, a minimizer of the left-hand side is also a minimizer of the right-hand side.*

Proof. On the one hand, the inequality $\geq$ trivially holds in (3.5). On the other hand, also the converse inequality holds, since every competitor $(R,S) \in \mathcal{R}_f \times \mathcal{S}$ for the right-hand side, can be modified, not increasing its energy, into a competitor for the minimum problem on the left-hand side. More specifically, let $(R,S) \in \mathcal{R}_f \times \mathcal{S}$ be such that $R + \partial S = T$ in $\mathcal{D}_0(\mathbb{R}^2)$, with T represented as in (3.3) and satisfying ($\mathrm{P_f}$); in particular $\partial S = T - R$ is a finite sum of Dirac deltas. By the Federer decomposition theorem for 1-currents (see Section 4.2.25 in [19]), we can write

$$S = \sum_{i=1}^{\infty} S_i, \qquad \text{in } \mathcal{D}_1(\mathbb{R}^2),$$

with $S_i \in \mathcal{S}$ for all $i \in \mathbb{N}$, and either $\partial S_i = 0$ (so S_i is a loop) or $\partial S_i = \delta_{z_i} - \delta_{w_i}$ for some $z_i \neq w_i, z_i, w_i \in \{x_k, y_k : k \in I\}$. If $\partial S_i = 0$, we set $\widehat{S}_i := 0$, i.e., we remove the loop. If $\partial S_i = \delta_{z_i} - \delta_{w_i}$ and $\mathrm{supp}(S_i) \cap (\mathbb{R}^2 \setminus \Omega) = \emptyset$, we set $\widehat{S}_i := [\![z_i w_i]\!]$ (the segment $\overline{z_i w_i}$ is not necessarily included in Ω). If $\partial S_i = \delta_{z_i} - \delta_{w_i}$ and $\mathrm{supp}(S_i) \cap (\mathbb{R}^2 \setminus \Omega) \neq \emptyset$, then, using ($\mathrm{P_f}$), we set $\widehat{S}_i := [\![z_i \widehat{z}_i]\!] + [\![\widehat{w}_i w_i]\!]$, where $\widehat{z}_i \in \{y_k : k \in I\}$ is a point on $\partial\Omega$ such that $d(z_i, \partial\Omega) = |z_i - \widehat{z}_i|$, and similarly, $\widehat{w}_i \in \{x_k : k \in I\}$ is a point on $\partial\Omega$ such that $d(w_i, \partial\Omega) = |w_i - \widehat{w}_i|$. Finally, if some $S_i = [\![z_i w_i]\!]$ is such that both z_i and w_i belong to $\partial\Omega$, we remove S_i, whereas if only one of them belongs to $\partial\Omega$, say $w_i \in \partial\Omega$, we replace S_i by $\widehat{S}_i := [\![\widehat{z}_i \widehat{w}_i]\!]$, where, again, $\widehat{w}_i \in \{x_k : k \in I\}$ is a point on $\partial\Omega$ such that $d(w_i, \partial\Omega) = |w_i - \widehat{w}_i|$.

Then $|\widehat{S}_i|_\Omega \leq |S_i|_\Omega$ for all $i \in \mathbb{N}$, and moreover, the support of

$$\widehat{S} := \sum_{i=1}^{\infty} \widehat{S}_i$$

consists of finitely many segments (possibly with repetitions) joining some point in $\{x_k : k \in I\}$ and some point in $\{y_k : k \in I\}$. Furthermore, $\partial\widehat{S} = \partial S$. From this remark, one

[5] The existence of a minimizer is guaranteed since the number of competitors is finite.

can easily define two sets $I_P, I_D \subseteq I$ of indices and an injective map $\tau\colon I_D \to I$ so that $\widehat{S} = S_\tau$, $R = R_\tau$, and it is checked that $|R_\tau|_\Omega + |S_\tau|_\Omega \le |R|_\Omega + |S|_\Omega$. This concludes the proof. ∎

Remark 3.2. As a consequence of the previous arguments, the minimum on the right-hand side of (3.5) can be taken among currents supported on $\overline{\Omega}$.

The following crucial fact is a result of regularity theory for minimal currents; since we were not able to find a specific reference, for the reader convenience we propose a direct proof, independent of regularity theory.

Proposition 3.3. *Let $T = \sum_{i=1}^{N}(\delta_{x_i} - \delta_{y_i}) \in X_f(\Omega)$. Then the infimum in (2.11), with T in place of Λ, is attained and there are minimizers $(R_{\min}, S_{\min}) \in \mathcal{R}_f \times \mathcal{S}$.*

Proof. The minimum problem on the right-hand side of (3.5) is attained, as a consequence of Lemma 3.1, and is trivially larger than or equal to $\|T\|_{\mathrm{flat},\alpha}$, see (2.11). We claim that actually equality holds, which will imply the thesis. To prove this, recalling (2.7), it is sufficient to show that

$$\min\left\{|R|_\Omega + \alpha^{-1}|S|_\Omega : (R,S) \in \mathcal{R}_f \times \mathcal{S}, T = R + \partial S\right\} \le \sup_{\substack{\varphi \in \mathrm{Lip}_0(\Omega) \\ \|\varphi\|_{\mathrm{Lip}_0,\alpha} \le 1}} \langle T, \varphi\rangle,$$

and this readily follows from Proposition A.2 in Appendix A. ∎

Now we prove that, for a general $\Lambda \in X(\Omega)$, the infimum on the right-hand side of (2.11) can be obtained infimizing just on pairs $(R,S) \in \mathcal{R}_f \times \mathcal{S}$.

Corollary 3.4 ($\|\cdot\|_{\mathrm{flat},\alpha}$ as an infimum over $\mathcal{R}_f \times \mathcal{S}$). *We have, for all $\Lambda \in X(\Omega)$,*

$$\|\Lambda\|_{\mathrm{flat},\alpha} = \inf\left\{|R|_\Omega + \alpha^{-1}|S|_\Omega : (R,S) \in \mathcal{R}_f \times \mathcal{S}, \Lambda = R + \partial S \text{ in } \mathcal{D}_0(\mathbb{R}^2)\right\}. \tag{3.6}$$

Proof. Given $\varepsilon > 0$, it is sufficient to show that there exist $R_\varepsilon \in \mathcal{R}_f$ and $S_\varepsilon \in \mathcal{S}$ such that $\Lambda = R_\varepsilon + \partial S_\varepsilon$ and

$$|R_\varepsilon|_\Omega + 2|S_\varepsilon|_\Omega \le \|\Lambda\|_{\mathrm{flat},\alpha} + \varepsilon.$$

Assuming $\Lambda = \sum_{i=1}^{\infty}(\delta_{x_i} - \delta_{y_i}) \in X(\Omega)$ is represented so that (P) is satisfied, select $N_\varepsilon \in \mathbb{N}$ in such a way that

$$\sum_{i=N_\varepsilon+1}^{\infty} |x_i - y_i| < \frac{\varepsilon}{3}. \tag{3.7}$$

Thus, for

$$\Lambda_\varepsilon := \sum_{i=N_\varepsilon+1}^{\infty} (\delta_{x_i} - \delta_{y_i}) \in \mathrm{Lip}_0(\Omega)' \quad \text{and} \quad T_\varepsilon := \sum_{i=1}^{N_\varepsilon}(\delta_{x_i} - \delta_{y_i}) \in X_f(\Omega),$$

we have

$$\|\Lambda_\varepsilon\|_{\mathrm{flat},\alpha} \le \frac{\varepsilon}{3} \quad \text{and} \quad \|T_\varepsilon\|_{\mathrm{flat},\alpha} \le \|\Lambda\|_{\mathrm{flat},\alpha} + \|\Lambda_\varepsilon\|_{\mathrm{flat},\alpha} \le \|\Lambda\|_{\mathrm{flat},\alpha} + \frac{\varepsilon}{3}. \tag{3.8}$$

By Proposition 3.3, there are integral currents $\widehat{R}_\varepsilon \in \mathcal{R}_f$ and $\widehat{S}_\varepsilon \in \mathcal{S}$, with $T_\varepsilon = \widehat{R}_\varepsilon + \partial \widehat{S}_\varepsilon$ in $\mathcal{D}_0(\mathbb{R}^2)$, such that

$$\|T_\varepsilon\|_{\mathrm{flat},\alpha} = |\widehat{R}_\varepsilon|_\Omega + 2|\widehat{S}_\varepsilon|_\Omega. \tag{3.9}$$

Setting $R_\varepsilon := \widehat{R}_\varepsilon$ and $S_\varepsilon := \widehat{S}_\varepsilon + \sum_{i=N_\varepsilon+1}^{\infty} [\![\overline{y_i x_i}]\!]$, one sees that $S_\varepsilon \in \mathcal{S}$, $\Lambda = R_\varepsilon + \partial S_\varepsilon$, and using (3.7), (3.9), and (3.8), that

$$|R_\varepsilon|_\Omega + 2|S_\varepsilon|_\Omega \le |\widehat{R}_\varepsilon|_\Omega + 2|\widehat{S}_\varepsilon|_\Omega + \frac{2\varepsilon}{3} = \|T_\varepsilon\|_{\mathrm{flat},\alpha} + \frac{2\varepsilon}{3} \le \|\Lambda\|_{\mathrm{flat},\alpha} + \varepsilon. \qquad \blacksquare$$

Proposition 3.5 (Existence of minimizers defining $\|\cdot\|_{\mathrm{flat},\alpha}$). *Let $\Lambda \in X(\Omega)$. Then the infimum in (2.11) is attained, and there are minimizers $R_{\min} \in \mathcal{D}_0(\Omega)$ and $S_{\min} \in \mathcal{D}_1(\Omega)$ which are integer multiplicity currents.*

Proof. Represent $\Lambda = \sum_{i=1}^{\infty}(\delta_{x_i} - \delta_{y_i})$ as in property (P). By (3.6), we can find a sequence $((R_k, S_k)) \subset \mathcal{R}_f \times \mathcal{S}$ (in particular, of integer multiplicity currents) with $\Lambda = R_k + \partial S_k$ in $\mathcal{D}_0(\Omega)$ for any $k \in \mathbb{N}$, and such that

$$\lim_{k\to+\infty} (|R_k|_\Omega + 2|S_k|_\Omega) = \|\Lambda\|_{\mathrm{flat},\alpha}.$$

By compactness (see Theorem 7.5.2 in [22]), up to a (not relabelled) subsequence, we know that $R_k \rightharpoonup R_{\min} \in \mathcal{D}_0(\Omega)$ and $S_k \rightharpoonup S_{\min} \in \mathcal{D}_1(\Omega)$ weakly as currents, and we have to prove that $R_{\min}$ and $S_{\min}$ can be chosen with integer multiplicity. Suppose $R_k = \sum_{i=1}^{m_k} \sigma_i \delta_{z_i}$ for some $m_k \in \mathbb{N}$, with $z_i \in \Omega$ and $\sigma_i \in \{-1, +1\}$; we may assume that there are no cancellations in the previous expression. We introduce points $w_i \in \partial\Omega$ so that $|z_i - w_i| = d(z_i, \partial\Omega)$, and write $R_k = \sum_{i=1}^{m_k} \sigma_i(\delta_{z_i} - \delta_{w_i})$ as a current in $\mathbb{R}^2$. In this way, $R_k = \partial \Sigma_k$, with $\Sigma_k = \sum_{i=1}^{m_k} \sigma_i [\![\overline{w_i z_i}]\!] \in \mathcal{S}$. We have

$$|R_k|_\Omega \le \|\Lambda\|_{\mathrm{flat},\alpha} + 1$$

for k large enough; since $|R_k|_\Omega = m_k$, we deduce that (m_k) is a bounded sequence. After passing to a not-relabelled subsequence, we have $R_k \rightharpoonup R_{\min} \in \mathcal{R}_f$ weakly in $\mathcal{D}_0(\Omega)$ as $k \to +\infty$. Moreover, the mass of Σ_k satisfies

$$|\Sigma_k|_\Omega \le m_k \,\mathrm{diam}(\Omega),$$

and is uniformly bounded in k. Since $\partial\Sigma_k = R_k$ in $\mathcal{D}_0(\Omega)$, also $\Sigma_k \rightharpoonup \Sigma$ weakly in $\mathcal{D}_1(\Omega)$, with Σ an integral current.

Now we know that $R_k + \partial S_k = \Lambda$ in $\mathcal{D}_0(\Omega)$. Writing $\Lambda = \partial T$, with $T = \sum_{i=1}^{\infty} [\![\overline{x_i y_i}]\!]$ an integer multiplicity current, we see that

$$\partial S_k = \partial T - \partial \Sigma_k \quad \text{for } k \text{ large enough},$$

and then $S_k + \Sigma_k - T \in \mathcal{D}_1(\Omega)$ is an integral current without boundary. By compactness, we can assume that the sequence $(S_k + \Sigma_k - T)$ weakly converges in $\mathcal{D}_1(\Omega)$ to an integral current Q without boundary. On the other hand, since $S_k \rightharpoonup S_{\min}$ weakly in $\mathcal{D}_1(\Omega)$, we conclude that $S_{\min} + \Sigma - T = Q$ is an integral current. In particular, $S_{\min} = Q - \Sigma + T$ is an integer multiplicity current. $\blacksquare$

3.3. Properties of minimizers

Here we prove a useful lemma which summarizes some properties of the minimizing partition $\{I_P, I_D\}$ and of the minimizing map τ on the left-hand side of (3.5).

Lemma 3.6 (Structure of minimizers of the combinatorial problem). *Let $T \in X_f(\Omega)$ be of the form (3.3) and satisfying ($\mathrm{P_f}$). Then there exist a disjoint partition $\{I_P, I_D\}$ of I and an injective map $\tau\colon I_D \to I$, minimizing the left-hand side of (3.5), for which, setting*

$$R_{\min} := \sum_{k\in I_P} \delta_{x_k} - \sum_{j\in I\setminus\tau(I_D)} \delta_{y_j} \in \mathcal{R}_f \quad \text{and} \quad S_{\min} := \sum_{k\in I_D} [\![\overline{y_{\tau(k)}\,x_k}]\!] \in \mathcal{S}, \tag{3.10}$$

so that $T = R_{\min} + \partial S_{\min}$, the following properties hold.

(a) *For all $k \in I_P$ and $j \in I\setminus\tau(I_D)$ for which $x_k \in \Omega$ and $y_j \in \Omega$, we have*

$$|x_k - y_j| \geq 1, \quad d(x_k, \partial\Omega) \geq \frac{1}{2} \quad \text{and} \quad d(y_j, \partial\Omega) \geq \frac{1}{2}. \tag{3.11}$$

Moreover, if $k \in I_D$ is such that either $x_k \in \Omega$ and $y_{\tau(k)} \in \partial\Omega$, or $x_k \in \partial\Omega$ and $y_{\tau(k)} \in \Omega$, then $\tau(k) = k$.

(b) *For all $k \in I_D$, the (relative) interior of the segment $\overline{y_{\tau(k)}\,x_k}$ is contained in Ω, and*

$$\begin{aligned} |x_k - y_{\tau(k)}| &\leq \min\{1, d(x_k,\partial\Omega) + d(y_{\tau(k)},\partial\Omega)\}, \\ |x_k - y_{\tau(k)}| &\leq \frac{1}{2} + \min\{d(x_k,\partial\Omega), d(y_{\tau(k)},\partial\Omega)\}. \end{aligned} \tag{3.12}$$

(c) *If $x_k \in \Omega \cap \operatorname{supp}(S_{\min})$ for some $k \in I_P$, then $x_k = x_h$ for some $h \in I_D$.*

(d) *If $y_j \in \Omega \cap \operatorname{supp}(S_{\min})$ for some $j \in I\setminus\tau(I_D)$, then $y_j = y_{\tau(k)}$ for some $k \in I_D$.*

(e) *If $k, h \in I_D$, $k \neq h$, and $\overline{y_{\tau(k)}\,x_k} \cap \overline{y_{\tau(h)}\,x_h} = \{r\}$, then either $r = y_{\tau(k)} = y_{\tau(h)}$ or $r = x_k = x_h$.*

(f) *If $k, h \in I_D$, $k \neq h$, and $\overline{y_{\tau(k)}\,x_k} \cap \overline{y_{\tau(h)}\,x_h}$ contains more than one point, then either $\overline{y_{\tau(k)}\,x_k} \cap \overline{y_{\tau(h)}\,x_h} = \overline{y_{\tau(k)}\,x_h}$ or $\overline{y_{\tau(k)}\,x_k} \cap \overline{y_{\tau(h)}\,x_h} = \overline{y_{\tau(h)}\,x_k}$.*

(g) *If the points in (3.10) contained in Ω are distinct and three by three not collinear, then the segments $\overline{y_{\tau(k)}\,x_k} \cap \Omega$, $k \in I_D$, are disjoint.*

(h) *$|S_{\min}|_\Omega = \sum_{k\in I_D} |x_k - y_{\tau(k)}|$, and in particular, $\operatorname{supp}(S_{\min}) = \bigcup_{k\in I_D} \overline{y_{\tau(k)}\,x_k}$.*

In words, (c) says that if $k \in I_P$ and $x_k \in \Omega$ intersects $\operatorname{supp}(S_{\min})$, the intersection happens in one extremum of the intervals composing $S_{\min}$, and similarly for y_j in (d). Item (e) says that if two intervals of $S_{\min}$ intersect at one point, this point must be an extremum of both. Item (f) says that if two intervals of $S_{\min}$ intersect at more than one point, then they cannot be contained one inside the other.

Proof. (a) Let us prove the first inequality[6] in (3.11). Suppose, to the contrary, that there exist $k \in I_P$ and $j \in I\setminus\tau(I_D)$ such that $x_k \in \Omega$, $y_j \in \Omega$ and $|x_k - y_j| < 1$. Define the

[6]We shall prove a stronger statement, namely the validity of (3.11) for any minimizing I_P, I_D and τ.

injective map $\varphi\colon I_D \cup \{k\} \to I$ as follows: $\varphi = \tau$ on I_D, and $\varphi(k) := j$. Then

$$R_\varphi = \sum_{h \in I_P \setminus \{k\}} \delta_{x_h} - \sum_{\iota \in I \setminus \varphi(I_D \cup \{k\})} \delta_{y_\iota} = R_{\min} - \delta_{x_k} - \delta_{y_j},$$

$$S_\varphi = \sum_{h \in I_D \cup \{k\}} [\![y_{\varphi(h)} x_h]\!] = S_{\min} + [\![y_j x_k]\!].$$

Thus,

$$\begin{aligned}|R_\varphi|_\Omega + 2|S_\varphi|_\Omega &\le |R_{\min}|_\Omega + 2|S_{\min}|_\Omega - |\delta_{x_k}| - |\delta_{y_j}| + 2|x_k - y_j| \\ &= |R_{\min}|_\Omega + 2|S_{\min}|_\Omega + 2(|x_k - y_j| - 1) < |R_{\min}|_\Omega + 2|S_{\min}|_\Omega,\end{aligned}$$

contradicting the minimality of $(R_{\min}, S_{\min})$.

Now, let us show the remaining part of assertion (a). Let $D^+ := \{k \in I_D : x_k \in \Omega, y_{\tau(k)} \in \partial\Omega\}$ and $D^- := \{k \in I_D : x_k \in \partial\Omega,\ y_{\tau(k)} \in \Omega\}$. For all $k \in D^+ \cup D^-$, define $\varphi(k) := k$, whereas $\varphi(k) := \tau(k)$ for all $k \in I_D \setminus (D^+ \cup D^-)$. It is easily checked that φ is injective, and that

$$\sum_{k \in I_D} |x_k - y_{\varphi(k)}| = \sum_{k \in D^+ \cup D^-} |x_k - y_k| + \sum_{k \in I_D \setminus (D^+ \cup D^-)} |x_k - y_{\tau(k)}| \le \sum_{k \in I_D} |x_k - y_{\tau(k)}|,$$

the inequality being true since, for $k \in D^+$ (and similarly for D^-), by ($\mathrm{P_f}$), y_k is a closest point on $\partial\Omega$ to x_k. In particular, replacing τ with φ we get a minimizing configuration satisfying the last statement in (a). In words, by assumption $x_k \in \Omega$ implies $y_{\tau(k)} \in \partial\Omega$, and $d(x_k, \partial\Omega) = |x_k - y_{\tau(k)}|$; we have shown that there are minimizers for which $d(x_k, \partial\Omega) = |x_k - y_k|$, so we are "connecting" x_k with y_k.

To conclude the proof of (a), we need to show the second and third inequalities in (3.11). Let $k \in I_P$, and suppose by contradiction that $d(x_k, \partial\Omega) = |x_k - y_k| < 1/2$. Let us extend τ on $I_D \cup \{k\}$ using $\varphi := \tau$ on I_D and $\varphi(k) := k$. Notice that this extension is well-defined, since $y_k \in \partial\Omega$ and the last statement of (a) is satisfied by τ. Also, in this case, the new partition with φ has smaller energy than the original one with τ, since

$$1 = |\delta_{x_k}|_\Omega > 2|x_k - y_{\varphi(k)}|,$$

and this is enough to prove that

$$|R_{\min}|_\Omega + 2|S_{\min}|_\Omega > |R_\varphi|_\Omega + 2|S_\varphi|_\Omega,$$

contradicting the minimality. In a similar manner, we prove the third inequality in (3.11).

(b) Let us start to prove that $\overline{y_{\tau(k)} x_k} \subset \overline{\Omega}$ for all $k \in I_D$. Suppose, to the contrary, that there exists $k \in I_D$ for which $\overline{y_{\tau(k)} x_k} \cap (\mathbb{R}^2 \setminus \overline{\Omega}) \neq \emptyset$, so that $|[\![y_{\tau(k)} x_k]\!]|_\Omega < |y_{\tau(k)} - x_k|$, and necessarily $|[\![y_{\tau(k)} x_k]\!]|_\Omega \ge d_\Omega(x_k, y_{\tau(k)})$. In such a case, $\tau(k) \neq k$ (by property ($\mathrm{P_f}$)), and so we set $\varphi(k) := k$ and $\varphi(j) := j$ for $j = \tau(k)$; moreover, we set $\varphi := \tau$ on the other indices. Owing to the last assertion in (a), φ is well-defined, and since

$$|x_k - y_k| + |x_j - y_j| \le |x_k - y_j|, \quad j = \tau(k),$$

it easily follows that the new partition and φ minimize (3.5). This concludes the proof of the first assertion in (b).

Let us prove the first inequality in (3.12). If $|x_k - y_{\tau(k)}| > d_\Omega(x_k, y_{\tau(k)})$, we modify the partition and τ as before, getting a contradiction with the minimality. If $|x_k - y_{\tau(k)}| > 1$, we erase k from I_D, and we find out that the new partition with τ replaced by its restriction on $I_D \setminus \{k\}$ realizes a smaller contribution, contradicting the minimality.

The last inequality in (3.12) is a consequence of the following argument. We may assume, without loss of generality, that $x_k, y_{\tau(k)} \in \Omega$ and that, by the first assertion in (b), the segment joining them has interior in Ω; by (a), we can also suppose $j := \tau(k) \notin I_D$. Hence we can delete k from I_D and add j to it, defining $\varphi(j) := j$ and $\varphi := \tau$ elsewhere. In such a case, by the minimality assumption, we obtain

$$1 + d(y_j, \partial\Omega) = |\delta_{x_k}| + 2|[\![y_{\varphi(j)} x_j]\!]|_\Omega \geq 2|[\![y_{\tau(k)} x_k]\!]|_\Omega = 2|x_k - y_{\tau(k)}|.$$

(c) Suppose there exists $k \in I_P$ with $x_k \in \Omega$, $x_k \in \operatorname{supp}(S_{\min}) \setminus \bigcup_{h \in I_D} x_h$; then necessarily x_k belongs to the relative interior of some segment $\overline{y_{\tau(j)} x_j}$, with $j \in I_D$ and $j \neq k$, so that $|x_j - y_{\tau(j)}| = |x_j - x_k| + |x_k - y_{\tau(j)}|$. Set $\tilde{I}_D = I_D \cup \{k\} \setminus \{j\}$ and let $\varphi\colon \tilde{I}_D \to I$ be the injective map such that $\varphi(i) := \tau(i)$ if $i \neq k$, $\varphi(k) := \tau(j)$. Now

$$2|x_j - y_{\tau(j)}| + |\delta_{x_k}| = 2|x_k - y_{\tau(j)}| + 2|x_j - x_k| + |\delta_{x_j}| > 2|x_k - y_{\varphi(k)}| + |\delta_{x_j}|,$$

implying that $|x_j - y_{\tau(j)}| > |x_k - y_{\varphi(k)}|$. Since $R_{\min}$ and R_φ have the same mass in Ω, the previous inequality readily gives

$$|R_{\min}|_\Omega + 2|S_{\min}|_\Omega > |R_\varphi|_\Omega + 2|S_\varphi|_\Omega,$$

contradicting the minimality of $(R_{\min}, S_{\min})$. In a similar manner, we prove (d).

(e) Suppose to the contrary that r belongs to $\overline{y_{\tau(k)} x_k} \setminus \{y_{\tau(k)}, x_k\}$. Set $\varphi\colon I_D \to I$, $\varphi(j) := \tau(j)$ if $j \neq k, h$, $\varphi(k) := \tau(h)$, $\varphi(h) := \tau(k)$. We have

$$\begin{aligned} |x_k - y_{\varphi(k)}| &= |x_k - y_{\tau(h)}| \leq |r - y_{\tau(h)}| + |x_k - r|, \\ |x_h - y_{\varphi(h)}| &= |x_h - y_{\tau(k)}| \leq |r - y_{\tau(k)}| + |x_h - r|, \end{aligned} \tag{3.13}$$

where at least one of these inequalities holds strictly, because the points $y_{\tau(k)}, x_k, y_{\tau(h)}, x_h$ are not collinear by construction. Summing the inequalities in (3.13), we get

$$|x_k - y_{\varphi(k)}| + |x_h - y_{\varphi(h)}| < |x_k - y_{\tau(k)}| + |x_h - y_{\tau(h)}|,$$

and this is enough to deduce that

$$|R_{\min}|_\Omega + 2|S_{\min}|_\Omega > |R_\varphi|_\Omega + 2|S_\varphi|_\Omega,$$

contradicting once again the minimality.

(f) If $\overline{y_{\tau(k)} x_k} \cap \overline{y_{\tau(h)} x_h}$ contains more than one point, it must contain a segment. In particular, we have to exclude the two cases: $\overline{y_{\tau(k)} x_k} \cap \overline{y_{\tau(h)} x_h} = \overline{x_k x_h}$ and $\overline{y_{\tau(k)} x_k} \cap \overline{y_{\tau(h)} x_h} = \overline{y_{\tau(k)} y_{\tau(h)}}$. Let us discuss the former (the latter being similar). In such a case, it is sufficient to set $\varphi(k) := h$, $\varphi(h) := k$, and $\varphi := \tau$ otherwise, and check that the map φ associated with the same partition provides

$$|R_{\min}|_\Omega + 2|S_{\min}|_\Omega > |R_\varphi|_\Omega + 2|S_\varphi|_\Omega,$$

contradicting the hypothesis.

Item (g) follows from (e) and (f), and (h) follows from the last assertion in (a) and the first in (b). ∎

4. Distributional Jacobian; maps with values in $\mathbb{S}^1$

If $u = (u_1, u_2) \in W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$, its distributional Jacobian determinant is the distribution $\mathrm{Det}(\nabla u) \in \mathcal{D}'(\Omega)$ defined by

$$\langle \mathrm{Det}(\nabla u), \varphi\rangle := \int_\Omega \lambda_u \cdot \nabla\varphi \, dx, \quad \forall \varphi \in \mathscr{C}_c^\infty(\Omega), \tag{4.1}$$

where

$$\begin{aligned}\lambda_u &:= \frac{1}{2}(u_1 \nabla^\perp u_2 - u_2 \nabla^\perp u_1) \\ &= \frac{1}{2}\Big(-u_1 \frac{\partial u_2}{\partial x_2} + u_2 \frac{\partial u_1}{\partial x_2}, u_1 \frac{\partial u_2}{\partial x_1} - u_2 \frac{\partial u_1}{\partial x_1}\Big) \in L^1(\Omega; \mathbb{R}^2),\end{aligned}$$

hence (cf. (2.1))

$$\mathrm{Det}(\nabla u) = -\mathrm{Div}\, \lambda_u \in \mathcal{D}'(\Omega).$$

Moreover, since $\lambda_u \in L^1(\Omega; \mathbb{R}^2)$, equality (4.1) extends to $\varphi \in \mathrm{Lip}_0(\Omega)$, so that

$$\mathrm{Det}(\nabla u) \in \mathrm{Lip}_0(\Omega)'.$$

It follows from the definition that the distributional Jacobian enjoys some well-known compactness properties. For instance, let $u \in W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$, let $(v_k) \subset W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$ be a bounded sequence in $L^\infty(\Omega; \mathbb{R}^2)$, and suppose $v_k \to u$ in $W^{1,1}(\Omega; \mathbb{R}^2)$. Then

$$\mathrm{Det}(\nabla v_k) \rightharpoonup \mathrm{Det}(\nabla u) \quad \text{in } \mathcal{D}'(\Omega). \tag{4.2}$$

Remark 4.1. The convergence in (4.2) can be strengthened into

$$\|\mathrm{Det}(\nabla v_k) - \mathrm{Det}(\nabla u)\|_{\mathrm{Lip}_0(\Omega)'} \to 0. \tag{4.3}$$

Indeed, take a subsequence (k_h); for any $\varphi \in \mathrm{Lip}_0(\Omega)$, write

$$\begin{aligned}\langle \mathrm{Det}(\nabla u) - \mathrm{Det}(\nabla v_{k_h}), \varphi\rangle &= \int_\Omega (\lambda_u - \lambda_{v_{k_h}}) \cdot \nabla\varphi \, dx \le \|\nabla\varphi\|_{L^\infty} \int_\Omega |\lambda_u - \lambda_{v_{k_h}}| \, dx \\ &\le \|\nabla\varphi\|_{L^\infty}\Big(C_1 \int_\Omega |\nabla u - \nabla v_{k_h}| \, dx + C_2 \int_\Omega |\nabla u \cdot (u - v_{k_h})| \, dx\Big).\end{aligned} \tag{4.4}$$

Since $(u - v_{k_h})$ tends to zero in $L^1(\Omega; \mathbb{R}^2)$, and since we can select a further subsequence (k_{h_l}) such that $(u - v_{k_{h_l}})$ tends to zero weakly-star in $L^\infty(\Omega; \mathbb{R}^2)$, we deduce that the limit of the right-hand side of (4.4) vanishes along the sub-subsequence, as $l \to +\infty$. In particular, taking the supremum of the left-hand side of (4.4) over $\varphi \in \mathrm{Lip}_0(\Omega)$ with $\|\varphi\|_{\mathrm{Lip}_0} \le 1$, we infer

$$\|\mathrm{Det}(\nabla v_{k_{h_l}}) - \mathrm{Det}(\nabla u)\|_{\mathrm{Lip}_0(\Omega)'} \to 0.$$

Thus, (4.3) follows from the Uryshon property.

4.1. Maps with values in $\mathbb{S}^1$

We collect here some useful tools and results, mostly on Sobolev maps taking values in $\mathbb{S}^1$. A large literature on this topic is available, e.g., following the results by Brezis and coauthors (see for instance [13] and references therein). Together with the Jacobian determinant, it is useful to introduce the notions of degree and lifting.

Definition 4.2 (Degree). Let $B_r \subset \mathbb{R}^2$ be a disc of radius $r > 0$, and let ν be the outer unit normal vector to ∂B_r. The degree of a map $u \in W^{1,1}(\partial B_r; \mathbb{S}^1)$ is defined as

$$\deg(u;\partial B_r) = \frac{1}{2\pi}\int_{\partial B_r}\left(u_1\frac{\partial u_2}{\partial\tau} - u_2\frac{\partial u_1}{\partial \tau}\right) d\mathcal{H}^1, \tag{4.5}$$

where $\tau := \nu^{\perp}$.

Notice that $\deg(u;\partial B_r) \in \mathbb{Z}$.

Definition 4.3 (Lifting). Let $u = (u_1, u_2) \in \mathrm{BV}(\Omega; \mathbb{S}^1)$. We say that $w \in \mathrm{BV}(\Omega)$ is a lifting of u if $e^{iw} = (\cos w, \sin w) = (u_1, u_2)$ a.e. in Ω.

The following result holds (see Section 6.2 in [20], or Théorème 0.1 and Remarque 0.1 in [17]).

Theorem 4.4. *Let $u \in \mathrm{BV}(\Omega; \mathbb{S}^1)$. Then there exists a lifting $w \in \mathrm{BV}(\Omega)$ of u such that*

$$\|w\|_{BV} \le 2\|u\|_{BV}.$$

If furthermore $u \in \mathrm{SBV}(\Omega; \mathbb{S}^1)$, then $w \in \mathrm{SBV}(\Omega)$.

Liftings w provided by Theorem 4.4 satisfy the following important feature: if $u \in W^{1,1}(\Omega; \mathbb{S}^1)$, then

$$\mathrm{Det}(\nabla u) = \frac{1}{2}\mathrm{Curl}(\nabla w) \quad \text{in } \mathcal{D}'(\Omega),$$

see (2.2). Indeed, for any $\varphi \in \mathcal{C}_c^\infty(\Omega)$,

$$\begin{aligned}\langle \mathrm{Det}(\nabla u), \varphi\rangle &= \frac{1}{2}\int_\Omega \left(u_1\nabla^\perp u_2 - u_2\nabla^\perp u_1\right)\cdot\nabla\varphi\, dx \\ &= \frac{1}{2}\int_\Omega \nabla^\perp w\cdot\nabla\varphi\, dx = \frac{1}{2}\langle \mathrm{Curl}(\nabla w), \varphi\rangle.\end{aligned} \tag{4.6}$$

Let $B = B_R(0) \supset\supset \Omega$ be an open disc, for some $R > 0$ big enough, and let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. We claim that there exists an extension $\bar u \in W^{1,1}(B; \mathbb{S}^1)$ of u. Indeed, let $w \in \mathrm{BV}(\Omega)$ be a lifting of u; since Ω has Lipschitz boundary, by [21], p. 162, there exists $\widehat{w} \in W^{1,1}(B \setminus \overline{\Omega}) \cap \mathrm{BV}(B)$, with trace $\widehat{w} \llcorner \partial\Omega = w \llcorner \partial\Omega$. If we set

$$\overline{w} := \begin{cases} w & \text{in } \Omega, \\ \widehat{w} & \text{in } B \setminus \overline{\Omega}, \end{cases} \tag{4.7}$$

the map

$$\bar u := e^{i\overline{w}} \tag{4.8}$$

is the map we are looking for. It is easy to see that, by construction, $\deg(\bar{u}, \partial B_r(0)) = 0$ for a.e. $r > 0$ with $\Omega \subset\subset B_r(0) \subset B$. Indeed, if r is so that $w \llcorner \partial B_r(0)$ belongs to $W^{1,1}(\partial B_r(0))$, denoting by $\widetilde{w}$ an arbitrary function in $W^{1,1}(B_r(0))$ with trace w on ∂B_r, by the Stokes theorem, one has

$$0 = \int_{B_r} \mathrm{Curl}(\nabla \widetilde{w})\, dx = \int_{\partial B_r} \nabla w \cdot \tau \, d\mathcal{H}^1 = \int_{\partial B_r} (u_1 \nabla u_2 - u_2 \nabla u_1) \cdot \tau \, d\mathcal{H}^1$$
$$= 2\pi \deg(u; \partial B_r(0)).$$

In what follows, we will need the following standard density result.

Theorem 4.5 (Density of $\mathcal{C}^\infty$ in $W^{1,1}(A;\mathbb{S}^1)$). *If $A \subset \mathbb{R}^2$ is a connected simply connected domain with smooth boundary, then the class*

$$\left\{v \in W^{1,1}(A;\mathbb{S}^1) : \exists n \in \mathbb{N},\ \exists\{a_1,\dots,a_n\} \subset A,\ v \in C^\infty(A \setminus \{a_1,\dots,a_n\};\mathbb{S}^1)\right\}$$

is dense in $W^{1,1}(A;\mathbb{S}^1)$. Furthermore,

$$\mathrm{Det}(\nabla v) = \pi \sum_{i=1}^{n} d_i \delta_{a_i}, \quad \forall v \in W^{1,1}(A;\mathbb{S}^1) \cap C^\infty(A \setminus \{a_1,\dots,a_n\};\mathbb{S}^1), \tag{4.9}$$

where $d_i = \deg(v; \partial B_r(a_i))$ for any $r > 0$ small enough.

Proof. See Theorem 4 with $k = 1$ in [11], and Lemma 2 in [12] for the second part of the statement. ∎

The next theorem is an extension of Theorems 3 and 3' in [14] to non-simply connected domains in $\mathbb{R}^2$. Even if it can be directly obtained from [14] and [12], for convenience we give a quick proof; for a more detailed discussion, we refer to Chapter 14 of [13]. Recall that the class $X(\Omega)$ is defined in (2.8).

Theorem 4.6 (Distributional Jacobian of $\mathbb{S}^1$-valued maps). *Let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$. Then*

$$\frac{1}{\pi}\mathrm{Det}(\nabla u) \in X(\Omega), \tag{4.10}$$

i.e., there exists a sequence $((x_i, y_i)) \subset \overline{\Omega} \times \overline{\Omega} \setminus \mathrm{Diag}^\star_\Omega$ such that $\sum_{i=1}^\infty d_\Omega(x_i, y_i) < +\infty$ and $\mathrm{Det}(\nabla u) = \pi \sum_{i=1}^\infty (\delta_{x_i} - \delta_{y_i})$.

Proof. We use an argument similar to that of Lemma 12' in [12]. Let $\bar{u} \in W^{1,1}(B;\mathbb{S}^1)$ be an extension of u as in (4.8), that satisfies (arguing as after formula (4.8)) $\deg(\bar{u}; \partial B_r(0)) = 0$ for suitable $r > 0$ big enough. Using Theorem 4.5, we can select a sequence

$$(u_k)_k \subset \left\{v \in W^{1,1}(B;\mathbb{S}^1) : \exists n \in \mathbb{N},\ \exists\{a_1,\dots,a_n\} \subset B,\ v \in C^\infty(B \setminus \{a_1,\dots,a_n\};\mathbb{S}^1)\right\}$$

converging to $\bar{u}$ in $W^{1,1}(B;\mathbb{R}^2)$. Now, denoting $\widetilde{B} = B_{2R}(0)$, we can further extend $\bar{u}$ and u_k on $\widetilde{B}$ in such a way that, for all $k > 0$, $\deg(\bar{u}; \partial B_r(0)) = \deg(u_k; \partial B_r(0)) = 0$ for a.e. $r \in (R, 2R)$, and so we can also rewrite (4.9) in the following way:

$$\frac{1}{\pi}\mathrm{Det}(\nabla u_k) = \sum_{i=1}^{n_k} (\delta_{x_i^k} - \delta_{y_i^k}), \tag{4.11}$$

for suitable (not necessarily distinct) points $x_i^k, y_i^k \in B_r(0)$ and $n_k \in \mathbb{N}$.

Moreover, since the condition $\deg(\bar{u}; \partial B_r(0)) = \deg(u_k; \partial B_r(0)) = 0$ holds for a.e. $r \in (R, 2R)$, we infer that $x_i^k, y_i^k \in \overline{B}_R(0)$. Owing to (4.3), we may suppose[7]

$$\|\mathrm{Det}(\nabla u_{k+1}) - \mathrm{Det}(\nabla u_k)\|_{\mathrm{flat},B} \le 1/2^k, \quad \forall k > 0. \tag{4.12}$$

As a result, we can write $\mathrm{Det}(\nabla \bar{u}) = \mathrm{Det}(\nabla u_1) + \sum_{k=1}^{\infty}(\mathrm{Det}(\nabla u_{k+1}) - \mathrm{Det}(\nabla u_k))$, the series being absolutely convergent in $\mathrm{Lip}_0(B)'$. Up to relabelling the indices in (4.11), we assume that for $k > 0$,

$$\mathrm{Det}(\nabla u_1) = \pi \sum_{i=1}^{m_1} (\delta_{x_i} - \delta_{y_i}) \quad \text{in } \mathrm{Lip}_0(B)', \tag{4.13}$$

$$\mathrm{Det}(\nabla u_{k+1}) - \mathrm{Det}(\nabla u_k) = \pi \sum_{i=m_k+1}^{m_{k+1}} (\delta_{x_i} - \delta_{y_i}) \quad \text{in } \mathrm{Lip}_0(B)', \tag{4.14}$$

in such a way that

$$\mathrm{Det}(\nabla \bar{u}) = \pi \sum_{i=1}^{\infty} (\delta_{x_i} - \delta_{y_i}) \quad \text{in } \mathrm{Lip}_0(B)'.$$

At the same time, restricting to $\mathrm{Lip}_0(\Omega)$ the above linear functionals originally defined on $\mathrm{Lip}_0(B)$, we can replace the preceding representations and assume that the points x_i and y_i in (4.13) and (4.14) belong to $\overline{\Omega}$, are of the form (3.3), and enjoy property (P). Up to a permutation of the points (y_i), one can further suppose that

$$\|\mathrm{Det}(\nabla u_1)\|_{\mathrm{flat}} = \sum_{i=1}^{m_1} d_\Omega(x_i, y_i), \quad \|\mathrm{Det}(\nabla u_{k+1}) - \mathrm{Det}(\nabla u_k)\|_{\mathrm{flat}} = \sum_{i=m_k+1}^{m_{k+1}} d_\Omega(x_i, y_i).$$

This, together with (4.12), implies $\|\mathrm{Det}(\nabla u)\|_{\mathrm{flat}} \le \sum_{i=1}^{\infty} d_\Omega(x_i, y_i) < +\infty$, which concludes the proof. ∎

5. Density results in $W^{1,1}(\Omega; \mathbb{S}^1)$

In this section, we want to show the following density result, which is an immediate consequence of Lemmas 5.3 and 5.5 below.

Proposition 5.1 (Density in $W^{1,1}(\Omega; \mathbb{S}^1)$). *Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. Then, for all $\varepsilon > 0$, there exists a map $u_\varepsilon \in W^{1,1}(\Omega; \mathbb{S}^1)$ with the following properties.*

(i) $\mathrm{Det}(\nabla u_\varepsilon) = \pi \sum_{i=1}^{N_\varepsilon} (\delta_{x_i} - \delta_{y_i})$ *in $\mathrm{Lip}_0(\Omega)'$ for some $N_\varepsilon \in \mathbb{N}$, with distinct and three by three not collinear points x_i, y_i in Ω.*

(ii) *There exist positive numbers $\rho_{x_i} < \varepsilon$ and $\rho_{y_i} < \varepsilon$, $i = 1, \dots, N$, such that the discs of the family $\{B_{\rho_{x_i}}(x_i), B_{\rho_{y_i}}(y_i) : x_i \in \Omega, y_i \in \Omega\}$ are contained in Ω, are pairwise disjoint, and*[8] *$u_\varepsilon = e^{i\theta_{x_i}}$ in $B_{\rho_{x_i}}(x_i)$, and $u_\varepsilon = e^{-i\theta_{y_i}}$ in $B_{\rho_{y_i}}(y_i)$.*

(iii) $\|u - u_\varepsilon\|_{W^{1,1}} + \|\mathrm{Det}(\nabla u) - \mathrm{Det}(\nabla u_\varepsilon)\|_{\mathrm{flat}} < \varepsilon$.

[7] We use that $\|\cdot\|_{\mathrm{flat},B}$ and $\|\cdot\|_{\mathrm{Lip}_0(B)'}$ are equivalent.

[8] Here and in the sequel, θ_x is the polar angular coordinate around x.

Recall the Definition 2.1 of the dipole map $w_{p,n}$, and set $v_{p,n} := e^{iw_{p,n}} \in W^{1,1}(\Omega;\mathbb{S}^1)$, which satisfies

$$\mathrm{Det}(\nabla v_{p,n}) = \pi(\delta_p - \delta_n).$$

Lemma 5.2 (Density: finite number of singular points). *Let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ and write, using also property (P),* $\mathrm{Det}(\nabla u) = \pi \sum_{i=1}^{\infty}(\delta_{x_i} - \delta_{y_i})$*, with $x_i, y_i \in \overline{\Omega}$, $x_i \neq y_i$, and $\sum_{i=1}^{\infty} |x_i - y_i| < +\infty$. Then, for all $\varepsilon > 0$, there exists a map $u_\varepsilon \in W^{1,1}(\Omega;\mathbb{S}^1)$ such that*

(i) $\mathrm{Det}(\nabla u_\varepsilon) = \pi \sum_{i=1}^{N_\varepsilon}(\delta_{x_i} - \delta_{y_i})$ *for some $N_\varepsilon \in \mathbb{N}$,*

(ii) $\|u - u_\varepsilon\|_{W^{1,1}} + \|\mathrm{Det}(\nabla u) - \mathrm{Det}(\nabla u_\varepsilon)\|_{\mathrm{flat}} < \varepsilon.$

Proof. Let $\eta > 0$ and choose $N_\eta \in \mathbb{N}$ so that $\sum_{i=N_\eta+1}^{\infty} |x_i - y_i| < \eta/2$. Given (x_i, y_i) with $i > N_\eta$, consider the dipole map $w_i := w_{x_i,y_i} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^2)$ in (2.3), and the cut-off function $\psi_i^\eta : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\psi_i^\eta(x) = \varrho\Big(\frac{1}{\eta_i}\, d(x, \overline{x_i y_i})\Big), \quad \text{with } \eta_i := 2^{-i}\eta, \tag{5.1}$$

where $d(x, \overline{x_i y_i}) = \mathrm{dist}(x, \overline{x_i y_i})$, $\varrho \in C^\infty([0,1])$ is non-increasing, $\rho \equiv 1$ in a right neighborhood of 0, $\rho \equiv 0$ in a left neighborhood of 1, and $|\varrho'| \le 2$. The support of ψ_i^η satisfies

$$|\mathrm{spt}(\psi_i^\eta)| \le \pi\eta_i^2 + 2\eta_i |x_i - y_i|, \quad \forall i > N_\eta. \tag{5.2}$$

By (2.4), one checks[9], for the approximate gradients, that there exists a constant $C > 0$ independent of η such that

$$\int_{\mathrm{spt}(\psi_i^\eta)} |\nabla w_i|\, dx \le C(\eta_i + |x_i - y_i|), \quad \forall i > N_\eta. \tag{5.3}$$

Let $w \in \mathrm{BV}(\Omega)$ be a lifting of u given by Theorem 4.4, and consider its extension $\overline{w}$ in B as in (4.7); substracting a phase contribution to u, we then define, in B,

$$w_\eta := \overline{w} - \sum_{i=N_\eta+1}^{\infty} w_i \psi_i^\eta \quad \text{and} \quad u_\eta := e^{iw_\eta} \in W^{1,1}(B;\mathbb{S}^1).$$

Let also $\bar{u} := e^{i\overline{w}}$; in particular, $\bar{u} = u$ in Ω. Setting $V_\eta := \bigcup_{i>N_\eta} \mathrm{spt}(\psi_i^\eta) \subset \mathbb{R}^2$, we infer, using (5.3) and (5.2),

$$\begin{aligned}
\int_{V_\eta} |\nabla u_\eta|\, dx = \int_{V_\eta} |\nabla w_\eta|\, dx &\le \int_{V_\eta} |\nabla \overline{w}|\, dx + \sum_{i=N_\eta+1}^{\infty} \Big(\frac{C}{\eta_i} |\mathrm{spt}(\psi_i^\eta)| + C(\eta_i + |x_i - y_i|)\Big) \\
&\le \int_{V_\eta} |\nabla \bar{u}|\, dx + C \sum_{i=N_\eta+1}^{\infty} (\eta_i + |x_i - y_i|) \\
&\le \int_{V_\eta \cap \Omega} |\nabla \bar{u}|\, dx + C \sum_{i=N_\eta+1}^{\infty} (\eta_i + |x_i - y_i|) + o_\eta(1),
\end{aligned} \tag{5.4}$$

where $o_\eta(1) \to 0$ as $\eta \to 0^+$. The presence of $\bar{u}$ is due to the fact that in general, $V_\eta \setminus \overline{\Omega}$ might be nonempty. But, since $|V_\eta \setminus \overline{\Omega}| \to 0$ as $\eta \to 0^+$, the last estimate in (5.4) holds.

[9]This estimate can be obtained integrating the right-hand side of (2.4) in the two discs $B_{\eta_i}(x_i)$ and $B_{\eta_i}(y_i)$, and estimating $|\nabla w_i|$ by C/η_i in the remaining part of $\mathrm{spt}(\psi_i^\eta)$.

From this and the definition of η_i in (5.1), we conclude

$$\|u - u_\eta\|_{W^{1,1}(\Omega,\mathbb{R}^2)} \le 2\int_{V_\eta} |\nabla u|\,dx + C\eta + o_\eta(1), \tag{5.5}$$

where we use that $u = u_\eta$ on $\Omega \setminus V_\eta$.

Now, we claim that

$$\mathrm{Det}(\nabla u_\eta) = \pi \sum_{i=1}^{N_\eta} (\delta_{x_i} - \delta_{y_i}), \tag{5.6}$$

which implies in turn that

$$\|\mathrm{Det}(\nabla u) - \mathrm{Det}(\nabla u_\eta)\|_{\mathrm{flat}} = \pi \Big\| \sum_{i=N_\eta+1}^{\infty} (\delta_{x_i} - \delta_{y_i}) \Big\|_{\mathrm{flat}} \le \pi \sum_{i=N_\eta+1}^{\infty} |x_i - y_i| < \frac{\pi\eta}{2}.$$

To show (5.6), for all $m > N_\eta$ define, in B,

$$f_m := \overline{w} - \sum_{i=N_\eta+1}^{m} w_i \psi_i^\eta \quad \text{and} \quad v_m := e^{if_m}.$$

Using an estimate similar to (5.4) and (5.5), we see that $v_m \to u_\eta$ in $W^{1,1}(\Omega;\mathbb{R}^2)$ as $m \to +\infty$, and therefore, owing to the same observation leading to (4.3),

$$\lim_{m\to+\infty} \|\mathrm{Det}(\nabla v_m) - \mathrm{Det}(\nabla u_\eta)\|_{\mathrm{Lip}_0(\Omega)'} = 0,$$

and also

$$\mathrm{Det}(\nabla v_m) \rightharpoonup \mathrm{Det}(\nabla u_\eta) \quad \text{in } \mathcal{D}'(\Omega). \tag{5.7}$$

On the other hand, $\mathrm{Det}(\nabla v_m) = \pi \sum_{i=1}^{N_\eta} (\delta_{x_i} - \delta_{y_i}) + \pi \sum_{i=m+1}^{\infty} (\delta_{x_i} - \delta_{y_i})$, and since the second term tends to zero in the flat distance, we conclude

$$\mathrm{Det}(\nabla v_m) \to \pi \sum_{i=1}^{N_\eta} (\delta_{x_i} - \delta_{y_i}) \quad \text{in } \mathrm{Lip}_0(\Omega)'. \tag{5.8}$$

In particular, from (5.7) and (5.8), claim (5.6) follows. From this and (5.5), it suffices to choose $\eta = \eta(\varepsilon)$ small enough to guarantee that (ii) holds. Hence setting $N_\varepsilon := N_\eta$ and $u_\varepsilon := u_\eta$, the thesis follows. ∎

Now we refine the approximation of Lemma 5.2.

Lemma 5.3 (Density: not collinear points). *Let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ be such that* $\mathrm{Det}(\nabla u) = \pi \sum_{i=1}^{N} (\delta_{x_i} - \delta_{y_i})$ *is a representation as in (3.3) satisfying property ($\mathrm{P_f}$) in Section 3.1, with $x_i, y_i \in \overline{\Omega}$, $x_i \neq y_i$, for $i = 1, \dots, N$. Then, for all $\varepsilon > 0$, there exists a map $u_\varepsilon \in W^{1,1}(\Omega;\mathbb{S}^1)$ such that*

(i) $\mathrm{Det}(\nabla u_\varepsilon) = \pi \sum_{i=1}^{N} (\delta_{x_i^\varepsilon} - \delta_{y_i^\varepsilon})$ *and the points x_i^ε, y_i^ε in Ω are distinct and three by three not collinear,*

(ii) $\|u - u_\varepsilon\|_{W^{1,1}} + \|\mathrm{Det}(\nabla u) - \mathrm{Det}(\nabla u_\varepsilon)\|_{\mathrm{flat}} < \varepsilon.$

Proof. Define

$$I^+ := \{i \in \{1,\dots,N\} : x_i \in \Omega\} \quad \text{and} \quad I^- := \{i \in \{1,\dots,N\} : y_i \in \Omega\}.$$

Fix $\eta > 0$. For all $i \in I^+$, let us choose $\widehat{x}_i, \widehat{y}_i \in \Omega$ with $\widehat{y}_i := x_i$ and in such a way that the points $\widehat{x}_i$, $i \in I^+$, are all distinct, three by three not collinear, and satisfy

$$\sum_{i \in I^+} |\widehat{x}_i - \widehat{y}_i| < \eta. \tag{5.9}$$

For all $i \in I^+$, let $\widehat{w}_i := w_{\widehat{x}_i,\widehat{y}_i}$ be the dipole map defined in (2.3), and let $\psi_i^\eta : \mathbb{R}^2 \to \mathbb{R}$ be the cut-off function given by

$$\psi_i^\eta(x) = \varrho\Big(\frac{1}{\eta} d(x, \overline{\widehat{x}_i \widehat{y}_i})\Big),$$

where ϱ is as in the proof of Lemma 5.2. In particular, ψ_i^η is Lipschitz continuous with Lipschitz constant $2/\eta$ and is supported in $V_i^\eta := \{x \in \mathbb{R}^2 : d(x, \overline{\widehat{x}_i \widehat{y}_i}) \le \eta\}$. Supposing that $\eta > 0$ is sufficiently small, we have $V_i^\eta \subset \Omega$. Now, using also (5.9), we notice that

$$|V_i^\eta| = \pi\eta^2 + 2\eta|\widehat{x}_i - \widehat{y}_i| \le C\eta^2, \quad \forall i \in I^+, \tag{5.10}$$

where $C > 0$ is a constant independent of i and η. Further, by (2.4) and (5.9), we deduce that there is a constant, still denoted by $C > 0$, and independent of η and i, such that

$$\int_{V_i^\eta} |\nabla \widehat{w}_i|\,dx \le C\eta, \quad \forall i \in I^+. \tag{5.11}$$

Similarly, for all $i \in I^-$ we choose $\widetilde{x}_i, \widetilde{y}_i \in \Omega$ with $\widetilde{x}_i := y_i$ and in such a way that the points $\widetilde{x}_i$, $i \in I^+$, and $\widetilde{y}_i$, $i \in I^-$, are all distinct, three by three not collinear, and satisfy

$$\sum_{i \in I^-} |\widetilde{x}_i - \widetilde{y}_i| < \eta. \tag{5.12}$$

In this case, we also introduce, for $i \in I^-$, the maps $\widetilde{w}_i := w_{\widetilde{x}_i,\widetilde{y}_i}$ and $\phi_i^\eta : \mathbb{R}^2 \to \mathbb{R}$, the latter defined as $\phi_i^\eta(x) := \max\{0, 1 - \frac{1}{\eta} d(x, \overline{\widetilde{x}_i \widetilde{y}_i})\}$, which enjoy the same features of ψ_i^η; in particular, the supports W_i^η of ϕ_i^η, $i \in I^-$, are contained in Ω and have Lebesgue measures bounded by $C\eta^2$. The same estimate as in (5.11) holds for $\widetilde{w}_i$.

Let us consider a lifting $w \in \mathrm{BV}(\Omega)$ of u provided by Theorem 4.4 and (4.7); we define

$$w_\eta := w + \sum_{i \in I^+} \psi_i^\eta\, \widehat{w}_i + \sum_{i \in I^-} \phi_i^\eta\, \widetilde{w}_i \quad \text{and} \quad v_\eta := e^{i w_\eta}.$$

Due to the fact that $w_\eta = w$ out of $A_\eta := (\bigcup_{i \in I^+} V_i^\eta) \cup (\bigcup_{i \in I^-} W_i^\eta)$, it is immediate that $v_\eta \to u$ in $L^1(\Omega; \mathbb{R}^2)$ as $\eta \to 0^+$. Since $|\nabla v_\eta| = |\nabla w_\eta|$ a.e. in Ω, we can estimate

$$|\nabla v_\eta| \le |\nabla w| + \frac{2}{\eta}\Big(\sum_{i \in I^+} \|\widehat{w}_i\|_{L^\infty} + \sum_{i \in I^-} \|\widetilde{w}_i\|_{L^\infty}\Big) + \sum_{i \in I^+} |\nabla \widehat{w}_i| + \sum_{i \in I^-} |\nabla \widetilde{w}_i|.$$

From this, in view of the fact that $v_\eta = u$ in $\Omega \setminus \big((\bigcup_{i\in I^+} V_i^\eta) \cup (\bigcup_{i\in I^-} W_i^\eta)\big)$, we conclude, using (5.11), that

$$\|\nabla v_\eta - \nabla u\|_{L^1} \le \|\nabla u\|_{L^1(A_\eta)} + C\,\frac{|A_\eta|}{\eta} + C\eta.$$

As the right-hand side is negligible as $\eta \to 0^+$ (see (5.10)), we conclude

$$\lim_{\eta\to 0^+} v_\eta = u \quad \text{in } W^{1,1}(\Omega;\mathbb{R}^2). \tag{5.13}$$

Furthermore, using (4.6), we readily see that

$$\begin{aligned}\mathrm{Det}(\nabla v_\eta) &= \frac{1}{2}\,\mathrm{Curl}(\nabla w_\eta) = \frac{1}{2}\Big(\mathrm{Curl}(\nabla w) + \sum_{i\in I^+}\mathrm{Curl}(\psi_i^\eta\,\widehat{w}_i) + \sum_{i\in I^-}\mathrm{Curl}(\phi_i^\eta\,\widetilde{w}_i)\Big)\\ &= \mathrm{Det}(\nabla u) + \sum_{i\in I^+}\mathrm{Det}(\nabla v_{\widehat{x}_i,\widehat{y}_i}) + \sum_{i\in I^-}\mathrm{Det}(\nabla v_{\widetilde{x}_i,\widetilde{y}_i}),\end{aligned}$$

which implies

$$\begin{aligned}\mathrm{Det}(\nabla u) - \mathrm{Det}(\nabla v_\eta) &= -\sum_{i\in I^+}\mathrm{Det}(\nabla v_{\widehat{x}_i,\widehat{y}_i}) - \sum_{i\in I^-}\mathrm{Det}(\nabla v_{\widetilde{x}_i,\widetilde{y}_i})\\ &= -\sum_{i\in I^+}(\delta_{\widehat{x}_i} - \delta_{\widehat{y}_i}) - \sum_{i\in I^-}(\delta_{\widetilde{x}_i} - \delta_{\widetilde{y}_i})\end{aligned}$$

in $\mathcal{D}'(\Omega)$. Thus, using (5.9) and (5.12), we get

$$\|\mathrm{Det}(\nabla u) - \mathrm{Det}(\nabla v_\eta)\|_{\mathrm{flat}} \le 2\eta.$$

In particular, from this and (5.13), setting $u_\varepsilon := v_\eta$ for $\eta > 0$ small enough, the thesis follows. ■

Remark 5.4. The noncollinearity condition will be used in the proof of Theorem 6.1 to guarantee the validity of condition (6.2).

The approximating maps in Lemma 5.3 can be suitably refined around the singular points as follows.

Lemma 5.5 (Density: behaviour near x_i, y_i). *Let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$ be such that* $\mathrm{Det}(\nabla u) = \pi\sum_{j=1}^n(\delta_{x_j} - \delta_{y_j})$*, with $x_j, y_j \in \overline{\Omega}$, $j = 1,\dots,n$. Let us assume that the points x_j, y_j which belong to Ω are distinct and three by three not collinear, as in the thesis of Lemma 5.3. Then, for all $\varepsilon > 0$, there exists a map $u_\varepsilon \in W^{1,1}(\Omega;\mathbb{S}^1)$ such that*

(i) *there exist positive numbers $\rho_{x_j} < \varepsilon$ and $\rho_{y_j} < \varepsilon$ such that the discs of the set $\{B_{\rho_{x_j}}(x_i), B_{\rho_{y_j}}(y_i) : x_j \in \Omega, y_i \in \Omega\}$ are contained in Ω, are pairwise disjoint, and we have $u_\varepsilon = e^{i\theta_{x_j}}$ in $B_{\rho_{x_j}}(x_j)$ and $u_\varepsilon = e^{-i\theta_{y_j}}$ in $B_{\rho_{y_j}}(y_j)$,*

(ii) $\|u - u_\varepsilon\|_{W^{1,1}} + \|\mathrm{Det}(\nabla u) - \mathrm{Det}(\nabla u_\varepsilon)\|_{\mathrm{flat}} < \varepsilon$.

Proof. Let $\{z_j : j = 1, \dots, N\}$ be the set, suitably relabelled, of those points among the x_j and the y_j which belong to Ω. Moreover, let $r > 0$ be small enough so that the discs $B_r(z_j)$ are contained in Ω and are pairwise disjoint. We can choose $r > 0$ arbitrarily small so that $u \llcorner \partial B_r(z_j) \in W^{1,1}(\partial B_r(z_j); \mathbb{S}^1)$ for all $j = 1, \dots, N$. We show how to modify u in one of these discs, say $B_r(z_1)$, and then proceed similarly for the other discs.

Let us assume, without loss of generality, that $z_1 = x_j = 0$ for some j (i.e., that z_1 is a positive pole at the origin), and write $B_r = B_r(0)$ in place of $B_r(x_j)$. Since in B_r we have $\mathrm{Det}(\nabla u) = \pi \delta_0$, it is not difficult to see that

$$\frac{1}{2} \int_{\partial B_s} \Big(u_1 \frac{\partial u_2}{\partial \tau} - u_2 \frac{\partial u_1}{\partial \tau} \Big) \, d\mathcal{H}^1 = \pi \deg(u; \partial B_r) = \pi \tag{5.14}$$

for all $s \in (0, r)$ such that $u \llcorner \partial B_s \in W^{1,1}(\partial B_s; \mathbb{S}^1)$. By the mean value theorem, we fix $d = d_j \in (r/2, r)$ so that

$$\int_{\partial B_d} |\nabla u| \, d\mathcal{H}^1 \le \frac{2}{r} \int_{r/2}^{r} \int_{\partial B_s} |\nabla u| \, d\mathcal{H}^1 ds = \frac{2}{r} \int_{B_r \setminus B_{r/2}} |\nabla u| \, dx, \tag{5.15}$$

and $u \llcorner \partial B_d \in W^{1,1}(\partial B_d; \mathbb{S}^1)$. Let $\theta_u \in \mathrm{BV}(\partial B_d)$ denote a lifting of $u \llcorner \partial B_d$ such that, owing to (5.14), θ_u has a unique jump point (say at $(d, 0) \in \partial B_d$) with $[\![\theta_u]\!] = 2\pi$. Consider a polar coordinate system (ρ, θ) around 0, and define $H \colon B_d \setminus \overline{B}_{d/2} \to \mathbb{R}$ as

$$H(x) := 2\theta_u\Big(d \frac{x}{|x|}\Big) \frac{|x| - d/2}{d} + 2\theta(x) \frac{d - |x|}{d}.$$

The function H has a jump of size 2π on the segment with endpoints $(d/2, 0)$ and $(d, 0)$. Also, $e^{iH} \in W^{1,1}(B_d \setminus \overline{B}_{d/2}; \mathbb{S}^1)$, and equals u on ∂B_d and $x/|x|$ on $\partial B_{d/2}$. We set

$$u_r(x) := \begin{cases} u(x) & \text{if } x \in \Omega \setminus B_d, \\ e^{iH(x)} & \text{if } x \in B_d \setminus \overline{B}_{d/2}, \\ x/|x| & \text{if } x \in B_{d/2}. \end{cases}$$

In particular, $u_r \in W^{1,1}(\Omega; \mathbb{S}^1)$. Let us estimate the gradient of H; we have

$$\begin{aligned} \nabla H(x) = {} & 2\dot{\theta}_u\Big(\frac{dx}{|x|}\Big)\left(\frac{\mathrm{Id} - \frac{x}{|x|} \otimes \frac{x}{|x|}}{|x|}\right)(|x| - d/2) \\ & + 2\theta_u\Big(\frac{dx}{|x|}\Big) \frac{x}{d|x|} + 2\nabla\theta(x) \frac{d - |x|}{d} - 2\theta(x) \frac{x}{d|x|}, \end{aligned}$$

where $\dot{\theta}_u$ denotes the (absolutely continuous part of the) derivative of θ_u. Therefore, using that $|x| \in (d/2, d)$ for $x \in B_d \setminus \overline{B}_{d/2}$, there is a constant $C > 0$ independent of d such that

$$|\nabla H(x)| \le 2 \left| \dot{\theta}_u\Big(\frac{dx}{|x|}\Big) \right| + \frac{C}{d}, \quad \text{for a.e. } x \in B_d \setminus \overline{B}_{d/2}.$$

On the other hand, since $e^{i\theta_u} = u$ on ∂B_d, we have $|\dot\theta_u(dx/|x|)| = |\nabla u(dx/|x|)|$, and integrating on $B_d \setminus \overline{B}_{d/2}$, we get

$$\begin{aligned}\int_{B_d\setminus \overline{B}_{d/2}} |\nabla u_r|\,dx &= \int_{B_d\setminus \overline{B}_{d/2}} |\nabla H|\,dx \le Cd + 2\int_{d/2}^{d}\int_{\partial B_s}\Big|\nabla u\Big(\frac{dx}{|x|}\Big)\Big|\,d\mathcal{H}^1(x)\,ds\\ &= Cd + C\int_{d/2}^{d}\int_{\partial B_d}|\nabla u|\,d\mathcal{H}^1\,ds \le Cd + C\int_{B_r\setminus \overline{B}_{r/2}}|\nabla u|\,dx,\end{aligned}$$

where we have used (5.15) and that $r/2 < d < r$ in the last inequality.

Now, applying a similar modification of u in the other discs centered at z_i, we can finally estimate the distance between u and u_r in $W^{1,1}(\Omega;\mathbb{R}^2)$, namely

$$\begin{aligned}&\|u-u_r\|_{L^1} \le n\pi r^2,\\ &\|\nabla u - \nabla u_r\|_{L^1} \le NCd + C\sum_{j=1}^{N}\int_{B_r(z_i)}|\nabla u|\,dx + \sum_{j=1}^{N}\int_{B_{d_j/2}(z_j)}\Big|\nabla\Big(\frac{x}{|x|}\Big)\Big|\,dx.\end{aligned}$$

Since r can be chosen arbitrarily small, the sum of the above right-hand sides can be bounded by ε, and for such r we denote $u_\varepsilon := u_r$. Observing that $\mathrm{Det}(\nabla u) = \mathrm{Det}(\nabla u_\varepsilon)$, the thesis follows by setting $\rho_{z_j} := d_j/2$, for all $j = 1,\dots,N$. ∎

6. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Recalling the definition of $\|\cdot\|_{\mathrm{flat},\alpha}$ in (1.2), we start with the following.

Theorem 6.1. *Let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$. Suppose that*

(i) *$\frac{1}{\pi}\mathrm{Det}(\nabla u) = \sum_{i=1}^{N}(\delta_{x_i} - \delta_{y_i}) =: T$ admits a representation in $\overline{\Omega}$ satisfying* (P$_\mathrm{f}$) *and such that the points x_i, y_i belonging to Ω are distinct and three by three not collinear*;

(ii) *there exists $R > 0$ such that the discs $B_R(x_i)$ and $B_R(y_j)$, with $x_i, y_j \in \Omega$, are contained in Ω and are pairwise disjoint, and we have that $u = e^{i\theta_{x_i}}$ in $B_R(x_i)$ and $u = e^{-i\theta_{y_j}}$ in $B_R(y_j)$.*

Then

$$\overline{\mathcal{A}}(u,\Omega) \le \int_\Omega \sqrt{1+|\nabla u|^2}\,dx + \|\mathrm{Det}(\nabla u)\|_{\mathrm{flat},\alpha}. \tag{6.1}$$

Proof. We need to exhibit[10] a sequence $(u_r) \subset C^1(\Omega;\mathbb{R}^2)$ converging to u in $L^1(\Omega;\mathbb{R}^2)$ such that $\liminf_{r\to 0^+}\mathcal{A}(u_r,\Omega)$ is less than or equal to the right-hand side of (6.1).

For the measure T, we consider currents $R_{\min} \in \mathcal{D}_0(\Omega)$ and $S_{\min} \in \mathcal{D}_1(\Omega)$ given by Lemma 3.6. After relabelling, we write

$$R_{\min} = \sum_{i=1}^{k}\sigma_i\delta_{z_i},\ \ \sigma_i \in \{-1,+1\},\ z_i \in \Omega,\ k \le N, \quad S_{\min} = \sum_{j\in J}[\![\overline{y_j x_j}]\!],\ \ J \subset \{1,\dots,N\},$$

[10] See (6.9) below.

with $T = R_{\min} + \partial S_{\min}$ (it may happen that $k = 0$, in which case we understand $R_{\min} = 0$, or that $J = \emptyset$, in which case $S_{\min} = 0$). By Lemma 3.6 (b), the segment $\overline{x_j y_j}$ is contained in Ω, with the only (possible) exception of an endpoint (thanks to condition (P$_\mathrm{f}$)). We will work in a disc $B \supset\supset \Omega$, that we fix from now on.

Take $r \in (0, R/2)$, and consider the set $\{B_{2r}(z_i) : i = 1, \dots, k\}$ (the z_i are among the x_j, y_j for $j \notin J$ and, being contained in Ω, satisfy assumption (ii)). These discs are contained in Ω, and the tubular neighborhoods

$$T_t(\overline{x_j y_j}) := \{x \in B : d(x, \overline{x_j y_j}) < t\}, \quad j \in J,$$

of $\overline{x_j y_j}$ are disjoint from this family of discs. Moreover, by hypothesis (i), due to non-collinearity, the segments $\overline{x_j y_j}$ are pairwise disjoint (see Lemma 3.6), and so for all $t > 0$ sufficiently small,

$$j_1, j_2 \in J,\ j_1 \neq j_2 \quad \Longrightarrow \quad T_t(\overline{x_{j_1} y_{j_1}}) \cap T_t(\overline{x_{j_2} y_{j_2}}) = \emptyset. \tag{6.2}$$

Set also

$$V_t := \bigcup_{j \in J} T_t(\overline{x_j y_j}). \tag{6.3}$$

If $k \geq 1$, for all $i = 1, \dots, k$, we fix a simple polygonal[11] curve γ_{z_i} starting at z_i and reaching the external boundary[12] of $\partial\Omega$. The curves γ_{z_i} can be chosen mutually disjoint, and disjoint from $\overline{V}_t$. Further, it is convenient to extend γ_{z_i} (keeping the same notation) in order to reach ∂B transversely. We set $T_{2r}(\gamma_{z_i}) := \{x \in B : d(x, \gamma_{z_i}) < 2r\}$, for $i = 1, \dots, k$, and observe that $B_{2r}(x_i) \subset T_{2r}(\gamma_{z_i})$ for all $i = 1, \dots, k$. If r is small enough, the elements of the family $\{T_{2r}(\gamma_{z_i}) : i = 1, \dots, k\}$ do not intersect each other, and moreover (choosing smaller t and r if necessary), $\overline{T_{2r}(\gamma_{z_i})} \cap \overline{V}_t = \emptyset$ for all $i = 1, \dots, k$. Consider also the connected curves $\gamma_{z_i}^{+,r}$ and $\gamma_{z_i}^{-,r}$, which run parallel to γ_{z_i} at distance r, defined as

$$\gamma_{z_i}^{\pm,r} := \{x \in T_{2r}(\gamma_{z_i}) \setminus B_r(z_i) : \overline{d}(x, \gamma_{z_i}) = \pm r\},$$

where $\overline{d}$ denotes a signed distance from γ_{z_i} (defined in a suitable neighborhood of γ_{z_i}). For every connected component $\partial_\ell\Omega$ of $\partial\Omega$ different from the external boundary ($\ell \in L$, with L some finite set of indices), we consider a simple polygonal curve $\omega_\ell \subset B$ connecting $\partial_\ell\Omega$ to ∂B, disjoint from $\overline{V}_t$, from $\bigcup_{i=1}^k \overline{T_{2r}(\gamma_{z_i})}$ and from $\partial_{\ell'}\Omega$, $\ell' \neq \ell$. Extending slightly ω_ℓ inside Ω_ℓ, where Ω_ℓ denotes the region outside Ω and enclosed by $\partial_\ell\Omega$, we assume that ω_ℓ starts at a point $\Delta_\ell \in \Omega_\ell$ with $B_{2r}(\Delta_\ell) \subset \Omega_\ell$, for all $\ell \in L$. Together with this, we consider the connected curves $\omega_\ell^{+,r}$ and $\omega_\ell^{-,r}$, which run parallel to ω_ℓ at distance r from each side, and join $\overline{B}_r(\Delta_\ell)$ with the external boundary:

$$\omega_\ell^{\pm,r} \subset \left\{x \in B \setminus B_r(\Delta_\ell) : \overline{d}(x, \omega_\ell) = \pm r\right\},$$

where, again, $\overline{d}$ denotes a signed distance from ω_ℓ (defined in a suitable neighborhood of ω_ℓ). We may assume, choosing smaller r and t if necessary, that the curves ω_ℓ and $\omega_\ell^{\pm,r}$ are pairwise disjoint and do not intersect $\overline{V}_t \cup \bigcup_{i=1}^k \overline{T_{2r}(\gamma_{z_i})} \cup \bigcup_{\ell' \neq \ell} \partial_{\ell'}\Omega$.

[11] I.e., not self-intersecting and obtained by a finite number of concatenations of segments.

[12] $\partial\Omega$ is, in general, not connected, and consists of a finite number of loops. The external boundary of $\partial\Omega$ is the loop whose interior contains all the others.

Finally, we define

$$B^{\pm} := B \setminus \Big[\bigcup_{i=1}^{k} \overline{B}_r(z_i) \cup \bigcup_{j\in J} \overline{x_j y_j} \cup \bigcup_{i=1}^{k} \gamma_{z_i}^{\pm,r} \cup \bigcup_{\ell\in L} \omega_\ell^{\pm,r} \cup \bigcup_{\ell\in L} B_r(\Delta_\ell) \Big].$$

Using our assumptions, it follows that B^+ and B^- are connected; however, they are not necessarily simply connected. By construction, for any closed simple Lipschitz curve $\alpha: \mathbb{S}^1 \to \Omega \cap B^{\pm}$ such that $u^\alpha := u \circ \alpha \in W^{1,1}(\mathbb{S}^1; \mathbb{S}^1)$, we have

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} (u_1^\alpha \nabla^\perp u_2^\alpha - u_2^\alpha \nabla^\perp u_1^\alpha) \cdot \dot\alpha \, ds = 0,$$

since the left-hand side is the degree of u on the boundary of the domain enclosed by the support of α, and such curves cannot enclose any connected component of $B \setminus \Omega$ due to the presence of $\bigcup_{\ell\in L} \omega_\ell$, and cannot enclose any single pole due to the presence of the γ_{z_i} (note that they can enclose some segment $\overline{x_j y_j}$). In particular, there exist two liftings[13] $w_\pm$ of u with

$$w_+ \in W^{1,1}(\Omega \cap B^+) \quad \text{and} \quad w_- \in W^{1,1}(\Omega \cap B^-).$$

For w_+ and w_-, we consider (not-relabelled) extensions $w_+ \in W^{1,1}(B^+)$ and $w_- \in W^{1,1}(B^-)$ as in (4.7). We are now going to suitably smoothen these liftings through a function w_r, that will allow us to eventually define the map u_r in (6.9).

We may assume that

$$\begin{aligned} & w_+ = w_- \\ & \text{on } B^+ \setminus \Big(\bigcup_{i=1}^{k} \overline{T_r(\gamma_{z_i})} \cup \bigcup_{\ell\in L} \overline{T_r(\omega_\ell)} \Big) = B^- \setminus \Big(\bigcup_{i=1}^{k} \overline{T_r(\gamma_{z_i})} \cup \bigcup_{\ell\in L} \overline{T_r(\omega_\ell)} \Big), \end{aligned} \tag{6.4}$$

where $T_r(\gamma_{z_i}) \subset \{x \in B : \overline{d}(x, \gamma_{z_i}) \in (-r, r)\}$ is the region enclosed by $\gamma_{z_i}^{+,r}$ and $\gamma_{z_i}^{-,r}$ and $T_r(\omega_\ell) := \{x \in B \setminus \bigcup_{\ell\in L} B_r(\Delta_\ell) : d(x, \omega_\ell) \in (-r, r)\}$ is the tubular neighborhood of ω_ℓ enclosed by $\omega_\ell^{+,r}$ and $\omega_\ell^{-,r}$. In addition, since the degree of u around z_i is σ_i, we see[14] that, for all $i = 1, \dots, k$,

$$[\![w_+]\!] = 2\pi\sigma_i \quad \mathcal{H}^1\text{-a.e. on } \gamma_{z_i}^{+,r} \quad \text{and} \quad [\![w_-]\!] = 2\pi\sigma_i \quad \mathcal{H}^1\text{-a.e. on } \gamma_{z_i}^{-,r}. \tag{6.5}$$

Furthermore,

$$[\![w_+]\!] = [\![w_-]\!] = 2\pi \quad \mathcal{H}^1\text{-a.e. on } \overline{x_j y_j}, \ \forall j \in J,$$

and

$$[\![w_\pm]\!] \in 2\pi\mathbb{Z} \quad \mathcal{H}^1\text{-a.e. on } \bigcup_{\ell\in L} \omega_\ell^{\pm,r}.$$

From (6.5), it follows that

$$w_+ = w_- + 2\pi\sigma_i \quad \text{a.e. in } T_r(\gamma_{z_i}), \ \forall i = 1, \dots, k. \tag{6.6}$$

[13] If $J = \emptyset$, i.e., no dipoles, and if Ω is simply connected, then we can take $w_+ = w_-$.

[14] The degree around a pole is computed using counterclockwise turns, and this implicitly determines an orientation of the jump of $w_\pm$.

Similarly, given $\ell \in L$, there exists $h_\ell \in \mathbb{Z}$ such that

$$w_+ = w_- + 2\pi h_\ell \quad \text{a.e. in } T_r(\omega_\ell). \tag{6.7}$$

Finally, we extend $w_\pm$ to 0 on $\bigcup_{i=1}^k \overline{B_r(z_i)} \cup \bigcup_{\ell \in L} \overline{B_r(\Delta_\ell)}$, and mollify $w_\pm$ using a kernel ϱ_r supported in $B_{r/4}(0)$. In particular, using also (6.6) and (6.7), we infer that the traces of the mollifications on γ_{z_i} and ω_ℓ satisfy

$$\begin{aligned} w_+ * \varrho_r &= w_- * \varrho_r + 2\pi\sigma_i \quad \mathcal{H}^1\text{-a.e. on } \gamma_{z_i}, \\ w_+ * \varrho_r &= w_- * \varrho_r + 2\pi h_\ell \quad \mathcal{H}^1\text{-a.e. on } \omega_\ell, \end{aligned}$$

and therefore, setting $B_r^- := \{x \in B : d(x, \partial B) > r\}$ and defining $w_r \colon B_r^- \setminus (\bigcup_{i=1}^k \gamma_{z_i} \cup \bigcup_{i=1}^k \overline{B_r(z_i)} \cup \bigcup_{\ell \in L} \omega_\ell \cup \bigcup_{\ell \in L} \overline{\Omega}_\ell) \to \mathbb{R}^2$ as

$$w_r := \begin{cases} w_+ * \varrho_r & \text{in } B_r^- \setminus (\bigcup_{i=1}^k T_r(\gamma_{z_i})) \setminus (\bigcup_{i=1}^k \overline{B_r(z_i)}) \setminus (\bigcup_\ell T_r(\omega_\ell)) \setminus (\bigcup_\ell \Omega_\ell), \\ w_- * \varrho_r & \text{in } \bigcup_{i=1}^k \{x \in T_r(\gamma_{z_i}) \setminus \overline{B_r(z_i)} : \bar{d}(x, \gamma_{z_i}) \in (0, r)\}, \\ w_+ * \varrho_r & \text{in } \bigcup_{i=1}^k \{x \in T_r(\gamma_{z_i}) \setminus \overline{B_r(z_i)} : \bar{d}(x, \gamma_{z_i}) \in (-r, 0)\}, \\ w_- * \varrho_r & \text{in } \bigcup_{\ell \in L} \{x \in T_r(\omega_\ell) \setminus \Omega_\ell : \bar{d}(x, \omega_\ell) \in (0, r)\} \\ w_+ * \varrho_r & \text{in } \bigcup_{\ell \in L} \{x \in T_r(\omega_\ell) \setminus \Omega_\ell : \bar{d}(x, \omega_\ell) \in (-r, 0)\}, \end{cases}$$

we see that $w_r \in C^\infty(B_r^- \setminus (\bigcup_{i=1}^k \gamma_{z_i} \cup \bigcup_{i=1}^k \overline{B_r(z_i)} \cup \bigcup_{\ell \in L} \omega_\ell) \cup \bigcup_{\ell \in L} \overline{\Omega}_\ell)$, and

$$\begin{aligned} [\![w_r]\!] &= 2\pi\sigma_i \quad \mathcal{H}^1\text{-a.e. on } \gamma_{z_i},\ i = 1, \dots, k, \\ [\![w_r]\!] &= 2\pi h_\ell \quad \mathcal{H}^1\text{-a.e. on } \omega_\ell,\ \ell \in L. \end{aligned} \tag{6.8}$$

Eventually, for all $i = 1, \dots, k$, by the assumptions on u and the choice of $r \in (0, R/2)$, we have $u(x) = e^{i\sigma_i \theta_{z_i}}$ for $x \in B_{2r}(z_i) \setminus \{z_i\}$ for suitable $\sigma_i \in \{\pm 1\}$.

Thus,

$$w_\pm - \sigma_i \theta_{z_i} \in 2\pi\mathbb{Z} \quad \text{in } B_{2r}(z_i) \setminus B_r(z_i).$$

Assuming, without loss of generality, that θ_{z_i} jumps on γ_{z_i} in $B_{2r}(z_i) \setminus B_r(z_i)$, by (6.4), for all $i = 1, \dots, k$ we find an integer ζ_i such that

$$w_+ = w_- = \sigma_i \theta_{z_i} + 2\pi\zeta_i \quad \text{in } B_{2r}(z_i) \setminus B_r(z_i) \setminus T_r(\gamma_{z_i}),$$

whereas in $B_{2r}(z_i) \cap T_r(\gamma_{z_i}) \setminus B_r(z_i)$, we have

$$\begin{aligned} w_+ &= \sigma_i \theta_{z_i} + 2\pi\zeta_i - 2\pi\sigma_i && \text{in } \{x : \bar{d}(x, \gamma_{z_i}) \in (0, r)\}, \\ w_+ &= \sigma_i \theta_{z_i} + 2\pi\zeta_i - 2\pi(\sigma_i + 1) && \text{in } \{x : \bar{d}(x, \gamma_{z_i}) \in (-r, 0)\}, \\ w_- &= \sigma_i \theta_{z_i} + 2\pi\zeta_i && \text{in } \{x : \bar{d}(x, \gamma_{z_i}) \in (0, r)\}, \\ w_- &= \sigma_i \theta_{z_i} + 2\pi(\zeta_i - 1) && \text{in } \{x : \bar{d}(x, \gamma_{z_i}) \in (-r, 0)\}. \end{aligned}$$

Therefore,

$$\begin{aligned} w_r - \sigma_i \theta_{z_i} * \varrho_r &\in 2\pi\mathbb{Z} && \text{in } (B_{5r/3}(z_i) \setminus B_{4r/3}(z_i)) \setminus T_r(\gamma_{z_i}), \\ w_r - \sigma_i \widehat{\theta}_{z_i} * \varrho_r &\in 2\pi\mathbb{Z} && \text{in } (B_{5r/3}(z_i) \setminus B_{4r/3}(z_i)) \cap T_r(\gamma_{z_i}), \end{aligned}$$

where $\sigma_i \widehat{\theta}_{z_i}$ is any lifting of $(x - z_i)/|x - z_i|$ which is continuous in $T_r(\gamma_{z_i})$.

We introduce a non-decreasing cut-off function $\psi\colon [0,2r] \to [0,1]$ of class C^1 such that $\psi = 0$ on $[0, 4r/3]$, $\psi = 1$ on $[5r/3, 2r]$, with $\psi' \leq 12/r$. Finally, we define

$$u_r(x) := \begin{cases} e^{iw_r(x)} & \text{if } x \in B_r^- \setminus \bigcup_{j=1}^k B_{2r}(z_j), \\ e^{iw_r(x)}\,\psi(|x-z_j|) & \text{if } x \in B_{2r}(z_j) \text{ for some } j = 1,\dots,k, \end{cases} \tag{6.9}$$

where we extend w_r in B to 0 outside its domain. In particular, we have that $u_r(x) = e^{i\sigma_i \widehat{\theta}_{z_i} * \varrho_r}\,\psi(|x-z_i|)$ for $x \in (B_{5r/3}(z_i) \setminus B_{4r/3}(z_i)) \cap T_r(\gamma_{z_i})$, for any $i = 1,\dots,k$. We also observe that if we suppose that the kernel ϱ_r is radial, a direct computation shows that $\widehat{\theta}_{z_i} * \varrho_r = \widehat{\theta}_{z_i}$ in $B_{5r/3}(z_i) \setminus B_{4r/3}(z_i)$. So

$$u_r(x) = e^{i\sigma_i \widehat{\theta}_{z_i}}\,\psi(|x-z_i|), \qquad \forall x \in (B_{5r/3}(z_i) \setminus B_{4r/3}(z_i)) \cap T_r(\gamma_{z_i}).$$

Now, u_r is of class C^1, $|u_r| \leq 1$, and it is straightforward to check that $u_r \to u$ pointwise almost everywhere in Ω as $r \to 0^+$. In particular, $\lim_{r\to 0^+} u_r = u$ in $L^1(\Omega;\mathbb{R}^2)$.

We are now in a position to estimate the graph area of the map u_r. In order to estimate it in $\Omega \setminus \bigcup_i \overline{B}_{2r}(x_i)$, it is convenient to consider a lifting

$$w \in W^{1,1}\Big(B \setminus \Big(\bigcup_{i=1}^k (\gamma_{z_i} \cup \overline{B}_r(z_i)) \cup \bigcup_{j\in J} \overline{x_j y_j}\Big) \cup \bigcup_{\ell\in L} (\omega_\ell \cup \overline{B}_r(\Delta_\ell))\Big),$$

which coincides with $w_\pm$ in the set in (6.4). Such a lifting $w \in \mathrm{BV}(\Omega \setminus \bigcup_{i=1}^k \overline{B}_r(z_i))$ satisfies

$$\begin{aligned} [\![w]\!] &= 2\pi\sigma_i && \mathcal{H}^1\text{-a.e. on } \gamma_{z_i},\ i = 1,\dots,k, \\ [\![w]\!] &= 2\pi && \mathcal{H}^1\text{-a.e. on } \overline{x_j y_j},\ j \in J, \\ [\![w]\!] &= 2\pi h_\ell && \mathcal{H}^1\text{-a.e. on } \gamma_{z_i},\ \ell \in L. \end{aligned}$$

Notice that $\lim_{r\to 0^+} w_r = w$ strictly in $\mathrm{BV}(V_t)$ (for t small enough as in (6.3)), since $w_r = w * \varrho_r$ on these sets. In particular, by classical results (see for instance Theorem 2.39 in [2]), one has

$$\begin{aligned} \int_{V_t} \sqrt{1+|\nabla u_r|^2}\,dx &= \int_{V_t} \sqrt{1+|\nabla w_r|^2}\,dx \to \int_{V_t} \sqrt{1+|\nabla w|^2}\,dx + |D^s w|(V_t) \\ &= \int_{V_t} \sqrt{1+|\nabla u|^2}\,dx + 2\pi \sum_{j\in J} |x_j - y_j|, \end{aligned} \tag{6.10}$$

as $r \to 0^+$. Concerning the integral over $\Omega \setminus V_t$, using that u_r takes values in $\mathbb{S}^1$ in $\Omega \setminus B_{2r}(z_i)$, we can estimate

(6.11)
$$\begin{aligned} &\int_{\Omega\setminus V_t} \sqrt{1+|\nabla u_r|^2+|\det(\nabla u_r)|^2}\,dx \leq \int_{\Omega\setminus V_t} \sqrt{1+|\nabla u_r|^2}\,dx + \int_{B_{2r}(z_i)} |\det(\nabla u_r)|\,dx \\ &= \int_{\Omega\setminus V_t\setminus B_{2r}(z_i)} \sqrt{1+|\nabla u_r|^2}\,dx + \int_{B_{2r}(z_i)} \sqrt{1+|\nabla u_r|^2}\,dx + \int_{B_{2r}(z_i)} |\det(\nabla u_r)|\,dx. \end{aligned}$$

Let us estimate the last term; so fix $i \in \{1, \ldots, k\}$, and assume, without loss of generality, that $\sigma_i = 1$. In $B_{2r}(z_i)$, we then have

$$u_r = (\cos(\theta_{z_i}), \sin(\theta_{z_i}))\, \psi(\rho_{z_i}),$$

where $(\rho_{z_i}, \theta_{z_i})$ is a polar coordinate system around z_i. Thus

$$\nabla u_r(\rho_{z_i}, \theta_{z_i}) = \begin{pmatrix} \psi'(\rho_{z_i}) \cos(\theta_{z_i}) & -\frac{\psi(\rho_{z_i})}{\rho_{z_i}} \sin(\theta_{z_i}) \\ \psi'(\rho_{z_i}) \sin(\theta_{z_i}) & \frac{\psi(\rho_{z_i})}{\rho_{z_i}} \cos(\theta_{z_i}) \end{pmatrix} \tag{6.12}$$

and therefore $\det(\nabla u_r(\rho_{z_i}, \theta_{z_i})) = \psi'(\rho_{z_i})\, \psi(\rho_{z_i})\, \rho_{z_i}^{-1}$, which gives

$$\int_{B_{2r}(z_i)} |\det(\nabla u_r)|\, dx = 2\pi \int_{4r/3}^{5r/3} \psi(\rho)\, \psi'(\rho)\, d\rho = \pi. \tag{6.13}$$

Moreover, (6.12) implies $|\nabla u_r| \leq C/|x - z_i|$ in $B_{2r}(z_i)$. In particular,

$$\int_{\bigcup_i B_{2r}(z_i)} \sqrt{1 + |\nabla u_r|^2}\, dx \to 0 \qquad \text{as } r \to 0^+.$$

Finally, due to the fact that $\nabla u_r \to \nabla u$ in $L^1(\Omega \setminus V_t; \mathbb{R}^{2\times 2})$, we infer

$$\int_{\Omega \setminus V_t \setminus \bigcup_i B_{2r}(z_i)} \sqrt{1 + |\nabla u_r|^2}\, dx \to \int_{\Omega \setminus V_t} \sqrt{1 + |\nabla u|^2}\, dx. \tag{6.14}$$

All in all, we have proved that the right-hand side in formula (6.11) tends to $k\pi + \int_{\Omega \setminus V_t} \sqrt{1 + |\nabla u|^2}\, dx$ as $r \to 0^+$. Thus, from (6.10) and (6.11), we get

$$\liminf_{r \to 0^+} \int_\Omega \sqrt{1 + |\nabla u_r|^2 + |\det(\nabla u_r)|^2}\, dx \leq \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + k\pi + 2\pi \sum_{j \in J} |x_j - y_j|,$$

which, in view of the results in Section 3 concerning $\|\cdot\|_{\mathrm{flat},\alpha}$, gives (6.1). ∎

We are now in a position to conclude the proof of Theorem 1.1 .

Proof of Theorem 1.1 . Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. In view of Proposition 5.1, we can pick a sequence $(u_k)_k \subset W^{1,1}(\Omega; \mathbb{S}^1)$ such that

(a) $\mathrm{Det}(\nabla u_k) = \pi \sum_{i=1}^{N_k} (\delta_{x_i^k} - \delta_{y_i^k})$ for some $N_k \in \mathbb{N}$, with each u_k satisfying (i) and (ii) of Proposition 5.1,

(b) $\|u - u_k\|_{W^{1,1}} + \|\mathrm{Det}(\nabla u) - \mathrm{Det}(\nabla u_k)\|_{\mathrm{flat}} < 1/k$, for all $k \in \mathbb{N}$.

Hence, owing to (a), we are in a position to apply Theorem 6.1 to each u_k, so that

$$\overline{\mathcal{A}}(u_k, \Omega) \leq \int_\Omega \sqrt{1 + |\nabla u_k|^2}\, dx + \|\mathrm{Det}(\nabla u_k)\|_{\mathrm{flat},\alpha}, \qquad \forall k > 0, \tag{6.15}$$

and therefore,

$$\begin{aligned} \overline{\mathcal{A}}(u, \Omega) &\leq \liminf_{k \to +\infty} \overline{\mathcal{A}}(u_k, \Omega) \leq \lim_{k \to +\infty} \left(\int_\Omega \sqrt{1 + |\nabla u_k|^2}\, dx + \|\mathrm{Det}(\nabla u_k))\|_{\mathrm{flat},\alpha} \right) \\ &= \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + \|\mathrm{Det}(\nabla u)\|_{\mathrm{flat},\alpha}. \end{aligned}$$

∎

7. On the countably subadditive interior envelope of $\overline{\mathcal{A}}$

As we have seen, the nonlocality of $\overline{\mathcal{A}}(u,\cdot)$ is unavoidable. Therefore, it seems interesting to consider the largest countably subadditive set function not larger than $\overline{\mathcal{A}}(u,\cdot)$, as defined in (1.6). We have the following integral representation result.

Proposition 7.1 ("Double" relaxation). *Let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$. Then*

$$\overline{\overline{\mathcal{A}}}(u,\Omega) = \int_\Omega \sqrt{1+|\nabla u|^2}\,dx. \tag{7.1}$$

Proof. Since $\overline{\mathcal{A}}(u,A) \geq \int_A \sqrt{1+|\nabla u|^2}\,dx$ for any open set $A \subseteq \Omega$, we only need to show the $\leq$ inequality in (7.1).

We already know, from Theorem 4.6, that $\Lambda := \frac{1}{\pi}\mathrm{Det}(\nabla u) = \sum_{i=1}^\infty (\delta_{x_i} - \delta_{y_i})$, with $\sum_{i=1}^\infty |x_i - y_i| < \infty$. Fix $\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ so that $\sum_{i=N_\varepsilon+1}^\infty |x_i - y_i| < \varepsilon$. Set

$$\Lambda_\varepsilon := \sum_{i=N_\varepsilon+1}^\infty (\delta_{x_i} - \delta_{y_i}).$$

Then, as $\|\Lambda_\varepsilon\|_{\mathrm{flat}} < \varepsilon$, we infer

$$\|\Lambda_\varepsilon\|_{\mathrm{flat},\alpha} < 2\varepsilon. \tag{7.2}$$

Let $\{z_k : k = 1,\dots,m_\varepsilon\}$ be the set of points in $\{x_k, y_k : k \leq N_\varepsilon\}$ which are contained in Ω. Choose mutually disjoint closed discs $\overline{B}_{2r}(z_k) \subset \Omega$ for $k = 1,\dots,m_\varepsilon$, and set

$$G_\varepsilon := \Omega \setminus \Big(\bigcup_{i=1}^{m_\varepsilon} \overline{B}_r(z_k)\Big);$$

notice that G_ε and $\overline{B}_{2r}(z_k)$ overlap on annuli of radii r and $2r$. By definition of $\overline{\overline{\mathcal{A}}}(u,\cdot)$, and using Theorem 1.1, we have

$$\begin{aligned}\overline{\overline{\mathcal{A}}}(u,\Omega) &\leq \overline{\mathcal{A}}(u,G_\varepsilon) + \sum_{k=1}^{m_\varepsilon} \overline{\mathcal{A}}(u,\overline{B}_{2r}(z_k)) \leq \int_{G_\varepsilon} \sqrt{1+|\nabla u|^2}\,dx + \|\mathrm{Det}(\nabla u)\|_{\mathrm{flat},\alpha,G_\varepsilon} \\ &+ \sum_{k=1}^{m_\varepsilon}\Big(\int_{\overline{B}_{2r}(z_k)} \sqrt{1+|\nabla u|^2}\,dx + \|\mathrm{Det}(\nabla u)\|_{\mathrm{flat},\alpha,\overline{B}_{2r}(z_k)}\Big).\end{aligned} \tag{7.3}$$

We claim that, for $r > 0$ sufficiently small,

$$\|\mathrm{Det}(\nabla u)\|_{\mathrm{flat},\alpha,G_\varepsilon} + \sum_{k=1}^{m_\varepsilon} \|\mathrm{Det}(\nabla u)\|_{\mathrm{flat},\alpha,\overline{B}_{2r}(z_k)} \leq 5\pi\varepsilon. \tag{7.4}$$

Since the discs $\overline{B}_{2r}(z_k)$ are mutually disjoint for $k \geq 1$, we see that

$$\|\Lambda_\varepsilon\|_{\mathrm{flat},\alpha,\bigcup_{k=1}^{m_\varepsilon}\overline{B}_{2r}(z_k)} = \sum_{k=1}^{m_\varepsilon} \|\Lambda_\varepsilon \llcorner \overline{B}_{2r}(z_k)\|_{\mathrm{flat},\alpha,\overline{B}_{2r}(z_k)}, \tag{7.5}$$

whereas, recalling (7.2),

$$\|\Lambda_\varepsilon\|_{\mathrm{flat},\alpha,G_\varepsilon} \leq \|\Lambda_\varepsilon\|_{\mathrm{flat},\alpha} < 2\varepsilon. \tag{7.6}$$

In $\overline{B}_{2r}(z_k)$, connecting z_k to $\partial B_{2r}(z_k)$ with a segment, we see that

$$\|h_k\,\delta_{z_k}\|_{\mathrm{flat},\alpha} \leq 2|h_k|r, \quad \text{for } k = 1,\dots,m_\varepsilon,$$

where $h_k \in \mathbb{Z}$ denotes the multiplicity of z_k in the distribution $\Lambda - \Lambda_\varepsilon = \sum_{k=1}^{N_\varepsilon}(\delta_{x_k} - \delta_{y_k})$. On the other hand, by construction, $(\Lambda - \Lambda_\varepsilon) \llcorner \overline{B}_{2r}(z_k) = h_k\,\delta_{z_k}$, and therefore,

$$\Lambda \llcorner \overline{B}_{2r}(z_k) = \Lambda_\varepsilon \llcorner \overline{B}_{2r}(z_k) + h_k\,\delta_{z_k},$$

which implies

$$\|\Lambda\|_{\mathrm{flat},\alpha,\overline{B}_{2r}(z_k)} \leq \|\Lambda_\varepsilon \llcorner \overline{B}_{2r}(z_k)\|_{\mathrm{flat},\alpha} + 2|h_k|r_k.$$

Summing over $k = 1,\dots,m_\varepsilon$, by (7.5) one gets

$$\begin{aligned}\sum_{k=1}^{m_\varepsilon}\|\Lambda\|_{\mathrm{flat},\alpha,\overline{B}_{2r}(z_k)} &\leq \sum_{k=1}^{m_\varepsilon}\|\Lambda_\varepsilon \llcorner \overline{B}_{2r}(z_k)\|_{\mathrm{flat},\alpha,\overline{B}_{2r}(z_k)} + \sum_{k=1}^{m_\varepsilon} 2|h_k|r \\ &= \|\Lambda_\varepsilon\|_{\mathrm{flat},\alpha,\bigcup_{k=1}^{m_\varepsilon}\overline{B}_{2r}(z_k)} + \sum_{k=1}^{m_\varepsilon} 2|h_k|r \leq 2\varepsilon + \sum_{k=1}^{m_\varepsilon} 2|h_k|r,\end{aligned} \tag{7.7}$$

where the last inequality follows from (7.2), since $\|\Lambda_\varepsilon\|_{\mathrm{flat},\alpha,\bigcup_{k=1}^{m_\varepsilon}\overline{B}_{2r}(z_k)} \leq \|\Lambda_\varepsilon\|_{\mathrm{flat},\alpha}$. From (7.6), we conclude

$$\|\mathrm{Det}(\nabla u)\|_{\mathrm{flat},\alpha,G_\varepsilon} + \sum_{k=1}^{m_\varepsilon}\|\mathrm{Det}(\nabla u)\|_{\mathrm{flat},\alpha,\overline{B}_{2r}(z_k)} \leq 4\pi\varepsilon + \pi\sum_{k=1}^{m_\varepsilon} 2|h_k|r,$$

for any $r > 0$ small enough, and (7.4) follows.

Now, from (7.3), we conclude that for every $r > 0$ sufficiently small, we have

$$\begin{aligned}\overline{\overline{\mathcal{A}}}(u,\Omega) &\leq \int_{G_\varepsilon}\sqrt{1+|\nabla u|^2}\,dx + \sum_{k=1}^{m_\varepsilon}\Big(\int_{\overline{B}_{2r}(z_k)}\sqrt{1+|\nabla u|^2}\,dx\Big) + 5\pi\varepsilon \\ &= \int_\Omega\sqrt{1+|\nabla u|^2}\,dx + \sum_{k=1}^{m_\varepsilon}\int_{B_{2r}(z_k)\setminus\overline{B}_r(z_k)}\sqrt{1+|\nabla u|^2}\,dx + 5\pi\varepsilon,\end{aligned}$$

which in turn (since $\nabla u \in L^1(\Omega;\mathbb{R}^{2\times 2})$) implies, letting $r \to 0^+$,

$$\overline{\overline{\mathcal{A}}}(u,\Omega) \leq \int_\Omega\sqrt{1+|\nabla u|^2}\,dx + 5\pi\varepsilon.$$

As ε is arbitrary, we have $\overline{\overline{\mathcal{A}}}(u,\Omega) \leq \int_\Omega\sqrt{1+|\nabla u|^2}\,dx$. This concludes the proof. ∎

As a direct consequence of Proposition 7.1, we get the following.

Corollary 7.2 (Integral representation). *Let $u \in W^{1,1}(\Omega;\mathbb{S}^1)$. Then the set function $E \mapsto \overline{\overline{\mathcal{A}}}(u,E)$ defines a Borel measure absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^2$, and coincides with*

$$\overline{\overline{\mathcal{A}}}(u,E) = \int_E\sqrt{1+|\nabla u|^2}\,dx \quad \textit{for all Borel sets } E \subseteq \Omega.$$

In order to prove Theorem 1.2, we now extend Theorem 6.1 to the case of open sets obtained from Ω by removing a finite set of points.

Theorem 7.3. *Let $C := \{c_1, \dots, c_M\}$ be a finite set of distinct points of Ω. Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ satisfy* (i) *and* (ii) *of Theorem* 6.1*, and suppose that*

$$
\begin{cases} \operatorname{dist}(x_i, \partial\Omega) \neq |x_i - c_k|, \\ \operatorname{dist}(y_i, \partial\Omega) \neq |y_i - c_k|, \end{cases} \qquad \forall k = 1, \dots, M, \ \forall i = 1, \dots, N, \tag{7.8}
$$

and that the points x_i, y_i, c_k, $i = 1, \dots, N$, $k = 1, \dots, M$, are three by three not collinear. Then

$$
\overline{\mathcal{A}}(u, \Omega \setminus C) \leq \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + \|\operatorname{Det}(\nabla u)\|_{\mathrm{flat},\alpha,\Omega \setminus C}.
$$

Proof. Fix $\bar\rho > 0$ small enough so that the discs $\overline{B}_{2\bar\rho}(c_k)$, $k = 1, \dots, M$, are contained in Ω, are mutually disjoint and, for each x_i or $y_i \in \Omega$, we have $x_i, y_i \in \Omega \setminus (\bigcup_{k=1}^M \overline{B}_{2\bar\rho}(c_k))$, respectively. For all $\rho \in (0, 2\bar\rho)$, let us denote $\Omega_\rho := \Omega \setminus (\bigcup_{k=1}^M \overline{B}_\rho(c_k))$, and let $R^\rho_{\min} \in \mathcal{R}_f(\Omega_\rho)$ and $S^\rho_{\min} \in \mathcal{S}(\Omega_\rho)$ be minimal currents as in Lemma 3.6, with $T = \frac{1}{\pi}\operatorname{Det}(\nabla u)$. In particular, $R^\rho_{\min} = \sum_{i=1}^h \sigma_i \delta_{z_i}$ (notice that, possibly reducing $\bar\rho > 0$, $R^\rho_{\min} = R_{\min}$ becomes independent of ρ), $S^\rho_{\min} = \sum_{j \in J} [\![\overline{p_j q_j}]\!]$, and there might be points $p_j = p_j^\rho$ or $q_j = q_j^\rho$ on $\partial B_\rho(c_k)$ for some j and k. However, since by assumption the points x_i, y_i, c_k are three by three not collinear, it is easy to see that the points in $\{p_j, q_j \in \partial B_\rho(c_k)$, for some $j, k\}$, if any, are distinct. In particular, the segments $\overline{p_j q_j}$, $j \in J$, are pairwise disjoint. Finally, as a consequence of (7.8), we may assume that if $\eta^\rho \in \partial B_\rho(c_k)$ is one of the points in the set $\{p_j, q_j \in \partial B_\rho(c_k)$ for some $j, k\}$, then

$$
\eta^\rho \to c_k \quad \text{as } \rho \to 0^+. \tag{7.9}
$$

Using $R_{\min}$ and $S^\rho_{\min}$, we can now consider the sequence $(u^\rho_r) \subset C^1(\Omega_\rho; \mathbb{R}^2)$ found in the proof of Theorem 6.1 (see (6.9)), with Ω replaced by Ω_ρ; so, for any $m \in \mathbb{N}$, we find $r_m \in (0, 2\bar\rho)$ small enough so that $v^\rho_m := u^\rho_{r_m}$ satisfies

$$
\begin{aligned} &\|v^\rho_m - u\|_{L^1(\Omega_\rho;\mathbb{R}^2)} \leq \frac{1}{m}, \\ &\mathcal{A}(v^\rho_m, \Omega_\rho) \leq \int_{\Omega_\rho} \sqrt{1 + |\nabla u|^2}\, dx + \|\operatorname{Det}(\nabla u)\|_{\mathrm{flat},\alpha,\Omega_\rho} + \frac{1}{m}. \end{aligned} \tag{7.10}
$$

Furthermore, as $\lim_{m\to+\infty} v^\rho_m = u$ in $W^{1,1}(\Omega_\rho; \mathbb{R}^2)$, we may also suppose

$$
\int_{\Omega_\rho \setminus \overline{\Omega}_{2\rho}} |\nabla v^\rho_m|\, dx \leq \int_{\Omega_\rho \setminus \overline{\Omega}_{2\rho}} |\nabla u|\, dx + \frac{1}{m}. \tag{7.11}
$$

Now we suitably modify v^ρ_m and extend it to $\Omega \setminus C$: for any $k = 1, \dots, M$, we pick a radius $s_k \in (\rho, 2\rho)$ so that

$$
\int_{\partial B_{s_k}(c_k)} |\nabla v^\rho_m|\, d\mathcal{H}^1 \leq \frac{1}{\rho} \int_{B_{2\rho}(c_k) \setminus \overline{B}_\rho(c_k)} |\nabla v^\rho_m|\, dx. \tag{7.12}
$$

Then we define

$$
\bar v^\rho_m(x) := \begin{cases} v^\rho_m(x) & \text{if } x \in \Omega_{s_k}, \\ v^\rho_m\left(c_k + s_k \frac{x - c_k}{|x - c_k|}\right) & \text{if } x \in B_{s_k}(c_k) \setminus \{c_k\}, \quad k = 1, \dots, M, \end{cases} \tag{7.13}
$$

where, with a little abuse of notation, we have denoted $\Omega_{s_k} := \Omega \setminus (\bigcup_{k=1}^M B_{s_k}(c_k))$. From (7.11), we get

$$\int_{\Omega_{s_k}\setminus\overline{\Omega}_{2\rho}} |\nabla \bar{v}_m^\rho|\, dx = \int_{\Omega_{s_k}\setminus\overline{\Omega}_{2\rho}} |\nabla v_m^\rho|\, dx \le \int_{\Omega_{\rho}\setminus\overline{\Omega}_{2\rho}} |\nabla v_m^\rho|\, dx \le \int_{\Omega_{\rho}\setminus\overline{\Omega}_{2\rho}} |\nabla u|\, dx + \frac{1}{m}\cdot$$

On the other hand, for any $k = 1, \dots, M$,

$$\int_{B_{s_k}(c_k)} |\nabla \bar{v}_m^\rho|\, dx = \int_0^{s_k} \int_{\partial B_s(c_k)} |\nabla \bar{v}_m^\rho|\, d\mathcal{H}^1\, ds = s_k \int_{\partial B_{s_k}(c_k)} |\nabla v_m^\rho|\, d\mathcal{H}^1,$$

where the last equality follows since, by definition, $\bar{v}_m^\rho$ is 0-homogeneous in $B_{s_k}(c_k)$, and the integral $\int_{\partial B_s(c_k)} |\nabla \bar{v}_m^\rho|\, d\mathcal{H}^1$ does not depend on $s \in (0, s_k)$. Using (7.12), and since $s_k \le 2\rho$, it then follows

$$\begin{aligned} \int_{\Omega\setminus\overline{\Omega}_{2\rho}} |\nabla \bar{v}_m^\rho|\, dx &= \int_{\Omega_{s_k}\setminus\overline{\Omega}_{2\rho}} |\nabla \bar{v}_m^\rho|\, dx + \sum_{k=1}^N \int_{B_{s_k}(c_k)} |\nabla \bar{v}_m^\rho|\, dx \\ &\le 2\int_{\Omega_{\rho}\setminus\overline{\Omega}_{2\rho}} |\nabla u|\, dx + \frac{2}{m}\cdot \end{aligned} \tag{7.14}$$

Furthermore, using that $|v_m^\rho| \le 1$, also $|\bar{v}_m^\rho| \le 1$, and it easily follows that

$$\limsup_{m\to+\infty} \|\bar{v}_m^\rho - u\|_{L^1(\Omega;\mathbb{R}^2)} \le \limsup_{m\to+\infty} \|\bar{v}_m^\rho - u\|_{L^1(\Omega\setminus\Omega_{2\rho};\mathbb{R}^2)} \le 8\pi M\rho^2. \tag{7.15}$$

Since in $\Omega_\rho \setminus \overline{\Omega}_{2\rho}$ the map v_m^ρ takes values in $\mathbb{S}^1$, using (7.10) and (7.14) we can estimate

$$\begin{aligned} \mathcal{A}(\bar{v}_m^\rho, \Omega\setminus C) &\le \mathcal{A}\Big(\bar{v}_m^\rho, \Omega \setminus \Big(\bigcup_{k=1}^M B_{s_k}(c_k)\Big)\Big) + \sum_{k=1}^M \pi s_k^2 + 2\int_{\Omega_{\rho}\setminus\Omega_{2\rho}} |\nabla u|\, dx + \frac{2}{m} \\ &\le \int_{\Omega_\rho} \sqrt{1+|\nabla u|^2}\, dx + \|\mathrm{Det}(\nabla u)\|_{\mathrm{flat},\alpha,\Omega_\rho} + 2\int_{\Omega_{\rho}\setminus\Omega_{2\rho}} |\nabla u|\, dx + \frac{3}{m}\cdot \end{aligned} \tag{7.16}$$

We have proved that for every $\rho > 0$ small enough, we can find a Lipschitz map $\bar{v}_m^\rho : \Omega \setminus C \to \mathbb{R}^2$ satisfying (7.16). In particular, choosing a sequence $\rho_h \searrow 0$, by a diagonal argument we find a sequence (v_h) of Lipschitz maps[15] on $\Omega \setminus C$ to $\mathbb{R}^2$ satisfying $v_h \to u$ in $L^1(\Omega;\mathbb{R}^2)$ (by (7.15)), and

$$\mathcal{A}(v_h, \Omega\setminus C) \le \int_{\Omega_{\rho_h}} \sqrt{1+|\nabla u|^2}\, dx + \|\mathrm{Det}(\nabla u)\|_{\mathrm{flat},\alpha,\Omega_{\rho_h}} + 2|Du|(\Omega_{\rho_h}\setminus\overline{\Omega}_{2\rho_h}) + \frac{3}{h}\cdot$$

Letting $h \to +\infty$, we use that $\mathrm{Det}(\nabla u)$ is a measure so that it is easy to see that, thanks to (2.7),

$$\|\mathrm{Det}(\nabla u)\|_{\mathrm{flat},\alpha,\Omega_{\rho_h}} \to \|\mathrm{Det}(\nabla u)\|_{\mathrm{flat},\alpha,\Omega\setminus C},$$

and the thesis easily follows. ∎

[15]Even if v_h is not C^1, by a density argument finding such a sequence is sufficient, see Proposition 3.5 in [3].

Using Theorem 7.3 and a density argument, we can prove the following.

Theorem 7.4. *Let C be a finite set of distinct points of Ω, and let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. Then*

$$\overline{\mathcal{A}}(u, \Omega \setminus C) \le \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + \|\mathrm{Det}(\nabla u)\|_{\mathrm{flat},\alpha,\Omega \setminus C}.$$

Proof. It is sufficient to argue along the lines of the proof of Theorem 1.1, replacing, in (6.15), $\|\cdot\|_{\mathrm{flat},\alpha}$ by $\|\cdot\|_{\mathrm{flat},\alpha,\Omega\setminus C}$. More specifically, in view of Proposition 5.1, we can pick a sequence $(u_k)_k \subset W^{1,1}(\Omega; \mathbb{S}^1)$ satisfying (a) and (b) of the proof of Theorem 1.1 (see the end of Section 6), and also (7.8). Applying Theorem 7.3 to each u_k, we obtain

$$\overline{\mathcal{A}}(u_k, \Omega \setminus C) \le \int_\Omega \sqrt{1 + |\nabla u_k|^2}\, dx + \|\mathrm{Det}(\nabla u_k)\|_{\mathrm{flat},\alpha,\Omega\setminus C}, \quad \forall k > 0.$$

By lower-semicontinuity, we conclude

$$\begin{aligned}\overline{\mathcal{A}}(u, \Omega) &\le \liminf_{k\to+\infty} \overline{\mathcal{A}}(u_k, \Omega) \le \lim_{k\to+\infty} \Big(\int_\Omega \sqrt{1+|\nabla u_k|^2}\, dx + \|\mathrm{Det}(\nabla u_k))\|_{\mathrm{flat},\alpha,\Omega\setminus C}\Big) \\ &= \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + \|\mathrm{Det}(\nabla u))\|_{\mathrm{flat},\alpha}.\end{aligned}$$

The equality is obtained since by (b), $u_k \to u$ in $W^{1,1}(\Omega; \mathbb{R}^2)$, $\|\mathrm{Det}(\nabla u) - \mathrm{Det}(\nabla u_k)\|_{\mathrm{flat}} \to 0$, and $\|\cdot\|_{\mathrm{flat},\alpha}$ is continuous in the flat metric. ∎

Theorem 7.5. *Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. Then for every $\varepsilon > 0$, there exists a finite set C_ε of points of Ω such that*

$$\overline{\mathcal{A}}(u, \Omega \setminus C_\varepsilon) \le \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + \varepsilon. \tag{7.17}$$

Proof. We know that $\frac{1}{\pi}\mathrm{Det}(\nabla u) = \sum_{i=1}^\infty (\delta_{x_i} - \delta_{y_i})$ with $\sum_{i=1}^\infty |x_i - y_i| < \infty$. Take $N_\varepsilon \in \mathbb{N}$ so that $\sum_{i=N_\varepsilon+1}^\infty |x_i - y_i| < \varepsilon/(2\pi)$, and let $C_\varepsilon := \{x_k \in \Omega : 1 \le k \le N_\varepsilon\} \cup \{y_k \in \Omega : 1 \le k \le N_\varepsilon\}$. Set $T := \mathrm{Det}(\nabla u) \llcorner (\Omega \setminus C_\varepsilon)$, so that $\|T\|_{\mathrm{flat},\alpha,\Omega\setminus C_\varepsilon} \le \varepsilon$. Then Theorem 7.4 implies (7.17). ∎

Using Theorem 7.5, we can positively answer to a modification of a conjecture by De Giorgi [18], adapted to the context of $\mathbb{S}^1$-valued Sobolev maps.

Corollary 7.6. *Let $u \in W^{1,1}(\Omega; \mathbb{S}^1)$. Then*

$$\overline{\overline{\mathcal{A}}}(u, \Omega) = \inf_{C\subset\Omega, \mathcal{H}^0(C)<+\infty} \overline{\mathcal{A}}(u, \Omega \setminus C).$$

Proof. From Theorem 7.5, we get

$$\inf_{C\subset\Omega, \mathcal{H}^0(C)<+\infty} \overline{\mathcal{A}}(u, \Omega \setminus C) \le \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx = \overline{\overline{\mathcal{A}}}(u, \Omega),$$

where the last equality follows from Corollary 7.2. On the other hand, for any finite set $C \subseteq \Omega$, we know [1] that

$$\overline{\mathcal{A}}(u, \Omega \setminus C) \ge \int_{\Omega\setminus C} \sqrt{1 + |\nabla u|^2}\, dx = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx = \overline{\overline{\mathcal{A}}}(u, \Omega). \qquad ∎$$

A. Appendix

In this appendix, we first collect a standard lemma, and next a proposition having an independent interest.

Lemma A.1. *Let* $\Lambda \in \mathrm{Lip}_0(\Omega)'$. *Then*

$$\sup_{\substack{\varphi\in C_c^1(\Omega)\\ \|\varphi\|_{\mathrm{Lip}_0,\alpha}\le 1}} \langle \Lambda,\varphi\rangle = \inf\left\{|R|_\Omega + \alpha^{-1}|S|_\Omega : (R,S)\in \mathcal{D}_0(\Omega)\times\mathcal{D}_1(\Omega),\ \Lambda = R+\partial S\right\}$$

$$= \sup_{\substack{\varphi\in \mathrm{Lip}_0(\Omega)\\ \|\varphi\|_{\mathrm{Lip}_0,\alpha}\le 1}} \langle \Lambda,\varphi\rangle. \tag{A.1}$$

Proof. We adapt the arguments of [19], see p. 367. Let $R\in\mathcal{D}_0(\Omega)$ and $S\in\mathcal{D}_1(\Omega)$ be such that $\Lambda = R+\partial S$ in $\mathcal{D}_0(\Omega)$. Then, as $\alpha = 1/2$,

$$|\langle\Lambda,\varphi\rangle| \le |R(\varphi)| + |S(d\varphi)| \le (|R|_\Omega + 2|S|_\Omega)\,\|\varphi\|_{\mathrm{Lip}_0,\alpha},\qquad \forall\varphi\in C_c^1(\Omega),$$

so the inequality $\le$ holds in the first line of (A.1). To prove the converse inequality, set

$$Y := \left\{(\varphi,\psi)\in C_c^1(\Omega)\times C_c^0(\Omega;\mathbb{R}^2)\right\},$$

endowed with the norm $\|(\varphi,\psi)\|_Y := \max\{\|\varphi\|_{L^\infty}, \frac{1}{2}\|\psi\|_{L^\infty}\}$, and define the linear injective operator

$$Q : C_c^1(\Omega)\to Y,\quad Q(\varphi) := (\varphi,\nabla\varphi),\quad \forall\varphi\in C_c^1(\Omega).$$

Since $Q(C_c^1(\Omega))\subset Y$, we have

$$\langle\Lambda, Q^{-1}(\varphi,\nabla\varphi)\rangle \le \|\Lambda\|_{\mathrm{flat},\alpha}\,\|(\varphi,\nabla\varphi)\|_Y,\qquad \forall\varphi\in C_c^1(\Omega),$$

and therefore we can extend the linear functional $\Lambda\circ Q^{-1}\colon Q(C_c^1(\Omega))\to\mathbb{R}$ to some linear functional $L\colon Y\to\mathbb{R}$ with

$$L(\varphi,\psi)\le \|\Lambda\|_{\mathrm{flat},\alpha}\,\|(\varphi,\psi)\|_Y,\qquad \forall(\varphi,\psi)\in Y. \tag{A.2}$$

Now we define

$$\begin{aligned} R(\varphi) &:= L(\varphi,0), && \forall\varphi\in C_c^1(\Omega),\\ S(\psi) &:= L(0,\psi), && \forall\psi\in C_c^0(\Omega;\mathbb{R}^2),\end{aligned}$$

so that, from (A.2), $\|(\varphi,\psi)\|_Y\le 1$ implies $R(\varphi)+S(\psi) = L(\varphi,\psi)\le\|\Lambda\|_{\mathrm{flat},\alpha}$. In particular, $R\in\mathcal{D}_0(\Omega)$ and $S\in\mathcal{D}_1(\Omega)$, and passing to the supremum,

$$|R|_\Omega + 2|S|_\Omega \le \|\Lambda\|_{\mathrm{flat},\alpha}.$$

Since $R(\varphi)+S(d\varphi) = L(\varphi,\nabla\varphi) = \langle\Lambda,\varphi\rangle$ for all $\varphi\in C_c^1(\Omega)$, it follows $\Lambda = R+\partial S$, and the first equality in (A.1) follows.

To show the second equality in (A.1), we first observe that, from the first equality,

$$\sup_{\substack{\varphi\in C_c^1(\Omega)\\ \|\varphi\|_{\mathrm{Lip}_0,\alpha}\le 1}} \langle \Lambda,\varphi\rangle \le \|\Lambda\|_{\mathrm{flat},\alpha},$$

and so $\le$ holds. On the other hand, if $R\in R_f$ and $S\in\mathcal{S}$ are such that $\Lambda = R+\partial S$ in $\mathcal{D}_0(\Omega)$, then

$$|\langle\Lambda,\varphi\rangle| = |R(\varphi)| + |S(d\varphi)| \le (|R|_\Omega + 2|S|_\Omega)\,\|\varphi\|_{\mathrm{Lip}_0,\alpha}, \quad \forall\varphi\in\mathrm{Lip}_0(\Omega),$$

so also the inequality $\le$ holds, thanks to Corollary 3.4. ■

The next result has been used in the proof of Proposition 3.3; it is based on Lemma 3.6.

Proposition A.2. *Let $T=\sum_{i=1}^n(\delta_{x_i}-\delta_{y_i})\in\mathcal{D}'(\Omega)$ be as in (3.3) and satisfying ($\mathrm{P_f}$), where $x_i, y_i\in\overline{\Omega}$, $x_i\ne y_i$. Let I_P, I_D, τ, $R_{\min}$ and $S_{\min}$ be as in Lemma 3.6. Then*

$$\exists\varphi\in\mathrm{Lip}_0(\Omega)\ \textit{ with }\ \|\varphi\|_{\mathrm{Lip}_0,\alpha}\le 1\ \textit{ such that }\ \langle T,\varphi\rangle = |R_{\min}|_\Omega + \alpha^{-1}|S_{\min}|_\Omega.$$

As a consequence, for all $k\in I_P$ and $j\in I\setminus\tau(I_D)$ with $x_k\in\Omega$ and $y_j\in\Omega$, we have $\varphi(x_k)=-\varphi(y_j)=1$, and for all $k\in I_D$, we have $\varphi(x_k)-\varphi(y_{\tau(k)})=\alpha^{-1}|x_k-y_{\tau(k)}|$. In particular,

$$\min\{|R|_\Omega+\alpha^{-1}|S|_\Omega : R\in\mathcal{R}_f,\ S\in\mathcal{S},\ T=R+\partial S\} = \max_{\substack{\phi\in\mathrm{Lip}_0(\Omega)\\ \|\phi\|_{\mathrm{Lip}_0,\alpha}\le 1}}\langle T,\phi\rangle.$$

Proof. Define

$$P^+ := \{k\in I_P : x_k\in\Omega\} \quad\text{and}\quad P^- := \{k\in I\setminus\tau(I_D) : y_k\in\Omega\}.$$

The function φ in the statement must satisfy $\varphi(x_k)=1$ for all $k\in P^+$, and $\varphi(y_k)=-1$ for all $k\in P^-$, and, recalling that $\alpha^{-1}=2$, also $\varphi(x_k)-\varphi(y_{\tau(k)})=2|x_k-y_{\tau(k)}|$ for all $k\in I_D$.

For any $k\in P^+$, we define $\phi_k(x):=1-2|x-x_k|$ for all $x\in\overline{\Omega}$, and set

$$\phi := \begin{cases} \max_{k\in P^+}\{\phi_k\} & \text{if } P^+\ne\emptyset,\\ -1 & \text{if } P^+=\emptyset.\end{cases}$$

Define also $\Psi_0:=\max\{d_0,\phi\}$, where $d_0(x):=\max\{-1,-2d(x,\partial\Omega)\}$ for all $x\in\overline{\Omega}$, i.e.,[16]

$$\Psi_0(x) = \max\{-1,-2d(x,\partial\Omega), 1-2|x-x_k|, k\in P^+\}, \quad \forall x\in\overline{\Omega}, \tag{A.3}$$

and observe that $\Psi_0\in\mathrm{Lip}_0(\Omega)$ with $\|\Psi_0\|_{\mathrm{Lip}_0,\alpha}\le 1$.

Using the minimality, and in particular (3.11), one verifies that $\Psi_0(x_k)=1$ for all $k\in P^+$. Let us check that $\Psi_0(y_k)=-1$ for all $k\in P^-$. If not, either $d_0(y_k)>-1$ or $\phi(y_k)>-1$, and both the two cases are excluded again by (3.11).

Now, we have to take into account the dipoles (i.e., $k\in I_D$) and the boundary values of Ψ_0. We divide the proof into three steps.

[16] Here we assume $P^+\ne\emptyset$, otherwise the quantity $1-2|x-x_k|$ does not appear as $\phi\equiv -1$.

Step 1. For all $k \in I_D$ with $y_{\tau(k)}, x_k \in \Omega$,

$$\Psi_0(x_k) \leq 1 \quad \text{and} \quad \Psi_0(y_{\tau(k)}) \leq 1 - 2|x_k - y_{\tau(k)}|. \tag{A.4}$$

Furthermore, if either $y_{\tau(k)} \in \partial\Omega$ or $x_k \in \partial\Omega$ for some $k \in I_D$, then

$$\text{either} \;\; \Psi_0(x_k) \leq 2d(x_k, \partial\Omega), \quad \text{or} \;\; \Psi_0(y_{\tau(k)}) = -2d(y_{\tau(k)}, \partial\Omega), \tag{A.5}$$

respectively.

Let us check (A.4). The first inequality follows since $\Psi_0 \leq 1$ on $\overline{\Omega}$. The second inequality is deduced as follows. By (A.3), if $\Psi_0(y_{\tau(k)}) = -2d(x_k, \partial\Omega)$, then we conclude by (3.12). If instead $\Psi_0(y_{\tau(k)}) = 1 - 2|x_h - y_{\tau(k)}|$ for some $h \in P^+$, then we conclude, since by minimality $|x_h - y_{\tau(k)}| \geq |x_k - y_{\tau(k)}|$ (where we have used $h \in P^+$).

Now, let us check (A.5). Assume that $y_{\tau(k)} \in \partial\Omega$ and, by (A.3), that for some $h \in P^+$, $\Psi_0(x_k) = 1 - 2|x_h - x_k|$. By the triangle inequality and (3.11), we have

$$\Psi_0(x_k) \leq 1 + 2|x_k - y_{\tau(k)}| - 2|x_h - y_{\tau(k)}| \leq 2|x_k - y_{\tau(k)}| = 2d(x_k, \partial\Omega).$$

Assume instead that $x_k \in \partial\Omega$. If, by contradiction, $\Psi_0(y_{\tau(k)}) > -2d(y_{\tau(k)}, \partial\Omega) = -2|y_{\tau(k)} - x_k|$, for some $h \in P^+$ we necessarily have $\Psi_0(y_{\tau(k)}) = 1 - 2|x_h - y_{\tau(k)}| > -2|y_{\tau(k)} - x_k|$. This contradicts the minimality of $R_{\min}$ and $S_{\min}$, because a direct check shows that

$$\begin{aligned} |R'|_\Omega + 2|S'|_\Omega &\leq |R_{\min}|_\Omega + 2|S_{\min}|_\Omega - 1 + 2|x_h - y_{\tau(k)}| - 2|x_k - y_{\tau(k)}| \\ &< |R_{\min}|_\Omega + 2|S_{\min}|_\Omega, \end{aligned}$$

where $R' := R_{\min} - \delta_{x_h}$ and $S' := S_{\min} - [\![\overline{y_{\tau(k)}\, x_k}]\!] + [\![\overline{y_{\tau(k)} x_h}]\!]$.

Eventually, we check that

$$\Psi_0 = 0 \quad \text{on } \partial\Omega. \tag{A.6}$$

Indeed, if $x \in \partial\Omega$ and $\Psi_0(x) = 1 - 2|x_k - x| > 0$ for some $k \in P^+$, then arguing as before, we can define $R' := R_{\min} - \delta_{x_k}$ and $S' := S_{\min} + [\![\overline{x x_k}]\!]$, and a direct check shows that $|R'|_\Omega + 2|S'|_\Omega < |R_{\min}|_\Omega + 2|S_{\min}|_\Omega$, against the minimality.

Before proceeding to the next step, for the sake of simplicity and without loss of generality, we relabel the indices and assume that $\tau\colon I_D \to I_D$ is the identity map, so that $I_P = I \setminus I_D = I \setminus \tau(I_D)$. The function φ, that will be constructed starting from Ψ_0 (see (A.15)), should satisfy $\varphi(x_k) = \Psi_0(x_k) = 1$ for all $k \in P^+$, $\varphi(y_k) = \Psi_0(y_k) = -1$ for all $k \in P^-$, and $\varphi(x_k) - \varphi(y_k) = 2|x_k - y_k|$ for every $k \in I_D$. To build such a φ, we apply a recursive procedure. We define, for all integers $m \geq 1$, the function Ψ_m as follows:

$$\Psi_m(x) := \max\{\Psi_{m-1}(x), \Phi_m(x)\}, \quad \forall x \in \overline{\Omega}, \tag{A.7}$$

where Φ_m is given by

$$\Phi_m(x) := \max_{k \in I_D}\{\phi_k^m(x)\}, \quad \phi_k^m(x) := \Psi_{m-1}(y_k) + 2|x_k - y_k| - 2|x_k - x|, \quad \forall x \in \overline{\Omega}.$$

Clearly, Φ_m is Lipschitz continuous with Lipschitz constant 2, for all $m \geq 1$.

Step 2. We claim that

(a) $\Psi_m(y_k) \le 1 - 2|x_k - y_k|$ and $\Psi_m(x_k) \le 1$ for all $k \in I_D$ with $x_k, y_k \in \Omega$;

(b) $\Psi_m(x_k) = 1$ and $\Psi_m(y_h) = -1$ for all $k \in P^+, h \in P^-$;

(c) $\Psi_m(x) = 0$ for $x \in \partial\Omega$;

(d) if, for some $k \in I_D$, either $y_k \in \partial\Omega$ or $x_k \in \partial\Omega$, then

$$\text{either } \Psi_m(x_k) \le 2d(x_k, \partial\Omega), \quad \text{or } \Psi_m(y_k) = -2d(y_k, \partial\Omega), \tag{A.8}$$

respectively.

Proof of (a). First we notice that, if $\{h_1, \dots, h_j\} \subseteq I_D$ with $h_i \ne h_{i'}$ for $i \ne i'$, then the minimality of $S_{\min}$ implies that

$$\sum_{i=1}^{j} |x_{h_i} - y_{h_i}| \le |x_{h_1} - y_{h_2}| + |x_{h_2} - y_{h_3}| + \cdots + |x_{h_j} - y_{h_1}|. \tag{A.9}$$

Now, by construction and definition of Φ_m, we can find a set of $r \ge 0$ indices (possibly $r = 0$) $0 = m_1 < \cdots < m_r \le m$, and indices $k_1, \dots, k_r \in I_D$ such that

$$\begin{aligned}
\Psi_m(y_k) &= \phi_k^m(y_k) = \Psi_{m_r}(y_{k_r}) + 2|x_{k_r} - y_{k_r}| - 2|x_{k_r} - y_k|,\\
\Psi_{m_r}(y_{k_r}) &= \phi_{k_r}^{m_r}(y_{k_r}) = \Psi_{m_{r-1}}(y_{k_{r-1}}) + 2|x_{k_{r-1}} - y_{k_{r-1}}| - 2|x_{k_{r-1}} - y_{k_r}|,\\
&\ \,\vdots\\
\Psi_{m_3}(y_{k_3}) &= \phi_{k_3}^{m_3}(n_{k_3}) = \Psi_{m_2}(y_{k_2}) + 2|x_{k_2} - y_{k_2}| - 2|x_{k_2} - y_{k_3}|,\\
\Psi_{m_2}(y_{k_2}) &= \phi_{k_2}^{m_2}(y_{k_2}) = \Psi_0(y_{k_1}) + 2|x_{k_1} - y_{k_1}| - 2|x_{k_1} - y_{k_2}|.
\end{aligned} \tag{A.10}$$

Notice that if $r = 0$, we simply have $\Phi_m(y_k) = \Phi_0(y_k)$, and (a) follows from (A.4) and thanks to $\Psi_0 \in \mathrm{Lip}_0(\Omega)$, $\|\Psi_0\|_{\mathrm{Lip}_0,\alpha} \le 1$. If $r > 0$, from (A.10) it follows that

$$\Psi_m(y_k) = \Psi_0(y_{k_1}) + 2\sum_{i=1}^{r} |x_{k_i} - y_{k_i}| - 2\sum_{i=1}^{r} |x_{k_i} - y_{k_{i+1}}|,$$

where we have set $y_{k_{r+1}} = y_k$. Now we have two cases:

(a1) $\Psi_0(y_{k_1}) = -2d(y_{k_1}, \partial\Omega)$;

(a2) $\Psi_0(y_{k_1}) = 1 - 2|x_h - y_{k_1}|$ for some $h \in P^+$.

In case (a1), we will show that

$$2\sum_{i=1}^{r} |x_{k_i} - y_{k_i}| - 2\sum_{i=1}^{r} |x_{k_i} - y_{k_{i+1}}| \le 1 + 2d(\partial\Omega, y_{k_1}) - 2|x_k - y_k|, \tag{A.11}$$

whereas in case (a2), we will show that

$$2\sum_{i=1}^{r} |x_{k_i} - y_{k_i}| - 2\sum_{i=1}^{r} |x_{k_i} - y_{k_{i+1}}| \le 2|x_h - y_{k_1}| - 2|x_k - y_k|, \tag{A.12}$$

and this will conclude (a).

First, in view of (3.12), we can assume that the indices $k_1, k_2, \dots, k_r$ are all distinct. Indeed, if $k_i = k_{i'}$ for some $i \ne i'$, then we can erase the indices $k_i, k_{i+1}, \dots, k_{i'-1}$, since $\sum_{j=i}^{i'-1} |x_{k_j} - y_{k_j}| - \sum_{j=i}^{i'-1} |x_{k_j} - y_{k_{j+1}}| \le 0$ by (3.12).

Therefore, assuming (a1), let us prove (A.11). Consider the point p on $\partial\Omega$ so that $|p - y_{k_1}| = d(y_{k_1}, \partial\Omega)$; then (A.11) is a consequence of the minimality of $R_{\min}$ and $S_{\min}$. Indeed, setting

$$R' := R_{\min} + \delta_{x_k} \quad \text{and} \quad S' := S_{\min} - \sum_{i=1}^{r+1} [\![\overline{x_{k_i} y_{k_i}}]\!] + \sum_{i=1}^{r} [\![\overline{x_{k_i} y_{k_{i+1}}}]\!] + [\![\overline{p y_{k_1}}]\!],$$

the inequality $|R'|_\Omega + 2|S'|_\Omega \geq |R_{\min}|_\Omega + 2|S_{\min}|_\Omega$ is equivalent to (A.11).

In case (a2), instead, to get (A.12), arguing as before, it suffices to write $|R'|_\Omega + 2|S'|_\Omega \geq |R_{\min}|_\Omega + 2|S_{\min}|_\Omega$, with

$$R' := R_{\min} + \delta_{y_{k_1}} + \delta_{x_h} \quad \text{and} \quad S' := S_{\min} - \sum_{i=1}^{r+1} [\![\overline{x_{k_i} y_{k_i}}]\!] + \sum_{i=1}^{r} [\![\overline{x_{k_i} y_{k_{i+1}}}]\!] + [\![\overline{x_h y_{k_1}}]\!].$$

So far, we have proved that $\Psi_m(y_k) \leq 1 - 2|x_k - y_k|$, for all $k \in I_D$. The fact that $\Psi_m(x_k) \leq 1$ for all $k \in I_D$ follows, since $\Psi_m \in \mathrm{Lip}_0(\Omega)$, with $\|\Psi_m\|_{\mathrm{Lip}_0,\alpha} \leq 1$.

Proof of (b). Let us check that $\Psi(p_h) \leq 1$ for all $h \in P^+$. To this aim, it is sufficient to observe that $\Phi_m(x) \leq 1$ for all $x \in \overline{\Omega}$, since $\phi_k^m(x) \leq \phi_k^m(x_k) = \Psi_{m-1}(y_k) + 2|x_k - y_k| \leq 1$, for all $k \in I_D$, in view of (a).

Let us check that $\Psi(x_h) = -1$ for all $h \in P^-$. Arguing as in (A.10), we can find a set of $r \geq 0$ indices $0 = m_1 < \cdots < m_r \leq m$, and indices $k_1, \ldots, k_r \in I_D$, such that

$$\Psi_m(y_h) = \Psi_0(y_{k_1}) + 2\sum_{i=1}^{r} |x_{k_i} - y_{k_i}| - 2\sum_{i=1}^{r} |x_{k_i} - y_{k_{i+1}}|,$$

where $k_{r+1} := h$. If $r = 0$, we readily conclude that $\Psi_m(y_h) = \Psi_0(y_h)$, and the thesis follows from the fact that $\Psi_0(y_h) = -1$ for all $h \in P^-$. Hence assume $r > 0$. Here we have two cases as well:

(b1) $\Psi_0(y_{k_1}) = 1 - 2|x_j - y_{k_1}|$ for some $j \in P^+$;

(b2) $\Psi_0(y_{k_1}) = -2d(y_{k_1}, \partial\Omega)$.

In the first case, the thesis is equivalent to

$$1 - 2|x_j - y_{k_1}| + 2\sum_{i=1}^{r} |x_{k_i} - y_{k_i}| - 2\sum_{i=1}^{r} |x_{k_i} - y_{k_{i+1}}| \leq -1. \tag{A.13}$$

As before, we might assume that the indices k_i are distinct. Writing $R' := R_{\min} + \delta_{x_h} - \delta_{y_j}$ and $S' := S_{\min} - \sum_{i=1}^{r} [\![\overline{x_{k_i} y_{k_i}}]\!] + \sum_{i=1}^{r} [\![\overline{x_{k_i} y_{k_{i+1}}}]\!] + [\![\overline{x_j y_{k_1}}]\!]$, (A.13) readily follows by the inequality $|R'|_\Omega + 2|S'|_\Omega \geq |R_{\min}|_\Omega + 2|S_{\min}|_\Omega$.

Now, if (b2) holds, we will conclude by showing that

$$-2d(y_{k_1}, \partial\Omega) + 2\sum_{i=1}^{r} |x_{k_i} - y_{k_i}| - 2\sum_{i=1}^{r} |x_{k_i} - y_{k_{i+1}}| \leq -1. \tag{A.14}$$

This is also obtained using the minimality of $R_{\min}$ and $S_{\min}$, setting $R' := R_{\min} + \delta_{y_h}$ and $S' := S_{\min} - \sum_{i=1}^{r} [\![\overline{x_{k_i} y_{k_i}}]\!] + \sum_{i=1}^{r} [\![\overline{x_{k_i} y_{k_{i+1}}}]\!] + [\![\overline{x_j y_{k_1}}]\!]$, where $x_j \in \partial\Omega$ is such that $|x_j - y_{k_1}| = d(y_{k_1}, \partial\Omega)$.

Proof of (c). To show this, fix $x \in \partial\Omega$. If $\Psi_m(x) = \Psi_0(x) = 0$, there is nothing to prove. If not, we can find a set of $r > 0$ indices $0 = m_1 < \cdots < m_r \le m$, and indices $k_1, \dots, k_r \in I_D$, such that

$$\Psi_m(x) = \Psi_0(y_{k_1}) + 2\sum_{i=1}^{r} |x_{k_i} - y_{k_i}| - 2\sum_{i=1}^{r} |x_{k_i} - y_{k_{i+1}}|,$$

where $y_{k_{r+1}} := x$. If $\Psi_0(y_{k_1}) = -2d(y_{k_1}, \partial\Omega)$, we show that

$$\sum_{i=1}^{r} |x_{k_i} - y_{k_i}| \le \sum_{i=1}^{r} |x_{k_i} - y_{k_{i+1}}| + d(y_{k_1}, \partial\Omega).$$

As usual, we might assume that the indices k_i are distinct; since $x \in \partial\Omega$, we have that $|x_{k_r} - x| \ge d(x_{r_k}, \partial\Omega)$, and so the previous inequality is obtained by minimality of $R_{\min}$ and $S_{\min}$, arguing similarly as in the preceding cases.

If instead $\Psi_0(y_{k_1}) = 1 - 2|x_h - y_{k_1}|$ for some $h \in P^+$, we reduce ourselves to prove that

$$1 + \sum_{i=1}^{r} |x_{k_i} - y_{k_i}| \le \sum_{i=1}^{r} |x_{k_i} - y_{k_{i+1}}| + |x_h - y_{k_1}|,$$

which is again implied by the minimality of $R_{\min}$ and $S_{\min}$.

Proof of (d). The first condition in (A.8) is a consequence of point (c) and the fact that $\Psi_m \in \mathrm{Lip}_0(\Omega)$, with $\|\Psi_m\|_{\mathrm{Lip}_0,\alpha} \le 1$. Let us prove the second condition. If $\Psi_m(y_k) = \Psi_0(y_k)$, then the thesis follows from (A.5); if not, we can find a sequence of $r > 0$ indices $0 = m_1 < \cdots < m_r \le m$, and indices $k_1, \dots, k_r \in I_D$, such that

$$\Psi_m(y_k) = \Psi_0(y_{k_1}) + 2\sum_{i=1}^{r} |x_{k_i} - y_{k_i}| - 2\sum_{i=1}^{r} |x_{k_i} - y_{k_{i+1}}|,$$

where $k_{r+1} := k$. Now, either $\Psi_0(y_{k_1}) = -2d(y_{k_1}, \partial\Omega)$ or $\Psi_0(y_{k_1}) = 1 - 2|x_h - y_{k_1}|$ for some $h \in P^+$. Again, assuming that the indices k_i are distinct, in the first case it is sufficient to observe that

$$d(y_k, \partial\Omega) + \sum_{i=1}^{r} |x_{k_i} - y_{k_i}| \le d(\partial\Omega, y_{k_1}) + \sum_{i=1}^{r} |x_{k_i} - y_{k_{i+1}}|,$$

which follows once more from the minimality of $R_{\min}$ and $S_{\min}$, since $d(y_k, \partial\Omega) = |x_k - y_k|$, and $x_k \in \partial\Omega$.

Assuming instead that $\Psi_0(y_{k_1}) = 1 - 2|x_h - y_{k_1}|$ for some $h \in P^+$, we can show that

$$1 + d(y_k, \partial\Omega) + \sum_{i=1}^{r} |x_{k_i} - y_{k_i}| \le \sum_{i=1}^{r} |x_{k_i} - y_{k_{i+1}}| + |x_h - y_{k_1}|,$$

using once again the minimality of $R_{\min}$ and $S_{\min}$.

Before passing to the next step, observe that since $\Psi_m \le \Psi_{m+1}$ for all $m \ge 0$, we can take the pointwise limit

$$\varphi := \lim_{m \to +\infty} \Psi_m, \tag{A.15}$$

and thus conditions (a), (b), (c), and (d) of Step 2 are still valid for φ. Furthermore, since Ψ_m is Lipschitz continuous with Lipschitz constant 2, we also have that $\Psi_m \to \varphi$ uniformly in $\overline{\Omega}$ by the Ascoli–Arzelà theorem. The next step concludes the proof of the proposition.

Step 3. The function φ satisfies

$$\varphi(x_k) = \varphi(y_k) + 2|x_k - y_k|, \quad \forall k \in I_D. \tag{A.16}$$

To see this, we define

$$\overline{\varphi} := \max_{k \in I_D}\{g_k\}, \quad g_k(x) := \varphi(y_k) + 2|x_k - y_k| - 2|x_k - x|, \quad \forall k \in I_D, \ \forall x \in \overline{\Omega}.$$

In order to prove (A.16), it is sufficient to show that $\varphi \geq \overline{\varphi}$; indeed, this implies $\varphi(x_k) \geq g_k(x_k) = \varphi(y_k) + 2|x_k - y_k|$ for all $k \in I_D$, and since the opposite inequality is guaranteed by the fact that φ is 2-Lipschitz, (A.16) follows. By the uniform convergence of the sequence (Ψ_m) to φ, for all $\varepsilon > 0$, we can find $m_\varepsilon \in \mathbb{N}$ so that $\Psi_m(x) + \varepsilon \geq \varphi(x)$ for all $x \in \overline{\Omega}$ and $m \in \mathbb{N}$ with $m \geq m_\varepsilon$. We compute

$$\begin{aligned} g_k(x) &= \varphi(y_k) + 2|x_k - y_k| - 2|x_k - x| \leq \varepsilon + \Psi_m(x) + 2|x_k - y_k| - 2|x_k - x| \\ &\leq \varepsilon + \Psi_{m+1}(x) \leq \varepsilon + \varphi(x), \end{aligned}$$

where the last but one inequality follows from the definition of Ψ_{m+1}. This implies that $\overline{\varphi}(x) \leq \varepsilon + \varphi(x)$ which, by the arbitrariness of $\varepsilon > 0$, implies the claim. ∎

Remark A.3. If one knows in advance the regularity result

$$\|T\|_{\mathrm{flat},\alpha} = \min\{|R|_\Omega + \alpha^{-1}|S|_\Omega : (R, S) \in \mathcal{R}_f \times \mathcal{S}, T = R + \partial S\},$$

since

$$\|T\|_{\mathrm{flat},\alpha} = \max_{\substack{\varphi \in \mathrm{Lip}_0(\Omega) \\ \|\varphi\|_{\mathrm{Lip}_0,\alpha} \leq 1}} \langle T, \varphi\rangle = \langle T, \overline{\varphi}\rangle,$$

it is not difficult to check that a maximizing $\overline{\varphi}$ satisfies the properties of the function φ in the proof of Proposition A.2.

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Giovanni Bellettini
Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena
Vía Roma 56, 53100 Siena;
Mathematics Section, The Abdus Salam International Centre for Theoretical Physics ICTP
Strada Costiera 11, 34151 Trieste, Italy;
giovanni.bellettini@unisi.it

Riccardo Scala
Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena
Vía Roma 56, 53100 Siena, Italy;
riccardo.scala@unisi.it

Giuseppe Scianna
Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena
Vía Roma 56, 53100 Siena, Italy;
giuseppe.scianna@unisi.it