Rev. Mat. Iberoam. 40 (2024), no. 6, 2325–2338
DOI 10.4171/RMI/1491

A characterization of homogeneous three-dimensional CR manifolds

Jih-Hsin Cheng, Andrea Malchiodi and Paul Yang

Abstract. We characterize homogeneous three-dimensional CR manifolds, in particular Rossi spheres, as critical points of a certain energy functional that depends on the Webster curvature and torsion of the pseudohermitian structure.

1. Introduction and statement of the results

Let (M,ξ) be a contact 3-manifold with contact structure ξ. A CR structure on (M,ξ) is an endomorphism $J\colon \xi \to \xi$ satisfying $J^2 = \mathrm{Id}$. With a choice of contact form θ (i.e., $\theta|_\xi = 0$, $\theta \wedge d\theta \neq 0$) such that $d\theta(\cdot, J\cdot) > 0$, (J,θ) is called a (strictly pseudoconvex) pseudohermitian structure. To such a structure one associates the (Tanaka–Webster) scalar curvature R and the torsion tensor A_{11} with norm $|A|^2_{J,\theta} := h^{1\bar{1}}h^{1\bar{1}}A_{11}A_{\bar{1}\bar{1}}$ (see for instance [13] for basic pseudohermitian geometry). A pseudohermitian automorphism is a diffeomorphism preserving the pseudohermitian structure. We call a pseudohermitian manifold homogeneous if the group of its pseudohermitian automorphisms acts transitively. On $(M,\xi) = (S^3,\hat{\xi})$, the standard contact 3-sphere, there exists a family of distinguished homogeneous pseudohermitian structures $(J_{(s)},\hat{\theta})$, called Rossi spheres, where $\hat{\theta}$ is the standard contact form on $(S^3,\hat{\xi})$. See Subsection 2.1 for a detailed description.

We recall that Rossi spheres ($s \neq 0$) are the simplest examples of non-embeddable CR 3-manifolds which (two to one) cover embeddable ones in $\mathbb{C}^3$, see [5], pp. 324-325. Apart from the non-embeddability property, Rossi spheres provide counterexamples in conformal pseudohermitian geometry. In relation to the problem of existence of minimizers for the CR Yamabe problem, each Rossi sphere $(J_{(s)},\hat{\theta})$, $s \neq 0$, has negative pseudohermitian mass, as defined in [7], for s close to 0, while the infimum of the CR Sobolev quotient coincides with the one for the standard 3-sphere ($s = 0$), but is not attained [8]. The notion of pseudo-Einstein contact form plays an important role in CR geometry. Geometrically, it is characterized by a volume-normalization condition (Theorem 3.3 in [4]), while analytically in dimension 3 it relates R to A_{11} in their first covariant

Mathematics Subject Classification 2020: 32V20 (primary); 53C17 (secondary).
Keywords: homogeneous CR 3-manifold, Rossi sphere, pseudohermitian structure, pseudo-Einstein contact form.

derivatives as follows:

$$R_{,1} = iA_{11,\bar{1}} \tag{1.1}$$

which is taken to be the definition of a pseudo-Einstein contact form in dimension 3 (see Definition 3.1 in [4]; cf. (3.1)). Equation (1.1) is useful in simplifying the expressions involving R and A_{11}. Among others, one can equate the Burns–Epstein invariant to the total Q'-curvature (up to a negative constant), see Theorem 1.2 on pp. 290–291 of [4].

In this paper, we exhibit a functional whose critical points (in both J and θ) characterize homogeneous pseudohermitian manifolds among pseudo-Einstein ones. Precisely, define the following energy functional:

$$E(J,\theta) := \int_M (R^2 - |A|^2_{J,\theta})\theta \wedge d\theta. \tag{1.2}$$

Our main result is as follows.

Theorem 1.1. *On a real analytic, closed (i.e., compact with no boundary) contact 3-manifold (M,ξ), suppose that (J,θ) is a real analytic, pseudo-Einstein critical point of (1.2). Then the universal cover $\tilde{M}$ of M with the structure naturally inherited from (ξ, J, θ), still denoted by the same notation, is homogeneous as a pseudohermitian 3-manifold.*

In the Riemannian case, a variational characterization of space forms was given in [10]. It is well known that homogeneous CR 3-manifolds have been classified by Cartan ([3]; see also [2]). In particular, when $(M,\xi) = (S^3, \hat{\xi})$, we have the following characterization for Rossi spheres.

Corollary 1.2. *On the standard contact 3-sphere $(S^3, \hat{\xi})$, it holds that (J,θ) is a real analytic, pseudo-Einstein critical point of (1.2) if and only if J is isomorphic to a Rossi sphere $J_{(s)}$ for some unique $s \le 0$, and θ is a constant multiple of $\hat{\theta}$.*

In Section 2, after a short review of Rossi spheres in Subsection 2.1 and basic variation formulas in Subsection 2.2, we derive the Euler–Lagrange equation (2.15) for the critical points of the energy functional $E(J,\theta)$. In Section 3, we show that from (2.15) and (J,θ) being pseudo-Einstein, it follows that (M,ξ,J,θ) (or (M,J,θ) with ξ omitted) is locally sub-symmetric through Proposition 3.1 and Lemma 3.2. In Section 3, we first prove Theorem 1.1. Then for $(M,\xi) = (S^3,\hat{\xi})$, we conclude Corollary 1.2 by a result in [2]. We remark that $\iota\colon (z^1,z^2) \to (iz^1,z^2)$ is a pseudohermitian isomorphism between $(J_{(s)},\hat{\theta})$ and $(J_{(-s)},\hat{\theta})$, see [8]. In Section 4, for its independent interest, we compute the second variation of the energy functional $E(J,\theta)$ at a critical point $(\hat{J},\hat{\theta})$.

Theorem 1.3. *With the notation above, the formulas for the second variations $\delta_J^2 E(\hat{J},\hat{\theta})$, $\delta_\theta\delta_J E(\hat{J},\hat{\theta})$ $(= \delta_J\delta_\theta E(\hat{J},\hat{\theta}))$ and $\delta_\theta^2 E(\hat{J},\hat{\theta})$ are given in (4.5), (4.7) and (4.8), respectively.*

We remark that there is no characterization in general on sign of the second variation. In fact, using for example individual Fourier components/modes for the variations of J and θ as in [1], it is possible to find deformations along which the second differential is either positive or negative at the standard S^3. See (A.8) and (A.12) for examples in Appendix A.

2. Rossi spheres, variations and energy functional

2.1. Rossi spheres

In this subsection, we give a short introduction to Rossi spheres. See Section 2.2 in [8] for more details. The standard contact form $\hat{\theta}$ on $S^3 := \{(z^1, z^2) \in C^2, |z^1|^2 + |z^2|^2 = 1\}$ reads as

$$\hat{\theta} = i(\bar{\partial} - \partial)(|z^1|^2 + |z^2|^2) = i \sum_{k=1}^{2} (z^k dz^{\bar{k}} - z^{\bar{k}} dz^k). \tag{2.1}$$

Note that $\hat{\theta}$ is SU(2)-invariant, where SU(2) acts on $\mathbb{C}^2$ in the canonical way. Dual to $\theta^1 = z^2 dz^1 - z^1 dz^2$, we have

$$Z_1 = z^{\bar{2}} \frac{\partial}{\partial z^1} - z^{\bar{1}} \frac{\partial}{\partial z^2}.$$

Note that Z_1 (respectively, $Z_{\bar{1}}$) is also SU(2)-invariant.

Consider the deformation of CR structures described by giving type (0, 1) vectors as follows:

$$Z_{\bar{1}(s)} = Z_{\bar{1}} + \frac{i E_{\bar{1}\bar{1}} s}{\sqrt{1 + |E_{11} s|^2}} Z_1,$$

where $E_{\bar{1}\bar{1}}$ is a deformation tensor associated to the CR structure J (cf. Subsection 2.2). Note that we use the notation $E_{\bar{1}\bar{1}}$ instead of $E_{\bar{1}}^{1}$ for convenience/simplicity; for a unitary frame/coframe, they are equal. The derivative of $Z_{\bar{1}(s)}$ in s reads as

$$\dot{Z}_{1(s)} = \frac{-i E_{11} Z_{\bar{1}}}{(1 + |E_{11}|^2 s^2)^{3/2}}. \tag{2.2}$$

We express Z_1 and $Z_{\bar{1}}$ in terms of $Z_{1(s)}$ and $Z_{\bar{1}(s)}$ as follows:

$$\begin{aligned} Z_1 &= i E_{11} s \sqrt{1 + |E_{11} s|^2}\, Z_{\bar{1}(s)} + (1 + |E_{11} s|^2)\, Z_{1(s)}, \\ Z_{\bar{1}} &= (-i)\, E_{\bar{1}\bar{1}}\, s \sqrt{1 + |E_{11} s|^2}\, Z_{1(s)} + (1 + |E_{11} s|^2)\, Z_{\bar{1}(s)}. \end{aligned} \tag{2.3}$$

Substituting the second equality of (2.3) into (2.2) gives

$$\dot{Z}_{1(s)} = \frac{-|E_{11}|^2 s \sqrt{1 + |E_{11} s|^2}\, Z_{1(s)} - i E_{11} (1 + |E_{11} s|^2)\, Z_{\bar{1}(s)}}{(1 + |E_{11}|^2 s^2)^{3/2}}. \tag{2.4}$$

Differentiating $J_{(s)} Z_{1(s)} = i Z_{1(s)}$ with respect to s, we get

$$\dot{J}_{(s)} Z_{1(s)} + J_{(s)} \dot{Z}_{1(s)} = i \dot{Z}_{1(s)}. \tag{2.5}$$

Substituting (2.4) into (2.5) and writing $\dot{J}_{(s)} = 2 E_{11}^{(s)} \theta^1_{(s)} \otimes Z_{\bar{1}(s)} +$ conjugate, we obtain

$$E_{11}^{(s)} = \frac{E_{11}}{\sqrt{1 + |E_{11} s|^2}}.$$

Therefore we have

$$\dot{E}_{11}^{(s)} = -E_{11}|E_{11}|^2 s(1+|E_{11}|^2 s^2)^{-3/2}.$$

For Rossi spheres, we take

$$E_{11} = i.$$

Observe that

$$\begin{aligned} Z_{1(s)} &= Z_1 + \frac{s}{\sqrt{1+s^2}} Z_{\bar{1}}, \\ Z_{\bar{1}(s)} &= Z_{\bar{1}} + \frac{s}{\sqrt{1+s^2}} Z_1 \end{aligned} \tag{2.6}$$

are SU(2)-invariant since both Z_1 and $Z_{\bar{1}}$ are SU(2)-invariant. Dual to (2.6), we have

$$\begin{aligned} \theta_{(s)}^1 &= (1+s^2)\theta^1 - s\sqrt{1+s^2}\,\theta^{\bar{1}}, \\ \theta_{(s)}^{\bar{1}} &= (1+s^2)\theta^{\bar{1}} - s\sqrt{1+s^2}\,\theta^1. \end{aligned} \tag{2.7}$$

Compute

$$i\theta_{(s)}^1 \wedge \theta_{(s)}^{\bar{1}} = (1+s^2)i\theta^1 \wedge \theta^{\bar{1}} = (1+s^2)\,d\theta, \tag{2.8}$$

where $d\theta = i\theta^1 \wedge \theta^{\bar{1}}$, i.e., $h_{1\bar{1}} = 1$. So from (2.8), it follows that

$$h_{1\bar{1}}^{(s)} = \frac{1}{1+s^2} \quad \text{and} \quad h_{(s)}^{1\bar{1}} := (h_{1\bar{1}}^{(s)})^{-1} = 1+s^2. \tag{2.9}$$

Suppose that the Webster curvature R of (J,θ) is a positive constant R_0 (see [13] and [14] for basic pseudohermitian geometry). Then we should take $\omega_1^1 = -iR_0\theta$ in the structure equation, so that $d\omega_1^1 = R_0\theta^1 \wedge \theta^{\bar{1}}$. For $\theta = \hat{\theta}$ (the standard contact form on S^3, see (2.1)), $R_0 = 1$, while $R = R_0$ for $\theta = \hat{\theta}/R_0$. Hence

$$\omega_1^1 = -iR_0\theta = -i\hat{\theta} = -2(z^{\bar{1}}\,dz^1 + z^{\bar{2}}\,dz^2), \tag{2.10}$$

where we have used that $z^{\bar{1}}\,dz^1 + z^{\bar{2}}\,dz^2 + \text{conjugate} = 0$ on S^3. We can then determine, from the structure equation for $(J_{(s)},\theta)$, that

$$\begin{aligned} \omega_{1(s)}^1 &= (-i)(1+2s^2)\,R_0\theta, \\ h_{(s)}^{1\bar{1}} A_{\bar{1}\bar{1}(s)} &= 2is\sqrt{1+s^2}\,R_0, \\ R_{(s)} &= (1+2s^2)\,R_0. \end{aligned} \tag{2.11}$$

It follows that $(J_{(s)},\theta)$ is pseudo-Einstein (see (3.1)), since $R_{(s),1(s)} = 0 = A_{11(s),\bar{1}(s)}$. For a pseudohermitian structure (J,θ), we recall that the sublaplacian Δ_b acting on a function u reads as

$$\Delta_b u = h^{1\bar{1}}(u_{,1\bar{1}} + u_{,\bar{1}1}) = u_{,1\bar{1}} + u_{,\bar{1}1} \quad \text{for a unitary frame (so } h^{1\bar{1}} = 1).$$

2.2. Basic formulas for variations in J and θ

In this subsection, we provide basic formulas for variations of a pseudohermitian manifold (M, J, θ). Write the variation of J as $\delta J = 2E$, $E = E_{11}\theta^1 \otimes Z_{\bar{1}}$ + conjugate (for a unitary coframe/frame θ^1/Z_1), $\delta\theta = 2h\theta$ meaning that there are a family of CR structures $J_{(t)}$ with $J_{(0)} = J$ and a family of contact forms $\theta_{(t)}$ with $\theta_{(0)} = \theta$, such that $\partial_t J_{(t)}|_{t=0} = 2E$ and $\partial_t \theta_{(t)}|_{t=0} = 2h\theta$. Denote by $\delta_J R$ (respectively, $\delta_\theta R$) $\partial_t R_{J_{(t)},\theta}|_{t=0}$ (respectively, $\partial_t R_{J,\theta_{(t)}}|_{t=0}$). Similarly, we use the notation $\delta_J A_{11}$ and $\delta_\theta A_{11}$. Recall (see [6], [7] or [1]) that

$$\begin{aligned} \delta_J R &= (iE_{11,\bar{1}\bar{1}} - A_{\bar{1}\bar{1}} E_{11}) + \text{conjugate}, \\ \delta_J A_{11} &= iE_{11,0}, \\ \delta_\theta R &= -2hR - 4\triangle_b h, \\ \delta_\theta A_{11} &= -2hA_{11} + 2i\,h_{,11}. \end{aligned} \tag{2.12}$$

Define the energy functional $E(J, \theta)$ for a pseudohermitian manifold (M, J, θ) by

$$E(J, \theta) := \int_M (R^2 - |A|^2_{J,\theta})\,\theta \wedge d\theta$$

(cf. (1.2)). We have the following first variation formulas for $E(J, \theta)$.

Proposition 2.1. *Suppose (M, J, θ) is a closed (compact with no boundary) pseudohermitian 3-manifold. Then we have, for a unitary frame Z_1 (and coframe θ^1),*

$$\delta_J E(J, \theta) = \int_M (2iR_{,\bar{1}\bar{1}} - 2RA_{\bar{1}\bar{1}} + iA_{\bar{1}\bar{1},0})\,E_{11} + \text{conjugate}, \tag{2.13}$$

$$\delta_\theta E(J, \theta) = \int_M \{-8\,\Delta_b R - 2i\,(A_{\bar{1}\bar{1},11} - A_{11,\bar{1}\bar{1}})\}h\,\theta \wedge d\theta, \tag{2.14}$$

where $\delta J = 2E$, $E = E_{11}\theta^1 \otimes Z_{\bar{1}}$ + conjugate, and $\delta\theta = 2h\theta$. So, the Euler–Lagrange equation for (J, θ) reads as

$$R_{,11} - \frac{i}{2}(A_{11,1\bar{1}} - A_{11,\bar{1}1}) = 0, \quad -4\,\Delta_b R - i(A_{\bar{1}\bar{1},11} - A_{11,\bar{1}\bar{1}}) = 0. \tag{2.15}$$

Proof. Making use of the first two formulas in (2.12) and the integration by parts, we get (2.13), while using the last two formulas in (2.12) and integrating by parts gives (2.14). Observe that $-iRA_{11} + \frac{1}{2}A_{11,0}$ equals $-\frac{i}{2}(A_{11,1\bar{1}} - A_{11,\bar{1}1})$ by the commutation relation $iA_{11,0} + 2RA_{11} = A_{11,1\bar{1}} - A_{11,\bar{1}1}$ ([14]). Together with (2.13) and (2.14), we conclude the proof of (2.15). ∎

A direct computation shows that each Rossi sphere $(S^3, J_{(s)}, \theta)$ is a solution to (2.15) by (2.11) and noting that $A_{11,0} = TA_{11} - 2\omega_1^1(T)A_{11}$. We notice that the coefficient of $|A|^2_{J,\theta}$ in the integrand of $E(J, \theta)$ is different from that in the integrand of the following energy functional:

$$\int (R^2 - 4|A|^2_{J,\theta})\,\theta \wedge d\theta,$$

which is known to be the total Q'-curvature (for pseudo-Einstein (J, θ)), whose critical points are spherical, see [4].

In order to compute the second variation, we need the formulas for $\delta_J R_{,\bar{1}\bar{1}} = \overline{\delta_J R_{,11}}$, $\delta_J A_{\bar{1}\bar{1},0} = \overline{\delta_J A_{11,0}}$, $\delta_\theta R_{,11}$, $\delta_\theta A_{11,0}$, $\delta_\theta(\Delta_b R)$ and $\delta_\theta A_{\bar{1}\bar{1},11}$. We compute these quantities at a pseudo-Einstein critical point $(\hat{J},\hat{\theta})$ of $E(J,\theta)$, where $\hat{R} =$ constant and $\hat{A}_{11,1} = \hat{A}_{11,\bar{1}} = 0$ (see Proposition 3.1). First we obtain

(2.16) $\delta_J R_{,11} = (\delta_J R)_{,11}$ at $(\hat{J},\hat{\theta}) \overset{(2.12)}{=} [iE_{11,\bar{1}\bar{1}} - \hat{A}_{\bar{1}\bar{1}}E_{11} - iE_{\bar{1}\bar{1},11} - \hat{A}_{11}E_{\bar{1}\bar{1}}]_{,11}$.

Recall that $A_{\bar{1}\bar{1},0} = TA_{11} - 2\omega_1^1(T)A_{11}$. So, using

$$\delta_J \omega_1^1 = i(A_{11}E_{\bar{1}\bar{1}} + A_{\bar{1}\bar{1}}E_{11})\theta - i(E_{11,\bar{1}}\theta^1 + E_{\bar{1}\bar{1},1}\theta^{\bar{1}}),$$

we have

(2.17) $\delta_J A_{11,0} = (\delta_J A_{11})_{,0} - 2(\delta\omega_1^1)(T)A_{11} = iE_{11,00} - 2i(A_{11}E_{\bar{1}\bar{1}} + A_{\bar{1}\bar{1}}E_{11})A_{11}$.

For variations in θ, we have the following basic formulas:

$$\begin{aligned} \delta\theta &= 2h\theta, \\ \delta_\theta Z_1 &= -hZ_1, \\ \delta_\theta T &= -2hT + 2ih_{,1}Z_{\bar{1}} - 2ih_{,\bar{1}}Z_1. \end{aligned} \tag{2.18}$$

Besides $\delta_\theta R$ and $\delta_\theta A_{11}$ in (2.12), we also have

$$\delta_\theta \omega_1^1 = 3h_{,1}\theta^1 - 3h_{,\bar{1}}\theta^{\bar{1}} + i(\triangle_b h)\theta. \tag{2.19}$$

Writing $R_{,11} = Z_1(Z_1 R) - \omega_1^1(Z_1)Z_1 R$, we compute $\delta_\theta(R_{,11})$ as follows:

(2.20)
$$\begin{aligned} \delta_\theta(R_{,11}) &= \delta_\theta Z_1(Z_1 R) + Z_1(\delta_\theta Z_1)R + Z_1 Z_1(\delta_\theta R) - (\delta_\theta\omega_1^1)(Z_1)Z_1 R \\ &\quad - \omega_1^1(\delta_\theta Z_1)Z_1 R - \omega_1^1(Z_1)(\delta_\theta Z_1)R - \omega_1^1(Z_1)Z_1(\delta_\theta R) \\ &= -4hR_{,11} - 8h_{,1}R_{,1} - 2h_{,11}R - 4(\triangle_b h)_{,11} \end{aligned}$$

by (2.18), (2.19) and (2.12). Similarly writing $A_{11,0} = TA_{11} - 2\omega_1^1(T)A_{11}$, we compute

$$\begin{aligned} \delta_\theta(A_{11,0}) &= (\delta_\theta T)A_{11} + T(\delta_\theta A_{11}) - 2(\delta_\theta\omega_1^1)(T)A_{11} - 2\omega_1^1(\delta_\theta T)A_{11} - 2\omega_1^1(T)\delta_\theta A_{11} \\ &= -4hA_{11,0} + 2ih_{,1}A_{11,\bar{1}} - 2ih_{,\bar{1}}A_{11,1} - 2h_{,0}A_{11} - 2i(\triangle_b h)A_{11} - 2ih_{,110} \end{aligned} \tag{2.21}$$

by (2.18), (2.19) and (2.12) again. For the second variation in θ, we also need to compute

$$\begin{aligned} \delta_\theta(\Delta_b R) &= \Delta_b(\delta_\theta R) \text{ at the critical points where } R = \hat{R} = \text{constant} \\ &= -2(\Delta_b h)\hat{R} - 4\Delta_b^2 h. \end{aligned} \tag{2.22}$$

and using (2.18), (2.12) and (2.19), we compute

$$\begin{aligned} \delta_\theta A_{\bar{1}\bar{1},1} &= (\delta_\theta Z_1)A_{\bar{1}\bar{1}} + (\delta_\theta A_{\bar{1}\bar{1}})_{,1} + 2(\delta_\theta\omega_1^1)(Z_1)A_{\bar{1}\bar{1}} + 2\omega_1^1(\delta_\theta Z_1)A_{\bar{1}\bar{1}} \\ &= -hA_{\bar{1}\bar{1},1} - 2h_{,1}A_{\bar{1}\bar{1}} - 2hA_{\bar{1}\bar{1},1} - 2ih_{,\bar{1}\bar{1}1} + 6h_{,1}A_{\bar{1}\bar{1}} \\ &= -3hA_{\bar{1}\bar{1},1} + 4h_{,1}A_{\bar{1}\bar{1}} - 2ih_{,\bar{1}\bar{1}1}. \end{aligned} \tag{2.23}$$

We then compute, at $(\hat{J},\hat{\theta})$ where $\hat{A}_{\bar{1}\bar{1},1} = 0$,

$$\delta_\theta A_{\bar{1}\bar{1},11} = (\delta_\theta A_{\bar{1}\bar{1},1})_{,1} = 4h_{,11}\hat{A}_{\bar{1}\bar{1}} - 2ih_{,\bar{1}\bar{1}11}. \tag{2.24}$$

3. Proofs of Theorem 1.1 and Corollary 1.2

Proposition 3.1. *Suppose (M,ξ) is a closed (compact with no boundary) contact 3-manifold. Suppose (J,θ) on (M,ξ) is pseudo-Einstein and a solution to (2.15). Then $R =$ constant, and $A_{11,1}=0$, $A_{11,\bar{1}}=0$.*

Proof. By the condition of (J,θ) being pseudo-Einstein, we have

$$R_{,1} = iA_{11,\bar{1}}. \tag{3.1}$$

Substituting (3.1) into the second equation of (2.15) gives $3\Delta_b R = 0$ by noting that $\Delta_b R = R_{,1\bar{1}} + R_{,\bar{1}1}$ $(h^{1\bar{1}} = h_{1\bar{1}} = 1)$. It follows that $R =$ constant, since M is closed. We now multiply the second equation of (2.15) by R and integrate over M with respect to the volume form $\theta\wedge d\theta$. After integrating by parts, we obtain

$$\int_M (-4|\nabla_b R|^2 + iR_{,11}A_{\bar{1}\bar{1}} - iR_{,\bar{1}\bar{1}}A_{11})\theta\wedge d\theta = 0, \tag{3.2}$$

where $|\nabla_b R|^2 := 2h^{1\bar{1}}R_{,1}R_{,\bar{1}} = 2R_{,1}R_{,\bar{1}}$. From the first equation of (2.15) and the commutation relation

$$iA_{11.0} = A_{11,1\bar{1}} - A_{11,\bar{1}1} - 2RA_{11},$$

it follows that

$$R_{,11} = iRA_{11} - \frac{1}{2}A_{11,0} = \frac{1}{2}i(A_{11,1\bar{1}} - A_{11,\bar{1}1}). \tag{3.3}$$

Substituting (3.3) into (3.2) gives

$$\int_M (-4|\nabla_b R|^2 + |A_{11,1}|^2 - |A_{11,\bar{1}}|^2)\theta\wedge d\theta = 0 \tag{3.4}$$

by integrating by parts. Now making use of $R =$ constant and (3.1) (so $A_{11,\bar{1}} = 0$) in (3.4), we get $\int_M |A_{11,1}|^2\theta\wedge d\theta = 0$, and hence $A_{11,\bar{1}} = 0$. We have completed the proof. ∎

Let τ denote the torsion tensor of the pseudohermitian connection ∇, i.e.,

$$\tau(U,V) := \nabla_U V - \nabla_V U - [U,V]$$

for any tangent vector fields U and V. Recall that the Reeb vector field T is the unique vector field such that $\theta(T) = 1$ and $d\theta(T,\cdot) = 0$. It is not hard to see from the formulas for Lie brackets in [13], p. 418, that for $Y, W \in \xi$,

$$\tau(Y,W) = d\theta(Y,W)T, \tag{3.5}$$

$$U(g_{J,\theta}(Y,W)) = g_{J,\theta}(\nabla_U Y, W) + g_{J,\theta}(Y, \nabla_U W), \tag{3.6}$$

where $g_{J,\theta}$ is the Levi metric defined by $g_{J,\theta}(Y,W) := d\theta(Y,JW)$ and U is any tangent vector.

Lemma 3.2. *On a pseudohermitian 3-manifold (M,ξ,J,θ), suppose $R =$ constant and $A_{11,1}=0$, $A_{11,\bar{1}}=0$. Then $\nabla_X R = 0$ and $\nabla_X\tau = 0$ for any $X\in\xi$.*

Proof. It is clear that $R=$ constant implies $\nabla_X R = X(R) = 0$. To prove that $\nabla_X \tau = 0$ for any $X \in \xi$, it is enough to show

$$(\nabla_X \tau)(Y,W)=0 \text{ for } Y,W \in \xi,\ \ (\nabla_X \tau)(Y,T)=(\nabla_X \tau)(T,Y)=0 \text{ and } (\nabla_X \tau)(T,T)=0.$$

We compute

$$\begin{aligned}\nabla_X(\tau(Y,W)) &\overset{(3.5)}{=} X(d\theta(Y,W))T + d\theta(Y,W)\nabla_X T\\ &= X(d\theta(Y,W))T \quad (\nabla T = 0 \text{ by equation (4.5) on p. 418 of [13]}).\end{aligned} \tag{3.7}$$

It follows from (3.7) and (3.5) that

$$\begin{aligned}(\nabla_X \tau)(Y,W) &= \nabla_X(\tau(Y,W)) - \tau(\nabla_X Y, W) - \tau(Y, \nabla_X W)\\ &= \{X(d\theta(Y,W)) - d\theta(\nabla_X Y, W) - d\theta(Y, \nabla_X W)\}T = 0.\end{aligned}$$

Here we have used that ∇ preserves ξ (see equation (4.5) in [13]), and applied (3.6) with W replaced by $-JW$ and the property that $\nabla \circ J = J \circ \nabla$ (see Proposition 3.1 (2) in [16], for instance). We next compute

$$\begin{aligned}(\nabla_X \tau)(Y,T) &= \nabla_X(\tau(Y,T)) - \tau(\nabla_X Y, T) - \tau(Y, \nabla_X T)\\ &= \nabla_X(\tau(Y,T)) - \tau(\nabla_X Y, T) \quad (\text{since } \nabla T = 0).\end{aligned} \tag{3.8}$$

Define the tensor

$$A := A_1^{\bar 1}\theta^1 \otimes Z_{\bar 1} + A_{\bar 1}^{1}\theta^{\bar 1} \otimes Z_1,$$

where $A_1^{\bar 1} = h^{1\bar 1}A_{11}$ and $A_{\bar 1}^{1}$ is the complex conjugate of $A_1^{\bar 1}$. Observe that (extending the defining domain of τ to complex tangent vectors by complex linearity)

$$\begin{aligned}\tau(Z_1,T) &= \nabla_{Z_1}T - \nabla_T Z_1 - [Z_1,T]\\ &= 0 - \omega_1^{1}(T)Z_1 - (A_1^{\bar 1}Z_{\bar 1} - \omega_1^{1}(T)Z_1) = -A_1^{\bar 1}Z_{\bar 1} = -A(Z_1)\end{aligned} \tag{3.9}$$

by [13], p. 418. In fact, we also have $\tau(fZ_1,T) = -A(fZ_1)$ for any complex function f. So it holds that

$$\tau(\nabla_X Z_1, T) = -A(\nabla_X Z_1). \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\nabla_X(\tau(Z_1,T)) - \tau(\nabla_X Z_1, T) = -\nabla_X(A(Z_1) + A(\nabla_X Z_1) = -(\nabla_X A)(Z_1) = 0 \tag{3.11}$$

by the condition $A_{11,1}=0$, $A_{11,\bar 1}=0$ due to Proposition 3.1. By taking complex conjugation, (3.11) also holds for $Z_{\bar 1}$ replacing Z_1. So the right-hand side of (3.8) vanishes. We have shown $(\nabla_X \tau)(Y,T) = 0$. Noting that $(\nabla_X \tau)(T,Y) = -(\nabla_X \tau)(Y,T) = 0$. Clearly $(\nabla_X \tau)(T,T) = 0$ since τ is skew-symmetric. We have completed the proof. ∎

We call a pseudohermitian automorphism ϕ of (M,ξ,J,θ) a *sub-symmetry* at a point x if $\phi(x) = x$ and $\phi_*|_{\xi_x} = -1$ ($\phi_* T_x = T_x$ hence the Reeb orbit through x is fixed by ϕ). A local sub-symmetry at a point x means that ϕ is only defined in a neighborhood of x.

Proof of Theorem 1.1. Suppose that (J,θ) is a pseudo-Einstein critical point of (1.2). By Proposition 2.1, (J,θ) satisfies the system (2.15), and hence from Proposition 3.1 it follows that

$$R = \text{constant}, \quad A_{11,1} = 0 \quad \text{and} \quad A_{11,\bar{1}} = 0.$$

By Lemma 3.2, we obtain that the curvature R and the torsion (tensor) τ are parallel along the (horizontal) direction of any contact vector. From the proof of Theorem 2.1 in [9] (noting that the Levi metric $g_{J,\theta}$ plays the role of the metric in the setting of [9]), for each point x we can find a local sub-symmetry ϕ_x (which is a local pseudohermitian automorphism) such that $\phi_x(x) = x$ and $\phi_x^2 = \mathrm{Id}$. Now lift ϕ_x to a local sub-symmetry $\tilde{\phi}_{\tilde{x}}$ on $\tilde{M}$, the universal cover of M, where $\tilde{x} \in \tilde{M}$ is a lift of x, i.e., $\pi(\tilde{x}) = x$, $\pi\colon \tilde{M} \to M$ is the natural projection. Since $\tilde{M}$ is simply connected, we can extend ϕ_x uniquely to a global pseudohermitian automorphism using the parabolic exponential map in p. 309 of [11] by a similar argument to that in pp. 252–255 of [12] (where the real analyticity is used) for extending an affine map. Observe that the fixed point set of $\tilde{\phi}_{\tilde{x}}$ is a Reeb orbit $F_{\tilde{x}}$ in $\tilde{M}$, $\{F_{\tilde{x}}\}_{\tilde{x}\in\tilde{M}}$ foliate $\tilde{M}$, and $\{\tilde{\phi}_{\tilde{x}}\}_{\tilde{x}\in\tilde{M}}$ permutes the Reeb orbits $\{F_{\tilde{x}}\}_{\tilde{x}\in\tilde{M}}$, since all the $\tilde{\phi}_{\tilde{x}}$ are pseudohermitian automorphisms. The sub-symmetry $\tilde{\phi}_{\tilde{x}}$ has the following properties:

$$\tilde{\phi}_{\tilde{x}}(\tilde{x}) = \tilde{x}, \quad (\tilde{\phi}_{\tilde{x}})_*|_{\xi_{\tilde{x}}} = -\mathrm{Id}, \quad \tilde{\phi}_{\tilde{x}}|_{F_{\tilde{x}}} = \mathrm{Id}, \quad \text{and} \quad \tilde{\phi}_{\tilde{x}}^2 = \mathrm{Id}, \quad \text{so } \tilde{\phi}_{\tilde{x}}^{-1} = \tilde{\phi}_{\tilde{x}}.$$

Let $\mathrm{Aut}_{\psi.h.}(\tilde{M},\xi,J,\theta)$ denote the group of all pseudohermitian automorphisms. We claim that

$$\mathrm{Aut}_{\psi.h.}(\tilde{M},\xi,J,\theta) \text{ acts on } (\tilde{M},\xi,J,\theta) \text{ transitively.} \tag{3.12}$$

Observe that $\tilde{M}$ is complete (meaning that it is complete as a metric space) by, for instance, Theorem 7.1 (b) in [15]. Next, given $p,q \in \tilde{M}$, we can find a Legendrian (horizontal) geodesic (with respect to the Levi-metric $g_{J,\theta}$) γ connecting p and q, parametrized by the arc length of $g_{J,\theta}$ by Theorem 7.1 (a) in [15]. Let $m \in \gamma$ be the middle point of the curve γ. It follows that $\tilde{\phi}_m$ maps p (respectively, q) to q (respectively, p). We have shown (3.12). That is, $(\tilde{M},\xi,J,\theta)$ is homogeneous as a pseudohermitian manifold. ∎

Remark 3.3. We notice that in [9], the authors make the assumption on homogeneity to classify all possible sub-symmetric spaces through a Lie-theoretic argument.

Proof of Corollary 1.2. By (2.11) (together with (2.9)), we verify that a Rossi sphere $(J_{(s)},\hat{\theta})$ is pseudo-Einstein and satisfies (2.15) by noting that $(\theta = \hat{\theta})$

$$A_{\bar{1}\bar{1}(s),0} = \hat{T}A_{\bar{1}\bar{1}(s)} - 2\omega_{1(s)}^{1}(\hat{T})A_{11(s)} = 0 - 2(-i)(1+2s^2)R_0A_{11(s)} = 2iR_{(s)}A_{11(s)}.$$

So, $(J_{(s)},\hat{\theta})$ is a pseudo-Einstein critical point of (1.2) in view of Proposition 2.1. Conversely, by Theorem 1.1, $(S^3,\hat{\xi},J,\theta)$ is homogeneous. In particular, it is a homogeneous CR 3-manifold. According to Cartan ([3], p. 69), the CR structure J must be left-invariant on SU(2) $(= S^3)$. By Proposition 5.1 (c) in [2], we conclude that J is isomorphic to a Rossi sphere $J_{(s)}$ for a unique $s \le 0$ (by comparing (2.7) with the coframe taken in the proof of Proposition 5.1 in [2], we get the parameter relation

$$\sqrt{t} = \sqrt{1+s^2} - s,$$

so $t \geq 1$ corresponds to $s \leq 0$, where t is strictly decreasing as a function of s). Moreover, that θ is SU(2)-invariant implies that θ is a constant multiple of $\hat{\theta}$. We have thus completed the proof. ∎

Remark 3.4. In [2], besides Rossi spheres, some other examples of homogeneous CR 3-manifolds are discussed. Let us write down R and A_{11} for two less known examples: $SL_2(\mathbb{R})$ and the Euclidean group $E_2 = SO_2 \rtimes \mathbb{R}^2$. For $SL_2(\mathbb{R})$, there are a family of homogeneous CR structures with parameter t (see Proposition 4.2 in [2]). With respect to a suitable unitary coframe in the proof of Proposition 4.2 in [2], we obtain ($t \neq 0, -1$)

$$R_{(t)} = -\frac{1+6t+t^2}{4|t|(1+t)} \quad \text{and} \quad A_{11(t)} = i\,\frac{(1-t)^2}{4|t|(1+t)}.$$

For E_2, there is a unique homogeneous CR structure up to $\mathrm{Aut}(E_2)$ (see Proposition 7.1 (a) in [2]). With respect to a suitable unitary coframe in the proof of Proposition 7.1 (a) in [2], we easily obtain that $R = 1/2$ and $A_{11} = i/2$.

4. Second variation: proof of Theorem 1.3

Starting from (2.13), we compute the second variation in J, at a critical point $(\hat{J}, \hat{\theta})$ where $(2iR_{,\bar{1}\bar{1}} - 2RA_{\bar{1}\bar{1}} + iA_{\bar{1}\bar{1},0}) = 0$:

$$\begin{aligned} &\delta_J^2 E(\hat{J},\hat{\theta}) \\ &= \int_M (2i\delta_J R_{,\bar{1}\bar{1}} - 2(\delta_J R)A_{\bar{1}\bar{1}} - 2R(\delta_J A_{\bar{1}\bar{1}}) + i\delta_J A_{\bar{1}\bar{1},0})\, E_{11} + \text{conjugate}. \end{aligned} \tag{4.1}$$

Applying (2.16), (2.12) and (2.17) to the right-hand side of (4.1), we obtain

$$\begin{aligned} &\delta_J^2 E(\hat{J},\hat{\theta}) \\ &= \int_M \left\{ \begin{array}{c} (-iE_{\bar{1}\bar{1},11} + iE_{11,\bar{1}\bar{1}} - \hat{A}_{\bar{1}\bar{1}}E_{11} - \hat{A}_{11}E_{\bar{1}\bar{1}})_{,\bar{1}\bar{1}} \\ -2i(E_{11,\bar{1}\bar{1}} - \hat{A}_{\bar{1}\bar{1}}E_{11})\hat{A}_{\bar{1}\bar{1}} + 2i(E_{\bar{1}\bar{1},11} - \hat{A}_{11}E_{\bar{1}\bar{1}})\hat{A}_{\bar{1}\bar{1}} \\ +2i\hat{R}E_{\bar{1}\bar{1},0} + E_{\bar{1}\bar{1},00} - 2\hat{A}_{\bar{1}\bar{1}}(\hat{A}_{\bar{1}\bar{1}}E_{11} + \hat{A}_{11}E_{\bar{1}\bar{1}}) \end{array} \right\} E_{11} \\ &\quad + \text{conjugate} \end{aligned} \tag{4.2}$$

(volume form $\hat{\theta} \wedge d\hat{\theta}$ omitted). The “slice condition" in [6], p. 235, for $\delta J = 2E$ reads as $B_{\hat{J}}E = 0$, i.e.,

$$iE_{11,\bar{1}\bar{1}} - \hat{A}_{11}E_{\bar{1}\bar{1}} = -iE_{\bar{1}\bar{1},11} - \hat{A}_{\bar{1}\bar{1}}E_{11}. \tag{4.3}$$

From the commutation relation $E_{\bar{1}\bar{1},1\bar{1}} - E_{\bar{1}\bar{1},\bar{1}1} = iE_{\bar{1}\bar{1},0} - 2\hat{R}E_{\bar{1}\bar{1}}$ and an integration by parts, it follows that

$$\int_M 2i\hat{R}E_{\bar{1}\bar{1},0}E_{11}\,\hat{\theta}\wedge d\hat{\theta} = \int_M \{4\hat{R}^2|E_{11}|^2 - 2\hat{R}|E_{\bar{1}\bar{1},1}|^2 + 2\hat{R}|E_{11,1}|^2\}\,\hat{\theta}\wedge d\hat{\theta}. \tag{4.4}$$

Making use of (4.3), (4.4) and integrating by parts again, we finally reduce (4.2) to

$$\tag{4.5} \delta_J^2 E(\hat{J},\hat{\theta}) = \int_M \left\{ \begin{array}{c} -2|E_{11,0}|^2 - 4\hat{R}|E_{\bar{1}\bar{1},1}|^2 + 4\hat{R}|E_{11,1}|^2 \\ +(8\hat{R}^2 - 4|\hat{A}_{11}|^2)|E_{11}|^2 \end{array} \right\}$$
$$+ \left[(2i+2)E_{11,\bar{1}}^2 - 2i\, E_{11,1}E_{\bar{1}\bar{1},1} + (2i-2)E_{11}^2 \hat{A}_{\bar{1}\bar{1}}\right]\hat{A}_{\bar{1}\bar{1}}$$
$$+ \text{conjugate of } [\ \cdots\ ]\hat{A}_{\bar{1}\bar{1}}.$$

(volume form $\hat{\theta} \wedge d\hat{\theta}$ omitted). Now we are going to compute

$$\delta_\theta \delta_J E(\hat{J},\hat{\theta})$$
$$\tag{4.6} = \int_M (2i\delta_\theta R_{,\bar{1}\bar{1}} - 2(\delta_\theta R)\hat{A}_{\bar{1}\bar{1}} - 2\hat{R}(\delta_\theta A_{\bar{1}\bar{1}}) + i\delta_\theta A_{\bar{1}\bar{1},0})E_{11} + \text{conjugate},$$

Substituting (2.20), (2.12) and (2.21) into (4.6) gives

$$\tag{4.7} \delta_\theta \delta_J E(\hat{J},\hat{\theta}) = \int_M \left\{ \begin{array}{c} (6\triangle_b h - 2ih_{,0})\hat{A}_{\bar{1}\bar{1}} \\ -8i(\triangle_b h)_{,\bar{1}\bar{1}} + 2h_{,\bar{1}\bar{1}0} \end{array} \right\} E_{11} + \text{conjugate}.$$

To compute $\delta_\theta^2 E(\hat{J},\hat{\theta})$, we apply (2.22), (2.24) to the δ_θ of (2.14):

$$\delta_\theta^2 E(\hat{J},\hat{\theta}) = \int_M \{-8\delta_\theta(\Delta_b R) - 2i\delta_\theta(A_{\bar{1}\bar{1},11} - A_{11,\bar{1}\bar{1}})\}h$$

to conclude via integrating by parts that

$$\tag{4.8} \delta_\theta^2 E(\hat{J},\hat{\theta}) = \int_M \left\{ \begin{array}{c} -16\hat{R}|\nabla_b h|^2 + 8i\hat{A}_{\bar{1}\bar{1}}(h_{,1})^2 - 8i\hat{A}_{11}(h_{,\bar{1}})^2 \\ +32(\triangle_b h)^2 - 8|h_{,11}|^2 \end{array} \right\} \hat{\theta} \wedge d\hat{\theta}.$$

For Rossi spheres $(S^3, J_{(s)}, \theta)$ with $\theta = \hat{\theta}/R_0$ (see Subsection 2.1), we compute via (2.11) that

$$R_{(s)}^2 - |A|_{J_{(s)},\theta}^2 = (1+2s^2)^2 R_0^2 - 4s^2(1+s^2)R_0^2 = R_0^2.$$

Together with $\hat{\theta} \wedge d\hat{\theta} = 8\,dv_{S^3}^{\mathrm{Eucl}}$ (recall (2.1) for $\hat{\theta}$, and $dv_{S^3}^{\mathrm{Eucl}}$ denotes the Euclidean volume form of S^3), we have

$$E(J_{(s)},\theta) = \int_{S^3} (R_{(s)}^2 - |A|_{J_{(s)},\theta}^2)\theta \wedge d\theta = \int_{S^3} R_0^2 \frac{\hat{\theta} \wedge d\hat{\theta}}{R_0^2} = \int_{S^3} 8 \cdot dv_{S^3}^{\mathrm{Eucl}} = 16\pi^2.$$

So, Rossi spheres are all critical points of $E(J,\theta)$ (as shown after the proof of Proposition 2.1) with the same energy.

A. Appendix

We give some examples for second variations in θ and J of E at the standard pseudohermitian 3-sphere $(S^3, J_{(0)}, \hat{\theta})$. Substituting $\hat{R} = 1$ and $\hat{A}_{11} = 0$ into (4.8) gives

$$\tag{A.1} \delta_\theta^2 E(J_{(0)},\hat{\theta}) = \int_{S^3} (-16|\nabla_b h|^2 + 32(\triangle_b h)^2 - 8|h_{,11}|^2)\hat{\theta} \wedge d\hat{\theta}.$$

Using integration by parts and the commutation relation $h_{,\bar{1}1\bar{1}} - h_{,\bar{1}\bar{1}1} = ih_{,\bar{1}0} - h_{,\bar{1}}$ (see formula (9) in [7]), we compute (volume form $\hat{\theta} \wedge d\hat{\theta}$ omitted)

$$\begin{aligned}\int_{S^3} h_{,11}h_{,\bar{1}\bar{1}} &= -\int_{S^3} h_{,1}(h_{,\bar{1}1\bar{1}} - ih_{,\bar{1}0} + h_{,\bar{1}}) \\ &= -\int_{S^3} h_{,1}(h_{,\bar{1}1\bar{1}} - ih_{,0\bar{1}} + h_{,\bar{1}}) \quad (\text{since } \hat{A}_{11} = 0) \\ &= \int_{S^3} h_{,1\bar{1}}h_{,\bar{1}1} - i\int_{S^3} h_{,1\bar{1}}h_{,0} - \int_{S^3} |h_{,1}|^2.\end{aligned} \tag{A.2}$$

Since $h_{,1\bar{1}} - h_{,\bar{1}1} = ih_{,0}$, we have

$$h_{,1\bar{1}} = \frac{1}{2}(\triangle_b h + ih_{,0}) \quad \text{and} \quad h_{,\bar{1}1} = \frac{1}{2}(\triangle_b h - ih_{,0}) \quad (\text{since } h \text{ is real}). \tag{A.3}$$

Substituting (A.3) into (A.2), we can reduce (A.2) to

$$\int_{S^3} h_{,11}h_{,\bar{1}\bar{1}} = \frac{1}{4}\int_{S^3} (\triangle_b h)^2 + \frac{3}{4}\int_{S^3} (h_{,0})^2 - \frac{i}{2}\int_{S^3} (\triangle_b h)h_{,0} - \frac{1}{2}\int_{S^3} |\nabla_b h|^2. \tag{A.4}$$

Let $H_{p,q,1}$ denote the restriction to S^3 of the space of the homogeneous complex harmonic polynomials of bidegree (p, q), where p is the holomorphic homogeneity and q the antiholomorphic one. Then for $f \in H_{p,q,1}$, one has

$$-\triangle_b f = \frac{1}{2}(pq + \frac{1}{2}(p+q))f \quad \text{and} \quad Tf = i\,\frac{(p-q)}{2}\,f \tag{A.5}$$

(see Proposition 2.2 on p. 10 of [1]; note that $\hat{\theta}$ is twice the contact form θ_0 in [1], p. 8, so the sublaplacian there is twice the sublaplacian here while T there is exactly the Reeb vector field here, i.e., $T = \hat{T}$). By (A.4), one reduces (A.1) to

$$\delta_\theta^2 E(J_{(0)}, \hat{\theta}) = 30\int_{S^3} (\triangle_b h)^2 - 6\int_{S^3} (h_{,0})^2 + 4i\int_{S^3} (\triangle_b h)h_{,0} - 12\int_{S^3} |\nabla_b h|^2. \tag{A.6}$$

Taking $h = f + \bar{f}$ in (A.5) and writing $\lambda = \frac{1}{2}(pq + \frac{1}{2}(p+q))$, $\mu = (p-q)/2$, we have $-\triangle_b h = \lambda(f + \bar{f})$ and $h_{,0} = Th = i\mu(f - \bar{f})$. Substituting these formulas into (A.6) and noting that $\int_{S^3} |\nabla_b h|^2 = -\int_{S^3} (\triangle_b h)h$, we reduce the right-hand side of (A.6) to

$$60\int_{S^3} \triangle_b f \triangle_b \bar{f} - 12\mu^2 \int_{S^3} f\bar{f} + 4i\int_{S^3} \triangle_b(f + \bar{f})i\mu(f - \bar{f}) - 24\lambda\int_{S^3} f\bar{f}. \tag{A.7}$$

Here we have used $\int_{S^3} f^2 = 0$ and $\int_{S^3} \bar{f}^2 = 0$. Using (A.5), we can further reduce (A.7) and conclude from (A.6) that

$$\delta_\theta^2 E(J_{(0)}, \hat{\theta}) = \int_{S^3} \left\{ \begin{array}{c} \frac{60}{4}(pq + \frac{1}{2}(p+q))^2 - \frac{12}{4}(p-q)^2 \\ -\frac{24}{2}(pq + \frac{1}{2}(p+q)) \end{array} \right\} |f|^2 \hat{\theta} \wedge d\hat{\theta} \tag{A.8}$$

for $\delta\theta = 2h\theta = 2(f + \bar{f})\hat{\theta}$, $f \in H_{p,q,1}$.

We now turn to compute $\delta_J^2 E(J_{(0)}, \hat{\theta})$ for $\delta J = 2E$, $E = E_{11}\theta^1 \otimes Z_{\bar{1}}$+conjugate with $E_{11} \in H_{p,q,1}$. Starting from (4.5) with $\hat{R} = 1$ and $\hat{A}_{11} = 0$, we have (volume form $\hat{\theta} \wedge d\hat{\theta}$ omitted)

$$\delta_J^2 E(J_{(0)}, \hat{\theta}) = \int_{S^3} (-2|E_{11,0}|^2 - 4|E_{\bar{1}\bar{1},1}|^2 + 4|E_{11,1}|^2 + 8|E_{11}|^2). \tag{A.9}$$

Via an integration by parts and the commutation relation $E_{11,\bar{1}1} - E_{11,1\bar{1}} = -iE_{11,0} - 2E_{11}$ (noting that $\hat{R} = 1$), we reduce the right-hand side of (A.9) to

$$\int_{S^3} (2E_{11,00} - 4iE_{11,0})\, E_{\bar{1}\bar{1}} \quad \text{(note that } 8|E_{11}|^2 \text{ is cancelled).} \tag{A.10}$$

We compute

$$\begin{aligned} E_{11,0} &= TE_{11} - 2\omega_1^1(T)E_{11} = TE_{11} - 2(-i)E_{11} \quad (\omega_1^1 = -i\hat{\theta} \text{ by } (2.10)) \\ &= i\left(\frac{p-q}{2}\right)E_{11} + 2iE_{11} = i\left(\frac{m}{2} + 2\right)E_{11}. \end{aligned} \tag{A.11}$$

Here we have written $m = p - q$. Applying (A.11) to (A.10), we finally obtain

$$\delta_J^2 E(J_{(0)}, \hat{\theta}) = -\frac{1}{2} m(m+4) \int_{S^3} |E_{11}|^2 \hat{\theta} \wedge d\hat{\theta} \tag{A.12}$$

(recall that $E_{11} \in H_{p,q,1}$, $m = p - q$).

Funding. J.-H. C. would like to thank the Ministry of Science and Technology of Taiwan for the support: grant no. MOST 110-2115-M-001-015, and the National Center for Theoretical Sciences for the constant support. A. M. is supported by the project *Geometric Problems with Loss of Compactness* from Scuola Normale Superiore. He is also a member of GNAMPA as part of INdAM. P. Y. acknowledges support from the NSF for the grant DMS 1509505.

References

[1] Afeltra, C., Cheng, J.-H., Malchiodi, A. and Yang, P.: On the variation of the Einstein–Hilbert action in pseudohermitian geometry. *J. Reine Angew. Math.* **813** (2024), 81–102. Zbl 07891384 MR 4780981

[2] Bor, G. and Jacobowitz, H.: Left-invariant CR structures on 3-dimensional Lie groups. *Complex Anal. Synerg.* **7** (2021), no. 3, article no. 23, 18 pp. Zbl 1495.32088 MR 4277409

[3] Cartan, E.: Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes. *Ann. Mat. Pura Appl.* **11** (1933), no. 1, 17–90. Zbl 0005.37401 MR 1553196

[4] Case, J. S. and Yang, P.: A Paneitz-type operator for CR pluriharmonic functions. *Bull. Inst. Math. Acad. Sin. (N.S.)* **8** (2013), no. 3, 285–322. Zbl 1316.32021 MR 3135070

[5] Chen, S.-C. and Shaw, M.-C.: *Partial differential equations in several complex variables*. AMS/IP Studies in Advanced Mathematics 19, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001. Zbl 0963.32001 MR 1800297

[6] Cheng, J.-H. and Lee, J. M.: The Burns–Epstein invariant and deformation of CR structures. *Duke Math. J.* **60** (1990), no. 1, 221–254. Zbl 0704.53028 MR 1047122

[7] Cheng, J.-H., Malchiodi, A. and Yang, P.: A positive mass theorem in three dimensional Cauchy–Riemann geometry. *Adv. Math.* **308** (2017), 276–347. Zbl 1360.32031 MR 3600060

[8] Cheng, J.-H., Malchiodi, A. and Yang, P.: On the Sobolev quotient of three-dimensional CR manifolds. *Rev. Mat. Iberoam.* **39** (2023), no. 6, 2017–2066. Zbl 07808561 MR 4671428

[9] Falbel, E. and Gorodski, C.: On contact sub-Riemannian symmetric spaces. *Ann. Sci. École Norm. Sup. (4)* **28** (1995), no. 5, 571–589. Zbl 0836.53032 MR 1341662

[10] Gursky, M. J. and Viaclovsky, J. A.: A new variational characterization of three-dimensional space forms. *Invent. Math.* **145** (2001), no. 2, 251–278. Zbl 1006.58008 MR 1872547

[11] Jerison, D. and Lee, J. M.: Intrinsic CR normal coordinates and the CR Yamabe problem. *J. Differential Geom.* **29** (1989), no. 2, 303–343. Zbl 0671.32016 MR 0982177

[12] Kobayashi, S. and Nomizu, K.: *Foundations of differential geometry. Vol I*. Interscience Publishers (a division of John Wiley), New York-London, 1963. Zbl 0119.37502 MR 0152974

[13] Lee, J. M.: The Fefferman metric and pseudohermitian invariants. *Trans. Amer. Math. Soc.* **296** (1986), no. 1, 411–429. Zbl 0595.32026 MR 0837820

[14] Lee, J. M.: Pseudo-Einstein structures on CR manifolds. *Amer. J. Math.* **110** (1988), no. 1, 157–178. Zbl 0638.32019 MR 0926742

[15] Strichartz, R. S.: Sub-Riemannian geometry. *J. Differential Geom.* **24** (1986), no. 2, 221–263. Zbl 0609.53021 MR 0862049

[16] Tanaka, N.: *A differential geometric study on strongly pseudo-convex manifolds*. Lectures in Mathematics 9, Department of Mathematics, Kyoto University, Kinokuniya Book Store Co., Ltd., Tokyo, 1975. Zbl 0331.53025 MR 0399517

Received September 3, 2023; revised April 3, 2024.

Jih-Hsin Cheng
Institute of Mathematics, Academia Sinica and National Center for Theoretical Sciences
6F, Astronomy-Mathematics Building, No. 1, Sec. 4, Roosevelt Road, Da-an, 106319 Taipei, Taiwan;
cheng@math.sinica.edu.tw

Andrea Malchiodi
Scuola Normale Superiore
Piazza Dei Cavalieri 7, 50126 Pisa, Italy;
andrea.malchiodi@sns.it

Paul Yang
Department of Mathematics, Princeton University
Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA;
yang@math.princeton.edu