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Non-existence of cusps for a free-boundary problem for water waves

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Abstract. In Varvaruca and Weiss (2011), Varvaruca and Weiss eliminate the existence of cusps for a free-boundary problem for two-dimensional water waves under assumptions that hold for solutions for which $\{u > 0\}$ is a "strip-like" domain in the sense of Varvaruca (2008). In this paper, it is proven that cusps do not exist in the natural setting for these free-boundary problems. In particular, non-strip-like domains are also allowed. This qualitative result follows from quantitative results which, roughly speaking, give lower bounds on the "slope" at which the free boundary approaches a stagnation point. This builds upon recent work on non-existence of cusps in McCurdy and Naples (2022) for local minimizers.

1. Introduction

In this note, we eliminate the existence of cusps in the free boundary for weak solutions of the following free-boundary problem:

$$\begin{aligned} \Delta u &= 0 && \text{in } \{u > 0\} \cap \Omega \subset \mathbb{R}^n, \\ |\nabla u(x_1, \dots, x_n)| &= |x_n|^\gamma && \text{on } \partial\{u > 0\} \cap \Omega \end{aligned} \tag{P_γ}$$

for $n = 2$ any $0 < \gamma$. In particular, we eliminate cusps under more natural assumptions (see Theorem 1.2) and prove some weak geometric conditions on the "slope" at which the free boundary of weak solutions must approach $\{x_n = 0\}$ (see Theorem 1.7).

1.1. Background

For $n = 2$, $\gamma = \frac{1}{2}$, the free-boundary problem (P_γ) has a history dating back to 1847 and the work of Stokes [8] on 2-dimensional inviscid, incompressible fluids acted upon by gravity with a free surface. Stokes studied the profiles of standing waves for such a fluid under the following: for $\Omega = [-1, 1] \times [-D, 1]$, $1 < D \leq \infty$, and imposed the physical boundary condition that u is constant on $[-1, 1] \times \{-D\}$. Stokes conjectured that there was a one-parameter family of solutions to (P_γ) called *stream functions* which were parametrized by wave height. This family of stream functions was conjectured to have

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a wave of maximal height for which the wave profile $\partial\{u > 0\}$ touches $\{x_2 = 0\}$ with angle $\frac{2\pi}{3}$. In honor of Stokes, this extremal wave is called the Stokes wave. Points in $\partial\{u > 0\} \cap \{x_2 = 0\}$ are called *stagnation points*. Under strong assumptions of symmetry, monotonicity, and graphicality of $\partial\{u > 0\}$, Toland [9] and McLeod [5] proved the existence of extreme periodic waves for $R = +\infty$ and $0 < R < \infty$ (i.e., for waves of infinite and finite depth). In 1982, Amick, Fraenkel, and Toland [1] and Plotnikov [6] both independently proved the Stokes conjecture on the aperture of $\{u > 0\}$, where it touches $\{x_2 = 0\}$ for this extremal wave under similar assumptions. See [2, 11] for more historical details and the derivation of (P_γ) from physical principles.

Solutions to the free boundary problem (P_γ) are critical points of the corresponding Alt–Caffarelli functional

$$J_\gamma(v) := \int_\Omega |\nabla v| + x_n^{2\gamma} \chi_{\{u>0\}} dx.$$

Because they are merely critical points and not minimizers, research on the Stokes wave has typically proceeded by analyzing *weak solutions* [7, 10]. These weak solutions assume a modicum of regularity to $\partial\{u > 0\}$ away from $\{x_n = 0\}$; see Definition 2.1. However, in recent work [11], Varvaruca and Weiss introduced the notion of a *variational solution* (see [11, Definition 3.1]). This variational notion of a solution allows Varvaruca and Weiss to employ geometric techniques to the study of the Stokes wave beyond the usual assumptions of symmetry, monotonicity, and graphicality of $\partial\{u > 0\}$. These geometric techniques are based upon a monotonicity formula analogous to the quantity in [12] for local minimizers of the Alt–Caffarelli functional J_0. In particular, this allowed them to obtain the following results.

Theorem 1.1 ([11, Lemma 4.2, Lemma 4.7, Proposition 5.5, Remark 5.9, Remark 6.2, Proposition 8.1, Theorem 10.2]). *Let $n \geq 2$, $\gamma = 1/2$, $\Omega \subset \mathbb{R}^n$ be open, and let u be a weak solution of (P_γ) such that locally u satisfies*

$$|\nabla u| \leq C_0 x_n^{\frac{1}{2}}. \tag{1.1}$$

Then, if we denote $\Sigma := \partial\{u > 0\} \cap \{x_n = 0\} \cap \Omega$, we may decompose Σ into the disjoint union $\Sigma = \Sigma_{\text{cusps}} \cup \Sigma_{\text{rect}} \cup \Sigma_{\text{iso}}$, where if we denote the weighted $Q^{2\gamma}$-density by

$$\theta^n_{Q^{2\gamma},\{u>0\}}(x) = \lim_{r\to 0^+} \frac{1}{\omega_n r^{n+2\gamma}} \int_{B_r(x)} Q^{2\gamma}(x)\chi_{\{u>0\}}(x) d\mathcal{H}^2(x),$$

then this density exists and we define

$$\Sigma_{\text{cusps}} := \left\{x \in \Sigma : \theta^n_{Q^{2\gamma},\{u>0\}}(x) = 0\right\}$$

$$\Sigma_{\text{good}} := \left\{x \in \Sigma : \theta^n_{Q^{2\gamma},\{u>0\}}(x) \in \left(0, \frac{1}{\omega_n r^{n+2\gamma}} \int_{B_r(x)} Q^{2\gamma}(x)\chi_{\{x_n\geq 0\}}(x) d\mathcal{H}^2(x)\right)\right\}$$

$$\Sigma_{\text{iso}} := \left\{x \in \Sigma : \theta^n_{Q^{2\gamma},\{u>0\}}(x) = \frac{1}{\omega_n r^{n+2\gamma}} \int_{B_r(x)} Q^{2\gamma}(x)\chi_{\{x_n\geq 0\}}(x) d\mathcal{H}^2(x)\right\}.$$

Furthermore, these sets satisfy the following properties.

(1) $\dim_{\mathcal{H}}(\Sigma_{\text{good}}) \leq n-2$, *and if* $n=2$, *then* Σ_{good} *is locally isolated.*

(2) Σ_{iso} *is closed, and if* $n=2$, *then* Σ_{iso} *is locally finite.*

We note that the assumption (1.1) is absolutely essential to the project of [11]. Assumption (1.1) provides the compactness necessary to geometric blow-up analysis. And it is also necessary in order to connect weak solutions and variational solutions, since any weak solution of (P_γ) which also satisfies (1.1) is a variational solution. See [11, Lemma 3.4] for details.

1.2. Main results

The main qualitative result of this paper is to eliminate the set Σ_{cusps}.

Theorem 1.2 (Main result). *Let* $n=2$ *and* $0<\gamma$. *Let* u *be a weak solution to* (P_γ) *such that for all* $(x,0)\in\Omega$ *there exists a neighborhood* $K\subset\subset\Omega$ *of* $(x,0)$ *and a constant* $C<\infty$ *(possibly depending upon* K*) such that*

$$|\nabla u(x_1,x_2)| \leq C|x_2|^\gamma \tag{1.2}$$

for all $(x_1,x_2)\in K$. *Then,* $\Sigma_{\text{cusps}}=\emptyset$.

Theorem 1.2 is inspired by and improves upon the following result from [11].

Theorem 1.3 ([11, Lemma 4.4]). *Let* $n=2$, $\gamma=\frac{1}{2}$, *and let* u *be a weak solution to* (P_γ) *satisfying*

$$|\nabla u(x_1,x_2)| \leq |x_2|^{\frac{1}{2}}. \tag{1.3}$$

Then, $\Sigma_{\text{cusps}}=\emptyset$.

Remark 1.4. The assumption (1.3) is much stronger than (1.2). In particular, it is not known whether or not weak solutions, in general, satisfy (1.3). For example, *local minimizers* of the corresponding Alt–Caffarelli functional J_γ are weak solutions to (P_γ) and satisfy (1.2) for a dimensional constant $1<C=C(n,\gamma)$ [3]. One might expect weak solutions, which are merely *critical points*, to be less well behaved than local minimizers. In [10], it is proven that if u is a solution to (P_γ) for any $0<\gamma$ and $\{u>0\}$ is a "strip-like" domain (see [10, Section 2.1]), then u satisfies (1.3) (see the proof of [10, Theorem 3.6]; in particular, it follows from the properties of the function Q in (4.19)). However, "strip-like" domains do not allow for air bubbles, and therefore, Theorem 1.3 only represents a partial solution to eliminating Σ_{cusps}.

The central improvement of this paper is to eliminate the existence of Σ_{cusps} under the more natural assumption (1.1) and only using the local properties a cusp must satisfy. In particular, this allows for $\{u>0\}$ which are not "strip-like" in the sense of [10] and for solutions to (P_γ) which do not satisfy the boundary conditions of wave equations.

The method of proof for Theorem 1.2 was initially inspired by the proof of Theorem 1.3. However, the improvement comes from a closer analysis of local cusp geometry using ideas introduced in [4].

In fact, Theorem 1.2 is a qualitative result which comes from a "quantitative" result on the geometry of the free boundary $\partial\{u > 0\}$. To state the result, we need to first define a family of rescalings and a height function, which will be central to helping us control the geometry of $\partial\{u > 0\}$ near $(x, 0) \in \partial\{u > 0\} \cap \Omega$.

Definition 1.5 (Rescalings). Let $0 < \gamma$, and let u be a weak solution to (P_γ) in the domain Ω. For any set $U \subset \Omega$, $(x, 0)$, and $0 < r$, we define the rescalings

$$U_{(x,0),r} := \frac{U - (x,0)}{r}.$$

If $(x_0, 0) \in \partial\{u > 0\} \cap \Omega$, then

$$u_{(x_0,0),r}(x_1, x_2) := \frac{u(rx_1 + x_0, rx_2)}{r^\gamma}$$

is a solution to (P_γ) in $\Omega_{(x_0,0),r}$ and

$$\{u_{(x_0,0),r} > 0\} = \{u > 0\}_{(x_0,0),r}.$$

If $(x, 0) \in \partial\{u > 0\} \cap \Omega$, then for any $0 < r$ there must be a component $\mathcal{O}_{(x,0),r}$ of $\{u_{(x,0),r} > 0\} \cap [-1, 1] \times [-1, 1]$ such that $(0, 0) \in \partial\mathcal{O}_{(x,0),r}$. Furthermore, it is clear that we can choose the components $\mathcal{O}_{(x,0),r}$ in a consistent manner such that, for all $0 < r_1 < r_2$,

$$\mathcal{O}_{(x,0),r_1} = (\mathcal{O}_{(x,0),r_2})_{(0,0),\frac{r_1}{r_2}}.$$

Definition 1.6 (Height function). Let $0 < \gamma$, and let u be a weak solution to (P_γ) such that $(x, 0) \in \partial\{u > 0\} \cap \Omega$. Then, for any $0 < r \leq 2^{-1}\operatorname{dist}((x, 0), \partial\Omega)$ and any component $\mathcal{O}_{(x,0),r}$ of $\{u_{(x,0),r} > 0\} \cap [-1, 1] \times [-1, 1]$ such that $(0, 0) \in \overline{\mathcal{O}_{(x,0),r}}$, we define the following function. For $0 < \rho \leq 1$, we define

$$\operatorname{Height}(\rho, \mathcal{O}_{(x,0),r}) := \min\{1, \sup\{|x_2| : (x_1, x_2) \in \mathcal{O}_{(0,0),r},\ |x_1| = \rho\}\}.$$

This function picks up the height of the component $\mathcal{O}_{(x,0),r}$ in the window $[-1, 1] \times [-1, 1]$ at distance ρ from $x = 0$ by checking its intersection with both $\{(x_1, x_2) : x_1 = \rho\}$ and $\{(x_1, x_2) : x_1 = -\rho\}$. Note that

$$\operatorname{Height}(\rho, \mathcal{O}_{(x,0),r}) = \rho \operatorname{Height}(1, \mathcal{O}_{(x,0),\rho r}) = \frac{1}{\rho r}\operatorname{Height}(\rho r, \mathcal{O}_{(x,0),1}).$$

Theorem 1.7 (Quantitative result). *Let $n = 2$ and $0 < \gamma$. Let u be a weak solution to (P_γ) with associated domain Ω. Suppose that*

$$(x, 0) \in \Omega \cap \partial\{u > 0\}$$

and u satisfies (1.2) in $B_r((x, 0)) \subset \Omega$ with constant $C < \infty$. Then, $\operatorname{Height}(1, \mathcal{O}_{(x,0),\rho}) \geq \frac{1}{6C}$ *for all $0 < \rho \leq r$.*

Remark 1.8. The proofs of Theorems 1.2, 1.3, and 1.7 are essentially restricted to $n = 2$. For recent results eliminating cusps in higher dimensions and arbitrary co-dimension, see [4] which obtained an analogous macroscopic geometric description of $\partial\{u > 0\}$ for *local minimizers* of an analogous Alt–Caffarelli functional J_γ in $n \geq 2$.

It is unknown whether or not Σ_{cusps} may be eliminated in $n \geq 3$. It is unknown whether or not Σ_{iso} may be eliminated in $n \geq 2$.

2. Preliminaries and reduction of Theorem 1.2 to Theorem 1.7

We begin by defining the appropriate notion of a solution to (P_γ).

Definition 2.1 (Weak solutions). Let $\Omega \subset \mathbb{R}^n$ and $0 < \gamma$. A function $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a *weak solution* to (P_γ) if u satisfies the following.

(1) $u \in C^0(\Omega)$, $u \geq 0$ in Ω.

(2) u is harmonic in $\{u > 0\} \cap \Omega$.

(3) For every $\tau > 0$, the topological free boundary, $\partial\{u > 0\} \cap \Omega \cap \{|x_n| > \tau\}$, can be decomposed into an $(n-1)$-dimensional $C^{2,\alpha}$-surface, denoted by $\partial_{\text{red}}\{u > 0\}$, which is relatively open in $\partial\{u > 0\}$, and a singular set with $\mathcal{H}^{n-1}$-measure zero.

(4) For any open neighborhood V containing a point

$$x_0 \in \Omega \cap \{|x_n| > \tau\} \cap \partial_{\text{red}}\{u > 0\},$$

the function $u \in C^1(V \cap \overline{\{u > 0\}})$ and satisfies $|\nabla u|^2 = x_n^{2\gamma}$ on $V \cap \partial_{\text{red}}\{u > 0\}$.

Remark 2.2. We note that, for physical reasons, the definition of a weak solution usually includes the assumption $u \equiv 0$ in $\Omega \cap \{x_n \leq 0\}$. However, (P_γ) is only a physical problem for $n = 2$ and $\gamma = 1/2$. In this note, we work without this assumption to allow a wider class of solutions.

In order to reduce Theorem 1.2 to the proof of Theorem 1.7, we need to following compactness result.

Lemma 2.3. *Let $n = 2$ and $0 < \gamma$. Let u be a weak solution which satisfies (1.2), $(x, 0) \in \Sigma$, and $0 < r_i \to 0$. Then, there is a $(1+\gamma)$-homogeneous function $u_\infty \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ and a subsequence such that $u_{(x,0),r_i} \to u_\infty$ in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$.*

Proof. The case $\gamma = 1/2$ is proven in [11, Lemma 4.1] under the physical assumptions on weak solutions in Remark 2.2. This argument holds *verbatim* for $0 < \gamma < \infty$. To wit, the assumption (1.2) implies that for $0 < R < \infty$ and all sufficiently large i (depending upon R) the functions $u_{(x,0),r_i}$ are uniformly bounded in $W^{1,2}(B_R(0))$. By Rellich–Kondrachov compactness and lower semicontinuity, it remains to show that if

$u_{(x,0),r_i} \rightharpoonup u_\infty$ in $W^{1,2}(B_R(0))$, then

$$\int_{\mathbb{R}^2} |\nabla u_\infty|^2 \eta d\mathcal{H}^2 \geq \limsup_{i\to\infty} \int_{\mathbb{R}^2} |\nabla u_{(x,0),r_i}|^2 \eta d\mathcal{H}^2$$

for all $\eta \in C_0^1(B_R(0))$. By assumption (1.2) and Arzela–Ascoli, u_∞ is continuous and $u_{(x,0),r_i} \to u_\infty$ uniformly in $B_R(0)$. Since $u_{(x,0),r_i}$ are harmonic in $\{u_{(x,0),r_i} > 0\}$, u_∞ is also harmonic in $\{u_\infty > 0\}$. Therefore, by integration by parts, we calculate

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla u_{(x,0),r_i}|^2 \eta d\mathcal{H}^2 &= -\int_{\mathbb{R}^2} u_{(x,0),r_i} \nabla u_{(x,0),r_i} \cdot \nabla \eta d\mathcal{H}^2 \\ &\to -\int_{\mathbb{R}^2} u_\infty \nabla u_\infty \cdot \nabla \eta d\mathcal{H}^2 = \int_{\mathbb{R}^2} |\nabla u_\infty|^2 \eta d\mathcal{H}^2.\end{aligned}$$

It remains to show that u_∞ is $(1+\gamma)$-homogeneous. This is proven in [3, Theorem 5.11] for $n = 2$, $k = 1$, and $\Gamma = \{x_2 = 0\}$ for local minimizers of J_γ. However, since weak solutions which satisfy (1.2) also satisfy the monotonicity formula in [3, Theorems 4.3 and 5.11] the argument demonstrating homogeneity holds for them as well. ∎

2.1. The reduction of Theorem 1.2 to Theorem 1.7

If we assume Theorem 1.7, then Theorem 1.2 will follow if it can be shown that if $(x,0) \in \Sigma_{\text{cusps}}$, then for any $0 < C_2 < 1$ there exists a radius $0 < r$ such that $\text{Height}(1, \mathcal{O}_{(x,0),r}) \leq C_2$. This follows from an argument analogous to [11, Lemma 4.4]. Consider Δu as a non-negative Radon measure supported on $\partial\{u > 0\}$. Then, Δu satisfies the following inequality:

$$\begin{aligned}\Delta u(\phi) &:= -\int_{\mathbb{R}^2} \nabla u \cdot \nabla \phi d\mathcal{H}^2 \quad \text{for } \phi \in C_c^\infty(\mathbb{R}^2) \\ \Delta u(U) &:= \sup\{\Delta u(\phi) : \phi \in C_c^\infty(U), |\phi|_\infty \leq 1\} \\ &\geq \int_{\partial_{\text{red}}\{u>0\}\cap U} |x_2|^\gamma d\sigma(x_1, x_2).\end{aligned}$$

Now, let $u_{(x,0),r}$ be the rescaling of u, and let $u_{(x,0),r}|_{\mathcal{O}_{(x,0),r}}$ be the piecewise function

$$u_{(x,0),r}|_{\mathcal{O}_{(x,0),r}} = \begin{cases} u_{(x,0),r} & \text{in } \mathcal{O}_{(x,0),r}, \\ 0 & \text{elsewhere.} \end{cases}$$

For $0 < r$ small enough, $u_{(x,0),r}|_{\mathcal{O}_{(x,0),r}}$ is a weak solution in $B_2(0,0)$, and hence, by Lemma 2.3, there exists a function $u_0 \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ such that $u_{(x,0),r}|_{\mathcal{O}_{(x,0),r}} \to u_0$ in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ as $r \to 0$. By the assumption that $\theta^2_{Q^{2\gamma},\mathcal{O}}(x,0) = 0$ and the fact that u_0 is homogeneous, we have that

$$u_0 \equiv 0 \quad \text{and} \quad \Delta u_0 \equiv 0.$$

Thus, $\Delta(u_{(x,0),r}|_{\mathcal{O}_{(x,0),r}}) \rightharpoonup \Delta u_0$ as $r \to 0$ and

$$\begin{aligned}
&\Delta u_{(x,0),r}|_{\mathcal{O}_{(x,0),r}}(B_2^2(0,0)) \\
&\geq \int_{\partial_{\mathrm{red}}\mathcal{O}_{(x,0),r}\cap B_2^2(0,0)} |y|^{\gamma}\, d\sigma \\
&\gtrsim \Big|\sup_{0<\rho<1}\{|x_2| : (x_1,x_2) \in \partial\mathcal{O}_{(x,0),r} \cap B_2^2(0,0), |x_1| = \rho\}\Big|^{2\gamma} \to 0. \qquad (2.1)
\end{aligned}$$

Thus, Theorem 1.2 follows from Theorem 1.7.

3. Proof of Theorem 1.7

3.1. Main geometric observations

The following geometric observation was inspired by the insight that if $(x,0) \in \Sigma_{\mathrm{cusps}}$, then $\partial\{u > 0\}$ must approach $\{x_2 = 0\}$ tangentially in the sense of (2.1). That is,

$$\theta^2_{Q^{2\gamma},\mathcal{O}}(x,0) = 0,$$

and (2.1) implies that $\mathrm{Height}(1, \mathcal{O}_{(x,0),r}) \to 0$ as $r \to 0$. Therefore, we may not expect a component of $\{u > 0\}$ which touches $(x,0)$ to be contained in a set of the form $\{(x_1,x_2) : |x_2| \geq m|x_1|\}$ in any neighborhood of $(x,0)$ for any $0 < m$. The content of the lemma below is that if we weaken this to consider sets of the form $\{(x_1,x_2) : |x_2| \geq m|x_1| - b\}$, then for appropriate choices of $0 < m, b$ we can find a neighborhood in which a large piece of $\partial\{u > 0\}$ is contained in such a set.

Lemma 3.1 (Main geometric observation). *Let $n = 2$, $0 < \gamma$, and let u be a weak solution to (P_γ) that satisfies (1.2). Assume that $(x,0) \in \Sigma_{\mathrm{cusps}}$ with $\delta = \mathrm{dist}((x,0), \partial\Omega)$. Let $\mathcal{O}$ be a component of $\{u > 0\} \cap B_\delta((x,0))$ such that $(x,0) \in \partial\mathcal{O}$. For any $0 < C_2 \leq \frac{1}{2}$, if there exists a radius $0 < r_0 \leq \delta$ such that*

$$\mathrm{Height}(1, \mathcal{O}_{(x,0),r_0}) \leq C_2,$$

then there exists a $0 < \rho \leq r_0$ such that $\mathrm{Height}(1, \mathcal{O}_{(x,0),\rho}) \leq C_2$ and one of the following conditions hold:

$$\mathrm{Height}(r, \mathcal{O}_{(x,0),\rho}) \geq 3\,\mathrm{Height}(1, \mathcal{O}_{(x,0),\rho})r - 2\,\mathrm{Height}(1, \mathcal{O}_{(x,0),\rho}) \qquad (3.1)$$

for all $r \in [2/3, 1]$, or

$$\mathrm{Height}(r, \mathcal{O}_{(x,0),\rho}) \geq -3\,\mathrm{Height}(1, \mathcal{O}_{(x,0),\rho})r - 2\,\mathrm{Height}(1, \mathcal{O}_{(x,0),\rho}) \qquad (3.2)$$

for all $r \in [-1, -2/3]$.

Remark 3.2. The lines described by equality in (3.1) and (3.2) are the lines which intersects the points $(\pm 1, \mathrm{Height}(1, \mathcal{O}_{(x,0),\rho}))$ and $(\pm 2/3, 0)$, respectively.

Proof. Let u, $0 < r_0$, and $0 < C_2 < 1/2$ be given. For ease of notation, we note that, by reflection across the x- and y-axes if necessary, we may assume that u attains $\mathrm{Height}(1, \mathcal{O}_{(x,0),r_0})$ in $\{(1, x_2) : x_2 \in \mathbb{R}_+\}$ and not $\{(1, x_2) : x_2 \in \mathbb{R}_-\}$ or $\{(-1, x_2) : x_2 \in \mathbb{R}\}$.

Let L_r be the line given by the graph of the function

$$x_2 = 3\,\mathrm{Height}(r, \mathcal{O}_{(x,0),r_0})x_1 - 2\,\mathrm{Height}(r, \mathcal{O}_{(x,0),r_0}).$$

We claim that we can find an $0 < \frac{1}{2} < r \leq 1$ such that

$$\mathrm{Height}(x_1, \mathcal{O}_{(x,0),r_0}) \geq 3\,\mathrm{Height}(r, \mathcal{O}_{(x,0),r_0})x_1 - 2\,\mathrm{Height}(r, \mathcal{O}_{(x,0),r_0})$$

for all $\frac{2}{3}r \leq x_1 \leq r$. If we can find such a radius, then $\rho = r \cdot r_0$ proves the lemma.

To prove the claim, we argue by contradiction. Let $r_1 = 1$. If r_i does not satisfy the claim, then there must exist a radius $\frac{2}{3}r_i < r < r_i$ such that

$$\mathrm{Height}(r, \mathcal{O}_{(x,0),r_0}) < 3\,\mathrm{Height}(r_i, \mathcal{O}_{(x,0),r_0})r - 2\,\mathrm{Height}(r_i, \mathcal{O}_{(x,0),r_0}). \tag{3.3}$$

Let $r_{i+1} \in [2r_i/3, r_i]$ be defined by

$$r_{i+1} := \inf\{r \in (2r_i/3, r_i) : (3.3) \text{ holds}\}.$$

Observe that by construction

$$\mathrm{Height}(r_{i+1}, \mathcal{O}_{(x,0),r_0}) < \mathrm{Height}(r_i, \mathcal{O}_{(x,0),r_0}). \tag{3.4}$$

If the inductively defined sequence $\{r_i\}_i$ does not terminate in finitely many steps with a radius which satisfies the claim, then $\{r_i\}_i$ forms a monotonically decreasing sequence in $[1/2, 1]$, and there is a limit point $r_\infty \in [\frac{1}{2}, 1]$ such that $r_i \to r_\infty$. By (3.4), there are two possibilities: either $\mathrm{Height}(r_\infty, \mathcal{O}_{(x,0),r_0}) > 0$ or $\mathrm{Height}(r_\infty, \mathcal{O}_{(x,0),r_0}) = 0$. The latter case contradicts the assumption that $\mathcal{O}_{(x,0),r_0}$ is a connected component which touches $(0, 0)$. Therefore, we may assume that $\mathrm{Height}(r_\infty, \mathcal{O}_{(x,0),r_0}) > 0$. We claim that $r_\infty = r$.

By the convergence of $\{r_i\}$ and the fact that $\{\mathrm{Height}(r_i, \mathcal{O}_{(x,0),r_0})\}_i$ also forms a Cauchy sequence, the sets $\{L_{r_i} \cap [-1, 1]^2\}$ converge in the Hausdorff metric on compact subsets to the set $L_{r_\infty} \cap [-1, 1]^2$. And since $0 < \mathrm{Height}(r_\infty, \mathcal{O}_{(x,0),r_0}) \leq 1/2$, we may estimate $\mathrm{slope}(L_{r_\infty}) \in (0, 2/3]$. Therefore, for any $0 < \delta$, there exists an $i(\delta) \in \mathbb{N}$ such that

$$\mathrm{dist}_{\mathcal{H}}(L_{r_\infty} \cap [-1, 1]^2, L_{r_j} \cap [-1, 1]^2) \leq \delta$$

for all $j \geq i(\delta)$. Therefore, if $r' \in [2r_{i(\delta)}/3, r_\infty]$ and

$$\big(3\,\mathrm{Height}(r_\infty, \mathcal{O}_{(x,0),r_0})r' - 2\,\mathrm{Height}(r_\infty, \mathcal{O}_{(x,0),r_0})\big) - \mathrm{Height}(r', \mathcal{O}_{(x,0),r_0}) \geq 4\delta > 0,$$

then

$$\big(3\,\mathrm{Height}(r_j, \mathcal{O}_{(x,0),r_0})r' - 2\,\mathrm{Height}(r_j, \mathcal{O}_{(x,0),r_0})\big) - \mathrm{Height}(r', \mathcal{O}_{(x,0),r_0}) \geq \delta > 0,$$

and by the minimality in the definition of r_{i+1}, for all $j \geq i(\delta)$ $r' \notin [2r_j/3, r_\infty)$. Letting $j \to \infty$, we may assume that $r' \notin (2r_\infty/3, r_\infty)$. Repeating the argument for $\delta \to 0$ shows that the claim holds. This proves the lemma. ∎

Using orthogonal projection, we obtain the following simple corollary.

Corollary 3.3. *Let $n = 2$, $0 < \gamma$, and let $u, (x, 0), \mathcal{O}, C_2, r_0, \rho$ be as in the statement of Lemma 3.1. Then,*

$$\int_{\partial\mathcal{O}_{(x,0),\rho}\cap[-1,1]^2} |x_2|^\gamma \, d\sigma \geq \int_{\mathcal{O}_{(x,0),\rho}\cap\partial[-1,1]^2} \frac{1}{6C_2}|x_2|^\gamma \, d\sigma.$$

Proof. By reflection, without loss of generality, we may assume that

$$\mathcal{O}_{(x,0),\rho} \cap [-1, 1]^2 \subset [0, 1]^2.$$

Define $\partial^+\mathcal{O}_{(x,0),\rho}$ to be the set

$$\begin{aligned}\partial^+&\mathcal{O}_{(x,0),\rho}\\ &:= \{(x_1, x_2) \in \partial\mathcal{O}_{(x,0),\rho} : x_2 = \operatorname{Height}(|x_1|, \mathcal{O}_{(x,0),\rho}),\ 2/3 \leq x_1 \leq 1\}\\ &\quad\cap \{(x_1, x_2) \in \partial\mathcal{O}_{(x,0),\rho} : x_2 \geq 3 \operatorname{Height}(1, \mathcal{O}_{(x,0),\rho})x_1 - 2 \operatorname{Height}(1, \mathcal{O}_{(x,0),\rho})\}.\end{aligned}$$

Let π_1 be an orthogonal projection onto $\{x_2 = 0\}$, and let π_2 be an orthogonal projection onto the line $\{x_1 = 1\}$. If f is the linear function such that $L_r = \operatorname{graph}_{\mathbb{R}}(f)$, define

$$\pi_{L_r} : \mathbb{R}^2 \to L_r$$

to be the function $\pi_{L_r}(x_1, x_2) = (x_1, f(x_1))$. Note that, for $C_2 \leq 1/2$, $|\nabla f| \leq 3/2$.

We observe that, for any set $U \subset \mathbb{R}^2$,

$$\begin{aligned}\mathcal{H}^1(U) &\geq \mathcal{H}^1(\pi_1(U))\\ &\geq \frac{1}{\sqrt{1 + (3/2)^2}}\mathcal{H}^1(\pi_{L_r}(U)) > \frac{1}{2}\mathcal{H}^1(\pi_{L_r}(U)).\end{aligned}$$

Then, (3.1), Definition 2.1 (3), and the choice of $C_2 \leq \frac{1}{2}$ imply that

$$\begin{aligned}\int_{\partial\mathcal{O}_{(x,0),\rho}\cap[-1,1]^2} |x_2|^\gamma \, d\sigma &\geq \int_{\partial^+\mathcal{O}_{(x,0),\rho}\cap[-1,1]^2} |x_2|^\gamma \, d\sigma\\ &\geq \frac{1}{2}\int_{\pi_{L_r}(\partial^+\mathcal{O}_{(x,0),\rho})\cap([\frac{2}{3},1]\times[0,1])} |x_2|^\gamma \, d\sigma\\ &\geq \frac{1}{2}\int_{\pi_2(L_r\cap([\frac{2}{3},1]\times[0,1]))} \sqrt{(3C_2)^{-2} + 1}|x_2|^\gamma \, d\sigma\\ &\geq \frac{1}{2}\int_{\mathcal{O}_{(x,0),r_2}\cap(\{1\}\times[0,1])} \frac{1}{3C_2}|x_2|^\gamma \, d\sigma.\end{aligned}$$

∎

3.2. Proof of Theorem 1.7

Let u satisfy (1.2) with constant $C < \infty$ in $B_r((x,0)) \subset \Omega$. We argue by contradiction. Let $0 < r_0$ be such that $0 < \text{Height}(1, \mathcal{O}_{(x,0),r_0}) \leq C_2 \leq 1/2$. Let $0 < \rho \leq r_0$ as in Lemma 3.1. We consider

$$V := \mathcal{O}_{(x,0),\rho} \cap \partial[-1,1]^2.$$

And note that

$$\int_V |x_2|^\gamma d\sigma \leq \int_{\partial D \cap [-1,1]^2} |x_2|^\gamma d\sigma$$

for any set D which is relatively open in $[-1,1]^2$ and satisfies $V \subset (\partial[-1,1]^2 \cap D)$.

Next, use Definition 2.1 (3), the divergence theorem, and (1.2) to calculate

$$\begin{aligned}\int_{\partial\mathcal{O}_{(x,0),\rho} \cap [-1,1]^2} |x_2|^\gamma d\sigma &\leq \Delta u\big(\overline{\mathcal{O}_{(x,0),\rho}} \cap [-1,1]^2\big) \\ &= \int_V \nabla u \cdot \vec{\eta} d\sigma \\ &\leq \int_V C|x_2|^\gamma d\sigma.\end{aligned}$$

But, by Corollary 3.3,

$$\int_{\partial\mathcal{O}_{(x,0),\rho} \cap [-1,1]^2} |x_2|^\gamma d\sigma \geq \int_{V_r} \frac{1}{6C_2} |x_2|^\gamma d\sigma.$$

Therefore, taking $6C_2 \leq \frac{1}{C}$, we have a contradiction. This proves Theorem 1.7.

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