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A combinatorial approach for computing the determinants of the generalized Vandermonde matrices

Edgar Enochs, Mohammad Reza Pournaki and Siamak Yassemi

E. Enochs is a professor emeritus of mathematics at the University of Kentucky, USA. He obtained his BSc from Louisiana State University and his PhD from the University of Notre Dame. He was an instructor at the University of Chicago (1958–1960), an assistant, associate and full professor at the University of South Carolina (1960–1967), and then a professor at the University of Kentucky (1967–2015). He was director of 52 PhD theses.

M. R. Pournaki is a professor of mathematics at Sharif University of Technology, Iran. He obtained his BSc and MSc from Sharif University of Technology and his PhD from the University of Tehran. He was at the Institute for Research in Fundamental Sciences (IPM) in Tehran for about 20 years as a student researcher, post doctoral research fellow and associate researcher.

S. Yassemi is a professor of mathematics at the University of Tehran, Iran. He obtained his PhD from Copenhagen University under supervision of H. B. Foxby. In 2018, he was elected by The World Academy of Sciences (TWAS) as a fellow member. In 2019, he was named Chevalier of the Ordre des Palmes Académiques for distinguished effort on extended multidimensional scientific cooperation.

1 Introduction

Alexandre-Théophile Vandermonde, a French musician and mathematician, was born in Paris in 1735. His first love was music and he pursued a music career. He turned to math-

Die Formel für die Determinante der Vandermonde-Matrizen ist ein Klassiker der linearen Algebra. Das Ergebnis hat zahlreiche Anwendungen in der Physik und in der Mathematik. Es scheint jedoch keine Verallgemeinerung dieser Formel für die Determinante der quadratischen Untermatrizen der Vandermonde-Matrix zu geben, zumindest wurde dies vermutet. Aber auch hier gäbe es viele nützliche Anwendungen für diese Determinanten. In der vorliegenden Arbeit wird nun ein kombinatorischer Ansatz zur Berechnung dieser Determinanten vorgeschlagen. Dieser Ansatz führt zu einer Vermutung über diese Determinanten im Allgemeinen. Um die Vermutung zu untermauern, wird gezeigt, dass sie für den kleinsten nichttrivialen Fall von 3×3 Untermatrizen gilt. Die Autoren hoffen, dass andere dazu motiviert werden, den höherdimensionalen Fall zu betrachten, gegebenenfalls mit natürlichen Einschränkungen.

ematics and worked with Bézout when he was 35 years old. He is remembered for his contributions to the theory of determinants. A Vandermonde matrix, named after him, is a matrix with the form

$$\begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix}.$$

The matrix is well known in linear algebra with various applications such as FFT in signal processing, polynomial interpolations, having close relationship with Frobenius formula in representation theory, etc. Finding a formula for the determinant of the Vandermonde matrix is, with the right approach, easy. This formula is known to be $\prod_{1\leq i<j\leq n}(x_j - x_i)$, which is a beautiful result with many applications in both physical and mathematical sciences. There are also some generalizations of the Vandermonde matrix, of which the most well known are

$$\begin{bmatrix} x_1^{k_1} & x_1^{k_2} & \cdots & x_1^{k_n} \\ x_2^{k_1} & x_2^{k_2} & \cdots & x_2^{k_n} \\ \vdots & \vdots & \vdots & \vdots \\ x_n^{k_1} & x_n^{k_2} & \cdots & x_n^{k_n} \end{bmatrix},$$

where the k_i's are integers satisfying $0 \leq k_1 < \cdots < k_n$. Computing the determinants of this kind of generalized Vandermonde matrices is an elusive problem, and there seems to be no generalizations of the above-mentioned formula for these determinants. However, again, there are many useful applications of these determinants when they can be found.

In this paper, we give a way to compute the determinants of such 3×3 generalized Vandermonde matrices. The solution involves a simplicial complex and suggests there may be a combinatorial approach for finding the determinants of larger such matrices. Indeed, our approach leads to a conjecture about these determinants in general. In support of the conjecture, we prove that it holds true for the smallest nontrivial dimensional case, that is, the 3×3 case. We hope others will be motivated to test the higher-dimensional case, perhaps with natural restrictions. For further study, we refer the reader to the paper of Kalman [3] which gives a clear introduction to the generalized Vandermonde matrices and their applications. Also, the book of Horn and Johnson [2] is useful for a detailed exposition of the theory of matrices.

2 A combinatorial viewpoint

In this section, we give a combinatorial way for computing the determinants of the 3×3 generalized Vandermonde matrices. In order to do this, we first prove a formula for the determinants of such 3×3 matrices, which involves a homogeneous polynomial $\alpha(x, y, z)$. Using this homogeneous polynomial, we approach a combinatorial way for computing the determinants of such 3×3 matrices. Finally, we give some questions and propose a conjecture about higher dimensions, where our results prove the conjecture for dimension three.

2.1 A formulaic approach

In the sequel, our matrices only involve three variables, and thus, for simplicity, we call these variables x, y and z. Therefore, given integers $1 \leq n < p$, our problem is to compute the determinant of the matrix

$$V = \begin{bmatrix} 1 & x^n & x^p \\ 1 & y^n & y^p \\ 1 & z^n & z^p \end{bmatrix}.$$

The following theorem leads to a combinatorial way for computing the determinant of the 3×3 generalized Vandermonde matrix V. It is worthwhile to mention that, from now on, the summations run over pairs or triples of *nonnegative* integers with some restrictions.

Theorem 2.1. *The determinant of the* 3×3 *generalized Vandermonde matrix* V *is equal to* $\det V = (y-x)(z-y)(z-x)\alpha(x,y,z)$*, where*

$$\alpha(x,y,z) = \sum_{\substack{i+j+k=n+p-2 \\ n-1 \leq i+j < p-1 \leq j+k}} y^j x^i z^i (z^{k-i-1} + z^{k-i-2}x + \cdots + zx^{k-i-2} + x^{k-i-1}).$$

Proof. To begin, we subtract the second row of V from the third, and then the first from the second. We get

$$\det V = \det \begin{bmatrix} 1 & x^n & x^p \\ 0 & y^n - x^n & y^p - x^p \\ 0 & z^n - y^n & z^p - y^p \end{bmatrix},$$

which, in turn, is equal to

$$\det V = \det \begin{bmatrix} y^n - x^n & y^p - x^p \\ z^n - y^n & z^p - y^p \end{bmatrix}.$$

Recalling that $n, p \geq 1$, the elements of the first row in the above determinant are divisible by $y - x$ and those of the second by $z - y$. Factoring these out, we see that the determinant of V is equal to

$$\det V = (y-x)(z-y) \det \begin{bmatrix} \sum_{i+j=n-1} x^i y^j & \sum_{i+j=p-1} x^i y^j \\ \sum_{j+k=n-1} y^j z^k & \sum_{j+k=p-1} y^j z^k \end{bmatrix}.$$

Note that the desired factor $z - x$ in the statement of the theorem is missing. Before proceeding the proof of Theorem 2.1, we need the following lemma.

Lemma 2.2. *For any integers* $n, p \geq 1$*, the following equality holds true:*

$$\Big(\sum_{i+j=n-1} x^i y^j \Big) \Big(\sum_{j+k=p-1} y^j z^k \Big) = \sum_{\substack{i+j+k=n+p-2 \\ i+j \geq n-1 \\ j+k \geq p-1}} x^i y^j z^k.$$

Proof. The terms of the product in the left-hand side of the above formula are of the form $x^i y^{j_1} y^{j_2} z^k$ with $i + j_1 = n - 1$ and $j_2 + k = p - 1$. Note that $x^i y^{j_1} y^{j_2} z^k = x^i y^{j_1 + j_2} z^k$ and let $j_1 + j_2 = j$. Thus, we get $x^i y^j z^k$, where

$$i + j + k = i + j_1 + j_2 + k = n - 1 + p - 1 = n + p - 2$$

and where $i + j \geq n - 1$ since $i + j \geq i + j_1 = n - 1$, and similarly, where $j + k \geq p - 1$.

Conversely, given a term $x^i y^j z^k$ with $i + j + k = n + p - 2$ and with $i + j \geq n - 1$ and $j + k \geq p - 1$, note that we must have $i \leq n - 1$. For if $i > n - 1$, then since $j + k \geq p - 1$, we get $i + j + k > n + p - 2$, while $i + j + k = n + p - 2$. Similarly, we must have $k \leq p - 1$. Thus, since $i \leq n - 1$ and $i + j \geq n - 1$, there exists $j_1 \leq j$ such that $i + j_1 = n - 1$. Let $j_2 = j - j_1$. This implies that

$$i + j_1 + j_2 + k = i + j + k = n + p - 2 = n - 1 + p - 1,$$

and thus, we get $j_2 + k = p - 1$.

By the above observation, we see that we get a bijective correspondence between the (i, j, k) with $i + j + k = n + p - 2$, $i + j \geq n - 1$ and $j + k \geq p - 1$, and pairs of pairs (i, j_1) and (j_2, k) such that $i + j_1 = n - 1$ and $j_2 + k = p - 1$. This says that we can rewrite the product $\left(\sum_{i+j=n-1} x^i y^j\right)\left(\sum_{j+k=p-1} y^j z^k\right)$ as

$$\sum_{\substack{i+j+k=n+p-2 \\ i+j \geq n-1 \\ j+k \geq p-1}} x^i y^j z^k,$$

which completes the proof of Lemma 2.2. ■

Now, we continue the proof of Theorem 2.1. By applying Lemma 2.2, we obtain

$$\det V = (y - x)(z - y)\Bigg(\sum_{\substack{i+j+k=n+p-2 \\ i+j \geq n-1 \\ j+k \geq p-1}} x^i y^j z^k - \sum_{\substack{i+j+k=n+p-2 \\ i+j \geq p-1 \\ j+k \geq n-1}} x^i y^j z^k\Bigg).$$

We want to subtract off the common terms in the difference appearing in the above equality. Recalling that $n < p$, we see that these are the $x^i y^j z^k$ with $i + j + k = n + p - 2$, $i + j \geq p - 1$ and $j + k \geq p - 1$. After subtracting the common terms and reversing the order in which we write our inequalities, we obtain

$$\det V = (y - x)(z - y)\Bigg(\sum_{\substack{i+j+k=n+p-2 \\ n-1 \leq i+j < p-1 \leq j+k}} x^i y^j z^k - \sum_{\substack{i+j+k=n+p-2 \\ n-1 \leq j+k < p-1 \leq i+j}} x^i y^j z^k\Bigg).$$

Note that if we have a term $x^i y^j z^k$ in the first sum of the difference appearing in the above equality, then we have the term $x^k y^j z^i$ in the second sum. Thus, our difference can be written as

$$\sum_{\substack{i+j+k=n+p-2 \\ n-1 \leq i+j < p-1 \leq j+k}} (x^i y^j z^k - x^k y^j z^i).$$

Our inequalities $i + j < p - 1 \leq j + k$ force the inequality $k - i > 0$. Thus, we may write the general term of the latter sum as follows:

$$\begin{aligned} x^i y^j z^k - x^k y^j z^i &= y^j (x^i z^k - x^k z^i) \\ &= y^j x^i z^i (z^{k-i} - x^{k-i}) \\ &= y^j x^i z^i (z - x)(z^{k-i-1} + z^{k-i-2} x + \cdots + z x^{k-i-2} + x^{k-i-1}). \end{aligned}$$

We now have the missing factor $z - x$ in each term, and thus, factoring it out, we see that $\det V = (y - x)(z - y)(z - x)\alpha(x, y, z)$, where $\alpha(x, y, z)$ is

$$\sum_{\substack{i+j+k=n+p-2 \\ n-1 \leq i+j < p-1 \leq j+k}} y^j x^i z^i (z^{k-i-1} + z^{k-i-2} x + \cdots + z x^{k-i-2} + x^{k-i-1}).$$

This completes the proof of Theorem 2.1. ■

2.2 A combinatorial approach

We now give a combinatorial way for computing the determinant of the 3×3 generalized Vandermonde matrix V. In order to do this, by Theorem 2.1, we need to focus on $\alpha(x, y, z)$, which is a homogeneous polynomial of degree $n + p - 3$. Suppose that $a x^i y^j z^k$ with $a \neq 0$ is a term of $\alpha(x, y, z)$. We substitute 1 for y in this term, and we get $a x^i z^k$. Conversely, if $a x^i z^k$, $a \neq 0$, is gotten in this manner, then we know j of the term $a x^i y^j z^k$ since $i + j + k = n + p - 3$. This means that if we find $\alpha(x, 1, z)$, then $\alpha(x, y, z)$ is completely known, and so is $\det V$. Thus, our purpose is indeed to give a combinatorial way for finding $\alpha(x, 1, z)$. In this direction, we consider

$$\alpha(x, y, z) = \sum_{\substack{i+j+k=n+p-2 \\ n-1 \leq i+j < p-1 \leq j+k}} y^j (z^{k-1} x^i + \cdots + z^i x^{k-1}),$$

and by substituting 1 for y, we get

$$\alpha(x, 1, z) = \sum_{\substack{i+j+k=n+p-2 \\ n-1 \leq i+j < p-1 \leq j+k}} (z^{k-1} x^i + \cdots + z^i x^{k-1}).$$

We want to express our inequalities in i and k alone by using the equality $i + j + k = n + p - 2$. To this end, consider $n - 1 \leq i + j$. This becomes $n - 1 \leq n + p - 2 - k$ or $k \leq p - 1$. In a similar manner, $i + j < p - 1$ becomes $i + j < (i + j + k - n + 2) - 1$ or $n - 1 < k$, and $p - 1 \leq j + k$ becomes $p - 1 \leq n + p - 2 - i$ or $i \leq n - 1$. Thus, our three inequalities are

$$i \leq n - 1 < k \leq p - 1.$$

Note that if i and k satisfy these latter inequalities, then

$$i + k \leq n - 1 + p - 1 = n + p - 2.$$

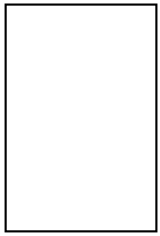

Figure 1. The rectangle regarding our indices

Thus, we obtain

$$\alpha(x,1,z) = \sum_{i\leq n-1<k\leq p-1} (z^{k-1}x^i + \cdots + z^i x^{k-1}).$$

Finally, by shifting the indices, we get

$$\alpha(x,1,z) = \sum_{i\leq n-1\leq k\leq p-2} (z^{k}x^i + \cdots + z^i x^{k}).$$

We are now in the position to think geometrically. From now on, the rectangles are considered with boundary and interior. By considering the (i,k)-plane, our indices with

$$0 \leq i \leq n-1 \leq k \leq p-2$$

correspond to the points with integer coordinates in the rectangle appearing in Figure 1. Note that this rectangle can be degenerate, that is, it can be a proper line segment (when $n-1=0$, $p-2>0$ or $p-2=n-1$, $n-1>0$) or even a point (when $n-1=p-2=0$). We are interested in computing $\alpha(x,1,z)$, which is the sum of polynomials $z^k x^i + \cdots + z^i x^k$, when the summation runs over all pairs (i,k) in our rectangle. Given integers $s,t\geq 0$, we let $C(s,t)$ be the coefficient of $x^s z^t$ in

$$\alpha(x,1,z) = \sum_{i\leq n-1\leq k\leq p-2} (z^{k}x^i + \cdots + z^i x^{k}).$$

Therefore, $C(s,t)$ is the number of (i,k) in our rectangle such that $x^s z^t$ is one of the terms of $z^k x^i + \cdots + z^i x^k$. Note that $\alpha(x,1,z)$ is completely determined by calculating the coefficients $C(s,t)$. Thus, our purpose would be in fact to give a combinatorial way for calculating each $C(s,t)$. In proceeding, we want to state this calculation in a geometric fashion.

We need to introduce a notion that plays an important role for us. Let a, b, c and d be nonnegative integers. Then, by the *lattice segment* $[(a,b),(c,d)]$, we mean the set of all points (s,t) with integer coordinates on the line segment from (a,b) to (c,d). For example, the lattice segment $[(3,0),(0,0)]$ is as follows:

$$[(3,0),(0,0)] = \{(3,0),(2,0),(1,0),(0,0)\}.$$

For our purpose, that is, for calculating each $C(s,t)$, we will be interested in such lattice segments having "slope" -1, and in particular, such lattice segments as

$$[(i,k),(k,i)] = \{(i,k),(i+1,k-1),\dots,(k-1,i+1),(k,i)\},$$

where $i \leq k$. (Note that (k,i) is the reflection of (i,k) through the line $i = k$.) By using this lattice segment, we are led to a combinatorial way for computing $C(s,t)$ given above. Indeed, $C(s,t)$ is equal to the number of lattice segments $[(i,k),(k,i)]$ with $(i,k) \in [0,n-1] \times [n-1,p-2]$ such that $(s,t) \in [(i,k),(k,i)]$. It is worthwhile to mention that this description of $C(s,t)$, together with noting that $(s,t) \in [(i,k),(k,i)]$ if and only if $(t,s) \in [(i,k),(k,i)]$, shows that $C(s,t)$ is symmetric in s and t, that is, $C(s,t) = C(t,s)$.

To illustrate, suppose that $n-1 = 2$ and $p-2 = 5$. We want to compute $C(3,3)$. We have $(1,5) \in [0,2] \times [2,5]$ and

$$[(1,5),(5,1)] = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}.$$

Thus, we get a contribution of one to $C(3,3)$ using the point $(1,5)$ in our rectangle. The only other point in the rectangle which makes such a contribution is $(2,4)$. Therefore, $C(3,3) = 2$. Hence, 2 is the coefficient of x^3z^3 in $\alpha(x,1,z)$.

2.3 A ladder to the higher dimensions

We need to elaborate in order to think about the higher dimensions. Given $1 \leq n \leq p-1$, we let R be the rectangle appearing in Figure 2. If we reflect R about the line $i = k$, then we get the rectangle R'. If we combine R and R', we get the two rectangles appearing in Figure 3. If we add the two skew dashed line segments, we then get a hexagon H with dashed boundary appearing in Figure 3. From now on, for simplicity, by a hexagon, we mean the set of all points with integer coordinates in this hexagon. Thus, our hexagon H

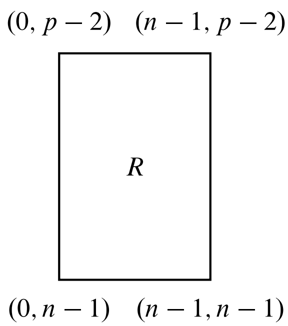

Figure 2. The rectangle R

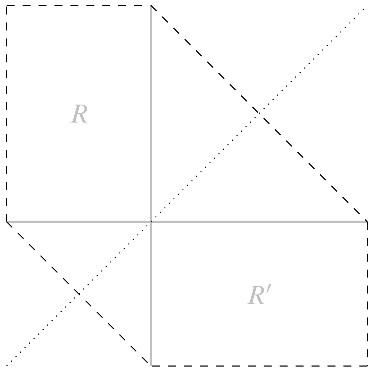

Figure 3. Combining the rectangles R and R'

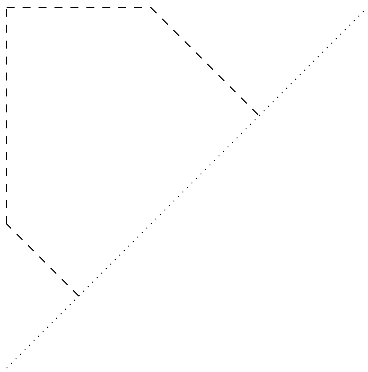

Figure 4. One half of the curve $C(s,t) = 1$

is indeed the union of all lattice segments $[(i,k),(k,i)]$ with $(i,k) \in R$, which, in turn, is equal to the set of all (s,t) with $C(s,t) \geq 1$. But we are interested in exploring the set of all (s,t) with $C(s,t) = 1$. It is not hard to see that, for all (s,t) on the parts of the boundary of H appearing in Figure 4, we have $C(s,t) = 1$. Using the symmetry of $C(s,t)$, we get $C(s,t) = 1$ for all (s,t) on the boundary of H. In fact, these are precisely all of the points (s,t) with $C(s,t) = 1$. Therefore, if we think of $C(s,t) = 1$ as a contour curve, we get that this curve is precisely the boundary of H.

Now that we have indicated how we argue, we will give other results without formal proofs. We recall that our purpose is to give a combinatorial description of $\alpha(x,y,z)$, via $\alpha(x,1,z)$, that possibly has some extension to the higher dimensions. Let us first make some comments about the rectangle R. By recalling the coordinates of R, we see that R is a square if and only if $(n-1)-0 = (p-2)-(n-1)$, so if and only if $p = 2n$. Also, note that R can be a degenerate rectangle, so a proper line segment (when $n-1 = 0$, $p-2 > 0$ or $p-2 = n-1$, $n-1 > 0$) or a point (when $n-1 = p-2 = 0$). In this case, H becomes a proper triangle (so here a degenerate hexagon) or a point.

We now consider the contour curve $C(s,t) = 2$. By sketching H, we see that $C(s,t) = 2$ holds true for all (s,t) which lie on the following lattice segments:

$$[(n-1,1),(1,n-1)], \qquad [(1,n-1),(1,p-3)],$$
$$[(1,p-3),(n-1,p-3)], \quad [(n-1,p-3),(p-3,n-1)].$$

Now, by using the symmetry of $C(s,t)$, we can argue that the graph of $C(s,t) = 2$ consists of these four lattice segments and their reflections through the line $i = k$. Thus, the graph of $C(s,t) = 2$ is the boundary of a hexagon contained in our original hexagon H. We can repeat and get the curves

$$C(s,t) = 1, \quad C(s,t) = 2, \quad \ldots,$$

which are the boundaries of the hexagons and each of which contained the hexagon corresponding to the previous curve. The procedure stops when we finally get that $C(s,t) = k$ corresponds to a degenerate hexagon, that is, a triangular or a point. In this stage, k is the maximum value of $C(s,t)$, and this function acquires its maximum k on all points of this degenerate hexagon. It can be argued that the maximum value of $C(s,t)$ is n in case $p \geq 2n$ and is $p-n$ in case $p \leq 2n$ (recall that $p = 2n$ only when R is a square).

We close the above discussion by giving two degenerate examples. If R is the lattice segment $[(0,0),(0,5)]$, then H is the triangle with vertices $(0,0), (0,5), (5,0)$. Thus, $C(s,t) = 1$ for all (s,t) in this triangle and $C(s,t) = 0$ otherwise. Therefore,

$$\alpha(x,1,z) = \sum_{i+k\leq 5} x^i z^k.$$

If R is the lattice segment $[(0,0),(0,0)]$, then we easily get $C(0,0) = 1$ and $C(s,t) = 0$ otherwise. Therefore,

$$\alpha(x,1,z) = 1.$$

Given our H and function $C(s,t)$, we can think of $C(s,t)$ as a height function. In this case, the points $(s,t,C(s,t))$ will form a simplicial complex. It follows from the earlier remarks that this complex will be a (maybe truncated) pyramid sitting on a face of height one. In this regard, we want to give some questions and propose a conjecture, which papers [1,4] add a little weight to them. We conclude the paper by stating these questions and our conjecture.

2.4 Some questions and a conjecture

In order to make the questions precise, one needs to generalize the above-mentioned notations to the higher dimensions. Let $i_1,\ldots,i_n$ and $j_1,\ldots,j_n$ be nonnegative integers ($n \geq 3$). Then $[(i_1,\ldots,i_n),(j_1,\ldots,j_n)]$ and $\alpha(x_1,\ldots,x_n)$ could be defined in a similar way as we defined them for the two-dimensional case and for the 3×3 generalized Vandermonde matrices. Now, let $C(i_1,\ldots,i_n)$ be the coefficient of $x_1^{i_1}\ldots x_n^{i_n}$ in the corresponding $\alpha(x_1,\ldots,x_n)$. By [4], the coefficients $C(i_1,\ldots,i_n)$ of $\alpha(x_1,\ldots,x_n)$ are all nonnegative (see also [1]). Also, in [1,4], the authors have provided a formula for $\sum C(s_1,\ldots,s_n)$, where the summation runs over all $(s_1,\ldots,s_n)$ in a suitable set, say H, as above. For this purpose, we raise the following questions.

Questions 2.3. *Let $n \geq 3$ be an integer and suppose that $C(i_1, \ldots, i_n)$ is the coefficient of $x_1^{i_1} \ldots x_n^{i_n}$ in the corresponding $\alpha(x_1, \ldots, x_n)$.*

(1) *Is the set of $(i_1, \ldots, i_n)$ with $C(i_1, \ldots, i_n) > 0$ convex? Here, convex means that if $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ are two such points, then all*

$$(s_1, \ldots, s_n) \in [(i_1, \ldots, i_n), (j_1, \ldots, j_n)]$$

satisfy $C(s_1, \ldots, s_n) > 0$. If this condition does not hold, is the set at least pathwise connected? Here, our paths are sequences of such lattice segments satisfying the obvious conditions (for example, concerning end points of adjacent segments).

(2) *The same question as above about the set of $(i_1, \ldots, i_n)$ with $C(i_1, \ldots, i_n) \geq 1$.*

(3) *Is the set of $(i_1, \ldots, i_n)$ with $C(i_1, \ldots, i_n) = 1$ pathwise connected? If so, are there simple such paths?*

(4) *Is there a way to find the largest value of $C(i_1, \ldots, i_n)$?*

(5) *If $C(s_1, \ldots, s_n) = k$ and $C(t_1, \ldots, t_n) = \ell$, where k and ℓ are nonnegative integers, does C take every integer value between k and ℓ?*

We now state our conjecture, but with some explanations. Let us state the conjecture for the following 4×4 generalized Vandermonde matrix, where $1 \leq k_1 < k_2 < k_3$ are some integers:

$$\begin{bmatrix} 1 & x_1^{k_1} & x_1^{k_2} & x_1^{k_3} \\ 1 & x_2^{k_1} & x_2^{k_2} & x_2^{k_3} \\ 1 & x_3^{k_1} & x_3^{k_2} & x_3^{k_3} \\ 1 & x_4^{k_1} & x_4^{k_2} & x_4^{k_3} \end{bmatrix}.$$

Let $\alpha(x_1, x_2, x_3, x_4)$ as usual, that is, the determinant of the above matrix divided by $\prod_{1 \leq i < j \leq 4}(x_j - x_i)$. Our object is to find the coefficients $C(i_1, i_2, i_3, i_4)$ of $x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$ in $\alpha(x_1, x_2, x_3, x_4)$. Let $k = k_1 + k_2 + k_3 - 6$, which is the degree of the homogeneous polynomial $\alpha(x_1, x_2, x_3, x_4)$. It is known that $C(i_1, i_2, i_3, i_4) \geq 0$ for all (i_1, i_2, i_3, i_4) (see [1,4]). Thus, we consider the set of (i_1, i_2, i_3, i_4) with $i_1 \geq 0$, $i_2 \geq 0$, $i_3 \geq 0$, $i_4 \geq 0$ and with $i_1 + i_2 + i_3 + i_4 = k$. Now, we can consider the *simplicial complex* C, where its vertices are the (i_1, i_2, i_3, i_4) in our set with $C(i_1, i_2, i_3, i_4) > 0$. Also, an *edge* in C will be formed by the following paths as long as the two end points are in C:

$$[(i_1 \pm 1, i_2, i_3, i_4), (i_1 \mp 1, i_2, i_3, i_4)],$$
$$[(i_1, i_2 \pm 1, i_3, i_4), (i_1, i_2 \mp 1, i_3, i_4)],$$
$$[(i_1, i_2, i_3 \pm 1, i_4), (i_1, i_2, i_3 \mp 1, i_4)],$$
$$[(i_1, i_2, i_3, i_4 \pm 1), (i_1, i_2, i_3, i_4 \mp 1)].$$

Let k be the maximum value of $C(i_1, i_2, i_3, i_4)$. We are interested in describing the set of (i_1, i_2, i_3, i_4) with

$$C(i_1, i_2, i_3, i_4) = 1,$$
$$\vdots$$
$$C(i_1, i_2, i_3, i_4) = k.$$

We may propose the following conjecture.

Conjecture 2.4. *Suppose that C is the above-mentioned simplicial complex. Then there are "spheres" $S_1, \ldots, S_{k-1}$ in C such that S_{j+1} is precisely the points in the interior of S_j connected by an edge with a point in S_j ($1 \leq j \leq k-2$). Moreover,*

(1) *S_j is the graph of $C(i_1, i_2, i_3, i_4) = j$ ($1 \leq j \leq k-1$) and*

(2) *T is the graph of $C(i_1, i_2, i_3, i_4) = k$, where T is the interior of S_{k-1}.*

If the conjecture is true, then to find $\alpha(x_1, x_2, x_3, x_4)$, we would need to know k, the maximum value of $C(i_1, i_2, i_3, i_4)$, and to know for which (i_1, i_2, i_3, i_4) we have $C(i_1, i_2, i_3, i_4) = 1$. We can propose an analogous conjecture for the $n \times n$ generalized Vandermonde matrices with $n \geq 4$. Our results show that the analogous conjecture for $n = 3$ is true. In order to see this, we need to add the third coordinate j so that $i + j + k = n + p - 2$. Then the conjecture would say there are *circles* $U_1, \ldots, U_{\ell-1}, V$, where ℓ is the maximum value satisfying the analogous condition. In fact, our circles are hexagons.

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Edgar Enochs
Department of Mathematics, University of Kentucky
Lexington, KY 40506-0027, USA
e.enochs@uky.edu

Mohammad Reza Pournaki
Department of Mathematical Sciences,
Sharif University of Technology
P.O. Box 11155-9415, Tehran, Iran
pournaki@ipm.ir

Siamak Yassemi
Department of Mathematics, Purdue University
West Lafayette, IN 47907, USA
syassemi@purdue.edu