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Short note

A new refinement of the Garfunkel–Bankoff inequality

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Abstract. This note introduces an innovative refinement of the Garfunkel–Bankoff inequality, further improving upon the Finsler–Hadwiger inequality and the Lukarevski–Marinescu inequality within a triangle.

1 Introduction

In the triangle ABC, we use the usual symbols, where A, B, and C denote the measures of the three angles, and a, b, and c represent the lengths of the sides opposite angles A, B, and C, respectively. Additionally, R, r, and s stand for the circumradius, inradius, and semiperimeter of triangle ABC.

In [3], J. Garfunkel proposed the following inequality:

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \tag{1}$$

The first proof of inequality (1) was provided by L. Bankoff in [1], so inequality (1) is also known as the Garfunkel–Bankoff inequality. It is worth noting that inequality (1) also represents an improvement of the Finsler–Hadwiger inequality (refer to [8,9])

$$a^2+b^2+c^2\geq 4\sqrt{3}S+(b-c)^2+(c-a)^2+(a-b)^2,$$

and at the same time, it serves as an equivalent form of the Kooi inequality (see [6])

$$s^2\leq\frac{R(4R+r)^2}{2(2R-r)}.$$

In [10], Wei-Dong Jiang proposed the following inequality:

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+\frac{r^2(R-2r)}{4R^2(R-r)}. \tag{2}$$

This represents a strengthened version of inequality (1), and therefore, inequality (2) is an improvement upon the Finsler–Hadwiger and Kooi inequalities. It is worth noting that M. Lukarevski and D. S. Marinescu also provided a refinement of Kooi's inequality (see [4–6]),

$$s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} - \frac{r^2(R-2r)}{4R}. \tag{3}$$

However, inequality (2) remains stronger than inequality (3) (see [10]).

In this article, we propose a new inequality that is an improvement of inequality (2) as follows:

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geq 2 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + \frac{r^2(R-2r)}{2(2R^2-r^2)(R-r)}. \tag{4}$$

Clearly, inequality (4) directly extends the Garfunkel–Bankoff inequality (inequality (1)). From inequality (4), we also have new improvements for the Finsler–Hadwiger and Lukarevski–Marinescu inequalities as follows:

- a new improvement of the Finsler–Hadwiger inequality,

$$a^2+b^2+c^2 \geq 4\sqrt{4 - \frac{2r}{R} + \frac{r^2(R-2r)}{2(2R^2-r^2)(R-r)}}\,S + (b-c)^2 + (c-a)^2 + (a-b)^2,$$

- a new improvement of the Lukarevski–Marinescu inequality (including the Kooi inequality),

$$s^2 \leq \frac{R(4R+r)^2}{2(2R-r)} - \frac{r^2(R-2r)}{2(2R^2-r^2)(R-r)}.$$

These extensions are all stronger than the extensions by Wei-Dong Jiang because inequality (4) is stronger than inequality (2), as it is evident from the fact that

$$\frac{r^2(R-2r)}{2(2R^2-r^2)(R-r)} \geq \frac{r^2(R-2r)}{4R^2(R-r)} \quad \text{or} \quad 4R^2 > 4R^2 - 2r^2.$$

2 Proof of inequality (4)

In this proof, we employ the following fundamental inequality within a triangle:

$$s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)}.$$

Historical references to the method of usage, as well as the nomenclature of this inequality, can be found in [7].

Proof. Using the well-known identities

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R}$$

and

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} = \frac{(4R+r)^2}{s^2} - 2,$$

inequality (4) is equivalent to

$$s^2 \leq \frac{2R(32R^5 + r^5 + 7Rr^4 + 6R^2r^3 - 30R^3r^2 - 16R^4r)}{16R^4 - 4r^4 + 10Rr^3 + R^2r^2 - 24R^3r}.$$

Using the fundamental inequality of a triangle (see [2,7])

$$s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)},$$

we must prove that

$$\begin{aligned} &2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)} \\ &\qquad \leq \frac{2R(32R^5 + r^5 + 7Rr^4 + 6R^2r^3 - 30R^3r^2 - 16R^4r)}{16R^4 - 4r^4 + 10Rr^3 + R^2r^2 - 24R^3r}. \end{aligned} \tag{5}$$

Putting $t = \frac{r}{R}$, we have $0 < t \leq \frac{1}{2}$. Inequality (5) is equivalent to

$$2 + 10t - t^2 + 2(1-2t)\sqrt{1-2t} \leq \frac{2(t^5 + 7t^4 + 6t^3 - 30t^2 - 16t + 32)}{-4t^4 + 10t^3 + t^2 - 24t + 16}.$$

This is true since

$$\begin{aligned} &\left[\frac{2(t^5 + 7t^4 + 6t^3 - 30t^2 - 16t + 32)}{-4t^4 + 10t^3 + t^2 - 24t + 16} - (2 + 10t - t^2)\right]^2 - [2(1-2t)\sqrt{1-2t}]^2 \\ &\quad = \frac{t^5(16t^7 + 96t^6 - 8t^5 - 504t^4 + 393t^3 + 724t^2 - 784t + 192)}{(-4t^4 + 10t^3 + t^2 - 24t + 16)^2} \\ &\quad = \frac{t^5(t+4)^2(2t-1)^2(2t+3)(2t^2 - 5t + 4)}{(-4t^4 + 10t^3 + t^2 - 24t + 16)^2} \geq 0 \end{aligned}$$

for $0 < t \leq \frac{1}{2}$. This completes our proof. ∎

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