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Short note

An improvement of the Weitzenböck inequality

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In a triangle ABC, with the sides a, b, c, the semi-perimeter, circumradius and inradius are denoted by s, R and r respectively; $\triangle$ is its area.

The celebrated Weitzenböck inequality can be stated as follows:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\triangle.$$

Equality holds if and only if the triangle is equilateral.

This was first proved in 1919 [2]. It has received considerable attention from researchers in the field of geometrical inequalities and has motivated a number of papers providing various sharpening, generalisations and improvements.

In this note, we consider the following problem: what is the greatest value of k such that the inequality

$$a^2 + b^2 + c^2 \geq 4\triangle\sqrt{k\left(\frac{R}{r} - 2\right) + 3} \tag{1}$$

holds?

Since

$$a^2 + b^2 + c^2 = 4\triangle(\cot A + \cot B + \cot C)$$

and

$$\frac{R}{r} = \frac{1}{4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}},$$

let

$$(A, B, C) = (\pi - 2x, x, x), \quad 0 < x < \pi.$$

Then (1) is changed to

$$2\cot x - \cot 2x \geq \sqrt{k\frac{1}{4\cos x\sin^2\frac{x}{2}} - 2k + 3}$$

Multiply both sides by x and let $x \to 0$; then $x\cot x \to 1$, $\frac{x}{\sin x} \to 1$. At this point, one can get

$$k \leq \frac{9}{4}.$$

The next theorem proves that (1) holds for $k = \frac{9}{4}$.

Theorem 1. *In $\triangle ABC$, we have*

$$a^2 + b^2 + c^2 \geq 4\sqrt{\frac{9R}{4r} - \frac{3}{2}}\triangle. \tag{2}$$

Equality holds if and only if the triangle is equilateral.

Proof. Using the basic identities in the triangle ABC,

$$a^2 + b^2 + c^2 = 2s^2 - 8Rr - 2r^2 \quad \text{and} \quad \triangle = rs,$$

it follows that (2) is equivalent to

$$s^4 - (17Rr - 4r^2)s^2 + (4Rr + r^2)^2 \geq 0.$$

This is true since

$$\begin{aligned} &s^4 - (17Rr - 4r^2)s^2 + (4Rr + r^2)^2 \\ &\quad = (s^2 - 16Rr + 5r^2)^2 \\ &\qquad + (15Rr - 6r^2)\left[s^2 - 16Rr + 5r^2 - \frac{r^2(R-2r)}{R-r}\right] \\ &\qquad + \frac{3r^3(4R-r)(R-2r)}{R-r} \\ &\quad \geq 0 \end{aligned}$$

follows from Gerretsen's inequality $s^2 \geq 16Rr - 5r^2$ and Euler's inequality $R \geq 2r$ and the following sharpened Gerretsen inequality [1, Theorem 2]:

$$16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r} \leq s^2 \leq 4R^2 + 4Rr + 3r^2 - \frac{r^2(R-2r)}{R-r}. \qquad \blacksquare$$

References

[1] M. Lukarevski, A new look at the fundamental triangle inequality, *Math. Mag.* **96** (2023), no. 2, 141–149. Zbl 07687517 MR 4570210

[2] R. Weitzenböck, Über eine Ungleichung in der Dreiecksgeometrie, *Math. Z.* **5** (1919), 137–146. MR 1544379

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