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Similarity of quadrilaterals as starting point for a geometric journey to orthocentric systems and conics

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(with an appendix by Ivan Izmestiev and Arseniy Akopyan)

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A short time ago, we read a book [4], and we came across a famous result, also described in [3, Theorem 7, p. 10]. Let $ABCD$ be a not cyclic quadrilateral and $A'B'C'D'$ – roughly spoken – the quadrilateral that results by constructing the four perpendicular bisectors of the sides of $ABCD$. In a similar way, $A''B''C''D''$ results from $A'B'C'D'$. Then $A''B''C''D''$ and $ABCD$ are homothetic. For the proof, it is important that $ABCD$ and $A''B''C''D''$ have equal angles at their vertices and that their diagonals make equal angles, too. Moreover, the sides and diagonals of $ABCD$ are even parallel to the sides and diago-

Ähnlichkeitssätze bei Dreiecken sind ein Klassiker der elementaren Geometrie. Diese Sätze lassen sich zum Teil leicht auf Vierecke übertragen, aber sobald die gegebenen Daten nicht einfach auf zwei Teildreiecke passen, wird es anspruchsvoller. In der vorliegenden Arbeit behandelt der Autor eine solche hinreichende Bedingung für die Ähnlichkeit von Vierecken. Auf dem Weg zum Ziel entdeckt er mit dynamischer Geometrie auch einen speziellen Kegelschnitt. Dieser kann auch ein Kreis sein. Die Argumente basieren in diesem Fall auf der Analyse eines Höhenschnittpunktvierecks, also eines Vierecks, bei dem jeder Punkt der Höhenschnittpunkt des Dreiecks der anderen drei Punkte ist; im Englischen *orthocentric system*. Mit projektiver Geometrie ergeben sich weitere Verallgemeinerungen und ein weiterer Kegelschnitt. Die Darstellung vermitteltet einen Einblick in den Erkenntnisprozess des Autors. Dadurch werden die Leserinnen und Leser auf eine spannende Reise durch die verschiedenen geometrischen Themenfelder mitgenommen.

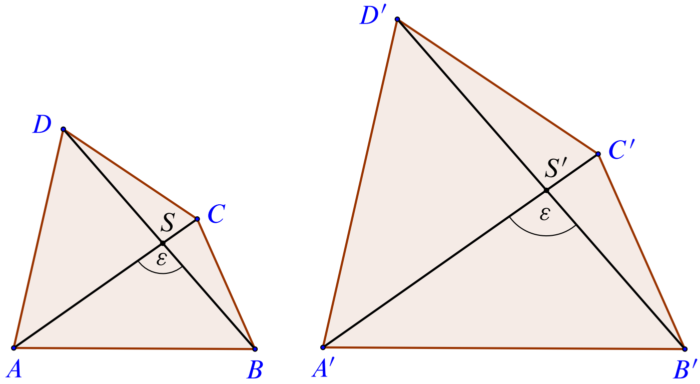


Figure 1. Similarity of quadrilaterals

nals of $A''B''C''D''$; this is crucial in the proof. Then we asked ourselves: could we prove the similarity of the quadrilaterals $ABCD$ and $A''B''C''D''$ also without using that they have parallel sides and diagonals, just using that the angles at the vertices and between their diagonals are equal?

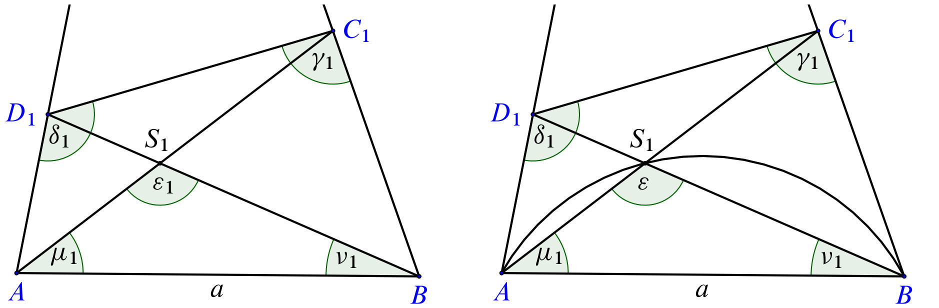
We could not find any reference for that particular problem, so we had to work for ourselves, and an interesting journey began. At first, we tried it with angle chasing and failed; another idea was to extend AB and DC to their intersection point (one gets two pairs of similar triangles), but also this did not work (how to prove that the factor of similarity is the same?). First, we formulate the corresponding theorem.

Theorem 1. *If two convex quadrilaterals $ABCD$ and $A'B'C'D'$ have the same angles at the vertices $(\alpha, \beta, \gamma, \delta)$ and the same angle between their diagonals ($\angle ASB = \varepsilon = \angle A'S'B'$, where S and S' are the intersection points of the diagonals), then they must be similar (Figure 1).*

Three different formulations of a proof. (1) We can fix the length of one side, say $AB = a$, and try to construct the quadrilateral: construct rays from A and B with the angles α, β; then choose an arbitrary dynamic point C_1 on the ray from B, and construct there the angle $\gamma_1 = \gamma$; this will yield D_1 with angle $\delta_1 = \delta$ and S_1 with angle ε_1, and probably, we will have $\varepsilon_1 \neq \varepsilon$ (Figure 2 (a)).

For proving the similarity in Theorem 1, we have to show that there is a unique position of C_1 that yields $\varepsilon_1 = \varepsilon$.

Here one can use a dynamic argument, also making use of the monotonicity and continuity of the process. Moving C_1 away from B obviously makes the angles μ_1, ν_1 bigger because also D_1 moves away from A; the straight line C_1D_1 is just translated; it does not change its direction – and therefore the angle ε_1 decreases (in the other direction, the changes of the angles are the other way round). Because of this monotonicity and the obviously underlying continuity (without proof), there is exactly one position of C_1 on the ray from B which yields $\angle AS_1B = \varepsilon$. In Figure 2, the angles α and β are acute, but nothing changes if one of them is or even both are obtuse.



(a) Arbitrary point C_1 on the ray from B (b) Arbitrary point S_1 on the circular arc

Figure 2

(2) One could also argue like follows. Construct the line segment $AB = a$, the two rays from A, B with the angles α, β, and the circular arc which corresponds to the given angle $\angle ASB = \varepsilon$. Choose an arbitrary point S_1 on this arc, which yields the points C_1, D_1 on the rays and the corresponding angles γ_1, δ_1. These will probably not equal the given angles γ, δ (Figure 2 (b)). Again, one can use a dynamic argument, monotonicity and continuity. Moving S_1 on the arc nearer to B obviously makes the angle μ_1 smaller, and C_1 moves nearer to B, too. Analogously, D_1 moves away from A, and hence the angle γ_1 is increasing and δ_1 is decreasing (when moving S_1 on the arc nearer to A, the changes of the angles are the other way round). Because of this monotonicity and the obviously underlying continuity, there exists exactly one position of S_1 on the arc which yields $\gamma_1 = \gamma$ and $\delta_1 = \delta$.

(3) A third version for arguing would be an indirect proof. Suppose that, for a quadrilateral $ABCD$, there exists a quadrilateral $A^*B^*C^*D^*$ which has the same angles at

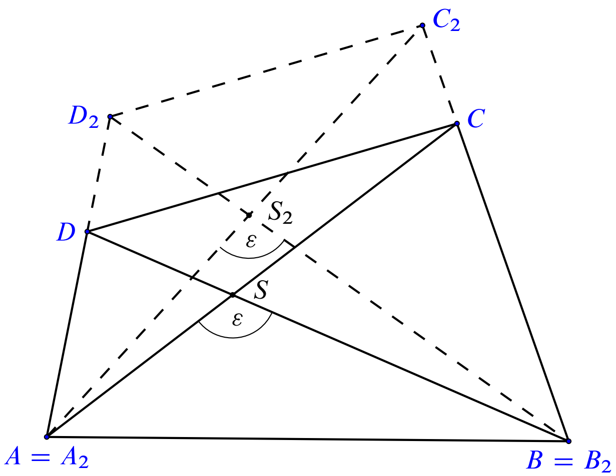


Figure 3. Indirect proof

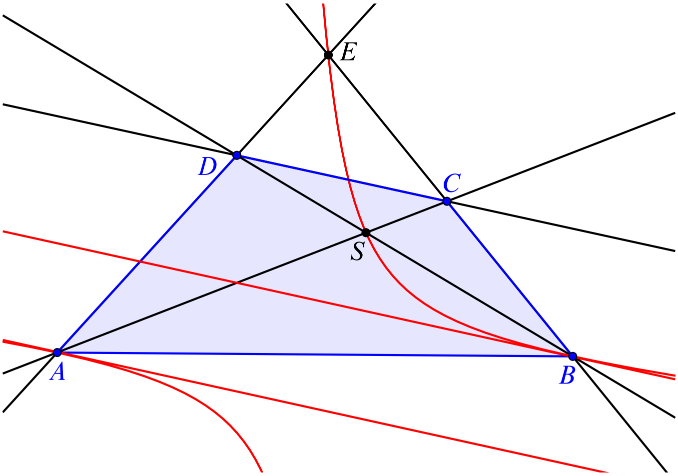


Figure 4. A hyperbola appears

the vertices ($\alpha, \beta, \gamma, \delta$) and between the diagonals (ε) but is not similar to $ABCD$. Then we can apply a special homothety to $A^*B^*C^*D^*$ yielding $A_2B_2C_2D_2$ with $|AB| = |A_2B_2|$. Then we draw the two quadrilaterals $ABCD$ and $A_2B_2C_2D_2$ in such a way that $A_2 = A$ and $B_2 = B$, and without loss of generality, we may assume $|AD_2| > |AD|$ and $|BC_2| > |BC|$. Let S be the intersection point of the diagonals in $ABCD$ and S_2 the one in $A_2B_2C_2D_2$.

Then S lies in the interior of the triangle $A_2B_2S_2$, and the equality $\angle ASB = \varepsilon = \angle A_2S_2B_2$ cannot hold (Figure 3). ■

Here, an interesting and still *unsolved problem* arises in a natural way (probably not easy to answer): *how to construct a quadrilateral with given values of* $\alpha, \beta, \gamma, \delta, \varepsilon$.

When moving the point C on the ray from B, our former colleague B. Schuppar (University of Dortmund, Germany) discovered (conjecture) that the locus of the intersection point S of the diagonals – when moving C on the ray from B; the angles at C and D do not change, i.e., CD is only translated "upward" or "downward" – seems to be a hyperbola through E and B with "center = midpoint of AB". Here, E is the intersection point of the straight lines AD and BC (Figure 4).

We tried to prove that within Euclidean geometry but failed. A sketch with GeoGebra showed that there seems to be nothing striking concerning its foci, asymptotes, and vertices. Then we got a hint per email from Arseniy Akopyan, and we looked in his book [1, p. 75]. There, Theorem 3.8 describes particularly our situation (the point Z mentioned in this theorem is in our case a point at infinity and corresponds to the direction of the straight line CD), and we learned that there is another striking phenomenon concerning the mentioned hyperbola: the tangents to the hyperbola at A and B are parallel to CD. That explains also why the midpoint of AB is the midpoint of the hyperbola. The proof of [1, Theorem 3.8] involves projective geometry, and applied to our problem, it proves the above-mentioned conjecture of B. Schuppar.

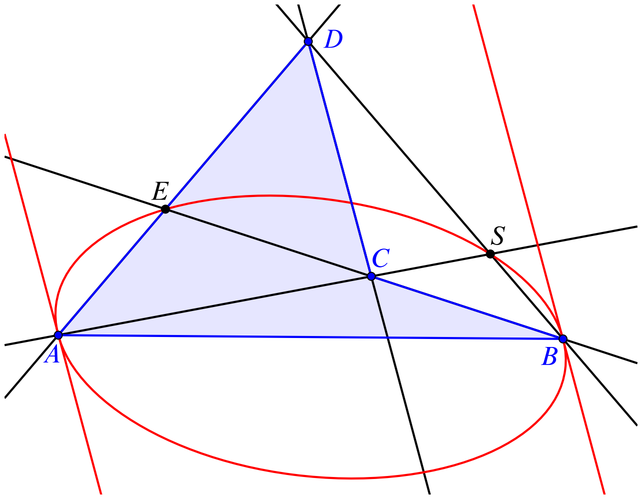


Figure 5. In case of concave quadrilaterals S moves on an ellipse

This would allow a sort of "construction" of the above-mentioned problem: S is the intersection point of the circular arc (Figure 2 (b)) and the hyperbola of Figure 4, but such a construction requires dynamic geometry software where one can construct such conics as an object and intersect them with other objects. Is there also a construction using only paper, pencil, ruler, compasses, and angle meter?

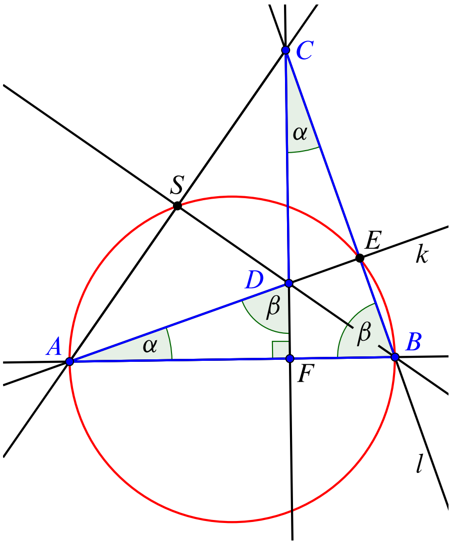
Before we have known these connections to conics and projective geometry, we explored the situation by playing around with the points A, B, C, D and discovered the following. If the angles at the vertices are changed so that the quadrilateral becomes concave, then the locus of the intersection point S of the diagonals (as C is moved on the straight line BE) is apparently an ellipse (Figure 5; the tangents at A and B are again parallel to CD).

Then we asked: can the locus of S also be a circle? The answer is *yes*, it can! This is clear after having checked the connections to projective geometry, but we did not have the hint from A. Akopyan to that time; therefore this was not so clear for us. We think it is worth mentioning that one can argue for that phenomenon without using projective geometry.

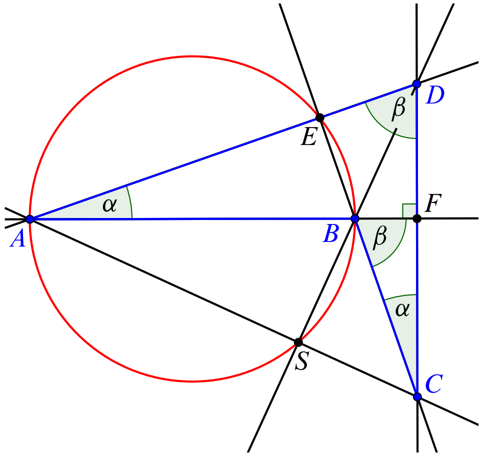
Below, we give a sufficient condition that yields a circle as the locus of S (conjecture: probably there are no other cases, i.e., this condition seems also to be necessary).

Theorem 2. *Let α, β be two acute angles with $\alpha + \beta = 90°$ and k, l two straight lines through $A \neq B$ which make the angles α, β with AB at the points A, B. The point C moves on l and the perpendicular from C to AB intersects k at D. Then the locus of the intersection point S of the diagonals AC and BD of the quadrilateral $ABCD$ is the circle with diameter AB.*

For a first proof, one just has to know the orthocenter of a triangle and Thales' theorem. We are well aware that Theorem 2 and its proof are not new geometrical knowledge, but



(a) Case 1



(b) Case 2

Figure 6

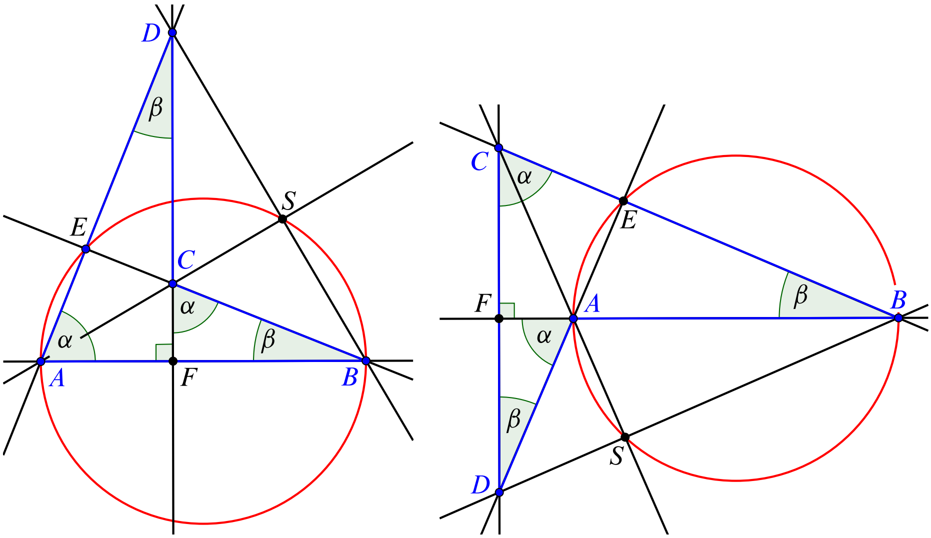
the way arriving at them may also be interesting for other readers. In any case, the four points A, B, C, D are a so-called orthocentric system or orthocentric quadrilateral (each point is the orthocenter of the triangle formed by the other three; all four possible triangles have the same circumradius and share the orthic triangle and the nine-point circle; see e.g. [2, p. 109].

Proof 1. Let E be the intersection point of the straight line l with the circle of diameter AB (then by Thales' theorem, we know $E \in k$) and F the foot of the perpendicular from C to AB. We have to distinguish four cases. In order to get better comprehensible figures, we assume in the first two cases $\alpha \leq \beta$; in Cases 3 and 4, we assume $\alpha \geq \beta$.

Case 1. C lies "above" E and CA intersects the "upper" semicircle; at D (somewhere on the line segment AE), there is the reflex angle $180° + \beta$ (Figure 6 (a)). Then we have right angles at E and F; thus D must be the orthocenter of $\triangle ABC$ and BD must intersect CA orthogonally at S. And by Thales' theorem, S must lie on the "upper" semicircle with diameter AB.

Case 2. C lies "underneath" B; then the angle at B in the quadrilateral is actually not β but the reflex angle $180° + \beta$; there is the acute angle β at D (Figure 6 (b)). This time, B must be the orthocenter of $\triangle ACD$, and again, we can conclude that BD intersects AC orthogonally at S somewhere on the "lower" semicircle with diameter AB.

Case 3. C lies somewhere between B and E and CA intersects the "upper" semicircle; at C, there is the reflex angle $180° + \alpha$ (Figure 7 (a)). Then we have right angles at E and F; thus C must be the orthocenter of $\triangle ABD$ and BD must intersect CA orthogonally at S somewhere on the "upper" semicircle with diameter AB.



(a) Case 3 (b) Case 4

Figure 7

Case 4. C lies above E and CA intersects the "lower" semicircle; at A, the angle is actually not α but the reflex angle $180° + \alpha$; instead, in this case, there is the acute angle β at D (Figure 7 (b)). Then we have right angles at E and F; thus A must be the orthocenter of $\triangle BCD$ and BD must intersect CA orthogonally at S on the "lower" semicircle with diameter AB. ■

Cases 1 and 3 on the one hand and Cases 2 and 4 on the other hand are essentially the same; one just has to interchange $\alpha \leftrightarrow \beta$ and $C \leftrightarrow D$. Actually, our first way of proving it was Proof 2 (see below) because, initially, we did not really consider the triangles; the quadrilaterals, some right angles, and the corresponding circles were primarily in our focus, and not triangles.

Proof 2. For a second proof, the basis of argumentation is the inscribed angle theorem. This proof provides an alternative to the usual proof concerning the orthocenter of a triangle. The inscribed angle theorem is a more advanced means than the phenomenon that the perpendicular bisectors of a triangle are concurrent (circumcenter O); thus it is quite natural that the usual version is preferred in almost all textbooks. We have seen that all four above cases reduce to the following: given a triangle ABC and two altitudes AE and CF which intersect in D, then BD is perpendicular to AC (Figure 8).

Because of the two right angles at E and F, we have the two circles c_1 and c_2, and we know $\varepsilon + \mu = 90°$. Then we apply the inscribed angle theorem one time in the circle c_2 and another time in the circle c_1, which yields $\angle DBE = \mu = \angle DAS$, and this, in turn, means that there must be a right angle at S. ■

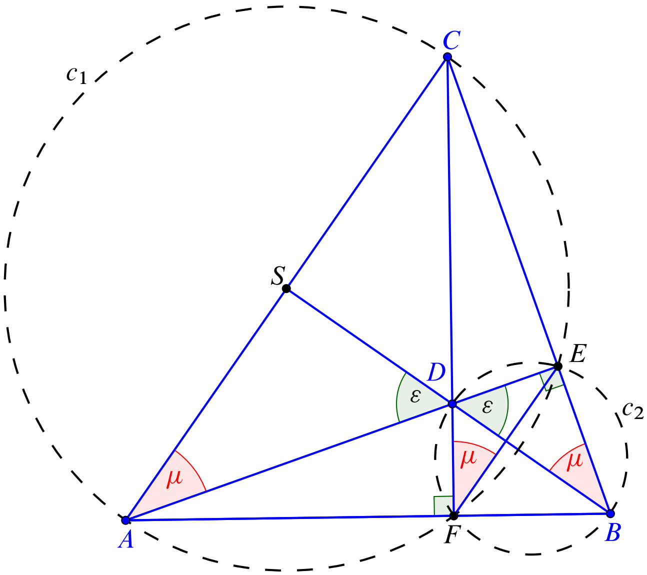


Figure 8. Proof with the inscribed angle theorem; this figure corresponds to Figure 6 (a) (for reasons of clarity, without the red circle with diameter AB)

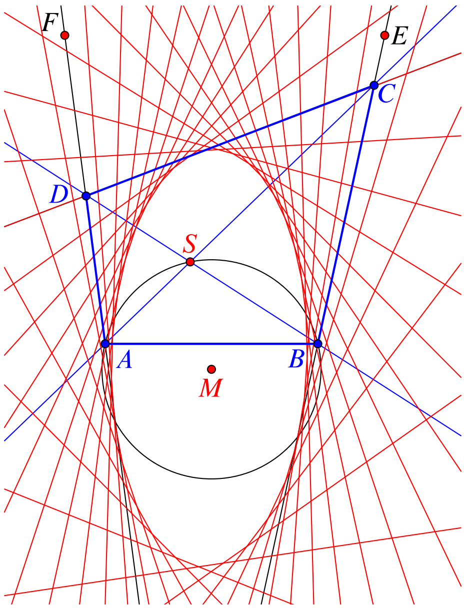


Figure 9. All the straight lines CD ($C \in BE$ and the corresponding point $D \in AF$) are tangent to an ellipse.

Remark. The proof of Akopyan and Zaslavsky [1, Theorem 3.8, p. 75, projective geometry] uses this very special situation of a circle as the generic case for proving the whole theorem. In projective geometry, all conics are equivalent.

Additionally, we observed other phenomena concerning conics. If the points A and B (side a) and the angles $\alpha, \beta, \varepsilon$ are fixed (ε denotes the angle between the diagonals; in Figure 9, the angles α, β are fixed via the points E and F: $\alpha = \angle BAF$ and $\beta = \angle ABE$), then we know that the intersection point S of the diagonals lies on a special circular arc (see Figure 2 (b) and Figure 9). If S is moved on this arc, one can see that all the corresponding straight lines CD are tangent to a conic; in case of Figure 9, it is an ellipse. Furthermore, the straight lines AD and BC are tangent to that conic, and the corresponding points of tangency are the intersection points (other than A, B) of these straight lines with the circle on which S is moving. We could not prove these findings, but they were proven by I. Izmestiev and A. V. Akopyan by means of projective geometry (many thanks to them! See the appendix; they formulated the theorem, lemmas, and proofs of this appendix especially for this publication and sent it to us per email; probably, there is no other reference in their publications where one can read them).

Conclusion

"Our tour" makes vivid the aspect of mathematics as a process. We started at a famous problem concerning quadrilaterals and asked a further question. We then worked out three possible formulations for a proof concerning a sufficient condition of two quadrilaterals to be similar, two of them using dynamic arguments and arguments involving monotonicity and continuity. With the help of dynamic geometry, we detected a – for us in the moment of detection – rather strange hyperbola. Not knowing the proper explanation for that, we went on exploring the situation with dynamic geometry, and then we saw ellipses on the screen. Wondering if we could manage even circles as the locus of the intersection point of the diagonals, we quickly found a condition for that. Although the proof is rather easy, we did not see immediately the crucial role of the orthocenter of triangles because, initially, we did not consider triangles. Rather, we concentrated on quadrilaterals, some right angles, and the corresponding circles. And finally, we understood the whole connection of the situation to conics better due to a hint by A. Akopyan and means of projective geometry. So we had an interesting journey through various fields of geometry, with new and interesting insights for us and perhaps for some readers, both on a rather elementary (similarity of quadrilaterals, locus of S is a circle: by means of the orthocenter of triangles and Thales' theorem or by means of the inscribed angle theorem which also can be seen as an alternative to the usual proof concerning the orthocenter of triangles) and on a more advanced level (by means of projective geometry). For readers interested to put further effort to this topic, this paper provides also an unsolved problem: how to construct a quadrilateral with given values of (1) angles at the vertices, (2) angle between the diagonals, and (3) one side, say a.

Appendix
(by Ivan Izmestiev and Arseniy Akopyan)

Theorem. *Let ℓ and m be two lines in the plane, and let $p \in \ell$ and $q \in m$ be two points different from the intersection point of ℓ and m. Further, let C be a conic passing through p and q. For any point $u \in C$, let r be the intersection point of the line pu with the line ℓ, and let s be the intersection point of the line qu with the line m. Then there is a conic C' such that, for any choice of $u \in C$, the line rs is tangent to C'. Besides, C' is tangent to the lines ℓ and m at their intersection points with C other than p and q (at p if C is tangent to ℓ, and at q if C is tangent to m).*

Lemma 1. *The map $\ell \to m$ sending r to s is a projective map.*

Proof. This map is the composition of maps

$$\ell \to p^* \to q^* \to m,$$

where p^* and q^* are the pencils of lines through p and q, respectively: a point $x \in \ell$ is sent to the line px; this line is sent to a line through q and the second intersection point of px with C; finally, a line through q is sent to its intersection point with m. Each of these three maps preserves cross-ratios (cross-ratio of lines in a pencil is defined using the sines of angles between lines): the first and the third map do so due to the sine law, the second map due to the inscribed angle theorem. Thus, the composition of all three maps also preserves the cross-ratios, so it is a projective map. ∎

Lemma 2. *If $f : \ell \to m$ is a projective map between two lines, then the lines connecting $x \in \ell$ with $f(x) \in m$ are all tangent to the same conic.*

Proof. Let us prove the dual statement: if g: $p^* \to q^*$ is a projective map between two pencils of lines, then the intersections of lines $y \ni p$ and $g(y) \ni q$ all lie on the same conic.

Choose three lines in p^* and intersect them with their images in q^*. There is a unique conic C which contains these three points and the points p and q. The map g': $p^* \to q^*$ constructed in the same way as the second map in the proof of Lemma 1 is projective and coincides with g on three points. Therefore, $g' = g$, and it follows that the intersection points $y \cap g(y)$ sweep C. ∎

The above theorem is a direct consequence of Lemmas 1 and 2. The only thing that remains to be shown is that the conic C' is tangent to ℓ and m at their intersection points with C different from p and q. This follows from moving the point u close to one of those intersection points: the line rs then tends to ℓ or m.

Acknowledgments. For the findings in the framework of projective geometry, we would like to thank Ivan Izmestiev and A. V. Akopyan.

Note added in proof. After this article was finished, we were informed that Norbert Hungerbühler and Juan Läuchli solved the problem of constructing a quadrilateral with given vertex angles and diagonal angle. Their solution will appear in this journal.

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