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Stability of Schur's iterates and fast solution of the discrete integrable NLS

Roman Bessonov and Pavel Gubkin

Abstract. We prove a sharp stability estimate for Schur iterates of contractive analytic functions in the open unit disk. We then apply this result in the setting of the inverse scattering approach and obtain a fast algorithm for solving the discrete integrable nonlinear Schrödinger equation (Ablowitz–Ladik equation) on the integer lattice, $\mathbb{Z}$. We also give a self-contained introduction to the theory of the nonlinear Fourier transform from the perspective of Schur functions and orthogonal polynomials on the unit circle.

1. Introduction

1.1. Schur's algorithm

The Schur class $\mathcal{S}(\mathbb{D})$ in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane, $\mathbb{C}$, consists of analytic functions F in $\mathbb{D}$ such that

$$\sup_{z \in \mathbb{D}} |F(z)| \leq 1.$$

For $F \in \mathcal{S}(\mathbb{D})$, we write $F \in \mathcal{S}_*(\mathbb{D})$ if F is not a finite Blaschke product. Take $F \in \mathcal{S}_*(\mathbb{D})$, set $F_0 = F$, and define the sequence $\{F_n\}_{n \geq 0}$ using Schur's algorithm:

$$zF_{n+1} = \frac{F_n - F_n(0)}{1 - \overline{F_n(0)} F_n}, \quad n \geq 0. \tag{1.1}$$

By construction and the Schwarz lemma, the resulting functions $F_0, F_1, F_2, \ldots$ will belong to the class $\mathcal{S}_*(\mathbb{D})$ as well. In the case where $F \in \mathcal{S}(\mathbb{D}) \setminus \mathcal{S}_*(\mathbb{D})$ is a Blaschke product of order $N \geq 0$, the same construction gives a finite sequence of Blaschke products $F_0, F_1, \ldots, F_N$ of orders $N, N-1, \ldots, 0$, correspondingly. In particular, F_N is a constant of unit modulus and the Schur's algorithm stops.

Note that $|F_n(0)| < 1$ for each $F \in \mathcal{S}_*(\mathbb{D})$, $n \geq 0$, by the maximum modulus principle. Therefore, each function $F \in \mathcal{S}_*(\mathbb{D})$ generates a sequence $\{F_n(0)\}_{n \geq 0} \subset \mathbb{D}$.

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These numbers are called the *recurrence coefficients* of F. It can be shown that the mapping

$$F \mapsto \{F_n(0)\}_{n\ge 0}$$

is a homeomorphism from $\mathcal{S}_*(\mathbb{D})$ with the topology of convergence on compact subsets of $\mathbb{D}$ onto the space of sequences $q\colon \mathbb{Z}_+ \to \mathbb{D}$ with the topology of element-wise convergence, see [18, Section 1.3.6]. Here, $\mathbb{Z}_+ = \mathbb{Z} \cap [0, +\infty)$. In particular, for every sequence $\{\alpha_n\}_{n\ge 0} \subset \mathbb{D}$, there exists a unique function $F \in \mathcal{S}_*(\mathbb{D})$ such that $\alpha_n = F_n(0)$ for every $n \in \mathbb{Z}_+$. In this paper, we study the stability of Schur's algorithm. We prove a sharp estimate for $|F_n(0) - G_n(0)|$ in terms of $F - G$ for functions $F, G \in \mathcal{S}_*(\mathbb{D})$ from the Szegő class, whose definition we now recall.

Let m denote the Lebesgue measure on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ normalised by $m(\mathbb{T}) = 1$. The following theorem can be found, e.g., in [18, Section 2.7.8].

Theorem 1.1 (Szegő theorem). *Let $F \in \mathcal{S}_*(\mathbb{D})$, and let $\{F_n(0)\}_{n\ge 0}$ be its recurrence coefficients. Then*

$$\int_{\mathbb{T}} \log(1 - |F|^2)\, dm = \log \prod_{n=0}^{\infty} (1 - |F_n(0)|^2),$$

where both sides are finite or infinite simultaneously.

We will refer to the functions $F \in \mathcal{S}_*(\mathbb{D})$ such that

$$\eta(F) = \prod_{n=0}^{\infty} (1 - |F_n(0)|^2) > 0, \tag{1.2}$$

as the *Schur functions of Szegő class.* Given any $r \in (0, 1)$, and an analytic function F in $\mathbb{D}$, we set

$$\|F\|_{L^2(r\mathbb{T})} = \Big(\int_{\mathbb{T}} |F(r\xi)|^2\, dm(\xi)\Big)^{1/2}.$$

Theorem 1.2. *Let F, G be Schur functions of Szegő class, and let $\eta(F), \eta(G) \ge \eta$ for some $\eta > 0$. Then, for every $r \in (0, 1)$ and $n \in \mathbb{Z}_+$, the estimate*

$$\|F_n - G_n\|_{L^2(r\mathbb{T})} \le C(\eta, r) r^{-n} \|F - G\|_{L^2(r\mathbb{T})}, \tag{1.3}$$

holds with the constant

$$C(\eta, r) = \exp\Big(\log \eta^{-1} \cdot \Big(2 + \frac{1}{1 - \sqrt{1-\eta}}\Big)\Big(\frac{4}{(1-r)^2} + 1\Big)\Big)$$

depending only on η, r.

The order of the exponential factor r^{-n} in Theorem 1.2 is sharp. Indeed, one can take $\delta \in (0,1)$ and set $F = \delta z^n$, $G = 0$. Then $F_n(z) = \delta$, $G_n(z) = 0$ for all $z \in \mathbb{D}$. So, we have $\|F_n - G_n\|_{L^2(r\mathbb{T})} = \|\delta\|_{L^2(r\mathbb{T})} = \delta$ and $\|F - G\|_{L^2(r\mathbb{T})} = \|\delta z^n\|_{L^2(r\mathbb{T})} = \delta r^n$ in this case. Since $\eta(F) = 1-\delta^2$, $\eta(G) = 1$ do not depend on n, a study for large n's shows that the order of growth r^{-n} in (1.3) cannot be improved within the Szegő class.

Theorem 1.2 can be used to estimate $|F_n(0) - G_n(0)|$ if we know that the Schur functions F, G are sufficiently close to each other in the disk $|z| \le r$. Indeed, by Bessel's inequality, we have

$$|F_n(0) - G_n(0)| \le \|F_n - G_n\|_{L^2(r\mathbb{T})},$$

because the system $\{z^k\}_{k\ge 0}$ is orthogonal in $L^2(r\mathbb{T})$. We want to emphasise that the constant $C(\eta, r)$ in Theorem 1.2 is uniform for functions $F \in \mathcal{S}_*(\mathbb{D})$ with the Szegő constant $\eta(F)$ separated from zero. This is the most important feature of (1.3) when it is compared with another stability result from inverse spectral theory – the Sylvester–Winebrenner theorem [21]. In the language of Schur functions, this theorem says that Schur's algorithm defines a homeomorphism in appropriate metric spaces.

Theorem 1.3 (Sylvester–Winebrenner theorem). *The mapping $F \mapsto \{F_n(0)\}_{n\ge 0}$ that takes a Schur function into the sequence of its recurrence coefficients is a homeomorphism from the metric space $X_+ = \{F \in \mathcal{S}_*(\mathbb{D}) : \eta(F) > 0\}$ with the metric*

$$\rho_s(F,G)^2 = -\int_{\mathbb{T}} \log\Big(1 - \Big|\frac{F-G}{1-\bar{F}G}\Big|^2\Big)\,dm$$

onto the metric space $\ell^2(\mathbb{Z}_+, \mathbb{D})$ of the square summable sequences $q\colon \mathbb{Z}_+ \to \mathbb{D}$ with the metric $\|q - \tilde{q}\|_{\ell^2}^2 = \sum_{n\in\mathbb{Z}_+} |q(n) - \tilde{q}(n)|^2$.

We prove this version of the Sylvester–Winebrenner theorem in Section 6. It is very natural to expect that the modulus of continuity of the homeomorphism in Theorem 1.3 is controlled by η on the subset of $F \in X_+$ with $\eta(F) > \eta > 0$. This is, however, not the case! See Proposition 6.12 below. On the other hand, the uniform character of estimate (1.3) will be crucial for the application of (1.3) to the discrete integrable nonlinear Schrödinger equation (Ablowitz–Ladik equation). Let us discuss it next.

1.2. AL: statement of the problem

Consider the defocusing Ablowitz–Ladik equation (AL) on the integer lattice, $\mathbb{Z}$,

$$\begin{cases} \frac{\partial}{\partial t} q(t,n) = i(1 - |q(t,n)|^2)(q(t,n-1) + q(t,n+1)), \\ q(0,n) = q_0(n), \quad n \in \mathbb{Z}. \end{cases} \tag{1.4}$$

The variable $t \in \mathbb{R}$ is considered as time, $n \in \mathbb{Z}$ is the discrete space variable. The Ablowitz–Ladik equation is the integrable model introduced in [1, 2] as a spatial discretization of the cubic nonlinear Schrödinger equation (NLS), see [3] for a general context and modern exposition. If we change variables to $u = e^{-2it}q$, then (1.4) becomes

$$\begin{aligned} i\frac{\partial}{\partial t}u(t,n) =& -(1-|u(t,n)|^2)(u(t,n-1)+u(t,n+1)) + 2u(t,n) \\ =& -u(t,n-1) + 2u(t,n) - u(t,n+1) \\ &+ |u(t,n)|^2(u(t,n-1)+u(t,n+1)), \end{aligned}$$

which is indeed a discretization of the continuous defocusing NLS equation,

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{\partial^2}{\partial x^2}u(t,x) + 2|u(t,x)|^2u(t,x), \quad x \in \mathbb{R}.$$

We are going to present a new solution method for (1.4) based on Schur's algorithm. The rate of its convergence will be estimated using Theorem 1.2. We deal with the following problem.

Problem 1.4. *Given $\varepsilon \in (0,1)$, $t \in \mathbb{R}$, $n_0 \in \mathbb{Z}$, and a sequence q_0 on $\mathbb{Z}$ such that*

$$\sup_{n\in\mathbb{Z}} |q_0(n)| \le 1, \qquad \prod_{n\in\mathbb{Z}}(1-|q_0(n)|^2) \ge \eta > 0,$$

evaluate the solution q of (1.4) at (t, n_0) with the absolute error at most ε.

The quantity $\prod_{n\in\mathbb{Z}}(1-|q(t,n)|^2) = \prod_{n\in\mathbb{Z}}(1-|q_0(n)|^2)$ is conserved under the flow of AL equation. So, it is a natural characteristic for results on stability/accuracy of solutions of AL equation.

We introduce the algorithm which solves Problem 1.4 in $O(\mathbf{n}\log^2\mathbf{n})$ operations, where $\mathbf{n} = t + \log\varepsilon^{-1}$. Thus, to have accuracy e^{-n} at the moment of time $t = 1$, one need to take at most $c_\eta n\log^2 n$ arithmetic operations for some constant $c_\eta > 0$ depending only on η. The basic Runge–Kutta scheme RK4 requires n steps ($\sim n \cdot k$ operations) for computing $u(1, j)$, $-k \le j \le k$, to guarantee accuracy $O(1/n^4)$ if we additionally assume that the impact of $u(t, j)$, $|j| \ge k$, is negligible for $0 \le t \le 1$.

1.3. AL: localization

Our solution method is a modification of the classical inverse scattering approach. From a bird's eye view, the standard procedure (see [24, Chapter 2]) of solving (1.4) by means of the inverse scattering theory (IST) is as follows. Given an initial datum

$q_0\colon \mathbb{Z} \to \mathbb{D}$, define the so-called *reflection coefficient* $\mathbf{r}_{q_0}$ at $z \in \mathbb{T}$ by

$$\mathbf{r}_{q_0}(z) = \frac{b(z)}{a(z)}, \quad \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} = \prod_{k\in\mathbb{Z}} \frac{1}{\sqrt{1-|q_0(k)|^2}} \cdot \begin{pmatrix} 1 & \overline{q_0(k)} z^{-k} \\ q_0(k) z^k & 1 \end{pmatrix}, \quad (1.5)$$

and find $q(t,\cdot)\colon \mathbb{Z} \to \mathbb{D}$ such that $\mathbf{r}_{q(t,\cdot)} = e^{-it(z+1/z)}\mathbf{r}_{q_0}$ on $\mathbb{T}$. It turns out that $q(t,\cdot)$ will solve (1.4) for an initial datum q_0, provided q_0 decays fast enough (say, $\sum_{k\in\mathbb{Z}} |q_0(k)| < \infty$). A fundamental problem appearing when one tries to solve (1.4) by IST with merely $\ell^2(\mathbb{Z},\mathbb{D})$ initial datum q_0 (i.e., for a general $q_0\colon \mathbb{Z} \to \mathbb{D}$ such that $\prod_{n\in\mathbb{Z}}(1-|q_0(n)|^2) > 0$) is that we can have $\mathbf{r}_{q_0} = \mathbf{r}_{\tilde{q}_0}$ for $q_0 \neq \tilde{q}_0$. This phenomenon was first observed by Volberg and Yuditskii in [25] on the level of Jacobi matrices, and then by Tao and Thiele [23] in the setting of the nonlinear Fourier transform, NLFT. It shows that when we pass to the reflection coefficients $\mathbf{r}_{q_0}$, $\mathbf{r}_{\tilde{q}_0}$, some information gets lost and there are no chances to solve (1.4) for $\ell^2(\mathbb{Z},\mathbb{D})$ initial data by using the IST approach directly. To overcame this difficulty (non-injectivity of NLFT), we first prove the following localization estimate.

Theorem 1.5. *Let $q_0\colon \mathbb{Z} \to \mathbb{D}$ be such that $\prod_{n\in\mathbb{Z}}(1-|q_0(n)|^2) \geq \eta$ for some $\eta > 0$ and let q be the solution of (1.4) for the initial datum q_0. Take $N \in \mathbb{Z}_+$, consider the sequence $q_{0,N}$ defined by*

$$q_{0,N}(n) = \begin{cases} q_0(n), & |n| \leq N, \\ 0, & |n| > N, \end{cases}$$

and let q_N be the corresponding solution of (1.4). Then, for $N \geq |j|$, $t > 0$ and all $r \in (0,1)$, we have

$$|q(t,j) - q_N(t,j)| \leq \frac{4e^{t/r} C(\eta,r)}{1-r} r^{N-|j|}, \quad (1.6)$$

where $C(\eta,r)$ is the function from Theorem 1.2.

Having in mind a possible future development of a parallel theory for continuous NLS equation, we use only "spectral" methods in the proof of Theorem 1.5. The reader interested in short and elementary proof of Theorem 1.5 by means of a direct approach could find it in Appendix A.

1.4. AL: compactly supported initial data

Having Theorem 1.5, it remains to solve (1.4) for compactly supported initial data $q_0\colon \mathbb{Z} \to \mathbb{D}$. This can be done by a variety of methods, both theoretically and numerically. In particular, the standard IST approach works in this case, but accuracy estimates for numerical schemes based on IST and ℓ^2-bounds are missed in the literature. Taking into account the non-injectivity of NLFT, we see that the problem is, in fact, fairly nontrivial: some distant compactly supported data q_0, $\tilde{q}_0$ correspond to almost identical reflection coefficients $\mathbf{r}_{q_0}$, $\mathbf{r}_{\tilde{q}_0}$. Indeed, it is enough to take different $q_0, \tilde{q}_0 \in \ell^2(\mathbb{Z}, \mathbb{D})$ with the same reflection coefficient and consider restrictions of q_0, $\tilde{q}_0$ to a large discrete interval $[-N, N]$. Then the corresponding reflection coefficients will almost coincide by continuity of NLFT. This phenomenon, when ignored, leads to instabilities. Below we describe a procedure that can be used to get the solution with prescribed accuracy.

Consider $q_0\colon \mathbb{Z} \to \mathbb{D}$ supported on $\mathbb{Z} \cap [0, \ell]$ for some $\ell \in \mathbb{Z}_+$. Note that q solves (1.4) if and only if $q(t, \cdot + j)$ solves (1.4) for the initial datum $q_0(\cdot + j)$. Therefore, we do not lose generality when we assume $\operatorname{supp} q_0 \subset [0, \ell]$. Moreover, it is easy to see that $q(t, n)$ solves (1.4) if and only if $\overline{q(-t, n)}$ solves (1.4) with the initial data $\overline{q_0}$. So, we can also assume that $t > 0$.

Consider the Fourier expansion of the inverse scattering multiplier $e^{it(z+1/z)}$:

$$e^{it(z+1/z)} = \sum_{k\in\mathbb{Z}} i^k J_k(2t) z^k, \quad z \in \mathbb{T}.$$

Here, J_k are the standard Bessel functions [7] of order k, i.e.,

$$J_k(2t) = i^{-k} \int_{\mathbb{T}} e^{it(z+1/z)} \bar{z}^k \, dm = \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m+k}}{m!(m+k)!}.$$

Let $P_{n,t} = \sum_{|k|\le n} i^k J_k(2t) z^k$ be the Laurent trigonometric polynomial of $e^{it(z+1/z)}$ of order n. Define the function $G_{n,t}$ by

$$G_{n,t} = (1 - \delta_{n,t}) z^n P_{n,t}, \quad \delta_{n,t} = \frac{t^n e^t}{n!}. \tag{1.7}$$

We will be interested in the situation when $n > ct$ with some $c > e$. In this case, this "$\delta_{n,t}$-correction" is very small but important: it places $G_{n,t}$ into Schur class. Given a sequence $q_0\colon \mathbb{Z} \to \mathbb{D}$ supported on $[0, \ell]$, define the coefficients a, b of q_0 by (1.5). Note that the product in (1.5) contains at most $\ell + 1$ nontrivial terms. One can check that a, $\bar{b}$ in (1.5) coincide on $\mathbb{T}$ with analytic polynomials in z of degree at most ℓ, and, moreover, $|\bar{b}(z)| < |a(z)|$ if $|z| \le 1$. Set $\mathbf{f}_{q_0} = \bar{b}/a$. The function $F_{n,0} = G_{n,t}\mathbf{f}_{q_0}$ is rational and belongs to the Schur class $\mathcal{S}_*(\mathbb{D})$ (see Proposition 3.1 below). Fix $j \in \mathbb{Z}$ and use Schur's algorithm (1.1) to find rational functions $F_{n,0}$, $F_{n,1}, F_{n,2}, \ldots, F_{n,n+j}, \ldots$ (Schur iterates of $F_{n,0}$). Set

$$\tilde{q}_n(t, j) = \begin{cases} F_{n,n+j}(0), & j \ge -n, \\ 0, & j < -n. \end{cases} \tag{1.8}$$

The following theorem shows that $\tilde{q}_n$ approximates the solution q of (1.4) with very high accuracy.

Theorem 1.6. *Let $t > 0$, and let $q_0\colon \mathbb{Z} \to \mathbb{D}$ be a sequence compactly supported on $\mathbb{Z}_+$. Assume that $\prod_{n\in\mathbb{Z}_+}(1-|q_0(n)|^2) \geq \eta$ for some $\eta > 0$. Then, the function $\tilde{q}_n$ in (1.8) satisfies*

$$|q(t,j) - \tilde{q}_n(t,j)| \leq 2^j C(\eta, 1/2) \frac{12e^{5t}}{\sqrt{2\pi n}} \Big(\frac{2et}{n}\Big)^n, \tag{1.9}$$

for all $n \in \mathbb{Z}_+$, $j \in \mathbb{Z}$, $t > 0$ such that $n + j \geq 0$, $n > t$, and $\delta_{n,t} < 1$, see (1.7). Here q is the solution of (1.4) and $C(\eta, r)$ is the function from Theorem 1.2.

Note that the right-hand side in (1.9) is very small when n is much larger than $2et$ and j is fixed. The estimate does not depend on the size of the support of q_0. In fact, Theorem 1.6 remains true if we assume only $\operatorname{supp} q_0 \subset [0, +\infty)$ and $\prod_0^{+\infty}(1-|q_0(n)|^2) > 0$. In this case, it is known that the product in (1.5) converges in Lebesgue measure on $\mathbb{T}$ (see Section 6) and defines coefficients a, b almost everywhere on $\mathbb{T}$. Moreover, $\mathbf{f}_{q_0} = \bar{b}/a$ will coincide with non-tangential values of a function of Schur class $\mathcal{S}_*(\mathbb{D})$. Then the $\tilde{q}_n(t,j)$ are well defined by (1.8), and (1.9) will hold for them.

1.5. AL: algorithm for Problem 1.4

Let us summarise the algorithm that solves Problem 1.4 based on Theorems 1.5 and 1.6. At first, one need to choose a window $\Delta = [n_0 - N, n_0 + N]$ where N is such that $|q(t,n_0) - q_N(t,n_0)| \leq \varepsilon/2$ for the exact solution q_N with the truncated initial datum $q_{0,N} = \chi_\Delta q_0$. Then, one need to shift $q_{0,N}$ by $n_0 - N$ to make it supported on $\mathbb{Z}_+ \cap [0, 2N]$ and use the algorithm described in Section 1.4 to find the approximate solution $\tilde{q}_n$ with accuracy $\varepsilon/2$ at $j = N$ for the shifted sequence. Taking $N = 5 + [4et + \log_2(C(\eta, 1/2)/\varepsilon)]$, $n = 2N$, we will get $|\tilde{q}_n(t,N) - q(t,n_0)| \leq \varepsilon$, see Section 5. In Section 5 we check that the whole procedure requires $O(\mathbf{n}\log^2\mathbf{n})$ operations for $\mathbf{n} = t + \log\varepsilon^{-1}$. In fact, the sequence $\tilde{q}_n$ approximates q with accuracy $O(\varepsilon)$ on the interval $[n_0 - N/2, n_0]$, not only at the point n_0. Considering the reflection of q_0 and applying the algorithm twice, one can construct approximation to q on $[n_0 - N/2, n_0 + N/2]$ in $O(\mathbf{n}\log^2\mathbf{n})$ operations.

1.6. AL: historical remarks and motivation

As a classical integrable model, the Ablowitz–Ladik equation has a well-developed theory in the periodic case [19, Chapter 11], [16, 17], in the finite case [9, 14], in the half-infinite case [12, 20], and on the whole lattice $\mathbb{Z}$, see [10, 11, 15, 24]. The paper [11] contains a historical overview and an extensive bibliography, including works following the original approach of Ablowitz and Ladik, who obtained a Lax pair for (1.4) by discretizing the Zakharov–Shabat Lax pair for the continuous NLS equation. Conversely, the references mentioned in this paragraph (and the results used in this paper) are mostly related to recent works that appeared after Nenciu and Simon [19, Chapter 11], [17] discovered a new Lax pair for this equation, making a connection to CMV matrices and orthogonal polynomials on the unit circle. The IST method as a tool for existence theorems for the Ablowitz–Ladik equation attracted a limited attention in the literature because the solvability of (1.4) for all initial data $q_0 \colon \mathbb{Z} \to \mathbb{D}$ can be easily obtained by means of a fixed point theorem (see Appendix A). However, the Ablowitz–Ladik equation is a perfect model for developing an accurate fast IST-based numerical scheme that can be later generalised for the continuous NLS equation.

1.7. The nonlinear Fourier transform

The last part of the paper can be regarded as the introduction to the theory of the nonlinear Fourier transform, NLFT. The main results in this area are due to Thiele and Tao, see the paper [23] or its extended version by Thiele, Tao, and Tsai [24], where the Ablowitz–Ladik equation appears in the setting of NLFT. The papers [23, 24] influenced much of the present work. We decided to give a short introduction to the theory of NLFT in the language of Schur functions and orthogonal polynomials to make the paper more self-contained. We hope that our arguments will be of independent interest for the orthogonal-polynomials community.

For $1 \le p < \infty$, let $\ell^p(\mathbb{Z}, \mathbb{D})$ be the set of sequences $q \colon \mathbb{Z} \to \mathbb{D}$ such that

$$\sum_{n \in \mathbb{Z}} |q(n)|^p < \infty.$$

We endow it with the usual distance $\|q_1 - q_2\|_{\ell^p} = (\sum_{n\in\mathbb{Z}} |q_1(n) - q_2(n)|^p)^{1/p}$. Note that $\ell^p(\mathbb{Z}, \mathbb{D})$ is not a linear space. Using formula (1.5), define the nonlinear Fourier transform (or the scattering map) by

$$\mathcal{F}_{\mathrm{sc}} \colon q \mapsto \mathbf{r}_q,$$

on the set $\ell^1(\mathbb{Z}, \mathbb{D})$. Here we consider $\mathcal{F}_{\mathrm{sc}}$ as the map from $\ell^1(\mathbb{Z}, \mathbb{D})$ to $L^\infty(\mathbb{T})$. Later on, the domain of $\mathcal{F}_{\mathrm{sc}}$ will be extended, while the target space will be changed to a narrower one. Define the metric space

$$X = \{h \in L^\infty(\mathbb{T}) : \|h\|_{L^\infty(\mathbb{T})} \le 1,\ \log(1 - |h|^2) \in L^1(\mathbb{T})\}, \tag{1.10}$$

with the Sylvester–Winebrenner metric ρ_s (see [21]) given by

$$\rho_s(h_1, h_2) = \sqrt{-\int_{\mathbb{T}} \log\Big(1 - \Big|\frac{h_1 - h_2}{1 - \bar{h}_1 h_2}\Big|^2\Big)\, dm}. \tag{1.11}$$

For $\delta \in [0, 1)$, denote $B[\delta] = \{h \in L^\infty(\mathbb{T}) : \|h\|_{L^\infty(\mathbb{T})} \le \delta\}$. We have $B[\delta] \subset X$ for every $\delta \in [0, 1)$. So, let us consider $B[\delta]$ as the subspace of X with the induced metric topology. As we will see below, $\mathcal{F}_{\rm sc}$ uniquely extends to the continuous map from $\ell^2(\mathbb{Z}, \mathbb{D})$ to X. Set $\mathcal{G}[\delta] = \mathcal{F}_{\rm sc}^{-1}(B[\delta])$ where $\mathcal{F}_{\rm sc}^{-1}(E)$ is the full preimage of a set E under the mapping $\mathcal{F}_{\rm sc}\colon \ell^2(\mathbb{Z}, \mathbb{D}) \to X$.

With this definitions at hand, we are ready to summarise the basic properties of the map $\mathcal{F}_{\rm sc}$.

Theorem 1.7. *The nonlinear Fourier transform $\mathcal{F}_{\rm sc}$ has the following properties:*

(1) *the map $\mathcal{F}_{\rm sc}$ extends uniquely to the continuous map $\mathcal{F}_{\rm sc}\colon \ell^2(\mathbb{Z}, \mathbb{D}) \to X$;*

(2) *the map $\mathcal{F}_{\rm sc}\colon \ell^2(\mathbb{Z}, \mathbb{D}) \to X$ is closed;*

(3) *we have $\mathcal{F}_{\rm sc}(q(\cdot - n)) = z^{-n}\mathcal{F}_{\rm sc}(q)$ for every $q \in \ell^2(\mathbb{Z}, \mathbb{D})$, $n \in \mathbb{Z}$;*

(4) *the map $\mathcal{F}_{\rm sc}\colon \ell^2(\mathbb{Z}, \mathbb{D}) \to X$ is surjective;*

(5) *the map $\mathcal{F}_{\rm sc}\colon \ell^2(\mathbb{Z}, \mathbb{D}) \to X$ is not injective;*

(6) *the map $\mathcal{F}_{\rm sc}\colon \mathcal{G}[\delta] \to B[\delta]$ is a homeomorphism for every $\delta \in (0, 1)$;*

(7) *if $q = q(t, n)$ is the solution of (1.4) with the initial datum $q_0 \in \mathcal{G}[\delta]$, then $q(t, \cdot) \in \mathcal{G}[\delta]$ for each $t \in \mathbb{R}$, and $q(t, \cdot) = \mathcal{F}_{\rm sc}^{-1}(e^{-it(z+1/z)}\mathcal{F}_{\rm sc}(q_0))$.*

Theorem 1.7 (2) is new. It implies, in particular, that $\mathcal{F}_{\rm sc}$ is a homeomorphism on the set of potentials $q \in \ell^2(\mathbb{Z}, \mathbb{D})$ that are completely determined by the reflection coefficient $\mathbf{r}_q$. Theorem 1.7 (7) is not proved in our paper (we did not find a sufficiently short argument), see [24] for the proof. Some ideas in the proof of Theorem 1.7 are due to S. Denisov; the authors would like to thank him for his contribution.

2. Schur's algorithm. Proof of Theorem 1.2

In this section we prove Theorem 1.2. For an analytic function F in $\mathbb{D}$, it will be convenient to set

$$M_F(r) = \max_{|z| \le r} |F(z)|.$$

At first, we prove the following lemma.

Lemma 2.1. *Let $F \in \mathcal{S}_*(\mathbb{D})$, and let F_k be its Schur iterates defined by* (1.1)*. Then for every $r \in [0, 1)$ we have*

$$\sum_{k=0}^{\infty} M_{F_k}^2(r) \le \frac{4}{(1-r)^2} \cdot \sum_{k=0}^{\infty} |F_k(0)|^2 \le \frac{4}{(1-r)^2} \cdot \log(\eta(F)^{-1}), \tag{2.1}$$

where $\eta(F)$ is defined by (1.2)*.*

Proof. Let us check the second inequality first. For $x \in (0,1)$, we have $(1-x)^{-1} \ge e^x$, therefore

$$\eta(F)^{-1} = \prod_{k\ge 0}(1-|F_k(0)|^2)^{-1} \ge \prod_{k\ge 0} e^{|F_k(0)|^2} = e^{\sum_{k\ge 0}|F_k(0)|^2},$$

which implies the required bound $\log(\eta(F)^{-1}) \ge \sum_{k\ge 0}|F_k(0)|^2$. Now, we focus on the first inequality in (2.1). Set $\alpha_j = F_j(0)$, $j \ge 0$. We will use the estimate in [18, (1.3.58)], which reads

$$|F(z)| \le 2\sum_{j=0}^{\infty}|\alpha_j||z|^j, \quad z \in \mathbb{D}.$$

Applying it to F_k in place of F for $|z| = r$, we get $M_{F_k}(r) \le 2\sum_{j=0}^{\infty}|\alpha_{k+j}|r^j$; hence,

$$\begin{aligned} M_{F_k}^2(r) &\le 4\Big(\sum_{j=0}^{\infty}|\alpha_{k+j}|r^{j/2}\cdot r^{j/2}\Big)^2 \\ &\le 4\sum_{j=0}^{\infty}|\alpha_{k+j}|^2 r^j \cdot \sum_{j=0}^{\infty} r^j = \frac{4}{1-r}\sum_{j=0}^{\infty}|\alpha_{k+j}|^2 r^j, \end{aligned}$$

by the Cauchy inequality. Summing up over $k \in \mathbb{Z}_+$, we get

$$\begin{aligned} \sum_{k=0}^{\infty} M_{F_k}^2(r) &= \frac{4}{1-r}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}|\alpha_{k+j}|^2 r^j \\ &= \frac{4}{1-r}\sum_{s=0}^{\infty}|\alpha_s|^2\sum_{j=0}^{s} r^j \le \frac{4}{(1-r)^2}\sum_{s=0}^{\infty}|\alpha_s|^2. \end{aligned}$$

This ends the proof. ■

Remark 2.2. Lemma 2.1 holds with a better (for small r) estimate with $1/(1-r)^4$ in place of $4/(1-r)^2$. To prove this, one needs to use the expression for F_k from [13, Theorem 8.70]. A study of the functions $F = \delta z^n$ for large n's and small δ's shows that the constant in Lemma 2.1 cannot be smaller than $1/1 - r^2$.

Proof of Theorem 1.2. Let $F, G \in \mathcal{S}_*(\mathbb{D})$. We have

$$z(F_1 - G_1) = \frac{F - F(0)}{1 - \overline{F(0)}F} - \frac{G - G(0)}{1 - \overline{G(0)}G} = \frac{P}{Q}. \tag{2.2}$$

Here, the numerator is

$$\begin{aligned} P &= (F - F(0))(1 - \overline{G(0)}G) - (G - G(0))(1 - \overline{F(0)}F) \\ &= [F - F(0) - G + G(0)] + FG(\overline{F(0)} - \overline{G(0)}) + [F(0)\overline{G(0)}G - \overline{F(0)}G(0)F]. \end{aligned}$$

We have

$$\begin{aligned} F(0)\overline{G(0)}G - \overline{F(0)}G(0)F &= F(0)\overline{G(0)}(G - F) + FF(0)(\overline{G(0)} - \overline{F(0)}) \\ &\quad + F\overline{F(0)}(F(0) - G(0)). \end{aligned}$$

It follows that

$$\begin{aligned} \|P\|_{L^2(r\mathbb{T})} \le{}& \|F - F(0) - G + G(0)\|_{L^2(r\mathbb{T})} + M_F(r)M_G(r)|F(0) - G(0)| \\ &+ |F(0)||G(0)|\|F - G\|_{L^2(r\mathbb{T})} + 2M_F(r)|F(0)||F(0) - G(0)|. \end{aligned}$$

For an analytic function H in $\mathbb{D}$, we have

$$|H(0)| \le \|H\|_{L^2(r\mathbb{T})}, \quad \|H - H(0)\|_{L^2(r\mathbb{T})} \le \|H\|_{L^2(r\mathbb{T})},$$

by orthogonality of the system $\{z^k\}_{k\ge 0}$. Applying this to $H = F - G$ and using $2xy \le x^2 + y^2$, we get

$$\begin{aligned} \|P\|_{L^2(r\mathbb{T})} &\le \|F - G\|_{L^2(r\mathbb{T})}(1 + M_F(r)M_G(r) + |F(0)||G(0)| + 2M_F(r)|F(0)|) \\ &\le \|F - G\|_{L^2(r\mathbb{T})}\Big(1 + \frac{3M_F(r)^2 + M_G(r)^2 + 3|F(0)|^2 + |G(0)|^2}{2}\Big). \end{aligned}$$

Since $|P|$ remains the same when we swap F, G, we also have

$$\|P\|_{L^2(r\mathbb{T})} \le \|F - G\|_{L^2(r\mathbb{T})}\Big(1 + \frac{M_F(r)^2 + 3M_G(r)^2 + |F(0)|^2 + 3|G(0)|^2}{2}\Big).$$

Taking a half-sum, we get

$$\|P\|_{L^2(r\mathbb{T})} \le \|F - G\|_{L^2(r\mathbb{T})}(1 + M_F^2(r) + M_G^2(r) + |F(0)|^2 + |G(0)|^2).$$

Further, for $z \in r\mathbb{T}$, we estimate the denominator Q in (2.2) as follows:

$$|Q(z)| = |(1 - \overline{G(0)}G)(1 - \overline{F(0)}F)| \ge (1 - |G(0)|M_G(r))(1 - |F(0)|M_F(r)),$$

where we use the fact that both brackets above are positive. Substituting the bounds for P, Q into (2.2) gives

$$r\|F_1 - G_1\|_{L^2(r\mathbb{T})} \le \|F - G\|_{L^2(r\mathbb{T})} \frac{1 + M_F^2(r) + M_G^2(r) + |F(0)|^2 + |G(0)|^2}{(1 - |G(0)|M_G(r))(1 - |F(0)|M_F(r))}.$$

The latter inequality applied to F_k and G_k in place of F, G for $k = 0, \dots, n-1$ implies

$$r^n \|F_n - G_n\|_{L^2(r\mathbb{T})} \le \|F - G\|_{L^2(r\mathbb{T})} \prod_{k=0}^{n-1} C_k, \tag{2.3}$$

for

$$C_k = \frac{1 + M_{F_k}^2(r) + M_{G_k}^2(r) + |F_k(0)|^2 + |G_k(0)|^2}{(1 - |G_k(0)|M_{G_k}(r))(1 - |F_k(0)|M_{F_k}(r))}.$$

It remains to estimate $\prod_{k=0}^{n-1} C_k$. For $\delta \in (0,1)$, denote by $c(\delta)$ the minimal positive number such that $\frac{1}{1-x} \le 1 + c(\delta)x$ for all $x \in (0,1)$ satisfying $1 - x^2 > \delta$. It is not difficult to check that

$$c(\delta) = \frac{1}{1 - \sqrt{1-\delta}} \in \Big[1, \frac{2}{\delta}\Big]. \tag{2.4}$$

Observe that

$$1 - |F_k(0)|^2 M_{F_k}(r)^2 \ge 1 - |F_k(0)|^2 \ge \prod_{m=0}^{+\infty} (1 - |F_m(0)|^2) = \eta(F) > \eta,$$

by our assumption. Then,

$$\frac{1}{1 - |F_k(0)|M_{F_k}(r)} \le 1 + c(\eta)|F_k(0)|M_{F_k}(r) \le 1 + \frac{c(\eta)}{2}(M_{F_k}^2(r) + |F_k(0)|^2).$$

A similar estimate holds for functions G_k. It follows that

$$\begin{aligned} C_k &\le (1 + M_{F_k}^2(r) + M_{G_k}^2(r) + |F_k(0)|^2 + |G_k(0)|^2) \\ &\quad \times \Big(1 + \frac{c(\eta)}{2}(M_{F_k}^2(r) + |F_k(0)|^2)\Big)\Big(1 + \frac{c(\eta)}{2}(M_{G_k}^2(r) + |G_k(0)|^2)\Big) \\ &\le \exp\Big(\Big(1 + \frac{c(\eta)}{2}\Big)(M_{F_k}^2(r) + M_{G_k}^2(r) + |F_k(0)|^2 + |G_k(0)|^2)\Big), \end{aligned}$$

where we used the elementary inequality $1 + x \le e^x$ three times. From Lemma 2.1, we get

$$\begin{aligned} \prod_{k=0}^{n-1} C_k &\le \exp\Big(\Big(1 + \frac{c(\eta)}{2}\Big)\Big(\sum_{k=0}^{n-1} M_{F_k}^2(r) + M_{G_k}^2(r) + |F_k(0)|^2 + |G_k(0)|^2\Big)\Big) \\ &\le \exp\Big(\Big(1 + \frac{c(\eta)}{2}\Big)\Big(\frac{8\log\eta^{-1}}{(1-r)^2} + 2\log\eta^{-1}\Big)\Big). \end{aligned}$$

Substituting the latter into (2.3) and the bound (2.4) imply (1.3) with

$$C(\eta, r) = \exp\Big(\log \eta^{-1} \cdot \Big(2 + \frac{1}{1 - \sqrt{1-\eta}}\Big)\Big(\frac{4}{(1-r)^2} + 1\Big)\Big).$$

This ends the proof. ∎

Remark 2.3. The function $C(\eta, r)$ is very large if η is not close to 1 or if r is close to 1. We have, e.g., $5 \cdot 10^{27} \le C(1/2, 1/2) \le 6 \cdot 10^{27}$, $10^6 \le C(4/5, 1/2) \le 2 \cdot 10^6$, and $9 \le C(24/25, 1/2) \le 10$. In [15], Killip, Ouyang, Visan, and Wu proved that the continuous NLS equation with arbitrary $L^2(\mathbb{R})$-initial data can be approximated by the solutions of equation (1.4). It is interesting to note that $\eta \to 1$ in their construction during approximation process.

3. Estimates for the multipliers. Proof of Theorem 1.6

Recall the definition (1.7) of $G_{n,t}$ and $P_{n,t}$ for $t > 0$:

$$P_{n,t} = \sum_{|k| \le n} i^k J_k(2t) z^k, \quad G_{n,t} = (1 - \delta_{n,t}) z^n P_{n,t}, \quad \delta_{n,t} = \frac{t^n e^t}{n!}.$$

In this section, we first prove a bound for $G_{n,t}$ and estimate the rate of convergence of $G_{n+1,t} - zG_{n,t}$ to zero. Then we prove Theorem 1.6. Throughout this section, we assume that $t > 0$.

Lemma 3.1. *Let $z \in \mathbb{T}$ and let $n > t > 0$ be such that $\delta_{n,t} < 1$ for $\delta_{n,t} = t^n e^t / n!$ from (1.7). Then we have $|G_{n,t}(z)| < 1$. In particular, for every $q_0 \in \ell^2(\mathbb{Z}, \mathbb{D})$ with* $\operatorname{supp} q_0 \subset \mathbb{Z}_+$, *we have $G_{n,t} \mathbf{f}_{q_0} \in \mathcal{S}_*(\mathbb{D})$ and the construction described in Section 1.4 is correct.*

Proof. We have

$$|P_{n,t}(z) - e^{it(z+1/z)}| = \Big|\sum_{|k|>n} i^k J_k(2t) z^k\Big| \le 2 \sum_{k>n} r^{-k} |J_k(2t)|, \quad |z| = r.$$

The standard estimate (see, e.g.,[7, p. 91]) $|J_\nu(2t)| \le |t|^\nu / \Gamma(\nu + 1)$ implies

$$|P_{n,t}(z) - e^{it(z+1/z)}| \le 2 \sum_{k>n} \frac{r^{-k} t^k}{k!} \le \frac{2t^n r^{-n} e^{t/r}}{(n+1)!} \le \frac{t^n r^{-n} e^{t/r}}{n!}, \tag{3.1}$$

$$|P_{n,t}(z)| \le |e^{it(z+1/z)}| + \frac{t^n r^{-n} e^{t/r}}{n!}. \tag{3.2}$$

In particular, for $z \in \mathbb{T}$ this gives $|P_{n,t}(z)| \le 1 + \delta_{n,t}$, where $\delta_{n,t} = t^n e^t/n!$ is from (1.7). Therefore, we have

$$|G_{n,t}(z)| = (1-\delta_{n,t})|P_{n,t}(z)| \le 1 - \delta_{n,t}^2 < 1, \quad z \in \mathbb{T},$$

where the factor $(1-\delta_{n,t})$ is positive by our assumption. For a compactly supported q_0 with $\operatorname{supp} q_0 \subset [0, \ell]$, it is not difficult to check that $\mathbf{f}_{q_0}$ is a Schur function by considering the partial products in (1.5) and using induction. For the general case, see formula (6.18) below. Then, we have $G_{n,t}\mathbf{f}_{q_0} \in \mathcal{S}_*(\mathbb{D})$ by construction. ■

Lemma 3.2. *Let n, t be as in Lemma 3.1. Then* $\max_{|z|=r} |G_{n,t}| \le e^{t/r}(r^n + 3\delta_{n,t})$ *for $r \in (0,1)$, and, moreover,*

$$\max_{|z|=r} |G_{n+1,t}(z) - zG_{n,t}(z)| \le S_n(t,r), \quad S_n(t,r) = 6\delta_{n,t}e^{t/r}.$$

Proof. Take $z \in \mathbb{D}$ such that $|z| = r$. By (1.7) and (3.2), we have

$$\begin{aligned}|z^{-n}G_{n,t}(z) - P_{n,t}(z)| = \delta_{n,t}|P_{n,t}(z)| &\le \delta_{n,t}\left(|e^{it(z+1/z)}| + \frac{t^n r^{-n} e^{t/r}}{n!}\right)\\ &\le \delta_{n,t}\left(e^{t(1/r-r)} + \frac{t^n r^{-n} e^{t/r}}{n!}\right)\\ &\le \delta_{n,t}(e^{t/r} + \delta_{n,t}r^{-n}e^{t/r}) = \delta_{n,t}e^{t/r}(1+\delta_{n,t}r^{-n}).\end{aligned}$$

Furthermore, we have

$$|z^{-n}G_{n,t}(z) - e^{it(z+1/z)}| \le |z^{-n}G_{n,t}(z) - P_{n,t}(z)| + |P_{n,t}(z) - e^{it(z+1/z)}|.$$

The last two estimates together with (3.1) imply

$$\begin{aligned}|z^{-n}G_{n,t}(z) - e^{it(z+1/z)}| &\le \delta_{n,t}e^{t/r}(1+\delta_{n,t}r^{-n}) + \frac{t^n r^{-n} e^{t/r}}{n!}\\ &\le \delta_{n,t}e^{t/r}(1+2r^{-n}).\end{aligned}$$

This gives

$$\max_{|z|=r} |G_{n,t}| \le r^n e^{t(1/r-r)} + 3\delta_{n,t}e^{t/r} \le e^{t/r}(r^n + 3\delta_{n,t}).$$

So, we have

$$\begin{aligned}\max_{|z|=r}|zG_{n,t}(z) - z^{n+1}e^{it(z+1/z)}| &\le \delta_{n,t}e^{t/r}r^{n+1}(1+2r^{-n})\\ &= \delta_{n,t}e^{t/r}(r^{n+1}+2r),\end{aligned}$$

and

$$\begin{aligned}\max_{|z|=r}|G_{n+1,t}(z) - z^{n+1}e^{it(z+1/z)}| &\le \delta_{n+1,t}e^{t/r}r^{n+1}(1+2r^{-(n+1)})\\ &\le \delta_{n,t}e^{t/r}(r^{n+1}+2),\end{aligned}$$

where we used the inequality $\delta_{n+1,t} \le \delta_{n,t}$ for $n > t > 0$. It remains to write

$$|G_{n+1,t}(z) - zG_{n,t}(z)| \le |G_{n+1,t}(z) - z^{n+1}e^{it(z+1/z)}| + |zG_{n,t}(z) - z^{n+1}e^{it(z+1/z)}|$$

and use the last two estimates. ∎

Lemma 3.3. *For every $n > 0$, $t > 0$, $r \in (0,1)$, we have*

$$\sum_{k\ge n} S_k(t,r)r^{-k} \le 6\delta_{n,t}e^{2t/r}\cdot r^{-n}.$$

Proof. For $n > t > 0$ we have $\delta_{n+1,t} = \delta_{n,t}t/(n+1)$; hence,

$$\sum_{k\ge n} S_k(t,r)r^{-k} \le 6\delta_{n,t}e^{t/r}r^{-n}\Big(1 + \frac{t/r}{n+1} + \frac{(t/r)^2}{(n+1)(n+2)} + \cdots\Big) \le 6\delta_{n,t}e^{t/r}r^{-n}e^{t/r} = 6\delta_{n,t}e^{2t/r}\cdot r^{-n}.$$

This is the required estimate. ∎

The following lemma will be proved in Section 6.

Lemma 3.4. *Suppose that $q \in \ell^2(\mathbb{Z},\mathbb{D})$ is such that $\operatorname{supp} q \subset \mathbb{Z}_+$ and let $\mathbf{f}_q$ be defined as in Section 1.4. Then the recurrence coefficients of $\mathbf{f}_q$ coincide with the sequence $\{q(k)\}_{k\ge 0}$.*

Proof of Theorem 1.6. Let $t > 0$, and let $q_0 \colon \mathbb{Z} \to \mathbb{D}$ be a sequence compactly supported on $\mathbb{Z}_+$. Assume that $\prod_{n\in\mathbb{Z}_+}(1-|q_0(n)|^2) \ge \eta$ for some $\eta > 0$. Define the functions $\mathbf{f}_{q_0} = \bar{b}/a$, $F_{n,0} = G_{n,t}\mathbf{f}_{q_0}$, and $F_{n,k}$ as in Section 1.4. Let also $\tilde{q}_n(t,j) = F_{n,n+j}(0)$, $j \ge -n$, $\tilde{q}_n(t,j) = 0$, $j < -n$, for $j \in \mathbb{Z}$. We are going to show that $\{\tilde{q}_n(t,j)\}_{n\ge 0}$ is a Cauchy sequence for each $j \in \mathbb{Z}$. Take two positive integers $n_2 > n_1 \ge -j$, fix $r \in (0,1)$ and consider the difference

$$\begin{aligned}|\tilde{q}_{n_2}(t,j) - \tilde{q}_{n_1}(t,j)| &= |F_{n_2,n_2+j}(0) - F_{n_1,n_1+j}(0)| \\ &\le \|F_{n_2,n_2+j} - F_{n_1,n_1+j}\|_{L^2(r\mathbb{T})} \\ &\le \sum_{k=n_1}^{n_2-1} \|F_{k+1,k+1+j} - F_{k,k+j}\|_{L^2(r\mathbb{T})}.\end{aligned}$$

Since $G_{n,t}$ is a contraction by Lemma 3.1, we have $|G_{k,t}\mathbf{f}_{q_0}| \le |\mathbf{f}_{q_0}|$ on $\mathbb{T}$; hence,

$$\min(\eta(G_{k+1,t}\mathbf{f}_{q_0}), \eta(zG_{k,t}\mathbf{f}_{q_0})) \ge \eta(\mathbf{f}_{q_0}) \ge \eta$$

for every k by Szegő Theorem 1.1 and our assumption. For a function $F \in \mathcal{S}_*(\mathbb{D})$, denote by $(F)_k$ the k-th Schur iterate of F (see (1.1), where $(F)_k$ are denoted by F_k). Note that $(F)_k = (zF)_{k+1}$. By Theorem 1.2, we have

$$\begin{aligned}\|F_{k+1,k+1+j} - F_{k,k+j}\|_{L^2(r\mathbb{T})} &= \|(G_{k+1,t}\mathbf{f}_{q_0})_{k+1+j} - (G_{k,t}\mathbf{f}_{q_0})_{k+j}\|_{L^2(r\mathbb{T})} \\ &= \|(G_{k+1,t}\mathbf{f}_{q_0})_{k+1+j} - (zG_{k,t}\mathbf{f}_{q_0})_{k+1+j}\|_{L^2(r\mathbb{T})} \\ &\le C(\eta,r)r^{-k-1-j}\|G_{k+1,t}\mathbf{f}_{q_0} - zG_{k,t}\mathbf{f}_{q_0}\|_{L^2(r\mathbb{T})} \\ &\le C(\eta,r)r^{-k-1-j}\|G_{k+1,t} - zG_{k,t}\|_{L^2(r\mathbb{T})}.\end{aligned}$$

Using Lemma 3.2 for $n_1 > t > 0$ such that $\delta_{n_1,t} < 1$, we can proceed as follows:

$$\begin{aligned}\|F_{k+1,k+1+j} - F_{k,k+j}\|_{L^2(r\mathbb{T})} &\le C(\eta,r)r^{-k-j-1}\max_{|z|=r}|G_{k+1,t} - zG_{k,t}| \\ &\le C(\eta,r)S_k(t,r)r^{-k-j-1}.\end{aligned}$$

From Lemma 3.3, we now see that

$$\begin{aligned}|\tilde{q}_{n_2}(t,j) - \tilde{q}_{n_1}(t,j)| &\le r^{-j-1}C(\eta,r)\cdot\sum_{k=n_1}^{\infty}S_k(t,r)r^{-k} \\ &\le 6C(\eta,r)\delta_{n_1,t}e^{2t/r}\cdot r^{-n_1-j-1}.\end{aligned}$$

Recall that $\delta_{n_1,t} = e^t t^{n_1}/n_1!$ decays very rapidly as $n_1 \to \infty$, thus, $\{\tilde{q}_n(t,j)\}_{n\ge -j}$ is a Cauchy sequence for every $j \in \mathbb{Z}$. Denote its limit by $\tilde{q}(t,\cdot)$. Letting $n_1 = n$ and taking the limit in as $n_2 \to +\infty$, we obtain

$$|\tilde{q}(t,j) - \tilde{q}_n(t,j)| \le 6C(\eta,r)\delta_{n,t}e^{2t/r}\cdot r^{-n-j-1}.$$

Taking $r = 1/2$ (any other $r \in (0,1)$ will do) and using $n! \ge \sqrt{2\pi n}(n/e)^n$, we get

$$|\tilde{q}(t,j) - \tilde{q}_n(t,j)| \le 6C(\eta,1/2)\frac{e^t t^n}{n!}e^{4t}2^{n+j+1} = 2^j C(\eta,1/2)\frac{12e^{5t}}{\sqrt{2\pi n}}\left(\frac{2et}{n}\right)^n,$$

where $n \in \mathbb{Z}_+$, $j \in \mathbb{Z}$, $t > 0$ are such that $n + j \ge 0$, $n > t > 0$, and $\delta_{n,t} < 1$.

It remains to show that $\tilde{q}(t,j) = q(t,j)$, i.e., that $\tilde{q}$ solves the Ablowitz–Ladik equation (1.4) with the initial datum q_0. By Theorem 1.7 (6) and Theorem 1.7 (7) (note that $q_0 \in \mathcal{G}[\delta]$, because q_0 is compactly supported) it is suffices to check that $\mathbf{r}_{\tilde{q}} = \mathbf{r}_q$, or, equivalently, $\mathbf{r}_{\tilde{q}} = e^{it(z+1/z)}\mathbf{r}_{q_0}$.

The sequence $\tilde{q}_n(t,\cdot - n)$ is supported on $\mathbb{Z}_+$, moreover, we have $\tilde{q}_n(t,j-n) = F_{n,j}(0)$ for $j \in \mathbb{Z}_+$. Let us denote the coefficients in (1.5) for q_0, $\tilde{q}_n(t,\cdot - n)$, by a, b, and $a_{n,0}$, $b_{n,0}$, respectively. We have $\mathbf{r}_{\tilde{q}_n(t,\cdot-n)} = b_{n,0}/a_{n,0}$ and $F_{n,0} = \mathbf{f}_{\tilde{q}_n(t,\cdot-n)} =$

$\overline{b_{n,0}}/a_{n,0}$, where the equality $F_{n,0} = \mathbf{f}_{\tilde{q}_n(t,\cdot-n)}$ holds by Lemma 3.4 because the recurrence coefficients of $F_{n,0}$ coincide with the sequence $\{\tilde{q}(t, j-n)\}_{j\ge 0}$. By (3.1), we also have

$$z^{-n}\overline{b_{n,0}}/a_{n,0} = z^{-n}F_{n,0} = z^{-n}G_{n,t}\mathbf{f}_{q_0} \to e^{it(z+1/z)}\mathbf{f}_{q_0} = e^{it(z+1/z)}\bar{b}/a$$

uniformly on $\mathbb{T}$. We will now use well-known properties of the coefficients a, b in (1.5). Namely, the functions a, $a_{n,0}$ are outer, have positive values at $z=0$, and satisfy $|a|^2 - |b|^2 = 1$, $|a_{n,0}|^2 - |b_{n,0}|^2 = 1$ on $\mathbb{T}$ (for the proof, see Section 6). The convergence $z^{-n}\overline{b_{n,0}}/a_{n,0} \to e^{it(z+1/z)}\bar{b}/a$ then implies $|a_{n,0}|^2 \to |a|^2$, $\log|a_{n,0}|^2 \to \log|a|^2$ uniformly on $\mathbb{T}$; hence, $a_{n,0} \to a$ in Lebesgue measure on $\mathbb{T}$ by the properties of outer functions (more precisely, by the weak continuity of the Hilbert transform, see a discussion next to formula (6.26)). It follows that $z^n b_{n,0} \to e^{-it(z+1/z)}b$ in Lebesgue measure on $\mathbb{T}$. Therefore,

$$\mathbf{r}_{\tilde{q}_n(t,\cdot)} = z^n \mathbf{r}_{\tilde{q}_n(t,\cdot-n)} = z^n b_{n,0}/a_{n,0} \to e^{-it(z+1/z)}b/a = e^{-it(z+1/z)}\mathbf{r}_{q(t,\cdot)}, \quad (3.3)$$

in Lebesgue measure on $\mathbb{T}$ (the first equality in (3.3) is Theorem 1.7 (3)). On the other hand, as $n \to +\infty$, the quantities

$$\int_{\mathbb{T}} \log(1 - |\mathbf{r}_{\tilde{q}_n(t,\cdot)}|^2)\,dm = \int_{\mathbb{T}} \log(|a_{n,0}|^{-2})\,dm = \log|a_{n,0}(0)|^{-2}$$

tend to

$$\log|a(0)|^{-2} = \int_{\mathbb{T}} \log(1 - |\mathbf{r}_q|^2)\,dm = \int_{\mathbb{T}} \log(1 - |e^{-it(z+1/z)}\mathbf{r}_q|^2)\,dm.$$

Then, taking into account (3.3), we see that $\mathbf{r}_{\tilde{q}_n(t,\cdot)} \to e^{-it(z+1/z)}\mathbf{r}_q$ in the metric space X by Proposition 6.10. Moreover, the quantities

$$\operatorname{esssup}_{\mathbb{T}}(|\mathbf{r}_{\tilde{q}_n(t,\cdot)}|^2) = 1 - \operatorname{esssup}_{\mathbb{T}} |a_{n,0}|^{-2}$$

are uniformly separated from 1 because $a_{n,0}$ converge uniformly on $\mathbb{T}$ to the bounded function a. Then continuity of the inverse NLFT map (i.e., Theorem 1.7 (6)) gives us the convergence of $\tilde{q}_n(t,\cdot)$ to $\mathcal{F}_{\rm sc}^{-1}(e^{-it(z+1/z)}\mathbf{r}_q)$ in a subspace $\mathcal{G}(\delta)$, $\delta \in (0,1)$, of the metric space $\ell^2(\mathbb{Z},\mathbb{D})$. Since the sequence $\tilde{q}_n(t,\cdot)$ converges elementwise to $\tilde{q}(t,\cdot)$ as $n \to +\infty$, we get $\tilde{q}(t,\cdot) = \mathcal{F}_{\rm sc}^{-1}(e^{-it(z+1/z)}\mathbf{r}_q)$ on $\mathbb{Z}$. Then, $\mathbf{r}_{\tilde{q}(t,\cdot)} = \mathcal{F}_{\rm sc}(\tilde{q}(t,\cdot)) = e^{-it(z+1/z)}\mathbf{r}_q$ almost everywhere on $\mathbb{T}$, and the proof is completed. ∎

Remark 3.5. In the proof of Theorem 1.6, we have used the fact that (1.4) is solvable for compactly supported initial data. This can be proved by a variety of methods, see Appendix A for a direct proof in a much more general situation. Theorem 1.7 (6) and Theorem 1.7 (7) guarantee that the solution will be determined by its reflection coefficient $\mathbf{r}_{q(t,\cdot)} = e^{-it(z+1/z)}\mathbf{r}_{q_0}$ at any moment of time $t \in \mathbb{R}$.

4. Localization. Proof of Theorem 1.5

The following lemma is well known, see, e.g., [18, (1.3.43)].

Lemma 4.1. *Let $F, G \in \mathcal{S}_*(\mathbb{D})$, and let F_k, G_k be their Schur iterates* (1.1)*. Assume that $F_k(0) = G_k(0)$ for $0 \le k \le n$. Then* $\max_{|z|=r} |F(z) - G(z)| \le 2r^{n+1}$.

Lemma 4.2. *Let $F, G \in \mathcal{S}(\mathbb{D})$ be such that* $\min(\eta(F), \eta(G)) \ge \eta$ *for some* $\eta > 0$. *Denote by F_k, G_k their Schur iterates* (1.1)*, and consider the solutions of* (1.4) *with the initial value*

$$q_{0,F} = \begin{cases} F_n(0), & n \ge 0, \\ 0, & n < 0, \end{cases} \qquad q_{0,G} = \begin{cases} G_n(0), & n \ge 0, \\ 0, & n < 0. \end{cases}$$

Denote them by q_F and q_G, respectively. Then, for every $n > t > 0$, $r \in (0, 1)$, the inequality

$$|q_F(t, j) - q_G(t, j)| \le r^{-j} e^{t/r} C(\eta, r) \sup_{|z|=r} |F(z) - G(z)|,$$

holds for all $j \in \mathbb{Z}$. Here $C(\eta, r)$ is the function from Theorem 1.2.

Proof. For a function $H \in \mathcal{S}_*(\mathbb{D})$, let us denote by $(H)_k$ its Schur iterates (1.1). By Theorem 1.6, we have

$$q_F(t, j) = \lim_{n\to\infty} (G_{n,t} F)_{n+j}(0), \quad q_G(t, j) = \lim_{n\to\infty} (G_{n,t} G)_{n+j}(0), \quad j \in \mathbb{Z}.$$

Therefore, we can apply Theorem 1.2 and the bound $|G_{n,t}| < e^{t/r}(r^n + 3\delta_{n,t})$ from Lemma 3.2 to get

$$\begin{aligned} |q_F(t, j) - q_G(t, j)| &\le \limsup_{n\to\infty} |(G_n F)_{n+j}(0) - (G_n G)_{n+j}(0)| \\ &\le \limsup_{n\to\infty} \|(G_{n,t} F)_{n+j} - (G_{n,t} G)_{n+j}\|_{L^2(r\mathbb{T})} \\ &\le \limsup_{n\to\infty} C(\eta, r) r^{-n-j} \|G_{n,t} F - G_{n,t} G\|_{L^2(r\mathbb{T})} \\ &\le \limsup_{n\to\infty} C(\eta, r) r^{-j} e^{t/r} (1 + 3\delta_{n,t} r^{-n}) \sup_{|z|=r} |F(z) - G(z)| \\ &= r^{-j} e^{t/r} C(\eta, r) \sup_{|z|=r} |F(z) - G(z)|, \end{aligned}$$

where we have the convergence $\delta_{n,t} r^{-n} \to 0$ as $n \to \infty$. ■

Proof of Theorem 1.5. Recall that $q_0 \colon \mathbb{Z} \to \mathbb{D}$ is such that

$$\prod_{n\in\mathbb{Z}} (1 - |q_0(n)|^2) \ge \eta > 0,$$

where the sequence $q_{0,N}$ is defined by

$$q_{0,N}(n) = \begin{cases} q_0(n), & |n| \le N, \\ 0, & |n| > N, \end{cases}$$

and q_N is the corresponding solution of (1.4) (see Remark 3.5). Let $C(\eta, r)$ be the function from Theorem 1.2. We want to prove the inequality

$$|q_{N+K}(t, j) - q_N(t, j)| \le \frac{4e^{t/r} C(\eta, r)}{1 - r} r^{N-|j|}, \quad K \in \mathbb{Z}_+. \tag{4.1}$$

Then $\{q_N(t, j)\}$ will be a Cauchy sequence for each t, j, and its limit, to be denoted by q, solves (1.4). This is easy to check if one rewrites (1.4) in the integral form,

$$q_N(T, j) - q_{0,N}(j) = i \int_0^T (1 - |q_N(t, j)|^2)(q_N(t, j - 1) + q_N(t, j + 1))\, dt$$

for $T \ge 0$, $j \in \mathbb{Z}$, fix $j \in \mathbb{Z}$, and take the limit using the Lebesgue dominated convergence theorem (with majorant 2). Then the relation $q_0 = \lim q_{0,N}$ and the uniqueness of the solution of (1.4) with a given initial datum q_0 give the claim, see Lemma A.2. Estimate (1.6) will follow from (4.1) by taking the limit as $K \to +\infty$. ■

For integer numbers $A \le B$, consider the sequences $q_{0,[A,B]}, \overleftarrow{q}_{0,[A,B]}$ in $\ell^2(\mathbb{Z}, \mathbb{D})$ defined by

$$q_{0,[A,B]}(j) = q_0(j + A)\mathbf{1}_{[0,B-A]}(j), \quad \overleftarrow{q}_{0,[A,B]}(j) = q_0(-j + B)\mathbf{1}_{[0,B-A]}(j),$$

where $\mathbf{1}_S$ is the indicator function of a set S. These sequences, both supported on $[0, B - A]$ and their entries, are symmetric on this segment. Denote the corresponding solutions of (1.4) by $q_{[A,B]}$ and $\overleftarrow{q}_{[A,B]}$. By the properties of (1.4), the symmetry relation

$$q_{[A,B]}(t, j) = \overleftarrow{q}_{[A,B]}(t, B - A - j), \quad t \in \mathbb{R}, j \in \mathbb{Z}, \tag{4.2}$$

holds for each $t \in \mathbb{R}$. Moreover, comparing this with the definition of q_N, we see that $q_N(t, j) = q_{[-N,N]}(t, j + N)$. Inequality (4.1) will follow by summing up a telescoping series if we check the estimate

$$|q_N(t, j) - q_{N+1}(t, j)| \le 4C(\eta, r)e^{t/r} r^{N-|j|}, \quad N \ge |j|.$$

In the new notation, the latter takes the form

$$|q_{[-N,N]}(t, j + N) - q_{[-N-1,N+1]}(t, j + N + 1)| \le 4C(\eta, r)e^{t/r} r^{N-|j|}, \tag{4.3}$$

where, again, $N \geq |j|$. For $A \leq B$ let $f_{[A,B]}$ and $\overleftarrow{f}_{[A,B]}$ be the Schur functions whose recurrence coefficients are $q_{0,[A,B]}|_{\mathbb{Z}_+}$ and $\overleftarrow{q}_{0,[A,B]}|_{\mathbb{Z}_+}$ respectively. The Schur functions $f_{[-N,N]}$, $f_{[-N,N+1]}$ have the same first $2N+1$ Schur coefficients. Hence, by Lemmas 4.1 and 4.2 we get

$$\begin{aligned}|q_{[-N,N]}(t,n) - q_{[-N,N+1]}(t,n)| &\leq r^{-n} C(\eta,r) e^{t/r} \sup_{|z|=r} |(f_{[-N,N]} - f_{[-N,N+1]})(z)| \\ &\leq 2C(\eta,r) e^{t/r} r^{2N-n+1},\end{aligned}$$

for all $n \geq 0$. Similarly, the functions $\overleftarrow{f}_{[-N,N+1]}$ and $\overleftarrow{f}_{[-N-1,N+1]}$ have coinciding first $2N+1$ Schur coefficients; therefore,

$$|\overleftarrow{q}_{[-N,N+1]}(t,n) - \overleftarrow{q}_{[-N-1,N+1]}(t,n)| \leq 2C(\eta,r) e^{t/r} r^{2N-n+1}, \quad n \geq 0. \tag{4.4}$$

Notice that

$$\begin{aligned}&|q_{[-N,N]}(t,n) - q_{[-N-1,N+1]}(t,n+1)| \\ &\quad \leq |q_{[-N,N]}(t,n) - q_{[-N,N+1]}(t,n)| + |q_{[-N,N+1]}(t,n) - q_{[-N-1,N+1]}(t,n+1)| \\ &\quad \leq 2C(\eta,r) e^{t/r} r^{2N-n+1} + |q_{[-N,N+1]}(t,n) - q_{[-N-1,N+1]}(t,n+1)|.\end{aligned}$$

By relation (4.2), the last term equals

$$|\overleftarrow{q}_{[-N,N+1]}(t,2N+1-n) - \overleftarrow{q}_{[-N-1,N+1]}(t,2N+1-n)|$$

which does not exceed

$$\cdots \leq 2C(\eta,r) e^{t/r} r^{2N-(2N+1-n)+1} = 2C(\eta,r) e^{t/r} r^{n},$$

where we used (4.4) in the first inequality. Therefore, we have

$$|q_{[-N,N]}(t,n) - q_{[-N-1,N+1]}(t,n+1)| \leq 2C(\eta,r) e^{t/r} (r^{2N-n+1} + r^{n}).$$

Substituting $n = j + N$ then gives

$$\begin{aligned}&|q_{[-N,N]}(t,j+N) - q_{[-N-1,N+1]}(t,j+N+1)| \\ &\quad \leq 2C(\eta,r) e^{t/r} (r^{N-j+1} + r^{j+N}) \\ &\quad \leq 4C(\eta,r) e^{t/r} r^{N-|j|},\end{aligned}$$

which is (4.3).

5. Complexity of the algorithm

In the introduction, we claimed that the algorithm in Section 1.5 takes $O(\mathbf{n}\log^2\mathbf{n})$ operations for $\mathbf{n} = t + \log\varepsilon^{-1}$. Here we prove this estimate.

Let $q_0 \in \ell^2(\mathbb{Z},\mathbb{D})$ be such that $\prod_{n\in\mathbb{Z}}(1-|q_0(n)|^2) \ge \eta > 0$, and let $t > 0$. Take $\varepsilon \in (0,1)$, set $r = 1/2$, and choose $N \in \mathbb{Z}_+$ such that the right-hand side in (1.6) does not exceed $\varepsilon/2$ at $j = 0$:

$$8e^{2t}C\Big(\eta,\frac{1}{2}\Big)2^{-N} \le \frac{1}{2}\varepsilon, \quad C\Big(\eta,\frac{1}{2}\Big) = \exp\Big(17\log\eta^{-1}\cdot\Big(2+\frac{1}{1-\sqrt{1-\eta}}\Big)\Big).$$

Since $8e^{2t}C(\eta,1/2)2^{-N} \le 2^{-N+4+3t}C(\eta,1/2)/2$, one can take any $N \ge 5 + [3t + \log_2(C(\eta,1/2)/\varepsilon)]$. Then, choose the window $\Delta = [n_0-N, n_0+N]$, truncate q_0 by setting $q_0 = 0$ on $\mathbb{Z}\setminus\Delta$, and shift q_0 by $n_0 - N$ to make it supported in $[0,2N]$. Denote the resulting sequence by $q_{0,[n_0-N,n_0+N]}$. Choose $n > t$ so that $\delta_{n,t} < 1$ and

$$2^jC(\eta,1/2)\frac{12e^{5t}}{\sqrt{2\pi n}}\Big(\frac{2et}{n}\Big)^n \le \frac{\varepsilon}{2}, \quad j = N.$$

Since we already have $8e^{2t}C(\eta,1/2)2^{-N} \le \varepsilon/2$, it suffices to choose n so that

$$2^{2N}e^{3t}\frac{12}{8\sqrt{2\pi n}}\Big(\frac{2et}{n}\Big)^n \le 1.$$

For $n \ge 2N \ge 8et \ge 5t$, we have

$$2^{2N}e^{3t}\frac{12}{8\sqrt{2\pi n}}\Big(\frac{2et}{n}\Big)^n \le 2^{2N+5t}\Big(\frac{2et}{n}\Big)^n \le 2^{5t}\Big(\frac{2et}{N}\Big)^{2N} \le \Big(\frac{4et}{N}\Big)^{2N} \le 1,$$

therefore, one can take $n = 2N$, $N = 5 + [4et + \log_2(C(\eta,1/2)/\varepsilon)]$. Note that, with this choice,

$$\delta_{n,t} = \frac{t^ne^t}{n!} \le \Big(\frac{te}{n}\Big)^ne^t \le \Big(\frac{te^2}{n}\Big)^n \le \Big(\frac{8et}{2N}\Big)^n < 1.$$

We see that, for $n = 2N$, $N = 5 + [4et + \log_2(C(\eta,1/2)/\varepsilon)]$, Theorem 1.6 applied to the sequence $q_{0,[n_0-N,n_0+N]}$ in place of q_0 will give a sequence $\tilde q_n$ approximating the corresponding solution $q_{[n_0-N,n_0+N]}$ with accuracy

$$|\tilde q_n(t,N) - q_{[n_0-N,n_0+N]}(t,N)| \le \frac{\varepsilon}{2}.$$

Then $|\tilde q_n(t,N+1) - q(t,n_0)| < \varepsilon$ and it remains to estimate the number of operations that are needed to construct $\tilde q_n(t,N)$ from q_0 for $n = 2N$.

Having $q_0, t_0, n_0, \varepsilon, \eta$, we set $N = 5 + [4et + \log_2 C(\eta,1/2)/\varepsilon]$ and define array $q_{0,[n_0-N,n_0+N]}$ of $2N+1$ elements. Then we use formula (1.5) to find a, b. This can be done either by a direct multiplication of $2N+1$ matrices in $O(N^2)$ operations or by using a dyadic divide-and-conquer multiplication algorithm together with the fast Fourier transform (FFT) in $O(N\log^2 N)$ operations. Next, we define the coefficients of the polynomials $P = G_{n,t}\bar{b}$ and $Q = a$ (two arrays of length $2n+1+2N+1$ and $2N+1$, respectively). This takes $O(N^2)$ operations in the naive realization of multiplications of polynomials or $O(N\log N)$ operations with FFT. Taking $n+N+1$ steps of Schur's algorithm for P/Q, we find $\tilde{q}_n(t,j)$ on $[0,N]$, which solves the problem. A straightforward realization of Schur's algorithm based on its definition requires $O(N^2) = O(\mathbf{n}^2)$ operations (recall that $\mathbf{n} = \log\varepsilon^{-1} + t$). This could be improved to $O(\mathbf{n}\log^2\mathbf{n})$ operations with a refined realization, see [4, Section 2.2]. Notice that the numerical experiments in [4] use arithmetic of real numbers, while the complexity estimate $O(\mathbf{n}\log^2\mathbf{n})$ given on [4, p. 192] holds for complex data. As the reader can see from the algorithm, the same $O(\mathbf{n}\log^2\mathbf{n})$ operations (with worse constant) are sufficient to find $\tilde{q}_n(t,N)$ on $[0,2N]$ and approximate $q(t,\cdot)$ with accuracy ε on the interval $[n_0 - N/2, n_0 + N/2]$, not just at the point n_0. It is also worth mentioning that the question of numerical stability (in our case – estimating round-off errors and taking into account issues related to arithmetic of long numbers) deserves a special consideration, it is neither treated in [4] nor in this paper.

Let us present some numerical results comparing our algorithm with the classical Runge–Kutta RK4 scheme, described, e.g., [6, Section 6.10]. Here is the computational setup: we consider q_0 on $[-100,100]$, where $q_0(n)$ for $|n| \le 100$ is chosen uniformly randomly in $\mathbb{D}$, and normalise it by constant to have

$$\prod_{|n|\le 100}(1-|q_0(n)|^2) = 0.96.$$

Then we find the smallest number of arithmetic operations needed to compute the approximate solution $\tilde{q}$ of (1.4) on $[-5,5]$ with the given relative accuracy

$$\varepsilon = \frac{\sum_{|j|\le 5}|\tilde{q}(t,j) - q(t,j)|}{\sum_{|j|\le 5}|q(t,j)|}$$

by the Runge–Kutta method and our method in the $O(\mathbf{n}^2)$ realization (without the fast Schur algorithm and the FFT polynomial multiplication). Denoting this smallest number of operations by $N_{\mathrm{RK4}}(t,\varepsilon)$ and $N_{G_{n,t}}(t,\varepsilon)$, correspondingly, we then compute the ratio $r(t,\varepsilon) = N_{G_{n,t}}(t,\varepsilon)/N_{\mathrm{RK4}}(t,\varepsilon)$. When we find $N_{\mathrm{RK4}}(t,\varepsilon)$ and $N_{G_{n,t}}(t,\varepsilon)$, the parameters of the algorithms are optimised: for the RK4 scheme, we optimised the step size Δt and the size of the window containing $[-5,5]$ to perform the numerical scheme. In our algorithm, we optimise n (the degree of multiplier $G_{n,t}$) and the size of the window (i.e., the number N from Section 1.5). Table 1 provides $r(t,\varepsilon)$ for some values of t and ε. The table shows that the proposed algorithm works faster for larger values of t and smaller values of ε.

t \ ε	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
1	1.15	0.89	0.63	0.38	0.22
2	0.90	0.48	0.32	0.20	0.11
3	0.79	0.46	0.25	0.15	0.09
4	0.61	0.35	0.22	0.11	0.07
5	0.59	0.33	0.19	0.11	0.06
6	0.53	0.31	0.18	0.09	0.06
7	0.50	0.29	0.16	0.09	0.05
8	0.49	0.29	0.16	0.09	0.05
9	0.48	0.25	0.14	0.08	0.05
10	0.40	0.24	0.13	0.07	0.04

Table 1. Comparison of the RK4 scheme and the proposed algorithm. The table contains $r(t,\varepsilon) = N_{G_{n,t}}(t,\varepsilon)/N_{\mathrm{RK4}}(t,\varepsilon)$, the average values for 30 random initial data q_0.

6. The nonlinear Fourier transform. Proof of Theorem 1.7

In this section we collect some basic facts about the nonlinear Fourier transform, NLFT. Some of them were used in the first part of the paper. The reader can find more information in the preprint [23] or in its extended version [24].

The exposition in this section is independent from the first part of the paper. Let us recall the definition of the NLFT map for the reader's convenience. For $p \ge 1$, define $\ell^p(\mathbb{Z},\mathbb{D})$ as a set of sequences $q\colon \mathbb{Z} \to \mathbb{D}$ satisfying $|q(n)| < 1$ for every $n \in \mathbb{Z}$ and $\sum_{n\in\mathbb{Z}} |q(n)|^p < \infty$. The set $\ell^p(\mathbb{Z}_+,\mathbb{D})$ is defined similarly with $\mathbb{Z}_+ = \mathbb{Z} \cap [0,+\infty)$. Take a sequence $q \in \ell^1(\mathbb{Z},\mathbb{D})$ and define a, b by

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} = \prod_{k\in\mathbb{Z}} \frac{1}{\sqrt{1-|q(k)|^2}} \cdot \begin{pmatrix} 1 & \overline{q(k)} z^{-k} \\ q(k) z^k & 1 \end{pmatrix}, \qquad z \in \mathbb{T}. \tag{6.1}$$

Here, the product $\prod_{k\in\mathbb{Z}} T_k$ of matrices T_k is understood as the limit

$$\lim_{n\to+\infty} T_{-n} T_{-n+1} \cdot \dots \cdot T_{n-1} T_n.$$

Assumption $q \in \ell^1(\mathbb{Z},\mathbb{D})$ guarantees that the product converges uniformly on $\mathbb{T}$. We will see in Section 6.2 that the product in (6.1) has the form $\left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right)$ for some a, b. The authors of [23] define NLFT as the map that sends q to the pair $(a \;\; b)$. We will use an equivalent definition and consider the so-called *reflection coefficient* $\mathbf{r}_q = b/a$ in place of $(a \;\; b)$. So, in our case, NLFT takes q into $\mathbf{r}_q$. In the next two subsections we define the reflection coefficient as an object of the theory of orthogonal polynomials on the unit circle. We also prove the equivalence of the two definitions of NLFT map.

6.1. Szegő measures and Szegő functions

Let μ be a probability measure supported on an infinite subset of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ of the complex plane, $\mathbb{C}$. For $n \in \mathbb{Z}_+$, denote by Φ_n the monic orthogonal polynomial of degree n generated by μ, and set $\Phi_n^* = z^n \overline{\Phi_n(1/\bar{z})}$. These polynomials satisfy the following relation:

$$\Phi_{n+1} = z\Phi_n - \bar{\alpha}_n \Phi_n^*, \quad n \geq 0, \quad \Phi_0 = 1, \tag{6.2}$$

where the *recurrence coefficients,* α_n, $n \geq 0$, lie in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Conversely, any sequence $\{\alpha_n\}_{n\geq 0} \subset \mathbb{D}$ gives rise to a unique probability measure μ on $\mathbb{T}$ whose closed support $\operatorname{supp}\mu$ contains infinitely many points. These two facts can be found in [18, Section 1.7]. The Schur function f of a probability measure μ on $\mathbb{T}$ is defined by

$$\frac{1 + zf(z)}{1 - zf(z)} = \int_{\mathbb{T}} \frac{1 + \bar{\xi}z}{1 - \bar{\xi}z}\, d\mu(\xi), \quad z \in \mathbb{D}. \tag{6.3}$$

Notice that (6.3) provides a bijective correspondence between Schur functions and measures on $\mathbb{T}$. Taking the real part on both sides of this equality, we get

$$\frac{1 - |zf(z)|^2}{|1 - zf(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \bar{\xi}z|^2}\, d\mu(\xi), \quad z \in \mathbb{D}. \tag{6.4}$$

From (6.3), (6.4), and the Schwarz lemma, we see that f indeed belongs to the Schur class $\mathcal{S}(\mathbb{D})$, i.e., it is analytic in $\mathbb{D}$ and satisfies $\sup_{z\in\mathbb{D}} |f(z)| \leq 1$. Recall that the Schur iterates of $f = f_0$ are defined by

$$zf_{n+1} = \frac{f_n - f_n(0)}{1 - \overline{f_n(0)} f_n}, \quad n \geq 0. \tag{6.5}$$

Geronimus's theorem says that the recurrence coefficients α_n in (6.2) coincide with the recurrence coefficients in Schur's algorithm: $\alpha_n = f_n(0)$, $n \geq 0$. See [18, Chapter 3] for the proof.

Let $\mu = w\, dm + \mu_{\mathbf{s}}$ be the Radon–Nikodym decomposition of μ into the absolutely continuous the and singular parts, where m is the Lebesgue measure on $\mathbb{T}$ normalised by $m(\mathbb{T}) = 1$. Denote by $\{\alpha_n\}$ the set of recurrence coefficients of the measure μ and let f be its Schur function. An extended version of Szegő theorem (Theorem 1.1) says that the conditions $\log w \in L^1(\mathbb{T})$, $\log(1 - |f|^2) \in L^1(\mathbb{T})$, $\{\alpha_n\} \in \ell^2(\mathbb{Z}_+, \mathbb{D})$ are equivalent, and, moreover,

$$\int_{\mathbb{T}} \log w(\xi)\, dm(\xi) = \int_{\mathbb{T}} \log(1 - |f(\xi)|^2)\, dm(\xi) = \log \prod_{n\geq 0} (1 - |\alpha_n|^2). \tag{6.6}$$

It is not difficult to see that the three quantities in (6.6) are defined for any triple μ, f, $\{\alpha_n\}$, but could be $-\infty$. In fact, Szegő theorem implies that the quantities in (6.6) are either finite (i.e., $> -\infty$) or not simultaneously. The measures of Szegő class

$$\mathrm{Sz}(\mathbb{T}) = \{\mu = w\,dm + \mu_{\mathbf{s}} : \mu(\mathbb{T}) = 1, \log w \in L^1(\mathbb{T})\}$$

and their orthogonal polynomials have many interesting properties that constitute the rich Szegő theory. We will use the part of this theory related to discrete scattering. For this, we will need the notion of the dual orthogonality measure, the Szegő function, and the dual Szegő function.

Consider a probability measure μ on $\mathbb{T}$ with infinite support. Let, as before, f denote the Schur function of μ. The *dual measure* μ_d is defined as the probability measure on $\mathbb{T}$ corresponding to the Schur function $-f$:

$$\int_{\mathbb{T}} \frac{1+\bar{\xi}z}{1-\bar{\xi}z}\,d\mu_d(\xi) = \frac{1+zf_d(z)}{1-zf_d(z)} = \frac{1-zf(z)}{1+zf(z)}, \quad z \in \mathbb{D}.$$

It is not difficult to check that if $\{\alpha_n\}_{n\ge 0}$ is the sequence of the recurrence coefficients of μ, then $\{-\alpha_n\}_{n\ge 0}$ is the sequence of the recurrence coefficients of μ_d. Monic orthogonal polynomials for μ_d will be denoted by Ψ_n. We will also need the normalised orthogonal polynomials for μ and μ_d:

$$\begin{aligned} \varphi_n &= \frac{\Phi_n}{\|\Phi_n\|_{L^2(\mu)}}, \qquad \varphi_n^* = \frac{\Phi_n^*}{\|\Phi_n^*\|_{L^2(\mu)}}, \\ \psi_n &= \frac{\Psi_n}{\|\Psi_n\|_{L^2(\mu_d)}}, \qquad \psi_n^* = \frac{\Psi_n^*}{\|\Psi_n^*\|_{L^2(\mu_d)}}. \end{aligned} \tag{6.7}$$

In fact,

$$\|\Phi_n\|^2_{L^2(\mu)} = \|\Phi_n^*\|^2_{L^2(\mu)} = \|\Psi_n\|^2_{L^2(\mu_d)} = \|\Psi_n^*\|^2_{L^2(\mu_d)} = \prod_{k=0}^{n-1}(1-|\alpha_k|^2), \tag{6.8}$$

for all $n \ge 1$, see [18, Chapter 3.2]. The Szegő function D_μ of a measure $\mu = w\,dm + \mu_{\mathbf{s}}$ from Szegő class $\mathrm{Sz}(\mathbb{T})$ is the outer function in the open unit disk $\mathbb{D}$ such that $D_\mu(0) > 0$ and $|D_\mu|^2 = w$ Lebesgue almost everywhere on $\mathbb{T}$ in the sense of nontangential boundary values. It can be defined by the formula

$$D_\mu(z) = \exp\Big(\frac{1}{2}\int_{\mathbb{T}} \frac{1+\bar{\xi}z}{1-\bar{\xi}z}\log w(\xi)\,dm(\xi)\Big), \quad z \in \mathbb{D}. \tag{6.9}$$

It follows from the Szegő theorem (see (6.6)) that $\mu \in \mathrm{Sz}(\mathbb{T})$ if and only if $\mu_d \in \mathrm{Sz}(\mathbb{T})$. We will denote the Szegő function of μ_d by D_{μ_d}. It is known that $\varphi_n^* \to D_\mu^{-1}$,

$\psi_n^* \to D_{\mu_d}^{-1}$ as $n \to \infty$ in $\mathbb{D}$, and

$$\frac{1+zf}{1-zf} = \lim_{n\to\infty} \frac{\Psi_n^*(z)}{\Phi_n^*(z)} = \lim_{n\to\infty} \frac{\psi_n^*(z)}{\varphi_n^*(z)} = \frac{D_{\mu_d}^{-1}(z)}{D_\mu^{-1}(z)}, \quad z \in \mathbb{D}, \tag{6.10}$$

see [18, Theorem 2.4.1 and Chapter 3.2]. In particular, we have

$$\mathrm{Re}(D_{\mu_d}^{-1}\overline{D_\mu^{-1}}) = \mathrm{Re}\Big(\frac{D_{\mu_d}^{-1}}{D_\mu^{-1}}\Big)|D_\mu|^{-2} = \frac{1-|zf|^2}{|1-zf|^2}|D_\mu|^{-2} = w|D_\mu|^{-2} = 1 \tag{6.11}$$

almost everywhere on $\mathbb{T}$ in the sense of non-tangential boundary values.

6.2. Reflection coefficients

Let us now define a reflection coefficient of a sequence q in $\ell^2(\mathbb{Z}, \mathbb{D})$. To simplify the notation, we set $q_n = q(n)$, $n \in \mathbb{Z}$. Consider the sequences $\{\alpha_n\}_{n\in\mathbb{Z}_+}$ and $\{\beta_n\}_{n\in\mathbb{Z}_+}$ from $\ell^2(\mathbb{Z}_+, \mathbb{D})$ defined by $\alpha_n = q_n$, for $n \ge 0$ and $\beta_0 = 0$, $\beta_n = -\overline{q_{-n}}$ for $n \ge 1$,

$$\begin{array}{cccccccc} & & & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \dots \\ \dots & q_{-3} & q_{-2} & q_{-1} & q_0 & q_1 & q_2 & q_3 & \dots \\ \dots & -\overline{\beta_3} & -\overline{\beta_2} & -\overline{\beta_1} & 0. & & & \end{array} \tag{6.12}$$

Define the measures μ^+, μ^- with the recurrence coefficients $\{\alpha_n\}_{n\ge 0}$, $\{\beta_n\}_{n\ge 0}$, respectively. Let also $\mu_d^\pm$ be the dual measures corresponding to $\mu^\pm$. Define the Wall analytic functions in $\mathbb{D}$ by

$$\mathfrak{a}^\pm = \frac{D_{\mu_d^\pm}^{-1} + D_{\mu^\pm}^{-1}}{2}, \quad \mathfrak{b}^\pm = \frac{D_{\mu_d^\pm}^{-1} - D_{\mu^\pm}^{-1}}{2z}. \tag{6.13}$$

The fact that $D_{\mu_d^\pm}^{-1}(0) = D_{\mu^\pm}^{-1}(0)$ follows from (6.10). Using (6.11), we obtain

$$|\mathfrak{a}^\pm|^2 - |\mathfrak{b}^\pm|^2 = \mathrm{Re}(D_{\mu_d^\pm}^{-1}\overline{D_{\mu^\pm}^{-1}}) = 1$$

Lebesgue almost everywhere on $\mathbb{T}$ in the sense of non-tangential boundary values. Also, we have

$$\frac{1+z\frac{\mathfrak{b}^\pm}{\mathfrak{a}^\pm}}{1-z\frac{\mathfrak{b}^\pm}{\mathfrak{a}^\pm}} = \frac{D_{\mu_d^\pm}^{-1}(z)}{D_{\mu^\pm}^{-1}(z)} = \frac{1+zf^\pm}{1-zf^\pm},$$

for the Schur functions $f^\pm$ of $\mu^\pm$; hence, $f^\pm = \mathfrak{b}^\pm/\mathfrak{a}^\pm$. On $\mathbb{T}$, we set

$$a = \mathfrak{a}^+\mathfrak{a}^- - \mathfrak{b}^+\mathfrak{b}^-, \quad b = \mathfrak{a}^-\overline{\mathfrak{b}^+} - \mathfrak{b}^-\overline{\mathfrak{a}^+}. \tag{6.14}$$

Below we will use the fact that a is defined by (6.14) not only on $\mathbb{T}$ but also on $\mathbb{D}$ and is analytic there. Note that $|a|^2 - |b|^2 = (|\mathfrak{a}^+|^2 - |\mathfrak{b}^+|^2)(|\mathfrak{a}^-|^2 - |\mathfrak{b}^-|^2) = 1$

almost everywhere on $\mathbb{T}$. Next, define the reflection coefficient, $\mathbf{r}_q$, of the sequence $q = \{q_n\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z}, \mathbb{D})$ by

$$\mathbf{r}_q = \frac{b}{a}. \tag{6.15}$$

It is possible to associate with q an operator on $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ in a way that will place the reflection coefficient $\mathbf{r}_q$ into the setting of a discrete scattering theory, see [24]. Our first proposition collects the properties of objects defined in the present section.

Proposition 6.1. *For every $q \in \ell^2(\mathbb{Z}, \mathbb{D})$ the functions $a, \mathfrak{a}^{\pm}$ are outer, $a(0) > 0$. The reflection coefficient $\mathbf{r}_q = b/a$ of q belongs to the unit ball of $L^\infty(\mathbb{T})$. It is completely determined by b, and, conversely, it determines the pair a, b uniquely.*

Proof. By definition and (6.10), we have

$$\begin{aligned} \mathfrak{a}^{\pm} &= \frac{1}{2} D_{\mu}^{-1} \cdot \Big(1 + \frac{1 + zf^{\pm}}{1 - zf^{\pm}}\Big), \\ a &= \mathfrak{a}^{+}\mathfrak{a}^{-}\Big(1 - \frac{\mathfrak{b}^{+}\mathfrak{b}^{-}}{\mathfrak{a}^{+}\mathfrak{a}^{-}}\Big) = \mathfrak{a}^{+}\mathfrak{a}^{-}(1 - f^{+}f^{-}). \end{aligned} \tag{6.16}$$

We know that $D_{\mu}^{-1}/2$ is outer and that both $1 + (1 + zf^{\pm})/(1 - zf^{\pm})$ and $1 - f^{+}f^{-}$ are analytic in $\mathbb{D}$ and have positive real part; hence, they are also outer, see [8, Corollary 4.8]. Therefore, $\mathfrak{a}^{\pm}$, a are outer as the products of outer functions. Next, $D_{\mu_d^{\pm}}(0) = D_{\mu^{\pm}}(0) > 0$; hence, the $\mathfrak{a}^{\pm}(0)$ are real and positive. We have $\beta_0 = 0$, therefore $f^{-}(0) = 0$ (recall Schur's algorithm (1.1)) and $\mathfrak{b}^{-}(0) = 0$. Thus, $a(0) = \mathfrak{a}^{+}(0)\mathfrak{a}^{-}(0) > 0$. From (6.15), we have

$$1 - |\mathbf{r}_q|^2 = \frac{|a|^2 - |b|^2}{|a|^2} = \frac{1}{|a|^2} \geq 0 \tag{6.17}$$

almost everywhere on $\mathbb{T}$. In particular, $\mathbf{r}_q$ belongs to the unit ball of $L^\infty(\mathbb{T})$. We proved that a is outer; hence, it is completely defined by $|a|$. Therefore, knowing the coefficient b, one can recover $|a| = \sqrt{1 + |b|^2}$ and a. In particular, the numerator b determines the whole fraction $\mathbf{r}_q = b/a$. Conversely, if the function $\mathbf{r}_q$ is given, then $|a|$ is defined by (6.17); hence, the pair a, b could be found from the fraction $\mathbf{r}_q = b/a$. ∎

The next proposition shows that (6.1) makes sense for all $q \in \ell^2(\mathbb{Z}, \mathbb{D})$, and, moreover, the definitions of a, b in (6.14), (6.1) are equivalent.

Proposition 6.2. *For every $q \in \ell^2(\mathbb{Z}, \mathbb{D})$, the product in (6.1) converges in Lebesgue measure on $\mathbb{T}$. Moreover, the functions a, b in (6.1) coincide with those in (6.14).*

Proof. Denote by $\Phi_{\pm,n}$, $\Psi_{\pm,n}$ the monic orthogonal polynomials of $\mu^{\pm}$ and $\mu_d^{\pm}$, and let $\varphi_{\pm,n}$, $\psi_{\pm,n}$ be the corresponding normalised polynomials, see (6.7). For each $n \ge 0$, $z \in \mathbb{T}$, we have

$$\prod_{k=0}^{n} \frac{1}{\sqrt{1-|\alpha_k|^2}} \cdot \prod_{k=0}^{n} \begin{pmatrix} 1 & \overline{\alpha_k} \bar{z}^k \\ \alpha_k z^k & 1 \end{pmatrix} = \begin{pmatrix} \frac{\psi^*_{+,n+1}+\varphi^*_{+,n+1}}{2} & \overline{\frac{\psi^*_{+,n+1}-\varphi^*_{+,n+1}}{2z}} \\ \frac{\psi^*_{+,n+1}-\varphi^*_{+,n+1}}{2z} & \overline{\frac{\psi^*_{+,n+1}+\varphi^*_{+,n+1}}{2}} \end{pmatrix}.$$

The proof is a routine verification of the identity

$$\begin{aligned} &\begin{pmatrix} \frac{\psi^*_{+,n+1}+\varphi^*_{+,n+1}}{2} & \overline{\frac{\psi^*_{+,n+1}-\varphi^*_{+,n+1}}{2z}} \\ \frac{\psi^*_{+,n+1}-\varphi^*_{+,n+1}}{2z} & \overline{\frac{\psi^*_{+,n+1}+\varphi^*_{+,n+1}}{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\psi^*_{+,n}+\varphi^*_{+,n}}{2} & \overline{\frac{\psi^*_{+,n}-\varphi^*_{+,n}}{2z}} \\ \frac{\psi^*_{+,n}-\varphi^*_{+,n}}{2z} & \overline{\frac{\psi^*_{+,n}+\varphi^*_{+,n}}{2}} \end{pmatrix} \frac{1}{\sqrt{1-|\alpha_n|^2}} \cdot \begin{pmatrix} 1 & \overline{\alpha_n} \bar{z}^n \\ \alpha_n z^n & 1 \end{pmatrix}, \end{aligned}$$

using relations (6.2) and (6.8). It is known that

$$\varphi^*_{\pm,n} \to D^{-1}_{\mu^{\pm}}, \quad \psi^*_{\pm,n} \to D^{-1}_{\mu^{\pm}_d}$$

in Lebesgue measure on $\mathbb{T}$, see [18, (2.4.34)]. Therefore, we have

$$\begin{aligned} &\prod_{k=0}^{\infty} \frac{1}{\sqrt{1-|q_k|^2}} \cdot \prod_{k=0}^{\infty} \begin{pmatrix} 1 & \overline{q_k} \bar{z}^k \\ q_k z^k & 1 \end{pmatrix} \\ &= \prod_{k=0}^{\infty} \frac{1}{\sqrt{1-|\alpha_k|^2}} \cdot \prod_{k=0}^{\infty} \begin{pmatrix} 1 & \overline{\alpha_k} \bar{z}^k \\ \alpha_k z^k & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathfrak{a}^+ & \overline{\mathfrak{b}^+} \\ \mathfrak{b}^+ & \overline{\mathfrak{a}^+} \end{pmatrix}, \end{aligned} \tag{6.18}$$

where the product converges in Lebesgue measure on $\mathbb{T}$. Recall that $\beta_k = -\overline{q_{-k}}$ for $k \ge 1$, $\beta_0 = 0$. We have

$$\begin{aligned} &\left(\prod_{k=-n}^{-1} \frac{1}{\sqrt{1-|q_k|^2}} \cdot \prod_{k=-n}^{-1} \begin{pmatrix} 1 & \overline{q_k} \bar{z}^k \\ q_k z^k & 1 \end{pmatrix} \right)^{-1} \\ &= \prod_{k=0}^{n} \left(\frac{1}{\sqrt{1-|\beta_k|^2}} \begin{pmatrix} 1 & -\beta_k z^k \\ -\overline{\beta_k} \bar{z}^k & 1 \end{pmatrix} \right)^{-1} \\ &= \prod_{k=0}^{n} \frac{1}{\sqrt{1-|\beta_k|^2}} \cdot \prod_{k=0}^{n} \begin{pmatrix} 1 & \beta_k z^k \\ \overline{\beta_k} \bar{z}^k & 1 \end{pmatrix}. \end{aligned}$$

Note that for each $k \geq 0$ we have

$$j_0 \begin{pmatrix} 1 & \beta_k z^k \\ \overline{\beta_k} \bar{z}^k & 1 \end{pmatrix} j_0 = \begin{pmatrix} 1 & \overline{\beta_k} \bar{z}^k \\ \beta_k z^k & 1 \end{pmatrix}, \quad j_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $\{\beta_k\}_{k \geq 0}$ coincides with the sequence of recurrence coefficients of μ^-. So, we obtain

$$j_0 \left(\prod_{k=-n}^{-1} \frac{1}{\sqrt{1-|q_k|^2}} \cdot \prod_{k=-n}^{-1} \begin{pmatrix} 1 & \overline{q_k} \bar{z}^k \\ q_k z^k & 1 \end{pmatrix} \right)^{-1} j_0 \to \begin{pmatrix} \mathfrak{a}^- & \overline{\mathfrak{b}^-} \\ \mathfrak{b}^- & \overline{\mathfrak{a}^-} \end{pmatrix},$$

where the convergence is in Lebesgue measure on $\mathbb{T}$. Equating the determinats on both sides of the above formula, we get $|\mathfrak{a}^-|^2 - |\mathfrak{b}^-|^2 = 1$. Taking the inverses, we then obtain

$$\prod_{k=-\infty}^{-1} \frac{1}{\sqrt{1-|q_k|^2}} \cdot \prod_{k=-\infty}^{-1} \begin{pmatrix} 1 & \overline{q_k} \bar{z}^k \\ q_k z^k & 1 \end{pmatrix} = j_0 \begin{pmatrix} \overline{\mathfrak{a}^-} & -\overline{\mathfrak{b}^-} \\ -\mathfrak{b}^- & \mathfrak{a}^- \end{pmatrix} j_0 = \begin{pmatrix} \mathfrak{a}^- & -\mathfrak{b}^- \\ -\overline{\mathfrak{b}^-} & \overline{\mathfrak{a}^-} \end{pmatrix}.$$

Eventually, we get

$$\begin{aligned} &\prod_{k=-\infty}^{\infty} \frac{1}{\sqrt{1-|q_k|^2}} \cdot \prod_{k=-\infty}^{\infty} \begin{pmatrix} 1 & \overline{q_k} \bar{z}^k \\ q_k z^k & 1 \end{pmatrix} \\ &\quad = \begin{pmatrix} \mathfrak{a}^- & -\mathfrak{b}^- \\ -\overline{\mathfrak{b}^-} & \overline{\mathfrak{a}^-} \end{pmatrix} \begin{pmatrix} \mathfrak{a}^+ & \overline{\mathfrak{b}^+} \\ \mathfrak{b}^+ & \overline{\mathfrak{a}^+} \end{pmatrix} = \begin{pmatrix} a & b \\ acb & \bar{a} \end{pmatrix}, \end{aligned}$$

with $a = \mathfrak{a}^+\mathfrak{a}^- - \mathfrak{b}^+\mathfrak{b}^-$, $b = \mathfrak{a}^-\overline{\mathfrak{b}^+} - \mathfrak{b}^-\overline{\mathfrak{a}^+}$, as claimed. ■

We can now prove Lemma 3.4 from Section 3.

Proof of Lemma 3.4. Propositions 6.2 and 6.1 imply that the definitions of a, b in (1.5) and (6.14) are equivalent. Note that, for $q \in \ell^2(\mathbb{Z}, \mathbb{D})$ with $\operatorname{supp} q \subset \mathbb{Z}_+$, we have $\mathfrak{a}^- = 1, \mathfrak{b}^- = 0$; hence,

$$\mathbf{f}_q = \frac{\bar{b}}{a} = \frac{\mathfrak{b}_+}{\mathfrak{a}_+} = f^+.$$

In particular, the recurrence coefficients of $\mathbf{f}_q$ coincide with those of f^+, μ^+, i.e., with the sequence $\{q(k)\}_{k \in \mathbb{Z}_+}$. ■

Proposition 6.3. *We have*

$$\mathbf{r}_{q(\cdot - n)} = z^{-n} \mathbf{r}_q$$

for every compactly supported $q \in \ell^2(\mathbb{Z}, \mathbb{D})$ and $n \in \mathbb{Z}$.

Proof. The product

$$\prod_{k\in\mathbb{Z}}\begin{pmatrix}1 & \overline{q(k-n)}z^{-k}\\ q(k-n)z^{k} & 1\end{pmatrix}$$

can be written in the form

$$\begin{aligned}\cdots &= \prod_{k\in\mathbb{Z}}\begin{pmatrix}1 & 0\\ 0 & z^{n}\end{pmatrix}\begin{pmatrix}1 & \overline{q(k-n)}z^{-(k-n)}\\ q(k-n)z^{k-n} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & z^{-n}\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 0 & z^{n}\end{pmatrix}\left[\prod_{k\in\mathbb{Z}}\begin{pmatrix}1 & \overline{q(k-n)}z^{-(k-n)}\\ q(k-n)z^{k-n} & 1\end{pmatrix}\right]\begin{pmatrix}1 & 0\\ 0 & z^{-n}\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 0 & z^{n}\end{pmatrix}\begin{pmatrix}a & b\\ \bar b & \bar a\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & z^{-n}\end{pmatrix} = \begin{pmatrix}a & bz^{-n}\\ \bar b z^{n} & \bar a\end{pmatrix}.\end{aligned}$$

Hence, $\mathbf{r}_{q(\cdot-n)} = bz^{-n}/a = z^{-n}\mathbf{r}_q$ by (6.15) and Proposition 6.2. ∎

Proposition 6.4. *There are $q_1 \neq q_2$ in $\ell^2(\mathbb{Z},\mathbb{D})$ such that $\mathbf{r}_{q_1} = \mathbf{r}_{q_2}$.*

Proof. Following [23], let us consider an imaginary-valued function $\mathfrak{b}$ on $\mathbb{T}$ of Smirnov class in the unit disk. One can take, say, $\mathfrak{b} = (1+z)/(1-z)$. Let $\mathfrak{a}$ be the outer function in $\mathbb{D}$ such that $|\mathfrak{a}|^2 - |\mathfrak{b}|^2 = 1$ almost everywhere on $\mathbb{T}$. The function $f = \mathfrak{b}/\mathfrak{a}$ is a Schur function of Szegő class. Indeed, $\log(1-|f|^2) = \log|a|^{-2}$ belongs to $L^1(\mathbb{T})$. Therefore, we can define the sequences $q_1, \tilde q_2 \in \ell^2(\mathbb{Z},\mathbb{D})$ by

$$\begin{array}{rccccccccl} n: & \dots & -3 & -2 & -1 & 0 & 1 & 2 & \dots & \\ q_1 = (& \dots & 0, & 0, & 0, & f_0(0), & f_1(0), & f_2(0), & \dots &),\\ \tilde q_2 = (& \dots & -\overline{f_2(0)}, & -\overline{f_1(0)}, & -\overline{f_0(0)}, & 0, & 0, & 0, & \dots &).\end{array}$$

For these sequences, we have

$$\mathfrak{a}^+_{q_1} = \mathfrak{a},\quad \mathfrak{b}^+_{q_1} = \mathfrak{b},\quad \mathfrak{a}^-_{q_1} = 1,\quad \mathfrak{b}^-_{q_1} = 0,\quad \mathfrak{a}^+_{\tilde q_2} = 1,\quad \mathfrak{b}^+_{\tilde q_2} = 0.$$

Furthermore, from the proof of Proposition 6.3, we obtain $\mathfrak{b}^-_{\tilde q_2} = z\mathfrak{b}$ and $\mathfrak{a}^-_{\tilde q_2} = \mathfrak{a}$. Therefore,

$$\begin{aligned} a_{q_1} &= \mathfrak{a}\cdot 1 - \mathfrak{b}\cdot 0 = \mathfrak{a}, & b_{q_1} &= 1\cdot\bar{\mathfrak{b}} - 0\cdot\bar{\mathfrak{a}} = \bar{\mathfrak{b}},\\ a_{\tilde q_2} &= 1\cdot\mathfrak{a} - 0\cdot z\mathfrak{b} = \mathfrak{a}, & b_{\tilde q_2} &= \mathfrak{a}\cdot\bar 0 - z\mathfrak{b}\cdot\bar 1 = -z\mathfrak{b}.\end{aligned}$$

Then $\mathbf{r}_{q_1} = \bar{\mathfrak{b}}/\mathfrak{a}$, $\mathbf{r}_{\tilde q_2} = -z\mathfrak{b}/\mathfrak{a}$, and, since $\mathfrak{b} = -\bar{\mathfrak{b}}$, we have $\mathbf{r}_{q_1} = z\mathbf{r}_{\tilde q_2}$ almost everywhere on $\mathbb{T}$. Note that $z\mathbf{r}_{\tilde q_2} = \mathbf{r}_{\tilde q_2(\cdot+1)}$ by Proposition 6.3. Now, set $q_2 = \tilde q_2(\cdot+1)$ and observe that $\mathbf{r}_{q_1} = \mathbf{r}_{q_2}$, while q_1, q_2 are supported on disjoint subsets of $\mathbb{Z}$, so $q_1 \neq q_2$. ∎

Proposition 6.5. *For every $q \in \ell^2(\mathbb{Z}, \mathbb{D})$, we have*

$$\int_{\mathbb{T}} \log(1 - |\mathbf{r}_q|^2)\, dm = -\log|a(0)|^2 = \log \prod_{n\in\mathbb{Z}} (1 - |q(n)|^2). \tag{6.19}$$

Proof. Take a sequence q in $\ell^2(\mathbb{Z}, \mathbb{D})$ and define $\{\alpha_n\}$, $\{\beta_n\}$, $\mu^\pm$, $\mathfrak{a}^\pm$, $\mathfrak{b}^\pm$, $f^\pm$, a and b as in the beginning of Section 6.2. From (6.17) and the mean value theorem, we get

$$\int_{\mathbb{T}} \log(1 - |\mathbf{r}_q|^2)\, dm = -\int_{\mathbb{T}} \log|a|^2\, dm = -\log|a(0)|^2.$$

In the proof of Proposition 6.1 we established $a(0) = \mathfrak{a}^+(0)\mathfrak{a}^-(0) = D^{-1}_{\mu^+}(0) D^{-1}_{\mu^-}(0)$. Let $w^\pm$ be the densities of the a.c. parts $\mu^\pm$ with respect to the Lebesgue measure on $\mathbb{T}$; then, from formula (6.9) and the Szegő theorem (6.6), it follows that

$$\begin{aligned} -\log a(0)^2 &= \int_{\mathbb{T}} \log w^+(\xi)\, dm(\xi) + \int_{\mathbb{T}} \log w^-(\xi)\, dm(\xi) \\ &= \log \prod_{n\geq 0} (1 - |\alpha_n|^2) + \log \prod_{n\geq 0} (1 - |\beta_n|^2) = \log \prod_{n\in\mathbb{Z}} (1 - |q_n|^2), \end{aligned}$$

as claimed. ∎

Proposition 6.6. *For every $q \in \ell^2(\mathbb{Z}, \mathbb{D})$, the functions $\mathfrak{a}^\pm/a$, $\mathfrak{b}^\pm/a$ belong to the unit ball of the Hardy class $H^2(\mathbb{D})$.*

Proof. Since a, $\mathfrak{a}^\pm$ are outer in $\mathbb{D}$ and $\mathfrak{b}^\pm$ are in the Smirnov class (see (6.13)), we need to show only that $\mathfrak{a}^\pm/a$ belong to the unit ball of $L^2(\mathbb{T})$. Denote, as before, $f^\pm = \mathfrak{b}^\pm/\mathfrak{a}^\pm$, and recall that $f^\pm$ are Schur functions. The function

$$h = \frac{1 - |f^- f^+|^2}{|1 - f^- f^+|^2} = \operatorname{Re}\Big(\frac{1 + f^- f^+}{1 - f^- f^+}\Big)$$

is positive and harmonic in $\mathbb{D}$, therefore, it coincides with the Poisson integral of a finite positive Borel measure on $\mathbb{T}$. Moreover, h is equal to the density of the absolutely continuous part of that measure almost everywhere on $\mathbb{T}$. Hence, $h \in L^1(\mathbb{T})$ (we borrowed this trick from [23]) and

$$\|h\|_{L^1(\mathbb{T})} = \int_{\mathbb{T}} \frac{1 - |f^- f^+|^2}{|1 - f^- f^+|^2}\, dm = \operatorname{Re}\Big(\frac{1 + f^-(0) f^+(0)}{1 - f^-(0) f^+(0)}\Big) = 1,$$

because $f^-(0) = 0$. On the other hand, by (6.16) we have

$$\frac{1}{|\mathfrak{a}^\pm|^2} = 1 - \frac{|\mathfrak{b}^\pm|^2}{|\mathfrak{a}^\pm|^2} = 1 - |f^\pm|^2, \qquad \frac{1}{|a|^2} = \frac{(1 - |f^+|^2)(1 - |f^-|^2)}{|1 - f^+ f^-|^2}.$$

almost everywhere on $\mathbb{T}$. This gives us

$$\left|\frac{\mathfrak{a}^{\pm}}{a}\right|^2 = \frac{(1-|f^+|^2)(1-|f^-|^2)}{(1-|f^{\pm}|^2)|1-f^+f^-|^2} = \frac{1-|f^{\mp}|^2}{|1-f^+f^-|^2} \le \frac{1-|f^-f^+|^2}{|1-f^-f^+|^2} = h.$$

Therefore, $\|\mathfrak{a}^{\pm}/a\|^2_{L^2(\mathbb{T})} \le \|h\|_{L^1(\mathbb{T})} = 1$, as claimed. ∎

The authors are grateful to S. Denisov for the argument based on (6.21) in the proof of proposition below.

Proposition 6.7. *Suppose $q_1, q_2 \in \ell^2(\mathbb{Z}, \mathbb{D})$ are such that $\mathbf{r}_{q_1} = \mathbf{r}_{q_2}$. If we suppose $\|\mathbf{r}_{q_{1,2}}\|_{L^\infty(\mathbb{T})} < 1$, then $q_1 = q_2$.*

Proof. Let $a_{1,2}$, $b_{1,2}$ be the coefficients in (6.14) corresponding to $\mathbf{r}_{q_1}$, $\mathbf{r}_{q_2}$, respectively. By Proposition 6.1, we have $a_1 = a_2$, $b_1 = b_2$, so we denote $a = a_{1,2}$, $b = b_{1,2}$. Then (6.14) gives four identities

$$a = \mathfrak{a}_k^+\mathfrak{a}_k^- - \mathfrak{b}_k^+\mathfrak{b}_k^-, \quad b = \mathfrak{a}_k^-\overline{\mathfrak{b}_k^+} - \mathfrak{b}_k^-\overline{\mathfrak{a}_k^+}, \quad k = 1, 2,$$

for the functions $\mathfrak{a}^{\pm}_{1,2}$, $\mathfrak{b}^{\pm}_{1,2}$ corresponding to q_1, q_2. Some simple algebra yields

$$\begin{pmatrix} \mathfrak{a}_k^+ & \mathfrak{b}_k^+ \\ \mathfrak{b}_k^- & \mathfrak{a}_k^- \end{pmatrix} \overline{\begin{pmatrix} \mathfrak{a}_k^+ & -\mathfrak{b}_k^- \\ -\mathfrak{b}_k^+ & \mathfrak{a}_k^- \end{pmatrix}} = \begin{pmatrix} |\mathfrak{a}_k^+|^2 - |\mathfrak{b}_k^+|^2 & -\mathfrak{a}_k^+\overline{\mathfrak{b}_k^-} + \mathfrak{b}_k^+\overline{\mathfrak{a}_k^-} \\ \mathfrak{b}_k^-\overline{\mathfrak{a}_k^+} - \mathfrak{a}_k^-\overline{\mathfrak{b}_k^+} & |\mathfrak{a}_k^-|^2 - |\mathfrak{b}_k^-|^2 \end{pmatrix} = \begin{pmatrix} 1 & \bar{b} \\ -b & 1 \end{pmatrix}, \tag{6.20}$$

almost everywhere on $\mathbb{T}$ for $k = 1, 2$. In particular, we have

$$\begin{pmatrix} \mathfrak{a}_1^+ & \mathfrak{b}_1^+ \\ \mathfrak{b}_1^- & \mathfrak{a}_1^- \end{pmatrix} \overline{\begin{pmatrix} \mathfrak{a}_1^+ & -\mathfrak{b}_1^- \\ -\mathfrak{b}_1^+ & \mathfrak{a}_1^- \end{pmatrix}} = \begin{pmatrix} \mathfrak{a}_2^+ & \mathfrak{b}_2^+ \\ \mathfrak{b}_2^- & \mathfrak{a}_2^- \end{pmatrix} \overline{\begin{pmatrix} \mathfrak{a}_2^+ & -\mathfrak{b}_2^- \\ -\mathfrak{b}_2^+ & \mathfrak{a}_2^- \end{pmatrix}}.$$

Inverting the matrices in the last equation, we obtain

$$\begin{pmatrix} \mathfrak{a}_2^+ & \mathfrak{b}_2^+ \\ \mathfrak{b}_2^- & \mathfrak{a}_2^- \end{pmatrix}^{-1} \begin{pmatrix} \mathfrak{a}_1^+ & \mathfrak{b}_1^+ \\ \mathfrak{b}_1^- & \mathfrak{a}_1^- \end{pmatrix} = \overline{\begin{pmatrix} \mathfrak{a}_2^+ & -\mathfrak{b}_2^- \\ -\mathfrak{b}_2^+ & \mathfrak{a}_2^- \end{pmatrix} \begin{pmatrix} \mathfrak{a}_1^+ & -\mathfrak{b}_1^- \\ -\mathfrak{b}_1^+ & \mathfrak{a}_1^- \end{pmatrix}^{-1}},$$

$$I := \frac{1}{a}\begin{pmatrix} \mathfrak{a}_2^- & -\mathfrak{b}_2^+ \\ -\mathfrak{b}_2^- & \mathfrak{a}_2^+ \end{pmatrix} \begin{pmatrix} \mathfrak{a}_1^+ & \mathfrak{b}_1^+ \\ \mathfrak{b}_1^- & \mathfrak{a}_1^- \end{pmatrix} = \frac{1}{a}\overline{\begin{pmatrix} \mathfrak{a}_2^+ & -\mathfrak{b}_2^- \\ -\mathfrak{b}_2^+ & \mathfrak{a}_2^- \end{pmatrix}}\,\overline{\begin{pmatrix} \mathfrak{a}_1^- & \mathfrak{b}_1^- \\ \mathfrak{b}_1^+ & \mathfrak{a}_1^+ \end{pmatrix}}. \tag{6.21}$$

Equating the $(1, 1)$ matrix elements in this identity, we get

$$\frac{\mathfrak{a}_1^+\mathfrak{a}_2^- - \mathfrak{b}_1^-\mathfrak{b}_2^+}{a} = \overline{\left(\frac{\mathfrak{a}_2^+\mathfrak{a}_1^- - \mathfrak{b}_2^-\mathfrak{b}_1^+}{a}\right)}.$$

Formula (6.17) and our assumption $\|\mathbf{r}_{q_{1,2}}\|_{L^\infty(\mathbb{T})} < 1$ imply that $a \in H^\infty(\mathbb{D})$. We now see from Proposition 6.6 that the functions

$$F_1 = \frac{\mathfrak{a}_1^+\mathfrak{a}_2^- - \mathfrak{b}_1^-\mathfrak{b}_2^+}{a}, \quad F_2 = \frac{\mathfrak{a}_2^+\mathfrak{a}_1^- - \mathfrak{b}_2^-\mathfrak{b}_1^+}{a}$$

belong to the Hardy space $H^1(\mathbb{D})$. Therefore, F_1 and F_2 are constant functions and

$$\overline{F_2} = F_1 = F_1(0) = \frac{\mathfrak{a}_1^+(0)\mathfrak{a}_2^-(0) - \mathfrak{b}_1^-(0)\mathfrak{b}_2^+(0)}{a(0)} = \frac{a(0)}{a(0)} = 1.$$

In other words, the $(1,1)$ coefficient of the matrix I in (6.21) is 1. Note that it coincides with the $(2,2)$ coefficient of I. Similarly, we use $\mathfrak{b}_1^{\pm}(0) = \mathfrak{b}_2^{\pm}(0) = 0$ and prove that the $(1,2)$, $(2,1)$ coefficients of I are 0 thus getting

$$\frac{1}{a}\begin{pmatrix} \mathfrak{a}_2^- & -\mathfrak{b}_2^+ \\ -\mathfrak{b}_2^- & \mathfrak{a}_2^+ \end{pmatrix}\begin{pmatrix} \mathfrak{a}_1^+ & \mathfrak{b}_1^+ \\ \mathfrak{b}_1^- & \mathfrak{a}_1^- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix} \mathfrak{a}_1^+ & \mathfrak{b}_1^+ \\ \mathfrak{b}_1^- & \mathfrak{a}_1^- \end{pmatrix} = \begin{pmatrix} \mathfrak{a}_2^+ & \mathfrak{b}_2^+ \\ \mathfrak{b}_2^- & \mathfrak{a}_2^- \end{pmatrix}.$$

It follows that $f_1^{\pm} = f_2^{\pm}$, which, in turn, is equivalent to $q_1 = q_2$ on $\mathbb{Z}$, because the recurrence coefficients of $f_{1,2}^{\pm}$ determine completely $q_{1,2}$ on $\mathbb{Z}_{\pm}$, see the beginning of Section 6.2. ■

6.3. Convergence in the space X

We first prove a version of the Sylvester–Winebrenner theorem [21] for Schur functions. We stated it as Theorem 1.3 in the introduction. Let us repeat it here.

Theorem 6.8 (Sylvester–Winebrenner theorem). *The mapping $f \mapsto \{f_n(0)\}_{n\geq 0}$ that takes a Schur function into the sequence of its recurrence coefficients is a homeomorphism from the metric space $X_+ = \{f \in \mathcal{S}_*(\mathbb{D}) : \eta(F) > 0\}$ with the metric given by*

$$\rho_s(f,g)^2 = -\int_{\mathbb{T}} \log\Big(1 - \Big|\frac{f-g}{1-\bar{f}g}\Big|^2\Big)\, dm$$

onto the metric space $\ell^2(\mathbb{Z}_+, \mathbb{D})$ of the square summable sequences $q\colon \mathbb{Z}_+ \to \mathbb{D}$ with the metric $\|q - \tilde{q}\|_{\ell^2}^2 = \sum_{n\in\mathbb{Z}_+} |q(n) - \tilde{q}(n)|^2$.

Proof. Assume that $f_n, f \in X_+$ are such that $\rho_s(f_n, f) \to 0$. Let q_n, q be the sequences of recurrence coefficients of f_n, f, respectively. By the Szegő theorem, we have $q_n, q \in \ell^2(\mathbb{Z}_+, \mathbb{D})$, and, moreover,

$$-\log\prod_{k\geq 0}(1 - |q_n(k)|^2) = \rho_s(f_n, 0) \to \rho_s(f, 0) = -\log\prod_{k\geq 0}(1 - |q(k)|^2).$$

The convergence $q_n \to q$ in $\ell^2(\mathbb{Z}_+, \mathbb{D})$ will follow if we check that $q_n(k) \to q(k)$ for each $k \in \mathbb{Z}_+$ (indeed, we then have $\sum_{k\geq N} |q_n(k)|^2 \to 0$ as $N \to +\infty$ uniformly in $n \in \mathbb{Z}_+$). To this end, note that the assumption $\rho_s(f_n, f) \to 0$ implies that the sequence $\{f_n\}$ converges to f in the Lebesgue measure on $\mathbb{T}$, and, since $|f_n| \le 1$, $|f| \le 1$ on $\mathbb{T}$, the functions f_n converge to f uniformly on compacts in $\mathbb{D}$. Now, the fact that $q_n(k) = (f_n)_k(0)$ tends to $(f)_k(0) = q(k)$ as $n \to +\infty$ for every $k \in \mathbb{Z}_+$ follows from the Schur's algorithm (6.5). We see that the mapping $f \mapsto q$ is continuous from X_+ to $\ell^2(\mathbb{Z}_+, \mathbb{D})$.

Turning to the inverse mapping, we introduce the quantities (see [21])

$$E(f, g) = -\int_{\mathbb{T}} \log(1 - \bar{f} g)\, dm, \quad E(f) = E(f, f). \tag{6.22}$$

We have

$$1 - \left|\frac{f - g}{1 - \bar{f} g}\right|^2 = \frac{(1 - |f|^2) \cdot (1 - |g|^2)}{|1 - \bar{f} g|^2};$$

hence,

$$\rho_s(f, g)^2 = E(f) + E(g) - 2 \operatorname{Re} E(f, g). \tag{6.23}$$

Suppose that q_n, q are sequences in $\ell^2(\mathbb{Z}_+, \mathbb{D})$ such that $q_n \to q$ in $\ell^2(\mathbb{Z}_+, \mathbb{D})$. Denote by f_n, f the Schur functions corresponding to these sequences. We have $f_n, f \in X_+$ by the Szegő theorem, see (6.6). Let us prove that $\rho_s(f_n, f) \to 0$ as $n \to +\infty$. Since $E(f_n) \to E(f)$ by the Szegő theorem, relation (6.23) shows that we only need to check that $E(f_n, f) \to E(f, f)$. We have

$$E(f_n, f) = -\int_{\mathbb{T}} \sum_{k \ge 0} \frac{(\bar{f}_n f)^k}{k}\, dm = -\int_{\mathbb{T}} \sum_{k=1}^{N} \frac{(\bar{f}_n f)^k}{k}\, dm - \int_{\mathbb{T}} \sum_{k=N+1}^{\infty} \frac{(\bar{f}_n f)^k}{k}\, dm,$$

and

$$\left|\int_{\mathbb{T}} \sum_{k=N+1}^{\infty} \frac{(\bar{f}_n f)^k}{k}\, dm\right| \le \int_{\mathbb{T}} \sum_{k=N+1}^{\infty} \frac{|\bar{f}_n f|^k}{k}\, dm \le \int_{\mathbb{T}} \sum_{k=N+1}^{\infty} \frac{|f|^k}{k}\, dm,$$

which tends to zero as $N \to +\infty$ by the Lebesgue dominated convergence theorem (the majorant is $\log(1/(1 - |f|)) \in L^1(\mathbb{T})$). Next, let us show that, for each $k \in \mathbb{Z}_+$, we have

$$\int_{\mathbb{T}} (\bar{f}_n f)^k\, dm \to \int_{\mathbb{T}} |f|^{2k}\, dm, \quad n \to +\infty. \tag{6.24}$$

The first m Taylor coefficients of f are polynomials in $q(0), \overline{q(0)}, \dots, q(m-1), \overline{q(m-1)}$ and similarly for f_n, see [18, (1.13.48) in Section 1.3]. Hence, the Taylor coefficients of f_n^k tend to those of f^k as $n \to \infty$. Rewrite the quantity in (6.24) as

$$\int_{\mathbb{T}} (\bar{f}_n f)^k \, dm = \sum_{m=0}^{\infty} \overline{c_m(f_n^k)} c_m(f^k)$$
$$= \sum_{m=0}^{M} \overline{c_m(f_n^k)} c_m(f^k) + \sum_{m=M+1}^{\infty} \overline{c_m(f_n^k)} c_m(f^k).$$

The second sum can be estimated using the Cauchy inequality by

$$\|f_n^k\|^2_{H^2(\mathbb{D})} \cdot \Big(\sum_{m=M+1}^{\infty} |c_m(f^k)|^2 \Big) \le \sum_{m=M+1}^{\infty} |c_m(f^k)|^2,$$

because $f_n^k \in \mathcal{S}(\mathbb{D})$ and consequently $\|f_n^k\|^2_{H^2(\mathbb{D})} \le 1$. Hence, it tends to 0 as $M \to \infty$. The first sum tends to $\sum_{m=0}^{M} |c_m(f^k)|^2$ as $n \to \infty$ and (6.24) follows. Relation (6.24) shows that $E(f_n, f) \to 0$, $\rho_s(f_n, f) \to 0$, and thus the mapping $q \mapsto f$ is continuous from $\ell^2(\mathbb{Z}_+, \mathbb{D})$ to the metric space X_+. ■

The following lemma is elementary. It is known as *Scheffé's lemma,* see [26, Section 5.10].

Lemma 6.9. *Let the measurable functions g, g_n on $\mathbb{T}$ be such that $g_j \to g$ in Lebesgue measure on $\mathbb{T}$ and $\|g_j\|_{L^1(\mathbb{T})} \to \|g\|_{L^1(\mathbb{T})}$ as $j \to \infty$. Then we have that $\|g - g_j\|_{L^1(\mathbb{T})} \to 0$.*

Proof. If $\|g\|_{L^1(\mathbb{T})} = 0$, then the lemma is trivial, otherwise we can reduce the statement of the lemma to the case $\|g_j\|_{L^1(\mathbb{T})} = \|g\|_{L^1(\mathbb{T})} = 1$ by changing g and g_n to $g/\|g\|_{L^1(\mathbb{T})}$ and $g_n/\|g_n\|_{L^1(\mathbb{T})}$ respectively. Consider any subsequence g_{n_k} of the sequence g_n. Let $g_{n_{k_j}}$ be its subsequence converging Lebesgue almost everywhere on $\mathbb{T}$. The limit of $g_{n_{k_j}}$ coincides with g Lebesgue almost everywhere on $\mathbb{T}$. To simplify the notation, we denote the new sequence $g_{n_{k_j}}$ by $\tilde{g}_j$. Let $\varepsilon > 0$. By Egorov's theorem and integrability of g, there is $K_\varepsilon \subset \mathbb{T}$ such that $m(K_\varepsilon) < \varepsilon$, $\|g\|_{L^1(K_\varepsilon)} < \varepsilon$ and $\tilde{g}_j \to g$ uniformly on $\mathbb{T} \setminus K_\varepsilon$. In particular, we have

$$\int_{\mathbb{T}\setminus K_\varepsilon} |\tilde{g}_j| \, dm \to \int_{\mathbb{T}\setminus K_\varepsilon} |g| \, dm \ge 1 - 2\varepsilon, \quad \limsup_{j\to\infty} \int_{K_\varepsilon} |\tilde{g}_j| \, dm \le 2\varepsilon.$$

Now, we only need to estimate $\limsup_{j\to\infty} \|g - \tilde{g}_j\|_{L^1(\mathbb{T})}$ from above by

$$\limsup_{j\to\infty} \|g - \tilde{g}_j\|_{L^1(\mathbb{T}\setminus K_\varepsilon)} + \limsup_{j\to\infty} \|\tilde{g}_j\|_{L^1(K_\varepsilon)} + \|g\|_{L^1(K_\varepsilon)} \le 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we see that $\tilde{g}_j \to g$ in $L^1(\mathbb{T})$. In other words, we have shown that any subsequence of g_n contains a subsequence converging to g in $L^1(\mathbb{T})$. Then $g_n \to g$ in $L^1(\mathbb{T})$ and the lemma follows. ■

Recall that the space X and the metric ρ_s on X are defined in (1.10) and (1.11). For $r \in X$, define the function $E(r)$ by (6.22).

Proposition 6.10. *Let $r_n, r \in X$. The following assertions are equivalent:*

(a) *r_n converges to r in X;*

(b) *r_n converges to r in Lebesgue measure on $\mathbb{T}$ and $\lim_{n\to+\infty} E(r_n) = E(r)$.*

Proof. Assume that $r_n \to r$ in X as $n \to +\infty$. The convergence in measure follows immediately. For all $n \geq 0$, we have $|1 - \overline{r_n} r| \geq 1 - |r|$ and $\log(1/(1 - |r|)) \in L^1(\mathbb{T})$. Hence, by the dominated convergence theorem, we have

$$E(r_n, r) = -\int_{\mathbb{T}} \log(1 - \overline{r_n} r)\, dm \to -\int_{\mathbb{T}} \log(1 - |r|^2)\, dm = E(r). \tag{6.25}$$

Thus, from (6.23) we see that

$$\begin{aligned} 0 &= \lim_{n\to+\infty} \rho_s(r_n, r)^2 = \lim_{n\to+\infty} (E(r) + E(r_n) - 2\operatorname{Re} E(r, r_n)) \\ &= \lim_{n\to+\infty} (E(r_n) - E(r)), \end{aligned}$$

which gives us the required assertion. On the other hand, if we assume (b), then (6.25) will follow by the same argument and similarly by (6.23) we will get

$$\lim_{n\to+\infty} \rho_s(r_n, r)^2 = \lim_{n\to+\infty} (E(r) + E(r_n) - 2\operatorname{Re} E(r, r_n)) = 0,$$

which is the convergence in X. ■

Proposition 6.11. *If $q_n \to q$ in $\ell^2(\mathbb{Z}, \mathbb{D})$, then $\mathbf{r}_{q_n} \to \mathbf{r}_q$ in X.*

Proof. We want to apply the criteria from Proposition 6.10. Convergence in $\ell^2(\mathbb{Z}, \mathbb{D})$ implies the convergence

$$\prod_{k\in\mathbb{Z}} (1 - |q_n(k)|^2) \to \prod_{k\in\mathbb{Z}} (1 - |q(k)|^2), \quad n \to \infty,$$

which yields $E(\mathbf{r}_{q_n}) \to E(\mathbf{r}_q)$ by Proposition 6.5. Thus, it is suffices to show only that $\mathbf{r}_{q_n} \to \mathbf{r}_q$ in Lebesgue measure on $\mathbb{T}$. Recall that, for every $q \in \ell^2(\mathbb{Z}, \mathbb{D})$, we have $f^{\pm} = \mathfrak{a}^{\pm}/\mathfrak{b}^{\pm}$, where $f^{\pm}$ are the Schur functions generated by q; hence,

$$\mathbf{r}_q = \frac{\mathfrak{a}^- \overline{\mathfrak{b}^+} - \mathfrak{b}^- \overline{\mathfrak{a}^+}}{\mathfrak{a}^+ \mathfrak{a}^- - \mathfrak{b}^+ \mathfrak{b}^-} = \frac{\overline{\mathfrak{a}^+}}{\mathfrak{a}^+} \frac{\overline{f^+} - f^-}{1 - f^+ f^-} = \exp(-2i\mathcal{H}(\log|\mathfrak{a}^+|)) \frac{\overline{f^+} - f^-}{1 - f^+ f^-}.$$

Here $\mathcal{H}$ denotes the Hilbert transform and we used the fact that $\mathfrak{a}^+$ is an outer function. Furthermore, we have $1/|\mathfrak{a}^+|^2 = 1 - |\mathfrak{b}^+|^2/|\mathfrak{a}^+|^2 = 1 - |f^+|^2$; hence,

$$\mathbf{r}_q = \exp(i\mathcal{H}(\log(1 - |f^+|^2)))\frac{\overline{f^+} - f^-}{1 - f^+ f^-}. \tag{6.26}$$

Similar formulae with Schur functions $f_n^\pm$ in place of $f^\pm$ hold for q_n. Theorem 6.8 implies the convergence $f_n^\pm \to f^\pm$ in Lebesgue measure on $\mathbb{T}$. Moreover, by the Szegő theorem, $\|1 - |f_n^+|^2\|_{L^1(\mathbb{T})} \to \|1 - |f^+|^2\|_{L^1(\mathbb{T})}$; hence, Lemma 6.9 can be applied to the functions

$$g_n = \log(1 - |f_n^+|^2), \quad g = \log(1 - |f^+|^2).$$

This gives the convergence of $\log(1 - |f_n^+|^2)$ to $\log(1 - |f^+|^2)$ in $L^1(\mathbb{T})$. Weak continuity of the Hilbert transform $\mathcal{H}$ (see [8, Section III.2]) then implies that the function $\exp(i\mathcal{H}(\log(1 - |f_n^+|^2)))$ converges in Lebesgue measure to the function $\exp(i\mathcal{H}(\log(1 - |f^+|^2)))$. From here and (6.26) we see that the functions $\mathbf{r}_{q_n}$ converge to $\mathbf{r}_q$ in Lebesgue measure on $\mathbb{T}$. ∎

The following proposition is not used in the proof of Theorem 1.7, but it explains how instabilities may arise in Schur's algorithm.

Proposition 6.12. *There is $\eta > 0$ such that the mapping $f \mapsto \{f_n(0)\}_{n\ge 0}$ taking a Schur function f into the sequence of its recurrence coefficients is not uniformly continuous with respect to the metrics in X_+, $\ell^2(\mathbb{Z}, \mathbb{D})$ on the subset of functions $f \in X_+$ satisfying $\eta(f) > \eta$.*

Proof. Take any $q \ne \tilde{q}$ in $\ell^2(\mathbb{Z}, \mathbb{D})$ such that $\mathbf{r}_q = \mathbf{r}_{\tilde{q}}$, see Proposition 6.4. Fix $\varepsilon > 0$ and use Proposition 6.11 to find a number $N(\varepsilon)$ such that we have $\rho_s(\mathbf{r}_{q_N}, \mathbf{r}_q) \le \varepsilon$, $\rho_s(\mathbf{r}_{\tilde{q}_N}, \mathbf{r}_{\tilde{q}}) \le \varepsilon$ for every $N \ge N(\varepsilon)$, where $q_N(k) = q(k)$, $\tilde{q}_N(k) = \tilde{q}(k)$ for $k \le N - 1$, and $q_N(k) = \tilde{q}_N(k) = 0$ for $k \ge N$. Next, shift these sequences to make them supported on $(-\infty, -1]$: define $q_{N,s}(k) = q_N(k + N)$, $\tilde{q}_{N,s}(k) = \tilde{q}_N(k + N)$ for $k \in \mathbb{Z}$. Let also $q_s = q(\cdot + N)$, $\tilde{q}_s = \tilde{q}(\cdot + N)$. We have

$$\rho_s(\mathbf{r}_{q_{N,s}}, \mathbf{r}_{\tilde{q}_{N,s}}) \le \rho_s(\mathbf{r}_{q_{N,s}}, \mathbf{r}_{q_s}) + \rho_s(\mathbf{r}_{q_s}, \mathbf{r}_{\tilde{q}_s}) + \rho_s(\mathbf{r}_{\tilde{q}_s}, \mathbf{r}_{\tilde{q}_{N,s}}) \le 2\varepsilon,$$

because

$$\rho_s(\mathbf{r}_{q_{N,s}}, \mathbf{r}_{q_s}) = \rho_s(\mathbf{r}_{q_N}, \mathbf{r}_q) \le \varepsilon,$$
$$\rho_s(\mathbf{r}_{q_s}, \mathbf{r}_{\tilde{q}_s}) = \rho_s(\mathbf{r}_q, \mathbf{r}_{\tilde{q}}) = 0,$$
$$\rho_s(\mathbf{r}_{\tilde{q}_{N,s}}, \mathbf{r}_{\tilde{q}_s}) = \rho_s(\mathbf{r}_{\tilde{q}_N}, \mathbf{r}_{\tilde{q}}) \le \varepsilon,$$

by Proposition 6.3 (it was proved for compactly supported q, but continuity in Proposition 6.4 extends it to whole space $\ell^2(\mathbb{Z}, \mathbb{D})$). On the other hand, $-\mathbf{r}_{q_{N,s}}$ and $-\mathbf{r}_{\tilde{q}_{N,s}}$

coincide on $\mathbb{T}$ with the Schur functions with the recurrence coefficients $\beta_N(n) = -\overline{q_{N,s}(-n)}$ and $\tilde{\beta}_N(n) = -\overline{\tilde{q}_{N,s}(-n)}$, $n \geq 0$, respectively; see (6.12), (6.14), and (6.15). Since the sequences $\{\beta_N(n)\}_{n\geq 0}$, $\{\tilde{\beta}_N(n)\}_{n\geq 0}$ are uniformly separated in $\ell^2(\mathbb{Z}_+, \mathbb{D})$ for large N, and we have $\rho_s(\mathbf{r}_{q_{N,s}}, \mathbf{r}_{\tilde{q}_{N,s}}) \leq 2\varepsilon$ for all $N \geq N(\varepsilon)$, the mapping in the statement of the proposition cannot be uniformly continuous. ∎

Proposition 6.13. *Let $q_n \in \ell^2(\mathbb{Z}, \mathbb{D})$ be such that $\mathbf{r}_{q_n} \to \mathbf{r}$ in X for some $\mathbf{r} \in X$. Then there is a subsequence q_{n_j} such that $q_{n_j} \to q$ in $\ell^2(\mathbb{Z}, \mathbb{D})$ and $\mathbf{r} = \mathbf{r}_q$.*

Proof. Since $\mathbf{r}_{q_n} \to \mathbf{r}$ in X, we know that $\mathbf{r}_{q_n} \to \mathbf{r}$ in Lebesgue measure on $\mathbb{T}$. Moreover, $E(\mathbf{r}_{q_n}) \to E(\mathbf{r})$ as $n \to +\infty$ by Proposition 6.10. Hence, Lemma 6.9 is applicable and we see that $\log(1 - |\mathbf{r}_{q_n}|^2)$ tends to $\log(1 - |\mathbf{r}|^2)$ in $L^1(\mathbb{T})$.

Consider the sequences $\mathfrak{a}_n^\pm$, $\mathfrak{b}_n^\pm$, $f_n^\pm = \mathfrak{a}_n^\pm/\mathfrak{b}_n^\pm$, a_n and b_n corresponding to q_n in the sense described at the beginning of Section 6.2. Furthermore, let A be an outer function in $\mathbb{D}$ with $A(0) > 0$ such that $|A|^{-2} = 1 - |\mathbf{r}|^2$ and $B = \mathbf{r}A$. From equation (6.17) and the definitions of a_n, b_n, A, B, we see that $a_n \to A$, $b_n \to B$ in Lebesgue measure on $\mathbb{T}$. Also we have $a_n \to A$ locally uniformly in $\mathbb{D}$. The functions $\mathfrak{a}_n^\pm/a_n$ are in the unit ball of $H^2(\mathbb{D})$ by Proposition 6.6; hence, one can choose a subsequence n_j, some functions $\tilde{\mathfrak{a}}^\pm$ and Schur functions $\tilde{f}^\pm$ such that

- $\mathfrak{a}_{n_j}^\pm \to \tilde{\mathfrak{a}}^\pm$ locally uniformly in $\mathbb{D}$;
- $\mathfrak{a}_{n_j}^\pm/a_{n_j} \to \tilde{\mathfrak{a}}^\pm/A$ locally uniformly in $\mathbb{D}$ and weakly in $H^2(\mathbb{D})$;
- $1/\mathfrak{a}_{n_j}^\pm \to 1/\tilde{\mathfrak{a}}^\pm$ locally uniformly in $\mathbb{D}$ and weakly in $H^2(\mathbb{D})$;
- $f_{n_j}^\pm \to \tilde{f}^\pm$ locally uniformly in $\mathbb{D}$.

With this choice of $\tilde{\mathfrak{a}}^\pm$, both functions $\tilde{\mathfrak{a}}^\pm/A$ and $A/\tilde{\mathfrak{a}}^\pm$ belong to the Smirnov class in $\mathbb{D}$; hence, the $\tilde{\mathfrak{a}}^\pm$ are outer functions. Put $\tilde{\mathfrak{b}}^\pm = \tilde{f}^\pm\tilde{\mathfrak{a}}^\pm$. Let q be defined in terms of recurrence coefficients of $\tilde{f}^\pm$ by

$$q(k) = \begin{cases} (\tilde{f}^+)_k(0), & k \geq 0, \\ -\overline{(\tilde{f}^-)_{-k}(0)}, & k < 0. \end{cases}$$

Note that $(\tilde{f}^-)_0(0) = (\tilde{f}^-)(0) = 0$ because $(f_n^-)(0) = 0$ for every n. We claim that $q_{n_j} \to q$ in $\ell^2(\mathbb{Z}, \mathbb{D})$. To prove this, introduce $\mathfrak{a}^\pm$, $\mathfrak{b}^\pm$, $f^\pm = \mathfrak{a}^\pm/\mathfrak{b}^\pm$, a, b as the objects from the beginning of Section 6.2 corresponding to q. It is clear that $f^\pm = \tilde{f}^\pm$. Let us show that

$$a = A, \quad b = B, \quad \mathfrak{a}^\pm = \tilde{\mathfrak{a}}^\pm, \quad \mathfrak{b}^\pm = \tilde{\mathfrak{b}}^\pm.$$

We have $f^\pm = \mathfrak{b}^\pm/\mathfrak{a}^\pm = \tilde{\mathfrak{b}}^\pm/\tilde{\mathfrak{a}}^\pm$ by construction, and the functions $\mathfrak{a}^\pm$, $\tilde{\mathfrak{a}}^\pm$ are outer (we do not know, however, if $1 - |f^\pm|^2 = |\tilde{\mathfrak{a}}^\pm|^{-2}$). Therefore, there are outer functions $s^{\pm}$ such that $\tilde{\mathfrak{a}}^{\pm} = s^{\pm}\mathfrak{a}^{\pm}$ and $\tilde{\mathfrak{b}}^{\pm} = s^{\pm}\mathfrak{b}^{\pm}$. It follows that

$$A = \tilde{\mathfrak{a}}^{+}\tilde{\mathfrak{a}}^{-} - \tilde{\mathfrak{b}}^{+}\tilde{\mathfrak{b}}^{-} = s^{+}s^{-}(\mathfrak{a}^{+}\mathfrak{a}^{-} - \mathfrak{b}^{+}\mathfrak{b}^{-}) = s^{+}s^{-}a.$$

almost everywhere on $\mathbb{T}$ because this relation holds in $\mathbb{D}$. Now, write formula (6.20) for q_{n_j} in the form

$$\begin{pmatrix} \mathfrak{a}_{n_j}^{+} & \mathfrak{b}_{n_j}^{+} \\ \mathfrak{b}_{n_j}^{-} & \mathfrak{a}_{n_j}^{-} \end{pmatrix} = \begin{pmatrix} 1 & \overline{b_{n_j}} \\ -b_{n_j} & 1 \end{pmatrix} \overline{\begin{pmatrix} \mathfrak{a}_{n_j}^{+} & -\mathfrak{b}_{n_j}^{-} \\ -\mathfrak{b}_{n_j}^{+} & \mathfrak{a}_{n_j}^{-} \end{pmatrix}^{-1}}$$
$$= \begin{pmatrix} 1 & \overline{b_{n_j}} \\ -b_{n_j} & 1 \end{pmatrix} \overline{\frac{1}{a_{n_j}} \begin{pmatrix} \mathfrak{a}_{n_j}^{-} & \mathfrak{b}_{n_j}^{-} \\ \mathfrak{b}_{n_j}^{+} & \mathfrak{a}_{n_j}^{+} \end{pmatrix}}.$$

Multiplying both sides by $1/a_{n_j}$, we get

$$\frac{1}{a_{n_j}} \begin{pmatrix} \mathfrak{a}_{n_j}^{+} & \mathfrak{b}_{n_j}^{+} \\ \mathfrak{b}_{n_j}^{-} & \mathfrak{a}_{n_j}^{-} \end{pmatrix} = \begin{pmatrix} 1/a_{n_j} & \overline{b_{n_j}}/a_{n_j} \\ -b_{n_j}/a_{n_j} & 1/a_{n_j} \end{pmatrix} \overline{\frac{1}{a_{n_j}} \begin{pmatrix} \mathfrak{a}_{n_j}^{-} & \mathfrak{b}_{n_j}^{-} \\ \mathfrak{b}_{n_j}^{+} & \mathfrak{a}_{n_j}^{+} \end{pmatrix}}. \tag{6.27}$$

By construction, we have $\mathfrak{a}_{n_j}^{\pm}/a_{n_j} \to \tilde{\mathfrak{a}}^{\pm}/A$, $\mathfrak{b}_{n_j}^{\pm}/a_{n_j} \to \tilde{\mathfrak{b}}^{\pm}/A$ weakly in H^2. We also have $\overline{b_{n_j}}/a_{n_j} \to \overline{B}/A$, $b_{n_j}/a_{n_j} \to B/A$, $1/a_{n_j} \to 1/A$ strongly in $L^2(\mathbb{T})$ by the dominated convergence theorem, because b_{n_j}/a_{n_j}, $\overline{b_{n_j}}/a_{n_j}$, $1/a_{n_j}$ are uniformly bounded and converge in Lebesgue measure on $\mathbb{T}$ to B/A. It follows that both sides of (6.27) converge weakly in $L^2(\mathbb{T})$. Taking the limit in (6.27), we obtain

$$\frac{1}{A} \begin{pmatrix} \tilde{\mathfrak{a}}^{+} & \tilde{\mathfrak{b}}^{+} \\ \tilde{\mathfrak{b}}^{-} & \tilde{\mathfrak{a}}^{-} \end{pmatrix} = \begin{pmatrix} 1/A & \overline{B}/A \\ -B/A & 1/A \end{pmatrix} \overline{\frac{1}{A} \begin{pmatrix} \tilde{\mathfrak{a}}^{-} & \tilde{\mathfrak{b}}^{-} \\ \tilde{\mathfrak{b}}^{+} & \tilde{\mathfrak{a}}^{+} \end{pmatrix}},$$

or, in equivalent form,

$$\begin{pmatrix} s^{+}\mathfrak{a}^{+} & s^{+}\mathfrak{b}^{+} \\ s^{-}\mathfrak{b}^{-} & s^{-}\mathfrak{a}^{-} \end{pmatrix} = \begin{pmatrix} 1 & \overline{B} \\ -B & 1 \end{pmatrix} \overline{\frac{1}{s^{+}s^{-}a} \begin{pmatrix} s^{-}\mathfrak{a}^{-} & s^{-}\mathfrak{b}^{-} \\ s^{+}\mathfrak{b}^{+} & s^{+}\mathfrak{a}^{+} \end{pmatrix}}.$$

Equation (6.20) written for q, a, b, $\mathfrak{a}^{\pm}$, $\mathfrak{b}^{\pm}$ says

$$\begin{pmatrix} \mathfrak{a}^{+} & \mathfrak{b}^{+} \\ \mathfrak{b}^{-} & \mathfrak{a}^{-} \end{pmatrix} = \begin{pmatrix} 1 & \bar{b} \\ -b & 1 \end{pmatrix} \overline{\frac{1}{a} \begin{pmatrix} \mathfrak{a}^{-} & \mathfrak{b}^{-} \\ \mathfrak{b}^{+} & \mathfrak{a}^{+} \end{pmatrix}}.$$

It follows that

$$\begin{pmatrix} s^{+} & 0 \\ 0 & s^{-} \end{pmatrix} \begin{pmatrix} 1 & \bar{b} \\ -b & 1 \end{pmatrix} \overline{\frac{1}{a} \begin{pmatrix} \mathfrak{a}^{-} & \mathfrak{b}^{-} \\ \mathfrak{b}^{+} & \mathfrak{a}^{+} \end{pmatrix}} = \begin{pmatrix} s^{+}\mathfrak{a}^{+} & s^{+}\mathfrak{b}^{+} \\ s^{-}\mathfrak{b}^{-} & s^{-}\mathfrak{a}^{-} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \overline{B} \\ -B & 1 \end{pmatrix} \overline{\frac{1}{s^{+}s^{-}a} \begin{pmatrix} s^{-}\mathfrak{a}^{-} & s^{-}\mathfrak{b}^{-} \\ s^{+}\mathfrak{b}^{+} & s^{+}\mathfrak{a}^{+} \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & \overline{B} \\ -B & 1 \end{pmatrix} \overline{\frac{1}{s^+s^-}\begin{pmatrix} s^- & 0 \\ 0 & s^+ \end{pmatrix}} \, \overline{\frac{1}{a}\begin{pmatrix} \mathfrak{a}^- & \mathfrak{b}^- \\ \mathfrak{b}^+ & \mathfrak{a}^+ \end{pmatrix}}.$$

From here, we get

$$\begin{aligned}
\begin{pmatrix} 1 & \bar{b} \\ -b & 1 \end{pmatrix} &= \begin{pmatrix} 1/s^+ & 0 \\ 0 & 1/s^- \end{pmatrix} \begin{pmatrix} 1 & \overline{B} \\ -B & 1 \end{pmatrix} \overline{\frac{1}{s^+s^-}\begin{pmatrix} s^- & 0 \\ 0 & s^+ \end{pmatrix}} \\
&= \begin{pmatrix} 1/s^+ & 0 \\ 0 & 1/s^- \end{pmatrix} \begin{pmatrix} 1 & \overline{B} \\ -B & 1 \end{pmatrix} \overline{\begin{pmatrix} 1/s^+ & 0 \\ 0 & 1/s^- \end{pmatrix}} \\
&= \begin{pmatrix} 1/|s^+|^2 & \overline{B}/s^+\overline{s^-} \\ -B/\overline{s^+}s^- & 1/|s^-|^2 \end{pmatrix}.
\end{aligned}$$

It follows that $|s^\pm|^2 = 1$. Recall that the $s^\pm$ are outer and $s^\pm(0) > 0$, therefore $s^\pm = 1$, $\tilde{\mathfrak{a}}^\pm = \mathfrak{a}^\pm$, $\tilde{\mathfrak{b}}^\pm = \mathfrak{b}^\pm$, $a = A$, $b = B$, and $\mathbf{r}_q = a/b = A/B = \mathbf{r}$. It remains to show that $q_{n_j} \to q$ in $L^2(\mathbb{T})$. Since the $f^\pm$ are locally uniform limits of the $f^\pm_{n_j}$ in $\mathbb{D}$, we have $\lim_{j\to+\infty} q_{n_j}(k) = q(k)$ for each $k \in \mathbb{Z}$ from Schur's algorithm (6.5) for $f^\pm$. Moreover, (6.19) and $a = A$ imply

$$\begin{aligned}
\log \prod_{k\in\mathbb{Z}} (1 - |q(k)|^2) = \log |a(0)|^{-2} = \log |A(0)|^{-2} &= \lim_{j\to+\infty} \log |a_{n_j}(0)|^{-2} \\
&= \lim_{j\to\infty} \log \prod_{k\in\mathbb{Z}} (1 - |q_{n_j}(k)|^2).
\end{aligned}$$

The last relation together with the elementwise convergence $\lim_{j\to+\infty} q_{n_j}(k) = q(k)$ gives $q_{n_j} \to q$ in the norm of $\ell^2(\mathbb{Z}, \mathbb{D})$. ■

Proposition 6.14. *The set $\mathcal{G} = \bigcup_{\delta\in[0,1)} \mathcal{G}[\delta]$ is dense in $\ell^2(\mathbb{Z}, \mathbb{D})$. In fact, we have $\ell^1(\mathbb{Z}, \mathbb{D}) \subset \mathcal{G}$. If $q \in \mathcal{G}$ and $\operatorname{supp} q \subset \mathbb{Z}_+$, then $\|\mathbf{f}_q\|_{L^\infty(\mathbb{T})} < 1$ for $\mathbf{f}_q = f^+$ (see Lemma 3.4).*

Proof. By Baxter's theorem (see [18, Chapter 5]), every measure μ with recurrence coefficients in $\ell^1(\mathbb{Z}_+, \mathbb{D})$ has its Szegő function, D_μ, in the Wiener algebra $W(\mathbb{T})$. It follows that $\mathfrak{a}^\pm$, $\mathfrak{b}^\pm$ are continuous and uniformly bounded on $\mathbb{T}$ if $q = \{q(n)\}_{n\in\mathbb{Z}}$ is in $\ell^1(\mathbb{Z}, \mathbb{D})$; hence, the function $a = \mathfrak{a}^+\mathfrak{a}^- - \mathfrak{b}^+\mathfrak{b}^-$ is uniformly bounded on $\mathbb{T}$ as well. Formula (6.17) then implies that $\mathbf{r}_q \in B[\delta]$, $q \in \mathcal{G}[\delta]$, for some $\delta \in [0, 1)$. The rest of the proposition is straightforward. ■

6.4. Proof of Theorem 1.7

Recall that the scattering map (or the nonlinear Fourier transform) is defined by

$$\mathcal{F}_{\mathrm{sc}}\colon q \mapsto \mathbf{r}_q,$$

on the set of sequences $\ell^2(\mathbb{Z}, \mathbb{D})$, see Proposition 6.2. Assertions (1) and (2) of the theorem are Propositions 6.11 and 6.13, respectively. Assertion (3) for compactly supported $q: \mathbb{Z} \to \mathbb{D}$ is Proposition 6.3. Since $\mathcal{F}_{\text{sc}}$ is continuous, assertion (3) then holds for all $q \in \ell^2(\mathbb{Z}, \mathbb{D})$. To prove assertion (4), consider potentials $q \in \ell^2(\mathbb{Z}, \mathbb{D})$ supported on $\mathbb{Z} \cap (-\infty, 0)$ and observe that Theorem 6.8 implies $X_+ \subset \mathcal{F}_{\text{sc}}(\ell^2(\mathbb{Z}, \mathbb{D}))$. Then, since the set $\mathcal{F}_{\text{sc}}(\ell^2(\mathbb{Z}, \mathbb{D}))$ is invariant under multiplication by z^n, $n \in \mathbb{Z}$, by assertion (3), the set $\mathcal{F}_{\text{sc}}(\ell^2(\mathbb{Z}, \mathbb{D}))$ contains trigonometric polynomials p such that $\|p\|_{L^\infty(\mathbb{T})} < 1$ of arbitrary degree. We claim that the set of such polynomials is dense in X. Indeed, one can approximate an arbitrary element of X by a sequence of continuous functions in the open unit ball of $L^\infty(\mathbb{T})$ using Lusin's theorem, and then uniformly approximate these continuous functions by Fejer means of their Fourier series. Since $\mathcal{F}_{\text{sc}}$ is a closed map, the fact that $\mathcal{F}_{\text{sc}}(\ell^2(\mathbb{Z}, \mathbb{D}))$ contains a dense subset of X implies $\mathcal{F}_{\text{sc}}(\ell^2(\mathbb{Z}, \mathbb{D})) = X$, and (4) follows. Assertion (5) is Proposition 6.4. To prove (6), note that $\mathcal{F}_{\text{sc}}(\mathcal{G}[\delta]) \subset B[\delta]$ by definition and $\mathcal{F}_{\text{sc}}(\mathcal{G}[\delta]) \supset B[\delta]$ because $\mathcal{F}_{\text{sc}}: \ell^2(\mathbb{Z}, \mathbb{D}) \to X$ is surjective. Thus, $\mathcal{F}_{\text{sc}}: \mathcal{G}[\delta] \to B[\delta]$ is a continuous surjection. By Proposition 6.7, this map is injective. Then $\mathcal{F}_{\text{sc}}: \mathcal{G}[\delta] \to B[\delta]$ is a closed continuous bijection between two topological spaces; hence, it is a homeomorphism, which is (6). Assertion (7) is not proved in our paper, the reader can find its proof at the end of [24, Chapter 2].

A. Appendix

Denote by $\ell^0(\mathbb{Z}, \mathbb{D})$ the set of all sequences $q = \{q_n\}_{n \in \mathbb{Z}}$ such that $|q_n| < 1$ for all $n \in \mathbb{Z}$. In this section we show that for every $q_0 \in \ell^0(\mathbb{Z}, \mathbb{D})$, the Ablowitz–Ladik equation (1.4) has a global unique solution.

Lemma A.1 (Boundedness [12, p. 4]). *If q solves (1.4) on $[0, t_0]$ for the initial data $q_0 \in \ell^0(\mathbb{Z}, \mathbb{D})$, then $q(t, \cdot) \in \ell^0(\mathbb{Z}, \mathbb{D})$ for all $t \in [0, t_0]$.*

Proof. Put $\rho_n(t)^2 = 1 - |q(t, n)|^2$, and assume that for some $n \geq 0$ there exists $t_1 \in [0, t_0]$ such that $\rho_n(t_1) = 0$ and $\rho_n(t) > 0$ for all $t \in (0, t_1)$. Then for all $t < t_1$ we have

$$\begin{aligned} 2\rho_n \rho_n' &= (\rho_n^2)' = -2\,\mathrm{Re}(q_n q_n') = -2\,\mathrm{Re}(q_n \cdot (i\rho_n^2(q_{n-1} + q_{n+1}))) \\ &= 2\rho_n^2\,\mathrm{Im}(q_n q_{n-1} + q_n q_{n+1}), \\ \rho_n' &= \rho_n\,\mathrm{Im}(q_n q_{n-1} + q_n q_{n+1}), \\ \rho_n(t) &= \rho_n(0) \exp\left[\int_0^t \mathrm{Im}(q_n q_{n-1} + q_n q_{n+1})\,ds\right]. \end{aligned}$$

If we now send t to t_1, the left-hand side will tend to 0, while the right-hand side will not, a contradiction. ∎

Lemma A.2 (Uniqueness [5, p. 20]). *If q, $\tilde{q}$ solve (1.4) on $[0, t]$ for some initial data $q_0 \in \ell^0(\mathbb{Z}, \mathbb{D})$, then $q = \tilde{q}$.*

Proof. Let $q(t, n)$ and $\tilde{q}(t, n)$ be two solutions for the same initial data q_0. We have

$$\begin{aligned}-i(q_n' - \tilde{q}_n') &= (1 - |q_n|^2)(q_{n-1} + q_{n+1}) - (1 - |\tilde{q}_n|^2)(\tilde{q}_{n-1} + \tilde{q}_{n+1}) \\ &= (q_{n-1} - \tilde{q}_{n-1}) + (q_{n+1} - \tilde{q}_{n+1}) - \\ &\quad - (|q_n|^2 q_{n-1} - |\tilde{q}_n|^2 \tilde{q}_{n-1}) - (|q_n|^2 q_{n+1} - |\tilde{q}_n|^2 \tilde{q}_{n+1}).\end{aligned}$$

By Lemma A.1, both $|q_n|$ and $|\tilde{q}_n|$ do not exceed 1; hence,

$$|q_n' - \tilde{q}_n'| \le 2|q_{n-1} - \tilde{q}_{n-1}| + 2|q_{n+1} - \tilde{q}_{n+1}| + 4|q_n - \tilde{q}_n|. \tag{A.1}$$

Therefore,

$$\begin{aligned}&(|q_n(t) - \tilde{q}_n(t)|^2)' = 2\,\mathrm{Re}((q_n - \tilde{q}_n)\overline{(q_n' - \tilde{q}_n')}), \\ &(|q_n(t) - \tilde{q}_n(t)|^2)' \le 12|q_n - \tilde{q}_n|^2 + 2|q_{n-1} - \tilde{q}_{n-1}|^2 + 2|q_{n+1} - \tilde{q}_{n+1}|^2\end{aligned}$$

Define

$$M(t) = \sum_{n \in \mathbb{Z}} \frac{|q_n(t) - \tilde{q}_n(t)|^2}{1 + n^2}.$$

We have $M(0) = 0$ and

$$M'(t) = \sum_{n \in \mathbb{Z}} \frac{(|q_n(t) - \tilde{q}_n(t)|^2)'}{1 + n^2} \le 20M(t).$$

Then Grönwall inequality gives $M(t) = 0$ for all $t \ge 0$; hence, q and $\tilde{q}$ coincide. ∎

Proposition A.3 (Existence [22, Section 1.1]). *For every $q_0 \in \ell^0(\mathbb{Z}, \mathbb{D})$, there exists a unique classical global solution q of* (1.4).

Proof. Uniqueness follows from Lemma A.2. Rewrite (1.4) in the integral form

$$q(t, n) = q_0(n) + \int_0^t i(1 - |q(s, n)|^2)(q(s, n-1) + q(s, n+1))\, ds, \quad n \in \mathbb{Z}. \tag{A.2}$$

Equation (1.4) and equation (A.2) are equivalent. Introduce the space of functions $Y = C([0, t] \times \mathbb{Z})$ where $t = 1/12$. For $u \in Y$, define the mapping

$$F(u)(t, n) = i(1 - |u(t, n)|^2)(u(t, n-1) + u(t, n+1)), \quad n \in \mathbb{Z}.$$

In this notation (A.2), becomes

$$q(t,n) = q_0(n) + \int_0^t F(q)(s,n)\,ds.$$

Further, consider

$$\Phi(u)(t,n) = q_0(n) + \int_0^t F(u)(s,n)\,ds, \quad n \in \mathbb{Z}.$$

Then solvability of (A.2) is equivalent to the existence of a fixed point for $\Phi\colon Y \mapsto Y$. Let us show that Φ is a contraction acting on the set $B_Y = \{u \in Y : \|u\|_Y \le 2\}$. Notice that

$$|F(u)(s,n)| \le 6\|u\|_Y, \quad s \le t, n \in \mathbb{Z},$$

$$|\Phi(u)(t,n)| \le |q_0(n)| + \int_0^t |F(u)(s,n)|\,ds \le 1 + 6t\|u\|_Y,$$

$$\|\Phi(u)\|_Y \le 1 + 6t\|u\|_Y.$$

In particular, if $u \in Y$, then $\Phi(u) \in Y$. Furthermore, from (A.1) we see that for $u, v \in Y$ we have

$$|\Phi(u)(t,n) - \Phi(v)(t,n)| \le \int_0^t |F(u)(s,n) - F(v)(s,n)|\,ds \le 6t\|u - v\|_Y.$$

We have $6t < 1$; hence, Φ is a contraction and (1.4) has a solution on $[0,t]$. By Lemma A.1, $q(t,\cdot)$ also satisfies $q(t,n) < 1$ for all $n \in \mathbb{Z}$, hence the fixed point algorithm can be applied to find the solution on the segment $[t, 2t]$. Iterating this procedure, we obtain the existence of a solution on $[0,\infty)$. A similar argument works for negative t, hence the proof is completed. ∎

The following proposition gives a proof of the convergence in Theorem 1.5 based on the idea from Lemma A.2.

Proposition A.4. *Take $q_0 \in \ell^0(\mathbb{Z},\mathbb{D})$ and let $q_{0,N}$, q, q_N be as in Theorem 1.5. Then, for $N \ge |j|$, $t > 0$ and all $r \in (0,1)$, we have*

$$|q(t,j) - q_N(t,j)| \le \frac{\sqrt{2}re^{10t/r^2}}{\sqrt{1-r^2}} r^{N-|j|}.$$

If we assume $\ell^2(\mathbb{Z},\mathbb{D})$*, then*

$$|q(t,j)-q_N(t,j)| \le re^{10t/r^2}\sqrt{\sum_{|m|>N}|q_0(m)|^2}\cdot r^{N-|j|}.$$

Proof. Set $M_N(t)=\sum_{m\in\mathbb{Z}}|q(t,m)-q_N(t,m)|^2r^{2|m|}$. At $t=0$, we have

$$M_N(0)=\sum_{|m|>N}|q_0(m)|^2r^{2|m|}\le\sum_{|m|>N}r^{2|m|}=\frac{2r^{2N+2}}{1-r^2}. \tag{A.3}$$

The inequalities similar to (A.1) give us $M_N'(t)\le 20r^{-2}M_N(t)$; hence,

$$|q(t,j)-q_N(t,j)|^2r^{2|j|}\le M_N(t)\le\exp(20r^{-2}t)M_N(0)=\frac{2e^{20t/r^2}r^{2N+2}}{1-r^2}.$$

The first part of the proposition follows. To establish the second inequality, we change the bound (A.3). We have

$$M_N(0)=\sum_{|m|>N}|q_0(m)|^2r^{2|m|}\le r^{2(N+1)}\sum_{|m|>N}|q_0(m)|^2.$$

Therefore,

$$\begin{aligned}|q(t,j)-q_N(t,j)|^2r^{2|j|}&\le M_N(t)\le\exp(20r^{-2}t)M_N(0)\\&=e^{20t/r^2}r^{2N+2}\sum_{|m|>N}|q_0(m)|^2,\end{aligned}$$

which concludes the proof. ■

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Roman Bessonov
St. Petersburg State University, Universitetskaya nab. 7-9, 199034 St. Petersburg;
St. Petersburg Department of Steklov Mathematical Institute, Fontanka, 27,
191023 St. Petersburg, Russia; bessonov@pdmi.ras.ru

Pavel Gubkin
St. Petersburg State University, Universitetskaya nab. 7-9, 199034 St. Petersburg;
St. Petersburg Department of Steklov Mathematical Institute, Fontanka, 27,
191023 St. Petersburg, Russia; gubkinpavel@pdmi.ras.ru