Rev. Mat. Iberoam. (Online first)
DOI 10.4171/RMI/1489

Published by EMS Press

Dualities for rational multi-particle Painlevé systems: Spectral versus Ruijsenaars

Ilia Gaiur and Vladimir Rubtsov

Abstract. The extension of the Painlevé–Calogero correspondence for the n-particle Inozemtsev systems raises to the multi-particle generalisations of the Painlevé equations. This extension may be obtained by the Hamiltonian reduction applied to the matrix Painlevé systems. Additionally, such procedure gives an isomonodromic formulation for these non-autonomous Hamiltonian systems. We provide dual systems for the rational multi-particle Painlevé systems (P_{I}, P_{II} and P_{IV}) using the Hamiltonian reduction. We describe this duality in terms of the spectral curve of a non-reduced system compared to the Ruijsenaars duality.

1. Introduction

The story of the *duality phenomenon* in the realm of multi-particle systems goes back to the ideas of Simon Ruijsenaars, which appeared 30 years ago. He raised the following question: to construct action-angle variables for both the A_n-type Calogero–Moser models as well as their "relativistic" or difference analogues (known now as the Ruijsenaars–Scheider systems). This story is well documented (see, e.g., [23]). The main tool in his approach is a commutation relation for the corresponding Lax matrix and some other explicit matrix function, both exhibited in the phase-space variables. Via the diagonalization of an original Lax matrix, he discovered that the auxiliary matrix can be interpreted as the Lax matrix of another multi-particle system, the position coordinates of which are given by the eigenvalues of the original Lax matrix, i.e., by the *action variables* of the first model. This duality can be described as follows: *the action variables of the first system are the position variables of the second, and vice versa*. The simplest example of such *Ruijsenaars duality* is exhibited in the self-duality phenomenon of the rational Calogero model. The self-duality of this model admits an easy geometric interpretation in terms of the Hamiltonian reduction of Kazhdan, Kostant and Sternberg (see [28]). We revisit their approach below, while adapting it for our goals.

The basic idea for other Calogero–Moser–Ruijsenaars–Scheider models comes also from a symplectic reduction. Starting with a "big phase space" (usually of a Lie algebra

Mathematics Subject Classification 2020: 34M56 (primary); 70H06 (secondary).
Keywords: isomondromic deformations, multi-particle systems, Hamiltonian reduction, moment map, Painlevé equations.

or a Lie group origin), we deal with two commuting families of "free canonical Hamiltonians". After choosing a suitable reduction, one can construct two "natural" models on the reduced phase space. "Free" Hamiltonians transform into non-trivial many-body Hamiltonians written in the variables which correspond to the particle positions. A natural map between these two models on the reduced phase space yields the "duality morphism".

Further, Gorsky and Nekrasov have developed the ideas of [28], using an infinite-dimensional setting for the reduction procedure from the *integrable sectors* of topological quantum fields: $N = 2$ Yang–Mills theory for the Calogero–Moser model, and the GWZNW theory for Ruijsenaars–Schneider models (see, e.g., [22]).

The duality ideas were formulated per se in the integrable systems realm in [19], where the authors specified the Ruijsenaars duality as the *action-coordinate* (*AC*) *duality* (see the description above), and proposed numerous examples. They focused their attention both on finite and infinite-dimensional phase spaces, as well as in the holomorphic setting. They proposed also (for the first time) to extend the phase space symmetry, considering the Poisson reduction for Poisson–Lie structures related to a *Heisenberg double*, getting the Fock–Rosly quadratic relations (well described in [3]). Recently [14], M. Fairon reviewed the duality phenomenon in the framework of the representation functor philosophy, using the double Poisson bracket approach.

We should note here that L. Feher, with various co-authors, put these ideas into a rigorous and explicit framework (see [15, 17, 18] and [16]). His students continue to investigate various aspects of the Ruijsenaars duality (see, e.g., [20]).

The authors of [19], apart from the AC-duality definition, introduced another important duality principle: the so-called *action-action* (AA) *duality*. This duality is transparent for the Seiberg–Witten theory, and maps the integrable system in itself. Locally, this map changes the action variable $I \to I_{\mathrm{dual}}$ via the canonical transformation generating a function S associated with the Lagrangian submanifold $I_{\mathrm{dual}} = \partial S/\partial I$ and, vice-versa, $I = \partial S_{\mathrm{dual}}/\partial I_{\mathrm{dual}}$ for the dual canonical transformation generating function S_{dual}.

We make an attempt to extend and to compare both the AC and the AA dualities in the framework of *non-autonomous Hamiltonian systems*, considering as an example a class of systems associated with the irregular Painlevé transcendents. More precisely, the other key ingredient of our work is a famous class of Painlevé equations and their matrix and non-commutative analogues.

A non-commutative Painlevé II equation first appeared in the work of second author and Retakh [41]. This abstract non-commutative equation generalizes the *matrix Painlevé II equation* proposed in [5] to the case of *non-commutative Painlevé time*. This generalization has (via a non-abelian Toda chain) analogue of "rational" solutions expressed via Hankel quasi-determinants, and allows an isomonodromic representation. It also appears in a non-commutative version of the Riemann–Hilbert problem [7]. Recently, a similar problem was addressed for the case of matrix Painlevé equations [10, 11], as well as non-commutative Painlevé equations with non-commutative time [9].

The multi-particle non-autonomous Hamiltonian systems, which generalize Painlevé equations from another point of view, were introduced (for Painlevé VI) by Manin [33] and Levin–Olshanetsky [32]. Manin has described a configuration space for Painlevé VI as a fibration (a pencil of elliptic curves) $\pi\colon \mathcal{E} \to B$ and its solutions as (multi-valued) sections of the fibration. He has given an interpretation of the correspondent Painlevé VI phase space as a (twisted) sheaf of the relative holomorphic one-forms on $\mathcal{E}$. This description gives the identification of the Painlevé VI moduli space with an affine space of certain special closed 2-forms on $\mathcal{E}$. If such form Ω from this space corresponds to the Painlevé VI equation, then the corresponding solutions are the leaves of the Lagrangian fibration of Ω. Manin has proposed a time-dependent Hamiltonian description of Painlevé VI as

$$\frac{dq}{d\tau} = \frac{\partial H}{\partial p}, \quad \frac{dp}{d\tau} = -\frac{\partial H}{\partial q},$$

where

$$H = \frac{p^2}{2} - \frac{1}{(2\pi)^2} \sum_{k=0}^{3} \alpha_k \, \wp(q + T_k/2, \tau).$$

Here, the T_k are points of order two, and $\wp$ denotes the Weierstrass $\wp$-function. Levin and Olshantetsky have given a “many-particle” generalization of Manin’s description by considerating the non-autonomous version of Hitchin integrable systems.

For other Painlevé transcendents, Takasaki [43] has computed (by the confluence procedure) the multi-particle *Painlevé–Calogero Hamiltonians*:

$$\begin{aligned}
\tilde{H}_{\mathrm{VI}}: &\quad \sum_{j=1}^{n} \Big(\frac{p_j^2}{2} + \sum_{\ell=0}^{3} g_\ell^2 \, \wp(q_j + \omega_\ell) \Big) + g^2 \sum_{j\neq k} (\wp(q_j - q_k) + \wp(q_j + q_k)), \\
\tilde{H}_{\mathrm{V}}: &\quad \sum_{j=1}^{n} \Big(\frac{p_j^2}{2} - \frac{\alpha}{\sinh^2(q_j/2)} - \frac{\beta}{\cosh^2(q_j/2)} + \frac{\gamma t}{2} \cosh(q_j) + \frac{\delta t^2}{8} \cosh(2q_j) \Big) \\
&\qquad + g^2 \sum_{j\neq k} \Big(\frac{1}{\sinh^2((q_j - q_k)/2)} + \frac{1}{\sinh^2((q_j + q_k)/2)} \Big), \\
\tilde{H}_{\mathrm{IV}}: &\quad \sum_{j=1}^{n} \Big(\frac{p_j^2}{2} - \frac{1}{2} \Big(\frac{q_j}{2} \Big)^6 - 2t \Big(\frac{q_j}{2} \Big)^4 - 2(t^2 - \alpha) \Big(\frac{q_j}{2} \Big)^2 + \beta \Big(\frac{q_j}{2} \Big)^{-2} \Big) \\
&\qquad + g^2 \sum_{j\neq k} \Big(\frac{1}{(q_j - q_k)^2} + \frac{1}{(q_j + q_k)^2} \Big), \\
\tilde{H}_{\mathrm{III}}: &\quad \sum_{j=1}^{n} \Big(\frac{p_j^2}{2} - \frac{\alpha}{4} \mathrm{e}^{q_j} + \frac{\beta t}{4} \mathrm{e}^{-q_j} - \frac{\gamma}{8} \mathrm{e}^{2q_j} + \frac{\delta t^2}{8} \mathrm{e}^{-2q_j} \Big) \\
&\qquad + g^2 \sum_{j\neq k} \frac{1}{\sinh^2((q_j - q_k)/2)}, \\
\tilde{H}_{\mathrm{II}}: &\quad \sum_{j=1}^{n} \Big(\frac{p_j^2}{2} - \frac{1}{2} \Big(q_j^2 + \frac{t}{2} \Big)^2 - \alpha q_j \Big) + g^2 \sum_{j\neq k} \frac{1}{(q_j - q_k)^2}, \\
\tilde{H}_{\mathrm{I}}: &\quad \sum_{j=1}^{n} \Big(\frac{p_j^2}{2} - 2q_j^3 - t q_j \Big) + g^2 \sum_{j\neq k} \frac{1}{(q_j - q_k)^2}.
\end{aligned}$$

Some of these Hamiltonian systems may be viewed as non-autonomous (deformed) versions of Inozemtsev systems (see [25]). This may be described as a multi-particle version of the so-called Painlevé–Calogero correspondence, see [32]: $\tilde{H}_{\mathrm{VI}}$ corresponds to the elliptic Inozemtsev system, $\tilde{H}_{\mathrm{V}}$ to the trigonometric case, and $\tilde{H}_{\mathrm{IV}}$ to the rational case. There are also two more Hamiltonians which have rational interaction potential: multi-particle Painlevé I and II. From now on, we refer to rational multi-particle Painlevé systems as P_{I}, P_{II} and P_{IV} multi-particle systems. In this work, we consider only rational multi-particle Painlevé models.

The second author (with M. Bertola and M. Cafasso) provided a scheme which connects the matrix Painlevé equations and Takasaki's Hamiltonians. They showed [8] that these multi-particle generalisations of the Painlevé systems may be obtained from the matrix Painlevé Hamiltonian systems via the Hamiltonian reduction procedure à la Kazhdan, Konstant and Sternberg [28], combined with some symplectic map for the reduced system. They also gave an isomonodromic formulation of multi-particle Painlevé using this reduction, answering Takasaki's question about the existence of a zero-curvature representation for these Hamiltonians. Recently, the authors of [6] have also described Bäcklund transformations and a Hamiltonian reduction scheme for several matrix Painlevé systems.

Here we briefly review our reduction procedure for the matrix Painlevé equations. Details are described in [8]. See also [4] for the basics of Hamiltonian group actions. We start with the isomonodromic problem of rank $2n$, which gives the matrix Painlevé equations (see [7, 26, 27])

$$\begin{cases} \dfrac{\partial}{\partial z}\Phi = \mathcal{A}(\mathbf{q}, \mathbf{p}, z, t)\,\Phi, \\[2mm] \dfrac{\partial}{\partial t}\Phi = \mathcal{B}(\mathbf{q}, \mathbf{p}, z, t)\,\Phi, \end{cases}$$

where $\mathbf{q}$ and $\mathbf{p}$ are unknown elements of $\mathfrak{gl}(n, \mathbb{C})$. The compatibility condition

$$\mathcal{A}_t - \mathcal{B}_z + [\mathcal{A}, \mathcal{B}] = 0$$

gives a non-autonomous Hamiltonian system with Hamiltonian $\operatorname{Tr} H(\mathbf{q}, \mathbf{p}, t)$ and symplectic form $\omega = \operatorname{Tr} \mathrm{d}\,\mathbf{p} \wedge \mathrm{d}\,\mathbf{q} = \sum_{i,j} \mathrm{d}\,p_{ij} \wedge \mathrm{d}\,q_{ji}$, where $H(q, p, t)$ is the corresponding Hamiltonian from the Okamoto list [40]:

$$\begin{aligned}
\mathrm{P_{VI}}: \quad H &= \frac{q(q-1)(q-t)}{t(t-1)}\Big[p^2 - \Big(\frac{\kappa_0}{q} + \frac{\kappa_1}{q-1} + \frac{\theta-1}{q-t}\Big)p + \frac{\kappa}{q(q-1)}\Big],\\
\mathrm{P_{V}}: \quad H &= \frac{q(q-1)^2}{t}\Big[p^2 - \Big(\frac{\kappa_0}{q} + \frac{\theta_1}{q-1} - \frac{\eta_1 t}{(q-1)^2}\Big)p + \frac{\kappa}{q(q-1)}\Big],\\
\mathrm{P_{IV}}: \quad H &= 2q\Big[p^2 - \Big(\frac{q}{2} + t + \frac{\kappa_0}{q}\Big)p + \frac{\theta_\infty}{2}\Big],\\
\mathrm{P_{III}}: \quad H &= \frac{q^2}{t}\Big[p^2 - \Big(\eta_\infty + \frac{\theta_0}{q} - \frac{\eta_0 t}{q^2}\Big)p + \frac{\eta_\infty(\theta_0+\theta_\infty)}{2q}\Big],\\
\mathrm{P_{II}}: \quad H &= \frac{p^2}{2} - \Big(q^2 + \frac{t}{2}\Big)p - \Big(\alpha + \frac{1}{2}\Big)q,\\
\mathrm{P_{I}}: \quad H &= \frac{p^2}{2} - 2q^3 - tq.
\end{aligned}$$

The phase space of matrix Painlevé equations is given by

$$M = (S, \omega), \quad S = \mathfrak{gl}(n, \mathbb{C}) \times \mathfrak{gl}(n, \mathbb{C}) \simeq \{(\mathbf{q}, \mathbf{p})\}, \quad \omega = \operatorname{Tr} \mathrm{d}\,\mathbf{p} \wedge \mathrm{d}\,\mathbf{q}. \tag{1.1}$$

The symplectic manifold M allows a group action of $\mathrm{GL}(n, \mathbb{C})$ by conjugation,

$$g \circ (\mathbf{q}, \mathbf{p}) = (\mathrm{Ad}_{g^{-1}}\mathbf{q}, \mathrm{Ad}_{g^{-1}}\mathbf{p}) = (g^{-1}\mathbf{q}g, g^{-1}\mathbf{p}g),$$

which is Hamiltonian, with moment map given by

$$\mathcal{M}(\mathbf{p}, \mathbf{q}) = [\mathbf{p}, \mathbf{q}]. \tag{1.2}$$

Since matrix Painlevé Hamiltonians are traces of rational Okamoto Hamiltonians, they are invariant under $\mathrm{GL}(n, \mathbb{C})$-action, and the moment map $\mathcal{M}$ gives first integrals. Finally, restricting to the level set of momentum,

$$[\mathbf{p}, \mathbf{q}] = \sqrt{-1}\, g(\mathbf{1} - v^{\mathsf{T}} \otimes v), \quad v = (1,\, 1,\, \ldots 1),$$

performing a diagonalization of the coordinate $\mathbf{q}$ and resolving the moment map condition for $\mathbf{p}$, we obtain the multi-particle Hamiltonian systems which coincide up to the canonical transformation with the Hamiltonians presented by Takasaki. The isomonodromic problem for the reduced system arises from the isomonodromic problem for the matrix Painlevé equations. The special gauge transformation sends the isomonodromic representation for the matrix Painlevé system to the isomonodromic representation for the reduced system.

The aim of this work is to introduce the dual Hamiltonian systems for P_{I}, P_{II} and P_{IV}. Briefly, the dual system is a multi-particle system obtained by the reduction from another point of the $\mathrm{GL}(n, \mathbb{C})$-orbit, where now $\mathbf{p}$ is diagonal. The isomonodromic representations are responsible for the spectral duality between the systems obtained by reduction, which differs from the Ruijsenaars (action-angle) duality.

For the matrix P_{II} and P_{I} systems, we also provide autonomous versions. Reduction of these autonomous systems give rise to integrable autonomizations of Takasaki Hamiltonians, and may be viewed as a further degeneration of the rational Inozemtsev system.

Let us briefly discuss the structure of this paper. In Section 2, we remind the basic objects of our study – two types of dualities for the integrable many-body systems. We illustrate both in the most simple and transparent cases. Then we interpret the spectral duality as a special case of the AA duality, and formulate it in the form of Theorem 2.1 (the results of this theorem are probably folklore, but we did not find them in the appropriate form).

Section 3 contains our main computational results. Here we obtain the dual Hamiltonians for the multi-particle Painlevé–Calogero IV, II and I systems, and discuss the self-duality property. It turns out that the Painlevé IV model has the property of self-duality, but the other two rational multi-particle Painlevé–Calogero Hamiltonians have not.

In Section 4, we study the autonomous avatars of rational multi-particle Painlevé–Calogero systems and their behaviour under reduction. This is a subject of the well-known and classical *Painlevé–Calogero correspondence* of Levin–Olshanetsky–Zabrodin–Zotov. We explicitly write the Lax representation for two non-commutative Painlevé models. We finish this section with a confluence procedure interpretation of the Inozemtsev system degeneration and its symplectic properties.

In Section 5, we observe a relation between two types of reduction for non-commutative integrable models (more precisely, for the matrix modified Korteweg–de Vries (MmKdV) equation, the spectral duality and the Painlevé–Calogero correspondence mappings).

In the final Section 6, we indicate a few possible new directions for the proposed duality exploration for the difference many-body systems (Ruijsenaars–Schneider models) and the q-Painlevé systems.

We have collected in Appendix A some technicalities related to an explicit computation of interactive terms for dual multi-particle Painlevé–Calogero Hamiltonians.

Remark 1.1. We will use the following notation: by the matrix Painlevé systems, we denote the Hamiltonian systems which were obtained by Kawakami in [26]. By the Calogero–Painlevé systems, we mean the multi-particle systems obtained in [8,43]. We use the term dual systems to denote multi-particle systems dual to Painlevé–Calogero which are introduced in Section 3.

2. Ruijsenaars duality and spectral duality

In this section, we review some basic facts about the Hamiltonian reduction and the Ruijsenaars duality for the many-body systems. We illustrate the Ruijsenaars duality in the simplest example of the self-dual rational Calogero system. Then we introduce a different kind of duality, which comes from the reduction of the matrix isospectral (or isomonodromic) systems. During this section and the rest of the paper, we work with a dense subset $\mathcal{S}$ of (1.1) which corresponds to the diagonalizable $\mathbf{p}$ and $\mathbf{q}$.

2.1. Ruijsenaars duality. Rational Calogero–Moser system

The rational Calogero–Moser system may be obtained considering the free particle Hamiltonian

$$H = \operatorname{Tr}\left(\frac{\mathbf{p}^2}{2}\right)$$

on the symplectic manifold M given by (1.1). The equations of the motion are

$$\begin{cases} \dot{\mathbf{p}} = 0, \\ \dot{\mathbf{q}} = \mathbf{p}. \end{cases}$$

Since H is invariant under conjugation, the moment map $\mathcal{M}$ given in (1.2) is a constant of motion. To obtain the rational Calogero–Moser system, we fix the level set of momentum by the following way:

$$[\mathbf{p}, \mathbf{q}] = \sqrt{-1}\, g(\mathbf{1} - v^{\mathsf{T}} \otimes v) = \mu_0, \quad v = (1, 1, \dots 1). \tag{2.1}$$

Such choice of the moment map allows us to reduce the dimension of the phase space from $2n^2$ to $2n$. It can be shown that the Ad_g orbit intersects the level set (2.1) at least

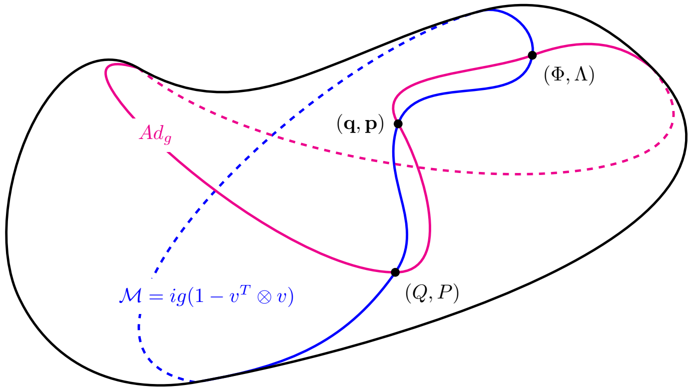

Figure 1. The intersection of the orbit of the point $(\mathbf{q}, \mathbf{p})$ with level set $\mathcal{M}$. The blue line means the level set of momentum, and the magenta line describes the orbit of group action.

at two extra points where q or p are diagonal (see Figure 1 for an illustration of this intersection). So we may find a matrix C such that

$$\mathbf{q} = CQC^{-1} \quad \text{and} \quad C^{-1}(\mathbf{1} - v^{\mathsf{T}} \otimes v)C = \mathbf{1} - v^{\mathsf{T}} \otimes v,$$

where $Q = \delta_{ij} q_i$. Acting by C on $\mathcal{M}$, we get

$$C^{-1}[\mathbf{p}, \mathbf{q}]C = [C^{-1}\mathbf{p}C, Q] = [P, Q] = \sqrt{-1}\, g(\mathbf{1} - v^{\mathsf{T}} \otimes v).$$

Resolving the moment map constraint, we obtain the entries of P:

$$P = p_i \delta_{ij} + (1 - \delta_{ij}) \frac{\sqrt{-1}\, g}{q_i - q_j}.$$

The reduced Hamiltonian and symplectic form turn to

$$H = \mathrm{Tr}\left(\frac{\mathbf{p}^2}{2}\right) = \mathrm{Tr}\left(\frac{P^2}{2}\right) = \sum_i \frac{p_i^2}{2} + \sum_{i<j} \frac{g^2}{(q_i - q_j)^2} \quad \text{and} \quad \omega = \sum_i \mathrm{d}\, p_i \wedge \mathrm{d}\, q_i,$$

and the equations of motion take the following form:

$$\begin{cases} \dot{P} = [P, F], \\ \dot{Q} = P + [Q, F], \end{cases} \qquad F = C^{-1}\dot{C}.$$

We see that P is a Lax matrix for the rational Calogero–Moser system, so the eigenvalues of P, $(I_1, I_2, \ldots, I_n)$, are the first integrals for the reduced system. We may also find a transition matrix $\tilde{C}$ for the eigenbasis of $\mathbf{p}$ such that

$$\tilde{C}^{-1}(\mathbf{1} - v^{\mathsf{T}} \otimes v)\tilde{C} = \mathbf{1} - v^{\mathsf{T}} \otimes v.$$

Making the same type of reduction, we obtain

$$\mathbf{p} = \tilde{C}\Lambda\tilde{C}^{-1}, \quad \Lambda = \delta_{ij} I_i, \quad \mathbf{q} = \tilde{C}\,\Phi\,\tilde{C}^{-1}, \quad \Phi = \delta_{ij}\,\phi_i + (1-\delta_{ij})\,\frac{\sqrt{-1}\,g}{I_i - I_j},$$

and the Hamiltonian reduces to

$$H = \operatorname{Tr}\Big(\frac{\mathbf{p}^2}{2}\Big) = \sum_j \frac{I_j^2}{2}, \quad \omega = \sum_j \mathrm{d}\,I_j \wedge \mathrm{d}\,\phi_j,$$

which gives the action-angle variables (I_i, ϕ_i). This coincides with the fact that the eigenvalues of P are constants of motion (actions). To describe the Ruijsenaars duality, we have to consider the flow generated by the dual free Hamiltonian

$$H^{(\text{dual})} = \operatorname{Tr}\Big(\frac{\mathbf{q}^2}{2}\Big).$$

It is obvious that the first set of reduced variables $\{q_1, q_2, \ldots, q_n;\ p_1, p_2, \ldots, p_n\}$ is the set of action-angle variables for the dual system. On the other hand, for the dual coordinates, we have

$$H^{(\text{dual})} = \sum_i \frac{\phi_i^2}{2} + \sum_{i<j} \frac{g^2}{(I_i - I_j)^2}.$$

So the action angle variables for the Hamiltonian $H = \operatorname{Tr}(\mathbf{p}^2/2)$ are the coordinates and the actions for the Hamiltonian $H^{(\text{dual})} = \operatorname{Tr}(\mathbf{q}^2/2)$, and vice versa. This duality is called the Ruijsenaars duality or the action-coordinate duality. Since the dual systems are the rational Calogero–Moser system in the variables (I_i, ϕ_i), the rational Calogero–Moser system is a self-dual system.

Further, we will use C for the transition matrix to the eigenbasis of $\mathbf{q}$, and $\tilde{C}$ to the eigenbasis of $\mathbf{p}$. There holds that

$$C^{-1}(\mathbf{1} - v^{\mathsf{T}} \otimes v)\,C = \tilde{C}^{-1}(\mathbf{1} - v^{\mathsf{T}} \otimes v)\,\tilde{C} = \mathbf{1} - v^{\mathsf{T}} \otimes v.$$

We also will use the following notation for the reduced coordinates:

$$Q = C^{-1}\mathbf{q}C = \delta_{ij}\,q_i, \quad P = C^{-1}\mathbf{p}\,C = \delta_{ij}\,p_i + (1-\delta_{ij})\,\frac{\sqrt{-1}\,g}{q_i - q_j},$$

and for the dual coordinates:

$$\Lambda = \tilde{C}^{-1}\mathbf{p}\,\tilde{C} = \delta_{ij}\,I_i, \quad \Phi = \tilde{C}^{-1}\,\mathbf{q}\,\tilde{C} = \delta_{ij}\,\phi_i + (1-\delta_{ij})\,\frac{\sqrt{-1}\,g}{I_i - I_j}.$$

The fact that the Ad_g orbit intersects the level set of the moment map in the points where $\mathbf{q}$ or $\mathbf{p}$ are diagonal may be rephrased in the following way: the transition matrices of the eigenbases for both $\mathbf{q}$ and $\mathbf{p}$ belong to the stabilizer G_{μ_0} of μ_0. So the previously described reduction gives the following symplectic quotient:

$$M_{\text{reduced}} = M//G_{\mu_0} = \mathcal{M}^{-1}(\mu_0)/G_{\mu_0}.$$

The dual systems in this situation may be viewed as a realisation of a symplectic quotient in different points of the orbit of the stabilizer.

2.2. Spectral duality

In contrast to the Ruijsenaars duality, we want to introduce another type of a duality which we call a spectral duality. This kind of duality is the special case of the action-action duality, as it is formulated in the following theorem.

Theorem 2.1. *Let $H(\mathbf{q},\mathbf{p})$ be a "rational" function (i.e., a trace of a polynomial in $\mathbf{q}$, $\mathbf{p}$, and their inverses) on the symplectic manifold $\mathcal{M} = \{(\mathbf{q},\mathbf{p}) \in \mathfrak{gl}(n,\mathbb{C}) \times \mathfrak{gl}(n,\mathbb{C})\}$, and let $\omega = \operatorname{Tr} \mathrm{d}\,\mathbf{p} \wedge \mathrm{d}\,\mathbf{q}$, which is invariant under the adjoint $\mathrm{GL}(n,\mathbb{C})$-action. Let $R(\mathcal{M})$ be a space of rational functions on $\mathcal{M}$ taking values in $\mathfrak{gl}(n,\mathbb{C})$, i.e., the elements of $R(\mathcal{M})$ are just polynomials in $\mathbf{q}$, $\mathbf{p}$ and their inverses. Let $H(p_i,q_i)$ and $H(I_i,\phi_i)$ be the dual multi-particle systems obtained by the reduction with respect to the moment map (1.2). If the Hamilton equations for $H(\mathbf{q},\mathbf{p})$ admit the isospectral or zero-curvature representation*

$$L_t = [L,M] \quad \text{or} \quad L_t - M_\lambda + [L,M] = 0,$$

with

$$L(\lambda), M(\lambda) \in R(\mathcal{M}) \otimes \mathfrak{gl}(n,\mathbb{C}) \otimes \mathbb{C}[\lambda^{\pm 1}],$$

then the dual systems $H(p_i,q_i)$ and $H(I_i,\phi_i)$ admit an isospectral (zero-curvature) representation given by the gauge transform for L,M operators. Furthermore, the spectral curves for both systems are the same:

$$\Gamma^{(p,q)}(\lambda,\mu) = \Gamma^{(I,\phi)}(\lambda,\mu) = \det(L(\lambda)-\mu) = 0.$$

Proof. Since the Hamiltonian $H(\mathbf{p},\mathbf{q})$ is invariant under the adjoint action, the moment map μ is a constant of motion with respect to the dynamics generated by $H(\mathbf{p},\mathbf{q})$. The moment map for the symplectic manifold $\mathcal{M}$ is $[\mathbf{p},\mathbf{q}]$. To obtain multi-particle reductions, we fix the moment map

$$[\mathbf{p},\mathbf{q}] = ig(\mathbf{1} - v^{\mathsf{T}} \otimes v)$$

and perform the diagonalization of $\mathbf{q}$ or $\mathbf{p}$ using the same arguments as before. To obtain the reduced Lax (or isomonodromic) representation, we do the gauge transformation for the eigenfunction of the isospectral (or isomonodromic) problem

$$\Psi = (C \otimes \mathrm{Id}_n)Y = hY,$$

where C is the transition matrix to the eigenbasis of $\mathbf{q}$ or $\mathbf{p}$. This leads to the following action on the L and M matrices:

$$L \to h^{-1}Lh, \quad M \to h^{-1}Mh - h^{-1}h_t.$$

Since L takes values in $R(\mathcal{M}) \otimes \mathfrak{gl}(n,\mathbb{C})$, it may be written as

$$L(\mathbf{q},\mathbf{p}) = \sum_{i,j=1}^{m} P_{ij}(\mathbf{q},\mathbf{p}) \otimes E_{ij},$$

where $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and P_{ij} are rational functions. Then the gauge transformation acts in the following way:

$$\begin{aligned} h^{-1}Lh &= \sum_{i,j=1}^{m} C^{-1}P_{ij}(\mathbf{q},\mathbf{p})C \otimes E_{ij} = \sum_{i,j=1}^{m} P_{ij}(C^{-1}\mathbf{q}C, C^{-1}\mathbf{p}C) \otimes E_{ij} \\ &= L(C^{-1}\mathbf{q}C, C^{-1}\mathbf{p}C), \end{aligned}$$

i.e., such gauge sends the L-operator to the Lax operator on the reduced space. To obtain the matrix M, we need to write explicitly $g^{-1}g_t$, which may be done using the reduced Hamilton equations (for a detailed proof, we refer to Lemma 2.5 in [8]).

Finally, an L-matrix for the reduced system is given by the conjugation of the L-matrix for the unreduced system, so they have the same eigenvalues and the same spectral curves. Since both the reduced (p, q) and the dual (I, ϕ) systems are obtained by this procedure, their spectral curves coincide. ■

Remark 2.2. The difference between the isospectral and the isomonodromic cases is that in the isospectral case the same spectral invariants are the integrals of motion for the reduced systems. For the isomonodromic duality, the invariants are not conserved quantities, since the dynamic is non-autonomous.

Here we provide a simple example of the described duality for the matrix harmonic oscillator, since the free particle Hamiltonian gives trivial result. In [37], the duality for the Calogero–Moser–Sutherland systems is described via the Hamiltonian reduction of the following Hamiltonian system:

$$H = \operatorname{Tr}\left(\frac{\mathbf{p}^2}{2} + \omega^2\frac{\mathbf{q}^2}{2}\right), \tag{2.2}$$

which we call the matrix harmonic oscillator. The construction of the dual system is more complicated than in the case of the rational Calogero–Moser system, because it requires a polar decomposition in order to change the phase space from $T^\star\mathfrak{g}$ to $T^\star G$, where G is a Lie group and $\mathfrak{g}$ is the corresponding Lie algebra. The obtained duality gives the Ruijsenaars duality between the Calogero–Moser–Sutherland and the rational Ruijsenaars–Sneider systems.

On the other hand, the Hamiltonian system given by (2.2) may be written as an isospectral deformation for the block-matrix L-operator:

$$\begin{cases} L\Psi = \lambda\Psi, \\ \Psi_t = M\Psi, \end{cases} \quad \text{with} \quad L = \begin{bmatrix} \mathbf{p} & \omega\mathbf{q} \\ \omega\mathbf{q} & -\mathbf{p} \end{bmatrix} \quad \text{and} \quad M = \frac{\omega}{2}\begin{bmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{bmatrix}.$$

The Lax equation $\dot{L} = [L, M]$ is the same as the Hamilton–Jacobi equations for the Hamiltonian (2.2) with symplectic form $\Omega = \operatorname{Tr} \mathrm{d}\,\mathbf{p} \wedge \mathrm{d}\,\mathbf{q}$. Reducing at Q and P, we get the system

$$H^{(\mathrm{red})} = \sum_i \left(\frac{p_i^2}{2} + \omega^2\frac{q_i^2}{2}\right) + \sum_{i<j}\frac{g^2}{(q_i - q_j)^2}.$$

The Lax pair for the reduced system is given by the gauge

$$\Psi = (C \otimes \mathrm{Id}_2)Y = hY,$$

which gives the following Lax pair:

$$\begin{cases} L^{(\mathrm{red})}Y = \mu Y, \\ Y_t = (h^{-1}Mh - h^{-1}h_t)Y, \end{cases} \qquad L^{(\mathrm{red})} = h^{-1}Lh = \begin{bmatrix} P & \omega Q \\ \omega Q & -P \end{bmatrix}.$$

Reducing at the dual value of the moment map gives the system

$$H^{(\text{dual})} = \sum_i \Big(\frac{I_i^2}{2} + \omega^2 \frac{\phi_i^2}{2} \Big) + \sum_{i<j} \frac{\omega^2 g^2}{(I_i - I_j)^2}$$

with the Lax representation

$$\Psi = (\tilde{C} \otimes \mathrm{Id}_2)\, \tilde{Y} = \tilde{h}\, \tilde{Y},$$
$$\begin{cases} L^{(\text{dual})} \tilde{Y} = \mu \tilde{Y}, \\ \tilde{Y}_t = (\tilde{h}^{-1} M \tilde{h} - \tilde{h}^{-1} \tilde{h}_t) \tilde{Y}, \end{cases} \qquad L^{(\text{dual})} = \tilde{h}^{-1} L \tilde{h} = \begin{bmatrix} \Lambda & \omega\Phi \\ \omega\Phi & -\Lambda \end{bmatrix}.$$

Since the Lax operators L, $L_{(\text{dual})}$ and $L_{(\text{red})}$ are conjugate to each other, the spectral curves for the unreduced, the reduced and the dual systems are the same. Indeed,

$$\begin{aligned} \Gamma(\lambda,\mu) &= \det(L-\mu) = \det(h)\det(L^{(\text{red})}-\mu)\det(h)^{-1} = \det(L^{(\text{red})}-\mu) \\ &= \Gamma^{(\text{red})}(\lambda,\mu) = \det(\tilde{h})\det(L^{(\text{dual})}-\mu)\det(\tilde{h})^{-1} = \Gamma^{(\text{dual})}(\lambda,\mu) = 0. \end{aligned}$$

In this case, there is no dependence on λ in the spectral curve equation $\Gamma(\lambda,\mu) = \Gamma(\mu)$, because we consider the Lax pair without a spectral parameter. In the general case, the spectral curve depends on both λ and μ. Since the spectral curves are the same,

$$\Gamma^{(\text{red})}(\lambda,\mu) = \Gamma^{(\text{dual})}(\lambda,\mu) = \Gamma(\lambda,\mu) = 0,$$

we call this correspondence the spectral duality. This duality may be viewed as follows: the Cauchy problem for an unreduced system fixes the coefficients of the spectral curve, and fixes the same data for the reduced and the dual systems.

In case of the free particle Hamiltonian $\mathrm{Tr}(\mathbf{p}^2/2)$, this spectral duality is trivial: we obtain the rational Calogero system and the free particle system which corresponds to the action angle variables. The consideration of the matrix harmonic oscillator is a non-trivial example of two systems which have the same spectral invariants.

The obtained reduced system is self-dual, in the sense that the symplectic map

$$I_i \to \omega q_i, \quad \phi_i \to -\frac{p_i}{\omega}$$

maps $H^{(\text{red})}$ to $H^{(\text{dual})}$, i.e.,

$$H^{(\text{dual})}(q_i, p_i) = H^{(\text{red})}\Big(-\frac{p_i}{\omega}, \omega q_i \Big).$$

This symplectic self-duality comes from the symmetry of the unreduced Hamiltonian. In the general case, the dual systems may not be self-dual.

A similar type of a duality was introduced in the works [35, 36, 46], which give classical-classical or classical-quantum duality for integrable systems. The most studied examples are the spectral duality between the Gaudin model and the Heisenberg chain, and the spectral self-dual Toda lattice. The main difference is that the spectral duality introduced by Morozov et. al. [35] interchanges λ and μ on a spectral curve. For example, the Gaudin model and the Heisenberg chain spectral curves are connected in the following way:

$$\Gamma^{(\text{Gaudin})}(\lambda,\mu)=\Gamma^{(\text{Heisenberg})}(\mu,\lambda).$$

This duality may be viewed as a Fourier transform between λ and μ. This Fourier transform duality was explored for the first time in a series of works by Adams, Harnad, Hurtubise and Previato, both for the isospectral and for the isomonodromic systems, in order to provide the description of the classical integrable systems in terms of the coadjoint orbits of a loop group action [1, 2, 24]. The main difference in our case is that we do not have this twist of spectral parameters, since our duality is not an analogue of the "Fourier" transform, but it originates from the different reductions of the non-reduced systems.

3. Dual Hamiltonians for the multi-particle Painlevé–Calogero systems

In this section, we provide dual Hamiltonian systems which are obtained from the Calogero–Painlevé IV, II and I systems by Hamiltonian reduction. Without loss of generality, we will use $p_1, p_2 \dots p_n$ and $q_1, q_2 \dots q_n$ for the dual coordinates, instead of $I_1, I_2 \dots I_n$ and $\phi_1, \phi_2 \dots \phi_n$, since in this case they do not correspond to the action-angle variables and we want to emphasize the self-duality in the case of P_{IV}. In this section and in the rest of the paper, we shall use isomonodromic pairs, which appeared before in [8, 26].

3.1. Painlevé–Calogero IV system

Proposition 3.1. *There is an anti-symplectic involution of the reduced phase space for the Calogero–Painlevé IV system such that the reduced Hamiltonian is self-dual.*

Proof. The isomonodromic linear problem for the fourth matrix Painlevé system reads

$$\begin{cases} \dfrac{\partial}{\partial\lambda}\Psi=\begin{bmatrix} -\dfrac{\mathbf{pq}}{\lambda} & \mathbf{qp}+\theta_0+\theta_1-\dfrac{\mathbf{pqp}+\theta_0\mathbf{p}}{\lambda} \\ 1+\dfrac{\mathbf{q}}{\lambda} & -\lambda+t+\dfrac{\mathbf{qp}+\theta_0}{\lambda} \end{bmatrix}\Psi, \\ \dfrac{\partial}{\partial t}\Psi=\begin{bmatrix} 0 & -\mathbf{qp}-\theta_0-\theta_1 \\ -1 & \lambda-\mathbf{q}-t \end{bmatrix}\Psi. \end{cases}$$

The compatibility conditions are

$$\text{(3.1)}\quad \begin{cases} \dot{\mathbf{q}}=[\mathbf{p},\mathbf{q}]_+-\mathbf{q}^2-t\,\mathbf{q}+\theta_0, \\ \dot{\mathbf{p}}=[\mathbf{p},\mathbf{q}]_+-\mathbf{p}^2+t\,\mathbf{p}+\theta_0+\theta_1, \end{cases}\qquad H=\operatorname{Tr}(\mathbf{pq}(\mathbf{p}-\mathbf{q}-t)+\theta_0\mathbf{p}-(\theta_0+\theta_1)\mathbf{q}).$$

In the dual coordinates

$$\Lambda=\operatorname{diag}(p_1,p_2,\dots,p_n)\quad\text{and}\quad\Phi=\operatorname{diag}(q_1,q_2,\dots,q_n)-\Big(\frac{ig}{p_i-p_j}\Big)_{i\neq j},$$

the Hamiltonian reads

$$H_{\mathrm{IV}}^{(\mathrm{dual})} = \sum_{i=1}^{n} \left[q_i p_i^2 - p_i q_i^2 - t q_i p_i + \theta_0 p_i - (\theta_0 + \theta_1) q_i \right] - g^2 \sum_{i<j} \frac{p_i + p_k}{(p_i - p_j)^2}. \tag{3.2}$$

Hence, the change of variables

$$q_i \to -p_i, \quad p_i \to -q_i, \quad \theta_0 \to -\theta_1, \quad \theta_1 \to -\theta_0$$

transforms $H_{\mathrm{IV}}^{(\mathrm{dual})}$ into

$$H_{\mathrm{IV}}^{(\mathrm{red})} = \sum_{i=1}^{n} \left[q_i p_i^2 - p_i q_i^2 - t p_i q_i + \theta_0 q_i - (\theta_0 + \theta_1) p_i \right] + g^2 \sum_{i<j} \frac{q_i + q_k}{(q_i - q_j)^2},$$

which is obtained by the reduction at (Q, P). The map provided above does change the sign of the symplectic form, so it is an anti-symplectic involution. This coincides with the fact that the P_{IV} Hamiltonian from (3.1) is invariant under the same change of variables in terms of $\mathbf{p}$ and $\mathbf{q}$. Indeed, after the first change of variables

$$\mathbf{p} \to -\mathbf{q}, \quad \mathbf{q} \to -\mathbf{p},$$

the Hamiltonian reads

$$H = \operatorname{Tr}\left(\mathbf{qp}(-\mathbf{q} + \mathbf{p} - t) - \theta_0 \mathbf{q} + (\theta_0 + \theta_1)\mathbf{p}\right).$$

Since trace is invariant under cyclic permutations, one may rewrite the obtained Hamiltonian as

$$\begin{aligned} H &= \operatorname{Tr}\left(-\mathbf{pq}^2 + \mathbf{pqp} - \mathbf{pq}t - \theta_0 \mathbf{q} + (\theta_0 + \theta_1)\mathbf{p}\right) \\ &= \operatorname{Tr}\left(\mathbf{pq}(\mathbf{p} - \mathbf{q} - t) - \theta_0 \mathbf{q} + (\theta_0 + \theta_1)\mathbf{p}\right). \end{aligned}$$

Now let us change the constants as follows:

$$\theta_0 \to \tilde{\theta}_0 + \tilde{\theta}_1 \quad \text{and} \quad \theta_1 \to -\tilde{\theta}_1.$$

This leads to

$$H = \operatorname{Tr}(\mathbf{pq}(\mathbf{p} - \mathbf{q} - t) + \tilde{\theta}_0 \mathbf{p} - (\tilde{\theta}_0 + \tilde{\theta}_1)\mathbf{q}).$$

As a consequence, the transformed Hamiltonian coincides with the Painlevé IV one after switching from $\tilde{\theta}_i$ to θ_i for $i = 1, 2$.

3.2. Painlevé–Calogero II system

We shall see that, in contrast to the precedent model, the dual Hamiltonian of the Calogero–Painlevé II system does not "survive" under this duality map.

Proposition 3.2. *The Painlevé–Calogero II system is* not *self-dual. Moreover, the dual system admits a* quadruple-wise *particle interaction, while in case of a* $\mathbf{q}$*-diagonal reduction, we obtain only a pair-wise particle interaction.*

Proof. The isomonodromic problem for the matrix Painlevé II system is given by

$$\begin{cases} \dfrac{\partial}{\partial\lambda}\Phi = \begin{bmatrix} i\lambda^2/2 + i\,\mathbf{q}^2 + i\,t/2 & \lambda\mathbf{q} - i\mathbf{p} - \theta/\lambda \\ \lambda\mathbf{q} + i\mathbf{p} - \theta/\lambda & -i\lambda^2/2 - i\,\mathbf{q}^2 - i\,t/2 \end{bmatrix}\Phi, \\ \\ \dfrac{\partial}{\partial t}\Phi = \begin{bmatrix} i\lambda/2 & \mathbf{q} \\ \mathbf{q} & -i\lambda/2 \end{bmatrix}\Phi, \end{cases}$$

the compatibility condition of which leads to the following equations:

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{p} \\ \dot{\mathbf{p}} = 2\mathbf{q}^3 + t\mathbf{q} + \theta \end{cases} \qquad H = \operatorname{Tr}\left(\frac{\mathbf{p}^2}{2} - \frac{1}{2}\left(\mathbf{q}^2 + \frac{t}{2}\right)^2 - \theta\mathbf{q}\right) \tag{3.3}$$

In the reduced coordinates P and Q, the Hamiltonian reads

$$H_{\mathrm{II}}^{(\mathrm{red})} = \sum_{i=1}^{n}\left[\frac{p_i^2}{2} - \frac{1}{2}\left(q_i^2 + \frac{t}{2}\right)^2 - \theta q_i\right] + g^2\sum_{j<k}\frac{1}{(q_j - q_k)^2}.$$

In the dual coordinates, the Hamiltonian turns into

$$\begin{aligned} H_{\mathrm{II}}^{(\mathrm{dual})} &= \sum_{i=1}^{n}\left[\frac{p_i^2}{2} - \frac{1}{2}\left(q_i^2 + \frac{t}{2}\right)^2 - \theta q_i\right] \\ &+ 2g^2\sum_{i<j}\frac{q_i^2 + q_iq_j + q_j^2 + t/2}{(p_i - p_j)^2} - g^4\sum_{i<j}\frac{1}{(p_i - p_j)^4} \\ &- \left(\sum_{i<j<k}\frac{2g^4}{(p_i - p_j)^2(p_j - p_k)^2} + \sum_{i<j<k<l}\frac{4g^4}{(p_i - p_j)(p_j - p_k)(p_k - p_l)(p_l - p_i)}\right). \end{aligned} \tag{3.4}$$

We describe all technical details of the explicit computations for the interaction terms in this Hamiltonian in Appendix A.

3.3. Painlevé–Calogero I system

The isomonodromic problem for the matrix P_{I} equation is given by

$$\begin{cases} \dfrac{\partial}{\partial\lambda}\Phi = \begin{bmatrix} \mathbf{p} & \lambda - \mathbf{q} \\ \lambda^2 + \lambda\mathbf{q} + \mathbf{q}^2 + t/2 & -\mathbf{p} \end{bmatrix}\Phi, \\ \\ \dfrac{\partial}{\partial t}\Phi = \begin{bmatrix} 0 & 1/2 \\ \lambda/2 + \mathbf{q} & 0 \end{bmatrix}\Phi, \end{cases} \tag{3.5}$$

and gives the following Hamiltonian system:

$$\begin{cases} \dot{\mathbf{q}} &= \mathbf{p}, \\ \dot{\mathbf{p}} &= \frac{3}{2}\mathbf{q}^2 + t/4, \end{cases} \qquad H = \operatorname{Tr}\left(\frac{\mathbf{p}^2}{2} - \frac{\mathbf{q}^3}{2} - \frac{t}{4}\mathbf{q}\right).$$

In the reduced coordinates Q and P, the Hamiltonian is

$$H_{\mathrm{I}}^{(\mathrm{red})} = \sum_{i=1}^{n} \left[\frac{p_i^2}{2} - \frac{q_i^3}{2} - \frac{t}{4} q_i \right] + g^2 \sum_{i<j} \frac{1}{(q_i - q_j)^2}.$$

Taking the dual coordinates Λ and Φ, we obtain

$$H_{\mathrm{I}}^{(\mathrm{dual})} = \sum_{i=1}^{n} \left[\frac{p_i^2}{2} - \frac{q_i^3}{2} - \frac{t}{4} q_i \right] - \frac{3g^2}{2} \sum_{j<i} \frac{q_i + q_j}{(p_i - p_j)^2}.$$

The obtained systems are not connected by the anti-symplectic involution. Computations of the cubic term are provided in Appendix A.

Remark 3.3. The self-dual Painlevé–Calogero IV system is completely integrable since its autonomous version is canonically equivalent to the rational Inozemtsev system, which is completely integrable, see [44]. We do not know if the dual Painlevé–Calogero I and II systems are completely integrable; this question requires further investigation, and will be addressed elsewhere.

4. The autonomous form of the multi-particle Painlevé equations. The Painlevé–Calogero correspondence

In this section, we write down the Lax pairs for the autonomous versions of the Hamiltonians of the non-commutative P_{I}, P_{II} and P_{IV}. This procedure the well-known Painlevé–Calogero correspondence for this type of equations. We use τ as a parameter which does not depend on time t further in the text.

Before diving into the examples, we provide a simple lemma about autonomization of the isomonodromic problems.

Lemma 4.1. *Let $A(z;t)$ and $B(z;t)$ be an isomonodromic pair,*

$$\frac{d}{dt} A - \frac{\partial}{\partial z} B + [A, B] = 0,$$

such that t is an isomonodromic deformation parameter. If the following cross-derivative vanishes,

$$\partial_t A - \partial_z B = 0, \tag{4.1}$$

then isomonodromic problem may be reduced to the Lax system by freezing the isomonodromic time.

Proof. This is a direct computation. ∎

4.1. Painlevé–Calogero II system

The autonomous form of the matrix P_{II} has form

$$
(4.2)\quad \begin{cases} L(\mathbf{q},\mathbf{p})\Phi = \mu\Phi, \\ L(\mathbf{q},\mathbf{p};\lambda) = \mathcal{A}(\mathbf{q},\mathbf{p},\lambda,\tau) = \begin{bmatrix} i\lambda^2/2 + i\mathbf{q}^2 + i\tau/2 & \lambda\mathbf{q} - i\mathbf{p} - \theta/\lambda \\ \lambda\mathbf{q} + i\mathbf{p} - \theta/\lambda & -i\lambda^2/2 - i\mathbf{q}^2 - i\tau/2 \end{bmatrix} \\ \dfrac{\partial}{\partial t}\Phi = M(\mathbf{q},\mathbf{p})\Phi, \\ M(\mathbf{q},\mathbf{p};\lambda) = \mathcal{B}(\mathbf{q},\mathbf{p},\lambda,\tau) = \begin{bmatrix} i\lambda/2 & \mathbf{q} \\ \mathbf{q} & -i\lambda/2, \end{bmatrix} \end{cases}
$$

$$
L_t + [L, M] = 0 \quad \Rightarrow \quad \begin{cases} \dot{\mathbf{q}} = A(\mathbf{q},\mathbf{p}) = \mathbf{p}, \\ \dot{\mathbf{p}} = B(\mathbf{q},\mathbf{p}) = 2\mathbf{q}^3 + \tau\mathbf{q} + \theta, \end{cases}
$$

$$
H = \mathrm{Tr}\left(\frac{\mathbf{p}^2}{2} - \frac{1}{2}\left(\mathbf{q}^2 + \frac{\tau}{2}\right)^2 - \theta\mathbf{q}\right).
$$

Since the phase space is the same as in the non-autonomous case, the moment map is the same as in the previous cases. Since the Hamiltonian in (4.2) is an autonomous version of the matrix Painlevé II Hamiltonian, it is also invariant under the group action, and one can obtain dual systems as a reduction at (P, Q) and (Φ, Λ) in the form

$$
H_{II}^{(\mathrm{red})} = \sum_{i=1}^{n}\left[\frac{p_i^2}{2} - \frac{1}{2}\left(q_i^2 + \frac{\tau}{2}\right)^2 - \theta q_i\right] + g^2\sum_{i<j}\frac{1}{(q_i - q_j)^2}
$$

and

$$
\begin{aligned}
H_{II}^{(\mathrm{dual})} &= \sum_{i=1}^{n}\left[\frac{p_i^2}{2} - \frac{1}{2}\left(q_i^2 + \frac{\tau}{2}\right)^2 - \theta q_i\right] \\
&+ 2g^2\sum_{i<j}\frac{q_i^2 + q_iq_j + q_j^2 + \tau/2}{(p_i - p_j)^2} - g^4\sum_{i<j}\frac{1}{(p_i - p_j)^4} \\
&- 2g^4\left(\sum_{i<j<k}\frac{1}{(p_i - p_j)^2(p_j - p_k)^2} + \sum_{i<j<k<l}\frac{2}{(p_i - p_j)(p_j - p_k)(p_k - p_l)(p_l - p_i)}\right),
\end{aligned}
$$

with the following Lax pair

$$
\begin{cases} \tilde{L}\Psi = \mu\Psi, \\ \frac{\partial}{\partial t}\Psi = (\tilde{M} - F \otimes \mathbf{1}_2)\Psi, \end{cases}
$$

where $\tilde{L}$ and $\tilde{M}$ are the images of initial (L, M)-pair after the gauge transformation which sends $(\mathbb{q}, \mathbb{p})$ to the reduced coordinates (Q, P) or the dual ones (Φ, Λ). The matrix F for the reduced and dual cases can be computed as

$$
(q_i - q_j)^2 F_{i,j}^{(\mathrm{red})} = ([A(Q, P), Q])_{i,j}, \quad (p_i - p_j)^2 F_{i,j}^{(\mathrm{dual})} = ([B(\Phi, \Lambda), \Lambda])_{i,j}
$$

$$
F_{j,j} = -\sum_{k\neq j} F_{j,k} + \frac{1}{n}\sum_{l\neq k} F_{l,k}
$$

The computations of matrices $F^{(\mathrm{red})}$ and $F^{(\mathrm{dual})}$ are similar to the computations provided in Lemma 2.5 of the work [8] by M. Bertola, M. Cafasso and V. Rubtsov. Explicitly, the L-matrices for the reduced and dual cases take the form

$$L^{(\mathrm{red})} = P\otimes\sigma_2+\sqrt{-1}\left(\frac{\lambda^2}{2}+Q^2+\frac{\tau}{2}\right)\otimes\sigma_3+\left(\lambda Q-\frac{\theta}{\lambda}\right)\otimes\sigma_1,$$
$$L^{(\mathrm{dual})} = \Lambda\otimes\sigma_2+\sqrt{-1}\left(\frac{\lambda^2}{2}+\Phi^2+\frac{\tau}{2}\right)\otimes\sigma_3+\left(\lambda \Phi-\frac{\theta}{\lambda}\right)\otimes\sigma_1,$$

where σ_1, σ_2 and σ_3 are the Pauli matrices.

4.2. Painlevé–Calogero I system

Freezing t in the isomonodromic problem (3.5), we obtain the following autonomous Hamiltonian system:

$$\begin{cases} L(\mathbf{q},\mathbf{p})\Phi=\mu\Phi \\ L(\mathbf{q},\mathbf{p};\lambda)=\mathcal{A}(\mathbf{q},\mathbf{p},\lambda,\tau)=\begin{bmatrix} \mathbf{p} & \lambda-\mathbf{q} \\ \lambda^2+\lambda\mathbf{q}+\mathbf{q}^2+\tau/2 & -\mathbf{p}\end{bmatrix} \\ \dfrac{\partial}{\partial t}\Phi=M(\mathbf{q},\mathbf{p})\Phi \\ M(\mathbf{q},\mathbf{p};\lambda)=\mathcal{B}(\mathbf{q},\mathbf{p},\lambda,\tau)=\begin{bmatrix} 0 & 1/2 \\ \lambda/2+\mathbf{q} & 0\end{bmatrix}\end{cases}$$

$$L_t+[L,M]=0 \quad\Rightarrow\quad \begin{cases} \dot{\mathbf{q}} &= A(\mathbf{p},\mathbf{q}) &= \mathbf{p} \\ \dot{\mathbf{p}} &= B(\mathbf{p},\mathbf{q}) &= \frac{3}{2}\mathbf{q}^2+\tau/4\end{cases}$$

$$H^{(\mathrm{aut})}=\mathrm{Tr}\left(\frac{\mathbf{p}^2}{2}-\frac{\mathbf{q}^3}{2}-\frac{\tau}{4}\mathbf{q}\right)$$

The reduced systems takes the form

$$H_{\mathrm{I}}^{(\mathrm{red})}=\sum_{i=1}^{n}\left[\frac{p_i^2}{2}-\frac{q_i^3}{2}-\frac{\tau}{4}q_i\right]+g^2\sum_{i<j}\frac{1}{(q_i-q_j)^2},$$
$$H_{I}^{(\mathrm{dual})}=\sum_{i=1}^{n}\left[\frac{p_i^2}{2}-\frac{q_i^3}{2}-\frac{\tau}{4}q_i\right]-\frac{3g^2}{2}\sum_{j<i}\frac{q_i+q_j}{(p_i-p_j)^2}.$$

The L-matrix has the following form:

$$L^{(\mathrm{red})}=P\otimes\sigma_3+(\lambda-Q)\otimes\sigma_++(\lambda^2+\frac{\tau}{2}+\lambda Q+Q^2)\otimes\sigma_-,$$
$$L^{(\mathrm{dual})}=\Lambda\otimes\sigma_3+(\lambda-\Phi)\otimes\sigma_++(\lambda^2+\frac{\tau}{2}+\lambda \Phi+\Phi^2)\otimes\sigma_-,$$

where σ_3, σ_+ and σ_- are the elements of the Cartan–Weyl basis of $\mathfrak{sl}(2,\mathbb{C})$.

4.3. The Painlevé–Calogero correspondence. The degeneration of the rational Inozemtsev system

A well-known confluence scheme for the Painlevé equations [39, 42] also holds for the matrix Painlevé systems. From the point of view of the reduced Painlevé–Calogero systems for the matrix Painlevé VI, V and IV systems, the confluence scheme is the non-autonomous version of the degeneration for the Inozemtsev Hamiltonian systems. In previous sections, we have shown that the corresponding autonomous multi-particle systems for the Painlevé–Calogero I and II systems may be written in the Lax form. According to Takasaki [43], the multi-particle Painlevé–Calogero I and II systems have to be a further degeneration of the rational Inozemtsev Hamiltonian system.

This degeneration may be obtained by the autonomization of the confluence scheme for the Painlevé equations, combined with the symplectic transformations given in [8]. In the case of the Painlevé–Calogero II and I systems, we obtain physical Hamiltonians $\sum_i p_i^2/2 + V(q_i, t)$ by reduction without any additional canonical transformations.

In this section, we will use $\mathbf{q}^{(\mathrm{IV})}$ and $\mathbf{q}^{(\mathrm{II})}$ for the non-commutative canonical coordinates for the matrix Painlevé IV and the matrix Painlevé II systems, respectively. We use same notation for the non-commutative moments and for the canonical coordinates of the reduced systems.

Theorem 4.2. *The confluence map from the matrix Painlevé IV system* (3.1) *to the matrix Painlevé II system* (3.3) *holds for the Painlevé–Calogero multi-particle systems, but fails for the dual systems* (3.2) *and* (3.4).

Proof. To provide the degeneration procedure from the matrix Painlevé IV systems to the matrix Painlevé II system we exploit the confluence formula i.e., a symplectic map for the extended phase space which contains a parameter ε. Taking the limit $\varepsilon \to 0$, we obtain the resulting confluent Hamiltonian.

To obtain (3.3) from (3.1), we use the following confluence symplectic map:

$$\begin{aligned}
&\mathbf{q}^{(\mathrm{IV})} = -\frac{1}{\varepsilon^3}\Big(\frac{1}{2} + \varepsilon^2 \mathbf{q}^{(\mathrm{II})}\Big), \quad \mathbf{p}^{(\mathrm{IV})} = -\varepsilon\big(\mathbf{p}^{(\mathrm{II})} + \big(\mathbf{q}^{(\mathrm{II})}\big)^2 + t^{(II)}/2\big),\\
&t^{(\mathrm{IV})} = \frac{1}{\varepsilon^3}\big(1 - \varepsilon^4 t^{(\mathrm{II})}\big), \quad \theta_0 = -\frac{1}{4\varepsilon^6}, \quad \theta_1 = -\theta + \frac{1}{4\varepsilon^6},\\
&H^{(\mathrm{II})} = -\varepsilon H^{(\mathrm{IV})} + \varepsilon^{-2}\frac{\theta}{2}.
\end{aligned} \tag{4.3}$$

We see that (4.3) is indeed a symplectic map

$$\begin{aligned}
\omega^{(\mathrm{IV})} &= \operatorname{Tr} \mathrm{d}\,\mathbf{p}^{(\mathrm{IV})} \wedge \mathrm{d}\,\mathbf{q}^{(\mathrm{IV})} = \operatorname{Tr}\big(\mathrm{d}\,\mathbf{p}^{(\mathrm{II})} \wedge \mathrm{d}\,\mathbf{q}^{(\mathrm{II})} + \mathrm{d}(\mathbf{q}^{(\mathrm{II})})^2 \wedge \mathrm{d}\,\mathbf{q}^{(\mathrm{II})}\big)\\
&= \operatorname{Tr} \mathrm{d}\,\mathbf{p}^{(\mathrm{II})} \wedge \mathrm{d}\,\mathbf{q}^{(\mathrm{II})} = \omega^{(\mathrm{II})},
\end{aligned}$$

since

$$\begin{aligned}
\operatorname{Tr}(\mathrm{d}\,\mathbf{q}^2 \wedge \mathrm{d}\,\mathbf{q}) &= \sum_{i,k,l} (\mathrm{d}(q_{ik}q_{kl}) \wedge \mathrm{d}\,q_{li})\\
&= \sum_{i,k,l} \underbrace{q_{ik}\,\mathrm{d}\,q_{kl} \wedge \mathrm{d}\,q_{li}}_{i=\alpha,k=\beta,l=j} + \underbrace{q_{kl}\,\mathrm{d}\,q_{ik} \wedge \mathrm{d}\,q_{li}}_{k=\alpha,l=\beta,i=j}\\
&= \sum_{\alpha,\beta} q_{\alpha\beta} \sum_j \big[\mathrm{d}\,q_{\beta j} \wedge \mathrm{d}\,q_{j\alpha} + \mathrm{d}\,q_{j\alpha} \wedge \mathrm{d}\,q_{\beta j}\big] = 0
\end{aligned}$$

and

$$\mathrm{d}H^{\mathrm{IV}} \wedge \mathrm{d}t^{\mathrm{IV}} = \mathrm{d}H^{\mathrm{II}} \wedge \mathrm{d}t^{\mathrm{II}}.$$

The function

$$-\varepsilon H^{(\mathrm{IV})} + \varepsilon^{-2}\,\frac{\theta}{2}$$

after the limit becomes

$$H = \operatorname{Tr}\Big(\frac{\mathbf{p}^2}{2} - \frac{1}{2}\Big(\mathbf{q}^2 + \frac{t}{2}\Big)^2 + \theta\mathbf{q}\Big),$$

after the transformation (4.3) and the limit $\varepsilon \to 0$. Here we strip off all superscripts in the Hamiltonian. The described transformation may be rewritten for the multi-particle Painlevé–Calogero IV system,

$$\begin{aligned} H_{\mathrm{IV}}^{(\mathrm{red})} = {} & \sum_{i=1}^{n} \big[q_i^{(\mathrm{IV})}(p_i^{(\mathrm{IV})})^2 - p_i^{(\mathrm{IV})}(q_i^{(\mathrm{IV})})^2 - t p_i^{(\mathrm{IV})} q_i^{(\mathrm{IV})} + \theta_0 q_i^{(\mathrm{IV})} - (\theta_0+\theta_1) p_i^{(\mathrm{IV})}\big] \\ & + g^2 \sum_{i<j} \frac{q_i^{(\mathrm{IV})} + q_k^{(\mathrm{IV})}}{(q_i^{(\mathrm{IV})} - q_j^{(\mathrm{IV})})^2}, \end{aligned}$$

in the following way:

$$q_i^{(\mathrm{IV})} = -\frac{1}{\varepsilon^3}\Big(\frac{1}{2} + \varepsilon^2 q_i^{(\mathrm{II})}\Big), \qquad p_i^{(\mathrm{IV})} = -\varepsilon\Big(p_i^{(\mathrm{II})} + (q_i^{(\mathrm{II})})^2 + \frac{t}{2}\Big)$$

with the same re-scaling for the constants, time, and Hamiltonian as in (4.3). The diagonal part of the Painlevé–Calogero IV Hamiltonian is a sum of uncoupled Painlevé IV Hamiltonians, so in the limit $\varepsilon \to 0$, the diagonal part transforms into the sum of uncoupled Painlevé II Hamiltonians. For the interaction term, we have

$$\begin{aligned} -\varepsilon g^2 \sum_{i<j} \frac{q_i^{(\mathrm{IV})} + q_j^{(\mathrm{IV})}}{(q_i^{(\mathrm{IV})} - q_j^{(\mathrm{IV})})^2} &= \sum_{i<j} \varepsilon^2 g^2 \frac{q_i^{(\mathrm{II})} + q_j^{(\mathrm{II})}}{(q_i^{(\mathrm{II})} - q_j^{(\mathrm{II})})^2} + \frac{g^2}{(q_i^{(\mathrm{II})} - q_j^{(\mathrm{II})})^2} \\ &\to \sum_{i<j} \frac{g^2}{(q_i^{(\mathrm{II})} - q_j^{(\mathrm{II})})^2}, \end{aligned}$$

so the Hamiltonian of the Painlevé–Calogero IV system transforms into the Hamiltonian of the Painlevé–Calogero II system. However, this reduction of the symplectic map (4.3) cannot be applied to the dual systems. Indeed, this confluence on the Painlevé–Calogero side is a consequence of the linearity of the transformation (4.3) with respect to $\mathbf{q}$. We see that the eigenspace of $\mathbf{q}^{(\mathrm{IV})}$ coincides with the eigenspace of $\mathbf{q}^{(\mathrm{II})}$, so we reduce from the same point of the $\mathrm{GL}(n,\mathbb{C})$-orbit. On the other hand, on the dual side we have that dual reduction (diagonalization of $\mathbf{p}^{(\mathrm{IV})}$) for the matrix Painlevé IV system implies transition to the eigenbasis of $\mathbf{p}^{(\mathrm{II})} + (\mathbf{q}^{(\mathrm{II})})^2$, which does not coincide with the dual reduction for the matrix Painlevé II system. So the dual systems are obtained from the different points of the $\mathrm{GL}(n,\mathbb{C})$-orbit. This fact is an obstacle for this degeneration for the dual systems. ■

Remark 4.3. Here we use the combination of the polynomial canonical transformation with the confluence transformation given in [39, 42], which is "linear" both for $\mathbf{q}$ and $\mathbf{p}$, and is given by

$$\begin{gathered}\mathbf{q}^{(\mathrm{IV})}=-\frac{1}{\varepsilon^{3}}\left(\frac{1}{2}+\varepsilon^{2}\tilde{\mathbf{q}}^{(\mathrm{II})}\right),\quad \mathbf{p}^{(\mathrm{IV})}=-\varepsilon\tilde{\mathbf{p}}^{(\mathrm{II})},\quad t^{(\mathrm{IV})}=\frac{1}{\varepsilon^{3}}\left(1-\varepsilon^{4}t^{(\mathrm{II})}\right)\\ \theta_0=-\frac{1}{4\varepsilon^{6}},\quad \theta_1=-\theta+\frac{1}{4\varepsilon^{6}},\quad \tilde{H}^{(\mathrm{II})}=-\varepsilon H^{(\mathrm{IV})}+\varepsilon^{-2}\frac{\theta}{2}.\end{gathered} \tag{4.4}$$

This transformation maps the Hamiltonian of the matrix Painlevé IV system to the following Hamiltonian:

$$\tilde{H}^{(\mathrm{II})}=\operatorname{Tr}\left(\frac{1}{2}\tilde{\mathbf{p}}\left(\tilde{\mathbf{p}}-2\tilde{\mathbf{q}}^{2}-t\right)-\theta\tilde{\mathbf{q}}\right), \tag{4.5}$$

the Painlevé–Calogero reduction of which takes the form

$$\sum_i\left[\frac{p_i^2}{2}-p_iq_i^2-\frac{t}{2}p_i-\theta p_i\right]+g^2\sum_{i<j}\frac{1}{(q_i-q_j)^2},$$

and the dual reduction is

$$\sum_i\left[\frac{p_i^2}{2}-p_iq_i^2-\frac{t}{2}p_i-\theta q_i\right]-g^2\sum_{i\neq j}\frac{p_i}{(p_i-p_j)^2}.$$

Since (4.4) is linear in $\mathbf{q}$ and $\mathbf{p}$, this confluence holds also for both the Painlevé–Calogero and the dual systems. The Hamiltonian system $\tilde{H}^{(\mathrm{II})}$ is canonically equivalent to (3.3), due to to the symplectic map

$$\tilde{\mathbf{q}}=\mathbf{q},\quad \tilde{\mathbf{p}}=\mathbf{p}+\mathbf{q}^{2}+t/2, \tag{4.6}$$

which is an obstacle for the degeneration of the dual systems. The degeneration to the matrix Painlevé I systems is given by the "linear" maps in $\mathbf{q}$ and $\mathbf{p}$ from the $\tilde{H}^{(\mathrm{II})}$ system. To obtain the confluence from $H^{(\mathrm{II})}$ given in (3.3), we need to use an inverse of (4.6) combined with the confluence which leaves the transformation linearity only for $\mathbf{q}$. So that degeneration to the matrix Painlevé I system holds for the Painlevé–Calogero systems, but not for the dual systems.

By fixing $t=\tau$, we get the autonomous version of the confluence scheme which provides the degeneration of the rational Inozemtsev system to the rational Calogero-Painlevé II and I systems.

The obtained multi-particle systems are further degenerations of the Inozemtsev systems. The arising dual systems look novel and need more detailed analysis. In case of the multi-particle Painlevé–Calogero II system, there is an interesting interpretation of this de-autonomization which arises from the symmetry reductions for the matrix modified Korteweg–de Vries equation. We discuss this connection in details in Section 5.

5. Matrix modified Korteweg–de Vries equation and the Painlevé–Calogero correspondence for the Painlevé II system

The matrix modified Korteweg–de Vries (MmKdV) equation has the following form [29]:

$$\mathbf{u}_t = \mathbf{u}_{xxx} + 3\,[\mathbf{u}_{xx}, \mathbf{u}] - 6\,\mathbf{u}\,\mathbf{u}_x\,\mathbf{u}. \tag{5.1}$$

In this section, we are going to show the connection between the traveling wave reduction of MmKdV and its self-similar reduction. The self-similar reduction is the following change of variables:

$$\mathbf{u}(x,t) = \frac{\mathbf{v}(z)}{(3t)^{1/3}}, \quad \text{with} \quad z = \frac{x}{(3t)^{1/3}}. \tag{5.2}$$

This change of variables leads to

$$\partial_t = -\frac{x}{(3t)^{4/3}}\partial_z \quad \text{and} \quad \partial_x = \frac{1}{(3t)^{1/3}}\partial_z,$$

and equation (5.1) turns into

$$\mathbf{v}_{zzz} + 3\,[\mathbf{v}, \mathbf{v}_{zz}] - 6\,\mathbf{v}\,\mathbf{v}_z\,\mathbf{v} + (z\,\mathbf{v})_z = 0. \tag{5.3}$$

Let us change the variable z to t, in order to return to our previous notation. It can be shown that every solution of the matrix Painlevé II equation solves equation (5.3) (but not vice versa), since (5.3) may be written, see [5, 21], as

$$(\partial_t + 3\,\mathrm{ad}_{\mathbf{v}} + 2\,\mathrm{ad}_{\mathbf{v}} \circ \partial_t^{-1} \circ \mathrm{ad}_{\mathbf{v}}) \circ (\mathbf{v}_{tt} - 2\,\mathbf{v}^3 + t\,\mathbf{v} + \theta) = 0, \tag{5.4}$$

where

$$\mathrm{ad}_{\mathbf{v}} \circ A = [\mathbf{v}, A].$$

After changing $t \to -t$ and $\theta \to -\theta$, equation (5.4) becomes the matrix Painlevé II equation. On the other hand, the travelling wave reduction of the MmKdV equation,

$$\mathbf{u}(x,t) = \mathbf{v}(z), \quad z = x - \omega t,$$

has the form

$$(\partial_z - 3\,\mathrm{ad}_{\mathbf{v}} + 2\,\mathrm{ad}_{\mathbf{v}} \circ \partial_z^{-1} \circ \mathrm{ad}_{\mathbf{v}}) \circ (\mathbf{v}_{zz} - 2\,\mathbf{v}^3 + \omega\mathbf{v} + \theta) = 0, \tag{5.5}$$

where ω is an arbitrary constant. Let us change z to t and ω to τ in order to show the connection with equation (5.5). The equation turns to

$$\mathbf{v}_{tt} + 2\,\mathbf{v}^3 + \tau\mathbf{v} + \theta = 0,$$

and can be written as the following Hamiltonian system:

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{p} \\ \dot{\mathbf{p}} = 2\mathbf{q}^3 + \tau\mathbf{q} + \theta \end{cases}, \quad H_{\mathrm{PII}}(\mathbf{q}, \mathbf{p}, \tau) = \operatorname{Tr}\Big(\frac{\mathbf{p}^2}{2} - \frac{1}{2}\Big(\mathbf{q}^2 + \frac{\tau}{2}\Big)^2 - \theta\mathbf{q}\Big).$$

Thus the matrix Painlevé II equation (which is the special case of the self-similar reduction of the MmKdV equation) is a τ-deformation of the special case of travelling wave reduction of the MmKdV equation. This Painlevé–Calogero correspondence can be lifted down to the multi-particle systems which are obtained by Hamiltonian reduction. Finally, we have the following diagram:

$$\begin{tikzcd}
H_{\mathrm{II}}(p_i, q_i, t) \arrow[rrrr, leftrightarrow, dotted, "\text{Painlevé–Calogero}", "\text{correspondence}"'] \arrow[dd, leftrightarrow, "\text{Spectral duality}"'] & & & & H_{\mathrm{II}}(p_i, q_i, \tau) \arrow[dd, leftrightarrow, "\text{Spectral duality}"] \\
& H_{\mathrm{II}}(\mathbf{p}, \mathbf{q}, t) \arrow[ul, "\mathbf{q}\text{-diag}"'] \arrow[dl, "\mathbf{p}\text{-diag}"] & \mathrm{MmKdV} \arrow[l, "\text{self-similar}", "\text{reduction}"'] \arrow[r, "\text{travelling wave}", "\text{reduction}"'] & H_{\mathrm{II}}(\mathbf{p}, \mathbf{q}, \tau) \arrow[ur, "\mathbf{q}\text{-diag}"] \arrow[dr, "\mathbf{p}\text{-diag}"'] & \\
H_{\mathrm{II}}^{(\mathrm{dual})}(p_i, q_i, t) \arrow[rrrr, leftrightarrow, dotted, "\text{Painlevé–Calogero}", "\text{correspondence}"'] & & & & H_{\mathrm{II}}^{(\mathrm{dual})}(p_i, q_i, \tau)
\end{tikzcd}$$

The following situation is a result of the existence of special symmetries of integrable PDEs, which are formulated in the next theorem.

Theorem 5.1. *For all integrable evolution PDEs which allow the self-similar reduction of the form*

$$u(x,t) = \frac{v(z)}{t^\beta}, \quad z = \frac{x}{t^\beta},$$

and also allow the travelling wave reduction

$$u = w(z), \quad z = x - \omega t,$$

the self-similar reduction is a ω-deformation of the travelling wave reduction.

Proof. By an integrable evolution PDE, we mean the following equation:

$$u_t + \partial_x F(u, u_x, u_{xx}, \dots) = 0.$$

The existence of the travelling wave solution means that the equation is invariant under translations, so F does not depend on x. The self-similar reduction means that there exist a solution of the form

$$u = \frac{v(z)}{t^\beta}, \quad \text{with} \quad z = \frac{x}{t^\alpha}.$$

The transformation above leads to the following expressions for partial derivatives:

$$u_t = -\frac{\beta}{t^{\beta+1}}\,(v + \frac{\alpha}{\beta} z v_z) \quad \text{and} \quad \partial_x^n u = \frac{1}{t^{\beta+n\alpha}}\,\frac{d^n v(z)}{dz^n}.$$

The equation transforms into

$$\frac{\beta}{t^{\beta+1}}\left(v + \frac{\alpha}{\beta}\, z v_z\right) - \frac{1}{t^\alpha}\,\frac{d}{dz} F(v/t^\beta, v_z/t^{\beta+\alpha}, \dots, v^{(n)}/t^{\beta+n\alpha}, \dots) = 0.$$

Existence of the self-similar reduction implies that the solution $v(z)$ does not depend on t, which means that the function F transforms in such a way that the t-variable factors out. This implies that F is quasi-homogeneous,

$$F(u, u_x, \dots) = \frac{F(v, v_z, \dots)}{t^{\beta+1-\alpha}},$$

and the reduction reads as

$$\beta\Big(v + \frac{\alpha}{\beta} z v_z\Big) - \partial_z F(v, v_z, \dots) = 0. \tag{5.6}$$

In the case when $\alpha = \beta$, one may integrate equation (5.6) and get

$$F(v, v_z, v_{zz}, \dots) - \beta z v = C.$$

Here, C is a constant of integration. On the other hand, the travelling wave reduction $u = w(z = x - \omega t)$ takes the form

$$F(v, v_z, v_{zz}, \dots) - \omega v = C$$

and we see that the self-similar reduction is the deformation of the travelling wave reduction with respect to ω, i.e., ω switches to βz. ∎

6. Further remarks and open questions

This paper is a small first step in the study of the dualities for non-autonomous many-particle systems. Our examples and computations raise a natural question of *Liouville and quantum integrability* for the obtained dual rational many-particle systems. Some of them are integrable ad hoc, but the integrability of other examples is not clear at all.

We note that in all our examples of the Hamiltonian reduction procedure for the Painlevé–Calogero Hamiltonian systems, we have always used the cotangent bundle of the Lie algebras $\mathfrak{gl}_n(\mathbb{C})$ for their phase space, and the momentum mapping is given by the matrix commutator $\mathcal{M} = [P, Q]$. At the same time, the Hamiltonian reductions for their Calogero–Moser prototypes can be described with various generalizations of this phase space and the momentum map: the group-like half-commutator $\mathcal{M} = PQP^{-1} - Q$ on T^*GL_n, or the "full"group-commutator $\mathcal{M} = PQP^{-1}Q^{-1}$ on the Heisenberg double $G \times G$. The existence of the isomonodromic systems on T^*GL_n and on the Heisenberg double $G \times G$ is an open problem which leads to the following question: how can we extend our dualities to the trigonometric and to the elliptic Painlevé–Calogero Hamiltonians? The Calogero–Moser–Ruijsenaars–Schneider duality table dictates a necessity of the existence for some "Ruijsenaars–Painlevé correspondence", while on the Painlevé side one should expect the appearance of some discrete or q-Painlevé systems. Some very encouraging computations show that there exists a (non-autonomous) canonical transformation which provides a direct formula linking the Hamiltonians $H_{\mathrm{PIII}}^{D_8}$ and $H_{\mathrm{Ruijsenaars}}^{\mathrm{rat}}$ and their multi-particle reduced forms illustrated the spectral duality in this case. This result and its further developments are the subject of a future paper.

Another strong evidence of the existence of this extension is also based on the one-to-one correspondence between the so-called BC_n Inozemtsev model with the Hamiltonian H_{BC_n} described above,

$$\sum_{j=1}^{n}\Big(\frac{p_j^2}{2}-\sum_{\ell=0}^{3}\kappa_\ell(\kappa_\ell+1)\,\wp(q_j+\omega_\ell)\Big)-\frac{\kappa(\kappa+1)}{2}\sum_{j\neq k}(\wp(q_j-q_k)+\wp(q_j+q_k)).$$

It is known, see [44], that the BC_n Inozemtsev model is a universal completely integrable model of quantum mechanics; in that paper, the correspondence between the BC_n Ruijsenaars–van Diejen systems and the BC_n Inozemtsev system was established. One should take into account the recent result of Takemura [45], who studied an analogue of the well-known relation between the Painlevé VI and the Heun differential systems for difference equations. He proposed, in particular, the correspondence between the elliptic difference Painlevé equations and the one-variable Ruijsenaars–van Diejen difference equation, regarded as a difference analogue of the Heun equation. He also proved that (degenerated) Rujisenaars–van Diejen operators of one variable are special cases of the linear q-difference equations related to certain q-Painlevé VI equations by a connection preserving deformation. Another interesting direction is to investigate the adaptation of the Inozemtsev limit to the non-autonomous multi-particle systems, in order to obtain new versions of the non-autonomous Toda lattice, in the same way as for autonomous systems (see [13, 47]).

Our next goal is to find and to study an appropriate general version of the Hamiltonian reduction and an analogue of the Ruijsenaars duality in the framework of this conjectural "Ruijsenaars–Painlevé correspondence", which should include the correspondence between the difference systems like the elliptic Ruijsenaars dual to the elliptic Calogero–Moser and the fabulous double-elliptic (DELL) ([34]) and the elliptic Rains Painlevé system and its various degenerations. The first interesting results in this direction were very recently obtained by Noumi, Ruijsenaars and Yamada in [38]: they showed that the 8-parameter elliptic Sakai difference Painlevé equation can be presented in a Lax-like form which can be specified as non-autonomous; this gives Schrödinger equation for the BC_1 8-parameter Ruijsenaars–van Diejen difference Hamiltonian.

Another intriguing and challenging problem is a transition of the described dualities to some close domains – such that the theory of (q-)special functions. We suppose that there are some interesting links between our *classical* dualities and their *quantum* counterparts, which we did not consider in this paper. In particular, we expect the existence of a close connection between the quantum version of our dualities and the dualities described by Koornwinder and Mazzocco [30] in the case of q-Askey scheme and the degenerate DAHA. We are going to clarify it and fill some gaps in one of the tables from [12]. In this direction, it would be highly ambitious and interesting to understand the conjectural correspondence for the trigonometric degeneration of the difference Ruijsenaars–Shneider systems and its duality with the q-Knizhnik–Zamolodchikov equations or, more generally, in the framework of the q-Langlands correspondence (see, e.g. [31]).

We finish the survey of possible applications of our computational observations with more general open questions related to the end of Section 5. This is the condition for symmetries of PDE which leads to the situation when one reduction is deformation of another. Besides for integrable PDEs lifting of this symmetries to the Lax pairs is also an essential issue in this problem. We also believe that these conditions for the self-similar reduction may be lifted to the non-commutative case, and to more general classes of equations.

A. Appendix: Calculation of interaction terms

In case of the multi-particle dual Painlevé systems, we obtain Hamiltonians which contain terms with more than two-particle interactions. This comes from the Q^3 and Q^4 terms in the matrix Hamiltonians, where Q is the rational Calogero Lax operator

$$Q = \delta_{ij} q_i + (1 - \delta_{ij}) \frac{\sqrt{-1} g}{p_i - p_j}.$$

In case of a fixed size of the matrix Q (the number of the particles), the calculation of the traces of any power of Q is a straightforward problem, which may be solved for example with the help of a computer algebra system. In case of an arbitrary size of Q, this computation is not a big problem either, but the process may be tedious.

Here we present a simple approach for a calculation of such interaction terms. We denote by n a number of particles (the size of Q), and by g a coupling constant. Since Q is a linear matrix function of g, the trace of Q^l is a polynomial in g of degree less than l:

$$\text{(A.1)} \qquad \operatorname{Tr} Q^l = \sum_{k=0}^{l} \frac{g^k}{k!} F_k, \quad \text{with} \quad F_k = \frac{d^k}{dg^k} \operatorname{Tr}(Q^l)\Big|_{g=0} = \operatorname{Tr}\Big(\frac{d^k}{dg^k} Q^l \Big|_{g=0} \Big).$$

To compute coefficients, we use the following technical lemma.

Lemma A.1. *The polynomial* $\operatorname{Tr} Q^l$ *is even for all* l*, i.e.,*

$$\operatorname{Tr} Q^l = \sum_{k=0}^{[l/2]} \frac{g^{2k}}{2k!} F_{2k},$$

where $[l/2]$ *is equal to* $l/2 - 1/2$ *in the case when* l *is odd, and to* $l/2$ *when* l *is even.*

Proof. In the upcoming calculations, we will use the following notation:

$$Q|_{g=0} = \delta_{ij}\, q_i = D \quad \text{and} \quad \frac{dQ}{dg} = (1 - \delta_{ij}) \frac{\sqrt{-1}}{p_i - p_j} = A.$$

Each odd coefficient F_{2r-1} of the polynomial (A.1) takes the form

$$\text{(A.2)} \qquad F_{2r-1} = \sum_{0 < i_1 < \cdots < i_{2r-1} \le l} \operatorname{Tr} E_{i_1, i_2 \dots i_{2r-1}},$$

where E is given by

$$E_{i_1, i_2 \dots, i_{2r-1}} = D^{i_1 - 1} A D^{i_2 - i_1 - 1} A \cdots D^{i_k - i_{k-1} - 1} A^{i_k} \cdots D^{i_{2r-1} - i_{2r-2} - 1} A D^{l - i_{2r-1}}$$

Since A is a symmetric matrix, D is a skew-symmetric matrix, and all the E's contain an odd number of D's, we have

$$(E_{i_1,i_2,\ldots,i_{2r-1}})^{\mathrm{T}} = -E_{l-i_{2r-1},i_{2r-2},\ldots,i_{2r-i_1}}$$

On the other hand, the trace is invariant under transpositions, so there are two cases. The first one is

$$E_{i_1,i_2,\ldots,i_{2r-1}} \neq E_{l-i_{2r-1},i_{2r-2},\ldots,i_{2r-i_1}},$$

so for each sequence $(i_1, i_2, \ldots, i_{2r-1})$ in the sum (A.2), we have the "mirror" sequence $(l - i_{2r-1}, l - i_{2r-2}, \ldots, l - i_{2r-i_1})$, such that

$$\operatorname{Tr} E_{i_1,i_2,\ldots,i_{2r-1}} + \operatorname{Tr} E_{l-i_{2r-1},i_{2r-2},\ldots,i_{2r-i_1}} = 0,$$

so they don not contribute to (A.1). The second case is

$$E_{i_1,i_2,\ldots,i_{2r-1}} = E_{l-i_{2r-1},i_{2r-2},\ldots,i_{2r-i_1}}.$$

In this case, $E_{i_1,i_2,\ldots,i_{2r-1}}$ is a skew-symmetric matrix, so it does not contribute to (A.2). ■

Finally, we provide the calculations of the traces for Q^3 and Q^4. In the case $l = 3$, we have

$$\begin{aligned}\operatorname{Tr} Q^3 &= \operatorname{Tr}(D^3) + 3g^2 \operatorname{Tr}(AAD) = \sum_{i=1}^{n} q_i^3 + 3g^2 \sum_{i\neq j} \frac{q_i}{(p_i - p_j)^2} \\ &= \sum_{i=1}^{n} q_i^3 + 3g^2 \sum_{i<j} \frac{q_i + q_j}{(p_i - p_j)^2}.\end{aligned}$$

For $l = 4$, the expansion takes the form

$$\operatorname{Tr} Q^4 = \operatorname{Tr}(D^4) + 2g^2 \left[2 \operatorname{Tr}(D^2 A^2) + \operatorname{Tr}(DADA)\right] + g^4 \operatorname{Tr}(A^4).$$

Here we have

$$\begin{aligned}\operatorname{Tr}(D^2 A^2) &= \sum_{i\neq j} \frac{q_i^2}{(p_i - p_j)^2} = \sum_{i<j} \frac{q_i^2 + q_j^2}{(p_i - p_j)^2}, \\ \operatorname{Tr}(DADA) &= \sum_{i\neq j} \frac{q_i q_j}{(p_i - p_j)^2} = 2\sum_{i<j} \frac{q_i q_j}{(p_i - p_j)^2}.\end{aligned}$$

The last g^4-term takes the form

$$\operatorname{Tr}(A^4) = \sum_{i\neq j\neq k\neq l\neq i} \frac{1}{(p_i - p_j)(p_j - p_k)(p_k - p_l)(p_l - p_i)},$$

and there are three possibilities: (a) $k = i, l = j$, (b) $l = j$, and (c) all indices i, j, k, l different. This leads to the following interaction term:

$$\begin{aligned}\operatorname{Tr}(A^4) = &\sum_{i<j} \frac{2}{(p_i - p_j)^4} + \sum_{i<j<k} \frac{4}{(p_i - p_j)^2 (p_j - p_k)^2} \\ &+ \sum_{i<j<k<l} \frac{8}{(p_i - p_j)(p_j - p_k)(p_k - p_l)(p_l - p_i)}.\end{aligned}$$

Acknowledgements. The authors are grateful to M. Cafasso, Yu. Chernyakov, M.Fairon, N. Nekrasov, E. Zenkevich and A. Zotov for helpful discussions. Our special thanks to M. Mazzocco, who carefully read our draft and made several useful remarks. I. Gaiur is grateful to M. Mazzocco for numerous and helpful discussions of the Hamiltonian reduction for autonomous and non-autonomous systems. V. Rubtsov thanks A. Alekseev and S. Hochloh for invitation to the Ascona conference on integrable systems and useful discussion. The authors also thank the Euler International Institute (Saint Petersburg) and organizers of the International Conference "Classical and Quantum Integrable Systems-2019", where the main results of this work were reported. I. Gaiur is also grateful to all participants and organizers of the seminar "Methods of classical and quantum integrable systems" at the Steklov Mathematical Institute (Moscow) for fruitful discussions.

V. Rubtsov acknowledges the France 2030 framework program Centre Henri Lebesgue ANR-11-LABX-0020-01 for creating an attractive mathematical environment. I. Gaiur acknowledges the University of Toronto, and especially Boris Khesin, for a productive environment.

Both authors are grateful to the anonymous referees for the useful remarks and spotlighting inaccuracies and misprints in the proofs.

Funding. Contribution of V. Rubtsov was done under support of the Russian Science Foundation (Project no. 23-41-00049).

References

[1] Adams, M. R., Harnad, J. and Hurtubise, J.: Dual moment maps into loop algebras. *Lett. Math. Phys.* **20** (1990), no. 4, 299–308. Zbl 0721.58025 MR 1077962

[2] Adams, M. R., Harnad, J. and Previato, E.: Isospectral Hamiltonian flows in finite and infinite dimensions. I. Generalized Moser systems and moment maps into loop algebras. *Comm. Math. Phys.* **117** (1988), no. 3, 451–500. Zbl 0659.58022 MR 0953833

[3] Audin, M.: Lectures on gauge theory and integrable systems. In *Gauge theory and symplectic geometry (Montreal, PQ, 1995)*, pp. 1–48. NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. 488, Kluwer Acad. Publ., Dordrecht, 1997. Zbl 0873.58016 MR 1461568

[4] Babelon, O., Bernard, D. and Talon, M.: *Introduction to classical integrable systems*. Cambridge Monogr. Math. Phys., Cambridge University Press, Cambridge, 2003. Zbl 1045.37033 MR 1995460

[5] Balandin, S. P. and Sokolov, V. V.: On the Painlevé test for non-abelian equations. *Phys. Lett. A* **246** (1998), no. 3-4, 267–272. Zbl 0946.34076 MR 1644158

[6] Bershtein, M., Grigorev, A. and Shchechkin, A.: Hamiltonian reductions in matrix Painlevé systems. *Lett. Math. Phys.* **113** (2023), no. 2, article no. 47, 51 pp. Zbl 1522.34115 MR 4576398

[7] Bertola, M. and Cafasso, M.: Fredholm determinants and pole-free solutions to the noncommutative Painlevé II equation. *Comm. Math. Phys.* **309** (2012), no. 3, 793–833. Zbl 1234.35171 MR 2885610

[8] Bertola, M., Cafasso, M. and Rubtsov, V.: Noncommutative Painlevé equations and systems of Calogero type. *Comm. Math. Phys.* **363** (2018), no. 2, 503–530. Zbl 1402.34092 MR 3851821

[9] Bobrova, I., Retakh, V., Rubtsov, V. and Sharygin, G.: A fully noncommutative analog of the Painlevé IV equation and a structure of its solutions. *J. Phys. A* **55** (2022), no. 47, article no. 475205, 30 pp. Zbl 1519.37077 MR 4516729

[10] Bobrova, I. A. and Sokolov, V. V.: On matrix Painlevé-4 equations. *Nonlinearity* **35** (2022), no. 12, 6528–6556. Zbl 1512.34165 MR 4512746

[11] Bobrova, I. A. and Sokolov, V. V.: On matrix Painlevé-4 equations. Part 2: Isomonodromic Lax pairs. Preprint, arXiv:2110.12159, 2021.

[12] Chekhov, L., Mazzocco, M. and Rubtsov, V.: Quantised Painlevé monodromy manifolds, Sklyanin and Calabi–Yau algebras. *Adv. Math.* **376** (2021), article no. 107442, 52 pp. Zbl 1506.14007 MR 4178914

[13] Chernyakov, Y. B.: Integrable systems obtained by a fusion of points from rational and elliptic Gaudin systems. *Theor. Math. Phys.* **141** (2004), no. 1, 1361–1380. Zbl 1178.37056 MR 2124509

[14] Fairon, M.: Morphisms of double (quasi-)Poisson algebras and action-angle duality of integrable systems. *Ann. H. Lebesgue* **5** (2022), 179–262. Zbl 1496.16045 MR 4391080

[15] Fehér, L. and Klimčík, C.: On the duality between the hyperbolic Sutherland and the rational Ruijsenaars–Schneider models. *J. Phys. A* **42** (2009), no. 18, article no. 185202, 13 pp. Zbl 1168.37016 MR 2591196

[16] Fehér, L. and Klimčík, C.: Poisson–Lie interpretation of trigonometric Ruijsenaars duality. *Comm. Math. Phys.* **301** (2011), no. 1, 55–104. Zbl 1208.53085 MR 2753671

[17] Fehér, L. and Klimčík, C.: Self-duality of the compactified Ruijsenaars–Schneider system from quasi-Hamiltonian reduction. *Nuclear Phys. B* **860** (2012), no. 3, 464–515. Zbl 1246.70016 MR 2909560

[18] Fehér, L. and Marshall, I.: The action-angle dual of an integrable Hamiltonian system of Ruijsenaars–Schneider–van Diejen type. *J. Phys. A* **50** (2017), no. 31, article no. 314004, 20 pp. Zbl 1371.81158 MR 3673476

[19] Fock, V., Gorsky, A., Nekrasov, N. and Rubtsov, V.: Duality in integrable systems and gauge theories. *J. High Energy Phys.* (2000), no. 7, article no. 28, 40 pp. Zbl 0965.81025 MR 1780352

[20] Gorbe, T. F.: *Integrable many-body systems of Calogero–Ruijsenaars type*. Ph.D. Thesis, Szegedi Tudomanyegyetem (Hungary), 2017. MR 4722933

[21] Gordoa, P. R., Pickering, A. and Zhu, Z. N.: On matrix Painlevé hierarchies. *J. Differential Equations* **261** (2016), no. 2, 1128–1175. Zbl 1352.34116 MR 3494393

[22] Gorsky, A. and Nekrasov, N.: Quantum integrable systems of particles as gauge theories. *Teoret. Mat. Fiz.* **100** (1994), no. 1, 97–103. Zbl 0854.58045 MR 1305791

[23] Gorsky, A. and Rubtsov, V.: Dualities in integrable systems: geometrical aspects. In *Integrable structures of exactly solvable two-dimensional models of quantum field theory (Kiev, 2000)*, pp. 173–198. NATO Sci. Ser. II Math. Phys. Chem. 35, Kluwer Acad. Publ., Dordrecht, 2001. Zbl 0993.81022 MR 1873571

[24] Harnad, J.: Dual isomonodromic deformations and moment maps to loop algebras. *Comm. Math. Phys.* **166** (1994), no. 2, 337–365. Zbl 0822.58018 MR 1309553

[25] Inozemtsev, V. I.: Lax representation with spectral parameter on a torus for integrable particle systems. *Lett. Math. Phys.* **17** (1989), no. 1, 11–17. Zbl 0679.70005 MR 0990577

[26] Kawakami, H.: Matrix Painlevé systems. *J. Math. Phys.* **56** (2015), no. 3, article no. 033503, 27 pp. Zbl 1322.34099 MR 3390928

[27] Kawakami, H.: Four-dimensional Painlevé-type equations associated with ramified linear equations I: Matrix Painlevé systems. *Funkcial. Ekvac.* **63** (2020), no. 1, 97–132. Zbl 1466.34081 MR 4117033

[28] Kazhdan, D., Kostant, B. and Sternberg, S.: Hamiltonian group actions and dynamical systems of Calogero type. *Comm. Pure Appl. Math.* **31** (1978), no. 4, 481–507. Zbl 0368.58008 MR 0478225

[29] Khalilov, F. A. and Khruslov, Ē. Y.: Matrix generalisation of the modified Korteweg–de Vries equation. *Inverse Problems* **6** (1990), no. 2, 193–204. Zbl 0709.35091 MR 1046162

[30] Koornwinder, T. H. and Mazzocco, M.: Dualities in the q-Askey scheme and degenerate DAHA. *Stud. Appl. Math.* **141** (2018), no. 4, 424–473. Zbl 1408.33032 MR 3879966

[31] Koroteev, P., Sage, D. S. and Zeitlin, A. M.: $(\mathrm{SL}(N), q)$-Opers, the q-Langlands correspondence, and quantum/classical duality. *Comm. Math. Phys.* **381** (2021), no. 2, 641–672. Zbl 1462.82013 MR 4207453

[32] Levin, A. M. and Olshanetsky, M. A.: Painlevé–Calogero correspondence. In *Calogero–Moser–Sutherland models (Montréal, QC, 1997)*, pp. 313–332. CRM Ser. Math. Phys., Springer, New York, 2000. MR 1843578

[33] Manin, Y. I.: Sixth Painlevé equation, universal elliptic curve, and mirror of $\mathbb{P}^2$. In *Geometry of differential equations*, pp. 131–151. Amer. Math. Soc. Transl. Ser. 2 186, American Mathematical Society, Providence, RI, 1998. Zbl 0948.14025 MR 1732409

[34] Mironov, A. and Morozov, A.: On the status of DELL systems. *Nuclear Phys. B* **999** (2024), article no. 116448, 25 pp. Zbl 07807834 MR 4689342

[35] Mironov, A., Morozov, A., Runov, B., Zenkevich, Y. and Zotov, A.: Spectral dualities in XXZ spin chains and five dimensional gauge theories. *J. High Energy Phys.* (2013), no. 12, article no. 034, 10 pp. Zbl 1342.81310 MR 3158957

[36] Mironov, A., Morozov, A., Runov, B., Zenkevich, Y. and Zotov, A.: Spectral duality between Heisenberg chain and Gaudin model. *Lett. Math. Phys.* **103** (2013), no. 3, 299–329. Zbl 1262.14040 MR 3019800

[37] Nekrasov, N.: On a duality in Calogero–Moser–Sutherland systems. Preprint, arXiv:hep-th/9707111, 2007.

[38] Noumi, M., Ruijsenaars, S. and Yamada, Y.: The elliptic Painlevé Lax equation vs. van Diejen's 8-coupling elliptic Hamiltonian. *SIGMA Symmetry Integrability Geom. Methods Appl.* **16** (2020), article no. 063, 16 pp. Zbl 1476.39021 MR 4120359

[39] Ohyama, Y. and Okumura, S.: A coalescent diagram of the Painlevé equations from the viewpoint of isomonodromic deformations. *J. Phys. A* **39** (2006), no. 39, 12129–12151. Zbl 1116.34072 MR 2266216

[40] Okamoto, K.: Polynomial Hamiltonians associated with Painlevé equations. I. *Proc. Japan Acad. Ser. A Math. Sci.* **56** (1980), no. 6, 264–268. Zbl 0476.34010 MR 0581468

[41] Retakh, V. and Rubtsov, V.: Noncommutative Toda chains, Hankel quasideterminants and the Painlevé II equation. *J. Phys. A* **43** (2010), no. 50, article no. 505204, 13 pp. Zbl 1211.37079 MR 2740382

[42] Takano, K.: Confluence processes in defining manifolds for Painlevé systems. *Tohoku Math. J. (2)* **53** (2001), no. 2, 319–335. Zbl 1021.34079 MR 1829984

[43] Takasaki, K.: Painlevé–Calogero correspondence revisited. *J. Math. Phys.* **42** (2001), no. 3, 1443–1473. Zbl 1016.34089 MR 1814699

[44] Takemura, K.: Quasi-exact solvability of Inozemtsev models. *J. Phys. A* **35** (2002), no. 41, 8867–8881. Zbl 1049.81062 MR 1946978

[45] Takemura, K.: Degenerations of Ruijsenaars–van Diejen operator and q-Painlevé equations. *J. Integrable Syst.* **2** (2017), no. 1, article no. xyx008, 27 pp. Zbl 1400.39019 MR 3710451

[46] Zabrodin, A. and Zotov, A.: Quantum Painlevé–Calogero correspondence. *J. Math. Phys.* **53** (2012), no. 7, article no. 073507, 19 pp. Zbl 1276.81083 MR 2985247

[47] Zotov, A. V. and Chernyakov, Y. B.: Integrable multiparticle systems obtained by means of the Inozemtsev limit. *Theor. Math. Phys.* **129** (2001), no. 2, 1526–1542. Zbl 1029.37037 MR 1904799

Received March 22, 2023; revised February 16, 2024.

Ilia Gaiur
Department of Mathematics, University of Toronto
40 St. George St., Toronto, ON, M5S 2E4, Canada;
iygayur@gmail.com

Vladimir Rubtsov
LAREMA, UMR 6093 du CNRS, Départment de Mathématics, Université d'Angers
2 Boulevard Lavoisier, 49045 Angers, France;
Institute for Information Transmission Problems, RAS
Bolshoy Karetny per. 19, build. 1, 127051 Moscow, Russia;
Institute for Geometry and Physics (IGAP)
Via Beirut 2, 34151 Trieste, Italy;
volodya@univ-angers.fr.