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Boutroux Ansatz for the Degenerate Third Painlevé Transcendents

by

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Abstract

For a general solution of the degenerate third Painlevé equation we show the Boutroux ansatz near the point at infinity. It admits an asymptotic representation in terms of the Weierstrass pe-function in cheese-like strips along generic directions. The expression is obtained by using isomonodromy deformation of a linear system governed by the degenerate third Painlevé equation.

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§1. Introduction

In the geometrical study of the spaces of initial values for Painlevé equations, Sakai [27] classified the third Painlevé equations into three types $P_{\mathrm{III}}(D_6)$, $P_{\mathrm{III}}(D_7)$ and $P_{\mathrm{III}}(D_8)$. For the types $P_{\mathrm{III}}(D_7)$ and $P_{\mathrm{III}}(D_8)$, Ohyama et al. [24] examined basic matters including τ-functions, irreducibility and the spaces of initial values. Equation $P_{\mathrm{III}}(D_8)$ is changed into a special case of $P_{\mathrm{III}}(D_6)$. Equation $P_{\mathrm{III}}(D_7)$ is called the degenerate third Painlevé equation or degenerate P_{III}, which may be normalised in the form

$$v_{\xi\xi} = \frac{v_\xi^2}{v} - \frac{v_\xi}{\xi} - \frac{2v^2}{\xi^2} + \frac{a}{\xi} + \frac{1}{v}$$

($v_\xi = dv/d\xi$) with $a \in \mathbb{C}$. The change of variables

$$2\xi = \epsilon b \tau^2, \quad v = \epsilon \tau u$$

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takes this equation to the equivalent equation discussed in [17, 18],

$$u_{\tau\tau} = \frac{u_\tau^2}{u} - \frac{u_\tau}{\tau} + \frac{1}{\tau}(-8\epsilon u^2 + 2ab) + \frac{b^2}{u}, \tag{1.1}$$

with $\epsilon = \pm 1$, $a \in \mathbb{C}$, $b \in \mathbb{R} \setminus \{0\}$, which governs isomonodromy deformation of the linear system (3.1). Using the isomonodromy system (3.1), Kitaev and Vartanian [17, 18] obtained asymptotic solutions of (1.1) as $\tau \to \pm\infty$, $\pm i\infty$ and $\tau \to \pm 0$, $\pm i0$, with connection formulas among them. Furthermore, for (1.1), a special meromorphic solution is studied in [16, 19] and a one-parameter family of trans-series solutions is given in [29].

As mentioned in [17, 29], in physical and geometrical applications, degenerate P_{III} appears in contexts independent of $P_{\mathrm{III}}(D_6)$, i.e. complete P_{III}, and its significant analytic properties are important. Indeed, the behaviours of solutions of (1.1) along real and imaginary axes [17, 18] are quite different from those for complete P_{III} [12]. For complete P_{III} of the sine-Gordon type, Novokshënov [22, 23] and [5, Chap. 16] provided an asymptotic representation of solutions in terms of the sn-function along generic directions near the point at infinity. It is meaningful to establish the counterpart of this expression for degenerate P_{III}.

In this paper we show the Boutroux ansatz [2] for degenerate P_{III}, i.e. present an elliptic asymptotic representation for a general solution along generic directions near the point at infinity. The main results are described in Section 2. As in Theorems 2.1 and 2.2, degenerate P_{III} admits a general solution written in terms of the Weierstrass $\wp$-function, and so does P_{I} ([7, 8, 14, 15]). On the other hand, for P_{II}, P_{IV}, $P_{\mathrm{III}}(D_6)$ (of sine-Gordon type) and P_{V}, elliptic asymptotic solutions are given by the sn-function ([5, 9, 10, 11, 15, 20, 21, 22, 23, 28, 30]). This fact reflects the position of degenerate P_{III}, i.e. $P_{\mathrm{III}}(D_7)$ in the degeneration scheme of the Painlevé equations [24, 25, 27].

For our purpose it is appropriate to treat an equation of the form

$$y'' = \frac{(y')^2}{y} - \frac{y'}{x} - 2y^2 + \frac{3a}{x} + \frac{1}{y} \tag{1.2}$$

($y' = dy/dx$), which comes from (1.1) via the substitution

$$\epsilon\tau u = (x/3)^2 y, \quad \epsilon b\tau^2 = 2(x/3)^3. \tag{1.3}$$

Equation (1.2) with $x = e^{i\phi}t$ governs isomonodromy deformation of the linear system

$$\frac{d\Psi}{d\lambda} = \frac{t}{3}\mathcal{B}(\lambda, t)\Psi, \tag{1.4}$$

with

$$\mathcal{B}(\lambda,t) = -ie^{i\phi}\lambda\sigma_3 + \begin{pmatrix} 0 & -2ie^{i\phi}y \\ \Gamma_0(t,y,y^t)/y & 0 \end{pmatrix} - \Big(\Gamma_0(t,y,y^t) + \frac{3}{2}(1+2ia)t^{-1}\Big)\lambda^{-1}\sigma_3 + 2e^{i\phi}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\lambda^{-2},$$

in which y and y^t are arbitrary complex parameters, and

$$\Gamma_0(t,y,y^t) = \frac{y^t}{y} - \frac{ie^{i\phi}}{y} - (1+3ia)t^{-1}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

As shown in Section 3, system (1.4) is a result of a transformation of system (3.1) treated in [17, 18]. The isomonodromy deformation of (3.1) is governed by equation (1.1), and solutions of (1.1) are related to the invariant monodromy data on the monodromy manifold for (3.1) defined by Stokes matrices and a connection matrix $G = (g_{ij}) \in \mathrm{SL}_2(\mathbb{C})$ for matrix solutions around $\mu = 0$ and $\mu = \infty$. System (1.4) admits the same monodromy manifold as of (3.1), which is described by the same Stokes matrices and G for suitably chosen matrix solutions (cf. Proposition 3.2), so that solutions of (1.1) and (1.2) correspond to the same monodromy data.

Applying WKB analysis we solve the direct monodromy problem for the linear system (1.4) in Section 5, and obtain key relations in Corollary 5.2 containing the monodromy data G and certain integrals, which lead to a solution of an inverse problem. Basic necessary materials for this calculation are summarised in Section 4. Asymptotic properties of these integrals are examined in Section 6 by the use of the ϑ-function, and from these formulas asymptotic forms in the main theorems are derived in Section 7. Then the justification as a solution of (1.2) is made along the lines of Kitaev [13, 15]. The final section is devoted to the Boutroux equations, which determine the modulus contained in the elliptic representation of solutions.

Throughout this paper we use the following symbols:

(1) σ_1, σ_2, σ_3 are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

(2) for complex-valued functions f and g, we write $f \ll g$ or $g \gg f$ if $f = O(|g|)$, and write $f \asymp g$ if $g \ll f \ll g$.

§2. Main results

To state our main results we give some explanations of necessary facts.

§2.1. Monodromy data

Isomonodromy system (3.1) admits the matrix solutions

$$Y_k^\infty(\mu) = (I + O(\mu^{-1}))\mu^{-(1/2+ia)\sigma_3} \exp(-i\tau\mu^2\sigma_3)$$

as $\mu \to \infty$ through the sector $|\arg \mu + \arg \tau^{1/2} - \pi k/2| < \pi/2$, and

$$X_k^0(\mu) = (i/\sqrt{2})\Theta_0^{\sigma_3}(\sigma_1 + \sigma_3 + O(\mu)) \exp(-i\sqrt{\tau\epsilon b}\mu^{-1}\sigma_3)$$

as $\mu \to 0$ through the sector $|\arg \mu - \arg(\tau\epsilon b)^{1/2} - \pi k| < \pi$, where $k \in \mathbb{Z}$ (see Section 3.2). Let the invariant Stokes matrices and a connection matrix be such that $Y_{j+1}^\infty(\mu) = Y_j^\infty(\mu)S_j^\infty$, $X_{j+1}^0(\mu) = X_j^0(\mu)S_j^0$ with $j \in \mathbb{Z}$ and that $Y_0^\infty(\mu) = X_0^0(\mu)G$. These are

$$S_0^\infty = \begin{pmatrix} 1 & 0 \\ s_0^\infty & 1 \end{pmatrix}, \quad S_1^\infty = \begin{pmatrix} 1 & s_1^\infty \\ 0 & 1 \end{pmatrix}, \quad S_0^0 = \begin{pmatrix} 1 & s_0^0 \\ 0 & 1 \end{pmatrix}$$

with (3.7), and $G = (g_{ij})$ with $g_{11}g_{22} - g_{12}g_{21} = 1$. The monodromy manifold $\mathcal{M}$ is given by $GS_0^\infty S_1^\infty \sigma_3 e^{\pi(i/2-a)\sigma_3} = S_0^0\sigma_1 G$, whose generic points are expressed by G [17, p. 1172]. Solutions $u(\tau)$ of (1.1) and $y(x)$ of (1.2) related via (1.3) correspond to the same monodromy data. As described in Remark 3.1, a change of the matrix solution basis induces an action on the monodromy data with G on $\mathcal{M}$, and each solution of (1.1) or (1.2) is parametrised by an orbit, or equivalence class, in the quotient of $\mathcal{M}$ under this action. In what follows, a solution corresponding to an orbit passing through G is simply called a *solution labelled by* G.

§2.2. Elliptic curve and Boutroux equations

For $A \in \mathbb{C}$ around $A = 3 \cdot 2^{2/3}$, the polynomial $4z^3 - Az^2 + 1$ has roots z_0, z_1 close to $2^{-1/3}$ and z_2 close to $-4^{-2/3}$, and especially, $z_0 = z_1 = 2^{-1/3}$, $z_2 = -4^{-2/3}$ when $A = 3 \cdot 2^{2/3}$. Let Π_+ and Π_- be the copies of $P^1(\mathbb{C}) \setminus ([\infty, z_2] \cup [z_0, z_1])$ and set $\Pi_A = \Pi_+ \cup \Pi_-$ glued along the cuts $[\infty, z_2]$ and $[z_0, z_1]$, where $\operatorname{Re} z \to -\infty$ along $[\infty, z_2]$. Then Π_A is the elliptic curve given by

$$w(A, z)^2 = 4z^3 - Az^2 + 1,$$

where the branch of $\sqrt{4z^3 - Az^2 + 1} := 2\sqrt{z - z_0}\sqrt{z - z_1}\sqrt{z - z_2}$ is chosen in such a way that $\operatorname{Re} \sqrt{z - z_j} \to +\infty$ as $z \to \infty$ along the positive real axis on the upper plane Π_+. The elliptic curve Π_A does not degenerate as long as $A \neq 3 \cdot 2^{2/3} e^{2\pi i m/3}$ $(m = 0, \pm 1)$, i.e. $4z^3 - Az^2 + 1$ has no double roots, and then we may define Π_A continuously.

As will be shown in Section 8, for any $\phi \in \mathbb{R}$, there exists $A_\phi \in \mathbb{C}$ with Π_{A_ϕ} such that, for every cycle $\mathbf{c}$ on Π_{A_ϕ},

$$\operatorname{Im} e^{i\phi} \int_{\mathbf{c}} \frac{w(A_\phi, z)}{z^2}\, dz = 0,$$

and that A_ϕ has the following properties (Proposition 8.15):

(1) for every ϕ, A_ϕ is uniquely determined;

(2) A_ϕ is continuous in $\phi \in \mathbb{R}$, and is smooth in $\phi \in \mathbb{R} \setminus \{k\pi/3 \mid k \in \mathbb{Z}\}$;

(3) $A_{\phi \pm 2\pi/3} = e^{\pm 2\pi i/3} A_\phi$, $A_{\phi+\pi} = A_\phi$, $A_{-\phi} = \overline{A_\phi}$;

(4) Π_{A_ϕ} degenerates if and only if $\phi = k\pi/3$ with $k \in \mathbb{Z}$, and then $A_0 = 3 \cdot 2^{2/3}$, $A_{\pm\pi/3} = e^{\mp 2\pi i/3} A_0$, $A_{\pm 2\pi/3} = e^{\pm 2\pi i/3} A_0$, $A_{\pm\pi} = A_0$.

In particular, for $0 < |\phi| < \pi/3$ let us consider A_ϕ for specified cycles. For A_ϕ close to $A_0 = 3 \cdot 2^{2/3}$, by Proposition 8.16, number the roots of $w(A_\phi, z)^2$ close to $2^{-1/3}$ in such a way that $\operatorname{Im} z_0 \le \operatorname{Im} z_1$ if $\phi > 0$ (respectively, $\operatorname{Im} z_1 \le \operatorname{Im} z_0$ if $\phi < 0$), and let the numbering be retained as long as coalescence does not occur. Then for $0 < |\phi| < \pi/3$ we have basic cycles $\mathbf{a}$ and $\mathbf{b}$ on Π_{A_ϕ}, which are drawn on Π_+ as in Figure 1. For $|\phi| < \pi/3$ the cycles $\mathbf{a}$ and $\mathbf{b}$ may be defined continuously on Π_{A_ϕ}, and the Boutroux equations are given by

$$\operatorname{Im} e^{i\phi} \int_{\mathbf{a}} \frac{w(A_\phi, z)}{z^2}\, dz = 0, \quad \operatorname{Im} e^{i\phi} \int_{\mathbf{b}} \frac{w(A_\phi, z)}{z^2}\, dz = 0, \tag{2.1}$$

admitting a unique solution A_ϕ. For $|\phi| < \pi/3$ the periods of Π_{A_ϕ} along $\mathbf{a}$ and $\mathbf{b}$ are defined by

$$\Omega^\phi_{\mathbf{a}} = \Omega_{\mathbf{a}} = \int_{\mathbf{a}} \frac{dz}{w(A_\phi, z)}, \quad \Omega^\phi_{\mathbf{b}} = \Omega_{\mathbf{b}} = \int_{\mathbf{b}} \frac{dz}{w(A_\phi, z)},$$

which satisfy $\operatorname{Im} \Omega_{\mathbf{b}}/\Omega_{\mathbf{a}} > 0$.

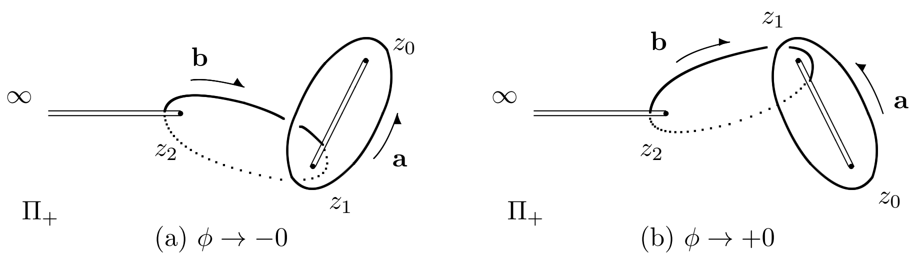

Figure 1. Cycles **a** and **b**

§2.3. Main theorems

Let $y(x) = y(G, x)$ be a solution of (1.2) labelled by the monodromy data $G = (g_{ij}) \in \mathrm{SL}_2(\mathbb{C})$. Then we have the following, in which $\wp(u; g_2, g_3)$ is the Weierstrass $\wp$-function satisfying $\wp_u^2 = 4\wp^3 - g_2\wp - g_3$ ([6, 31]):

Theorem 2.1. *Suppose that* $0 < \phi < \pi/3$ *and that* $g_{11}g_{12}g_{22} \neq 0$. *Then*

$$y(x) = \wp(i(x - x_0^+) + O(x^{-\delta}); g_2(A_\phi), g_3(A_\phi)) + \frac{A_\phi}{12}$$

as $x = te^{i\phi} \to \infty$ *through the cheese-like strip*

$$S(\phi, t_\infty, \kappa_0, \delta_0) = \big\{x = te^{i\phi} \bigm| \mathrm{Re}\, t > t_\infty,\ |\mathrm{Im}\, t| < \kappa_0\big\} \setminus \bigcup_{\sigma \in \mathcal{P}(x_0^+)} \big\{|x - \sigma| < \delta_0\big\},$$

with

$$\mathcal{P}(x_0^+) = \big\{\sigma \bigm| \wp(i(\sigma - x_0^+); g_2(A_\phi), g_3(A_\phi)) = \infty\big\} = \big\{x_0^+ - i\Omega_{\mathbf{a}}\mathbb{Z} - i\Omega_{\mathbf{b}}\mathbb{Z}\big\}.$$

Here, δ *is some positive number,* κ_0 *a given positive number,* δ_0 *a given small positive number,* $t_\infty = t_\infty(\kappa_0, \delta_0)$ *a sufficiently large number depending on* (κ_0, δ_0), *and*

$$g_2(A_\phi) = \frac{A_\phi^2}{12}, \quad g_3(A_\phi) = \frac{A_\phi^3}{216} - 1,$$

$$-ix_0^+ \equiv \frac{i}{2\pi}\Big(\Omega_{\mathbf{a}} \log \frac{g_{12}}{g_{22}} - \Omega_{\mathbf{b}}(\log(g_{11}g_{22}) - \pi i)\Big) - ia\Omega_0 \quad \mathrm{mod}\ \Omega_{\mathbf{a}}\mathbb{Z} + \Omega_{\mathbf{b}}\mathbb{Z}$$

with

$$\Omega_0 = \int_\infty^{0^+} \frac{dz}{w(A_\phi, z)},$$

in which 0^+ *denotes* $0 \in \Pi_+$ *and the contour* $[\infty, 0^+] \subset \Pi_+$ *contains the line from* $-\infty$ *to* z_2 *along the upper shore of the cut* $[\infty, z_2]$.

Theorem 2.2. *Suppose that* $-\pi/3 < \phi < 0$ *and that* $g_{11}g_{21}g_{22} \neq 0$. *Then* $y(x)$ *admits an asymptotic representation of the same form as in Theorem 2.1 with the phase shift*

$$-ix_0^- \equiv \frac{-i}{2\pi}\Big(\Omega_{\mathbf{a}} \log \frac{g_{21}}{g_{11}} + \Omega_{\mathbf{b}}(\log(g_{11}g_{22}) - \pi i)\Big) - ia\Omega_0 \quad \mathrm{mod}\ \Omega_{\mathbf{a}}\mathbb{Z} + \Omega_{\mathbf{b}}\mathbb{Z}.$$

Remark 2.1. From a relation in the proof of Theorem 2.1 we have an expression for $y'(x)$ for $0 < \phi < \pi/3$ and $-\pi/3 < \phi < 0$ of the form

$$\frac{iy'(x) + 1}{2y(x)^2} = \wp(i(x - \hat{x}_0^\pm) + O(x^{-\delta}); g_2(A_\phi), g_3(A_\phi)) + \frac{A_\phi}{12},$$

respectively, where $i\hat{x}_0^\pm = ix_0^\pm + \Omega_0$.

Remark 2.2. The phase shifts in the theorems above are represented by $g_{11}g_{22}$, g_{21}/g_{11} and g_{12}/g_{22}, which are invariants under an action on G in Remark 3.1.

The expressions of $y(x)$ in Theorems 2.1 and 2.2 are determined by A_ϕ and $x_0 = x_0^+$ for $0 < \phi < \pi/3$, $= x_0^-$ for $-\pi/3 < \phi < 0$. Since $\Omega_{\mathbf{a},\mathbf{b}}$ and Ω_0 depend on A_ϕ, these may be denoted by $\Omega^\phi_{\mathbf{a},\mathbf{b}}$ and Ω^ϕ_0, respectively. To emphasise this fact, write

$$y(x) = P(A_\phi, x_0(G, \Omega^\phi_{\mathbf{a}}, \Omega^\phi_{\mathbf{b}}, \Omega^\phi_0); x)$$

for $0 < |\phi| < \pi/3$.

For ϕ such that $|\phi - 2m\pi/3| < \pi/3$ $(m \in \mathbb{Z})$, set $\Omega^\phi_{\mathbf{a},\mathbf{b}} = e^{2m\pi i/3}\Omega^{\phi-2m\pi/3}_{\mathbf{a},\mathbf{b}}$. The period, say $\Omega^\phi_{\mathbf{a}}$, may be expressed by the integral on Π_+,

$$\Omega^\phi_{\mathbf{a}} = \int_{e^{2m\pi i/3}\mathbf{a}} \frac{dz}{w(A_\phi, z)} = \int_{e^{2m\pi i/3}\mathbf{a}} \frac{dz}{w(e^{2m\pi i/3}A_{\phi-2m\pi/3}, z)}$$
$$= e^{2m\pi i/3}\int_{\mathbf{a}} \frac{d\zeta}{w(A_{\phi-2m\pi/3}, \zeta)} = e^{2m\pi i/3}\Omega^{\phi-2m\pi/3}_{\mathbf{a}} \quad (z = e^{2m\pi i/3}\zeta).$$

Furthermore, for $|\phi - 2m\pi/3| < \pi/3$ set $\Omega^\phi_0 = e^{2m\pi i/3}\Omega^{\phi-2m\pi/3}_0$. The following provides an analytic continuation of $y(x)$ beyond the sector $|\phi| < \pi/3$:

Theorem 2.3. *Suppose that* $0 < \phi - 2m\pi/3 < \pi/3$ *(respectively,* $-\pi/3 < \phi - 2m\pi/3 < 0$*) for* $m \in \mathbb{Z} \setminus \{0\}$*. Then* $y(x)$ *admits the expression*

$$y(x) = y(G, x) = P(A_\phi, x_0(G^{(m)}, \Omega^\phi_{\mathbf{a}}, \Omega^\phi_{\mathbf{b}}, \Omega^\phi_0); x)$$

as $x = te^{i\phi} \to \infty$ *through the strip* $S(\phi, t_\infty, \kappa_0, \delta_0)$ *with* $\mathcal{P}(x_0(G^{(m)}, \Omega^\phi_{\mathbf{a}}, \Omega^\phi_{\mathbf{b}}, \Omega^\phi_0))$*, if* $g^{(m)}_{11}g^{(m)}_{12}g^{(m)}_{22} \ne 0$ *(respectively,* $g^{(m)}_{11}g^{(m)}_{21}g^{(m)}_{22} \ne 0$*), where*

$$G^{(m)} = \begin{cases} (S^0_0\sigma_1)^m G\sigma_3^m e^{(m\pi/3)(a-i/2)\sigma_3} & \text{if } m \ge 1, \\ (\sigma_1 S^0_0)^n G\sigma_3^n e^{(n\pi/3)(i/2-a)\sigma_3} & \text{if } m = -n \le -1. \end{cases}$$

Remark 2.3. The matrix $G^{(m)}$ has another expression of the form

$$G^{(m)} = \begin{cases} G(S^\infty_0 S^\infty_1 \sigma_3 e^{\pi(i/2-a)\sigma_3})^m \sigma_3^m e^{(m\pi/3)(a-i/2)\sigma_3} & \text{if } m \ge 1, \\ G(\sigma_3 e^{\pi(a-i/2)\sigma_3} S^\infty_1 S^\infty_0)^n \sigma_3^n e^{(n\pi/3)(i/2-a)\sigma_3} & \text{if } m = -n \le -1. \end{cases}$$

§2.4. Examples

For simplicity suppose that $\epsilon = 1$ and $b = 2$ in equation (1.1). Let $G = (g_{ij})$ with $g_{11}g_{22} - g_{12}g_{21} = 1$ be the monodromy data in Kitaev–Vartanian [17, 18], which coincide with ours above. Suppose that $g_{11}g_{12}g_{21}g_{22} \ne 0$. Then [17, Thm. 3.1], [18, Thms. 2.1 and 2.3] with $\varepsilon_1 = \varepsilon_2 = 0$ provide general solutions of (1.1) as in the following examples, in which we write $l(g_{11}g_{22}) = i(2\pi)^{-1}\log(g_{11}g_{22})$.

Example 2.1. If $|\mathrm{Re}\, l(g_{11}g_{22})| < 1/6$, equation (1.1) admits a solution of the form

$$u(\tau) = 2^{-1/3}\tau^{1/3} + 2^{1/2}3^{-1/4}e^{3\pi i/4}l(g_{11}g_{22})^{1/2}\cosh(\chi(\tau)),$$
$$\chi(\tau) = i2^{1/3}3^{3/2}\tau^{2/3} + l(g_{11}g_{22})\log(2^{1/3}3^{3/2}\tau^{2/3}) + \gamma(g_{11}g_{22}, g_{12}/g_{22}) + o(\tau^{-\tilde{\delta}})$$

as $\tau \to +\infty$, where $\gamma(g_{11}g_{22}, g_{12}/g_{22})$ is a constant expressed by $(g_{11}g_{22}, g_{12}/g_{22})$, and $\tilde{\delta}$ is some positive number.

Example 2.2. For $\mathrm{Re}\, l(g_{11}g_{22}) \in (0, 1)$, equation (1.1) admits a solution of the form

$$u(\tau) = 2^{-1/3}\tau^{1/3}\Big(1 - \frac{3}{2\sin^2(\tilde{\chi}(\tau)/2)}\Big)$$
$$= 2^{-1/3}\tau^{1/3}\frac{\sin(\tilde{\chi}(\tau)/2 - \chi_0)\sin(\tilde{\chi}(\tau)/2 + \chi_0)}{\sin^2(\tilde{\chi}(\tau)/2)},$$

with

$$\chi_0 = -\pi/2 + (i/2)\log(2 + \sqrt{3}),$$
$$\tilde{\chi}(\tau) = 2^{1/3}3^{3/2}\tau^{2/3} + l_*(g_{11}g_{22})\log(2^{1/3}3^{3/2}\tau^{2/3}) + \gamma_*(g_{ij}) + o(\tau^{-\tilde{\delta}})$$

as $\tau \to +\infty$ in a strip $|\mathrm{Im}\, \tau^{2/3}| \ll 1$. Here, $l_*(g_{11}g_{22}) = (2\pi)^{-1}\log(-g_{11}g_{22})$ $(\in \mathbb{R})$ if $\mathrm{Re}\, l(g_{11}g_{22}) = 1/2$, and $= -i(l(g_{11}g_{22}) - 1/2)$ otherwise; and $\gamma_*(g_{ij})$ is a constant expressed by $(l_*(g_{11}g_{22}), g_{11}g_{12}, g_{21}g_{22})$ if $\mathrm{Re}\, l(g_{11}g_{22}) = 1/2$, and by $(l(g_{11}g_{22}), g_{11}g_{12})$ otherwise.

By the change of variables $\tau^2 = (x/3)^3$, $\tau u = (x/3)^2 y$, these solutions are taken to solutions of (1.2) on the positive real axis. Proposition 3.2 guarantees the transfer between solutions of (1.1) and (1.2) with labels. Observing $g_{11}g_{12} = g_{11}g_{22}\cdot g_{12}/g_{22}$ and $g_{21}g_{22} = g_{11}g_{22}\cdot g_{21}/g_{11}$, and applying Theorems 2.1 and 2.2, we have elliptic representations of these solutions for $-\pi/3 < \phi < 0$ and $0 < \phi < \pi/3$.

In the case where $g_{12} = 0$ or $g_{21} = 0$, [17, Thms. 3.2 and 3.3] with $\varepsilon_1 = \varepsilon_2 = 0$ give one-parameter solutions as follows:

Example 2.3. Suppose that g_{21} or $g_{12} = 0$ and that $g_{11}g_{22} = 1$. Then (1.1) admits

$$u(\tau) = 2^{-1/3}\tau^{1/3} + \frac{(s_0^0 - ie^{-\pi a})c_*^{ia}}{2 \cdot 3^{1/4}\pi^{1/2}}\exp(\epsilon_* i(2^{1/3}3^{3/2}\tau^{2/3} + k_*\pi/4))(1 + o(\tau^{-\tilde{\delta}})),$$

as $\tau \to +\infty$. Here, $s_0^0 - ie^{-\pi a} = g_{12}/g_{22}$, $c_* = 2 - \sqrt{3}$, $\epsilon_* = -1$, $k_* = -1$ if $g_{21} = 0$; and $s_0^0 - ie^{-\pi a} = -g_{21}/g_{11}$, $c_* = 2 + \sqrt{3}$, $\epsilon_* = 1$, $k_* = 3$ if $g_{12} = 0$.

If $g_{11}g_{22}g_{12} \neq 0$, $g_{21} = 0$ (respectively, $g_{11}g_{22}g_{21} \neq 0$, $g_{12} = 0$), Theorem 2.1 for $0 < \phi < \pi/3$ (respectively, Theorem 2.2 for $-\pi/3 < \phi < 0$) applies to the corresponding solution of (1.2). In the case, say $g_{21} = 0$, this solution is represented by the $\wp$-function for $0 < \phi < \pi/3$, and is truncated for $-\pi < \phi < 0$.

§3. Isomonodromy deformation and monodromy data

§3.1. Isomonodromy deformation

Equation (1.1) governs isomonodromy deformation of the linear system

$$(3.1)\quad \frac{dU}{d\mu} = \mathcal{U}(\mu,\tau)U,$$

$$\mathcal{U}(\mu,\tau) = -2i\tau\mu\sigma_3 + 2\tau\begin{pmatrix} 0 & 2i\epsilon e^{i\varphi} \\ -(\epsilon/4)e^{-i\varphi}(u^\tau/u - 1/\tau - i\varphi_\tau) & 0\end{pmatrix}$$
$$-\frac{1}{\mu}\Big(ia + \frac{\tau}{2}(u^\tau/u - i\varphi_\tau)\Big)\sigma_3 + \frac{1}{\mu^2}\begin{pmatrix} 0 & 2\epsilon e^{i\varphi}(ia - i\tau\varphi_\tau/2) \\ -iue^{-i\varphi} & 0\end{pmatrix},$$

with $\varphi_\tau = (d/d\tau)\varphi = 2a/\tau + b/u$, i.e. the monodromy data remain invariant under small change of τ if and only if $u^\tau = (d/d\tau)u$ holds and $u(\tau)$ solves (1.1) [17, Props. 1.1, 1.2 and 2.1]. Let us change (3.1) into system (1.4) associated with (1.2). After the transformation

$$U = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2}\end{pmatrix}\begin{pmatrix} \sqrt{\epsilon}\tau^{3/4} & 0 \\ 0 & \tau^{-3/4}/\sqrt{\epsilon}\end{pmatrix}\widehat{U}, \quad \mu = \sqrt{2/\kappa}\,\tau^{1/2}\hat{\mu},$$

put

$$\tau^2 = \kappa\xi, \quad \tau u = q/\epsilon, \quad u^\tau + u/\tau = 2(\epsilon\kappa)^{-1}q^\xi, \quad \widehat{U} = \begin{pmatrix} (2/\kappa)^{1/4} & 0 \\ 0 & (2/\kappa)^{-1/4}\end{pmatrix}V,$$

with κ chosen so that $\epsilon\kappa b = 2$. Then (3.1) becomes

$$\frac{dV}{d\hat{\mu}} = \mathcal{V}(\hat{\mu},\xi)V,$$

$$\mathcal{V}(\hat{\mu},\xi) = -4i\xi\hat{\mu}\sigma_3 + \begin{pmatrix} 0 & 4i \\ -\xi(2\xi q^\xi/q - 2(1+ia) - 2i\xi/q) & 0\end{pmatrix}$$
$$-\frac{1}{\hat{\mu}}\Big(\xi\frac{q^\xi}{q} - \frac{1}{2} - \frac{i\xi}{q}\Big)\sigma_3 - \frac{i}{\hat{\mu}^2}\begin{pmatrix} 0 & 1/q \\ q & 0\end{pmatrix}.$$

The further change of variables

$$V = \begin{pmatrix} -i/\sqrt{q} & 0 \\ 0 & i\sqrt{q} \end{pmatrix} \Psi, \quad q = (x/3)^2 y, \quad \xi = (x/3)^3,$$
$$q^\xi = y^x + 2y/x, \quad (x/3)\hat{\mu} = \lambda/2$$

with $x = te^{i\phi}$ and $y^x = e^{-i\phi} y^t$ takes the system above to (1.4):

$$\frac{d\Psi}{d\lambda} = \frac{t}{3} \mathcal{B}(\lambda, t) \Psi,$$

whose right-hand side is written in the form

$$\begin{aligned} \mathcal{B}(\lambda, t) &= b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3, \\ b_1 &= -(i/2)(2e^{i\phi} y + i\Gamma_0(t, y, y^t) y^{-1}) + 2ie^{i\phi}\lambda^{-2}, \\ b_2 &= (1/2)(2e^{i\phi} y - i\Gamma_0(t, y, y^t) y^{-1}), \\ b_3 &= -ie^{i\phi}\lambda - (\Gamma_0(t, y, y^t) + 3(1/2 + ia)t^{-1})\lambda^{-1}, \\ \Gamma_0(t, y, y^t) &= \frac{y^t}{y} - \frac{ie^{i\phi}}{y} - \frac{1 + 3ia}{t}. \end{aligned} \tag{3.2}$$

In the linear systems above, u, u^τ; q, q^ξ; y, y^t are arbitrary complex parameters or functions, and $2(\epsilon\kappa)^{-1} q^\xi = u^\tau + u/\tau$, $q^\xi = y^x + 2y/x$ and $y^x = e^{-i\phi} y^t$ are compatible with their derivatives.

Proposition 3.1. *System (1.4) admits the isomonodromy property if and only if $y^t = (d/dt)y$ holds and $y = y(e^{i\phi} t) = y(x)$ solves equation (1.2).*

§3.2. Monodromy data

For each $j \in \mathbb{Z}$ system (1.4) admits the matrix solutions

$$\widehat{Y}_j^\infty(\lambda) = (I + O(\lambda^{-1}))\lambda^{-(1/2+ia)\sigma_3} \exp(-(i/6)e^{i\phi} t\lambda^2\sigma_3) \tag{3.3}$$

as $\lambda \to \infty$ through the sector $|\arg \lambda + \phi/2 - j\pi/2| < \pi/2$, and

$$\widehat{Y}_j^0(\lambda) = (i/\sqrt{2})(\sigma_1 + \sigma_3 + O(\lambda)) \exp(-(2i/3)e^{i\phi} t\lambda^{-1}\sigma_3) \tag{3.4}$$

as $\lambda \to 0$ through the sector $|\arg \lambda - \phi - j\pi| < \pi$. The Stokes matrices are such that

$$\widehat{Y}_{j+1}^\infty(\lambda) = \widehat{Y}_j^\infty(\lambda)\hat{S}_j^\infty, \quad \widehat{Y}_{j+1}^0(\lambda) = \widehat{Y}_j^0(\lambda)\hat{S}_j^0,$$

and the connection matrix $\widehat{G} = (\hat{g}_{ij})$ is defined by

$$\widehat{Y}_0^\infty(\lambda) = \widehat{Y}_0^0(\lambda)\widehat{G}, \quad \hat{g}_{11}\hat{g}_{22} - \hat{g}_{12}\hat{g}_{21} = 1. \tag{3.5}$$

The Stokes matrices satisfy

$$\hat{S}^{\infty}_{k+2} = \sigma_3 e^{-\pi(a-i/2)\sigma_3} \hat{S}^{\infty}_{k} e^{\pi(a-i/2)\sigma_3} \sigma_3, \quad \hat{S}^{0}_{k} = \sigma_1 \hat{S}^{0}_{k+1} \sigma_1,$$

for $k \in \mathbb{Z}$, and the monodromy manifold is given by

$$\widehat{G} \hat{S}^{\infty}_0 \hat{S}^{\infty}_1 \sigma_3 e^{\pi(i/2-a)\sigma_3} = \hat{S}^0_0 \sigma_1 \widehat{G}$$

with

$$\hat{S}^{\infty}_0 = \begin{pmatrix} 1 & 0 \\ \hat{s}^{\infty}_0 & 1 \end{pmatrix}, \quad \hat{S}^{\infty}_1 = \begin{pmatrix} 1 & \hat{s}^{\infty}_1 \\ 0 & 1 \end{pmatrix}, \quad \hat{S}^{0}_0 = \begin{pmatrix} 1 & \hat{s}^{0}_0 \\ 0 & 1 \end{pmatrix}.$$

These monodromy data and their relations are obtained by the same argument as in [17, Sect. 2].

Let $G = (g_{ij})$ be the monodromy data for system (3.1) given in [17, 18]. This connection matrix is defined by

$$Y^{\infty}_0(\mu) = X^0_0(\mu) G.$$

Here, $Y^{\infty}_0(\mu)$ and $X^0_0(\mu)$ are matrix solutions of system (3.1) as follows:

$$Y^{\infty}_k(\mu) = (I + O(\mu^{-1}))\mu^{-(1/2+ia)\sigma_3} \exp(-i\tau\mu^2\sigma_3)$$

as $\mu \to \infty$ through the sector $|\arg \mu + \arg \tau^{1/2} - \pi k/2| < \pi/2$, and

$$X^0_k(\mu) = (i/\sqrt{2}) \Theta_0^{\sigma_3} (\sigma_1 + \sigma_3 + O(\mu)) \exp(-i\sqrt{\tau \epsilon b}\, \mu^{-1} \sigma_3),$$
$$\Theta_0 = (\epsilon b)^{1/4} \tau^{-1/4} (-u e^{-i\varphi}/\tau)^{-1/2}$$

as $\mu \to 0$ through the sector $|\arg \mu - \arg(\tau\epsilon b)^{1/2} - \pi k| < \pi$ [17, Prop. 2.2]. Furthermore, Stokes matrices are defined by

$$Y^{\infty}_{j+1}(\lambda) = Y^{\infty}_j(\lambda) S^{\infty}_j, \quad X^0_{j+1}(\lambda) = X^0_j(\lambda) S^0_j,$$

and the monodromy manifold $\mathcal{M}$ for (3.1) is given by

$$GS^{\infty}_0 S^{\infty}_1 \sigma_3 e^{\pi(i/2-a)\sigma_3} = S^0_0 \sigma_1 G \tag{3.6}$$

with

$$S^{\infty}_0 = \begin{pmatrix} 1 & 0 \\ s^{\infty}_0 & 1 \end{pmatrix}, \quad S^{\infty}_1 = \begin{pmatrix} 1 & s^{\infty}_1 \\ 0 & 1 \end{pmatrix}, \quad S^{0}_0 = \begin{pmatrix} 1 & s^{0}_0 \\ 0 & 1 \end{pmatrix}.$$

For $k \in \mathbb{Z}$,

$$S^{\infty}_{k+2} = \sigma_3 e^{-\pi(a-i/2)\sigma_3} S^{\infty}_{k} e^{\pi(a-i/2)\sigma_3} \sigma_3, \quad S^{0}_{k} = \sigma_1 S^{0}_{k+1} \sigma_1. \tag{3.7}$$

A generic point on $\mathcal{M}$ is represented by G; indeed, if $g_{11}g_{22} \neq 0$, then s^{∞}_0, s^{∞}_1 and s^0_0 are written in terms of g_{ij} [17, p. 1172].

Remark 3.1. Instead of the matrix solutions $(Y_j^\infty(\mu), X_j^0(\mu))$, we may take

$$(Y_{j,*}^\infty(\mu), X_{j,*}^0(\mu)) := (c^{\sigma_3/2} Y_j^\infty(\mu) c^{-\sigma_3/2}, c^{\sigma_3/2} X_j^0(\mu))$$

with any $c \in \mathbb{C} \setminus \{0\}$, in which $Y_{j,*}^\infty(\mu) = (I + O(\mu^{-1}))Y_j^\infty(\mu)$. Then the connection formula $Y_0^\infty(\mu) = X_0^0(\mu)G$ becomes $Y_{0,*}^\infty(\mu) = X_{0,*}^0(\mu)Gc^{-\sigma_3/2}$, which induces the action

$$\mathbf{ac}\colon (S_0^\infty, S_1^\infty, S_0^0, G) \mapsto (c^{\sigma_3/2} S_0^\infty c^{-\sigma_3/2}, c^{\sigma_3/2} S_1^\infty c^{-\sigma_3/2}, S_0^0, Gc^{-\sigma_3/2})$$

on $\mathcal{M}$. As shown in [26, Sect. 3.5], each solution of (1.1) corresponds to an orbit by the action **ac**, and the quotient of $\mathcal{M}$ consisting of these orbits is a nonsingular affine cubic surface $V_a(\mathcal{M}) \subset \mathbb{C}^3$ parametrised by a. Then $g_{11}g_{22}$, g_{21}/g_{11}, g_{12}/g_{22} are invariants under **ac**, and two of them may be coordinates of a generic point on $V_a(\mathcal{M})$.

As in [17, Thms. 3.1, 3.2, 3.3] and [18, Thms. 2.1, 2.2, 2.3], a solution of (1.1) labelled by G is parametrised by $g_{11}g_{22}$, g_{12}/g_{22}, g_{21}/g_{11}, provided that (3.1) is an isomonodromy system governed by (1.1). The following relation suggests that we are allowed to use the same monodromy invariants in parametrising our solutions of (1.2) as in [17] and [18] (cf. Examples 2.1, 2.2, 2.3):

Proposition 3.2. *Let* $(Y_0^{\infty,*}(\lambda), \hat{Y}_0^0(\lambda)) = (\hat{Y}_0^\infty(\lambda)\Theta_{0,*}^{-\sigma_3}, \hat{Y}_0^0(\lambda))$ *be a pair of matrix solutions of* (1.4) *near* $\lambda = \infty$ *and* 0, *where* $\Theta_{0,*} = \Theta_0 c_0^{1/2+ia}$ *with* $c_0 = (3/2)\sqrt{\epsilon b}\tau^{1/2}x^{-1}$. *Then, for this pair, the corresponding Stokes matrices and connection matrix coincide with* S_0^∞, S_1^∞, S_0^0 *and* G *for* $(Y_0^\infty(\mu), X_0^0(\mu))$ *of* (3.1).

Proof. Note that (3.1) is changed into (1.4) by the transformation $U = \Theta_0^{\sigma_3}\Psi$, $\mu = c_0\lambda$ with $c_0 = (3/2)\sqrt{\epsilon b}\tau^{1/2}x^{-1}$. Set $Y_0^{\infty,*}(\lambda) = \hat{Y}_0^0(\lambda)G^*$. Then

$$\begin{aligned}(\Theta_0^{-\sigma_3} Y_0^\infty(c_0\lambda), \Theta_0^{-\sigma_3} X_0^0(c_0\lambda)) &= (\hat{Y}_0^\infty(\lambda)\Theta_0^{-\sigma_3} c_0^{-(1/2+ia)\sigma_3}, \hat{Y}_0^0(\lambda)) \\ &= (Y_0^{\infty,*}(\lambda), \hat{Y}_0^0(\lambda))\end{aligned}$$

solves (1.4). Insertion of this into $Y_0^\infty(\mu) = X_0^0(\mu)G$ yields $G = G^*$. Let $S_0^{\infty,*}$, $S_1^{\infty,*}$ and $S_0^{0,*}$ be the Stokes matrices for $(Y_0^{\infty,*}(\lambda), \hat{Y}_0^0(\lambda))$. Then the equation of the monodromy manifold is

$$G S_0^{\infty,*} S_1^{\infty,*} \sigma_3 e^{\pi(i/2-a)\sigma_3} = S_0^{0,*}\sigma_1 G,$$

which yields the entries of $S_0^{\infty,*}$, $S_1^{\infty,*}$ and $S_0^{0,*}$ in terms of g_{ij} coinciding with those of S_0^∞, S_1^∞ and S_0^0 derived from (3.6) as in [17, p. 1172]. This completes the proof. □

Remark 3.2. We have $G = \widehat{G}\Theta_0^{-\sigma_3} c_0^{-(1/2+ia)\sigma_3} = \widehat{G}\Theta_{0,*}^{-\sigma_3}$, $S_m^\infty = \Theta_{0,*}^{\sigma_3}\hat{S}_m^\infty\Theta_{0,*}^{-\sigma_3}$ and $S_m^0 = \hat{S}_m^0$.

Equation (3.6) of the monodromy manifold may be extended.

Proposition 3.3. *For* $m = 1, 2, 3, \dots,$

$$GS_0^\infty S_1^\infty \cdots S_{2m-2}^\infty S_{2m-1}^\infty \sigma_3^m e^{m\pi(i/2-a)\sigma_3} = S_0^0 \cdots S_{m-1}^0 \sigma_1^m G,$$
$$GS_{-1}^\infty S_{-2}^\infty \cdots S_{-2m+1}^\infty S_{-2m}^\infty \sigma_3^m e^{m\pi(a-i/2)\sigma_3} = S_{-1}^0 \cdots S_{-m}^0 \sigma_1^m G.$$

Proof. Recall the relations $Y_k^\infty(\mu) = \sigma_3 Y_{k+2}^\infty(\mu e^{\pi i})\sigma_3 e^{-\pi(a-i/2)\sigma_3}$ and $X_k^0(\mu) = \sigma_3 X_{k+1}^0(\mu e^{\pi i})\sigma_1$ given by [17, eqn. (24)]. Then

$$\begin{aligned} Y_0^\infty(\mu) S_0^\infty S_1^\infty \cdots S_{2m-2}^\infty S_{2m-1}^\infty &= Y_{2m}^\infty(\mu) = \sigma_3 Y_{2(m-1)}^\infty(\mu e^{-\pi i})\sigma_3 e^{\pi(a-i/2)\sigma_3} \\ &= \cdots = \sigma_3^m Y_0^\infty(\mu e^{-m\pi i})\sigma_3^m e^{m\pi(a-i/2)\sigma_3}, \\ Y_0^0(\mu) S_0^0 \cdots S_{m-1}^0 &= Y_m^0(\mu) = \sigma_3 Y_{m-1}^0(\mu e^{-\pi i})\sigma_1 \\ &= \cdots = \sigma_3^m Y_0^0(\mu e^{-m\pi i})\sigma_1^m. \end{aligned}$$

Using $Y_0^\infty(\mu) = Y_0^0(\mu)G$ and $Y_0^\infty(\mu e^{-m\pi i}) = Y_0^0(\mu e^{-m\pi i})G$, we have

$$\begin{aligned} Y_0^0(\mu) G S_0^\infty S_1^\infty \cdots S_{2m-2}^\infty S_{2m-1}^\infty &= \sigma_3^m Y_0^0(\mu e^{-m\pi i}) G \sigma_3^m e^{m\pi(a-i/2)\sigma_3} \\ &= Y_0^0(\mu) S_0^0 \cdots S_{m-1}^0 \sigma_1^m G \sigma_3^m e^{m\pi(a-i/2)\sigma_3}, \end{aligned}$$

which implies the first relation. □

The formulas above are also written as follows:

Proposition 3.4. *For* $m = 1, 2, 3, \dots,$

$$GS_0^\infty S_1^\infty \cdots S_{2m-2}^\infty S_{2m-1}^\infty = (S_0^0\sigma_1)^m G\sigma_3^m e^{m\pi(a-i/2)\sigma_3},$$
$$GS_{-1}^\infty S_{-2}^\infty \cdots S_{-2m+1}^\infty S_{-2m}^\infty = (\sigma_1 S_0^0)^m G\sigma_3^m e^{m\pi(i/2-a)\sigma_3}.$$

Proof. By (3.7), $S_{j-1}^0\sigma_1^j = \sigma_1 S_{j-2}^0\sigma_1^{j-1} = \cdots = \sigma_1^{j-1}S_0^0\sigma_1$, and hence

$$S_0^0 \cdots S_{m-1}^0 \sigma_1^m G = (S_0^0\sigma_1)^m G, \quad S_{-1}^0 \cdots S_{-m}^0 \sigma_1^m G = (\sigma_1 S_0^0)^m G.$$

Combining these with Proposition 3.3, we have the desired result. □

§4. WKB analysis

§4.1. Turning points and Stokes graphs

Let us examine the characteristic roots $\pm\mu = \pm\mu(t,\lambda)$ of $\mathcal{B}(t,\lambda)$, the turning points, i.e. the roots of μ, and the Stokes graph, which are used in calculating monodromy data for system (1.4). The characteristic roots are given by

$$\mu^2 = b_1^2 + b_2^2 + b_3^2$$
$$= -e^{2i\phi}\lambda^2 + e^{2i\phi}a_\phi\lambda^{-2} - 4e^{2i\phi}\lambda^{-4} + 3ie^{i\phi}(1+2ia)t^{-1} \tag{4.1}$$

with

$$a_\phi = a_\phi(t) = e^{-2i\phi}\Big(\frac{y^t}{y} + \frac{1}{2t}\Big)^2 + 4y + \frac{1}{y^2} - 3ie^{-i\phi}(1+2ia)\frac{1}{ty}. \tag{4.2}$$

The Stokes graph consists of the Stokes curves and the vertices: each Stokes curve is defined by $\mathrm{Re}\int_{\lambda_*}^{\lambda}\mu(\lambda)\,d\lambda = 0$ with a turning point λ_*, and the vertices are turning points or singular points $\lambda = 0, \infty$. Here, $\mu(\lambda)$ is considered on a two-sheeted Riemann surface glued along cuts with ends of turning points or singular points.

First suppose that $\phi = 0$. If $a_0 = a_{\phi=0} = 3\cdot 2^{2/3}$, then

$$\mu(\infty,\lambda)^2|_{\phi=0} = -\lambda^2 + a_0\lambda^{-2} - 4\lambda^{-4} = -\lambda^{-4}(\lambda^2 - 2^{1/3})^2(\lambda^2 + 4^{2/3}).$$

This means that $\mu(t,\lambda)$ admits six turning points λ_0, λ_1, λ_0', λ_1', λ_2, λ_2' such that λ_0 and λ_1 coalesce at $2^{1/6}$, λ_0' and λ_1' at $-2^{1/6}$ as $t\to\infty$, and that λ_2 and λ_2' approach $\pm 2^{2/3}i$, respectively. The Stokes graph with $\phi = 0$ is used in [17, Sect. 4]. (Note that a solution $y(x)$ of (1.2) for $x = t > 0$ corresponds to $u(\tau)$ satisfying (1.1) for $\tau > 0$ if $\epsilon b > 0$.) The limit Stokes graph with $t = \infty$ is as in Figure 2(c) and $\mu(\lambda)$ is defined on the two-sheeted Riemann surface $\mathcal{R}_0$ glued along, say $[\lambda_2, e^{\pi i/2}0]\cup[\lambda_2', e^{-\pi i/2}0]$.

The limit Stokes graph for the isomonodromy system (1.4) is considered to reflect the Boutroux equations (2.1). When ϕ increases or decreases, the limit turning points for λ_0 and λ_1 move according to the solution A_ϕ of the Boutroux equations (2.1). By Proposition 8.16, for ϕ close to 0, the double turning point at $2^{1/6}$ is resolved into two simple turning points such that $\mathrm{Im}\,\lambda_0 > 0 > \mathrm{Im}\,\lambda_1$, $\mathrm{Re}\,\lambda_0 < 2^{1/6} < \mathrm{Re}\,\lambda_1$ if $\phi > 0$, and that $\mathrm{Im}\,\lambda_0 < 0 < \mathrm{Im}\,\lambda_1$, $\mathrm{Re}\,\lambda_0 < 2^{1/6} < \mathrm{Re}\,\lambda_1$ if $\phi < 0$. As will be shown in Proposition 8.15, for $0 < |\phi| < \pi/3$ the coalescence of turning points does not occur, and then topological properties of the limit Stokes graph remain invariant. Every turning point is simple, and the two-sheeted Riemann surface $\mathcal{R}_\phi$ of $\mu(\lambda)$ is glued along the cuts $[\lambda_0,\lambda_1]$, $[\lambda_0',\lambda_1']$ and $[\lambda_2, e^{(\pi-\phi)i/2}0]\cup[\lambda_2', e^{-(\pi+\phi)i/2}0]$. The Stokes graph lies on the upper sheet of $\mathcal{R}_\phi$. For $-\pi/3 < \phi < 0$ and $0 < \phi < \pi/3$, the limit Stokes graphs are as in Figures 2(b) and (d), in which each limit turning point with $t = \infty$ is also denoted by λ_ι or λ_ι'. In our calculation, for $0 < |\phi| < \pi/3$, we use the Stokes curve from 0 to ∞ passing through λ_0 and λ_1 appearing as a resolution of the double turning point. For a technical reason, the cut $[\lambda_0,\lambda_1]$ on the upper sheet is made in such a way

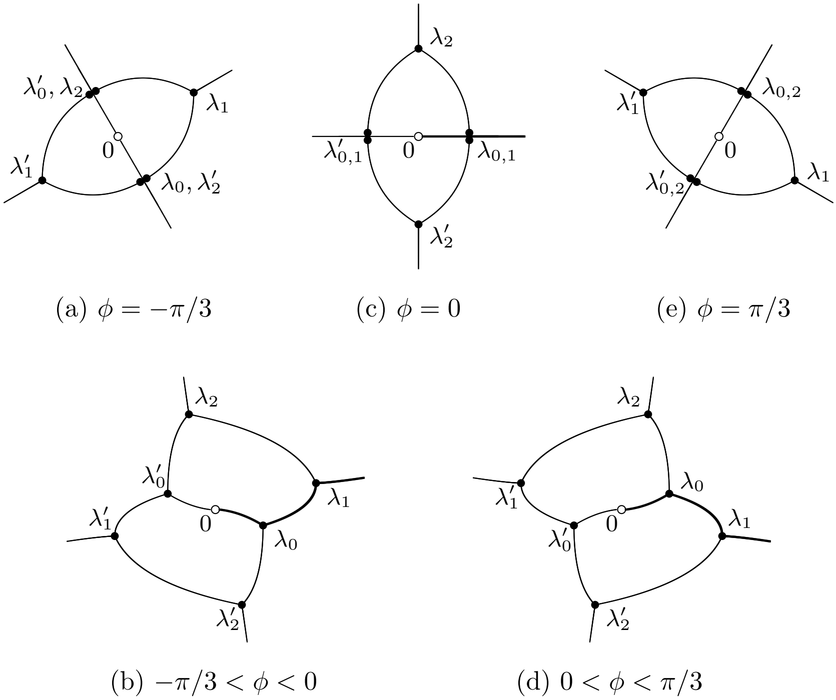

Figure 2. Limit Stokes graphs for $|\phi| \le \pi/3$

that the Stokes curve $(\lambda_0, \lambda_1)^\sim$ is located along the lower shore (respectively, the upper shore) of the cut if $0 < \phi < \pi/3$ (respectively, $-\pi/3 < \phi < 0$), and the cut $[0, \lambda_2]$ in such a way that the cut $[\lambda_0, \lambda_1]$ is located on the right-hand side of $[0, \lambda_2]$ (cf. Figures 3, 4, 5).

Let us set

$$\mu(t, \lambda) = ie^{i\phi}\lambda^{-2}\sqrt{4 - a_\phi\lambda^2 + \lambda^6 - 3ie^{-i\phi}(1 + 2ia)\lambda^4 t^{-1}}.$$

This square root is defined as the product of the form

$$\begin{aligned} -ie^{-i\phi}\lambda^2\mu(\infty, \lambda) &= 2\sqrt{(1 - \lambda_{0,\infty}^{-2}\lambda^2)(1 - \lambda_{1,\infty}^{-2}\lambda^2)(1 - \lambda_{2,\infty}^{-2}\lambda^2)} \\ &= 2\sqrt{1 - \lambda_{0,\infty}^{-2}\lambda^2}\sqrt{1 - \lambda_{1,\infty}^{-2}\lambda^2}\sqrt{1 - \lambda_{2,\infty}^{-2}\lambda^2} \end{aligned}$$

with $\lambda_{j,\infty} = \lambda_j(\infty)$ satisfying $\lambda_{0,\infty}^2\lambda_{1,\infty}^2\lambda_{2,\infty}^2 = -4$, in which the branch of each minor square root is fixed in such a way that $\sqrt{1 - \lambda_{j,\infty}^{-2}\lambda^2} \to 1$ as $\lambda \to 0$ on the upper sheet. Then $\mu(t,\lambda) \to -ie^{i\phi}\lambda + O(1)$ as $\lambda \to \infty$ and $\mu(t,\lambda) \to 2ie^{i\phi}\lambda^{-2} + O(1)$ as $\lambda \to 0$ on the upper sheet.

An unbounded domain $D \subset \mathcal{R}_\phi$ is called a canonical domain if, for each $\lambda \in D$, there exist contours $C_\pm(\lambda) \subset D$ terminating in λ such that

$$\mathrm{Re}\int_{\lambda_-}^{\lambda} \mu(\lambda)\,d\lambda \to -\infty \quad \left(\text{respectively, } \mathrm{Re}\int_{\lambda_+}^{\lambda} \mu(\lambda)\,d\lambda \to +\infty\right)$$

as $\lambda_- \to \infty$ along $C_-(\lambda)$ (respectively, as $\lambda_+ \to \infty$ along $C_+(\lambda)$) (see [4], [5, p. 242]). The interior of a canonical domain contains exactly one Stokes curve, and its boundary consists of Stokes curves.

§4.2. WKB solution

The following WKB solution will be used in our calculus:

Proposition 4.1. *In the canonical domain whose interior contains a Stokes curve issuing from the turning point λ_0 or λ_1, system (1.4) with $\mathcal{B}(\lambda,t)$ given by (3.2) admits an asymptotic solution expressed as*

$$\Psi_{\mathrm{WKB}}(\lambda) = T(I + O(t^{-\delta}))\exp\left(\int_{\tilde{\lambda}_*}^{\lambda} \Lambda(\tau)\,d\tau\right), \quad T = \begin{pmatrix} 1 & \frac{b_3-\mu}{b_1+ib_2} \\ \frac{\mu-b_3}{b_1-ib_2} & 1 \end{pmatrix}$$

outside suitable neighbourhoods of zeros of $b_1 \pm ib_2$ as long as $|\lambda - \lambda_\iota| \gg t^{-2/3+(2/3)\delta}$ ($\iota = 0,1,2$). Here, δ is an arbitrary number such that $0 < \delta < 1$, $\tilde{\lambda}_*$ is a base point near λ_0 or λ_1, and

$$\Lambda(\lambda) = \frac{t}{3}\mu(t,\lambda)\sigma_3 - \mathrm{diag}\, T^{-1}T_\lambda.$$

Proof. This is shown by using $\mu = -ie^{i\phi}\lambda + O(1)$ near $\lambda = \infty$, and $= 2ie^{i\phi}\lambda^{-2} + O(1)$ near $\lambda = 0$ (cf. [5, Thm. 7.2], [28, Prop. 3.8]). □

Remark 4.1. In the proposition above

$$\begin{aligned}\mathrm{diag}\, T^{-1}T_\lambda &= \frac{1}{2\mu(\mu+b_3)}(i(b_1b_2' - b_1'b_2)\sigma_3 + (b_3\mu' - b_3'\mu)I) \\ &= \frac{1}{4}\left(1 - \frac{b_3}{\mu}\right)\frac{\partial}{\partial\lambda}\log\frac{b_1+ib_2}{b_1-ib_2}\sigma_3 + \frac{1}{2}\frac{\partial}{\partial\lambda}\log\frac{\mu}{\mu+b_3}I,\end{aligned}$$

where $b_1' = (\partial/\partial\lambda)b_1$.

§4.3. Local solution around a turning point

Near turning points the WKB solution above fails in expressing asymptotic behaviour. In the neighbourhood of λ_ι, system (1.4) is reduced to

$$\frac{dW}{d\zeta} = \begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix} W, \tag{4.3}$$

which has the solutions ${}^T(\mathrm{Ai}(\zeta), \mathrm{Ai}_\zeta(\zeta))$, ${}^T(\mathrm{Bi}(\zeta), \mathrm{Bi}_\zeta(\zeta))$ with the Airy function $\mathrm{Ai}(\zeta)$ and $\mathrm{Bi}(\zeta) = e^{-\pi i/6}\mathrm{Ai}(e^{-2\pi i/3}\zeta)$ ([1, 3]). Then we have the following solution near each simple turning point ([5, Thm. 7.3], [28, Prop. 3.9]):

Proposition 4.2. *For each simple turning point λ_ι ($\iota = 0, 1, 2$) write $c_k = b_k(\lambda_\iota)$, $c'_k = (b_k)_\lambda(\lambda_\iota)$ ($k = 1, 2, 3$), and suppose that c_k, c'_k are bounded and $c_1 \pm ic_2 \ne 0$. Let $\hat{t} = 2(2\kappa_c)^{-1/3}(c_1 - ic_2)(t/3)^{1/3}$ with $\kappa_c = c_1c'_1 + c_2c'_2 + c_3c'_3$. Then system (1.4) admits a matrix solution of the form*

$$\Phi_\iota(\lambda) = T_\iota(I + O(t^{-\delta'}))\begin{pmatrix} 1 & 0 \\ 0 & \hat{t}^{-1} \end{pmatrix} W(\zeta), \quad T_\iota = \begin{pmatrix} 1 & -\frac{c_3}{c_1+ic_2} \\ -\frac{c_3}{c_1-ic_2} & 1 \end{pmatrix},$$

in which $\lambda - \lambda_\iota = (2\kappa_c)^{-1/3}(t/3)^{-2/3}(\zeta + \zeta_0)$ with $|\zeta_0| \ll t^{-1/3}$, as long as $|\zeta| \ll t^{(2/3-\delta')/3}$, i.e. $|\lambda - \lambda_\iota| \ll t^{-2/3+(2/3-\delta')/3}$. Here, δ' is an arbitrary number such that $0 < \delta' < 2/3$, and $W(\zeta)$ solves system (4.3) having canonical solutions $W_\nu(\zeta)$ ($\nu \in \mathbb{Z}$) such that

$$W_\nu(\zeta) = \zeta^{-(1/4)\sigma_3}(\sigma_3 + \sigma_1)(I + O(\zeta^{-3/2}))\exp((2/3)\zeta^{3/2}\sigma_3)$$

as $\zeta \to \infty$ through the sector $|\arg\zeta - (2\nu - 1)\pi/3| < 2\pi/3$, and that $W_{\nu+1}(\zeta) = W_\nu(\zeta)S_\nu$ with

$$S_1 = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, \quad S_{\nu+1} = \sigma_1 S_\nu \sigma_1.$$

Remark 4.2. Putting $\lambda - \lambda_\iota = (2\kappa_c)^{-1/3}(e^{2\pi i/3})^{2j}(t/3)^{-2/3}(\zeta + \zeta_0)$, $j \in \{0, \pm 1\}$, we have an expression of $\Phi_\iota(\lambda)$ with $\hat{t} = 2(2\kappa_c)^{-1/3}(e^{2\pi i/3})^{2j}(c_1 - ic_2)(t/3)^{1/3}$.

§5. Calculation of the connection matrix

We calculate the connection matrix $\widehat{G} = (\hat{g}_{ij})$ given by (3.5) as a solution of the direct monodromy problem by applying WKB analysis to system (1.4). Suppose that $a_\phi(t)$ is given by (4.2) with a pair of arbitrary functions $(y, y^t) = (y(t), y^t(t))$ not necessarily solving (1.2), and that

$$a_\phi(t) = A_\phi + \frac{B_\phi(t)}{t}, \quad B_\phi(t) \ll 1 \tag{5.1}$$

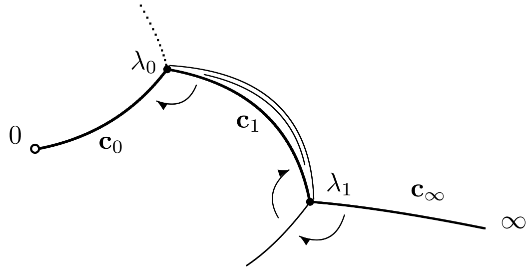

Figure 3. Stokes curve for $0 < \phi < \pi/3$

for $t \in S_\phi(t'_\infty, \kappa_1, \delta_1)$ with given $\kappa_1 > 0$, small given $\delta_1 > 0$ and sufficiently large $t'_\infty > 0$. Here, A_ϕ is a solution of the Boutroux equations (2.1), and

$$S_\phi(t'_\infty, \kappa_1, \delta_1) = \big\{ t \;\big|\; \operatorname{Re} t > t'_\infty,\ |\operatorname{Im} t| < \kappa_1,\ |y(t)| + |y^t(t)| + |y(t)|^{-1} < \delta_1^{-1} \big\}.$$

Let $0 < \phi < \pi/3$. We calculate the analytic continuation of the matrix solution near $\lambda = \infty$ along the Stokes curve consisting of

$$\mathbf{c}_\infty = (\infty, \lambda_1)^\sim, \quad \mathbf{c}_1 = (\lambda_1, \lambda_0)^\sim, \quad \mathbf{c}_0 = (\lambda_0, 0)^\sim$$

starting from ∞ and terminating at 0 on the upper sheet of the Riemann surface $\mathcal{R}_\phi$ of $\mu(\infty, \lambda)$ as in Figure 3. Under supposition (5.1), these curves $\mathbf{c}_0$, $\mathbf{c}_1$, $\mathbf{c}_\infty$ lie within the distance $O(t^{-1})$ from the limit Stokes graph. Recall that the curve $\mathbf{c}_1$ is located along the lower shore of the cut $[\lambda_0, \lambda_1]$.

In the WKB solution, write $\Lambda(\lambda)$ in the component-wise form $\Lambda(\lambda) = \Lambda_3(\lambda) + \Lambda_I(\lambda)$ with

$$\Lambda_3(\lambda) = \frac{t}{3}\mu(t,\lambda)\sigma_3 - \operatorname{diag} T^{-1}T_\lambda|_{\sigma_3}\sigma_3, \quad \Lambda_I(\lambda) = -\operatorname{diag} T^{-1}T_\lambda|_I I,$$

in which $\operatorname{diag} T^{-1}T_\lambda|_{\sigma_3}\sigma_3 \in \mathbb{C}\sigma_3$, $\operatorname{diag} T^{-1}T_\lambda|_I I \in \mathbb{C}I$. In Propositions 4.1 and 4.2, if $\delta = \delta' = 2/9 - \varepsilon$ with any ε such that $0 < \varepsilon < 2/9$, then both propositions are applicable in the annulus

$$\mathcal{A}^\iota_\varepsilon \colon t^{-2/3+(2/3)(2/9-\varepsilon)} \ll |\lambda - \lambda_\iota| \ll t^{-2/3+(2/3)(2/9+\varepsilon/2)}$$

$(\iota = 0, 1)$. In what follows we set $\delta = 2/9 - \varepsilon$, and write $c_k = b_k(\lambda_0)$, $d_k = b_k(\lambda_1)$ $(k = 1, 2, 3)$.

(1) Let $\Psi_\infty(\lambda)$ along $\mathbf{c}_\infty = (\infty, \lambda_1)^\sim$ be a WKB solution by Proposition 4.1, and let $Y_0^{\infty,*}(\lambda) = \widehat{Y}_0^{\infty}(\lambda)\Theta_{0,*}^{-\sigma_3}$ be given by (3.3) and Proposition 3.2. Set

$Y_0^{\infty,*}(\lambda)\Theta_{0,*}^{\sigma_3} = \widehat{Y}_0^{\infty}(\lambda) = \Psi_\infty(\lambda)\Gamma_\infty$. Using $\mu(t,\lambda) = -ie^{i\phi}\lambda - \frac{3}{2}(1+2ia)t^{-1}\lambda^{-1} + O(\lambda^{-3})$ along $\mathbf{c}_\infty$, and $\mu - b_3 \ll \lambda^{-1}$ as $\lambda \to \infty$, we have

$$\begin{aligned}\Gamma_\infty &= \Psi_\infty(\lambda)^{-1}\widehat{Y}_0^{\infty}(\lambda) = \Psi_\infty(\lambda)^{-1}Y_0^{\infty,*}(\lambda)\Theta_{0,*}^{\sigma_3}\\ &= \exp\Big(-\int_{\tilde\lambda_1}^{\lambda}\Lambda(\tau)\,d\tau\Big)T^{-1}(I+O(|t^{-\delta}|+|\lambda|^{-1}))\\ &\quad\times\exp\Big(-\frac{1}{6}(ie^{i\phi}t\lambda^2+3(1+2ia)\log\lambda)\sigma_3\Big)\\ &= C_3(\tilde\lambda_1)c_I(\tilde\lambda_1)(I+O(t^{-\delta}))\\ &\quad\times\exp\Big(-\lim_{\substack{\lambda\to\infty\\ \lambda\in\mathbf{c}_\infty}}\Big(\int_{\lambda_1}^{\lambda}\Lambda_3(\tau)\,d\tau+\frac{1}{6}(ie^{i\phi}t\lambda^2+3(1+2ia)\log\lambda)\sigma_3\Big)\Big),\end{aligned}$$

in which $C_3(\tilde\lambda_1) = \exp(\int_{\lambda_1}^{\tilde\lambda_1}\Lambda_3(\tau)\,d\tau)$, $c_I(\tilde\lambda_1) = \exp(-\int_{\tilde\lambda_1}^{\infty}\Lambda_I(\tau)\,d\tau)$, and $\tilde\lambda_1 \in \mathbf{c}_\infty$, $\tilde\lambda_1 - \lambda_1 \asymp t^{-1}$.

(2) For $\Psi_\infty(\lambda)$ and for $\Phi_1^+(\lambda)$ given by Proposition 4.2 in the annulus $\mathcal{A}_\varepsilon^1$ around λ_1, set $\Psi_\infty(\lambda) = \Phi_1^+(\lambda)\Gamma_{1+}$ along $\mathbf{c}_\infty$. Suppose that the curve $(2\kappa_d)^{1/3}(\lambda - \tilde\lambda_1) = (t/3)^{-2/3}(\zeta + O(t^{-1/3}))$, $\kappa_d = d_1d_1' + d_2d_2' + d_3d_3'$ with $\lambda \in \mathbf{c}_\infty$ enters the sector $|\arg\zeta - 7\pi/3| < 2\pi/3$ (the other cases are similarly treated by Remark 4.2). Write $K^{-1} = 2(2\kappa_d)^{-1/3}(d_1 - id_2)$. Then, by Propositions 4.1 and 4.2,

$$\begin{aligned}\Gamma_{1+} &= \Phi_1^+(\lambda)^{-1}\Psi_\infty(\lambda)\\ &= W(\zeta)^{-1}\begin{pmatrix}1 & 0\\ 0 & (t/3)^{-1/3}K\end{pmatrix}^{-1}(I+O(t^{-\delta}))\begin{pmatrix}1 & -\frac{d_3}{d_1+id_2}\\ -\frac{d_3}{d_1-id_2} & 1\end{pmatrix}^{-1}\\ &\quad\times\begin{pmatrix}1 & \frac{b_3-\mu}{b_1+ib_2}\\ \frac{\mu-b_3}{b_1-ib_2} & 1\end{pmatrix}(I+O(t^{-\delta}))\exp\Big(\int_{\tilde\lambda_1}^{\lambda}\Lambda(\tau)\,d\tau\Big)\\ &= W(\zeta)^{-1}\begin{pmatrix}1 & \frac{d_3}{d_1+id_2}\\ \frac{(t/3)^{1/3}\mu}{2K(d_1-id_2)} & \frac{(t/3)^{1/3}\mu}{2Kd_3}\end{pmatrix}(I+O(t^{-\delta}))\exp\Big(\int_{\tilde\lambda_1}^{\lambda}\Lambda(\tau)\,d\tau\Big)\end{aligned}$$

for $\lambda \in \mathcal{A}_\varepsilon^1 \cap \mathbf{c}_\infty$, where $(\mu - b_3)/(b_1 \pm ib_2) = (\mu - d_3)/(d_1 \pm id_2) + O(\eta)$, $\eta = \lambda - \tilde\lambda_1$. Since $\mu = (2\kappa_d)^{1/2}\eta^{1/2}(1+O(\eta)) = 2K(d_1 - id_2)(t/3)^{-1/3}\zeta^{1/2}(1+O(\eta))$, we have

$$\Gamma_{1+} = \exp\Big(\int_{\tilde\lambda_1}^{\lambda}\Lambda(\tau)\,d\tau - \frac{2}{3}\zeta^{3/2}\sigma_3\Big)\zeta^{1/4}(I+O(t^{-\delta}))\begin{pmatrix}1 & 0\\ 0 & -\frac{d_1-id_2}{d_3}\end{pmatrix}.$$

By $\Lambda_3(\lambda) = ((2\kappa_d)^{1/2}(t/3)\eta^{1/2}(1+O(\eta)) + O(\eta^{-1/2}))\sigma_3$ and $\Lambda_I(\lambda) = (-\eta^{-1}/4 + O(\eta^{-1/2}))I$ (cf. Remark 4.1) for $\eta = \lambda - \tilde\lambda_1$, $\lambda \in \mathcal{A}_\varepsilon^1 \cap \mathbf{c}_\infty$,

$$\Gamma_{1+} = (\tilde\zeta_1)^{1/4}(I+O(t^{-\delta}))C_3(\tilde\lambda_1)^{-1}\begin{pmatrix}1 & 0\\ 0 & -\frac{d_1-id_2}{d_3}\end{pmatrix}$$

with suitably chosen $\tilde{\zeta}_1 \asymp \tilde{\lambda}_1 - \lambda_1$.

(3) Let $\Phi_1^-(\lambda)$ be the solution by Proposition 4.2 near $\mathbf{c}_1 = (\lambda_1, \lambda_0)^\sim$, and set $\Phi_1^+(\lambda) = \Phi_1^-(\lambda)\Gamma_{1*}$, where $\Phi_1^+(\lambda)$ is the analytic continuation along an arc in $\mathcal{A}_\varepsilon^1$ in the clockwise direction. Then by Proposition 4.2,

$$\Gamma_{1*} = \Phi_1^-(\lambda)^{-1}\Phi_1^+(\lambda) = S_2 S_3 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}.$$

(4) For $\Phi_1^-(\lambda)$ and the WKB solution $\Psi_1^-(\lambda)$ along $\mathbf{c}_1$, set $\Phi_1^-(\lambda) = \Psi_1^-(\lambda)\Gamma_{1-}$. Then, supposing the curve $(2\kappa_d)^{1/3}(\lambda - \tilde{\lambda}_1') = (t/3)^{-2/3}(\zeta + O(t^{-1/3}))$ with $\lambda \in \mathbf{c}_1$ to be in the sector $|\arg\zeta - \pi| < 2\pi/3$, we have, for $\tilde{\lambda}_1' \in \mathbf{c}_1$, $|\tilde{\lambda}_1' - \lambda_1| \asymp t^{-1}$,

$$\begin{aligned}
\Gamma_{1-} &= \Psi_1^-(\lambda)^{-1}\Phi_1^-(\lambda) \\
&= \exp\Big(-\int_{\tilde{\lambda}_1'}^{\lambda} \Lambda(\tau)\,d\tau\Big)(I + O(t^{-\delta}))\begin{pmatrix} 1 & \frac{b_3-\mu}{b_1+ib_2} \\ \frac{\mu-b_3}{b_1-ib_2} & 1 \end{pmatrix}^{-1} \\
&\quad \times \begin{pmatrix} 1 & -\frac{d_3}{d_1+id_2} \\ -\frac{d_3}{d_1-id_2} & 1 \end{pmatrix}(I + O(t^{-\delta}))\begin{pmatrix} 1 & 0 \\ 0 & (t/3)^{-1/3}\widetilde{K} \end{pmatrix} W(\zeta) \\
&= \exp\Big(\frac{2}{3}\zeta^{3/2}\sigma_3 - \int_{\tilde{\lambda}_1'}^{\lambda} \Lambda(\tau)\,d\tau\Big)\zeta^{-1/4}(I + O(t^{-\delta}))\begin{pmatrix} 1 & 0 \\ 0 & -\frac{d_3}{d_1-id_2} \end{pmatrix},
\end{aligned}$$

where $\widetilde{K}^{-1} = 2(2\kappa_d)^{-1/3}(d_1 - id_2)$. This yields

$$\Gamma_{1-} = (\tilde{\zeta}_1')^{-1/4}(I + O(t^{-\delta}))C_3'(\tilde{\lambda}_1')\begin{pmatrix} 1 & 0 \\ 0 & -\frac{d_3}{d_1-id_2} \end{pmatrix}$$

with $C_3'(\tilde{\lambda}_1') = \exp(\int_{\lambda_1}^{\tilde{\lambda}_1'} \Lambda_3(\tau)\,d\tau)$ for some $\tilde{\zeta}_1' \asymp \tilde{\lambda}_1' - \lambda_1$.

(5) For $\Psi_1^-(\lambda)$ and the WKB solution $\Psi_0^+(\lambda)$ along $\mathbf{c}_1$ near λ_0, set $\Psi_1^-(\lambda) = \Psi_0^+(\lambda)\Gamma_{01}$. Then, for $\tilde{\lambda}_0 \in \mathbf{c}_1$, $\tilde{\lambda}_0 - \lambda_0 \asymp t^{-1}$,

$$\begin{aligned}
\Gamma_{01} &= \Psi_0^+(\lambda)^{-1}\Psi_1^-(\lambda) \\
&= \exp\Big(-\int_{\tilde{\lambda}_0}^{\lambda} \Lambda(\tau)\,d\tau\Big)T^{-1}(I + O(t^{-\delta}))T\exp\Big(\int_{\tilde{\lambda}_1'}^{\lambda} \Lambda(\tau)\,d\tau\Big) \\
&= C_3'(\tilde{\lambda}_1')^{-1}C_3''(\tilde{\lambda}_0)c_I(\tilde{\lambda}_1', \tilde{\lambda}_0)\exp\Big(-\int_{\lambda_0}^{\lambda_1} \Lambda_3(\tau)\,d\tau\Big),
\end{aligned}$$

where $C_3''(\tilde{\lambda}_0) = \exp(\int_{\lambda_0}^{\tilde{\lambda}_0} \Lambda_3(\tau)\,d\tau)$, $c_I(\tilde{\lambda}_1', \tilde{\lambda}_0) = \exp(-\int_{\tilde{\lambda}_0}^{\tilde{\lambda}_1'} \Lambda_I(\tau)\,d\tau)$.

(6) For $\Psi_0^+(\lambda)$ and for $\Phi_0^+(\lambda)$ given by Proposition 4.2 in the annulus $\mathcal{A}_\varepsilon^0$ around λ_0, set $\Psi_0^+(\lambda) = \Phi_0^+(\lambda)\Gamma_{0+}$. Then, by the same argument as in (2) above, we have

$$\Gamma_{0+} = \Phi_0^+(\lambda)^{-1}\Psi_0^+(\lambda) = (\tilde{\zeta}_0)^{1/4}(I + O(t^{-\delta}))C_3''(\tilde{\lambda}_0)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -\frac{c_1 - ic_2}{c_3} \end{pmatrix}$$

for some $\tilde{\zeta}_0 \asymp \tilde{\lambda}_0 - \lambda_0$.

(7) Let $\Phi_0^-(\lambda)$ be the solution by Proposition 4.2 near $\mathbf{c}_0 = (\lambda_0, 0)^\sim$, and set $\Phi_0^+(\lambda) = \Phi_0^-(\lambda)\Gamma_{0*}$, where $\Phi_0^+(\lambda)$ is the analytic continuation along an arc in $\mathcal{A}_\varepsilon^0$ in the clockwise direction. Then by Proposition 4.2,

$$\Gamma_{0*} = \Phi_0^-(\lambda)^{-1}\Phi_0^+(\lambda) = S_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}.$$

(8) For $\Phi_0^-(\lambda)$ and the WKB solution $\Psi_0(\lambda)$ along $\mathbf{c}_0$, set $\Phi_0^-(\lambda) = \Psi_0(\lambda)\Gamma_{0-}$. By the same argument as in (4), we have

$$\Gamma_{0-} = \Psi_0(\lambda)^{-1}\Phi_0^-(\lambda) = (\tilde{\zeta}_0')^{-1/4}(I + O(t^{-\delta}))\widehat{C}_3(\tilde{\lambda}_0')\begin{pmatrix} 1 & 0 \\ 0 & -\frac{c_3}{c_1 - ic_2} \end{pmatrix}$$

with $\widehat{C}_3(\tilde{\lambda}_0') = \exp(\int_{\lambda_0}^{\tilde{\lambda}_0'} \Lambda_3(\tau)\,d\tau)$ for some $\tilde{\zeta}_0' \asymp \tilde{\lambda}_0' - \lambda_0$.

(9) For $\Psi_0(\lambda)$ and $\widehat{Y}_0^0(\lambda)$ given by (3.4), set $\Psi_0(\lambda) = \widehat{Y}_0^0(\lambda)\Gamma_0$. Then

$$\begin{aligned}\Gamma_0 &= \widehat{Y}_0^0(\lambda)^{-1}\Psi_0(\lambda) \\ &= \exp\Big(\frac{2i}{3}e^{i\phi}t\lambda^{-1}\sigma_3\Big)\frac{\sqrt{2}}{i}(\sigma_1 + \sigma_3)^{-1}(I + O(|t^{-\delta}| + |\lambda|))T\exp\Big(\int_{\tilde{\lambda}_0'}^{\lambda}\Lambda(\tau)\,d\tau\Big).\end{aligned}$$

Note that $\mu(t, \lambda) = 2ie^{i\phi}\lambda^{-2} + O(1)$ as $\lambda \to 0$ along $\mathbf{c}_0$. Since

$$(\sigma_1 + \sigma_3)^{-1}\lim_{\lambda\to 0}T(\lambda) = \frac{1}{2}(\sigma_1 + \sigma_3)\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \sigma_3,$$

we have

$$\Gamma_0 = \widehat{C}_3(\tilde{\lambda}_0')^{-1}\hat{c}_I(\tilde{\lambda}_0')(\sigma_3 + O(t^{-\delta}))\exp\Big(\lim_{\substack{\lambda\to 0 \\ \lambda\in\mathbf{c}_0}}\Big(\int_{\lambda_0}^{\lambda}\Lambda_3(\tau)\,d\tau + \frac{2i}{3}e^{i\phi}t\lambda^{-1}\sigma_3\Big)\Big)$$

with $\hat{c}_I(\tilde{\lambda}_0') = -\sqrt{2}i\exp(\int_{\tilde{\lambda}_0'}^{0}\Lambda_I(\tau)\,d\tau)$.

Collecting the matrices above, we have the connection matrix

$$\begin{aligned}
\widehat{G} &= G\Theta_{0,*}^{\sigma_3} = \widehat{Y}_0^0(\lambda)^{-1} Y_0^{\infty,*}(\lambda)\Theta_{0,*}^{\sigma_3} = \widehat{Y}_0^0(\lambda)^{-1}\widehat{Y}_0^{\infty}(\lambda) \\
&= \Gamma_0\Gamma_{0-}\Gamma_{0\,*}\Gamma_{0+}\Gamma_{01}\Gamma_{1-}\Gamma_{1*}\Gamma_{1+}\Gamma_{\infty} \\
&= \epsilon_+ i(\sigma_3 + O(t^{-\delta}))\exp(J_0\sigma_3)\begin{pmatrix}1 & 0\\ 0 & -c_0^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ -i & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -c_0\end{pmatrix} \\
&\quad \times \exp(-J_1\sigma_3)\begin{pmatrix}1 & 0\\ 0 & -d_0^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ -i & 1\end{pmatrix}\begin{pmatrix}1 & -i\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -d_0\end{pmatrix}\exp(-J_\infty\sigma_3) \\
&= \epsilon_+(I + O(t^{-\delta})) \\
&\quad \times \begin{pmatrix} i\exp(J_0 - J_1 - J_\infty) & -d_0\exp(J_0 - J_1 + J_\infty) \\ (c_0^{-1}\exp(-J_1) + d_0^{-1}\exp(J_1))\exp(-J_0 - J_\infty) & ic_0^{-1}d_0\exp(-J_0 - J_1 + J_\infty)\end{pmatrix}
\end{aligned}$$

if $0 < \phi < \pi/3$, where $\epsilon_+^2 = 1$, $c_0 = (c_1 - ic_2)/c_3$, $d_0 = (d_1 - id_2)/d_3$, and

$$J_0\sigma_3 = \lim_{\substack{\lambda\to 0\\ \lambda\in\mathbf{c}_0}}\left(\int_{\lambda_0}^{\lambda}\Lambda_3(\tau)\,d\tau + \frac{2i}{3}e^{i\phi}t\lambda^{-1}\sigma_3\right), \tag{5.2}$$

$$J_1\sigma_3 = \int_{\lambda_0}^{\lambda_1}\Lambda_3(\tau)\,d\tau \quad (\text{along } \mathbf{c}_1), \tag{5.3}$$

$$J_\infty\sigma_3 = \lim_{\substack{\lambda\to\infty\\ \lambda\in\mathbf{c}_\infty}}\left(\int_{\lambda_1}^{\lambda}\Lambda_3(\tau)\,d\tau + \frac{1}{6}(ie^{i\phi}t\lambda^2 + 3(1+2ia)\log\lambda)\sigma_3\right). \tag{5.4}$$

In the case $-\pi/3 < \phi < 0$, from the analytic continuation along the Stokes curves as in Figure 4, it follows that

$$\begin{aligned}
\widehat{G} &= \epsilon_- i(\sigma_3 + O(t^{-\delta}))\exp(J_0\sigma_3)\begin{pmatrix}1 & 0\\ 0 & -c_0^{-1}\end{pmatrix}\begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -c_0\end{pmatrix} \\
&\quad \times \exp(-\hat{J}_1\sigma_3)\begin{pmatrix}1 & 0\\ 0 & -d_0^{-1}\end{pmatrix}\begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ i & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -d_0\end{pmatrix}\exp(-J_\infty\sigma_3) \\
&= \epsilon_-(I + O(t^{-\delta})) \\
&\quad \times \begin{pmatrix} -ic_0d_0^{-1}\exp(J_0 + \hat{J}_1 - J_\infty) & (c_0\exp(\hat{J}_1) + d_0\exp(-\hat{J}_1))\exp(J_0 + J_\infty) \\ -d_0^{-1}\exp(-J_0 + \hat{J}_1 - J_\infty) & -i\exp(-J_0 + \hat{J}_1 + J_\infty)\end{pmatrix}.
\end{aligned}$$

Here, $\epsilon_-^2 = 1$, and

$$\hat{J}_1\sigma_3 = \int_{\lambda_0}^{\lambda_1}\Lambda_3(\tau)\,d\tau \quad (\text{along } \hat{\mathbf{c}}_1), \tag{5.5}$$

in which $\hat{\mathbf{c}}_1$ is a curve joining λ_0 to λ_1 located along the upper shore of the cut on the upper sheet of $\mathcal{R}_\phi$. Thus we have the following proposition:

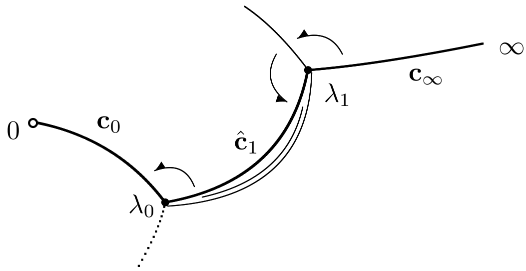

Figure 4. Stokes curve for $-\pi/3 < \phi < 0$

Proposition 5.1. *Let $c_0 = (c_1 - ic_2)/c_3$, $d_0 = (d_1 - id_2)/d_3$ with $c_k = b_k(\lambda_0)$, $d_k = b_k(\lambda_1)$ for $k = 1, 2, 3$. If $0 < \phi < \pi/3$, then*

$$\widehat{G} = \epsilon_+(I + O(t^{-\delta}))$$
$$\times \begin{pmatrix} i\exp(J_0 - J_1 - J_\infty) & -d_0\exp(J_0 - J_1 + J_\infty) \\ (c_0^{-1}\exp(-J_1) + d_0^{-1}\exp(J_1))\exp(-J_0 - J_\infty) & ic_0^{-1}d_0\exp(-J_0 - J_1 + J_\infty) \end{pmatrix},$$

and, if $-\pi/3 < \phi < 0$, then

$$\widehat{G} = \epsilon_-(I + O(t^{-\delta}))$$
$$\times \begin{pmatrix} -ic_0d_0^{-1}\exp(J_0 + \hat{J}_1 - J_\infty) & (c_0\exp(\hat{J}_1) + d_0\exp(-\hat{J}_1))\exp(J_0 + J_\infty) \\ -d_0^{-1}\exp(-J_0 + \hat{J}_1 - J_\infty) & -i\exp(-J_0 + \hat{J}_1 + J_\infty) \end{pmatrix}.$$

Here, $\epsilon_\pm^2 = 1$, and J_0, J_1, $\hat{J}_1$, J_∞ are integrals given by (5.2) through (5.5).

From the proposition above with $\widehat{G} = G\Theta_{0,*}^{\sigma_3}$, $G = (g_{ij})$ (Remark 3.2), we derive key relations.

Corollary 5.2. *If $0 < \phi < \pi/3$ and $g_{11}g_{12}g_{22} \neq 0$, then*

$$g_{11}g_{22} = -c_0^{-1}d_0(1 + O(t^{-\delta}))\exp(-2J_1),$$
$$\frac{g_{12}}{g_{22}} = ic_0(1 + O(t^{-\delta}))\exp(2J_0).$$

If $-\pi/3 < \phi < 0$ and $g_{11}g_{21}g_{22} \neq 0$, then

$$g_{11}g_{22} = -c_0d_0^{-1}(1 + O(t^{-\delta}))\exp(2\hat{J}_1),$$
$$\frac{g_{21}}{g_{11}} = -ic_0^{-1}(1 + O(t^{-\delta}))\exp(-2J_0).$$

§6. Asymptotic properties of monodromy data

§6.1. Expressions of J_0, J_1 and $\hat{J}_1$

To examine asymptotic properties of J_0, J_1 and $\hat{J}_1$, we make the change of variables $\lambda^{-2} = z$. Then, by (4.1) and (4.2), $\mu(t,\lambda)$ becomes

$$\begin{aligned}\mu(t,\lambda)\,d\lambda &= \Big(-\frac{e^{2i\phi}}{z} + e^{2i\phi}a_\phi z - 4e^{2i\phi}z^2 + 3ie^{i\phi}(1+2ia)t^{-1}\Big)^{1/2}\frac{(-z^{-3/2})}{2}\,dz \\ &= \Big(-\frac{i}{2}e^{i\phi}\frac{w(z)}{z^2} - \frac{3}{4}(1+2ia)t^{-1}\frac{1}{zw(z)} + O(t^{-2}w(z)^{-3})\Big)\,dz\end{aligned}$$

with $w(z)^2 = w(a_\phi, z)^2 = 4z^3 - a_\phi z^2 + 1$, for z such that $w(z) \gg 1$. The turning points λ_0, λ_1, λ_2 and 0 on $\mathcal{R}_\phi$ are mapped to

$$z_0 = \lambda_0^{-2}, \quad z_1 = \lambda_1^{-2}, \quad z_2 = \lambda_2^{-2}$$

and ∞, respectively, on the elliptic curve Π_{a_ϕ} for $w(a_\phi, z)$ constructed in the same way as in the case of Π_{A_ϕ} in Section 2.2. The branch of $\mu(t,\lambda)$ is compatible with that of $w(a_\phi, z)$. Suppose that Π_{a_ϕ} is equipped with the cycles $\mathbf{a}$ and $\mathbf{b}$ as in Section 2.2. Then the inverse image of the cycle $\mathbf{a}$ is a closed curve $\mathbf{a}_\lambda$ surrounding the cut $[\lambda_0, \lambda_1]$ anticlockwise (see Figure 5).

Since

$$\int \frac{w(z)}{z^2}\,dz = 2\frac{w(z)}{z} - a_\phi\int\frac{dz}{w(z)} + 3\int\frac{dz}{z^2 w(z)},$$

we have

$$\begin{aligned}\mu(t,\lambda)\,d\lambda = -ie^{i\phi}\frac{w(z)}{z} + \frac{i}{2}e^{i\phi}a_\phi\frac{dz}{w(z)} - \frac{3i}{2}e^{i\phi}\frac{dz}{z^2w(z)} \\ - \frac{3}{4}(1+2ia)t^{-1}\frac{dz}{zw(z)} + O(t^{-2}w(z)^{-3})\,dz,\end{aligned}$$

in which $w(z)/z = 2z^{1/2} + O(z^{-1/2})$ as $z \to \infty$. Hence

$$\begin{aligned}&\lim_{\substack{\lambda\to 0\\ \lambda\in\mathbf{c}_0}}\Big(\int_{\lambda_0}^{\lambda}\mu(t,\tau)\,d\tau + 2ie^{i\phi}\lambda^{-1}\Big) \\ &\quad= -\frac{i}{4}e^{i\phi}a_\phi\int_{\mathbf{b}}\frac{dz}{w(z)} + \frac{3i}{4}e^{i\phi}\int_{\mathbf{b}}\frac{dz}{z^2w(z)} + \frac{3}{8}(1+2ia)t^{-1}\int_{\mathbf{b}}\frac{dz}{zw(z)} \\ &\qquad + O(t^{-2}) \\ &\quad= \frac{i}{4}e^{i\phi}\int_{\mathbf{b}}\frac{w(z)}{z^2}\,dz + \frac{3}{8}(1+2ia)t^{-1}\int_{\mathbf{b}}\frac{dz}{zw(z)} + O(t^{-2}),\end{aligned} \tag{6.1}$$

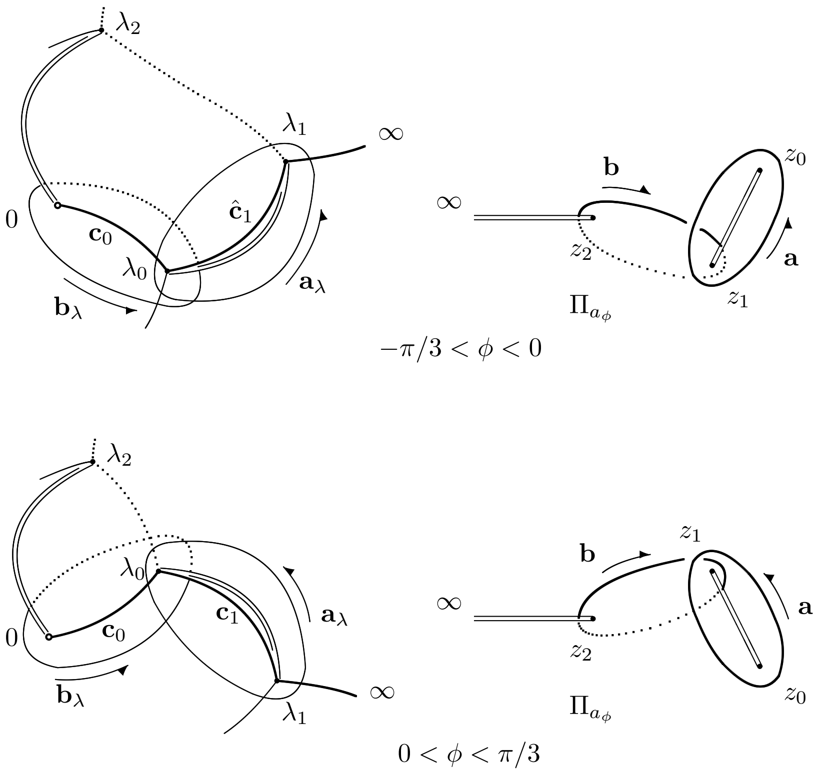

Figure 5. Correspondence of the cycles under the map $z = \lambda^{-2}$

and

$$\int_{\lambda_0(\mathbf{c}_1)}^{\lambda_1} \mu(t,\tau)\,d\tau, \quad -\int_{\lambda_0(\hat{\mathbf{c}}_1)}^{\lambda_1} \mu(t,\tau)\,d\tau$$

$$= \frac{i}{4} e^{i\phi} a_\phi \int_{\mathbf{a}} \frac{dz}{w(z)} - \frac{3i}{4} e^{i\phi} \int_{\mathbf{a}} \frac{dz}{z^2 w(z)} - \frac{3}{8}(1+2ia)t^{-1} \int_{\mathbf{a}} \frac{dz}{z w(z)} + O(t^{-2})$$

$$= -\frac{i}{4} e^{i\phi} \int_{\mathbf{a}} \frac{w(z)}{z^2}\,dz - \frac{3}{8}(1+2ia)t^{-1} \int_{\mathbf{a}} \frac{dz}{z w(z)} + O(t^{-2}), \tag{6.2}$$

in which $\int_{\lambda_0(\mathbf{c})}^{\lambda_1}$ denotes the integral along the contour $\mathbf{c}$. By Remark 4.1,

$$\Lambda_3(t,\lambda) = \Big(\frac{t}{3}\mu(t,\lambda) - \operatorname{diag} T^{-1}T_\lambda|_{\sigma_3}\Big)\sigma_3, \tag{6.3}$$

$$\operatorname{diag} T^{-1}T_\lambda|_{\sigma_3} = \frac{1}{4}\Big(1 - \frac{b_3}{\mu}\Big)\frac{\partial}{\partial\lambda}\log\frac{b_1 + ib_2}{b_1 - ib_2}.$$

To calculate J_0, J_1 and $\hat{J}_1$, it is necessary to know $\operatorname{diag} T^{-1}T_\lambda|_{\sigma_3}$ in addition to (6.1) and (6.2). Note that, by (3.2),

$$b_1 = 2ie^{i\phi}\lambda^{-2} - iK_+, \quad b_2 = K_-, \quad b_3 = -ie^{i\phi}\lambda - K_0\lambda^{-1},$$

with $K_\pm = e^{i\phi}y \pm \frac{1}{2}iy^{-1}\Gamma_0(t,y,y^t)$, $K_0 = \Gamma_0(t,y,y^t) + \frac{3}{2}(1+2ia)t^{-1}$, $\Gamma_0(t,y,y^t) = y^t y^{-1} - ie^{i\phi}y^{-1} - (1+3ia)t^{-1}$. Setting $z_\pm = e^{-i\phi}(K_+ \pm K_-)/2$, i.e.

$$z_+ = y, \quad z_- = (i/2)e^{-i\phi}y^{-1}\Gamma_0(t,y,y^t), \tag{6.4}$$

and $\lambda^{-2} = z$, we have

$$b_1 - ib_2 = 2ie^{i\phi}(z - z_+), \quad b_1 + ib_2 = 2ie^{i\phi}(z - z_-). \tag{6.5}$$

By (4.1), $\mu^2 = -e^{2i\phi}\lambda^2 w(z)^2 + O(t^{-1})$, which implies $\mu = ie^{i\phi}\lambda(w(z) + O(t^{-1}z))$ on the upper sheet of Π_{a_ϕ}, and hence

$$\frac{b_3}{\mu} = -ie^{-i\phi}\frac{b_3}{\lambda}\Big(\frac{1}{w(z)} + O(t^{-1}z^{-2})\Big),$$

where $b_3/\lambda = -K_0 z - ie^{i\phi}$ satisfies $(b_3/\lambda)(z_\pm) = -(\mu/\lambda)(z_\pm) = -ie^{i\phi}w(z_\pm) + O(t^{-1})$, since $\mu(z_\pm)^2 = (b_1 - ib_2)(b_1 + ib_2)(z_\pm) + b_3(z_\pm)^2 = b_3(z_\pm)^2$ by (6.5). These facts combined with (6.5) yield

$$\begin{aligned}
\operatorname{diag} T^{-1}T_\lambda|_{\sigma_3}\, d\lambda &= \frac{1}{4}\Big(1 - \frac{b_3}{\mu}\Big)\frac{d}{d\lambda}\log\frac{b_1 + ib_2}{b_1 - ib_2}\, d\lambda \\
&= \frac{1}{4}\Big(1 - \frac{b_3}{\mu}\Big)\frac{d}{dz}\log\frac{b_1 + ib_2}{b_1 - ib_2}\, dz \\
&= \frac{1}{4}\Big(1 + ie^{-i\phi}\frac{b_3}{\lambda}\Big(\frac{1}{w(z)} + O(t^{-1}z^{-2})\Big)\Big)\Big(\frac{1}{z - z_-} - \frac{1}{z - z_+}\Big)\, dz \\
&= -\frac{1}{4}\Big(\frac{1}{z - z_+} - \frac{1}{z - z_-} + \Big(\frac{w(z_+)}{z - z_+} - \frac{w(z_-)}{z - z_-}\Big)\frac{1}{w(z)} \\
&\qquad + O(t^{-1}z^{-2})\Big)\, dz,
\end{aligned}$$

which implies

$$\lim_{\substack{\lambda\to 0\\ \lambda\in\mathbf{c}_0}}\int_{\lambda_0}^{\lambda}\operatorname{diag} T^{-1}T_\lambda|_{\sigma_3}\, d\lambda = \frac{1}{4}\log\frac{z_0 - z_+}{z_0 - z_-} + \frac{1}{8}\int_{\mathbf{b}}\Big(\frac{w(z_+)}{z - z_+} - \frac{w(z_-)}{z - z_-}\Big)\frac{dz}{w(z)} + O(t^{-1}).$$

Here, by (6.5), $c_0^2 = (c_1 - ic_2)^2/c_3^2 = -(c_1 - ic_2)/(c_1 + ic_2) = -(z_0 - z_+)/(z_0 - z_-)$ and $\log((z_0 - z_+)/(z_0 - z_-)) = \log(-c_0^2) = 2\log(ic_0)$. Similarly,

$$\begin{aligned}
&-\int_{\lambda_0(\mathbf{c}_1)}^{\lambda_1}\operatorname{diag} T^{-1}T_\lambda|_{\sigma_3}\, d\lambda + \frac{1}{2}\log(c_0 d_0^{-1}), \quad \int_{\lambda_0(\hat{\mathbf{c}}_1)}^{\lambda_1}\operatorname{diag} T^{-1}T_\lambda|_{\sigma_3}\, d\lambda - \frac{1}{2}\log(c_0 d_0^{-1}) \\
&\quad = \frac{1}{8}\int_{\mathbf{a}}\Big(\frac{w(z_+)}{z - z_+} - \frac{w(z_-)}{z - z_-}\Big)\frac{dz}{w(z)} + O(t^{-1}).
\end{aligned}$$

Insertion of (6.1), (6.2) and the relations above into (5.2), (5.5) with (6.3) provides the expressions of J_0, J_1 and $\hat{J}_1$. Then by Corollary 5.2 we have the following proposition:

Proposition 6.1. *Let*

$$W(z) = \Big(\frac{w(z_+)}{z - z_+} - \frac{w(z_-)}{z - z_-}\Big)\frac{1}{w(z)}.$$

(1) *Suppose that* $g_{11}g_{22} \neq 0$, $g_{12}/g_{22} \neq 0$. *For* $0 < \phi < \pi/3$,

$$\log \frac{g_{12}}{g_{22}} = \frac{ie^{i\phi}t}{6}\int_{\mathbf{b}} \frac{w(z)}{z^2}\,dz - \frac{1}{4}\int_{\mathbf{b}} W(z)\,dz + \frac{1}{4}(1+2ia)\int_{\mathbf{b}} \frac{dz}{zw(z)} + O(t^{-\delta}),$$

$$\log(g_{11}g_{22}) = \frac{ie^{i\phi}t}{6}\int_{\mathbf{a}} \frac{w(z)}{z^2}\,dz - \frac{1}{4}\int_{\mathbf{a}} W(z)\,dz + \frac{1}{4}(1+2ia)\int_{\mathbf{a}} \frac{dz}{zw(z)} + \pi i + O(t^{-\delta}).$$

(2) *Suppose that* $g_{11}g_{22} \neq 0$, $g_{21}/g_{11} \neq 0$. *For* $-\pi/3 < \phi < 0$,

$$\log \frac{g_{21}}{g_{11}} = -\frac{ie^{i\phi}t}{6}\int_{\mathbf{b}} \frac{w(z)}{z^2}\,dz + \frac{1}{4}\int_{\mathbf{b}} W(z)\,dz - \frac{1}{4}(1+2ia)\int_{\mathbf{b}} \frac{dz}{zw(z)} + O(t^{-\delta}),$$

$$\log(g_{11}g_{22}) = \frac{ie^{i\phi}t}{6}\int_{\mathbf{a}} \frac{w(z)}{z^2}\,dz - \frac{1}{4}\int_{\mathbf{a}} W(z)\,dz + \frac{1}{4}(1+2ia)\int_{\mathbf{a}} \frac{dz}{zw(z)} + \pi i + O(t^{-\delta}).$$

Remark 6.1. In the proposition above,

$$\frac{ie^{i\phi}t}{6}\int_{\mathbf{a},\mathbf{b}} \frac{w(z)}{z^2}\,dz = -\frac{ie^{i\phi}a_\phi t}{6}\int_{\mathbf{a},\mathbf{b}} \frac{dz}{w(z)} + \frac{ie^{i\phi}t}{2}\int_{\mathbf{a},\mathbf{b}} \frac{dz}{z^2 w(z)}.$$

§6.2. Expressions by the ϑ-function

For $w(z)^2 = w(a_\phi, z)^2 = 4z^3 - a_\phi z^2 + 1$, the differential equation $(dz/du)^2 = w(a_\phi, z)^2$ defines the Weierstrass $\wp$-function

$$z = \wp(u; g_2, g_3) + \frac{a_\phi}{12}, \quad g_2 = \frac{a_\phi^2}{12}, \quad g_3 = -1 + \frac{a_\phi^3}{216}.$$

The periods of $\wp(u; g_2, g_3)$ are

$$\omega_{\mathbf{a}} = \int_{\mathbf{a}} \frac{dz}{w(a_\phi, z)}, \quad \omega_{\mathbf{b}} = \int_{\mathbf{b}} \frac{dz}{w(a_\phi, z)}, \quad \tau = \frac{\omega_{\mathbf{b}}}{\omega_{\mathbf{a}}}, \quad \operatorname{Im}\tau > 0,$$

where $\mathbf{a}$ and $\mathbf{b}$ are the cycles on the elliptic curve $\Pi_{a_\phi} = \Pi_+ \cup \Pi_-$ for $w(a_\phi, z)$ in Section 6.1 (cf. Figure 5). The ϑ-function $\vartheta(z,\tau) = \vartheta(z)$ is defined by

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2 + 2\pi i z n},$$

and we set

$$\nu = \frac{1+\tau}{2}$$

(cf. [6, 31]). For z, $\tilde{z} \in \Pi_{a_\phi} = \Pi_+ \cup \Pi_-$, let

$$F(\tilde{z}, z) = \frac{1}{\omega_{\mathbf{a}}} \int_{\tilde{z}}^{z} \frac{dz}{w(z)} = \frac{1}{\omega_{\mathbf{a}}} \int_{\infty}^{z} \frac{dz}{w(z)} - \frac{1}{\omega_{\mathbf{a}}} \int_{\infty}^{\tilde{z}} \frac{dz}{w(z)}.$$

For any $z_0 \in \Pi_{a_\phi}$ denote the projections of z_0 on the respective sheets by $z_0^+ = (z_0, w(z_0)) = (z_0, w(z_0^+))$ and $z_0^- = (z_0, -w(z_0)) = (z_0, -w(z_0^+))$. If $z_0 \in \Pi_+$ (respectively, $z_0 \in \Pi_-$), then $z_0^{\pm} \in \Pi_{\pm}$ (respectively, $z_0^{\pm} \in \Pi_{\mp}$).

Proposition 6.2. *For any* $z_0 \in \Pi_{a_\phi}$,

$$\frac{dz}{(z-z_0)w(z)} = \frac{1}{w(z_0^+)} d\log \frac{\vartheta(F(z_0^+, z)+\nu, \tau)}{\vartheta(F(z_0^-, z)+\nu, \tau)} - g_0(z_0)\frac{dz}{w(z)},$$

$$g_0(z_0) = \frac{w'(z_0^+)}{2w(z_0^+)} - \frac{1}{\omega_{\mathbf{a}}}\frac{1}{w(z_0^+)}\Big(\pi i + \frac{\vartheta'}{\vartheta}(F(z_0^-, z_0^+)+\nu, \tau)\Big).$$

Proof. For $z_0 = \wp(u_0) + a_\phi/12 \in \Pi_{a_\phi}$ let $u_0^{\pm}$ be such that $z_0^{\pm} = \wp(u_0^{\pm}) + a_\phi/12$. Then

$$\begin{aligned}
\frac{dz}{(z-z_0)w(z)} &= \frac{du}{\wp(u) - \wp(u_0)} \\
&= \frac{1}{w(z_0^+)}\Big(\zeta(u-u_0^+) - \zeta(u-u_0^-) + \zeta(u_0^+ - u_0^-) - \frac{1}{2}w'(z_0^+)\Big)\, du \\
&= \frac{1}{w(z_0^+)} d\log\frac{\sigma(u-u_0^+)}{\sigma(u-u_0^-)} + \frac{1}{w(z_0^+)}\Big(\zeta(u_0^+ - u_0^-) - \frac{1}{2}w'(z_0^+)\Big)\, du.
\end{aligned}$$

From

$$\begin{aligned}
d\log\frac{\sigma(u-u_0^+)}{\sigma(u-u_0^-)} &= -\frac{2\eta_{\mathbf{a}}}{\omega_{\mathbf{a}}}(u_0^+ - u_0^-)\, du + d\log\frac{\vartheta(F(z_0^+, z)+\nu, \tau)}{\vartheta(F(z_0^-, z)+\nu, \tau)}, \\
\zeta(u_0^+ - u_0^-) &= \frac{\sigma'}{\sigma}(u_0^+ - u_0^-) \\
&= \frac{2\eta_{\mathbf{a}}}{\omega_{\mathbf{a}}}(u_0^+ - u_0^-) + \frac{\pi i}{\omega_{\mathbf{a}}} + \frac{1}{\omega_{\mathbf{a}}}\frac{\vartheta'}{\vartheta}(F(z_0^-, z_0^+)+\nu, \tau)
\end{aligned}$$

with $F(z_0^{\pm}, z) = \omega_{\mathbf{a}}^{-1}\int_{z_0^{\pm}}^{z} dz/w(z)$, the desired formula follows. □

Observe that

$$\begin{aligned}
&\log \vartheta(F(z_0^{\pm}, z) + \nu, \tau)|_{\mathbf{a}} = 0, \\
&\log \frac{\vartheta(F(z_0^{+}, z) + \nu, \tau)}{\vartheta(F(z_0^{-}, z) + \nu, \tau)}\Big|_{\mathbf{b}} = \log \frac{\vartheta(F(z_0^{+}, z_{\mathbf{b}}) + \tau + \nu, \tau)\vartheta(F(z_0^{-}, z_{\mathbf{b}}) + \nu, \tau)}{\vartheta(F(z_0^{-}, z_{\mathbf{b}}) + \tau + \nu, \tau)\vartheta(F(z_0^{+}, z_{\mathbf{b}}) + \nu, \tau)} \\
&\qquad = \log \exp\bigl(-\pi i(2(F(z_0^{+}, z_{\mathbf{b}}) + \nu) + \tau)\bigr) \\
&\qquad\quad + \log \exp\bigl(\pi i(2(F(z_0^{-}, z_{\mathbf{b}}) + \nu) + \tau)\bigr) \\
&\qquad = 2\pi i F(z_0^{-}, z_0^{+})
\end{aligned}$$

for $z_{\mathbf{b}} \in \mathbf{b} \cap (\Pi_+)^{\mathrm{cl}} \cap (\Pi_-)^{\mathrm{cl}}$, since $\vartheta(z \pm \tau, \tau) = e^{-\pi i(\tau \pm 2z)}\vartheta(z, \tau)$, where $(\Pi_+)^{\mathrm{cl}}$ denotes the closure of Π_+. Then

$$\begin{aligned}
&\int_{\mathbf{a}} \frac{dz}{(z - z_0)w(z)} = -g_0(z_0)\omega_{\mathbf{a}}, \\
&\int_{\mathbf{b}} \frac{dz}{(z - z_0)w(z)} = \frac{2\pi i}{w(z_0^{+})} F(z_0^{-}, z_0^{+}) + \tau \int_{\mathbf{a}} \frac{dz}{(z - z_0)w(z)}.
\end{aligned}$$

Differentiation of both sides with respect to z_0 at $z_0 = 0$ yields

$$\int_{\mathbf{b}} \frac{dz}{z^2 w(z)} = \frac{4\pi i}{\omega_{\mathbf{a}}} + \tau \int_{\mathbf{a}} \frac{dz}{z^2 w(z)}.$$

Using these formulas we have the following proposition:

Proposition 6.3. *For $W(z)$ as in Proposition 6.1 and for $z_{\pm}$ by (6.4),*

$$\begin{aligned}
\int_{\mathbf{a}} W(z)\, dz &= -(w(z_+)g_0(z_+) - w(z_-)g_0(z_-))\omega_{\mathbf{a}} \\
&= -\frac{1}{2}(w'(z_+^{+}) - w'(z_-^{+}))\omega_{\mathbf{a}} \\
&\quad + \frac{\vartheta'}{\vartheta}(F(z_+^{-}, z_+^{+}) + \nu, \tau) - \frac{\vartheta'}{\vartheta}(F(z_-^{-}, z_-^{+}) + \nu, \tau), \\
\biggl(\int_{\mathbf{b}} - \tau \int_{\mathbf{a}}\biggr) W(z)\, dz &= 2\pi i(F(z_+^{-}, z_+^{+}) - F(z_-^{-}, z_-^{+})),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbf{a}} \frac{dz}{z w(z)} = -g_0(0^{+})\omega_{\mathbf{a}}, \quad g_0(0^{+}) = \frac{1}{\omega_{\mathbf{a}}}\Bigl(\pi i + \frac{\vartheta'}{\vartheta}(F(0^{-}, 0^{+}) + \nu, \tau)\Bigr), \\
&\biggl(\int_{\mathbf{b}} - \tau \int_{\mathbf{a}}\biggr) \frac{dz}{z w(z)} = -2\pi i F(0^{-}, 0^{+}), \\
&\biggl(\int_{\mathbf{b}} - \tau \int_{\mathbf{a}}\biggr) \frac{dz}{z^2 w(z)} = \frac{4\pi i}{\omega_{\mathbf{a}}}.
\end{aligned}$$

Remark 6.2. In the proposition above, the first formula is rewritten in the form

$$\int_{\mathbf{a}} W(z)\,dz = 2\Big(\frac{\vartheta'}{\vartheta}\Big(\frac{1}{2}F(z_+^-, z_+^+) + \nu, \tau\Big) - \frac{\vartheta'}{\vartheta}\Big(\frac{1}{2}F(z_-^-, z_-^+) + \nu, \tau\Big)\Big).$$

The right-hand side is obtained by comparing the poles of $(\vartheta'/\vartheta)(\frac{1}{2}F(z^-, z^+)+\nu, \tau)$ with those of $-\frac{1}{2}w'(z^+) + (\vartheta'/\vartheta)(F(z^-, z^+) + \nu, \tau)$ on Π_{a_ϕ}, and showing that the difference is a constant (see also [15, pp. 117–119]).

§6.3. Expression of $B_\phi(t)$

Let us write the quantity $B_\phi(t)$ in terms of

$$\Omega_{\mathbf{a}} = \int_{\mathbf{a}} \frac{dz}{w(A_\phi, z)}, \qquad \Omega_{\mathbf{b}} = \int_{\mathbf{b}} \frac{dz}{w(A_\phi, z)},$$
$$\mathcal{J}_{\mathbf{a}} = \int_{\mathbf{a}} \frac{w(A_\phi, z)}{z^2}\,dz, \quad \mathcal{J}_{\mathbf{b}} = \int_{\mathbf{b}} \frac{w(A_\phi, z)}{z^2}\,dz$$

with $w(A_\phi, z) = \sqrt{4z^3 - A_\phi z^2 + 1}$ and $\mathbf{a}$, $\mathbf{b}$ on $\Pi_{A_\phi} = \Pi_+^* \cup \Pi_-^* = \lim_{a_\phi(t)\to A_\phi} \Pi_{a_\phi}$. By (5.1) the cycles $\mathbf{a}$ and $\mathbf{b}$ on Π_{a_ϕ} may be regarded as those on Π_{A_ϕ}, and are independent of t for sufficiently large t.

Let $0 < \phi < \pi/3$. By Proposition 6.3, the integral $\int_{\mathbf{a}} W(z)\,dz$ is expressed in terms of $\vartheta_*(\pm) = (\vartheta'/\vartheta)(\frac{1}{2}F(z_\pm^-, z_\pm^+) + \nu, \tau)$ (Remark 6.2) or $w'(z_\pm^+)$ and $(\vartheta'/\vartheta)(F(z_\pm^-, z_\pm^+) + \nu, \tau)$, in which

$$F(z_\pm^-, z_\pm^+) = \frac{1}{\omega_{\mathbf{a}}} \int_{z_\pm^-}^{z_\pm^+} \frac{dz}{w(a_\phi, z)} = \frac{2}{\omega_{\mathbf{a}}} \int_{\infty}^{z_\pm^+} \frac{dz}{w(a_\phi, z)}.$$

Note that $\int_{\mathbf{a}} W(z)\,dz$ has no poles or zeros in $S_\phi(t'_\infty, \kappa_1, \delta_1)$. Indeed, if, say $\vartheta_*(+)$ or $\vartheta_*(-) = \infty$ at $t = t_*$, then z_+ or $z_- = \infty$, and hence t_* is a pole or a zero of $y(t)$, or a pole of $y^t(t)$, which is excluded from $S_\phi(t'_\infty, \kappa_1, \delta_1)$. Consider $z_\pm = z_\pm(t)$ (cf. (6.4)) moving on the elliptic curve Π_{a_ϕ} crossing **a**- and **b**-cycles, and then $F(z_\pm^-, z_\pm^+) = 2p_\pm(t) + 2q_\pm(t)\tau + O(1)$ with $p_\pm(t)$, $q_\pm(t) \in \mathbb{Z}$. This implies the boundedness of $\mathrm{Re}(\vartheta'/\vartheta)(\frac{1}{2}F(z_\pm^-, z_\pm^+) + \nu, \tau)$ or $\mathrm{Re}(\vartheta'/\vartheta)(F(z_\pm^-, z_\pm^+) + \nu, \tau)$ in $S_\phi(t'_\infty, \kappa_1, \delta_1)$, and hence the modulus of $\mathrm{Re} \int_{\mathbf{a}} W(z)\,dz$ is uniformly bounded in $S_\phi(t'_\infty, \kappa_1, \delta_1)$. Note that, by (5.1),

$$\begin{aligned}\frac{1}{z^2}(w(a_\phi, z) - w(A_\phi, z)) &= \frac{1}{z^2}\Big(\sqrt{4z^3 - a_\phi z^2 + 1} - \sqrt{4z^3 - A_\phi z^2 + 1}\Big)\\ &= -\frac{t^{-1}B_\phi(t)}{2w(A_\phi, z)}\big(1 + O(t^{-1}B_\phi(t))\big).\end{aligned}$$

By using this and Proposition 6.3, the second formula in Proposition 6.1(1) is written in the form

$$\log(g_{11}g_{22}) = \frac{ie^{i\phi}t}{6}\int_{\mathbf{a}}\Big(\frac{w(A_\phi,z)}{z^2} - \frac{t^{-1}B_\phi(t)}{2w(A_\phi,z)}\Big)\,dz \\ -\frac{1}{4}\int_{\mathbf{a}} W(z)\,dz - \frac{1}{4}(1+2ia)g_0(0^+)\omega_{\mathbf{a}} + \pi i + O(t^{-\delta}),$$

which implies

$$ie^{i\phi}\Big(t\mathcal{J}_{\mathbf{a}} - \frac{\Omega_{\mathbf{a}}}{2}B_\phi(t)\Big) \\ = \frac{3}{2}\int_{\mathbf{a}} W(z)\,dz + \frac{3}{2}(1+2ia)g_0(0^+)\omega_{\mathbf{a}} + 6\log(g_{11}g_{22}) - 6\pi i + O(t^{-\delta}).$$

Recall that $G = \widehat{G}\Theta_{0,*}^{-\sigma_3} = (g_{ij})$, $g_{ij} = g_{ij}(t)$ is a solution of the direct monodromy problem. Suppose that

$$(6.6)\qquad |\log(g_{11}g_{22})| \ll 1, \quad |\log(g_{12}/g_{22})| \ll 1 \quad \text{in } S_\phi(t'_\infty,\kappa_1,\delta_1).$$

By the Boutroux equations (2.1), $\operatorname{Im} e^{i\phi}\Omega_{\mathbf{a}}B_\phi(t)$ is bounded as $e^{i\phi}t\to\infty$ through $S_\phi(t'_\infty,\kappa_1,\delta_1)$. By using the first formula of Proposition 6.1(1), we have

$$ie^{i\phi}\Big(t\mathcal{J}_{\mathbf{b}} - \frac{\Omega_{\mathbf{b}}}{2}B_\phi(t)\Big) \\ = \frac{3}{2}\int_{\mathbf{b}} W(z)\,dz + \frac{3}{2}(1+2ia)(2\pi i F(0^-,0^+) + g_0(0^+)\omega_{\mathbf{b}}) \\ + 6\log\frac{g_{12}}{g_{22}} + O(t^{-\delta}),$$

in which $\int_{\mathbf{b}} W(z)\,dz$ admits a similar expression in terms of the ϑ-function with $\hat{\tau} = (-\omega_{\mathbf{a}})/\omega_{\mathbf{b}}$. This implies the boundedness of $\operatorname{Im} e^{i\phi}\Omega_{\mathbf{b}}B_\phi(t)$. Then we have $|B_\phi(t)| \le C_0$ for some $C_0 > 0$ in $S_\phi(t'_\infty,\kappa_1,\delta_1)$. The implied constant of $B_\phi(t) \ll 1$ in (5.1) may be supposed to be greater than $2C_0$, which causes no changes in the subsequent equations by choosing t'_∞ larger if necessary, and hence the boundedness of $B_\phi(t)$ has been shown independently of (5.1) under (6.6). The case $-\pi/3 < \phi < 0$ is similarly treated under the supposition

$$(6.7)\qquad |\log(g_{11}g_{22})| \ll 1, \quad |\log(g_{21}/g_{11})| \ll 1 \quad \text{in } S_\phi(t'_\infty,\kappa_1,\delta_1).$$

Remark 6.3. The argument above also works under a weaker condition, say $B_\phi(t) \ll t^{(1-\delta)/2}$. The supposition $B_\phi(t) \ll 1$ in (5.1) guarantees that each turning point is located within the distance $O(t^{-1})$ from its limit one, which enables us to use the limit Stokes graph in the WKB analysis.

Proposition 6.4. *Suppose that* $0<\phi<\pi/3$ *and* (6.6) *(respectively,* $-\pi/3<\phi<0$ *and* (6.7)*). Then, in* $S_\phi(t'_\infty,\kappa_1,\delta_1)$*,* $B_\phi(t)$ *is bounded, and*

$$\begin{aligned} ie^{i\phi}\Big(t\mathcal{J}_{\mathbf{a}}-\frac{\Omega_{\mathbf{a}}}{2}B_\phi(t)\Big) &= \frac{3}{2}\int_{\mathbf{a}} W(z)\,dz+\frac{3}{2}(1+2ia)g_0(0^+)\omega_{\mathbf{a}} \\ &\quad +6\log(g_{11}g_{22})-6\pi i+O(t^{-\delta}), \\ &= 3\Big(\frac{\vartheta'}{\vartheta}\Big(\frac{1}{2}F(z_+^-,z_+^+)+\nu,\tau\Big)-\frac{\vartheta'}{\vartheta}\Big(\frac{1}{2}F(z_-^-,z_-^+)+\nu,\tau\Big)\Big) \\ &\quad +\frac{3}{2}(1+2ia)g_0(0^+)\omega_{\mathbf{a}}+6\log(g_{11}g_{22})-6\pi i+O(t^{-\delta}), \\ g_0(0^+) &= \frac{1}{\omega_{\mathbf{a}}}\Big(\pi i+\frac{\vartheta'}{\vartheta}(F(0^-,0^+)+\nu,\tau)\Big). \end{aligned}$$

Remark 6.4. Conversely, (5.1) implies (6.6) and (6.7).

The following fact guarantees the possibility of limitation with respect to a_ϕ:

Proposition 6.5. *Under the same supposition as in Proposition* 6.4*, we have*

$$\Big(\int_{z_+^-}^{z_+^+}-\int_{z_-^-}^{z_-^+}\Big)\frac{dz}{w(a_\phi,z)}=\Big(\int_{z_+^-}^{z_+^+}-\int_{z_-^-}^{z_-^+}\Big)\frac{dz}{w(A_\phi,z)}+O(t^{-1})$$

uniformly in $z_+^\pm$*,* $z_-^\pm$ *as* $te^{i\phi}\to\infty$ *through* $S_\phi(t'_\infty,\kappa_1,\delta_1)$*.*

Proof. To show this proposition we note the lemma below, which follows from the relations

$$\int\frac{w}{z^2}\,dz=-\frac{180}{A_\phi^2}\int w\,dz+\Big(\frac{108}{A_\phi^2}-A_\phi\Big)\int\frac{dz}{w}-\frac{w}{z}-\frac{6}{A_\phi}w+\frac{72}{A_\phi^2}zw,$$

$$\Omega_{\mathbf{a}}J_{\mathbf{b}}-\Omega_{\mathbf{b}}J_{\mathbf{a}}=-\frac{A_\phi^2}{15}\pi i,\quad J_{\mathbf{a},\mathbf{b}}=\int_{\mathbf{a},\mathbf{b}} w\,dz,$$

with $w=w(A_\phi,z)$, the latter equality being obtained in the same way as in the proof of Legendre's relation [6, 31].

Lemma 6.6. $\Omega_{\mathbf{a}}\mathcal{J}_{\mathbf{b}}-\Omega_{\mathbf{b}}\mathcal{J}_{\mathbf{a}}=12\pi i$.

From the boundedness of $B_\phi(t)$ it follows that $\omega_{\mathbf{a},\mathbf{b}}=\Omega_{\mathbf{a},\mathbf{b}}+O(t^{-1})$. By Propositions 6.1, 6.3 and Remark 6.1, in the case $0<\phi<\pi/3$, we have

$$\begin{aligned} &\log(g_{12}/g_{22})-\tau\log(g_{11}g_{22}) \\ &\quad=\Big(\int_{\mathbf{b}}-\tau\int_{\mathbf{a}}\Big)\Big(\frac{ie^{i\phi}t}{6}\cdot\frac{w(a_\phi,z)}{z^2}-\frac{1}{4}W(z)+\frac{1+2ia}{4zw(a_\phi,z)}\Big)\,dz-\tau\pi i+O(t^{-\delta}) \end{aligned}$$

$$= -\frac{2\pi e^{i\phi}t}{\omega_{\mathbf{a}}} - \frac{\pi i}{2}(F(z_+^-, z_+^+) - F(z_-^-, z_-^+)) + O(1)$$
$$= -\frac{2\pi e^{i\phi}t}{\omega_{\mathbf{a}}} - \pi i\Big(p(t) + \frac{\omega_{\mathbf{b}}}{\omega_{\mathbf{a}}}q(t)\Big) + O(1) = \Upsilon \ll 1,$$

with $p(t) = p_+(t) - p_-(t)$, $q(t) = q_+(t) - q_-(t) \in \mathbb{Z}$, since $F(z_\pm^-, z_\pm^+) = 2p_\pm(t) + 2q_\pm(t)\tau$, $p_\pm$, $q_\pm \in \mathbb{Z}$. Set $e^{i\phi}\mathcal{J}_{\mathbf{a}}t/6 + \pi q(t) = X$, $e^{i\phi}\mathcal{J}_{\mathbf{b}}t/6 - \pi p(t) = Y$, where $|\operatorname{Im} X|$ and $|\operatorname{Im} Y|$ are bounded by the Boutroux equations (2.1). Then, by Lemma 6.6 and $\omega_{\mathbf{a},\mathbf{b}} = \Omega_{\mathbf{a},\mathbf{b}} + O(t^{-1})$,

$$\omega_{\mathbf{a}}\Upsilon = -2\pi e^{i\phi}t - i(e^{i\phi}t(\Omega_{\mathbf{a}}\mathcal{J}_{\mathbf{b}} - \Omega_{\mathbf{b}}\mathcal{J}_{\mathbf{a}})/6 + \omega_{\mathbf{b}}X - \omega_{\mathbf{a}}Y) + O(1)$$
$$= -i(\omega_{\mathbf{b}}X - \omega_{\mathbf{a}}Y) + O(1) \ll 1$$

with $\operatorname{Im}(\omega_{\mathbf{b}}/\omega_{\mathbf{a}}) > 0$, which implies $|X|$, $|Y| \ll 1$, and hence

$$\pi p(t) = e^{i\phi}\mathcal{J}_{\mathbf{b}}t/6 + O(1), \quad \pi q(t) = -e^{i\phi}\mathcal{J}_{\mathbf{a}}t/6 + O(1).$$

Since $w(a_\phi, z)^{-1} - w(A_\phi, z)^{-1} = (z^2/2)w(A_\phi, z)^{-3}B_\phi(t)t^{-1} + O(t^{-2})$, we have

$$\Big|\Big(\int_{z_+^-}^{z_+^+} - \int_{z_-^-}^{z_-^+}\Big)\Big(\frac{1}{w(a_\phi, z)} - \frac{1}{w(A_\phi, z)}\Big)\,dz\Big|$$
$$\ll \Big|\Big(\int_{z_+^-}^{z_+^+} - \int_{z_-^-}^{z_-^+}\Big)\frac{z^2 B_\phi(t)t^{-1}}{w(A_\phi, z)^3}\,dz\Big| + |t^{-1}|$$
$$\ll \Big|t^{-1}\Big(\int_{z_+^-}^{z_+^+} - \int_{z_-^-}^{z_-^+}\Big)\frac{z^2\,dz}{w(A_\phi, z)^3}\Big| + |t^{-1}|$$
$$\ll |t^{-1}|\,|p(t)j_{\mathbf{a}} + q(t)j_{\mathbf{b}}| + |t^{-1}|$$
$$\ll |\mathcal{J}_{\mathbf{b}}j_{\mathbf{a}} - \mathcal{J}_{\mathbf{a}}j_{\mathbf{b}}| + |t^{-1}| = 2|(\partial/\partial A_\phi)(\mathcal{J}_{\mathbf{b}}\Omega_{\mathbf{a}} - \mathcal{J}_{\mathbf{a}}\Omega_{\mathbf{b}})| + |t^{-1}| \ll t^{-1},$$

where $j_{\mathbf{a},\mathbf{b}} = \int_{\mathbf{a},\mathbf{b}} z^2 w(A_\phi, z)^{-3}\,dz$. This completes the proof of the proposition. □

§7. Proofs of the main theorems

§7.1. Proofs of Theorems 2.1 and 2.2

Suppose that $0 < \phi < \pi/3$. Let $G = (g_{ij}) \in \mathrm{SL}_2(\mathbb{C})$ be a given matrix with $g_{11}g_{12}g_{22} \neq 0$ in the inverse monodromy problem. Then

$$\log(g_{12}/g_{22}) - \tau\log(g_{11}g_{22})$$
$$= \Big(\int_{\mathbf{b}} - \tau\int_{\mathbf{a}}\Big)\Big(\frac{ie^{i\phi}t}{6}\cdot\frac{w(a_\phi, z)}{z^2} - \frac{1}{4}W(z) + \frac{1+2ia}{4zw(a_\phi, z)}\Big)\,dz - \tau\pi i + O(t^{-\delta})$$

$$= -\frac{2\pi e^{i\phi}t}{\omega_{\mathbf{a}}} - \frac{\pi i}{2}(F(z_+^-, z_+^+) - F(z_-^-, z_-^+)) - \frac{\pi i}{2}(1+2ia)F(0^-, 0^+) \\ - \tau\pi i + O(t^{-\delta})$$

(cf. the proof of Proposition 6.5). By Proposition 6.5, replacing a_ϕ with A_ϕ, we have

$$\log(g_{12}/g_{22}) - \tau\log(g_{11}g_{22}) = -\frac{2\pi e^{i\phi}t}{\Omega_{\mathbf{a}}} - \frac{\pi i}{2}(F_{A_\phi}(z_+^-, z_+^+) - F_{A_\phi}(z_-^-, z_-^+)) \\ - \frac{\pi i}{2}(1+2ia)F_{A_\phi}(0^-, 0^+) - \frac{\Omega_{\mathbf{b}}}{\Omega_{\mathbf{a}}}\pi i + O(t^{-\delta})$$

with $F_{A_\phi}(\tilde{z}, z) = \Omega_{\mathbf{a}}^{-1}\int_{\tilde{z}}^{z} dz/w(A_\phi, z)$. Note that

$$F_{A_\phi}(z_+^-, z_+^+) - F_{A_\phi}(z_-^-, z_-^+) = 2(F_{A_\phi}(\infty, z_+^+) - F_{A_\phi}(\infty, z_-^+)), \\ F_{A_\phi}(0^-, 0^+) = 2F_{A_\phi}(\infty, 0^+),$$

and let $\wp(u) = \wp(u; g_2, g_3)$ with $g_2 = \frac{1}{12}A_\phi^2$, $g_3 = \frac{1}{216}A_\phi^3 - 1$. Let us set

$$u_+ = \Omega_{\mathbf{a}}F_{A_\phi}(\infty, z_+^+), \quad u_- = \Omega_{\mathbf{a}}F_{A_\phi}(\infty, z_-^+), \quad \text{i.e. } z_\pm^+ = \wp(u_\pm) + \frac{A_\phi}{12},$$

to write

$$u_+ - u_- = 2ie^{i\phi}t + \frac{i}{\pi}\Big(\Omega_{\mathbf{a}}\log\frac{g_{12}}{g_{22}} - \Omega_{\mathbf{b}}\log(g_{11}g_{22})\Big) \\ - \Omega_{\mathbf{b}} - (1+2ia)\Omega_{\mathbf{a}}F_{A_\phi}(\infty, 0^+) + O(t^{-\delta}). \tag{7.1}$$

By the addition theorem for the $\wp$-function,

$$\wp(u_+ + u_-) = -\wp(u_+) - \wp(u_-) + \frac{1}{4}\Big(\frac{\wp'(u_+) - \wp'(u_-)}{\wp(u_+) - \wp(u_-)}\Big)^2 \\ = -z_+^+ - z_-^+ + \frac{A_\phi}{6} + \frac{1}{4}\Big(\frac{w(z_+^+) - w(z_-^+)}{z_+^+ - z_-^+}\Big)^2.$$

By (6.4), $z_+ = y$ and $z_- = (i/2)e^{-i\phi}y^{-1}\Gamma_0(t, y, y^t)$ satisfy

$$z_+^+ + z_-^+ = e^{-i\phi}K_+, \\ w(z_\pm^+) = ie^{-i\phi}(b_3/\lambda)(z_\pm^+) = 1 - ie^{-i\phi}\Gamma_0(t, y, y^t)z_\pm^+ + O(t^{-1}),$$

and hence

$$\wp(u_+ + u_-) = -e^{-i\phi}K_+ + \frac{A_\phi}{6} + \frac{1}{4}(ie^{-i\phi}\Gamma_0(t, y, y^t) + O(t^{-1}))^2 \\ = \frac{A_\phi}{6} - \frac{1}{4}(4e^{-i\phi}K_+ + e^{-2i\phi}\Gamma_0(t, y, y^t)^2) + O(t^{-1}) \\ = -\frac{A_\phi}{12} + O(t^{-1}),$$

since $4e^{-i\phi}K_+ + e^{-2i\phi}\Gamma_0(t,y,y^t)^2 = a_\phi + O(t^{-1})$. This implies

$$u_+ + u_- = \int_\infty^{0^+} \frac{dz}{w(A_\phi,z)} + O(t^{-1}) = \Omega_{\mathbf{a}} F_{A_\phi}(\infty,0^+) + O(t^{-1}). \tag{7.2}$$

From (7.1) and (7.2) with $\Omega_{\mathbf{a}} F_{A_\phi}(\infty,0^+) = \Omega_0$, it follows that

$$\begin{aligned} u_+ &= \int_\infty^{z_+^+} \frac{dz}{w(A_\phi,z)} \\ &= ie^{i\phi}t + \frac{i}{2\pi}\Big(\Omega_{\mathbf{a}}\log\frac{g_{12}}{g_{22}} - \Omega_{\mathbf{b}}\log(g_{11}g_{22})\Big) - \frac{\Omega_{\mathbf{b}}}{2} - ia\Omega_0 + O(t^{-\delta}), \\ u_- &= \int_\infty^{z_-^+} \frac{dz}{w(A_\phi,z)} \\ &= -ie^{i\phi}t - \frac{i}{2\pi}\Big(\Omega_{\mathbf{a}}\log\frac{g_{12}}{g_{22}} - \Omega_{\mathbf{b}}\log(g_{11}g_{22})\Big) + \frac{\Omega_{\mathbf{b}}}{2} + (1+ia)\Omega_0 + O(t^{-\delta}), \end{aligned}$$

which leads us to the asymptotic expressions of Theorem 2.1 and Remark 2.1.

Justification. The justification of $y(x)$ as a solution of (1.2) is made along the lines of [15, pp. 105–106, 120–121]. Let $\mathcal{G} = (g_{12}/g_{22}, g_{11}g_{22})$ be a given point such that $g_{11}g_{12}g_{22} \neq 0$ on the monodromy manifold for (1.4). In addition to $y(x)$ obtained above, we have the following expression for $B_\phi(t)$ from Proposition 6.4:

Proposition 7.1. *In $S_\phi(t'_\infty, \kappa_1, \delta_1)$,*

$$\begin{aligned} &ie^{i\phi}\Big(t\mathcal{J}_{\mathbf{a}} - \frac{\Omega_{\mathbf{a}}}{2}B_\phi(t)\Big) \\ &\quad = 3\Big(\frac{\vartheta'}{\vartheta}(\Omega_{\mathbf{a}}^{-1}i(x-x_0^+)+\nu,\tau_\Omega) + \frac{\vartheta'}{\vartheta}\big(\Omega_{\mathbf{a}}^{-1}(i(x-x_0^+)-\Omega_0)+\nu,\tau_\Omega\big)\Big) \\ &\qquad + \frac{3}{2}(1+2ia)g_0(0^+)\Omega_{\mathbf{a}} + 6\log(g_{11}g_{22}) - 6\pi i + O(t^{-\delta}) \end{aligned}$$

with $x = e^{i\phi}t$, $\tau_\Omega = \Omega_{\mathbf{b}}/\Omega_{\mathbf{a}}$.

The equation about u_+ and the proposition above provide the leading term expressions

$$y_{\mathrm{as}} = y_{\mathrm{as}}(\mathcal{G},t) = \wp(i(e^{i\phi}t - x_0^+); g_2(A_\phi), g_3(A_\phi)) + \frac{A_\phi}{12}$$

and $(B_\phi)_{\mathrm{as}} = (B_\phi)_{\mathrm{as}}(\mathcal{G},t)$ without $O(t^{-\delta})$, where x_0^+ depends on $(g_{12}/g_{22}, g_{11}g_{22})$. Taking (4.2) and (5.1) into account, we set

$$y_{\mathrm{as}}^t = -\frac{y_{\mathrm{as}}}{2}t^{-1} + ie^{i\phi}\sqrt{4y_{\mathrm{as}}^3 - A_\phi y_{\mathrm{as}}^2 + 1 - (3ie^{-i\phi}(1+2ia) + (B_\phi)_{\mathrm{as}}y_{\mathrm{as}})y_{\mathrm{as}}t^{-1}},$$

where the branch of the square root is chosen in such a way that y^t_{as} is compatible with $(\partial/\partial t)y_{\mathrm{as}}$. Then $(y_{\mathrm{as}}, y^t_{\mathrm{as}}) = (y_{\mathrm{as}}(\mathcal{G},t), y^t_{\mathrm{as}}(\mathcal{G},t))$ fulfils (5.1) with $B_\phi(t) = (B_\phi)_{\mathrm{as}}(\mathcal{G},t)$ in the domain $\hat{S}(\phi, t_\infty, \kappa_0, \delta_2) = \{t \mid \mathrm{Re}\, t > t_\infty,\ |\mathrm{Im}\, t| < \kappa_0\} \setminus \bigcup_{i\sigma\in Z_0}\{|t - e^{-i\phi}\sigma| < \delta_2\}$ with $Z_0 = \{ix_0^+ + \Omega_{\mathbf{a}}\mathbb{Z} + \Omega_{\mathbf{b}}\mathbb{Z}\} \cup \{ix_0^+ + \Omega_0 + \Omega_{\mathbf{a}}\mathbb{Z} + \Omega_{\mathbf{b}}\mathbb{Z}\} \cup \{ix_0^+ + \xi_0 \mid \wp(\xi_0) = -A_\phi/12\}$. Let $\mathcal{G}_{\mathrm{as}}(t)$ be the monodromy data for system (1.4) containing $(y_{\mathrm{as}}, y^t_{\mathrm{as}})$. As a result of the WKB analysis for the direct monodromy problem we have $\|\mathcal{G}_{\mathrm{as}}(t) - \mathcal{G}\| \ll t^{-\delta}$, which holds uniformly in a neighbourhood of $\mathcal{G}$. Then the justification scheme of Kitaev [13] applies to our case. Using the maximal modulus principle in each neighbourhood of $i\sigma = ix_0^+ + \{\Omega_0 + \Omega_{\mathbf{a}}\mathbb{Z} + \Omega_{\mathbf{b}}\mathbb{Z}\} \cup \{\xi_0 \mid \wp(\xi_0) = -A_\phi/12\}$, we obtain Theorem 2.1. Theorem 2.2 is proved by the same argument as above.

§7.2. Proof of Theorem 2.3

Let (1.4) with $y^t = (d/dt)y$ be an isomonodromy system. Equation (1.2), system (1.4) and the function a_ϕ with $y^t = (d/dt)y$ remain invariant under the substitution

$$\phi = \tilde{\phi} + 2m\pi/3, \quad y = e^{2m\pi i/3}\tilde{y}, \quad x = e^{2m\pi i/3}\tilde{x},$$
$$\lambda = e^{2m\pi i/3}\tilde{\lambda}, \quad a_\phi = e^{2m\pi i/3}a_{\tilde{\phi}}.$$

To show the theorem we use this symmetry (cf. [14]). Let ϕ be such that $0 < |\phi - 2m\pi/3| < \pi/3$. Then a new system with respect to $(\tilde{\lambda}, \tilde{y}, \tilde{x}, \tilde{\phi})$ is an isomonodromy system for $0 < |\tilde{\phi}| < \pi/3$. Denote by $G^{(m)}$ a connection matrix as the matrix monodromy data for the system governed by $\tilde{y}(\tilde{x}) = e^{-2m\pi i/3}y(x) = e^{-2m\pi i/3}y(e^{2m\pi i/3}\tilde{x})$. We would like to know the relation between $G^{(m)}$ and G. The matrix solutions of the new system are

$$\widetilde{Y}_j^\infty(\tilde{\lambda}) \sim \tilde{\lambda}^{-(1/2+ia)\sigma_3} \exp(-(i/6)e^{i\tilde{\phi}}t\tilde{\lambda}^2\sigma_3)$$

as $\tilde{\lambda} \to \infty$ through the sector $|\arg\tilde{\lambda} + \tilde{\phi}/2 - j\pi/2| < \pi/2$, and

$$\widetilde{Y}_j^0(\tilde{\lambda}) \sim (i/\sqrt{2})(\sigma_1 + \sigma_3)\exp(-(2i/3)e^{i\tilde{\phi}}t\tilde{\lambda}^{-1}\sigma_3)$$

as $\tilde{\lambda} \to 0$ through the sector $|\arg\tilde{\lambda} - \tilde{\phi} - j\pi| < \pi$. The connection matrix $G^{(m)}$ is defined by $\widetilde{Y}_0^{\infty,*}(\tilde{\lambda}) = \widetilde{Y}_0^\infty(\tilde{\lambda})\Theta_{0,*}^{-\sigma_3} = \widetilde{Y}_0^0(\tilde{\lambda})G^{(m)}$. Note that $\widetilde{Y}_0^\infty(\tilde{\lambda})$ and $\widetilde{Y}_0^0(\tilde{\lambda})$ are also expressed as

$$\widetilde{Y}_0^\infty(\tilde{\lambda}) = \widetilde{Y}_0^\infty(e^{-2m\pi i/3}\lambda) \sim \lambda^{-(1/2+ia)\sigma_3}\exp(-(i/6)e^{i\phi}t\lambda^2\sigma_3)e^{(2m\pi i/3)(1/2+ia)\sigma_3}$$

in the sector $|\arg\lambda + \phi/2 - m\pi| < \pi/2$, and that

$$\widetilde{Y}_0^0(\tilde{\lambda}) = \widetilde{Y}_0^0(e^{-2m\pi i/3}\lambda) \sim (i/\sqrt{2})(\sigma_1 + \sigma_3)\exp(-(2i/3)e^{i\phi}t\lambda^{-1}\sigma_3)$$

in the sector $|\arg\lambda - \phi| < \pi$. Then we have $\widetilde{Y}^0_0(\tilde\lambda) = \widehat{Y}^0_0(\lambda)$ and, if $m \ge 1$,

$$\begin{aligned}\widetilde{Y}^\infty_0(\tilde\lambda) &= \widehat{Y}^\infty_{2m}(\lambda)e^{(2m\pi i/3)(1/2+ia)\sigma_3}\\ &= \widehat{Y}^\infty_0(\lambda)\hat S^\infty_0\hat S^\infty_1\cdots\hat S^\infty_{2m-2}\hat S^\infty_{2m-1}e^{(2m\pi i/3)(1/2+ia)\sigma_3},\end{aligned}$$

which implies, by Remark 3.2,

$$G^{(m)} = GS^\infty_0 S^\infty_1\cdots S^\infty_{2m-2}S^\infty_{2m-1}e^{(2m\pi i/3)(1/2+ia)\sigma_3}.$$

This combined with Proposition 3.4 yields the expression of $G^{(m)}$ for $m \ge 1$ as in the theorem. Note that $P_*(u,A) = \wp(u; g_2(A), g_3(A)) + \frac{1}{12}A$ solves $(P_u)^2 = 4P^3 - AP^2 + 1$. Then

$$P_*(u,A) = e^{-2\pi i/3}P_*(e^{2\pi i/3}u, e^{2\pi i/3}A) = e^{2\pi i/3}P_*(e^{-2\pi i/3}u, e^{-2\pi i/3}A).$$

By Theorems 2.1 and 2.2, $\tilde y(\tilde x) = e^{-2m\pi i/3}y(x)$ for $0 < |\phi - 2m\pi/3| = |\tilde\phi| < \pi/3$ is represented by

$$e^{-2m\pi i/3}y(x) = \tilde y(\tilde x) = P_*\big(i(\tilde x - x_0(G^{(m)}, \Omega^{\tilde\phi}_{\mathbf a}, \Omega^{\tilde\phi}_{\mathbf b}, \Omega^{\tilde\phi}_0)) + O(x^{-\delta}); A_{\tilde\phi}\big).$$

Using the relation above, we have

$$\begin{aligned}y(x) &= e^{2m\pi i/3}P_*\big(i(\tilde x - x_0(G^{(m)}, \Omega^{\tilde\phi}_{\mathbf a}, \Omega^{\tilde\phi}_{\mathbf b}, \Omega^{\tilde\phi}_0)) + O(x^{-\delta}); A_{\tilde\phi}\big)\\ &= P_*\big(i(x - e^{2m\pi i/3}x_0(G^{(m)}, \Omega^{\tilde\phi}_{\mathbf a}, \Omega^{\tilde\phi}_{\mathbf b}, \Omega^{\tilde\phi}_0)) + O(x^{-\delta}); A_\phi\big)\\ &= P_*\big(i(x - x_0(G^{(m)}, \Omega^{\phi}_{\mathbf a}, \Omega^{\phi}_{\mathbf b}, \Omega^{\phi}_0)) + O(x^{-\delta}); A_\phi\big),\end{aligned}$$

which is denoted by $P(A_\phi, x_0(G^{(m)}, \Omega^\phi_{\mathbf a}, \Omega^\phi_{\mathbf b}, \Omega^\phi_0); x)$ as in the theorem.

§8. Modulus A_ϕ and the Boutroux equations

Recall the elliptic curve Π_A for $w(A,z)^2 = 4z^3 - Az^2 + 1$ defined in Section 2.2. For a given $\phi \in \mathbb{R}$ we would like to examine the modulus $A_\phi \in \mathbb{C}$ such that, for every cycle $\mathbf{c} \subset \Pi_{A_\phi}$,

$$\operatorname{Im} e^{i\phi}\int_{\mathbf c}\frac{w(A_\phi, z)}{z^2}\,dz = 0.$$

First, for $|\phi| \le \pi/3$, let us consider A_ϕ satisfying the Boutroux equations

$$(\mathrm{BE})_\phi \qquad \operatorname{Im} e^{i\phi}I_{\mathbf a}(A_\phi) = 0, \quad \operatorname{Im} e^{i\phi}I_{\mathbf b}(A_\phi) = 0,$$

where $\mathbf a$, $\mathbf b$ denote the basic cycles given in Section 2.2 and

$$I_{\mathbf a,\mathbf b}(A) = \int_{\mathbf a,\mathbf b}\frac{w(A,z)}{z^2}\,dz = \int_{\mathbf a,\mathbf b}\frac{1}{z^2}\sqrt{4z^3 - Az^2 + 1}\,dz.$$

It is easy to see that $w(A,z)^2 = 4z^3 - Az^2 + 1$ has double roots z_0, z_1 if and only if

$$A = 3 \cdot 2^{2/3}, \quad z_0, z_1 = 2^{-1/3}, \quad z_2 = -4^{-2/3};$$
$$A = 3 \cdot 2^{2/3} e^{\pm 2\pi i/3}, \quad z_0, z_1 = 2^{-1/3} e^{\pm 2\pi i/3}, \quad z_2 = -4^{-2/3} e^{\pm 2\pi i/3}.$$

Example 8.1. When $\phi = 0$, we have $I_{\mathbf{a}}(3 \cdot 2^{2/3}) = 0$, $I_{\mathbf{b}}(3 \cdot 2^{2/3}) = -2^{4/3}3^{3/2}$. Indeed,

$$\begin{aligned} I_{\mathbf{b}}(3 \cdot 2^{2/3}) &= 2 \int_{-4^{-2/3}}^{2^{-1/3}} \frac{2}{z^2} \big(i\sqrt{2^{-1/3} - z}\big)^2 \sqrt{z + 4^{-2/3}}\, dz \\ &= -4^{2/3} \int_{-1}^{2} \frac{(2-t)\sqrt{t+1}}{t^2}\, dt, \end{aligned}$$

in which the residue of the integrand at $z = 0$ vanishes.

Note that $\mathbf{a}$ is a cycle enclosing the cut $[z_0, z_1]$. In accordance with [14, Sect. 7] we begin with the following:

Proposition 8.1. *Suppose that* $\operatorname{Im} I_{\mathbf{a}}(A) = 0$. *Then* $A \in \mathbb{R}$.

Proof. Set

$$J_{\mathbf{a}}(A) = \int_{-\mathbf{a}} \frac{1}{z^2} v(A,z)\, dz$$

with $v(A,z) = \sqrt{4z^3 + Az^2 - 1} = -iw(A,-z)$. Since $I_{\mathbf{a}}(A) = -iJ_{\mathbf{a}}(A)$, the supposition means $J_{\mathbf{a}}(A) \in i\mathbb{R}$. In this proof, to simplify the description, we write $v(A,z) = v_A(z)$, $v(\bar{A},z) = v_{\bar{A}}(z)$ and $v(A,z) \pm v(\bar{A},z) = (v_A \pm v_{\bar{A}})(z)$. Then

$$0 = J_{\mathbf{a}}(A) + \overline{J_{\mathbf{a}}(A)} = J_{\mathbf{a}}(A) + J_{\bar{\mathbf{a}}}(\bar{A}) = J_{\mathbf{a}}(A) - J_{\mathbf{a}}(\bar{A}) = (A - \bar{A}) \int_{-\mathbf{a}} \frac{dz}{(v_A + v_{\bar{A}})(z)}.$$

The polynomials $v_A(z)^2$ and $v_{\bar{A}}(z)^2$ have the roots $-z_0$, $-z_1$, $-z_2$, and $-\overline{z_0}$, $-\overline{z_1}$, $-\overline{z_2}$, respectively. The algebraic functions $(v_A \pm v_{\bar{A}})(z)$ may be considered on the two-sheeted Riemann surface glued along the cuts $[-z_0, -z_1]$, $[-\overline{z_0}, -\overline{z_1}]$, $[-z_2, -\overline{z_2}] \cup [-\infty, -\operatorname{Re} z_2]$ (cf. Figure 6). The cycle $-\mathbf{a}$ may be supposed to enclose both cuts $[-z_0, -z_1]$, $[-\overline{z_0}, -\overline{z_1}]$, and the cycles as in Figure 6(a.1) and (a.2) may be deformed into contours consisting of horizontal and vertical lines and enclosing the cuts $[-z_2, -\overline{z_2}] \cup [-\infty, -\operatorname{Re} z_2]$ clockwise as in Figure 6(a*.1) and (a*.2), respectively. Possible extension of this contour is caused by further movement of $-z_0$, $-z_1$ and $-z_2$, and is given by adding horizontal and vertical lines located in the symmetric position with respect to the real axis. To show $A \in \mathbb{R}$ it is sufficient to verify that, under the supposition $A - \bar{A} \neq 0$,

$$J = \int_{-\mathbf{a}} \frac{dz}{(v_A + v_{\bar{A}})(z)} \neq 0.$$

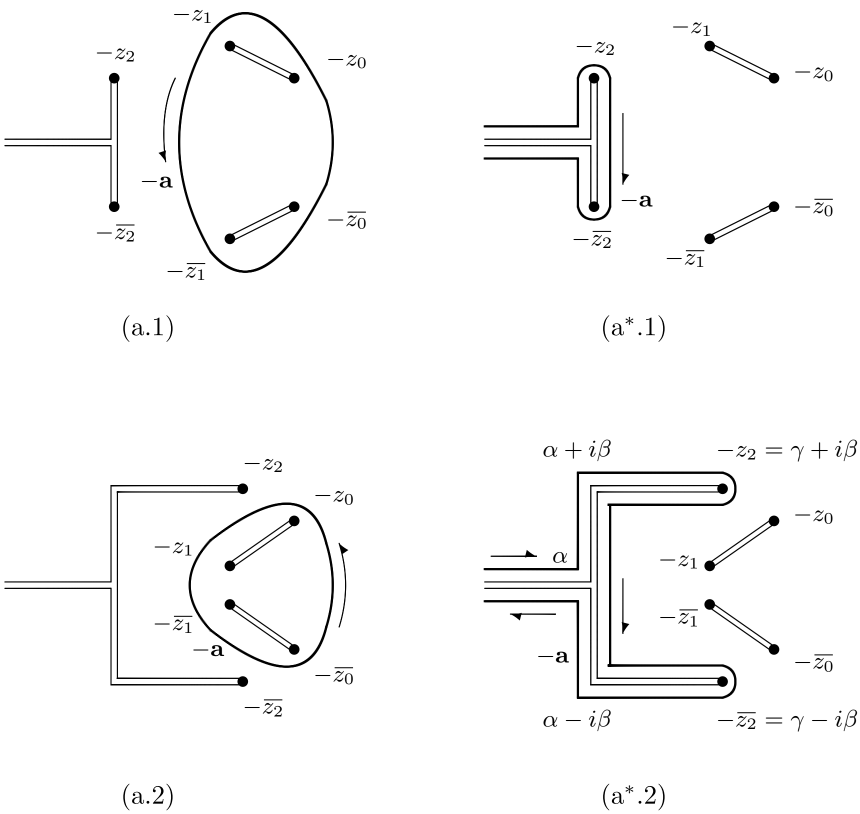

Figure 6. Modification of the cycle $-\mathbf{a}$

Let us compute this integral along the contour $-\mathbf{a}$, say as in Figure 6(a*.2) with vertices $\alpha \pm i\beta$, $\gamma \pm i\beta$ such that $-z_2, -\overline{z_2} = \gamma \pm i\beta$, in which $\alpha \le \gamma$, $\beta \ge 0$, and α may be supposed to be $\alpha < 0$.

The integral J is decomposed into three parts: $J = J_0 + J_{\text{hor}} + J_{\text{ver}}$ with the real line part

$$J_0 = 2\int_{-\infty}^{\alpha} \frac{dz}{(v_A + v_{\bar{A}})(z)},$$

the horizontal part $J_{\text{hor}} = J^+_{\text{hor}} + J^-_{\text{hor}}$, where

$$J^+_{\text{hor}} = \int_{\alpha}^{\gamma} \frac{ds}{(v_A + v_{\bar{A}})(s + i\beta)} + \int_{\gamma}^{\alpha} \frac{ds}{(-v_A + v_{\bar{A}})(s + i\beta)},$$
$$J^-_{\text{hor}} = \int_{\alpha}^{\gamma} \frac{ds}{(-v_A + v_{\bar{A}})(s - i\beta)} + \int_{\gamma}^{\alpha} \frac{ds}{(-v_A - v_{\bar{A}})(s - i\beta)},$$

and the vertical part $J_{\rm ver}=J^+_{\rm ver}+J^-_{\rm ver}$, where

$$J^+_{\rm ver}=\int_0^{\beta}\frac{i\,dt}{(v_A+v_{\bar A})(\alpha+it)}+\int_{\beta}^{0}\frac{i\,dt}{(-v_A+v_{\bar A})(\alpha+it)},$$
$$J^-_{\rm ver}=\int_0^{-\beta}\frac{i\,dt}{(-v_A+v_{\bar A})(\alpha+it)}+\int_{-\beta}^{0}\frac{i\,dt}{(-v_A-v_{\bar A})(\alpha+it)}.$$

Then we have

$$J_{\rm hor}=\frac{2}{A-\bar A}\Bigl(\int_\alpha^\gamma\frac{v_A(s+i\beta)}{(s+i\beta)^2}\,ds-\int_\alpha^\gamma\frac{v_{\bar A}(s-i\beta)}{(s-i\beta)^2}\,ds\Bigr)\in\mathbb{R}$$

and

$$J_{\rm ver}=\frac{2i}{A-\bar A}\Bigl(\int_0^\beta\frac{v_A(\alpha+it)}{(\alpha+it)^2}\,dt+\int_0^\beta\frac{v_{\bar A}(\alpha-it)}{(\alpha-it)^2}\,dt\Bigr)\in\mathbb{R},$$

and hence $J_{\rm hor}+J_{\rm ver}\in\mathbb{R}$. Furthermore, observing, for $-t=x-\alpha$, $t\ge 0$, $\alpha<0$,

$$v_A(x)=(-4(t-\alpha)^3+(\operatorname{Re}A+i\operatorname{Im}A)(t-\alpha)^2-1)^{1/2}=i(g(t)-ih(t))^{1/2},$$
$$g(t)=4(t-\alpha)^3-\operatorname{Re}A\cdot(t-\alpha)^2+1,\quad h(t)=\operatorname{Im}A\cdot(t-\alpha)^2,$$

we have

$$\frac{1}{2}J_0=\int_{-\infty}^{\alpha}\frac{dx}{(v_A+v_{\bar A})(x)}=-\frac{i}{\sqrt2}\int_0^\infty\frac{dt}{\sqrt{g(t)+\sqrt{g(t)^2+h(t)^2}}}\in i\mathbb{R}\setminus\{0\},$$

which implies $J\ne 0$ under the supposition $A-\bar A\ne 0$. In the case where extension by horizontal or vertical lines occurs, the contributions from these parts to J are integrals analogous to $J_{\rm hor}$ or $J_{\rm ver}$, and $J\ne 0$ are similarly shown. □

Let us examine $I_{\mathbf a}(A)$ for $A\in\mathbb{R}$. It is easy to see that $w(A,z)^2$ has real roots $z_2<z_1<z_0$ if $A>3\cdot 2^{2/3}$. Then $I_{\mathbf a}(A)\in i\mathbb{R}\setminus\{0\}$. For $A=3\cdot2^{2/3}$ we have $z_2<z_1=z_0=2^{-1/3}$, and then $I_{\mathbf a}(3\cdot 2^{2/3})=0$.

Suppose that $A<3\cdot 2^{2/3}$. The roots of $w(A,z)^2$ are $\alpha\pm i\beta$ and z_2 with α, β, $z_2\in\mathbb{R}$, and $\mathbf a$ is a cycle enclosing the cut $[\alpha-i\beta,\alpha+i\beta]$. Then $I_{\mathbf a}(A)\in i\mathbb{R}$, since $\overline{I_{\mathbf a}(A)}=-I_{\mathbf a}(A)$, and the integral

$$I_{\mathbf a}(A)=2i\int_{-\beta}^{\beta}\frac{w(A,\alpha+it)}{(\alpha+it)^2}\,dt=4i\int_0^\beta\operatorname{Re}\frac{w(A,\alpha+it)}{(\alpha+it)^2}\,dt$$

satisfies, for $A<3\cdot 2^{2/3}$,

$$\frac{\partial}{\partial A}\Bigl(\frac1i I_{\mathbf a}(A)\Bigr)=2\int_0^\beta\operatorname{Re}w(A,\alpha+it)^{-1}\,dt=\sqrt2\int_0^\beta\frac{\sqrt{g_*+\sqrt{g_*^2+h_*^2}}}{\sqrt{g_*^2+h_*^2}}\,dt>0,$$

where

$$g_* = g_*(t) = \operatorname{Re} w(A, \alpha + it)^2 = 4(\alpha^3 - 3\alpha t^2) - A(\alpha^2 - t^2) + 1,$$
$$h_* = h_*(t) = \operatorname{Im} w(A, \alpha + it)^2 = 4(-t^3 + 3\alpha^2 t) - 2A\alpha t.$$

This implies $I_{\mathbf{a}}(A) \in i\mathbb{R} \setminus \{0\}$ for $A < 3 \cdot 2^{2/3}$.

The fact above combined with Proposition 8.1 leads us to the following:

Proposition 8.2. *If $\phi = 0$, then the Boutroux equations $(\mathrm{BE})_0$ admit a unique solution $A_0 = 3 \cdot 2^{2/3}$.*

Corollary 8.3. *For every $A \in \mathbb{C}$, $(I_{\mathbf{a}}(A), I_{\mathbf{b}}(A)) \neq (0, 0)$.*

Proof. If $I_{\mathbf{a}}(A) = 0$, then $A \in \mathbb{R}$ by Proposition 8.1. By Proposition 8.2 and Example 8.1, $A = 3 \cdot 2^{2/3}$ and $I_{\mathbf{b}}(A) \neq 0$. □

Proposition 8.4. *Suppose that, for A_ϕ solving $(\mathrm{BE})_\phi$ with $0 < |\phi| \le \pi/3$, the elliptic curve Π_{A_ϕ} degenerates. Then $\phi = \pi/3$ or $-\pi/3$ and $A_{\pm\pi/3} = 3 \cdot 2^{2/3} e^{\mp 2\pi i/3}$.*

Proof. When Π_{A_ϕ} degenerates, $A_\phi = 3 \cdot 2^{2/3} e^{2k\pi i/3}$, $k = 0, \pm 1$. Suppose that $A_\phi = 3 \cdot 2^{2/3} e^{2\pi i/3}$, and that the roots of $w(A_\phi, z)^2$ are $z_0 = z_1$ and $z_2 \neq z_0, z_1$. Then

$$e^{i\phi} \int_{z_0}^{z_2} \frac{1}{z^2} \sqrt{4z^3 - A_\phi z^2 + 1}\, dz = e^{i(\phi - 2\pi/3)} \int_{\zeta_0}^{\zeta_2} \frac{1}{\zeta^2} \sqrt{4\zeta^3 - 3 \cdot 2^{2/3} \zeta^2 + 1}\, d\zeta \neq 0$$

with $\zeta_{0,2} = z_{0,2} e^{-2\pi i/3} \in \{2^{-1/3}, -4^{-2/3}\}$ is real valued, which implies $\phi = -\pi/3$. Similarly, if $A_\phi = 3 \cdot 2^{2/3} e^{-2\pi i/3}$, then $\phi = \pi/3$. □

Proposition 8.5. *If $\phi = \pm\pi/3$, then the Boutroux equations $(\mathrm{BE})_{\pm\pi/3}$ admit a unique solution $A_{\pm\pi/3} = 3 \cdot 2^{2/3} e^{\mp 2\pi i/3}$.*

Proof. For $\phi = \pi/3$, $(\mathrm{BE})_{\pi/3}$ are equivalent to

$$e^{\pi i/3} \int_{\mathbf{c}} \frac{1}{z^2} \sqrt{4z^3 - A_{\pi/3} z^2 + 1}\, dz \in \mathbb{R}$$

for every cycle $\mathbf{c}$ on $\Pi_{A_{\pi/3}}$, which is written as $(\mathrm{BE})_0$ with $\phi = 0$,

$$e^{\pi i} \int_{\mathbf{c} e^{2\pi i/3}} \frac{1}{\zeta^2} \sqrt{4\zeta^3 - e^{2\pi i/3} A_{\pi/3} \zeta^2 + 1}\, d\zeta \in \mathbb{R} \quad (z = e^{-2\pi i/3} \zeta).$$

Then by Proposition 8.2, $e^{2\pi i/3} A_{\pi/3} = 3 \cdot 2^{2/3}$ is a unique solution of $(\mathrm{BE})_{\pi/3}$. □

The function $h(A) = I_{\mathbf{a}}(A)/I_{\mathbf{b}}(A)$ [21, Appx. I] is useful in examining A_ϕ.

Proposition 8.6. *Suppose that* $A \in \mathbb{C}$.

(1) *If* A *solves* (BE)$_\phi$ *for some* $\phi \in \mathbb{R}$ *and* $I_{\mathbf{b}}(A) \neq 0$, *then* $h(A) \in \mathbb{R}$.

(2) *If* $h(A) \in \mathbb{R} \setminus \{0\}$, *then, for some* $\phi \in \mathbb{R}$, A *solves* (BE)$_\phi$.

Proof. Suppose that $h(A) = \rho \in \mathbb{R}$, and write $I_{\mathbf{a}}(A) = u + iv$, $I_{\mathbf{b}}(A) = U + iV$ with u, v, U, $V \in \mathbb{R}$. Then $u = \rho U$, $v = \rho V$, and hence $v/u = V/U = -\tan\phi$ for some $\phi \in [-\pi/2, \pi/2]$. This implies $\operatorname{Im} e^{i\phi} I_{\mathbf{a}}(A) = \operatorname{Im} e^{i\phi} I_{\mathbf{b}}(A) = 0$. □

Proposition 8.7. *The set* $\{A \in \mathbb{C} \mid A$ solves (BE)$_\phi$ for some $\phi \in \mathbb{R}\}$ *is bounded.*

Proof. The roots of $w(A, z)$ are z_0, $z_1 \sim \pm A^{-1/2}$, $z_2 \sim A/4$ if A is large. Then

$$\begin{aligned} \int_{z_2}^{z_0} \frac{1}{z^2} w(A, z)\, dz &\sim \int_{A/4}^{A^{-1/2}} \frac{1}{z^2} \sqrt{4z^3 - Az^2 + 1}\, dz \\ &\sim iA^{1/2} \int_1^{4A^{-3/2}} \frac{1}{t} \sqrt{1 - t}\, dt \\ &\sim iA^{1/2}(2 + \log(2A^{-3/2})) \sim -\frac{3i}{2} A^{1/2} \log A \end{aligned}$$

and

$$\begin{aligned} \int_{z_0}^{z_1} \frac{1}{z^2} w(A, z)\, dz &\sim \int_{-A^{-1/2}}^{A^{-1/2}} \frac{1}{z^2} \sqrt{4z^3 - Az^2 + 1}\, dz \\ &\sim A^{1/2} \int_{-1}^{1} \frac{1}{t^2} \sqrt{1 - t^2}\, dt \\ &\sim \pi A^{1/2}. \end{aligned}$$

This implies $h(A) \notin \mathbb{R}$ if A is sufficiently large, which completes the proof. □

The following fact is used in discussing solutions of (BE)$_\phi$.

Let $0 < |\phi| < \pi/3$, and write

$$I_{\mathbf{a}}(A) = u(A) + iv(A), \quad I_{\mathbf{b}}(A) = U(A) + iV(A).$$

Note that A solves (BE)$_\phi$ if and only if

$$\begin{aligned} \operatorname{Im} e^{i\phi} I_{\mathbf{a}}(A) &= u(A)\sin\phi + v(A)\cos\phi = 0, \\ \operatorname{Im} e^{i\phi} I_{\mathbf{b}}(A) &= U(A)\sin\phi + V(A)\cos\phi = 0, \end{aligned}$$

that is,

$$u(A)\tan\phi + v(A) = 0, \quad U(A)\tan\phi + V(A) = 0. \tag{8.1}$$

Then, by the Cauchy–Riemann equations, the Jacobian for (8.1) with $A = x + iy$ is written as

$$\det J(\phi, A) = \det \begin{pmatrix} u_x \tan\phi + v_x & u_y \tan\phi + v_y \\ U_x \tan\phi + V_x & U_y \tan\phi + V_y \end{pmatrix}$$
$$= (1 + \tan^2\phi)(v_x V_y - v_y V_x)$$
$$= -\frac{1}{4}(1 + \tan^2\phi)|\Omega_{\mathbf{a}}(A)|^2 \operatorname{Im} \frac{\Omega_{\mathbf{b}}(A)}{\Omega_{\mathbf{a}}(A)}, \tag{8.2}$$

where $\Omega_{\mathbf{a}}(A)$ and $\Omega_{\mathbf{b}}(A)$ are periods of the elliptic curve $w(A, z)$. For $0 < |\phi| < \pi/3$, (d/dt)(8.1) with $t = \tan\phi$ is written as

$$J(\phi, A) \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} + \begin{pmatrix} u(A) \\ U(A) \end{pmatrix} \equiv \mathbf{o}.$$

Then we have

$$(x'(t), y'(t)) \neq (0, 0) \quad \text{and} \quad (d/d\phi)A = (x'(t) + iy'(t)) \cos^{-2}\phi \neq 0 \tag{8.3}$$

for $0 < |\phi| < \pi/3$.

Proposition 8.8. *Suppose that, for some ϕ_0 such that $0 < |\phi_0| < \pi/3$, A_{ϕ_0} solves (BE)$_{\phi_0}$. Then there exists a trajectory T_0: $A = \chi(\phi_0, \phi)$ for $0 < |\phi| < \pi/3$ with the properties*

(1) $\chi(\phi_0, \phi_0) = A_{\phi_0}$*;*
(2) *for each ϕ, $A = \chi(\phi_0, \phi)$ solves (BE)$_\phi$;*
(3) *$\chi(\phi_0, \phi)$ is smooth for $0 < |\phi| < \pi/3$.*

Proof. Since the Jacobian (8.2) satisfies $\det J(\phi_0, A_{\phi_0}) \in \mathbb{R} \setminus \{0\}$, there exists a local trajectory $A = \chi_{\mathrm{loc}}(\phi_0, \phi)$ having the properties (1), (2) and (3) above for $|\phi - \phi_0| < \delta$, where δ is sufficiently small. Since (8.2) is in $\mathbb{R} \setminus \{0\}$ for $0 < |\phi| < \pi/3$, $\chi_{\mathrm{loc}}(\phi_0, \phi)$ may be extended to the interval $0 < |\phi| < \pi/3$. □

Proposition 8.9. *The trajectory T_0: $A = \chi(\phi_0, \phi)$ given above may be extended to $|\phi| \le \pi/3$ such that $\chi(\phi_0, \phi)$ is continuous in ϕ and that $\chi(\phi_0, 0) = A_0 = 3 \cdot 2^{2/3}$, $\chi(\phi_0, \pm\pi/3) = A_{\pm\pi/3} = 3 \cdot 2^{2/3} e^{\mp 2\pi i/3}$.*

Proof. To show that $\chi(\phi_0, \phi) \to A_0$ as $\phi \to 0$, suppose to the contrary that there exists a sequence $\{\phi_\nu\}$ such that $\phi_\nu \to 0$ and that $\chi(\phi_0, \phi_\nu)$ does not converge to A_0. There exists a subsequence $\{\phi_{\nu(n)}\}$ such that $\chi(\phi_0, \phi_{\nu(n)}) \to A_0'$ for some $A_0' \neq A_0$, since, by Proposition 8.7, the trajectory T_0 for $0 < |\phi| < \pi/3$ is bounded. Then we have $\operatorname{Im} I_{\mathbf{a}}(A_0') = \operatorname{Im} I_{\mathbf{b}}(A_0') = 0$, which contradicts the uniqueness of a solution of (BE)$_0$. Hence $\chi(\phi_0, \phi)$ is extended to $\phi = 0$ and is continuous in a neighbourhood of $\phi = 0$. By Proposition 8.5, it is possible to extend $\chi(\phi_0, \phi)$ to $\phi = \pm\pi/3$ by the same argument. □

Lemma 8.10. $h'(A) = -6\pi i I_{\mathbf{b}}(A)^{-2}$.

Proof. From $I'_{\mathbf{a},\mathbf{b}}(A) = -\Omega_{\mathbf{a},\mathbf{b}}/2$ and Lemma 6.6, the conclusion follows. □

Corollary 8.11. *If $I_{\mathbf{b}}(A) \neq 0, \infty$, then $h(A)$ is conformal around A.*

By Example 8.1, $h(A)$ is conformal at $A_0 = 3 \cdot 2^{2/3}$ and $h(A_0) = 0$. By Lemma 8.10,

$$h(A) = h'(A_0)(A - A_0) + o(A - A_0) = -\frac{\pi i}{2^{5/3} \cdot 3^2}(A - A_0) + o(A - A_0)$$

around $A = A_0$. By Proposition 8.6, for a sufficiently small $\varepsilon > 0$, the inverse image of $(-\varepsilon, 0) \cup (0, \varepsilon)$ under $h(A)$ is a trajectory $T_{0-} \cup T_{0+}$: $A = \chi_0^{\pm}(\phi)$ solving (BE)$_\phi$, and is expressed as

$$\chi_0^{\pm}(\phi) = A_0 + \gamma_0(\phi) i + o(\gamma_0(\phi)), \tag{8.4}$$

near $\phi = 0$, where $\gamma_0(\phi) \in \mathbb{R}$ is continuous in ϕ and $\gamma_0(0) = 0$.

The fact above implies that there exists a local trajectory solving (BE)$_\phi$ near $\phi = 0$. From this, a trajectory for $|\phi| \leq \pi/3$ as in Proposition 8.9 may be obtained. Furthermore, if two trajectories $\chi_1(\phi)$ and $\chi_2(\phi)$ solving (BE)$_\phi$ satisfy $\chi_1(\phi_0) = \chi_2(\phi_0)$ for some ϕ_0 such that $0 < |\phi_0| < \pi/3$, then (8.2) or the conformality of $h(A)$ at $A = A_0$ implies $\chi_1(\phi) \equiv \chi_2(\phi)$. Thus we have the following:

Proposition 8.12. *There exists a trajectory $A = A_\phi$ for $|\phi| \leq \pi/3$ with the properties*

(1) *for each ϕ, A_ϕ is a unique solution of (BE)$_\phi$;*
(2) *A_ϕ is smooth in ϕ for $0 < |\phi| < \pi/3$ and continuous in ϕ for $|\phi| \leq \pi/3$.*

For any cycle $\mathbf{c}$, it is easy to see that

$$e^{i\phi} \int_{\mathbf{c}} \frac{1}{z^2} w(A_\phi, z)\, dz = e^{i(\phi \mp 2\pi/3)} \int_{e^{\mp 2\pi i/3}\mathbf{c}} \frac{1}{\zeta^2} w(e^{\mp 2\pi i/3} A_\phi, \zeta)\, d\zeta,$$
$$e^{i\phi} \int_{\mathbf{c}} \frac{1}{z^2} w(A_\phi, z)\, dz = -e^{i(\phi + \pi)} \int_{\mathbf{c}} \frac{1}{\zeta^2} w(A_\phi, \zeta)\, d\zeta,$$

which yields the following:

Proposition 8.13. *Set $A_{\phi \mp 2\pi/3} = e^{\mp 2\pi i/3} A_\phi$ for $|\phi| \leq \pi/3$. Then for $|\phi| \leq \pi$, A_ϕ is a unique solution of (BE)$_\phi$. Furthermore, $A_{\phi + \pi} = A_\phi$, $A_{-\phi} = \overline{A_\phi}$.*

Let us examine the properties of A_ϕ in more detail. Note that the trajectory $A = A_\phi = x + iy$ for $|\phi| < \pi/3$ satisfies $h(A_\phi) \in \mathbb{R}$. Then, by (8.3),

$$\frac{d}{dt}h(A_\phi) = (x'(t) + iy'(t))(-6\pi i)I_{\mathbf{b}}(A_\phi)^{-2} \in \mathbb{R} \setminus \{0\}$$

with $t = \tan\phi$ for $0 < |\phi| < \pi/3$. Setting $I_{\mathbf{b}}(A_\phi)^{-1} = P + iQ$, we have

$$-\frac{1}{6\pi}\,\mathrm{Im}\,\frac{d}{dt}h(A_\phi) = x'(t)(P^2 - Q^2) - 2y'(t)PQ = 0.$$

If $x'(t_0) = 0$ for some $t_0 = \tan(\phi_0) \neq 0, \pm\infty$, then $PQ = 0$, and hence $I_{\mathbf{b}}(A_{\phi_0}) \in i\mathbb{R}\setminus\{0\}$ or $\mathbb{R}\setminus\{0\}$. This is impossible for $0 < |\phi| < \pi/3$, which implies $x'(t) \neq 0$ for $0 < |\phi| < \pi/3$. Since $A_{\pm\pi/3} = A_0 e^{\mp 2\pi i/3}$, we have $x'(t) < 0$ for $0 < \phi < \pi/3$ and $x'(t) > 0$ for $-\pi/3 < \phi < 0$. If $y'(t_0) = 0$ for some t_0 with $0 < |\phi_0| < \pi/3$, then $P^2 - Q^2 = 0$, i.e. $I_{\mathbf{b}}(A_{\phi_0})^{-1} = P(1 \pm i)$, implying $\phi_0 = \pm\pi/4$. Note that $PQ < 0$, $|P| > |Q|$ for $-\pi/4 < \phi < 0$ and that $PQ > 0$, $|P| > |Q|$ for $0 < \phi < \pi/4$. It follows that $y'(t) < 0$ for $0 < |\phi| < \pi/4$.

Proposition 8.14. *The trajectory $A_\phi = x(t) + iy(t)$ with $t = \tan\phi$ has the properties*

(1) *$x'(t) > 0$ for $-\pi/3 < \phi < 0$, and $x'(t) < 0$ for $0 < \phi < \pi/3$;*
(2) *$y'(t) < 0$ for $0 < |\phi| < \pi/4$ and $y'(\tan(\pm\pi/4)) = 0$.*

Thus we have the following proposition:

Proposition 8.15. *For every $\phi \in \mathbb{R}$ there exists a trajectory $A = A_\phi$ with the properties*

(1) *for each ϕ, A_ϕ is a unique solution of $(\mathrm{BE})_\phi$;*
(2) *$A_{\phi\pm 2\pi/3} = e^{\pm 2\pi i/3}A_\phi$, $A_{\phi+\pi} = A_\phi$, $A_{-\phi} = \overline{A_\phi}$;*
(3) *$A_0 = 3\cdot 2^{2/3}$, $A_{\pm\pi/3} = 3\cdot 2^{2/3}e^{\mp 2\pi i/3}$;*
(4) *A_ϕ is continuous in $\phi \in \mathbb{R}$, and smooth in $\phi \in \mathbb{R}\setminus\{m\pi/3 \mid m \in \mathbb{Z}\}$.*

Figure 7 is a rough drawing of the trajectory of A_ϕ.

By Proposition 8.14, when $|\phi|$ is sufficiently small, the location of the turning points may be examined. Small variance of A_ϕ around $\phi = 0$ is given by $A_\phi = A_0 + \delta_\phi$ with δ_ϕ having the properties (1) $\delta_\phi \to 0$ as $\phi \to 0$; (2) $\mathrm{Re}\,\delta_\phi \le 0$; (3) $\mathrm{Im}\,\delta_\phi \ge 0$ if $\phi \le 0$ and $\mathrm{Im}\,\delta_\phi \le 0$ if $\phi \ge 0$. Then the roots $z_0, z_1 = 2^{-1/3}$ and $z_2 = -4^{-2/3}$ of $w(A_0, z)^2$ vary in such a way that

$$z_0 = 2^{-1/3} + \varrho + O(\varrho^2), \quad z_1 = 2^{-1/3} - \varrho + O(\varrho^2), \quad z_2 = -4^{-2/3} + O(\varrho^2)$$

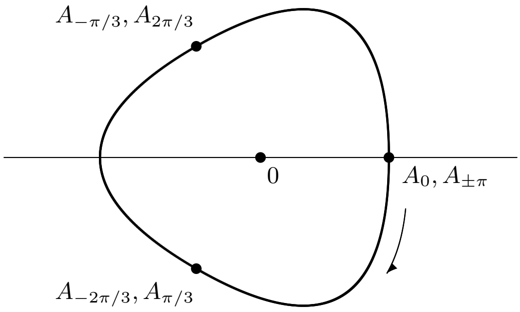

Figure 7. Trajectory of A_ϕ for $|\phi| \le \pi$

with $\varrho = 2^{-2/3} \cdot 3^{-1/2} \delta_\phi^{1/2}$. Indeed, insertion of $z_0 = 2^{-1/3} + \varrho_+$, $z_1 = 2^{-1/3} + \varrho_-$, $z_2 = -4^{-2/3} + \varrho_2$ into $z_0 + z_1 + z_2 = A_\phi/4$, $z_1 z_2 + z_2 z_0 + z_0 z_1 = 0$, $z_0 z_1 z_2 = -1/4$ yields

$$p + \varrho_2 = \delta_\phi/4, \quad p + 4\varrho_2 + 2^{4/3} q = O(p\varrho_2), \quad p - 2\varrho_2 + 2^{1/3} q = O(|\varrho_2|(|p| + |q|))$$

with $p = \varrho_+ + \varrho_-$, $q = \varrho_+ \varrho_-$, from which the estimates above follow. Thus we have the following:

Proposition 8.16. *If $|\phi|$ is sufficiently small, the turning points λ_k and $z_k = \lambda_k^{-2}$ $(k = 0, 1, 2)$ are represented as*

$$\lambda_0 = 2^{1/6} - \varepsilon_\phi e^{i\theta_\phi} + O(\varepsilon_\phi^2), \quad \lambda_1 = 2^{1/6} + \varepsilon_\phi e^{i\theta_\phi} + O(\varepsilon_\phi^2), \quad \lambda_2 = 2^{2/3} i + O(\varepsilon_\phi^2),$$
$$z_0 = 2^{-1/3} + 2^{1/2} \varepsilon_\phi e^{i\theta_\phi} + O(\varepsilon_\phi^2), \quad z_1 = 2^{-1/3} - 2^{1/2} \varepsilon_\phi e^{i\theta_\phi} + O(\varepsilon_\phi^2),$$
$$z_2 = -4^{-2/3} + O(\varepsilon_\phi^2).$$

Here, ε_ϕ and θ_ϕ fulfil

(1) *$\varepsilon_\phi > 0$ and $\varepsilon_\phi \to 0$ as $\phi \to 0$; and*

(2) *$\theta_\phi \to \pi/4$ as $\phi \to 0$ with $\phi < 0$, and $\theta_\phi \to -\pi/4$ as $\phi \to 0$ with $\phi > 0$.*

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