Publ. RIMS Kyoto Univ. **60** (2024), 699–721
DOI 10.4171/PRIMS/60-4-2

Counting Divisorial Contractions with Centre a cA_n-Singularity

by

Erik Paemurru

Abstract

First, we simplify the existing classification due to Kawakita and Yamamoto of 3-dimensional divisorial contractions with centre a cA_n-singularity, also called a compound A_n singularity. Next, we describe the global algebraic divisorial contractions corresponding to a given local analytic equivalence class of divisorial contractions with centre a point. Finally, we consider divisorial contractions of discrepancy at least 2 to a fixed variety with centre a cA_n-singularity. We show that if there exists one such divisorial contraction, then there exist uncountably many such divisorial contractions.

Mathematics Subject Classification 2020: 14E30 (primary); 14E05, 14J30 (secondary).
Keywords: weighted blow-up, terminal singularity, Mori theory.

§1. Introduction

The minimal model program and the Sarkisov program give a general framework for the birational classification of algebraic varieties, a central problem in algebraic geometry. Morphisms called divisorial contractions play a major role in both the minimal model program and the Sarkisov program. Therefore, classifying divisorial contractions is a fundamental problem.

A *divisorial contraction* is a proper birational morphism $\varphi\colon Y \to X$ between terminal algebraic varieties such that the exceptional locus of φ is a prime divisor and $-K_Y$ is φ-ample. The explicit classification of 3-dimensional divisorial contractions where the centre is a point has been completed, except when the centre is a cD_n or a cE_n-singularity and the discrepancy is 1. The case where the centre is a non-Gorenstein point has been done in [Hay99, Hay00, Hay05, Kaw05, Kaw12, Kaw96] and the Gorenstein case in [Kaw01, Kaw02, Kaw03], [Kaw05, Thm. 1.2] and [Yam18].

Communicated by Y. Namikawa. Received November 1, 2022. Revised February 23, 2023.

E. Paemurru: Department of Mathematics, University of Miami, Coral Gables, FL 33146, USA; current affiliation: Mathematik und Informatik, Universität des Saarlandes, 66123 Saarbrücken, Germany;
e-mail: paemurru@math.uni-sb.de

Above, the divisorial contractions are classified up to *local analytic equivalence*, meaning that we allow local analytic changes on X around P and on Y around the exceptional locus. Since the local analytic germ of a $\mathbb{Q}$-factorial variety can be non-factorial, the classification is given more generally for $\mathbb{Q}$-Gorenstein varieties with terminal singularities without requiring $\mathbb{Q}$-factoriality. If a morphism $\varphi\colon Y \to X$ is a divisorial contraction in this sense, without requiring $\mathbb{Q}$-factoriality, and if X is $\mathbb{Q}$-factorial, then we automatically find that Y is $\mathbb{Q}$-factorial, since the prime exceptional divisor is Cartier.

In this paper we focus on cA_n-singularities, also called compound A_n singularities, meaning that a general section through the point defines the surface A_n-singularity; see Definition 3.7. Compound A_n singularities are the simplest 3-dimensional terminal hypersurface singularities. There is an ongoing project with the goal of showing that all smooth Fano 3-folds are obtainable by deformations from singular toric 3-folds with cA_n-singularities.[1]

The classification due to Hayakawa, Kawakita, Kawamata and Yamamoto gives a list of weighted blow-ups such that every divisorial contraction is locally analytically equivalent to at least one member of the list. One way to improve the classification is to find which members of the classification lists give locally analytically equivalent blow-ups:

Problem 1.1. Describe the local analytic equivalence classes of 3-dimensional divisorial contractions with centre a point.

This is roughly what was asked in [Cor00, Prob. 3.8].

We have solved Problem 1.1 for cA_n-singularities in Theorem 6.1 and Lemma 6.2. This can drastically simplify the classification, as the complicated family in Theorem 3.10(3) reduces to just one simple case, Theorem 6.1(3).

The next step is to classify divisorial contractions globally algebraically:

Problem 1.2. Describe all global algebraic blow-ups up to isomorphism over the base that are locally analytically equivalent to a given weighted blow-up.

We have solved Problem 1.2 completely in Corollary 5.6. The global algebraic classification has applications in birational rigidity, finding birational relations and computing Sarkisov links; see [AK16, AZ16, Oka14, Oka18, Oka20, Pae20].

To prove that a given morphism is a divisorial contraction of a certain type, for example when computing Sarkisov links, it is best to have a classification list

[1] A. Corti and H. Ruddat, Smoothing toroidal crossing Fano threefolds with admissible log singularities, unpublished.

Singularity	Isomorphism over base	Local analytic equivalence
Smooth point	Uncountable	Countable
cA_n, only discr 1	n	$\lceil n/2 \rceil$
cA_n, admits discr > 1	Uncountable	Finite

Table 1. Counting divisorial contractions with centre a cA_n-point

where the conditions are as mild and as easy to check as possible. One way to phrase this is the following:

Problem 1.3. Describe an algorithm to determine whether a given weighted blow-up is locally analytically equivalent to a given member of the classification list.

We have solved Problem 1.3 for cA_n-singularities in Theorem 6.1 and Lemma 6.2. It is straightforward to determine the weight of a power series and it is algorithmic to check the singularity type of a simple singularity (Definition 3.6). To check whether a given singularity is of type A_n, D_n, E_6, E_7 or E_8, see [AGZV85, "16.2 The determinator 1.–9_k."] or [GLS07, Thms. I.2.48, I.2.51 and I.2.53]. It can be computed using a computer algebra system, for example Singular [DGPS22].

On the other hand, to prove local properties or local inequalities such as [KOPP24, Thm. 1.2], it helps to have a list which is as specific as possible, containing only a few members. Even though the classification lists in the literature contain uncountable families of weighted blow-ups, a countably infinite list, or even a finite list in certain cases, may suffice.

Problem 1.4. Given a variety X and a point P, determine whether there exist finitely many, countably many or uncountably many divisorial contractions to X with centre P, depending on the singularity type of P, where the counting is up to local analytic equivalence and up to global algebraic isomorphism over X.

We have solved Problem 1.4 for cA_n-singularities in Theorem 6.5. We have also included the case of smooth points in Table 1. By the proof of Theorem 6.5, the cardinalities up to local biholomorphism around the exceptional loci over the base are the same as up to global algebraic isomorphism over the base in the case of cA_n-points and smooth points.

As an example application of such results, [Oka20] uses the specific counts of divisorial contractions of a given type described in [Hay99] (such as [Hay99, Thm. 6.4]) to prove birational birigidity of varieties.

Table 1 raises the following questions:

Question 1.5. Let X be a 3-dimensional variety and $P \in X$ a singular point. Do there exist only finitely many divisorial contractions to X with centre P up to local analytic equivalence?

Question 1.6. Let X be a 3-dimensional variety and $P \in X$ a point. Is it true that there exist uncountably many divisorial contractions to X with centre P up to isomorphism over X if and only if there exists a divisorial contraction to X with centre P with discrepancy greater than 1?

Regarding Question 1.6, it is known that there exist only finitely many divisorial contractions of discrepancy at most 1 to a fixed variety; see Proposition 6.4. By [Pae20, §6] we expect there to be only finitely many divisorial contractions of ordinary type that are $(r_1, r_2, a, 1)$-blow-ups with centre a cA_n-singularity, even if the discrepancy a is greater than 1, as long as the inequalities $a \leq r_1 \leq r_2$ are satisfied. This does not answer Question 1.6 negatively; see Theorem 6.5.

The proofs in this paper rely on the concept of *weight-respecting maps*; see Definition 4.1, which is comparable to the equivalence relation "$\sim$" defined in [Hay99, "3.7 Weighted valuations"] for 3-dimensional index ≥ 2 terminal singularities embedded as hypersurfaces in orbifolds.

§2. Meaning of classification

The classification due to Hayakawa, Kawakita, Kawamata and Yamamoto is a classification list, as defined in Definition 2.1, except that it does not satisfy item (1) if the discrepancy of φ is 1 and the centre is either a cD or a cE point.

Definition 2.1. A set L is called a *classification list* if it consists of pairs $(\boldsymbol{w}, Z)$, where $\boldsymbol{w} := (w_1, \dots, w_5) \in ((1/m)\mathbb{Z})^5$ is a vector of positive rational numbers and Z is a codimension-2 complete intersection complex analytic space with an isolated singularity at the origin $\mathbf{0}$ inside an orbifold $\mathbb{C}^5/\mathbb{Z}_m$, such that

(1) for every 3-dimensional divisorial contraction φ with centre a point, φ is locally analytically equivalent (Definition 3.3) to the $\boldsymbol{w}$-blow-up of Z for some $(\boldsymbol{w}, Z)$ in L,

(2) for every pair $(\boldsymbol{w}, Z)$ in L, there exist a $\mathbb{Q}$-Gorenstein variety X with terminal singularities and a point $P \in X$ such that (X, P) is locally biholomorphic to $(Z, \mathbf{0})$, and

(3) for every $\mathbb{Q}$-Gorenstein variety X with terminal singularities and point $P \in X$, if (X, P) is locally biholomorphic to $(Z, \mathbf{0})$ for some $(\boldsymbol{w}, Z)$ in L, then there exists a divisorial contraction to X which is locally analytically equivalent to the $\boldsymbol{w}$-blow-up of Z.

$\boldsymbol{w}$	Conditions	f	wt f	Sing	Discr
$(1,a,b)$	$(a,b)=1, a\leq b$			sm	$a+b$
$(r_1,r_2,a,1)$	$r_1\leq r_2$, $a \mid r_1+r_2$, $(r_1,a)=(r_2,a)=1$, $a(n+1)=r_1+r_2$	$xy+g(z,t)$, $\operatorname{wt} g=r_1+r_2$	r_1+r_2	cA_n	a
$(1,5,3,2)$		$xy+z^2+t^3$	6	A_2	4
$(4,3,2,1)$		$x^2+y^2+z^3+xt^2$	6	E_6	3

Table 2. Local analytic equivalence classes of divisorial contractions, cA_n-singularities

The divisorial contraction in item (3) can be constructed using either Proposition 5.1 or Corollary 5.6.

Given a classification list L and two pairs $(\boldsymbol{w},Z)$ and $(\boldsymbol{w}',Z')$, Problem 1.1 asks us to determine when the weighted blow-ups of Z and Z' are locally analytically equivalent. For cA_n-points, we prove their local analytic equivalence if $\boldsymbol{w}=\boldsymbol{w}'$ in Lemma 6.2. It should not be difficult to prove in the case where there are only finitely many such divisorial contractions, which happens for example when the discrepancy is at most 1; see Proposition 6.4. See [Hay99, Hay00] for explicit descriptions and counts of minimal discrepancy divisorial contractions with centre a non-Gorenstein point.

Definition 2.2. We say a classification list L is a *nice classification list* if for every two pairs $(\boldsymbol{w},Z)$ and $(\boldsymbol{w}',Z')$ in L, the $\boldsymbol{w}$-blow-up of Z and the $\boldsymbol{w}'$-blow-up of Z' are locally analytically equivalent if and only if Z and Z' are biholomorphic around the origins and $\boldsymbol{w}=\boldsymbol{w}'$.

Finding a nice classification list, if it exists, would solve Problem 1.1.

Question 2.3. Does there exist a nice classification list?

By Theorem 6.1 and Lemma 6.2, the answer to Problem 2.3 is yes in the case of cA_n-points. We give two nice classification lists L and L' for cA_n-singularities and smooth points, corresponding to columns 1, 2, 3 and columns 1, 2, 4, 5 of Table 2, respectively. The final column "Discr" in Table 2 gives the discrepancy of the divisorial contraction.

For cA_n-singularities, the classification list contains only weighted blow-ups such that X is embedded locally analytically as a hypersurface $\mathbb{V}(f)$ in $\mathbb{C}^4$. We can also embed it as a codimension-2 complete intersection $\mathbb{V}(f, x_5)$ in $\mathbb{C}^5$ choosing the weight w_5 to be any positive integer. In such cases we write only the first four variables x, y, z, t and their weights w_1, w_2, w_3, w_4. Similarly, $\mathbb{C}^3$ can be embedded in $\mathbb{C}^5$ by $\mathbb{V}(x_4, x_5)$ with any positive integer weights w_4, w_5 for x_4, x_5, so we give only the first three weights.

The first nice classification list for cA_n-singularities and smooth points is given by

$$L := L_1 \cup L_2 \cup L_3 \cup L_4, \tag{2.3.1}$$

where

$$\begin{aligned}
L_1 &:= \big\{((1,a,b),\mathbb{C}^3) \bigm| a \text{ and } b \text{ are coprime positive integers, } a \le b\big\},\\
L_2 &:= \big\{\big((r_1,r_2,a,1),\mathbb{V}(xy+g(z,t))\big) \bigm| \operatorname{wt} g = r_1+r_2\big\},\\
L_3 &:= \big\{((1,5,3,2),\mathbb{V}(xy+z^2+t^3))\big\},\\
L_4 &:= \big\{((4,3,2,1),\mathbb{V}(x^2+y^2+z^3+xt^2))\big\},
\end{aligned}$$

where in L_2 the convergent power series $g \in \mathbb{C}\{z,t\}$ defines an isolated singularity at the origin and r_1, r_2 and a are positive integers that satisfy $r_1 \le r_2$, a divides $r_1 + r_2$, a is coprime to both r_1 and r_2, and $a(n+1) = r_1 + r_2$. The polynomial $xy + g(z,t)$ with $\operatorname{wt} g = r_1 + r_2$ appears in [Kaw02, Thm. 1.1] and the polynomial $xy + z^2 + t^3$ appears in [Kaw03, Thm. 1.13], whereas the complicated condition of [Yam18, Thm. 2.6] is simplified in Theorem 6.1 to $x^2 + y^2 + z^3 + xt^2$.

The second nice classification list for cA_n-singularities and smooth points is given by

$$L' := L_1 \cup L'_2 \cup L'_3 \cup L'_4,$$

where

$$\begin{aligned}
L'_2 &:= \big\{((r_1,r_2,a,1),\mathbb{V}(f)) \bigm| (\mathbb{V}(f),\mathbf{0}) \text{ is a } cA_n\text{-singularity},\ \operatorname{wt} f = r_1+r_2\big\},\\
L'_3 &:= \big\{((1,5,3,2),\mathbb{V}(f)) \bigm| (\mathbb{V}(f),\mathbf{0}) \text{ is an } A_2\text{-singularity},\ \operatorname{wt} f = 6\big\},\\
L'_4 &:= \big\{((4,3,2,1),\mathbb{V}(f)) \bigm| (\mathbb{V}(f),\mathbf{0}) \text{ is an } E_6\text{-singularity},\ \operatorname{wt} f = 6\big\},
\end{aligned}$$

where the convergent power series $f \in \mathbb{C}\{x,y,z,t\}$ defines an isolated singularity at the origin and in L_2 the positive integers r_1, r_2 and a satisfy the same conditions as for the first classification list. The singularities cA_n, A_2 and E_6 are defined in Definitions 3.6 and 3.7.

§3. Preliminaries

Notation 3.1. Let $\mathbb{C}$ denote the complex numbers. A *variety*, short for algebraic variety, is defined to be an integral separated scheme of finite type over $\mathbb{C}$. All morphisms of varieties are defined over $\mathbb{C}$. The $\mathbb{C}$-algebra of power series that are absolutely convergent in a neighbourhood of the origin is denoted $\mathbb{C}\{\boldsymbol{x}\}$, short for $\mathbb{C}\{x_1,\dots,x_n\}$. The complex space, short for complex analytic space, corresponding to a variety X is denoted X^{an}. A *singularity* is defined to be a complex space germ (see [GLS07, Def. I.1.47]). If I is an ideal of regular functions on a variety or an ideal of holomorphic functions on a complex space, then $\mathbb{V}(I)$ denotes the zero locus of I. If I is an ideal of holomorphic function germs on a complex space germ (X,P), then $(\mathbb{V}(I),P)$ denotes the (possibly non-reduced) subgerm defined by I (see [GLS07, §I.1.4]).

Given a convergent power series $f\in\mathbb{C}\{\boldsymbol{x}\}\coloneqq\mathbb{C}\{x_1,\dots,x_n\}$ we define the *multiplicity* of f, denoted mult f, by

$$\operatorname{mult} f \coloneqq \min\bigl\{i_1+\dots+i_n \bigm| x_1^{i_1}\cdot\ldots\cdot x_n^{i_n} \text{ has non-zero coefficient in } f\bigr\}.$$

Given positive integer weights $w_1,\dots,w_n$ for variables $x_1,\dots,x_n$ we define the *weight* of f, denoted wt f, by

$$\operatorname{wt} f \coloneqq \min\bigl\{w_1i_1+\dots+w_ni_n \bigm| x_1^{i_1}\cdot\ldots\cdot x_n^{i_n} \text{ has non-zero coefficient in } f\bigr\}$$

if f is non-zero, and we define $\operatorname{wt}0=\infty$ otherwise. We denote the quasihomogeneous weight d part of f by $f_{\mathrm{wt}=d}$. The *quadratic part* of f is defined to be the homogeneous degree 2 part of f. The *quadratic rank* of f is defined to be the rank of the symmetric matrix M with complex coefficients such that the quadratic part of f is equal to $\boldsymbol{x}^TM\boldsymbol{x}$ where $\boldsymbol{x}$ is the vector $(x_1,\dots,x_n)$.

We denote the *Jacobian ideal* $(\partial f/\partial x_1,\dots,\partial f/\partial x_n)\subseteq\mathbb{C}\{\boldsymbol{x}\}$ of f by $j(f)$. We remind that the *Milnor algebra* of f is the $\mathbb{C}$-algebra $\mathbb{C}\{\boldsymbol{x}\}/j(f)$. We say that a set S of monomials of $\mathbb{C}[\boldsymbol{x}]\coloneqq\mathbb{C}\{x_1,\dots,x_n\}$ is a *monomial spanning set* for a $\mathbb{C}$-algebra $\mathbb{C}\{\boldsymbol{x}\}/J$ if the set $\{s+J\mid s\in S\}$ generates the $\mathbb{C}$-vector space $\mathbb{C}\{\boldsymbol{x}\}/J$, and we say S is a *monomial basis* for the $\mathbb{C}$-algebra $\mathbb{C}\{\boldsymbol{x}\}/J$ if $\{s+J\mid s\in S\}$ is a basis for the $\mathbb{C}$-vector space $\mathbb{C}\{\boldsymbol{x}\}/J$. By Zorn's lemma, every $\mathbb{C}$-algebra $\mathbb{C}\{\boldsymbol{x}\}/J$ has a (possibly infinite) monomial basis.

Definition 3.2. A *divisorial contraction* is a proper birational morphism $\varphi\colon Y\to X$ between normal $\mathbb{Q}$-Gorenstein varieties with terminal singularities such that

(1) the exceptional locus of φ is a prime divisor and
(2) $-K_Y$ is φ-ample.

Definition 3.3 ([Pae21, Def. 2.14]). Let $\varphi\colon Y\to X$ and $\varphi'\colon Y'\to X'$ be birational morphisms of varieties (or bimeromorphic holomorphisms of complex analytic spaces). We say that an isomorphism $X\to X'$ *lifts* if there exists an isomorphism $Y \cong Y'$ such that the diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} \\ X & \longrightarrow & X' \end{array}$$

commutes. We say that φ and φ' are *equivalent* if there exists an isomorphism $X \cong X'$ that lifts. We say φ and φ' are *locally equivalent* if there exist isomorphic open subsets $U \subseteq X$ and $U' \subseteq X'$ containing the centres of the morphisms φ and φ' such that the restrictions $\varphi|_{\varphi^{-1}U} \colon \varphi^{-1}U \to U$ and $\varphi'|_{\varphi'^{-1}U'} \colon \varphi'^{-1}U' \to U'$ are equivalent.

If we consider the complex space corresponding to a variety or when we wish to emphasise that we are working in the category of complex spaces, then we say *analytically equivalent* or *locally analytically equivalent*.

Definition 3.4. Let n be a positive integer and let $\boldsymbol{w} = (w_1, \ldots, w_n)$ be positive integers, called the weights of the blow-up. Define a $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ by $\lambda \cdot (u, x_1, \ldots, x_n) = (\lambda^{-1}u, \lambda^{w_1}x_1, \ldots, \lambda^{w_n}x_n)$ and define T by the geometric quotient $(\mathbb{C}^{n+1} \setminus \mathbb{V}(x_1, \ldots, x_n))/\mathbb{C}^*$ (or its analytification). Then the map $\varphi \colon T \to \mathbb{C}^n$, $[u, x_1, \ldots, x_n] \mapsto (u^{w_1}x_1, \ldots, u^{w_n}x_n)$ is called the ***w****-blow-up of* $\mathbb{C}^n$. If $Z \subseteq \mathbb{C}^n$ is a closed subvariety (or a closed complex subspace $Z \subseteq D$ where $D \subseteq \mathbb{C}^n$ is open) and $\widetilde{Z}$ is the closure of $\varphi^{-1}(Z \setminus \{\mathbf{0}\})$ in T (in $\varphi^{-1}D$), then the restriction $\varphi|_{\widetilde{Z}} \colon \widetilde{Z} \to Z$ is called the ***w****-blow-up of* Z. Let $\rho \colon Y \to X$ be a surjective birational morphism of varieties (or a surjective bimeromorphic holomorphism of complex spaces). Given an open subset $U \subseteq X$ containing the centre of ρ and an isomorphism $U \cong X' \subseteq \mathbb{C}^n$ taking a point $P \in X$ to the origin $\mathbf{0}$, the map ρ is called the ***w****-blow-up of* X *at* P if the restriction $\rho|_{\rho^{-1}U} \colon \rho^{-1}U \to U$ is equivalent, through the given isomorphism $U \cong X'$, to the $\boldsymbol{w}$-blow-up of X'.

Remark 3.5. We make the following remarks on Definition 3.4:

(a) A weighted blow-up crucially depends both on the isomorphism $U \cong X'$ and a choice of coordinates $x_1, \ldots, x_n$, even though it is not explicit in the notation.

(b) Replacing $\boldsymbol{w}$ by $(w_1/g, \ldots, w_n/g)$ in Definition 3.4, where g is the greatest common divisor of $w_1, \ldots, w_n$, gives an isomorphic blow-up over X.

(c) By [CLS11, Thm. 5.1.11], the weighted blow-up of an affine space in Definition 3.4 coincides with the toric description of subdividing a cone in [KM92, Prop.-Def. 10.3].

Definition 3.6. A *simple hypersurface singularity*, also known as an *ADE-singularity*, is a complex space germ (X, P) isomorphic to $(\mathbb{V}(f), \mathbf{0})$, where $f \in$

$\mathbb{C}\{x_1,\dots,x_n\}$ is one of the following:

$$\begin{aligned}
A_k &: x_1^{k+1}+x_2^2+\cdots+x_n^2, \quad k\geq 1,\\
D_k &: x_1(x_2^2+x_1^{k-2})+x_3^2+\cdots+x_n^2, \quad k\geq 4,\\
E_6 &: x_1^3+x_2^4+x_3^2+\cdots+x_n^2,\\
E_7 &: x_1(x_1^2+x_2^3)+x_3^2+\cdots+x_n^2,\\
E_8 &: x_1^3+x_2^4+x_3^2+\cdots+x_n^2,
\end{aligned}$$

where n is at least 1 in the case A_k and at least 2 in all other cases.

Definition 3.7. Let k be a positive integer. A *compound A_k-singularity*, denoted cA_k, is a complex space germ isomorphic to $(\mathbb{V}(xy+g),\mathbf{0})\subseteq(\mathbb{C}^4,\mathbf{0})$, where $g\in\mathbb{C}\{z,t\}$ has multiplicity $k+1$.

We state the known classification of divisorial contractions to both smooth points and cA_n-points.

Theorem 3.8 ([Kaw01, Thm. 1.1]). *Let P be a smooth point of a 3-dimensional $\mathbb{Q}$-Gorenstein variety X with terminal singularities. Let $\varphi\colon Y\to X$ be a surjective birational morphism with centre P. Then φ is a divisorial contraction if and only if φ is locally analytically equivalent to the $(1,a,b)$-blow-up of $\mathbb{A}^3$, where a and b are coprime positive integers.*

Remark 3.9. By [CLS11, Lem. 11.4.10] the discrepancy of the $(1,a,b)$-blow-up of $\mathbb{A}^3$ is $a+b$.

Theorem 3.10. *Let n be a positive integer. Let P be a cA_n-point of a $\mathbb{Q}$-Gorenstein variety with terminal singularities. Let $\varphi\colon Y\to X$ be a surjective birational morphism with centre P. Then φ is a divisorial contraction if and only if one of the following holds:*

(1) *φ is locally analytically equivalent to the $(r_1,r_2,a,1)$-blow-up of $\mathbb{V}(f)$ at $\mathbf{0}$ where $f\in\mathbb{C}\{x,y,z,t\}$ is such that*

  (1a) *r_1, r_2 and a are positive integers such that $r_1\leq r_2$, $a(n+1)=r_1+r_2$, a divides r_1+r_2, a is coprime to both r_1 and r_2 and*

  (1b) *$f=xy+g(z,t)$, where $\operatorname{wt} g=r_1+r_2$,*

(2) *$n=1$ and φ is locally analytically equivalent to the $(1,5,3,2)$-blow-up of $\mathbb{V}(f)$ at $\mathbf{0}$ where $f\in\mathbb{C}\{x,y,z,t\}$ is such that*

  (2a) *$f=xy+z^2+t^3$,*

(3) *$n = 2$ and φ is locally analytically equivalent to the $(4,3,2,1)$-blow-up of $\mathbb{V}(f)$ at $\mathbf{0}$ where $f \in \mathbb{C}\{x,y,z,t\}$ is such that*

(3a) *$f = x^2 + y^2 + 2cxy + 2xp(z,t) + 2cyp_{\mathrm{wt}=3}(z,t) + z^3 + g(z,t)$, where $c \in \mathbb{C} \setminus \{-1,1\}$, $\operatorname{wt} g \geq 6$, the power series p contains only monomials of weights 2 and 3 for the weights $(4,3,2,1)$, the coefficient of t^2 is non-zero in p and $\deg g(z,1) \leq 2$.*

Conditions (1b), (2a) and (3a) are the same as in [Kaw02, Thm. 1.1], [Kaw03, Thm. 1.13] and [Yam18, Thm. 2.6], except for the small difference in that the condition that $z^{(r_1+r_2)/a}$ *has a non-zero coefficient in* f is replaced by the equivalent condition $a(n+1) = r_1 + r_2$. We simplify Theorem 3.10 in Theorem 6.1. In particular, we show that we can replace item (3a) with the much simpler condition $f = x^2 + y^2 + z^3 + xt^2$. This polynomial appears in [Kaw03, Exa. 6.8].

Remark 3.11. By [Hay99, §3.9] the discrepancy in Theorem 3.10 in cases (1), (2) and (3) is respectively a, 4 and 3.

§4. Weight-respecting maps

The main tools we use in this paper are weight-respecting maps (see Definition 4.1) and some classical theorems from singularity theory in weight-respecting form (see Lemmas 4.3, 4.5 and Corollary 4.7).

For Definition 4.1 and Lemma 4.2, let n and m be positive integers. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_m)$ denote the coordinates on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Choose positive integer weights for $\boldsymbol{x}$ and $\boldsymbol{y}$.

Definition 4.1 ([Pae21, Def. 4.1]). Let $X \subseteq \mathbb{C}^n$ and $X' \subseteq \mathbb{C}^m$ be complex analytic spaces. We say that a biholomorphic map $\psi\colon X \to X'$ taking $\mathbf{0}$ to $\mathbf{0}$ is *weight respecting* if denoting its inverse by θ, we can locally analytically around the origins write $\psi = (\psi_1, \ldots, \psi_m)$ and $\theta = (\theta_1, \ldots, \theta_n)$, where for all i and j, the power series $\psi_j \in \mathbb{C}\{\boldsymbol{x}\}$ and $\theta_i \in \mathbb{C}\{\boldsymbol{y}\}$ satisfy $\operatorname{wt}(\psi_j) \geq \operatorname{wt}(y_j)$ and $\operatorname{wt}(\theta_i) \geq \operatorname{wt}(x_i)$.

Compare Definition 4.1 with the equivalence relation "$\sim$" defined in [Hay99, "3.7 Weighted valuations"] for 3-dimensional index ≥ 2 terminal singularities embedded as hypersurfaces in orbifolds.

Lemma 4.2 ([Pae21, Cor. 4.4]). *If a biholomorphism from $X \subseteq \mathbb{C}^n$ to $X' \subseteq \mathbb{C}^m$ taking $\mathbf{0}$ to $\mathbf{0}$ is weight respecting, then it lifts to the weighted blown-up spaces.*

Compare Lemma 4.2 with [Hay99, Lem. 5.6].

We recall that two convergent power series $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$ are said to be *right equivalent* if there exists a biholomorphic map germ $\varphi\colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$

such that $g = f \circ \varphi$, and are said to be *contact equivalent* if there exist a biholomorphic map germ $\varphi\colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ and a unit $u \in \mathbb{C}\{x_1, \ldots, x_n\}$ such that $g = u(f \circ \varphi)$ (this is [GLS07, Def. I.2.9]).

It is a standard result that every convergent power series is right equivalent to a polynomial (see the theorem in [AGZV85, §6.3] or [GLS07, Cor. I.2.24]). By following the standard proof we show that, unsurprisingly, there also exists a weight-respecting right equivalence.

Lemma 4.3. *Let n be a positive integer. Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ define an isolated singularity at the origin with Milnor number μ. Then for every integer $N \geq \mu + 1$, there exists an automorphism Ψ of $\mathbb{C}\{x_1, \ldots, x_n\}$ such that*

(1) *$\Psi(f)$ is equal to the truncation of f up to degree N and*
(2) *for every $i \in \{1, \ldots, n\}$ the truncation of $\Psi(x_i)$ up to degree $N - \mu$ is equal to x_i.*

Proof. Denote $\mathbb{C}\{\boldsymbol{x}\} \coloneqq \mathbb{C}\{x_1, \ldots, x_n\}$ and let $\mathfrak{m} \coloneqq \sum x_i \mathbb{C}\{\boldsymbol{x}\}$ be the maximal ideal. The ideal $\mathfrak{m}^\mu$ is contained in the Jacobian ideal

$$j(f) \coloneqq \sum \frac{\partial f}{\partial x_i} \mathbb{C}\{\boldsymbol{x}\}$$

of f by the proof of the lemma in [AGZV85, §5.5]. Let $h \in \mathfrak{m}^{N+1}$. Define $F \in \mathbb{C}\{\boldsymbol{x}\}[t]$ by $F \coloneqq f + th$. Below we construct a biholomorphic map germ $\psi\colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ by following the proof of [GLS07, Thm. I.2.23].

First, we show that for every $t_0 \in \mathbb{C}$ we have the equality of ideals

$$j(f) \cdot \mathbb{C}\{\boldsymbol{x}, t - t_0\} = \sum \frac{\partial F}{\partial x_i} \mathbb{C}\{\boldsymbol{x}, t - t_0\}, \tag{4.3.1}$$

where we consider $F = f + t_0 h + (t - t_0)h$ as an element of $\mathbb{C}\{\boldsymbol{x}, t - t_0\}$. Since $\mathfrak{m}^\mu$ is inside $j(f)$, we find the equality of ideals

$$j(f) \cdot \mathbb{C}\{\boldsymbol{x}, t - t_0\} = \sum \frac{\partial F}{\partial x_i} \mathbb{C}\{\boldsymbol{x}, t - t_0\} + \mathfrak{m} \cdot j(f) \cdot \mathbb{C}\{\boldsymbol{x}, t - t_0\}. \tag{4.3.2}$$

Applying the Nakayama lemma ([GLS07, Prop. B.3.6]) to equation (4.3.2) gives equation (4.3.1).

Now, equation (4.3.1) implies that

$$h \in (\mathfrak{m}^{N+1-\mu} \cdot \mathbb{C}\{\boldsymbol{x}, t - t_0\}) \cdot \left(\sum \frac{\partial F}{\partial x_i} \mathbb{C}\{\boldsymbol{x}, t - t_0\} \right).$$

By [GLS07, Thm. I.2.22(2) and Rem. I.2.22.1], for every $t_0 \in \mathbb{C}$ there exists an open neighbourhood $U_{t_0} \subseteq \mathbb{C}$ of t_0 such that for every $t' \in U_{t_0}$ there exists a biholomorphic map germ $\psi_{t'}\colon (\mathbb{C}^n, \mathbf{0}) \to (\mathbb{C}^n, \mathbf{0})$ such that $x_i \circ \psi_{t'} - x_i \in \mathfrak{m}^{N+1-\mu}$ and $(f + t'h) \circ \psi_{t'} = f + t_0 h$. Since the interval $[0, 1]$ is compact, there exist finitely many biholomorphic map germs $\psi_{t_1}, \ldots, \psi_{t_k}$ such that the composition $\psi := \psi_{t_k} \circ \cdots \circ \psi_{t_1}$ satisfies $f \circ \psi = f + h$ and $x_i \circ \psi - x_i \in \mathfrak{m}^{N+1-\mu}$.

Finally, choosing h to be the negative of the sum of the degree $> N$ parts of f and choosing Ψ to be the precomposition by ψ proves the lemma. □

The lemma in [AGZV85, §12.6] is useful for computing normal forms of singularities. Here we give a weight-respecting version.

Lemma 4.4. *Let n be a positive integer and let $\boldsymbol{w} := (w_1, \ldots, w_n)$ be positive integer weights for the variables $x_1, \ldots, x_n$. Let $f \in \mathbb{C}\{\boldsymbol{x}\}$ define an isolated singularity at the origin. Let f_0 denote the least weight non-zero quasihomogeneous part of f. Choose a monomial spanning set $S \subseteq \mathbb{C}[\boldsymbol{x}]$ for the Milnor algebra $\mathbb{C}\{\boldsymbol{x}\}/j(f_0)$ of f_0. Then there exists an automorphism Ψ of $\mathbb{C}\{\boldsymbol{x}\}$ of the following form: for all i: $\Psi(x_i) = x_i + g_i$, where each $g_i \in \mathbb{C}\{\boldsymbol{x}\}$ is either zero or satisfies $\operatorname{wt} g_i > \operatorname{wt} x_i$, such that every monomial of $\mathbb{C}[\boldsymbol{x}]$ with weight greater than $\operatorname{wt} f_0$ that does not belong to S has coefficient zero in $\Psi(f)$.*

Proof. We find an automorphism Ψ' of $\mathbb{C}[\![\boldsymbol{x}]\!]$ satisfying the conditions of the lemma following the proof in [AGZV85, §12.6]. Next, we define an automorphism $\widehat{\Psi}$ of $\mathbb{C}\{\boldsymbol{x}\}$ by letting $\widehat{\Psi}(x_i)$ be the truncation of $\Psi'(x_i)$ up to some high enough degree N. The automorphism $\widehat{\Psi}$ satisfies the conditions of the lemma, except that there might be monomials of weight greater than N that have a non-zero coefficient in $\widehat{\Psi}(f)$. Now using Lemma 4.3 we find a suitable Ψ. □

Lemma 4.5. *Let $n \geq 2$ be an integer and let $\boldsymbol{w} = (w_1, \ldots, w_n)$ be positive integer weights for variables $\boldsymbol{x} = (x_1, \ldots, x_n)$. Let $f \in \mathbb{C}\{\boldsymbol{x}\}$ be such that the coefficient of x_1x_2 is non-zero in f and $\operatorname{wt} x_1x_2 = \operatorname{wt} f$. Then there exists a weight-respecting automorphism Ψ of $\mathbb{C}\{\boldsymbol{x}\}$ such that the only monomial that belongs to the ideal (x_1, x_2) and has non-zero coefficient in $\Psi(f)$ is x_1x_2.*

Proof. See the proof of [Pae21, Prop. 4.6]. □

Remark 4.6. In the case where f defines an isolated singularity at the origin, Lemma 4.5 follows also from Lemma 4.4.

One easy corollary of Lemma 4.4 is the following:

Corollary 4.7. *Let the variables $\boldsymbol{x} = (x_1, \ldots, x_n)$ have positive integer weights $\boldsymbol{w} = (w_1, \ldots, w_n)$. Let the least weight non-zero quasihomogeneous part f_0 of $f \in \mathbb{C}\{\boldsymbol{x}\}$ be one of the five forms described in Definition 3.6. Then there is a weight-respecting automorphism Ψ of $\mathbb{C}\{\boldsymbol{x}\}$ such that $\Psi(f) = f_0$.*

Proof. Use Lemma 4.4 with a set S that does not contain any elements of weight greater than wt f. □

§5. From analytic to algebraic category

In Proposition 5.1, we show how to extend blow-ups along points with possibly non-reduced structure (equivalently, blow-ups along coherent ideal sheaves with cosupport a point) from the analytic category to the algebraic. Proposition 5.1 was explained to me by Masayuki Kawakita.

Proposition 5.1. *Let X be a variety and $\mathcal{J}$ a coherent $\mathcal{O}_{X^{\mathrm{an}}}$-ideal sheaf with cosupport a point, where X^{an} is the analytification of X. Then there exists a coherent $\mathcal{O}_X$-ideal sheaf $\mathcal{I}$ such that its analytification is $\mathcal{J}$.*

Proof. Since the cosupport of $\mathcal{J}$ is a point P, there exists a positive integer k such that the kth power of the maximal ideal of $\mathcal{O}_{X^{\mathrm{an}},P}$ is in the stalk $\mathcal{J}_P$. Proposition 5.1 follows. □

We give an alternative construction in Corollary 5.6(a), which describes the divisorial contraction as a weighted blow-up. Corollary 5.6(a) is a modification of [Pae21, Lem. 4.9] which was used for explicitly constructing weighted blow-ups of affine hypersurfaces with a cA_n-point.

Construction 5.2. Let U be an affine variety $\operatorname{Spec}(\mathbb{C}[x_1,\dots,x_n]/I)$ containing the point $\mathbb{V}(x_1,\dots,x_n)$, for some ideal $I\subseteq\mathbb{C}[x_1,\dots,x_n]$. Assign positive integer weights $w_1,\dots,w_m$ to the variables $y_1,\dots,y_m$ of $\mathbb{C}^m$ and assign weights $1,\dots,1$ to the variables $x_1,\dots,x_n$.

Let $\psi\colon(U^{\mathrm{an}},\mathbf{0})\to(Z,\mathbf{0})$ be a local biholomorphism to a complex space $Z\subseteq\mathbb{C}^m$ containing the origin. Define the variety $\widehat{U}$ by

$$\widehat{U}\colon\mathbb{V}(I+(\psi_1^{<w_1}-y_1,\dots,\psi_m^{<w_m}-y_m))\subseteq\mathbb{A}^{n+m}:=\operatorname{Spec}\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_m],$$

where $\psi_j^{<w_j}\in\mathbb{C}[x_1,\dots,x_n]$ denotes the truncation of the jth coordinate power series of ψ up to order w_j-1. Note that $\widehat{U}$ is isomorphic to U.

Proposition 5.3. *In Construction 5.2, the local biholomorphism $(\widehat{U}^{\mathrm{an}},\mathbf{0})\to(Z,\mathbf{0})$ given by the composition of $y_j\mapsto y_j+\psi_j^{<w_j}-\psi_j$ and the projection to $\mathbb{C}^m$ is weight respecting.*

Proof. The local biholomorphism $y_j\mapsto y_j+\psi_j^{<w_j}-\psi_j$ with inverse $y_j\mapsto y_j-\psi_j^{<w_j}+\psi_j$ is clearly weight respecting. The projection to $\mathbb{C}^m$ is given by

$$(x_1,\dots,x_n,y_1,\dots,y_m)\mapsto(y_1,\dots,y_m),$$

with inverse

$$(\theta_1, \ldots, \theta_n, y_1, \ldots, y_m) \leftarrow\!\shortmid (y_1, \ldots, y_m),$$

where $\theta_i \in \mathbb{C}\{y_1, \ldots, y_m\}$ are convergent power series with constant term zero. We see that for all i, either $\operatorname{wt}\theta_i \geq \operatorname{wt} x_i = 1$ or $\theta_i = 0$. This shows that the projection to $\mathbb{C}^m$ is weight respecting. □

Remark 5.4. If any of the weights w_j were zero in Construction 5.2, then the truncation $\psi_j^{<w_j}$ might not be a polynomial.

Lemma 5.5. *Let $Y_1 \to X$ and $Y_2 \to X$ be birational morphisms of varieties. Then Y_1 and Y_2 are isomorphic over X if and only if the analytifications Y_1^{an} and Y_2^{an} are locally biholomorphic over X^{an} around the exceptional loci.*

Proof. "$\Longrightarrow$". The isomorphism $Y_1 \to Y_2$ induces a biholomorphism $Y_1^{\mathrm{an}} \to Y_2^{\mathrm{an}}$.

"$\Longleftarrow$". The local biholomorphism extends to a unique biholomorphism $Y_1^{\mathrm{an}} \to Y_2^{\mathrm{an}}$ over X^{an}. Now, it suffices to show that if f is a rational map of varieties such that its analytification is holomorphic, then f is a morphism of varieties. For this, it suffices to show that if f is a rational function on an affine variety $Z = \operatorname{Spec} A$ such that its analytification f^{an} is holomorphic, then $f \in A$. For this, we follow the argument in [JS19].

First, we show that f is integral over A. Let $\bar{A}$ be the integral closure of A in its field of fractions. Using the inclusions $\operatorname{Frac} A \to \operatorname{Frac} \bar{A}$ and $\mathcal{O}_{(\operatorname{Spec} A)^{\mathrm{an}}} \to \mathcal{O}_{(\operatorname{Spec} \bar{A})^{\mathrm{an}}}$, we see that f is a rational function on $\operatorname{Spec} \bar{A}$ and f^{an} is a holomorphic function on $(\operatorname{Spec} \bar{A})^{\mathrm{an}}$. Therefore, f^{an} is bounded on every small analytic neighbourhood of any point of $(\operatorname{Spec} \bar{A})^{\mathrm{an}}$. Therefore, the order of vanishing of f along every prime divisor D of $\operatorname{Spec} \bar{A}$ is non-negative. Since $\operatorname{Spec} \bar{A}$ is normal, we find $f \in \bar{A}$.

By [JK20, Prop. 2.2], A is integrally closed in $\mathcal{O}_{Z^{\mathrm{an}}}(Z^{\mathrm{an}})$. Since f is holomorphic, we have $f \in \mathcal{O}_{Z^{\mathrm{an}}}(Z^{\mathrm{an}})$, and since f is integral over A, we find $f \in A$. □

Corollary 5.6. *Let X be a variety, $P \in X$ a closed point and $U \subseteq X$ an affine open containing P. Let $W \to Z$ be a weighted blow-up of complex spaces with centre a point $Q \in Z$ such that (X^{an}, P) is locally biholomorphic to (Z, Q). Then,*

(a) *the construction in Proposition 5.3 gives a weighted blow-up $Y \to X$ that is locally analytically equivalent to $W \to Z$;*

(b) *every blow-up $Y \to X$ that is locally analytically equivalent to $W \to Z$ is isomorphic over X to a blow-up $\widehat{Y} \to X$ given in Construction 5.2 for some ψ.*

Proof. (a) It suffices to consider the case where Z is a complex subspace of $\mathbb{C}^m$ and Q is the origin. Using Proposition 5.3, we find an isomorphism $U \to \widehat{U} \subseteq \mathbb{A}^{n+m}$

and a choice of weights for the variables $x_1, \dots, x_n, y_1, \dots, y_m$ of $\mathbb{A}^{n+m}$ such that the weighted blow-up of $\widehat{U} \subseteq \mathbb{A}^{n+m}$ is locally analytically equivalent to $W \to Z$ by Lemma 4.2. By gluing, we find a weighted blow-up $Y \to X$ which is locally analytically equivalent to $W \to Z$.

(b) Let $X^{\mathrm{an}} \to Z$ be local biholomorphism that lifts to the blown-up spaces. The construction in Proposition 5.3 gives a weighted blow-up $\widehat{Y} \to \widehat{X}$, an isomorphism $\widehat{X} \to X$ and a weight-respecting local biholomorphism $\widehat{X}^{\mathrm{an}} \to Z$. Since both $X^{\mathrm{an}} \to Z$ and $\widehat{X}^{\mathrm{an}} \to Z$ lift to the blown-up spaces, $\widehat{X}^{\mathrm{an}} \to X^{\mathrm{an}}$ locally lifts to blown-up spaces. By Lemma 5.5, the isomorphism $\widehat{X} \to X$ lifts to the blown-up spaces. □

Example 5.7 shows that Proposition 5.1 and Corollary 5.6(a) cannot always be true when we blow up a positive-dimensional closed complex subspace.

Example 5.7. Let X be a $\mathbb{Q}$-factorial 3-fold with a unique singular point P which is an *ordinary double point*, meaning a singularity isomorphic to $(\mathbb{V}(xy+zt), \mathbf{0}) \subseteq (\mathbb{C}^4, \mathbf{0})$. Then locally analytically there exists a small resolution φ, the blow-up of the divisor $\mathbb{V}(x, z)$ of $\mathbb{V}(xy + zt)$ with exceptional locus a curve. On the other hand, since X is $\mathbb{Q}$-factorial, there is no proper birational morphism $Y \to X$ from a smooth variety Y which is locally analytically equivalent to φ.

§6. Counting divisorial contractions

We show that we can simplify Theorem 3.10.

Theorem 6.1. *Theorem 3.10 remains true if we*

(1) *replace item (1b) with "*$\operatorname{wt} f = r_1 + r_2$*",*
(2) *replace item (2a) with "*$(\mathbb{V}(f), \mathbf{0})$ *is an* A_2*-singularity and* $\operatorname{wt} f = 6$*" and*
(3) *replace item (3a) with either "*$f = x^2 + y^2 + z^3 + xt^2$*" or with "*$(\mathbb{V}(f), \mathbf{0})$ *is an* E_6*-singularity and* $\operatorname{wt} f = 6$*".*

Proof. (1) Follows from [Pae21, Prop. 4.6].

(2) Let $f \in \mathbb{C}\{x, y, z, t\}$ be such that $\operatorname{wt} f = 6$ and $(\mathbb{V}(f), \mathbf{0})$ is an A_2-singularity.

If the coefficient of yt is non-zero and the coefficient of xy is zero in f, then after a suitable coordinate change of the form $t \mapsto by + cz$, where b and c are complex numbers, the coefficients of y^2 and yz will be zero in f. This coordinate change is weight respecting. Since f has quadratic rank 3, after scaling, the quadratic part will be $yt + z^2$. Now $(\mathbb{V}(f), \mathbf{0})$ is an A_2-singularity if and only if the coefficient of x^3 is non-zero, which cannot happen since $\operatorname{wt} f = 6$.

Therefore, since the quadratic rank of f is 3, the coefficient of xy is non-zero. After a suitable coordinate change of the form $x \mapsto x + by + cz + dt$, where a, b, c, d are complex numbers, the coefficients of y^2, yz and yt will be zero. This coordinate change is weight respecting. Now the coefficient of z^2 must be non-zero. After scaling, the quadratic part of f will be $xy + z^2$. We see that $(\mathbb{V}(f), \mathbf{0})$ is an A_2-singularity if and only if the coefficient of t^3 is non-zero.

After scaling, the least weight non-zero quasihomogeneous part of f with respect to weights $\boldsymbol{w} = (3, 3, 3, 2)$ will be $xy + z^2 + t^3$. Corollary 4.7 gives a weight-respecting automorphism Ψ such that $\Psi(f) = xy + z^2 + t^3$.

(3) To begin, we show that $f \in \mathbb{C}\{x, y, z, t\}$ satisfying item (3a) of Theorem 3.10 defines an E_6-singularity at the origin. Let Ψ be the automorphism of $\mathbb{C}\{x, y, z, t\}$ given by composing $x \mapsto x/\sqrt{1 - c^2}$ with $y \mapsto y - c(x + p_{\mathrm{wt}=3})$. Defining $p', g' \in \mathbb{C}\{z, t\}$ by $p = 1/2(\sqrt{1 - c^2})p' + c^2 p_{\mathrm{wt}=3}$ and $g = g' + c^2 + p^2_{\mathrm{wt}=3} - z^3$, we find that $\Psi(f)$ is equal to $x^2 + y^2 + xp' + g'$, where $\operatorname{wt} p'$ is 2, $\operatorname{wt} g'$ is 6 and all monomials of weight greater than 3 have coefficient zero in p'. Let Φ be the coordinate change $x \mapsto x - p'/2$ composed with a suitable scaling of the variable t. Then the least weight non-zero quasihomogeneous part of $\Phi(\Psi(f))$ will be $x^2 + y^2 + z^3 + t^4$ under the weights $\boldsymbol{w} = (6, 6, 4, 3)$. Using the lemma in [AGZV85, §12.6] or Corollary 4.7 we find that $\Phi(\Psi(f))$ is right equivalent to $x^2 + y^2 + z^3 + t^4$, proving that f defines an E_6-singularity at the origin.

Now let $f \in \mathbb{C}\{x, y, z, t\}$ be such that $\operatorname{wt} f = 6$ and $(\mathbb{V}(f), \mathbf{0})$ is an E_6-singularity.

First, we show that the coefficient of xz in f is zero. We recall that the quadratic rank of a convergent power series defining a 3-dimensional E_6-singularity is 2. If the coefficient of xz is non-zero, since f has quadratic rank 2, the coefficient of y^2 must be zero. After a suitable coordinate change of the form $z \mapsto ax + by + cz$, where a, b and c are complex numbers and c is non-zero, the quadratic part of f will be xz. By Lemma 4.5, after a weight-respecting coordinate change f will be of the form $xz + h$, where $h \in \mathbb{C}\{y, t\}$. If the coefficient of $y^2 t$ is non-zero, then by [GLS07, Thm. I.2.51] $(\mathbb{V}(f), \mathbf{0})$ is either a D_k-singularity or a non-isolated singularity, a contradiction. So the coefficient of $y^2 t$ is zero. Since $(\mathbb{V}(f), \mathbf{0})$ is a cA_2-singularity, after scaling the 3-jet of f will be $xz + y^3$. If the coefficient d of yt^3 is non-zero, then Corollary 4.7 with $\boldsymbol{w} := (9, 6, 9, 4)$ and $f_0 := xz + y^3 + dyt^3$ implies that $(\boldsymbol{V}(f), \mathbf{0})$ is an E_7-singularity, a contradiction. Therefore, the coefficient of yt^3 is zero. Now [GLS07, Thm. I.2.55(2)] shows that $(\mathbb{V}(f), \mathbf{0})$ is not a simple singularity, a contradiction.

Second, we show that the coefficient of y^2 is non-zero. If the coefficient of y^2 is zero, then the coefficient of xy must be non-zero. After a suitable coordinate change of the form $y \mapsto ax + y + h$, where $a \in \mathbb{C}$ and $h \in \mathbb{C}\{x, y, z, t\}$ has multiplicity at least 2 or is zero, the only monomial that is divisible by x and has non-zero coefficient in f will be xy. Now the weight of f is at least 5 and the monomials of weight 5 that have a non-zero coefficient in f are in the set $\{zt^3, t^5\}$. After a suitable coordinate change of the form $x \mapsto x + h'$, where $h' \in \mathbb{C}\{y, z, t\}$ has multiplicity at least 2 or is zero, the only monomial in the ideal (x, y) that has non-zero coefficient in f will be xy. The weight of f is still at least 5 and the monomials of weight 5 are still in the set $\{zt^3, t^5\}$. Since $(\mathbb{V}(f), \mathbf{0})$ is a cA_2-singularity, the coefficient of z^3 is non-zero. If the coefficient of zt^3 is non-zero, then Corollary 4.7 with $\boldsymbol{w} := (9, 9, 6, 4)$ and $f_0 := axy + bz^3 + czt^3$, where $a, b, c \in \mathbb{C}$ are non-zero, shows that $(\mathbb{V}(f), \mathbf{0})$ is an E_7-singularity, a contradiction. So the coefficient of zt^3 is zero. If the coefficient of t^5 is non-zero, then Corollary 4.7 with $\boldsymbol{w} := (15, 15, 5, 3)$ and $f_0 := axy + bz^3 + ct^5$, where $a, b, c \in \mathbb{C}$ are non-zero, shows that $(\boldsymbol{V}(f), \mathbf{0})$ is an E_8-singularity, a contradiction. Therefore, $f - axy$ belongs to the ideal (z, t^2) of $\mathbb{C}\{z, t\}$. By [GLS07, Thm. I.2.55(2)] $(\mathbb{V}(f), \mathbf{0})$ is not a simple singularity, a contradiction.

Next, we show that the coefficient of xt^2 is non-zero. If the coefficient of xt^2 is zero, then after a suitable linear weight-respecting coordinate change the quadratic part of f will be $xy + y^2$. Now, after a suitable weight-respecting coordinate change of the form $y \mapsto y + h$, where $h \in \mathbb{C}\{x, y, z, t\}$ has multiplicity at least 2 or is non-zero, followed by an application of Lemma 4.3, the only monomial with non-zero coefficient in f that is divisible by x will be xy. After a suitable coordinate change of the form $x \mapsto x + h'$, where $h' \in \mathbb{C}\{x, y, z, t\}$, the only monomial in the ideal that has non-zero coefficient in f will be xy and the weight of f will still be 6. By [GLS07, Thm. I.2.55(2)] $(\mathbb{V}(f), \mathbf{0})$ is not a simple singularity, a contradiction.

Now, after a suitable linear weight-respecting coordinate change, the quadratic part of f will be $x^2 + y^2$. Using a suitable weight-respecting coordinate change of the form $x \mapsto x + h$ and $y \mapsto y + h'$, where $h, h' \in \mathbb{C}\{x, y, z, t\}$, followed by an application of Lemma 4.3, the power series f will have the form

$$f = x^2 + y^2 + xp + g, \tag{6.1.1}$$

where $p \in \mathbb{C}\{z, t\}$ has only monomials of weight 2 and 3, the coefficient of t^2 in p is 1 and the coefficient of z^3 in $g \in \mathbb{C}\{z, t\}$ is 1.

Finally, we show that there exists a weight-respecting automorphism Ψ of $\mathbb{C}\{x, y, z, t\}$ such that $\Psi(f) = x^2 + y^2 + z^3 + xt^2$, where f is given by equation (6.1.1). The least weight non-zero quasihomogeneous part of $g - p^2/4$ under the weights $(4, 3)$ is $z^3 - t^4/4$. By Corollary 4.7 there exists an automorphism Φ of $\mathbb{C}\{z, t\}$ such that $\Phi(g - p^2/4)$ is equal to $z^3 - t^4/4$, $\Phi(z) - z$ is in the ideal $(z, t)^2$ and $\Phi(t) - t$ is in the ideal (z, t^2). So Φ is weight respecting with respect to weights $(2, 1)$. We find that

$$(\Phi(p) - t^2)(\Phi(p) + t^2) = 4(\Phi(g) - z^3) \tag{6.1.2}$$

is either zero or has weight at least 6. The term $\Phi(p) + t^2$ has weight 2 since the coefficient of t^2 is 2. Therefore, $\Phi(p) - t^2$ is either zero or has weight at least 4. Applying Φ^{-1} to equation (6.1.2), we find that $\Phi^{-1}(t^2) - p$ is also either zero or has weight at least 4. Now it suffices to choose Ψ to be

$$\Psi(x) := x + \frac{t^2 - \Phi(p)}{2}, \quad \Psi(y) := y, \quad \Psi(z) := \Phi(z), \quad \Psi(t) := \Phi(t). \qquad \square$$

Lemma 6.2. *Let n be a positive integer. Let P be a cA_n-point of a $\mathbb{Q}$-Gorenstein variety X with terminal singularities. Then any two divisorial contractions to X with centre P are locally analytically equivalent if they are either*

(1) *both of type* (1) *with the same weights* $(r_1, r_2, a, 1)$*,*
(2) *both of type* (2) *or*
(3) *both of type* (3)*.*

Proof. Case (1) is [Pae21, Prop. 4.7], case (2) is clear and case (3) follows from Theorem 6.1(3). $\square$

We describe conditions for the existence of divisorial contractions to X with centre P of types (1), (2) and (3) of Theorem 3.10.

Lemma 6.3. *Let P be a cA_n-point of a $\mathbb{Q}$-Gorenstein variety X with terminal singularities.*

(a) *If there exists a divisorial contraction of type* (1) *to X with centre P which is an $(r_1, r_2, a, 1)$-blow-up, then for all $a' \in \{1, \dots, a\}$ and for all $r'_1 \in \{1, \dots, a'(n+1) - 1\}$ such that a' is coprime to both r'_1 and $r'_2 := a'(n+1) - r'_1$ there exists a divisorial contraction of type* (1) *which is an $(r'_1, r'_2, a', 1)$-blow-up.*
(b) *There is a positive integer N such that there is no divisorial contraction of type* (1) *to X with centre P which is an $(r_1, r_2, a, 1)$-blow-up where $a > N$.*
(c) *If $n = 1$, then there exists a divisorial contraction of type* (1) *which is an $(r_1, r_2, a, 1)$-blow-up if and only if (X^{an}, P) is an A_k-singularity where $k \geq a$.*
(d) *If $n = 1$, then there exists a divisorial contraction of type* (2) *if and only if (X^{an}, P) is the A_2-singularity.*
(e) *If $n = 2$, then there exists a divisorial contraction of type* (1) *with $a = 2$ if and only if (X^{an}, P) is not a simple singularity.*

(f) *If* $n = 2$, *then there exists a divisorial contraction of type* (3) *if and only if* (X^{an}, P) *is an* E_6*-singularity.*

Proof. (a) If f is of the form $xy + g$ and the weight of $g \in \mathbb{C}\{z, t\}$ is $r_1 + r_2$ with respect to the weights $(r_1, r_2, a, 1)$, then the weight of g is also $r'_1 + r'_2$ with respect to the weights $(r'_1, r'_2, a', 1)$.

(b) By [GLS07, Cor. I.2.18] or [AGZV85, §12.2] the Milnor number of $xy + g_{\mathrm{wt}=r_1+r_2}$ is at least $n((n + 1)a - 1)$. On the other hand, the isolated singularity (X^{an}, P) has finite Milnor number.

(e) By [GLS07, Thm. I.2.55(2)] a cA_2-singularity $(\mathbb{V}(f), \mathbf{0})$, where f is in $\mathbb{C}\{x, y, z, t\}$, is not contact simple if and only if there is an automorphism Ψ of $\mathbb{C}\{x, y, z, t\}$ such that $\Psi(f) = xy + g(z, t)$, where g is in the ideal z, t^2 of $\mathbb{C}\{z, t\}$.

Parts (c), (d) and (f) follow from the definition of simple singularities (Definition 3.6). □

It is known that there are only finitely many divisorial contractions with discrepancy at most 1; see [Kaw05, below Thm. 1.2]. I have added a proof here since I have not found a proof in the literature. The precise statement is as follows:

Proposition 6.4. *Let X be a $\mathbb{Q}$-Gorenstein variety with terminal singularities. Then there are only finitely many divisorial contractions to X with discrepancy at most* 1.

Proof. Let $f : Y \to X$ be a resolution of singularities with exceptional locus of pure codimension 1. Let v be the valuation on the function field $\mathbb{C}(X)$ given by the exceptional divisor of a divisorial contraction to X. Then v is equal to the valuation given by a prime divisor D on a normal variety Z with a proper birational morphism $Z \to Y$. The centre of D on Y is necessarily contained in an exceptional prime divisor of f. We see that if the discrepancy of D is at most 1, then the centre of D on Y necessarily coincides with an exceptional prime divisors of f. So v is equal to the valuation given by one of the finitely many exceptional prime divisor of f. The proposition follows from the fact that any two divisorial contractions whose exceptional divisors define the same valuation are isomorphic over X; see [Kaw01, Lem. 3.4]. □

Theorem 6.5. *Let n be a positive integer. Let P be a point of a $\mathbb{Q}$-Gorenstein variety X with terminal singularities. We count the number of divisorial contractions to X with centre P.*

(a) *If* (X^{an}, P) *is smooth, then there are uncountably many divisorial contractions up to isomorphism over* X *and countably many up to local analytic equivalence.*

(b) *If* (X^{an}, P) *is a* cA_n*-singularity that admits only discrepancy* 1 *divisorial contractions, then there are exactly* n *divisorial contractions up to isomorphism over* X *and exactly* $\lceil n/2 \rceil$ *up to local analytic equivalence, where* $\lceil r \rceil$ *denotes the smallest integer greater than or equal to the real number* r*.*

(c) *If* (X^{an}, P) *is a* cA_n*-singularity that admits a divisorial contraction with discrepancy* ≥ 2*, then there are uncountably many divisorial contractions up to isomorphism over* X *and finitely many up to local analytic equivalence.*

Proof. (a) By Theorem 3.8 there are countably many divisorial contractions up to local analytic equivalence. Since the automorphism Ψ of $\mathbb{C}\{x, y, z\}$ given by $z \mapsto z + ax$, where $a \in \mathbb{C}$ is non-zero, does not lift to an isomorphism of the blown-up spaces when performing a $(1, 1, 2)$-blow-up, there are uncountably many divisorial contractions up to isomorphism over X.

(b) Similarly to the proof of [Hay99, Thm. 6.4], we can show that there are exactly n local analytic germs of divisorial contractions up to isomorphism over X^{an}. Note that the last sentence in the statement of [Hay99, Thm. 6.4] contains a typo: it should say, "Furthermore, there are exactly k divisors with discrepancies $1/m$ over X" (the symbol k is missing). The global algebraic divisorial contractions are constructed using Proposition 5.1 or Corollary 5.6. To see that there are exactly $\lceil n/2 \rceil$ divisorial contractions up to local analytic equivalence, note that $(x, y, z, t) \mapsto (y, x, z, t)$ is weight respecting with respect to the weights $(r_1, r_2, a, 1)$ and $(r_2, r_1, a, 1)$.

(c) It follows from Lemma 6.3(b) and Lemma 6.2 that there are only finitely many divisorial contractions up to local analytic equivalence.

If (X^{an}, P) is not an E_6-singularity, then there exists a divisorial contraction of type (1) of Theorem 6.1 with $a > 1$. By Lemma 6.3(a) there exists a divisorial contraction with $r_1 = 1$. Let $f \in \mathbb{C}\{x, y, z, t\}$ be as in item (1b). For any $c \in \mathbb{C}$ there exists an automorphism Φ_c of $\mathbb{C}\{x, y, z, t\}$ that fixes f given by

$$\Phi_c(x) \coloneqq x, \quad \Phi_c(y) \coloneqq y + h, \quad \Phi_c(z) \coloneqq z + cx, \quad \Phi_c(t) \coloneqq t,$$

where $h \in \mathbb{C}\{x, y, z, t\}$ depends on f. Each automorphism Φ_c defines a divisorial contraction of the analytic germ (X^{an}, P), naming composing the divisorial contraction to X with the precomposition with Φ_c. The composition $\Phi_{c'} \circ \Phi_c^{-1}$ is weight respecting with respect to weights $(1, r_2, a, 1)$ if and only if $c = c'$. We can check on the affine patch $t \neq 0$ of the $(1, r_2, a, 1)$-blown-up space that the biholomorphic map germ corresponding to $\Phi_{c'} \circ \Phi_c^{-1}$ lifts to an isomorphism of the blown-up spaces if and only if $c = c'$. Thus there are uncountably many analytic germs of $(1, r_2, a, 1)$-blow-ups to X with centre P. By Proposition 5.1 or Corollary 5.6, each such analytic germ extends to a divisorial contraction to X with centre P.

If (X^{an}, P) is an E_6-singularity, then for any complex number $v \in \mathbb{C}$, any square root w of $1 - v^2$ and any $u \in \{-1, 1\}$, the automorphism $\Psi_{u,v,w}$ of $\mathbb{C}\{x, y, z, t\}$ given by

$$\Psi_{u,v,w}(x) \coloneqq vx + wy + (v-1)t^2/2, \qquad \Psi_{u,v,w}(z) \coloneqq z,$$
$$\Psi_{u,v,w}(y) \coloneqq uwx - uvy + uwt^2/2, \qquad \Psi_{u,v,w}(t) \coloneqq t$$

fixes $x^2 + y^2 + z^3 + xt^2$. Note that $\Psi_{u',v',w'} \circ \Psi_{u,v,w}^{-1}$ is weight respecting with respect to weights $(4, 3, 2, 1)$ if and only if $v' = v$ and $w' = uu'w$. We can check that the biholomorphic map germ corresponding to $\Psi_{u',v',w'} \circ \Psi_{u,v,w}^{-1}$ lifts to an isomorphism of the blown-up spaces if and only if $v' = v$ and $w' = uu'w$. Similarly to the previous case, this shows that there are uncountably many divisorial contractions of type (3) to X with centre P. □

Acknowledgements

I would like to thank Masayuki Kawakita, Takuzo Okada and Yuki Yamamoto for useful discussions. I would like to thank Ziquan Zhuang for noticing a mistake in an earlier draft. Supported by the Simons Investigator Award HMS, National Science Fund of Bulgaria, National Scientific Program "Excellent Research and People for the Development of European Science" (VIHREN), Project No. KP-06-DV-7.

References

[AK16] H. Ahmadinezhad and A.-S. Kaloghiros, Non-rigid quartic 3-folds, Compos. Math. **152** (2016), 955–983. Zbl 1388.14043 MR 3505644

[AZ16] H. Ahmadinezhad and F. Zucconi, Mori dream spaces and birational rigidity of Fano 3-folds, Adv. Math. **292** (2016), 410–445. Zbl 1348.14036 MR 3464026

[AGZV85] V. I. Arnol'd, S. M. Guseĭn-Zade and A. N. Varchenko, *Singularities of differentiable maps. Vol. I*, Monographs in Mathematics 82, Birkhäuser Boston, Boston, MA, 1985. Zbl 1290.58001 MR 0777682

[Cor00] A. Corti, Singularities of linear systems and 3-fold birational geometry, in *Explicit birational geometry of 3-folds*, London Mathematical Society Lecture Note Series 281, Cambridge University Press, Cambridge, 2000, 259–312. Zbl 0960.14017 MR 1798984

[CLS11] D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics 124, American Mathematical Society, Providence, RI, 2011. Zbl 1223.14001 MR 2810322

[DGPS22] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann, SINGULAR 4-3-0 – A computer algebra system for polynomial computations, http://www.singular.uni-kl.de, 2022, visited on 8 August 2024.

[GLS07] G.-M. Greuel, C. Lossen and E. Shustin, *Introduction to singularities and deformations*, Springer Monographs in Mathematics, Springer, Berlin, 2007. Zbl 1125.32013 MR 2290112

[Hay99] T. Hayakawa, Blowing ups of 3-dimensional terminal singularities, Publ. Res. Inst. Math. Sci. **35** (1999), 515–570. Zbl 0969.14008 MR 1710753

[Hay00] T. Hayakawa, Blowing ups of 3-dimensional terminal singularities. II, Publ. Res. Inst. Math. Sci. **36** (2000), 423–456. Zbl 1017.14006 MR 1781436

[Hay05] T. Hayakawa, Divisorial contractions to 3-dimensional terminal singularities with discrepancy one, J. Math. Soc. Japan **57** (2005), 651–668. Zbl 1083.14013 MR 2139726

[JK20] A. Javanpeykar and R. Kucharczyk, Algebraicity of analytic maps to a hyperbolic variety, Math. Nachr. **293** (2020), 1490–1504. Zbl 1469.32006 MR 4135823

[JS19] A. Javanpeykar and D. E. Speyer, Rational functions on reduced complex varieties that extend to global holomorphic functions, https://mathoverflow.net/q/333799 (version: 2019-06-12), visited on 8 August 2024.

[Kaw01] M. Kawakita, Divisorial contractions in dimension three which contract divisors to smooth points, Invent. Math. **145** (2001), 105–119. Zbl 1091.14007 MR 1839287

[Kaw02] M. Kawakita, Divisorial contractions in dimension three which contract divisors to compound A_1 points, Compositio Math. **133** (2002), 95–116. Zbl 1059.14020 MR 1918291

[Kaw03] M. Kawakita, General elephants of three-fold divisorial contractions, J. Amer. Math. Soc. **16** (2003), 331–362. See https://www.kurims.kyoto-u.ac.jp/~masayuki/Website/Documents/erratum-03.pdf for erratum, visited 8 August 2024. Zbl 1059.14021 MR 1949163

[Kaw05] M. Kawakita, Three-fold divisorial contractions to singularities of higher indices, Duke Math. J. **130** (2005), 57–126. Zbl 1091.14008 MR 2176548

[Kaw12] M. Kawakita, Supplement to classification of threefold divisorial contractions, Nagoya Math. J. **206** (2012), 67–73. Zbl 1256.14014 MR 2926484

[Kaw96] Y. Kawamata, Divisorial contractions to 3-dimensional terminal quotient singularities, in *Higher-dimensional complex varieties (Trento, 1994)*, De Gruyter, Berlin, 1996, 241–246. Zbl 0894.14019 MR 1463182

[KM92] J. Kollár and S. Mori, Classification of three-dimensional flips, J. Amer. Math. Soc. **5** (1992), 533–703. Zbl 0773.14004 MR 1149195

[KOPP24] I. Krylov, T. Okada, E. Paemurru and J. Park, $2n^2$-inequality for cA_1 points and applications to birational rigidity, Compos. Math. **160** (2024), 1551–1595. Zbl 07869026 MR 4751526

[Oka14] T. Okada, Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections, Proc. Lond. Math. Soc. (3) **109** (2014), 1549–1600. Zbl 1349.14056 MR 3293158

[Oka18] T. Okada, Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections, II, J. Reine Angew. Math. **738** (2018), 73–129. Zbl 1397.14025 MR 3794889

[Oka20] T. Okada, Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections, III, Kyoto J. Math. **60** (2020), 825–893. Zbl 1458.14050 MR 4134351

[Pae20] E. Paemurru, Birational models of terminal sextic double solids, PhD thesis, Loughborough University, 2020, https://doi.org/10.26174/thesis.lboro.13169225.v1, visited on 8 August 2024.

[Pae21] E. Paemurru, Birational geometry of sextic double solids with a compound A_n singularity, arXiv:2101.00501v3, [v1] 2021, [v3] 2024.

[Yam18] Y. Yamamoto, Divisorial contractions to cDV points with discrepancy greater than 1, Kyoto J. Math. **58** (2018), 529–567. Zbl 1419.14018 MR 3843389