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Bound States in Soft Quantum Layers

by

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Abstract

We develop a general approach to study three-dimensional Schrödinger operators with confining potentials depending on the distance to a surface. The main idea is to apply parallel coordinates based on the surface but outside its cut-locus in the Euclidean space. If the surface is asymptotically planar in a suitable sense, we give an estimate on the location of the essential spectrum of the Schrödinger operator. Moreover, if the surface coincides up to a compact subset with a surface of revolution with strictly positive total Gauss curvature, it is shown that the Schrödinger operator possesses an infinite number of discrete eigenvalues.

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§1. Introduction

Consider a non-relativistic quantum particle propagating in the vicinity of an unbounded surface Σ in $\mathbb{R}^3$. Spectral properties of the *hard-wall* idealisation, where the Hamiltonian is identified with the Dirichlet Laplacian in the tubular neighbourhood called *layer*

$$\Omega_a := \left\{x \in \mathbb{R}^3 : \operatorname{dist}(x, \Sigma) < a\right\}, \tag{1}$$

were first analysed by Duclos et al. in the pioneering work [10]. While the essential spectrum is stable under local perturbations of the straight layer $\mathbb{R}^2 \times (-a, a)$,

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the most interesting result of the study is the existence of *bound states*, i.e. discrete eigenvalues. This highly non-trivial property for unbounded domains was established in [10] under rather restrictive geometric and topological conditions about Σ. However, the subsequent works of Carron et al. [6] and notably of Lu et al. [32, 31, 33, 34, 28] have demonstrated that the existence of discrete spectra due to bending is indeed a robust phenomenon. See also [16, 5, 29, 30, 21, 17] for quantitative properties of the eigenvalues and eigenfunctions, and [19, 9, 8, 35, 27] for layers over non-smooth surfaces.

To allow for quantum tunnelling, Exner and Kondej [15] introduced a *leaky* realisation of the confinement to Σ by considering the singular Schrödinger operator $-\Delta + \alpha\delta_\Sigma$ in $L^2(\mathbb{R}^3)$, where δ_Σ is the Dirac delta function and $\alpha < 0$. Under suitable geometric assumptions about the surface Σ, the authors demonstrated the existence of discrete spectra in the regime of *large confinement*, i.e. $\alpha \to -\infty$. The robust existence of the discrete eigenvalues for *all* negative α is stated as an open problem in [12, Sect. 7.5]. Spectral analysis of related models can be found in [3, 2, 4, 18, 1].

The purpose of the present paper is to investigate the existence of discrete spectra in yet another realisation of the confinement, namely when the particle Hamiltonian is identified with the Schrödinger operator

$$H := -\Delta + V \quad \text{in } L^2(\mathbb{R}^3), \tag{2}$$

where V is a *regular* potential modelling a force which constrains the particle to the tubular neighbourhood Ω_a. Extending the terminology of Exner [13, 14] for analogous models when the submanifold is a curve to the present case of surfaces, we call these realisations *soft layers*. In this case, there exists only a general asymptotic spectral analysis by adiabatic methods of Wachsmuth and Teufel [37] (see also [22]), from which it follows that the discrete spectrum will exist for *deep and narrow* confining potentials V (in agreement with the leaky model above).

From a more general perspective, the hard-wall and leaky realisations fall into the unifying scheme (2) provided that we *formally* set $V_{\text{hard}} := \infty\chi_{\mathbb{R}^3\setminus\Omega_a}$ and $V_{\text{leaky}} := \alpha\delta_\Sigma$. This can be made rigorous by considering the Dirichlet boundary conditions for the Laplacian in $L^2(\Omega_a)$ or by defining (2) by means of the sesquilinear form, respectively. As a matter of fact, the present approach yields new results for the leaky layers too, namely the robust existence of discrete eigenvalues for *all* negative α, solving in this way the open problem of [12, Sect. 7.5], at least in a special class of rotationally symmetric geometries. The hard-wall layers could also be treated simultaneously, but our technique does not bring anything new in this case (except for the explicit observation missing in [10, 6] that there is an *infinite* number of eigenvalues in hard-wall layers with appropriate rotationally symmetric ends).

Before stating our main results, let us informally summarise the characteristic hypotheses. The surface Σ is assumed to be smooth and orientable, the Gauss curvature of Σ is integrable (see (6)) and Σ is *asymptotically planar* in the sense that both the Gauss and mean curvatures vanish at infinity of Σ (see (15)). More restrictively, we assume that Σ is *asymptotically cut-locus planar* (see (16)). As usual in the theory of quantum waveguides, we always assume that the tubular neighbourhood (1) does not overlap itself with some positive a (see (9) and (10)). Finally, Σ is assumed to contain a cylindrically symmetric end with positive total Gauss curvature (i.e. the integral of the Gauss curvature is positive), whose asymptotic cut-locus is known explicitly (see (30)) and whose parallel curvature admits a power-like decay (see (32)).

The confining potential $V : \mathbb{R}^3 \to \mathbb{R}$ is assumed to be an essentially bounded function or the leaky realisation $V_{\text{leaky}} := \alpha\delta_\Sigma$ with $\alpha \in \mathbb{R}$. In the former case, we assume that the support of V is contained in the closure of Ω_a and that the profile does not vary along Σ. More specifically, if n is a unit normal vector field along Σ and $p \in \Sigma$, we assume

$$\boxed{W(t) := V(p + n(p)t) \text{ is independent of } p \quad \& \quad \operatorname{supp} W \subset [-a, a].} \tag{3}$$

This is certainly the case for leaky layers too, because δ_Σ is zero range and α is assumed to be a constant. The corresponding one-dimensional operator

$$T := -\partial_t^2 + W(t) \quad \text{in } L^2(\mathbb{R})$$

with form domain $H^1(\mathbb{R})$ (the sum should be understood as the form sum in the leaky case) has the essential spectrum covering $[0, \infty)$ in both cases. We assume that W is attractive in the sense that

$$\boxed{T \text{ possesses at least one negative eigenvalue.}} \tag{4}$$

This hypothesis holds in the leaky case if, and only if, α is a negative constant. In general, a sufficient condition to guarantee (4) is that $\int_\mathbb{R} W < 0$ (which particularly involves negative potentials). Moreover, it is easy to design potentials which simultaneously satisfy $\int_\mathbb{R} W \geq 0$ and (4) (e.g., it is enough to consider the strong coupling regime of any W possessing a negative minimum; see [20, Thm. 4]). Let $E_1 < 0$ denote the lowest discrete eigenvalue of T.

Our main result reads as follows.

Theorem 1. *Let Σ be an orientable smooth surface which is asymptotically cut-locus planar (16) and admits an integrable Gauss curvature (6). Let the tubular* *neighbourhood* (1) *not overlap itself with some positive a, i.e.* (9) *and* (10) *hold. Let V be an essentially bounded function (or the distribution $\alpha\delta_\Sigma$ with $\alpha < 0$) satisfying* (3) *and* (4). *Then*

$$\inf \sigma_{\mathrm{ess}}(H) \geq E_1.$$

Moreover, if Σ coincides up to a compact subset with a cylindrically symmetric surface with positive total Gauss curvature and satisfying the extra hypotheses (30) *and* (32), *then H possesses an infinite number (counting multiplicities) of discrete eigenvalues below E_1, and in this case* $\inf \sigma_{\mathrm{ess}}(H) = E_1$.

A special circumstance is of course when Σ coincides with the cylindrically symmetric end. Then a canonical example of the surface satisfying all the hypotheses is the paraboloid of revolution (and other surfaces obtained by revolving polynomially growing curves). Another typical example is the family of surfaces Σ_θ obtained by revolving the planar curve of [26] (see Figure 1),

$$\Gamma_\theta(s) := \begin{cases} \left(R\sin\frac{s}{R}, R\left(1-\cos\frac{s}{R}\right)\right) & \text{if } s \in \left[0, \frac{\theta}{2}R\right), \\ \left(\left(s-\frac{\theta}{2}R\right)\cos\frac{\theta}{2} + R\sin\frac{\theta}{2}, \right. & \\ \qquad \left.\left(s-\frac{\theta}{2}R\right)\sin\frac{\theta}{2} + R\left(1-\cos\frac{\theta}{2}\right)\right) & \text{if } s \in \left[\frac{\theta}{2}R, \infty\right), \end{cases} \tag{5}$$

along the second axis in $\mathbb{R}^3$, where $R > 0$ and $\theta \in [0, \pi]$. Hence Σ_θ is the union of a spherical cap and a conical end; it is a plane if $\theta = 0$, while the end becomes cylindrical if $\theta = \pi$. Of course, Σ_θ is not smooth (unless $\theta = 0$), but it is piecewise smooth (in fact, piecewise analytic).

The surface Σ_θ can be regarded as a regularised version of the conical geometry considered in [19, 9, 35, 11]. Indeed, Theorem 1 applies to Σ_θ with $\theta \in (0, \pi)$, confirming in this way the results of the precedent works. In fact, not necessarily rotationally symmetric cones are considered in [35, 11], and moreover the accumulation rate of the eigenvalues is derived there. On the other hand, the strength of the present work is that we go substantially beyond the conical geometries, so the present paper can be considered as a generalisation of [35, 11].

The present paper can be also considered as a generalisation of our precedent work [26] to three dimensions. Indeed, our modus operandi is again rooted in developing the method of parallel coordinates based on Σ involving the *cut-locus* of Σ. Unfortunately, the common limitation of the method is that we can consider special surfaces only: those for which the cut-locus is known explicitly. However, the unprecedented novelty with respect to [26] is that an *asymptotic* knowledge of the cut-locus is enough in the present work. This enables us to cover more

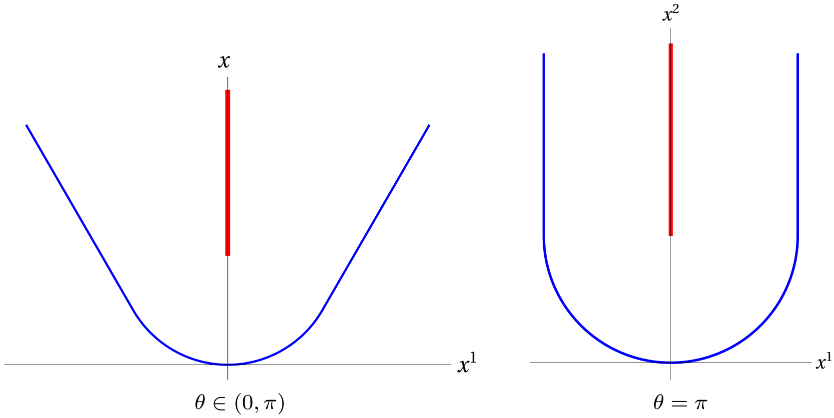


Figure 1. The piecewise smooth curve (5) (symmetrically extended to $s \in \mathbb{R}$) and its cut-locus (red).

general geometries. What is more, the present proof of the existence of a discrete spectrum exhibits another important novelty with respect to the previous work: The argument requires a careful choice of the trial function *localised at infinity*. This is indeed a huge difference with respect to [26], where the trial function was the standard one, i.e. essentially a constant localised everywhere. This phenomenon is closely related to the existence of the *intrinsic* Gauss curvature K for surfaces, while there are just extrinsic curvatures for curves. It was noticed in [10] that a more refined choice of trial functions is necessary for layers over surfaces with *positive* total Gauss curvature. Furthermore, the unconventional choice of the trial function enables one to conclude that there is actually an *infinite* number of discrete eigenvalues.

The paper is organised as follows. In Section 2 we develop a general approach to soft and leaky quantum layers; in particular we introduce a useful parametrisation of $\mathbb{R}^3$ involving the cut-locus of Σ. In Section 3 we consider the special situation of cylindrically symmetric layers. Theorem 1 is proved in Section 4.

§2. Parallel coordinates

Let us first develop a general approach to study soft quantum layers. The first part of our approach (before speaking about the *cut-locus*) is rather standard and we refer to [6, 29] for similar geometric preliminaries.

Let Σ be a connected orientable smooth surface in $\mathbb{R}^3$. We are particularly interested in non-compact complete surfaces, but Σ can alternatively be compact in these geometric preliminaries. The induced metric of Σ will be denoted by g. Introducing the standard notation $|g| := \det(g)$, the surface element of Σ reads $\mathrm{d}\Sigma = |g|^{1/2}\, \mathrm{d}s^1 \wedge \mathrm{d}s^2$, where (s^1, s^2) is a local coordinate system of Σ. The orientation of Σ is specified by a globally defined unit normal vector field $n \in \Sigma \to \mathbb{S}^2$. For any $p \in \Sigma$, we introduce the *Weingarten map*

$$L \colon T_p\Sigma \to T_p\Sigma \colon \{\xi \mapsto -\mathrm{d}n(\xi)\}.$$

The eigenvalues k_1, k_2 of L are called the *principal curvatures* of Σ. They are defined only locally on Σ, but the *Gauss curvature* $K := \det(L) = k_1 k_2$ and the *mean curvature* $M := \frac{1}{2}\operatorname{tr}(L) = \frac{1}{2}(k_1 + k_2)$ are globally defined smooth functions on Σ. The relationship of L with the *second fundamental form* h of Σ is through the formula $L^\mu{}_\nu = g^{\mu\rho} h_{\rho\nu}$, where $h = h_{\mu\nu}\, \mathrm{d}s^\mu\, \mathrm{d}s^\nu$, $g = g_{\mu\nu}\, \mathrm{d}s^\mu\, \mathrm{d}s^\nu$ and, as usual, $g^{\mu\nu}$ denote the entries of the inverse matrix $(g_{\mu\nu})^{-1}$. Here we adopt the Einstein summation convention, the range of Greek and Latin indices being $1, 2$ and $1, 2, 3$, respectively.

The characteristic hypothesis of this work is that the Gauss curvature is integrable:

$$\boxed{K \in L^1(\Sigma).} \tag{6}$$

Then the *total Gauss curvature* $\mathcal{K} := \int_\Sigma K\, \mathrm{d}\Sigma$ is well defined. The quantity $\mathcal{K}$ plays an important role in the global geometry of Σ. In fact, by the celebrated Gauss–Bonnet theorem (see, e.g., [25, Sect. 6.3]), $\mathcal{K}$ is a topological invariant for closed surfaces. By [23], the hypothesis (6) implies that Σ is conformally equivalent to a closed surface from which a finite number of points have been removed.

Let us consider the *normal exponential map*

$$\Phi \colon \Sigma \times \mathbb{R} \to \mathbb{R}^3 \colon \{(p, t) \mapsto p + n(p)t\}. \tag{7}$$

It gives rise to *parallel* (or *Fermi*) "coordinates" (p, t) based on Σ. The metric G induced by (7) has a block form

$$G = g \circ (I - tL)^2 + \mathrm{d}t^2, \tag{8}$$

where I denotes the identity map on $T_p\Sigma$. Consequently,

$$|G| := \det(G) = |g|[\det(I - tL)]^2 = |g|[(1 - tk_1)(1 - tk_2)]^2 = |g|(1 - 2Mt + Kt^2)^2.$$

The map Φ is standardly used in the theory of quantum layers as a convenient parametrisation of the tubular neighbourhood Ω_a introduced in (1). It follows by the inverse function theorem that the restricted map $\Phi \colon \Sigma \times (-a, a) \to \Omega_a$ is a local diffeomorphism provided that

$$\boxed{0 < a < (\max\{\|k_1\|_\infty, \|k_2\|_\infty\})^{-1}} \tag{9}$$

(with the convention that the right-hand side equals ∞ if the principal curvatures are identically equal to zero). Of course, to be able to satisfy this inequality with a *positive* a, it is necessary to assume that the Gauss and mean curvatures are globally bounded functions (this is automatically satisfied for compact Σ). The crucial requirement that the tubular neighbourhood (1) "does not overlap itself" precisely means that $\Phi\colon \Sigma \times (-a,a) \to \Omega_a$ is a (global) diffeomorphism, which is ensured by assuming in addition to (9) the ad hoc requirement that

$$\boxed{\Phi \restriction \Sigma \times (-a,a) \text{ is injective.}} \tag{10}$$

Then $(\Sigma \times (-a,a), G)$ is an embedded submanifold of $\mathbb{R}^3$ and $\Omega_a = \Phi(\Sigma \times (-a,a))$ indeed has the geometrical meaning of the set of points in $\mathbb{R}^3$ squeezed between two parallel hypersurfaces at the distance a from Σ. Indeed, within Ω_a, one observes that $p \mapsto \Phi(p,t)$ is an embedded surface parallel to Σ at the distance $|t|$ for any fixed $t \in (-a,a)$, while $t \mapsto \Phi(p,t)$ is a straight line (i.e. a geodesic in $\mathbb{R}^3$) orthogonal to Σ at any fixed point $p \in \Sigma$.

Now we go beyond the standard approach to quantum layers by extending the parallel coordinates Φ from Ω_a to the whole space $\mathbb{R}^3$. We are inspired by [36, Appx. 1]. Define the *cut-radius* maps $c_\pm\colon \Sigma \to (0,\infty]$ by the property that the segment $t \mapsto \Phi(p,t)$ for positive (respectively, negative) t minimises the distance from Σ if, and only if, $t \in [0, c_+(p))$ (respectively, $t \in (-c_-(p), 0]$). The cut-radius maps are known to be continuous. The *cut-locus*

$$\operatorname{Cut}(\Sigma) := \big\{\Phi(p, c_+(p)) : p \in \Sigma\big\} \cup \big\{\Phi(p, -c_-(p)) : p \in \Sigma\big\} \tag{11}$$

is a closed subset of $\mathbb{R}^3$ of measure zero (see, e.g., [7, Chap. III]). The map Φ, when restricted to the set

$$U := \big\{(p,t) \in \Sigma \times \mathbb{R} : -c_-(p) < t < c_+(p)\big\} \tag{12}$$

is a diffeomorphism onto $\Phi(U) = \mathbb{R}^3 \setminus \operatorname{Cut}(\Gamma)$. Obviously, one has the inclusion

$$\operatorname{Cut}(\Sigma) \supset \operatorname{Cut}_0(\Sigma) := \big\{\Phi(p,t) : 1 - 2M(p)t + K(p)t^2 = 0\big\}, \tag{13}$$

where $\operatorname{Cut}_0(\Sigma)$ is called the *conjugate locus* of Σ (points where the Jacobian of Φ vanishes).

Outside the cut-locus, we have the usual coordinates of quantum layers. If (s^1, s^2) is a local coordinate system of Σ, then (s^1, s^2, t) is a natural local coordinate system of $\Phi(U)$. With respect to the corresponding coordinate frame, the metric G admits the matrix representation

$$(G_{ij}) = \begin{pmatrix} (G_{\mu\nu}) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with } G_{\mu\nu} = g_{\mu\rho}(\delta^\rho_\sigma - tL^\rho{}_\sigma)(\delta^\sigma_\nu - tL^\sigma{}_\nu). \tag{14}$$

In particular, the volume element of $\Phi(U)$ is given by

$$dv := (1 - 2Mt + Kt^2)\, d\Sigma \wedge dt.$$

In agreement with [10], we say that a non-compact surface Σ is *asymptotically planar* if the Gauss and mean curvatures vanish at infinity, which we schematically write as

$$\boxed{K, M \xrightarrow{\infty} 0.} \tag{15}$$

Recall that a function f, defined on a non-compact manifold Σ, is said to *vanish at infinity* if, given any positive number ε, there exists a compact subset $K \subset \Sigma$ such that $|f| < \varepsilon$ on $\Sigma \setminus K$. Similarly, we say that f *diverges at infinity* and write $f \xrightarrow{\infty} \infty$ if, given any positive number ε, there exists a compact subset $K \subset \Sigma$ such that $|f| > \varepsilon^{-1}$ on $\Sigma \setminus K$. In parallel with (15), we say that Σ is *asymptotically cut-locus planar* if

$$\boxed{c_\pm \xrightarrow{\infty} \infty.} \tag{16}$$

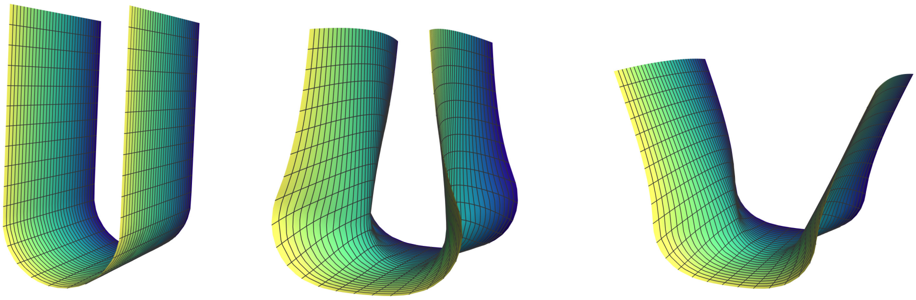
Note that (16) implies (15) due to the inclusion (13). On the other hand, we expect that the reverse implication does not hold in general.

Conjecture 1. *There exists a connected surface such that* (15) *holds but* (16) *is violated.*

For possibly disconnected surfaces this is obvious (think about two parallel planes), but constructing an explicit connected example seems difficult (see Figure 2 for a partial attempt). A sufficient condition to ensure hypothesis (16) as a consequence of (15) is given by surfaces of revolution (cf. Lemma 4 below).

Now we turn from geometric to analytic preliminaries. Recall our Hamiltonian H given in (2). If $V : \mathbb{R}^3 \to \mathbb{R}$ is an essentially bounded function (as is indeed the case for the soft realisation of the confinement), then H can be introduced as an ordinary operator sum of the self-adjoint Laplacian with domain $H^2(\mathbb{R}^3)$ and the maximal operator of multiplication generated by V. The associated closed form reads

$$h[u] := \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V|u|^2, \quad \operatorname{dom} h := H^1(\mathbb{R}^3). \tag{17}$$



(a) asymptotically neither cut-locus planar nor planar

(b) asymptotically planar (along x^2) but not cut-locus planar

(c) asymptotically cut-locus planar (along x^2)

Figure 2. Towards the proof of Conjecture 1. Surface (a) is the curve of Figure 1 with $\theta = \pi$ and $R = 1$ translated along axis x^3. Then it is not asymptotically cut-locus planar because the distance between the flat parts equals the constant 2 as $x^2 \to \infty$. Surface (b) is obtained from (a) by taking the radius R in (5) dependent on both s and x^3, namely $R(s, x^3) := 1+(x^3)^2/(1+s^2)$. Then (b) is not asymptotically cut-locus planar either (the distance between the modified flat parts remains 2 as $x^2 \to \infty$ and $x^3 = 0$), while $K, M \to 0$ as $x^2 \to \infty$ and x^3 is *fixed* (the challenge is to have (15) globally). Surface (c) is the curve of Figure 1 with $\theta = \frac{5}{7}\pi$ and R as for (b); then $c_\pm \to \infty$ as $x^2 \to \infty$ and x^3 is fixed.

If V is the distribution of the leaky type $V_{\text{leaky}} := \alpha\delta_\Sigma$, it is simplest to start with the form (17), where the second term should be interpreted as $\alpha \int_\Sigma |u|^2$. Again, it is a well-defined and closed form under our standing hypothesis (9) and (10). In either case, H can be *defined* as the self-adjoint operator associated with h (with the properly interpreted second integral) via the representation theorem [24, Thm. VI.2.1].

Finally, we express H in the parallel coordinates. This is achieved by means of the unitary map $\mathcal{U}\colon L^2(\mathbb{R}^3) \to L^2(U, \mathrm{d}v)$ defined by $\mathcal{U}u := u \circ \Phi$. Then $\widehat{H} := \mathcal{U}H\mathcal{U}^{-1}$ is the operator associated with the quadratic form $\hat{h}[\psi] := h[\mathcal{U}^{-1}\psi]$ with $\operatorname{dom}\hat{h} := \mathcal{U}\operatorname{dom} h$. Explicitly, using the block-diagonal form of the metric (8), one has

$$\hat{h}[\psi] = \int_U \overline{\partial_\mu\psi} G^{\mu\nu}\partial_\nu\psi \,\mathrm{d}v + \int_U |\partial_t\psi|^2 \,\mathrm{d}v + \int_U W(t)|\psi|^2 \,\mathrm{d}v, \tag{18}$$

where $(G^{\mu\nu}) := (G_{\mu\nu})^{-1}$. Hereafter, the last integral should be interpreted as $\alpha \int_\Sigma |\psi(\cdot, 0)|^2 \,\mathrm{d}\Sigma$ in the case of leaky layers. In the sense of distributions, the operator $\widehat{H}$ associated with $\hat{h}$ acts as

$$\widehat{H} = -|G|^{-1/2}\partial_\mu |G|^{1/2} G^{\mu\nu}\partial_\nu - |G|^{-1/2}\partial_t |G|^{1/2}\partial_t + W.$$

Here, W is absent in the case of leaky layers, the influence of the Dirac interaction being realised by appropriate transmission conditions imposed on Σ in the operator domain. We shall not need to specify this condition, for it is enough to work on the level of forms for our purposes.

§3. Rotationally symmetric layers

The parallel coordinates of the previous section enable one to transfer the geometrically complicated action of the operator H into the coefficients of the transformed operator $\widehat{H}$. The problem is that even the form domain $\operatorname{dom} \hat{h}$ is not easy to identify because of the boundary conditions on ∂U. An objective of this paper is to point out that there exists a special class of surfaces for which this is feasible because of more precise information about the cut-locus. These are surfaces of revolution, so here we consider layers which are invariant with respect to rotations around a fixed axis in $\mathbb{R}^3$.

Let $r, z \colon [0,\infty) \to \mathbb{R}$ be smooth functions such that $r(s) > 0$ for all $s > 0$, $r(0) = 0 = z(0)$, $r'(0) = 1$ and

$$r'(s)^2 + z'(s)^2 = 1 \tag{19}$$

for all $s \geq 0$. The last identity implies that the planar curve $\Gamma(s) := (r(s), z(s))$ is unit-speed (see Figure 3). We consider the smooth surface of revolution Σ obtained by revolving Γ around the second axis:

$$\Sigma := \big\{(r(s)\cos\vartheta, r(s)\sin\vartheta, z(s)) : (s,\vartheta) \in [0,\infty) \times [0, 2\pi)\big\}. \tag{20}$$

We use the following natural parametrisation of Σ:

$$p \colon (0,\infty) \times (0,2\pi) \to \mathbb{R}^3 \colon \big\{(s,\vartheta) \mapsto (r(s)\cos\vartheta, r(s)\sin\vartheta, z(s))\big\}, \tag{21}$$

which gives rise to geodesic polar "coordinates" (s,ϑ) on Σ. With respect to these coordinates, the induced metric $g_{\mu\nu} := \partial_\mu p \cdot \partial_\nu p$, where the dot denotes the scalar product in $\mathbb{R}^3$, reads

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{22}$$

In particular, the surface element of Σ reads $\mathrm{d}\Sigma := r(s)\,\mathrm{d}s \wedge \mathrm{d}\vartheta$. Because of the availability of a unique chart p^{-1} (which covers the whole Σ except for the curve Γ,

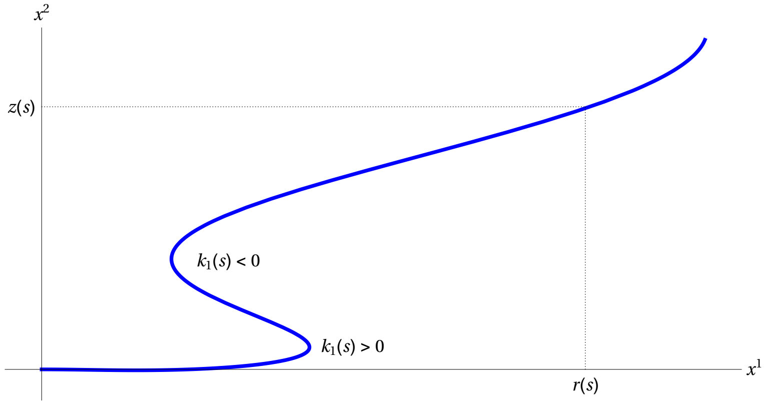


Figure 3. The geometry of the generating curve Γ.

which is a set of measure zero relative to Σ), we shall consider the geometric objects of Σ as functions of (s, ϑ) rather than points of Σ.

With respect to the surface normal

$$n(s, \vartheta) := (-z'(s)\cos\vartheta, -z'(s)\sin\vartheta, r'(s)), \tag{23}$$

we have the following formulae for the second fundamental form $h_{\mu\nu} := -\partial_\mu n \cdot \partial_\nu p$ and the Weingarten tensor $L^\mu{}_\nu = g^{\mu\rho} h_{\rho\nu}$:

$$\begin{aligned} (h_{\mu\nu}) &= \begin{pmatrix} r'z'' - r''z' & 0 \\ 0 & rz' \end{pmatrix}, \\ (L^\mu{}_\nu) &= \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \quad k_1 := r'z'' - r''z', k_2 := \frac{z'}{r}. \end{aligned} \tag{24}$$

Consequently, the layer metric (14) is actually diagonal. The principal curvatures k_1 and k_2 will be called the *meridian* and *parallel* curvatures, respectively. Because of the rotational symmetry, the curvatures are independent of ϑ, so we suppress this variable from the arguments to simplify the notation.

Differentiating (19), we obtain the identity $r'r'' + z'z'' = 0$. Using it in the definition of $K = k_1 k_2$ with help of the formulae (24) for the principal curvatures, we arrive at the Jacobi equation

$$r'' + Kr = 0, \tag{25}$$

subject to initial conditions $r(0) = 0$ and $r'(0) = 1$. The differential equation (25) has important consequences. First, we have the following upper bound on the Jacobian r.

Lemma 1. *Assume* (6). *Then there exists a positive constant* C_Σ *such that*

$$\forall\, s \geq 0, \quad r(s) \leq C_\Sigma s.$$

Proof. Integrating (25), we obtain the uniform bound

$$r'(s) = 1 - \int_0^s (Kr)(s)\,\mathrm{d}s \leq 1 + \frac{\|K\|_{L^1(\Sigma)}}{2\pi} =: C_\Sigma$$

for all $s \geq 0$. Integrating this inequality, we obtain the desired claim. □

Second, (25) implies the Gauss–Bonnet theorem

$$\mathcal{K} = 2\pi[1 - r'(\infty)], \quad \text{where } r'(\infty) := \lim_{s\to\infty} r'(s). \tag{26}$$

Note that the limit is well defined as a consequence of this equality and hypothesis (6). Necessarily,

$$0 \leq \mathcal{K} \leq 2\pi. \tag{27}$$

Here, the non-negativity follows from (19), while the upper bound is valid due to the positivity of r. If the total Gauss curvature $\mathcal{K}$ is positive, we get important information on the parallel curvature.

Lemma 2. *Assume* (6) *with* $\mathcal{K} > 0$. *There exist positive numbers* δ *and* s_0 *such that*

$$\forall s \geq s_0, \quad \frac{\delta}{r(s)} \leq |k_2(s)| \leq \frac{1}{r(s)}.$$

Proof. The lemma is due to [10, Lem. 6.1]; we repeat the proof to make the presentation self-contained. By (27), (26) and the assumption $\mathcal{K} > 0$, one has $0 \leq r'(\infty) < 1$. It follows that there exist $\delta' \in (0, \frac{1}{2})$ and $s_0 > 0$ such that $-\delta' \leq r'(s) \leq 1 - \delta'$ for all $s \geq s_0$. Then the desired claim follows from the definition of k_2. □

It follows from Lemmata 1 and 2 that k_2 is not integrable, i.e., $k_2 \notin L^1((0,\infty))$. On the other hand, the meridian curvature k_1 is integrable, which follows from the smoothness of r, z and the following estimates:

$$\infty > \frac{\|K\|_{L^1(\Sigma)}}{2\pi} \geq \int_{s_0}^\infty |k_1(s)k_2(s)| r(s)\,\mathrm{d}s \geq \delta \int_{s_0}^\infty |k_1(s)|\,\mathrm{d}s. \tag{28}$$

Consequently, $M \notin L^1((0,\infty))$. This is the essence of our subsequent analysis: even if M may decay at infinity, it is not negligible in the integral sense there.

Since Σ is rotationally symmetric, the cut-radius maps $c_\pm$ do not depend on the angular variable, so we may suppress it from the argument. The map Φ defined in (7), when restricted to the open set

$$U := \{(s,\vartheta,t) \in (0,\infty)\times(0,2\pi)\times\mathbb{R} : -c_-(s) < t < c_+(s)\}, \tag{29}$$

is a diffeomorphism onto $\Phi(U) = \mathbb{R}^3 \setminus \{(x^1,0,x^3) : x^1 \geq 0,\ x^3 \in \mathbb{R}\}$. Since the latter coincides with $\mathbb{R}^3$ up to a set of measure zero, the map Φ can be used as a parametrisation of $\mathbb{R}^3$.

The basic hypothesis (9) is obviously satisfied (with a positive a) due to (15) and smoothness of Σ. The global requirement (10) must still be satisfied ad hoc due to possible self-crossings of Γ. The following observation shows, however, that it is actually enough to exclude the self-crossings only locally.

Lemma 3. *Assume (6) and (15). Then $r(s) \to \infty$ as $s \to \infty$.*

Proof. If (6) holds with $\mathcal{K} > 0$, then the result follows from Lemma 2 (note that K, M vanish at infinity if, and only if, k_1, k_2 vanish at infinity). If $\mathcal{K} = 0$, then $r'(\infty) = 1$. But then $r(s) = r(s_0) + \int_{s_0}^{s} r' \geq \frac{1}{2}(s - s_0)$ for all sufficiently large $s_0 < s$. Fixing s_0 and sending s to ∞, we get the desired claim. $\square$

Our main hypothesis for rotationally symmetric layers is that the cut-locus of Σ asymptotically coincides with the upper part of the axis of symmetry:

$$\boxed{\exists R_0 > 0, \quad \mathrm{Cut}(\Sigma) \setminus B_{R_0}(0) = \{(0,0,x^3) : x^3 \geq R_0\}.} \tag{30}$$

Lemma 4. *Assume (6) with $\mathcal{K} > 0$, (15) and (30). Then there exists $s_0 > 0$ such that, for all $s \geq s_0$, $k_1(s) \geq 0$, $k_2(s) > 0$, $c_-(s) = \infty$, $c_+(s) = 1/k_2(s)$ and $k_1(s) \leq k_2(s)$.*

Proof. The crucial observation is that $r(s) \to \infty$ as $s \to \infty$ as a consequence of Lemma 2 and (15) (or see Lemma 3 directly). At the same time, $z(s) \to \pm\infty$ as $s \to \infty$ because $z'(\infty)^2 = 1 - r'(\infty)^2 > 0$, where the inequality is implied by $\mathcal{K} > 0$ and (26). Then $c_-(s) = \infty$ and $c_+(s) = 1/k_2(s)$ for all sufficiently large s. Indeed, by (30) and the symmetry, for all sufficiently large s and any $\vartheta \in [0, 2\pi)$, the curve $\gamma(t) := p(s,\vartheta) + n(s,\vartheta)t$ does not intersect the ball $B_{R_0}(0)$, so the only intersection must be with the semi-axis $\{(0,0,x^3) : x^3 \geq R_0\}$. Moreover, $\gamma^1(t) = 0$ implies $t = r/z' = 1/k_2$. This argument also excludes the possibility $z(s) \to -\infty$ as $s \to \infty$, because otherwise the curve γ would intersect the negative semi-axis $\{(0,0,x^3) : x^3 \leq -R_0\}$ for all sufficiently large s. Consequently, $z'(\infty) > 0$, so the parallel curvature $k_2(s)$ is positive for all sufficiently large s. We claim that the meridian curvature $k_1(s)$ is non-negative for all sufficiently large s (see Figure 3).

Indeed, if it is not the case, then there exist large positive numbers $s_1 < s_2$ such that the graph of the curve Γ is strictly concave on (s_1, s_2), implying the existence of a cut-locus of Γ to the right of the curve (when traced according to the arc-length parameter), therefore violating (30). Finally, if $k_1(s) > k_2(s)$, then $1/k_1(s) < 1/k_2(s) = c_+(s)$, implying a contradiction that there exists a conjugate point outside the cut-locus. □

There are many surfaces of revolution satisfying (30). For instance, if $s \mapsto \Gamma(s)$ is a convex graph, then $c_-(s) = \infty$ for all $s \geq 0$, so there is no cut-locus "outside" Σ (i.e. for negative t). To show that the cut-locus satisfies (30) "inside" Σ (i.e. for positive t), one can employ the geometric interpretation of the principal curvatures (minimal and maximal values of the normal curvatures of all the curves passing through a given point). Then it is easy to verify that (30) can be achieved as a consequence (15) (still assuming (6) with $\mathcal{K} > 0$). In particular, the paraboloid of revolution satisfies (30) (as well as (6) with $\mathcal{K} = 2\pi$ and (15)).

To be even more explicit, let us consider the family of surfaces Σ_θ obtained by revolving the planar curve (5). The cut-locus of Σ_θ is the semi-axis $\{(0,0,x^3) : x^3 \geq R\}$ if $\theta \in (0,\pi]$, while it is empty if $\theta = 0$. Of course, Σ_θ is not smooth (unless $\theta = 0$), but it is piecewise smooth (in fact, piecewise analytic). The total Gauss curvature of Σ_θ reads

$$\mathcal{K}_\theta = 2\pi(1 - \cos\tfrac{\theta}{2}), \tag{31}$$

so the hypotheses of Lemma 2 are met whenever $\theta \in (0,\pi)$.

Finally, we make a hypothesis about a sufficient decay of the parallel curvature at infinity:

$$\boxed{\exists\, \epsilon > 0, \quad k_2(s) = O(s^{-\epsilon}) \text{ as } s \to \infty.} \tag{32}$$

This condition (with $\epsilon = 1$) is easily verified for Σ_θ whenever $\theta \in [0,\pi)$. It also holds for the paraboloid of revolution (with $\epsilon = 1/2$) and other surfaces obtained by revolving polynomially growing curves.

§4. The proofs

We assume that the potential V is either the distribution $V_{\mathrm{leaky}} := \alpha\delta_\Sigma$ with $\alpha < 0$ or it is an essentially bounded function satisfying (3) and (4). Let $E_1 < 0$ denote the lowest discrete eigenvalue of T. The variational characterisation yields

$$E_1 = \inf_{\substack{\xi \in H^1(\mathbb{R}) \\ \xi \neq 0}} \frac{\displaystyle\int_{\mathbb{R}} |\xi'(t)|^2 \, \mathrm{d}t + \int_{\mathbb{R}} W(t)|\xi(t)|^2 \, \mathrm{d}t}{\displaystyle\int_{\mathbb{R}} |\xi(t)|^2 \, \mathrm{d}t}. \tag{33}$$

In the leaky case, the integral $\int_{\mathbb{R}} W(t)|\xi(t)|^2\,\mathrm{d}t$ should be interpreted as $\alpha|\xi(0)|^2$, in which case, explicitly, $E_1 = -\frac{\alpha^2}{4}$. It is well known that E_1 is simple and that the corresponding eigenfunction ξ_1 can be chosen to be positive. We additionally choose the eigenfunction to be normalised to 1 in $L^2(\mathbb{R})$, i.e., $\|\xi_1\|_{L^2(\mathbb{R})} = 1$. Explicitly, $\xi_1(t) = \sqrt{|\alpha|/4}e^{\frac{\alpha}{2}|t|}$ in the leaky case. In any case, one knows that $\xi_1 \in H^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and that the following identities hold true:

$$\xi_1(t) = N_\pm e^{\mp\sqrt{-E_1}t} \quad \text{for every } \pm t > a, \tag{34}$$

where $N_\pm$ are positive constants.

Remark 1. In principle, the assumption $\operatorname{supp} W \subset [-a, a]$ of (3) could be relaxed to a decay of W at infinity. Then the asymptotics (34) could be replaced by Agmon-type estimates.

First of all, we locate the essential spectrum of H (assuming $\operatorname{supp} W \subset [-a, a]$ or the leaky setting). As an auxiliary quantity, in parallel with (33), we consider

$$E_1^b := \inf_{\substack{\xi \in H^1((-b,b)) \\ \xi \neq 0}} \frac{\displaystyle\int_{-b}^{b} |\xi'(t)|^2\,\mathrm{d}t + \int_{-b}^{b} W(t)|\xi(t)|^2\,\mathrm{d}t}{\displaystyle\int_{-b}^{b} |\xi(t)|^2\,\mathrm{d}t}, \tag{35}$$

where $b > a$. Of course, E_1^b is the lowest eigenvalue of the operator T restricted to $(-b, b)$, subject to Neumann boundary conditions.

Lemma 5. *One has*

$$\lim_{b\to\infty} E_1^b = E_1.$$

Proof. In the leaky case, E_1^b solves the implicit equation $2\sqrt{-E} = -\alpha\coth(\sqrt{-E}b)$, from which the convergence can easily be deduced. In the regular case, let us assume $b > a$. By using ξ_1 (or, more precisely, its restriction to $(-b, b)$) as a trial function in (35), it is easy to see that

$$E_1^b \leq E_1. \tag{36}$$

To get the opposite estimate in the limit, let ξ_1^b be the positive minimiser of (35) normalised to 1 in $L^2((-b, b))$. We extend it to the whole line by setting

$$\tilde{\xi}_1^b(t) := \begin{cases} \xi_1^b(t) & \text{if } |t| < b, \\ \xi_1^b(\pm b)\exp(\mp\sqrt{-E_1}(t \mp b)) & \text{if } \pm t \geq b. \end{cases}$$

Since $\tilde{\xi}_1^b \in H^1(\mathbb{R})$, we use it as a trial function in (33) and obtain

$$E_1 \leq \frac{E_1^b + \frac{1}{2}\sqrt{-E_1}[\xi_1^b(-b)^2 + \xi_1^b(b)^2]}{1 + \frac{1}{2}\frac{1}{\sqrt{-E_1}}[\xi_1^b(-b)^2 + \xi_1^b(b)^2]}.$$

It remains to notice that $\xi_1^b(\pm b) \to 0$ as $b \to \infty$. To see it, we employ the explicit solution

$$\xi_1^b(t) = \xi_1^b(\pm b)\cosh\big(\sqrt{-E_1^b}(t \mp b)\big) \quad \text{for } a \leq \pm t \leq b.$$

Take $t = \pm a$ and use that the value $\xi_1^b(\pm a)$ can be estimated by the $H^1\big((-a,a)\big)$ norm of ξ_1^b as follows:

$$\xi_1^b(\pm a)^2 \leq \int_{-a}^{a} |\xi_1^{b\prime}(t)|^2 \, \mathrm{d}t + C_a \int_{-a}^{a} |\xi_1^b(t)|^2 \, \mathrm{d}t,$$

where explicitly $C_a := 1 + (2a)^{-1}$. In turn, the right-hand side can be estimated by using the identity

$$\int_{-b}^{b} |\xi_1^{b\prime}(t)|^2 \, \mathrm{d}t + \int_{-b}^{b} W(t)|\xi_1^b(t)|^2 \, \mathrm{d}t = E_1^b \int_{-b}^{b} |\xi_1^b(t)|^2 \, \mathrm{d}t.$$

Consequently, $\xi_1^b(\pm a)^2 \leq \|W\|_\infty + C_a$. Finally, we obtain the exponential decay

$$\xi_1^b(\pm b) \leq \frac{\sqrt{\|W\|_\infty + C_a}}{\cosh\big(\sqrt{-E_1^b}(a \mp b)\big)} \xrightarrow[b\to\infty]{} 0.$$

This concludes the proof of the lemma. $\square$

The following theorem does not require that Σ is a surface of revolution.

Theorem 2. *Let Σ be an orientable smooth surface which is asymptotically cut-locus planar* (16)*. Let the tubular neighbourhood* (1) *not overlap itself with some positive a, i.e.* (9) *and* (10) *hold. Let V satisfy* (3) *and* (4)*. Then*

$$\inf \sigma_{\mathrm{ess}}(H) \geq E_1.$$

Proof. Fixing a point $p_0 \in \Sigma$ and giving any positive number R, we divide the surface Σ into two parts $\Sigma_{\mathrm{int}} := \Sigma \cap B_R(p_0)$ and $\Sigma_{\mathrm{ext}} := \Sigma \setminus \overline{B_R(p_0)}$, where $B_R(p_0)$ is the geodesic ball of radius R centred at p_0. Correspondingly, we divide the set U into two parts:

$$U_{\mathrm{int}} := \big\{(p,t) \in \Sigma_{\mathrm{int}} \times \mathbb{R} : -c_-(p) < t < c_+(p)\big\},$$
$$U_{\mathrm{ext}} := \big\{(p,t) \in \Sigma_{\mathrm{ext}} \times \mathbb{R} : -c_-(p) < t < c_+(p)\big\}.$$

The interior part is further subdivided into two subparts:

$$U_{\text{int},1} := \big\{(p,t) \in \Sigma_{\text{int}} \times \mathbb{R} : -a < t < a\big\},$$
$$U_{\text{int},2} := \big\{(p,t) \in \Sigma_{\text{int}} \times \mathbb{R} : -c_-(p) < t < -a \ \vee \ a < t < c_+(p)\big\}.$$

Analogously, we subdivide U_{ext} into two subparts:

$$U_{\text{ext},1} := \big\{(p,t) \in \Sigma_{\text{ext}} \times \mathbb{R} : -b < t < b\big\},$$
$$U_{\text{ext},2} := \big\{(p,t) \in \Sigma_{\text{ext}} \times \mathbb{R} : -c_-(p) < t < -b \ \vee \ b < t < c_+(p)\big\},$$

where $b > 0$. By (16), we can assume $b > a$ by choosing R large enough. Define

$$c_R^\pm := 1 \pm 2\|M\|_R b \pm \|K\|_R b^2 \quad \text{with } \|\cdot\|_R := \|\cdot\|_{L^\infty(\Sigma_{\text{ext}})}.$$

Since (16) implies (15), given any (large) $b > a$, there exists (large) R such that c_R^- is positive.

We consider the auxiliary operator $\widehat{H}^N$ which is obtained from $\widehat{H}$ by imposing an extra Neumann condition (i.e. no condition on the level of sesquilinear forms) on the boundaries of the subsets described above. More specifically, $\widehat{H}^N = \widehat{H}^N_{\text{int},1} \oplus \widehat{H}^N_{\text{int},2} \oplus \widehat{H}^N_{\text{ext}}$, where $\widehat{H}^N_{\text{ext}} = \widehat{H}^N_{\text{ext},1} \oplus \widehat{H}^N_{\text{ext},2}$ is the self-adjoint operator associated with the form $\hat{h}^N_{\text{ext}}$ in $L^2(U_{\text{ext}}, \mathrm{d}v)$ defined by

$$\hat{h}^N_{\text{ext}}[\psi] := \int_{U_{\text{ext}}} \overline{\partial_\mu \psi} G^{\mu\nu} \partial_\nu \psi \, \mathrm{d}v + \int_{U_{\text{ext}}} |\partial_t \psi|^2 \, \mathrm{d}v + \int_{U_{\text{ext}}} W(t) |\psi|^2 \, \mathrm{d}v,$$
$$\operatorname{dom} \hat{h}^N_{\text{ext}}[\psi] := \big\{\psi \restriction \hat{h}_{\text{ext}} : \psi \in \operatorname{dom} \hat{h}\big\},$$

and similarly for the other operators. Obviously, $\operatorname{dom} \hat{h}^N \supset \operatorname{dom} \hat{h}$, therefore $\widehat{H}^N \leq \widehat{H}$ in the sense of quadratic forms, so, by the minimax principle, it is enough to show that $\inf \sigma_{\text{ess}}(\widehat{H}^N) \geq E_1$. Since $\operatorname{supp} W \subset [-a, a]$ due to (3), the operator $\widehat{H}^N_{\text{int},2}$ is non-negative, so the inequality $\inf \sigma_{\text{ess}}(\widehat{H}^N_{\text{int},2}) \geq E_1$ is trivial. At the same time, $\widehat{H}^N_{\text{int},1}$ is an operator with compact resolvent, so it does not contribute to the essential spectrum of $\widehat{H}^N$ (one has $\inf \sigma_{\text{ess}}(\widehat{H}^N_{\text{int},1}) = \infty$ by the minimax principle). It remains to estimate the essential spectrum of $\widehat{H}^N_{\text{ext}}$.

For every $\psi \in \operatorname{dom} \hat{h}^N_{\text{ext}}$,

$$\begin{aligned}
\hat{h}^N_{\text{ext}}[\psi] &\geq \int_{U_{\text{ext}}} |\partial_t \psi|^2 \, \mathrm{d}v + \int_{U_{\text{ext}}} W(t) |\psi|^2 \, \mathrm{d}v \\
&\geq \int_{U_{\text{ext},1}} |\partial_t \psi|^2 \, \mathrm{d}v + \int_{U_{\text{ext},1}} W(t) |\psi|^2 \, \mathrm{d}v \\
&\geq c_R^- \int_{U_{\text{ext},1}} |\partial_t \psi|^2 \, \mathrm{d}\Sigma \, \mathrm{d}t + \int_{U_{\text{ext},1}} W(t) |\psi|^2 \, \mathrm{d}v
\end{aligned}$$

$$\geq c_R^- E_1^b \int_{U_{\mathrm{ext},1}} |\psi|^2 \,\mathrm{d}\Sigma\,\mathrm{d}t - c_R^- \int_{U_{\mathrm{ext},1}} W(t)|\psi|^2 \,\mathrm{d}\Sigma\,\mathrm{d}t + \int_{U_{\mathrm{ext},1}} W(t)|\psi|^2 \,\mathrm{d}v$$
$$\geq \frac{c_R^-}{c_R^+} E_1^b \|\psi\|^2_{L^2(U_{\mathrm{ext}},\mathrm{d}v)} - c_R^- \int_{U_{\mathrm{ext},1}} W(t)|\psi|^2 \,\mathrm{d}\Sigma\,\mathrm{d}t + \int_{U_{\mathrm{ext},1}} W(t)|\psi|^2 \,\mathrm{d}v,$$

where the fourth inequality employs the variational definition of E_1^b (cf. (35)) with the help of Fubini's theorem and the fact that $\operatorname{supp} W \subset [-a, a]$. Since

$$\int_{U_{\mathrm{ext},1}} W(t)|\psi|^2 \,\mathrm{d}v \geq \int_{U_{\mathrm{ext},1}} c_R(t) W(t)|\psi|^2 \,\mathrm{d}\Sigma\,\mathrm{d}t,$$

where $c_R(t) := c_R^-$ if $W(t) \geq 0$ and $c_R(t) := c_R^+$ if $W(t) < 0$, we get

$$\hat{h}^N_{\mathrm{ext}}[\psi] \geq \frac{c_R^-}{c_R^+} E_1^b \|\psi\|^2_{L^2(U_{\mathrm{ext}},\mathrm{d}v)} - (c_R^+ - c_R^-) \int_{U_{\mathrm{ext},1}} \chi_{\{W(t)<0\}}(t) |W(t)|\, |\psi|^2 \,\mathrm{d}\Sigma\,\mathrm{d}t$$
$$\geq \left(\frac{c_R^-}{c_R^+} E_1^b - \frac{c_R^+ - c_R^-}{c_R^-} \|W\|_\infty \right) \|\psi\|^2_{L^2(U_{\mathrm{ext}},\mathrm{d}v)}.$$

In summary,

$$\inf \sigma_{\mathrm{ess}}(\widehat{H}) \geq \inf \sigma_{\mathrm{ess}}(\widehat{H}^N_{\mathrm{ext}}) \geq \inf \sigma(\widehat{H}^N_{\mathrm{ext}}) \geq \frac{c_R^-}{c_R^+} E_1^b - \frac{c_R^+ - c_R^-}{c_R^-} \|W\|_\infty.$$

Since $c_R^-/c_R^+ \to 1$ as $R \to \infty$ due to (15) (which is a consequence of (16)), we obtain $\inf \sigma_{\mathrm{ess}}(\widehat{H}) \geq E_1^b$. Finally, the arbitrariness of b and Lemma 5 yield that $\inf \sigma_{\mathrm{ess}}(\widehat{H}) \geq E_1$. □

We leave as an open problem whether $\sigma_{\mathrm{ess}}(H) \supset [E_1, \infty)$ under the hypotheses of Theorem 2.

Now we turn to the existence of bound states. We heavily rely on results in Section 3 for rotationally symmetric layers. In particular, recall that, under the hypotheses of the following theorem, the surface Jacobian $r(s)$ is bounded from above by a multiple of s (Lemma 1) and diverges as $s \to \infty$ (Lemma 3); the parallel curvature k_2 behaves like r^{-1} (Lemma 2); the cut-radius maps satisfy $c_-(s) = \infty$ and $c_+(s) = k_2(s)^{-1}$ for all sufficiently large s (Lemma 4); and thus (16) follows as a consequence of (15).

Theorem 3. *Let Σ be a surface of revolution given by (20) and satisfying (6) with $\mathcal{K} > 0$ and (15). Let V satisfy (3) with some positive a and (4). Assume in addition (30) and (32). Then H possesses an infinite number (counting multiplicities) of discrete eigenvalues below E_1.*

Proof. In view of Theorem 2, to establish the existence of a discrete eigenvalue of H, it is enough to show that $\inf \sigma(H) < E_1$. By the minimax principle, it is thus enough to find a trial function $\psi \in \operatorname{dom} Q := \operatorname{dom} \hat{h}$ such that $Q[\psi] := \hat{h}[\psi] - E_1\|\psi\|^2 < 0$, where $\|\cdot\|$ denotes the norm in $L^2(U, \mathrm{d}v)$. Recall that, in the rotationally symmetric case, we have

$$\begin{aligned} \hat{h}[\psi] &= \int_U |\partial_s\psi(s,\vartheta,t)|^2 \frac{1-k_2(s)t}{1-k_1(s)t}\,\mathrm{d}\Sigma\,\mathrm{d}t + \int_U \frac{|\partial_\vartheta\psi(s,\vartheta,t)|^2}{r(s)^2}\frac{1-k_1(s)t}{1-k_2(s)t}\,\mathrm{d}\Sigma\,\mathrm{d}t \\ &\quad + \int_U |\partial_t\psi(s,\vartheta,t)|^2(1-k_1(s)t)(1-k_2(s)t)\,\mathrm{d}\Sigma\,\mathrm{d}t \qquad (37) \\ \|\psi\|^2 &= \int_U |\psi(s,\vartheta,t)|^2(1-k_1(s)t)(1-k_2(s)t)\,\mathrm{d}\Sigma\,\mathrm{d}t, \end{aligned}$$

where $\mathrm{d}\Sigma = r(s)\,\mathrm{d}s \wedge \mathrm{d}\vartheta$ and U is given by (29).

For every real ε, we introduce a ϑ-independent trial function

$$\psi_{n,\varepsilon}(s,\vartheta,t) := \varphi_n(s)\xi_1(t) + \varepsilon\phi_n(s)t\xi_1(t),$$

where the sequence $\{\varphi_n\}_{n=2}^\infty$ is defined by

$$\varphi_n(s) := \begin{cases} 0 & \text{if } s \in [0,n), \\ \dfrac{\log(s/n)}{\log n} & \text{if } s \in [n,n^2), \\ \dfrac{\log(n^3/s)}{\log n} & \text{if } s \in [n^2,n^3), \\ 0 & \text{if } s \in [n^3,\infty), \end{cases} \qquad \text{and} \qquad \phi_n(s) := \frac{\varphi_n(s)}{s}.$$

Note that the supports of φ_n and ϕ_n tend to infinity as $n \to \infty$. We always assume that n is so large that the asymptotic properties of Lemma 4 hold. Proceeding as in [26, Lem. 1] (see also below), one can verify that $\psi_{n,\varepsilon} \in \operatorname{dom} Q$. Then

$$Q[\psi_{n,\varepsilon}] = Q[\varphi_n\xi_1] + 2\varepsilon Q(\phi_n t\xi_1, \varphi_n\xi_1) + \varepsilon^2 Q[\phi_n t\xi_1]. \qquad (38)$$

We make the decomposition $Q = Q_1 + Q_2$, where $Q_1[\psi] := \int_U \overline{\partial_\mu\psi}G^{\mu\nu}\partial_\nu\psi\,\mathrm{d}v$ is just the first line of (37).

$\boxed{Q_1}$ One has

$$Q_1[\varphi_n\xi_1] = \int_U \varphi_n'(s)^2\xi_1(t)^2 f(s,t)\,\mathrm{d}\Sigma\,\mathrm{d}t \quad \text{with } f(s,t) := \frac{1-k_2(s)t}{1-k_1(s)t} \geq 0.$$

Since

$$\partial_t f(\cdot,t) = \frac{k_1-k_2}{(1-k_1t)^2} \leq 0,$$

where the inequality holds due to Lemma 4, one has $\|f\|_\infty = 1$. Consequently,

$$\begin{aligned} Q_1[\varphi_n \xi_1] &\le \int_U \varphi_n'(s)^2 \xi_1(t)^2 \, \mathrm{d}\Sigma \, \mathrm{d}t \\ &\le 2\pi C_\Sigma \int_0^\infty \varphi_n'(s)^2 s \, \mathrm{d}s \\ &= \frac{4\pi C_\Sigma}{\log n}, \end{aligned}$$

where the second inequality holds due to Lemma 1 and the normalisation of ξ_1. Similarly,

$$Q_1[\phi_n t \xi_1] \le \frac{4\pi C C_\Sigma}{n^2 \log n}, \quad \text{where } C := \int_{\mathbb{R}} \xi_1(t)^2 t^2 \, \mathrm{d}t,$$

where the extra decay n^{-2} comes from the bound $s \ge n$ on the support of ϕ_n. The mixed term $Q_1(\phi_n t\xi_1, \varphi_n \xi_1)$ tends to zero as $n \to \infty$ by using these estimates and the Schwarz inequality. In summary,

$$\lim_{n\to\infty} Q_1[\psi_{n,\varepsilon}] = 0.$$

$\boxed{Q_2, \text{ order } \varepsilon^0}$ One has

$$\begin{aligned} Q_2[\varphi_n \xi_1] &= \int_U \varphi_n(s)^2 \xi_1'(t)^2 (1 - 2M(s)t + K(s)t^2) \, \mathrm{d}\Sigma \, \mathrm{d}t \\ &\quad + \int_U W(t) \varphi_n(s)^2 \xi_1(t)^2 (1 - 2M(s)t + K(s)t^2) \, \mathrm{d}\Sigma \, \mathrm{d}t \\ &\quad - E_1 \int_U \varphi_n(s)^2 \xi_1(t)^2 (1 - 2M(s)t + K(s)t^2) \, \mathrm{d}\Sigma \, \mathrm{d}t \\ &= \int_U |\varphi_n(s)|^2 \xi_1(t) \xi_1'(t) (2M(s) - 2K(s)t) \, \mathrm{d}\Sigma \, \mathrm{d}t \\ &\quad + \int_\Sigma |\varphi_n(s)|^2 \big[\xi_1(t)\xi_1'(t)(1 - 2M(s)t + K(s)t^2)\big]_{t=-c_-(s)}^{t=c_+(s)} \, \mathrm{d}\Sigma \\ &= \int_U |\varphi_n(s)|^2 |\xi_1(t)|^2 K(s) \, \mathrm{d}\Sigma \, \mathrm{d}t \\ &\quad + \int_\Sigma |\varphi_n(s)|^2 \big[\xi_1(t)^2 (M(s) - K(s)t) \\ &\qquad + \xi_1(t)\xi_1'(t)(1 - 2M(s)t + K(s)t^2)\big]_{t=-c_-(s)}^{t=c_+(s)} \, \mathrm{d}\Sigma. \end{aligned} \tag{39}$$

Here, the first equality follows by an integration by parts and the identity $-\xi_1'' + W\xi_1 = E_1\xi_1$. The second equality is a result of yet another integration by parts after writing $2\xi_1\xi_1' = (\xi_1^2)'$. Recall that the support of φ_n tends to infinity as

$n \to \infty$. Then the resulting integral over U vanishes as $n \to \infty$ due to (6) and the dominated convergence theorem. What is more, the resulting integral over Σ vanishes as $n \to \infty$, for the exponential decay of ξ_1 dominates all the other functions that appear there. More specifically, evaluating at $-c_-(s) = -\infty$ (recall Lemma 4) does not contribute. Recalling that $c_+ = 1/k_2$, one has $(1 - 2Mc_+ + Kc_+^2) = (1 - k_1 c_+)(1 - k_2 c_+) = 0$ and $M - Kc_+ = \frac{1}{2}(k_2 - k_1) \leq 1$ (because the curvatures vanish at infinity), so it is enough to estimate (recall (34))

$$\begin{aligned}\left|\int_\Sigma |\varphi_n(s)|^2 \xi_1(c_+(s))^2 \,\mathrm{d}\Sigma\right| &\leq 2\pi \int_n^{n^3} \xi_1(c_+(s))^2 r(s)\,\mathrm{d}s \\ &\leq 2\pi N_+^2 \int_n^{n^3} e^{-2\sqrt{-E_1}c_+(s)} c_+(s)\,\mathrm{d}s \\ &\leq 2\pi N_+^2 (n^3 - n) e^{-2\sqrt{-E_1}c_+(n)} c_+(n^3),\end{aligned}$$

where the first inequality employs $r \leq c_+$. The upper bound vanishes as $n \to \infty$ due to (32). In summary,

$$\lim_{n\to\infty} Q_2[\varphi_n \xi_1] = 0.$$

$\boxed{Q_2\text{, order } \varepsilon^1}$ Proceeding as in (39), we have

$$\begin{aligned}&Q_2(\phi_n t\xi_1, \varphi_n \xi_1) \\ &\quad= \int_U \phi_n(s)\varphi_n(s)(t\xi_1(t))'\xi_1'(t)(1 - 2M(s)t + K(s)t^2)\,\mathrm{d}\Sigma\,\mathrm{d}t \\ &\qquad+ \int_U W(t)\phi_n(s)\varphi_n(s)t\xi_1(t)\xi_1(t)(1 - 2M(s)t + K(s)t^2)\,\mathrm{d}\Sigma\,\mathrm{d}t \\ &\qquad- E_1 \int_U \phi_n(s)\varphi_n(s)t\xi_1(t)\xi_1(t)(1 - 2M(s)t + K(s)t^2)\,\mathrm{d}\Sigma\,\mathrm{d}t \\ &\quad= \int_U \phi_n(s)\varphi_n(s)t\xi_1(t)\xi_1'(t)(2M(s) - 2K(s)t)\,\mathrm{d}\Sigma\,\mathrm{d}t \\ &\qquad+ \int_\Sigma \phi_n(s)\varphi_n(s)\big[t\xi_1(t)\xi_1'(t)(1 - 2M(s)t + K(s)t^2)\big]_{t=-c_-(s)}^{t=c_+(s)}\,\mathrm{d}\Sigma \\ &\quad= -\int_U \phi_n(s)\varphi_n(s)\xi_1(t)^2(M(s) - 2K(s)t)\,\mathrm{d}\Sigma\,\mathrm{d}t \\ &\qquad+ \int_\Sigma \phi_n(s)\varphi_n(s)\big[t\xi_1(t)^2(M(s) - K(s)t) \\ &\qquad\qquad+ t\xi_1(t)\xi_1'(t)(1 - 2M(s)t + K(s)t^2)\big]_{t=-c_-(s)}^{t=c_+(s)}\,\mathrm{d}\Sigma.\end{aligned}$$

Again, the terms containing K in the resulting integral over U and the resulting integral over Σ vanish as $n \to \infty$. Consequently, as $n \to \infty$,

$$\begin{aligned} Q_2(\phi_n t\xi_1, \varphi_n \xi_1) &= -\int_U \phi_n(s)\varphi_n(s)\xi_1(t)^2 M(s)\,\mathrm{d}\Sigma\,\mathrm{d}t + o(1) \\ &= -\frac{1}{2}\int_U \phi_n(s)\varphi_n(s)\xi_1(t)^2 k_2(s)\,\mathrm{d}\Sigma\,\mathrm{d}t + o(1), \end{aligned}$$

where the second equality follows from (28) and the dominated convergence theorem. Here, employing Lemma 2,

$$\begin{aligned} \int_U \phi_n(s)\varphi_n(s)\xi_1(t)^2 k_2(s)\,\mathrm{d}\Sigma\,\mathrm{d}t &\geq 2\pi\delta \int_0^\infty \frac{\varphi_n(s)^2}{s} \int_{-c_-(s)}^{c_+(s)} \xi_1(t)^2\,\mathrm{d}t\,\mathrm{d}s \\ &= 2\pi\delta \int_0^\infty \frac{\varphi_n(s)^2}{s} \left(1 - \int_{c_+(s)}^\infty \xi_1(t)^2\,\mathrm{d}t\right)\mathrm{d}s \\ &= 2\pi\delta \int_0^\infty \frac{\varphi_n(s)^2}{s} \left(1 - \frac{\xi_1(c_+(s))^2}{2\sqrt{-E_1}}\right)\mathrm{d}s. \end{aligned}$$

Here, the inequality is due to Lemma 2, the first equality employs the normalisation of ξ_1 and the last equality is due to the asymptotics (34). Since $c_+(s) \to \infty$ as $s \to \infty$, one has

$$Q_2(\phi_n t\xi_1, \varphi_n \xi_1) \leq -\pi\delta \int_0^\infty \frac{\varphi_n(s)^2}{s}\,\mathrm{d}s + o(1)$$

as $n \to \infty$. It remains to compute

$$\int_0^\infty \frac{\varphi_n(s)^2}{s}\,\mathrm{d}s = \frac{2}{3}\log n. \tag{40}$$

In summary,

$$Q_2(\phi_n t\xi_1, \varphi_n \xi_1) \leq -c_1 \log n + o(1) \quad \text{as } n \to \infty,$$

where $c_1 := \frac{2}{3}\pi\delta$ is positive.

$\boxed{Q_2,\ \text{order } \varepsilon^2}$ Integrating by parts as above, we have

$$\begin{aligned} Q_2[\phi_n t\xi_1] = &\int_U \phi_n(s)^2 (t\xi_1(t))'(t\xi_1(t))'(1 - 2M(s)t + K(s)t^2)\,\mathrm{d}\Sigma\,\mathrm{d}t \\ &+ \int_U W(t)\phi_n(s)^2 t\xi_1(t) t\xi_1(t)(1 - 2M(s)t + K(s)t^2)\,\mathrm{d}\Sigma\,\mathrm{d}t \\ &- E_1 \int_U \phi_n(s)^2 t\xi_1(t) t\xi_1(t)(1 - 2M(s)t + K(s)t^2)\,\mathrm{d}\Sigma\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= \int_U \phi_n(s)^2 t\xi_1(t)(t\xi_1(t))'(2M(s) - 2K(s)t)\,\mathrm{d}\Sigma\,\mathrm{d}t \\
&\quad - 2\int_U \phi_n(s)^2 t\xi_1(t)\xi_1'(t)(1 - 2M(s)t + K(s)t^2)\,\mathrm{d}\Sigma\,\mathrm{d}t \\
&\quad + \int_\Sigma \phi_n(s)^2 \big[t\xi_1(t)(t\xi_1(t))'(1 - 2M(s)t + K(s)t^2)\big]_{t=-c_-(s)}^{t=c_+(s)}\,\mathrm{d}\Sigma \\
&= \int_U \phi_n(s)^2 t^2\xi_1(t)^2 K(s)\,\mathrm{d}\Sigma\,\mathrm{d}t \\
&\quad + \int_U \phi_n(s)^2 \xi_1(t)^2(1 - 4M(s)t + 3K(s)t^2)\,\mathrm{d}\Sigma\,\mathrm{d}t \\
&\quad + \int_\Sigma \phi_n(s)^2 \big[\xi_1(t)^2(t - 2M(s)t^2 + K(s)t^3) \\
&\qquad\qquad + t^2\xi_1(t)^2(M(s) - K(s)t)\big]_{t=-c_-(s)}^{t=c_+(s)}\,\mathrm{d}\Sigma \\
&\quad + \int_\Sigma \phi_n(s)^2 \big[t\xi_1(t)(t\xi_1(t))'(1 - 2M(s)t + K(s)t^2)\big]_{t=-c_-(s)}^{t=c_+(s)}\,\mathrm{d}\Sigma.
\end{aligned}
$$

As above, we conclude that

$$
\begin{aligned}
Q_2[\phi_n t\xi_1] &= \int_U \phi_n(s)^2\xi_1(t)^2(1 - 4M(s)t)\,\mathrm{d}\Sigma\,\mathrm{d}t + o(1) \\
&= \int_U \phi_n(s)^2\xi_1(t)^2(1 - 2k_2(s)t)\,\mathrm{d}\Sigma\,\mathrm{d}t + o(1) \\
&\le \int_U \phi_n(s)^2\xi_1(t)^2\,\mathrm{d}\Sigma\,\mathrm{d}t + o(1)
\end{aligned}
$$

as $n \to \infty$. Here (using Lemma 1),

$$
\begin{aligned}
\int_U \phi_n(s)^2\xi_1(t)^2\,\mathrm{d}\Sigma\,\mathrm{d}t &\le 2\pi C_\Sigma \int_0^\infty \frac{\varphi_n(s)^2}{s^2}\, s \int_{-c_-(s)}^{c_+(s)} \xi_1(t)^2\,\mathrm{d}t\,\mathrm{d}s \\
&= 2\pi C_\Sigma \int_0^\infty \frac{\varphi_n(s)^2}{s}\left(1 - \int_{c_+(s)}^\infty \xi_1(t)^2\,\mathrm{d}t\right)\mathrm{d}s \\
&= 2\pi C_\Sigma \int_0^\infty \frac{\varphi_n(s)^2}{s}\left(1 - \frac{\xi_1(c_+(s))^2}{2\sqrt{-E_1}}\right)\mathrm{d}s.
\end{aligned}
$$

Consequently,

$$
Q_2[\phi_n t\xi_1] \le c_2 \log n + o(1) \quad \text{as } n \to \infty,
$$

where $c_2 := \frac{2}{3} 2\pi C_\Sigma$.

$\boxed{Q}$ Putting the results together, we finally arrive at

$$
Q[\psi_{n,\varepsilon}] \le (-2\varepsilon c_1 + \varepsilon^2 c_2)\log n + o(1) \quad \text{as } n \to \infty. \tag{41}
$$

Obviously, it is possible to choose ε positive and sufficiently small so that $Q[\psi_{n,\varepsilon}]$ is negative for all sufficiently large n. (Note that $\|\psi_{n,\varepsilon}\| \to \infty$ as $n \to \infty$, so the result (41) does not contradict the fact that Q is bounded from below.)

$\boxed{1 \mapsto \infty}$ The argument above, together with Theorem 2, demonstrates that there is at least one discrete eigenvalue (below E_1). To realise that the same argument actually shows that there is an infinite number (counting multiplicities) of discrete eigenvalues, it is enough to notice that we have constructed a non-compact sequence of trial functions. Indeed, $\{\psi_{n,\varepsilon}\}_{n=2}^{\infty}$ certainly contains an infinite subsequence of functions with mutually disjoint supports. □

Theorem 1 from the introduction is a combination of Theorems 2 and 3, as well as the observation that the trial function from the proof of Theorem 3 "does not feel" what happens on any compact subset of $\mathbb{R}^3$. In fact, the perturbation of Σ can be a conical surface as in [11]. At the same time, the essential spectrum of H is stable under changes of Σ on a compact set. The extra property that $\inf \sigma_{\mathrm{ess}}(H) = E_1$ in the last part of Theorem 1 follows from the general inequality $\inf \sigma_{\mathrm{ess}}(H) \geq E_1$ and the fact that there is an infinite number of discrete eigenvalues, which can accumulate to the lowest point of the essential spectrum only.

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