# On the Geography of Slopes of Fibrations

by

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## Abstract

The slopes of fibrations of genus $g \geq 2$ have strict lower (resp., upper) bound, namely $\lambda_m(g)$ (resp., $\lambda_M(g)$). In this paper, we show that if $g \neq 3$, then each rational number $r \in [\lambda_m(g), \lambda_M(g)]$ can occur as the slope of some fibration of genus $g$. A similar result is also true for $g = 3$ and $r \in [\lambda_m(3), 9]$.

## §1. Introduction

Let $X$ be a minimal nonsingular projective surface of general type over $\mathbb{C}$ and its Chern numbers be $c_1^2(X)$, $c_2(X)$, $\chi(\mathcal{O}_X)$ with $c_1^2(X) + c_2(X) = 12\chi(\mathcal{O}_X)$. The geography problem of surfaces asks for which pair of integers $(a, b)$ does there exist $X$ such that $c_1^2(X) = a$ and $c_2(X) = b$? We refer to [BHPV04, Chap. VII, §8] for general existence results. One of them, due to Sommese [So84], states that every rational point in $[1/5, 3]$ can occur as the slope $c_1^2(X)/c_2(X)$ of some surface $X$.

It is natural to study the geography problem for fibrations, which is an open problem in [AK02, §1.1].

Let $S$ be a nonsingular projective surface and $C$ be a nonsingular projective curve of genus $b$. Let $f \colon S \to C$ be a relatively minimal fibration of genus $g$. One

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has the following well-known relative invariants:

$$(1.1) \quad \begin{cases} \chi_f = \chi(\mathcal{O}_S) - (g-1)(b-1), \\ e_f = c_2(S) - 4(g-1)(b-1), \\ K_f^2 = c_1^2(S) - 8(g-1)(b-1). \end{cases}$$

When $g \geq 2$, $K_f^2 = 0$ if and only if $\chi_f = 0$ if and only if $f$ is a locally trivial fibration. For a locally nontrivial fibration $f$, the slope of $f$ is defined as

$$\lambda_f = K_f^2 / \chi_f.$$

The following geography problem of fibrations has been considered by Xiao, and he obtained a partial result when $g = 2$ (see [Xi85, Thm. 2.9] and [Xi92, Thm. 4.3.5]).

**Question 1.1.** *Let $g \geq 2$ be an integer. For what kinds of nonnegative integers $(\chi, K, b)$, can we find a fibration $f \colon S \to C$ of genus $g$ with $\chi_f = \chi$, $K_f^2 = K$, and $g(C) = b$?*

Xiao pointed out that the existence of fibrations with high slope is interesting for the difficulty of construction, and he raised an open problem in [Xi92] to find examples of hyperelliptic fibrations with maximal slopes. Though the latter problem is solved ([Mo98, LT13]), the existence of fibrations with high slopes is still confusing.

In this note, we will focus on Question 1.1 for slopes of fibrations.

**Question 1.2.** *Given $r > 0$ and $g \geq 2$, is there a fibration $f \colon S \to C$ of genus $g$ with $\lambda_f = r$?*

It is known that ([Xi87, CH88]) $\lambda_f \geq \lambda_m(g) := \frac{4g-4}{g}$ for any $g \geq 2$, and the equality can be reached. If $g \geq 3$, then $\lambda_f \leq 12$. Moreover, the equality holds here if and only if $f$ is a Kodaira fibration, that is, $f$ has no singular fibers but it has variable moduli. See [GD-H91, Za95] for explicit Kodaira fibrations. If $f$ is hyperelliptic, Xiao ([Xi92]) showed that $\lambda_f \leq \lambda_M^h(g)$, where

$$\lambda_M^h(g) := \begin{cases} 12 - \dfrac{8g+4}{g^2} & \text{if } g \text{ is even}, \\ 12 - \dfrac{8g+4}{g^2-1} & \text{if } g \text{ is odd}. \end{cases}$$

Let

$$\lambda_M(g) := \begin{cases} 7 & \text{if } g = 2, \\ 12 & \text{if } g \geq 3; \end{cases}$$

then our main result is as follows.

**Theorem 1.3.** *If $g \neq 3$, then, for each rational number $r \in [\lambda_m(g), \lambda_M(g)]$, there exists a fibration of genus $g$ with slope $r$.*

Now we outline the proof of the above theorem. Firstly, we take a semistable fibration $f \colon S \to C$ with slope $\lambda_f$ such that $f$ is composed of a fibration $\varphi \colon P \to C$ and a generically finite morphism $\Pi \colon S \dashrightarrow P$ of degree 2 (which may be a rational map; see Section 3 for details). Secondly, after taking a cyclic cover $\pi_1 \colon C_1 \to C$ of degree $d$ branching over noncritical points of $\varphi$, we get a pullback fibration $f_1 \colon S_1 \to C_1$ (resp., $\varphi_1 \colon P_1 \to C_1$) of $f$ (resp., $\varphi$) with respect to $\pi_1$. Thirdly, after adding $k$ general fibers of $\varphi_1$ to the branch locus of $\Pi_1 \colon S_1 \dashrightarrow P_1$, we construct a modified general finite map $\Pi'_1 \colon S'_1 \to P_1$ and a fibration $f'_1 \colon S'_1 \to C_1$:
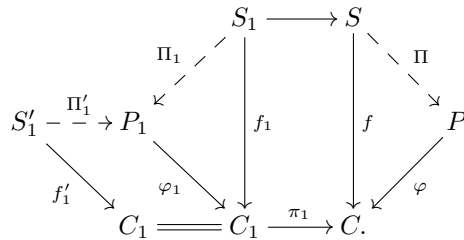


Finally, we will show that the slopes of $f'_1$ can take every rational number in some interval by varying $d$ and $k$. It is shown that ([Ta10]) relative invariants are the sum of modular invariants and the Chern numbers of singular fibers; see Section 2. We will use these formulas to calculate relative invariants of $f'_1$, which is convenient. The relation between modular invariants of $f'_1$ and those of $f$ is the motivation for our construction.

Our method is also available when $\Pi$ is cyclic.

But for $g = 3$, we cannot give a complete answer due to the lack of Kodaira fibrations of genus 3 with generically cyclic cover $\Pi$. However, we have the following partial result by the same method.

**Theorem 1.4.** *If $g = 3$, then, for each rational number $r \in [\lambda_m(3), 9]$, there exists a fibration of genus $3$ with slope $r$.*

Note that, for a fibration $f$ of genus 3 coming from a cyclic triple cover over a ruled surface, [CT06] shows $\lambda_f \leq 9$. We will give an example (Example 2.6) to show that this upper bound is sharp in this case.

Our proof is based on that of Sommese's result; see [So84]. The difference is that we construct a fibered surface with new *singular* fibers by modifying the branch locus. The latter trick has been used by Xiao [Xi92] and Chen [Ch87].

By our construction, it is easy to see that lots of the fibrations we obtain are not semistable. So it is interesting to consider the geography problem of slopes of semistable fibrations, and we raise the following conjecture.

**Conjecture 1.5.** *Let $g \geq 2$ be an integer. For each rational number $r \in [\lambda_m(g), \lambda_M(g)]$, there exists a semistable fibration of genus $g$ with slope $r$.*

## §2. Preliminaries

### §2.1. The Chern numbers of singular fibers

Denote by $F$ a singular fiber of $f\colon S \to C$ over $p \in C$. Let $\pi\colon \widetilde{C} \to C$ be a semistable reduction of $F$, i.e., $\pi$ is ramified over $p = f(F)$ and some noncritical points of $f$, and the fibers of the pullback fibration $\tilde{f}\colon \tilde{S} \to \widetilde{C}$ over $\pi^{-1}(p)$ are semistable. Here $\tilde{f}\colon \tilde{S} \to \widetilde{C}$ is the unique relatively minimal birational model of $X \times_C \widetilde{C} \to \widetilde{C}$.

**Definition 2.1** ([Ta10, Def. 1.1]). The Chern numbers of $F$ are defined as follows:

$$(2.1) \qquad c_1^2(F) = K_f^2 - \frac{1}{d}K_{\tilde{f}}^2, \quad c_2(F) = e_f - \frac{1}{d}e_{\tilde{f}}, \quad \chi_F = \chi_f - \frac{1}{d}\chi_{\tilde{f}},$$

where $d$ is the degree of $\pi$.

Note that $\frac{1}{d}K_{\tilde{f}}^2$, $\frac{1}{d}e_{\tilde{f}}$, and $\frac{1}{d}\chi_{\tilde{f}}$ are independent of the choice of the semistable reduction $\pi$, so the above definition is well defined (see [Ta96, Lem. 2.3] or [Ta10] for more details).

If $\chi_F \neq 0$, then we define

$$\lambda_F \triangleq c_1^2(F)/\chi_F.$$

In order to compute these invariants, we introduce some numerical invariants of the singularity of a curve. Let $B$ be a curve on $S$ which is a nonzero effective divisor.

**Definition 2.2.** A *partial resolution* of the singularities of $B$ is a sequence of blowups $\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_r\colon \hat{S} \to S$,

$$(\hat{S}, \sigma^*B) = (S_r, B_r) \xrightarrow{\sigma_r} S_{r-1} \xrightarrow{\sigma_{r-1}} \cdots \xrightarrow{\sigma_2} (S_1, B_1) \xrightarrow{\sigma_1} (S_0, B_0) = (S, B),$$

satisfying the following conditions:

  (i)  $B_{r,\mathrm{red}}$ has at worst ordinary double points as its singularities;
  (ii) $B_i = \sigma_i^* B_{i-1}$ is the total transform of $B_{i-1}$;

(iii) $\sigma_i$ is the blowup of $S_{i-1}$ at a singular point $(B_{i-1,\text{red}}, p_{i-1})$ which is not an ordinary double point.

Now let $\sigma \colon \hat{S} \to S$ be the partial resolution of the singularities of $F$. Then $\sigma^* F$ is called the *normal crossing model* of $F$. A $(-1)$-curve in a fiber is called *redundant* if it meets the other components in at most two points. It is obvious that a redundant $(-1)$-curve can be contracted without introducing singularities worse than ordinary double points. After a finite number of such contractions, we will finally obtain a normal crossing fiber $\overline{F}$ containing no redundant $(-1)$-curves. We call $\overline{F}$ the *minimal normal crossing model* of $F$ (see [LuT13, Def. 2.2]). So $\overline{F}$ is determined uniquely by $F$.
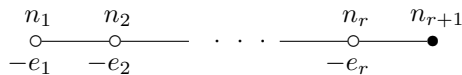
In Definition 2.2, we denote by $m_i$ the multiplicity of $B_{i,\text{red}}$ at $p_i$. If $q \in B_{r,\text{red}}$ is a double point, and the two local components of $(B_r, q)$ have multiplicities $a_q$ and $b_q$, then we define $[a_q, b_q] := \frac{\gcd(a_q, b_q)^2}{a_q b_q}$. In particular, if $B_{\text{red}}$ has only one singular point $p = p_0$, then we define

$$\alpha_p = \sum_{i=1}^{r}(m_i - 2)^2, \quad \beta_p = \sum_{q \in B_r}[a_q, b_q],$$

where $q$ runs over all of the double points of $B_{r,\text{red}}$. These two invariants are independent of the resolution. Let

$$\alpha_B := \sum_{p \in B}\alpha_p, \quad \beta_B := \sum_{p \in B}\beta_p.$$

**Definition 2.3.** Let $\overline{F}$ be the minimal normal crossing model of $F$, and let $G(\overline{F})$ be the dual graph of $\overline{F}$. A *Hirzebruch–Jung (H-J) branch* of rational curves in $G(\overline{F})$ is the following subgraph:

$$
\begin{array}{ccccccc}
n_1 & n_2 & & & n_r & n_{r+1} \\
\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!-\!\!- & \cdots & -\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\bullet \\
-e_1 & -e_2 & & & -e_r
\end{array}
$$

In the above dual graph, $\overset{n_i}{\underset{-e_i}{\circ}}$ denotes a smooth rational curve $\Gamma_i$ with $\Gamma_i^2 = -e_i$ whose multiplicity in $\overline{F}$ is $n_i$. Moreover, $\bullet$ denotes either an irrational curve, or a smooth rational curve meeting at 3 or more points with the other components. For convenience, in the following context, we omit the subscript $(-e)$ whenever $e = 2$.

Let $\beta_F^-$ be the total contribution of all the H-J branches in $G(\overline{F})$ to $\beta_F$, and $\beta_F^+ := \beta_F - \beta_F^-$. Denote by $\mu_F$ the sum of the Milnor numbers of the singularities
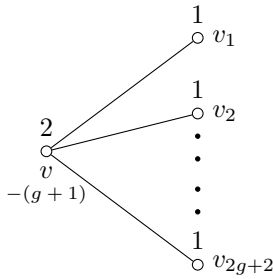
Figure 1. Hyperelliptic fiber $F_h$     Figure 2. Fiber $F_g$ of genus $g \geq 4$
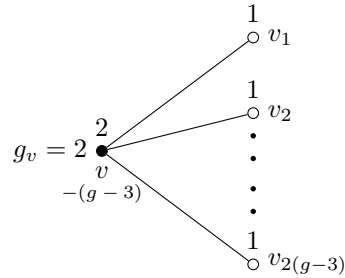
of $F_{\mathrm{red}}$. Let $N_F = g - p_a(F_{\mathrm{red}})$. By [Ta96, Ta10], we know that

$$(2.2) \quad \begin{cases} c_1^2(F) = 4N_F + F_{\mathrm{red}}^2 + \alpha_F - \beta_F^-, \\ c_2(F) = 2N_F + \mu_F - \beta_F^+, \\ 12\chi_F = 6N_F + F_{\mathrm{red}}^2 + \alpha_F + \mu_F - \beta_F. \end{cases}$$

The following examples will be used in our proof.

**Example 2.4.** Let $F_h$ (resp., $F_g$) be a singular fiber of a fibration of genus $g$ with the following dual graph. Here we assume that all the irreducible curves in $F_h$ (resp., $F_g$) have smooth support; see Figures 1 and 2.

Then $F_h$ and $F_g$ are both minimal normal crossing. One can obtain their Chern numbers as in Table 1. For similar detailed computation, we refer to [LT13].

**Example 2.5.** Let $F_0$ (resp., $F_1$, $F_\infty$, $F$) be a singular fiber of a fibration of genus 3, and $\overline{F}_0$ (resp., $\overline{F}_1$, $\overline{F}_\infty$, $\overline{F}$) be its minimal normal crossing model. Suppose the dual graphs of these fibers are as shown in Figure 3.

From a straightforward computation, we obtain the Chern numbers of these singular fibers as in Table 2.

| Fibers | $N_F$ | $F_{\mathrm{red}}^2$ | $\beta_F^-$ | $\beta_F^+$ | $\alpha_F$ | $\mu_F$ | $c_1^2(F)$ | $\chi_F$ | $\lambda_F$ |
|--------|-------|----------------------|-------------|-------------|------------|---------|------------|----------|-------------|
| $F_h$ | $g$ | $-(g+1)$ | $g+1$ | $0$ | $0$ | $2g+2$ | $2g-2$ | $\frac{g}{2}$ | $\frac{4g-4}{g}$ |
| $F_g$ | $g-2$ | $-(g-3)$ | $g-3$ | $0$ | $0$ | $2(g-3)$ | $2g-2$ | $\frac{g-2}{2}$ | $\frac{4g-4}{g-2}$ |

Table 1. The Chern numbers of $F_h$ and $F_g$

(a) $\overline{F}_0$    (b) $\overline{F}_1$
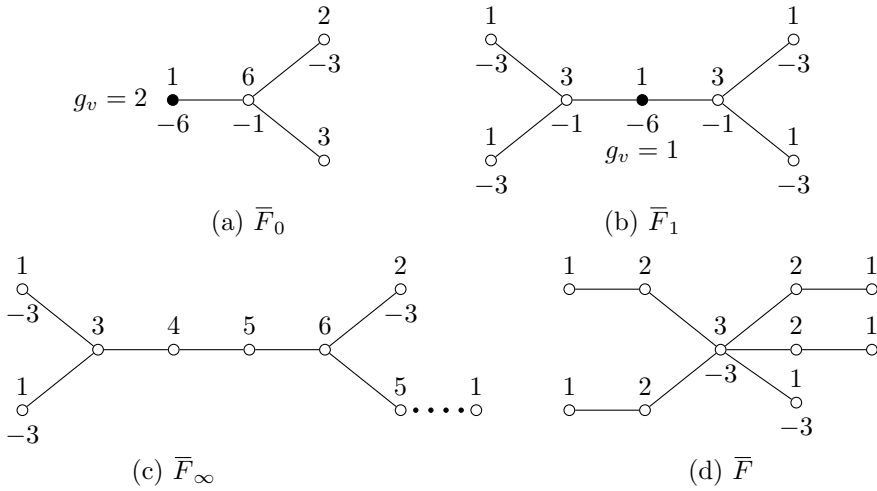
(c) $\overline{F}_\infty$    (d) $\overline{F}$

Figure 3. The dual graphs of singular fibers of genus 3

## §2.2. Modular invariants

The fibration $f: S \to C$ induces a moduli map $J: C \to \overline{\mathcal{M}}_g$ from the base curve $C$ to the moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g$. Let $\lambda$ be the Hodge divisor class of $\overline{\mathcal{M}}_g$, $\delta$ be the boundary divisor class, and $\kappa = 12\lambda - \delta$. Then there are three fundamental *modular invariants* of $f$ defined as follows (see [Ta10]):

$$\kappa(f) = \deg J^*\kappa, \quad \lambda(f) = \deg J^*\lambda, \quad \delta(f) = \deg J^*\delta.$$

If $f$ is semistable, then

$$(2.3) \qquad \kappa(f) = K_f^2, \quad \delta(f) = e_f, \quad \lambda(f) = \chi_f.$$

| Fibers | $N_F$ | $F_{\mathrm{red}}^2$ | $\beta_F^-$ | $\beta_F^+$ | $\alpha_F$ | $\mu_F$ | $c_1^2(F)$ | $c_2(F)$ | $\chi_F$ |
|---|---|---|---|---|---|---|---|---|---|
| $F_0$ | 0 | 0 | $\frac{5}{6}$ | $\frac{1}{6}$ | 1 | 2 | $\frac{1}{6}$ | $\frac{11}{6}$ | $\frac{1}{6}$ |
| $F_1$ | 0 | 0 | $\frac{4}{3}$ | $\frac{2}{3}$ | 2 | 8 | $\frac{2}{3}$ | $\frac{22}{3}$ | $\frac{2}{3}$ |
| $F_\infty$ | 3 | $-5$ | $\frac{11}{6}$ | $\frac{1}{6}$ | 0 | 11 | $\frac{31}{6}$ | $\frac{101}{6}$ | $\frac{11}{6}$ |
| $F$ | 3 | $-4$ | 3 | 0 | 0 | 9 | 5 | 15 | $\frac{5}{3}$ |

Table 2. The Chern numbers of singular fibers for $g = 3$

The modular invariants satisfy the *base change property*, i.e., if $\tilde{f}$ is the pullback fibration of $f$ with respect to $\pi$ of degree $d$, then

$$(2.4) \qquad \kappa(\tilde{f}) = d \cdot \kappa(f), \quad \delta(\tilde{f}) = d \cdot \delta(f), \quad \lambda(\tilde{f}) = d \cdot \lambda(f).$$

Let $F_1, \ldots, F_s$ be all the singular fibers of $f$. It is proved in [Ta94, Ta96] that

$$(2.5) \qquad \begin{cases} K_f^2 = \kappa(f) + \displaystyle\sum_{i=1}^s c_1^2(F_i), \\ e_f = \delta(f) + \displaystyle\sum_{i=1}^s c_2(F_i), \\ \chi_f = \lambda(f) + \displaystyle\sum_{i=1}^s \chi_{F_i}. \end{cases}$$

**Example 2.6.** By using results in Example 2.5, we will find a semistable fibration $f$ of genus 3 with slope $\lambda_f = 9$, which derives from a cyclic covering over a ruled surface (see Section 3 for details).

Let $\{C_\lambda\}_{\lambda \in \mathbb{P}^1}$ be a pencil of curves in $\mathbb{P}^2$ defined by $F - \lambda G = 0$, where

$$F = Y^3 Z + X^2(X^2 - Z^2), \quad G = Z^2(X^2 - Z^2).$$

After blowing up the base points of $\{C_\lambda\}$, we obtain a relatively minimal fibration $h \colon S_0 \to \mathbb{P}^1$, where $S_0$ is a rational surface with 16 exceptional curves and

$$c_1^2(S_0) = -7, \quad \chi(\mathcal{O}_{S_0}) = 1, \quad c_2(S_0) = 19.$$

By (1.1), we have

$$K_h^2 = 9, \quad \chi_h = 3, \quad e_h = 27.$$

All the singular fibers of $h$ are $F_0$, $F_1$, $F_\infty$, where $F_\lambda := h^{-1}(\lambda)$. The dual graphs of the minimal normal crossing models $\overline{F}_i$ are as in Figure 3(a)–(c) respectively.

Thus, by (2.5) and Table 2, we have

$$\kappa(h) = 3, \quad \delta(h) = 1, \quad \lambda(h) = \frac{1}{3}.$$

It is easy to see that $h$ is also a fibration of genus 3 from a cyclic triple covering over a ruled surface

$$\varphi_0 \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1, \quad (x, \lambda) \to \lambda,$$

defined by

$$y^3 = (\lambda - x^2)(x^2 - 1).$$

After a semistable reduction, the pullback semistable fibration $f\colon S \to C$ is a cyclic triple covering over a ruled surface. By (2.3) and (2.4), the slope of $f$ is

$$\lambda_f = \frac{\kappa(h)}{\lambda(h)} = 9.$$

### §2.3. Resolution of cyclic coverings

In our paper, a cyclic covering of degree $n$ is a finite covering $\pi\colon Y \to X$ on a smooth algebraic surface $X$ such that the function field $\mathbb{C}(Y)$ of $Y$ is a cyclic extension of $\mathbb{C}(X)$ of degree $n$. The branch locus $R$ of $\pi$ satisfies $R \equiv nL$ for some effective divisor $L$ on $X$.

For the reader's convenience, we will recall some basic facts on cyclic coverings. We refer to [BHPV04, Ta91] for more details.

By a sequence of blowups $\sigma\colon \overline{X} \to X$ at the singular points of $R$, the pullback of $R$ in $\overline{X}$ is a normal crossing divisor. The normalization $\overline{Y}$ of $\overline{X} \times_X Y$ is a cyclic covering $\bar{\pi}\colon \overline{Y} \to \overline{X}$ whose branch locus $\overline{R}$ is a normal crossing divisor contained in the pullback of $R$ in $\overline{X}$. Thus $\overline{Y}$ admits at worst Hirzebruch–Jung singularities which can be resolved directly by Hirzebruch's method, where the resolution is denoted by $h\colon Y' \to \overline{Y}$. Finally, we get the resolution morphism $\sigma' := \bar{\sigma} \circ h\colon Y' \to Y$, which is called *Jung's resolution* (see [PP11] or [LuT14]). Thus $\pi' := \bar{\pi} \circ h\colon Y' \to \overline{X}$ is a generically cyclic covering:

$$
\begin{array}{ccccc}
Y' & \xrightarrow{\ h\ } & \overline{Y} & \xrightarrow{\ \bar{\sigma}\ } & Y \\
 & {}_{\pi'}\searrow & {}_{\bar{\pi}}\downarrow & & \downarrow{}_{\pi} \\
 & & \overline{X} & \xrightarrow{\ \sigma\ } & X.
\end{array}
$$

If $n = 2$ or $3$, then $\pi$ is totally ramified on $R$ and there is a *canonical resolution* $\sigma\colon \overline{X} \to X$ such that the branch locus $\overline{R}$ of $\bar{\pi}$ is smooth and hence $\overline{Y}$ is a smooth surface, i.e., $Y' = \overline{Y}$ (see [BHPV04, Chap. III, Sect. 7] and [Ta91, Sect. II]). However, $h$ is not usually an identity map whenever $n > 3$.

The branch locus $R$ of a cyclic covering $\pi\colon Y \to X$ can be written as follows:

$$R = A_1 + 2A_2 + \cdots + (n-1)A_{n-1},$$

where each $A_i$ is a reduced effective divisor and contains no component of the other $A_j$.

Let $\varphi\colon X \to C$ be a fibration of genus $g_0$. The cyclic covering $\pi$ gives a fibration $\bar{\varphi} = \varphi \circ \sigma \circ \pi'\colon Y' \to C$ of genus $g$ (not necessarily relatively minimal). Let $\Gamma$ be a smooth fiber of $\varphi$ satisfying that

   (i)  $\Gamma$ is a component of $A_1$;

(ii) $\Gamma$ meets transversely with other irreducible components of $R$ at $p_1, \ldots, p_m$.
It is easy to see that $m = (\sum_{i=1}^{n-1} A_i)\Gamma$.

The pullback $F := \pi'^*(\sigma^*(\Gamma))$ of $\Gamma$ is the corresponding fiber of $\bar{\varphi}$.

Now we will illustrate how to obtain the fiber $F$ by the following examples. Since we only use cyclic coverings of degree $n \leq 3$ in the proofs of our theorems, such cyclic coverings will be investigated in detail first.

**Example 2.7.** We assume that $n = 2$. In this case, $R$ is reduced. Since $R \equiv 2L$ and $\Gamma^2 = 0$, $m = R\Gamma$ is even.

To illustrate the canonical resolution, a thin line is used to represent a component contained in $R$ and a dashed line is used to represent a component not contained in $R$:
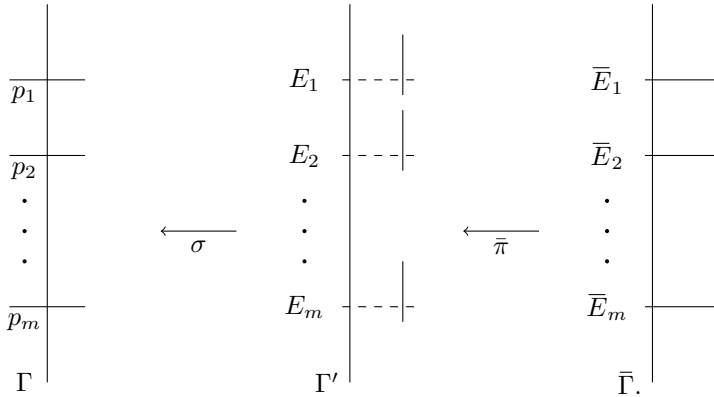


From the canonical resolution, one has
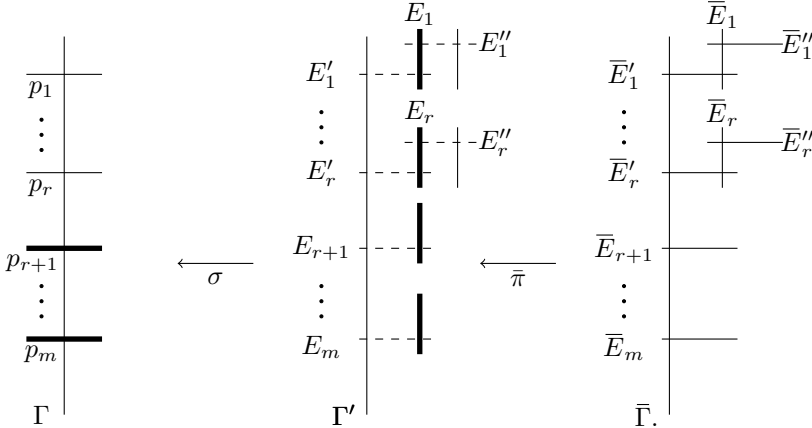
$$\sigma^*\Gamma = \Gamma' + E_1 + \cdots + E_m,$$

where $\Gamma'$ is the strict transform of $\Gamma$ with $(\Gamma')^2 = -m$ and the $E_i$ are $(-1)$-curves respectively. Note that $\bar{\pi}^*\Gamma' = 2\bar{\Gamma}$ and $\bar{\pi}^*E_i = \bar{E}_i$ where $\bar{\Gamma}$ is a smooth curve of genus $g_0$ with $\bar{\Gamma}^2 = -\frac{m}{2}$ and the $\bar{E}_i$ are $(-2)$-curves respectively. Therefore we get

$$F = 2\bar{\Gamma} + \sum_{i=1}^{m} \bar{E}_i.$$

From the Hurwitz formula, $g = 2g_0 - 1 + \frac{m}{2}$.

**Example 2.8.** Assume that $n = 3$. In this case, $R = A_1 + 2A_2$. We assume that $\Gamma$ meets transversely with other components of $A_1$ (resp., $A_2$) at $p_1, \ldots, p_r$ (resp., $p_{r+1}, \ldots, p_m$). One has $r + m \equiv 0 \pmod 3$ by $R \equiv 3L$ and $\Gamma^2 = 0$.

In the following canonical resolution, a thin (resp., thick) line is used to represent a component contained in $A_1$ (resp., $A_2$) and a dashed line is used to represent a component not contained in $R$:



We have

$$\sigma^*\Gamma = \Gamma' + \sum_{i=1}^{r}(E_i + 2E_i' + E_i'') + \sum_{i=r+1}^{m} E_i,$$

where $\Gamma'$ is the strict transform of $\Gamma$ with $(\Gamma')^2 = -m - r$, $E_1, \ldots, E_r$ are $(-3)$-curves, and other exceptional curves are all $(-1)$-curves respectively.

One has $\bar{\pi}^*\Gamma' = 3\bar{\Gamma}$, $\bar{\pi}^*E_i = 3\bar{E}_i$ $(i = 1, \ldots, r)$, $\bar{\pi}^*E_i = \bar{E}_i$ $(i = r+1, \ldots, m)$, $\bar{\pi}^*E_i' = \bar{E}_i'$, and $\bar{\pi}^*E_i'' = \bar{E}_i''$, where $\bar{\Gamma}$ is a smooth curve of genus $g_0$ with $\bar{\Gamma}^2 = -\frac{m+r}{3}$, the $\bar{E}_i$ are $(-1)$-curves $(i = 1, \ldots, r)$, and other components of $F$ are all $(-3)$-curves respectively. Therefore we get

$$F = 3\bar{\Gamma} + \sum_{i=1}^{r}(3\bar{E}_i + 2\bar{E}_i' + \bar{E}_i'') + \sum_{i=r+1}^{m} \bar{E}_i.$$

The minimal normal-crossing model of $F$ can be obtained by contracting $\bar{E}_1, \ldots, \bar{E}_r$. From the Hurwitz formula, $g = 3g_0 - 2 + m$.

**Example 2.9.** In the case of $n > 3$, since $\Gamma$ intersects with $R$ transversely, $\Gamma' := \sigma^*\Gamma$ is also the strict transform of $\Gamma$ in $\bar{X}$. So $\Gamma'$ is a smooth fiber of the fibration $\varphi \circ \sigma \colon \bar{X} \to C$ of genus $g_0$ which passes through $m$ nodes of $\bar{R}$ (i.e., the pullbacks of the intersections $p_i$). Therefore, $F = n\bar{\Gamma} + V$ where $\bar{\Gamma}$ is the pullback component of $\Gamma'$ in $Y'$ and $V$ is an effective divisor supported on the exceptional set of the resolution $h$.

We can describe $V$ more precisely:

Let $p$ be an intersection of $A_i$ and $\Gamma'$; then $\bar{\pi}$ is defined locally by $z^n = xy^i$ in a neighborhood of $p$ where $x$ (resp., $y$) is the local coordinate of $A_i$ (resp., $\Gamma'$). Thus $\bar{\pi}$ is totally ramified over $p$ and the exceptional set of $\bar{\pi}^{-1}(p)$ in $Y'$ is a Hirzebruch–Jung chain with a unique component $B_p$ meeting transversely with $\bar{\Gamma}$. One can find a unique effective divisor $\Theta_p$ supported on this H-J chain satisfying that $\Theta_p B_p = -n$ and $\Theta_p B = 0$ for the other component $B$ of the H-J chain (see [BHPV04, Chap. III, Sect. 5]).

Therefore, from the Zariski lemma, one has $V = \sum_p \Theta_p$, where $p$ runs over all the nodes of $\bar{R}$ lying on $\Gamma'$.

## §3. Construction and slopes

Let $\varphi\colon P \to C$ be a fibration and $\Pi\colon S_0 \to P$ be a cyclic cover with branch locus $R$. By Jung's resolution $\sigma$, one can get a smooth surface $\bar{S}$ with a generically finite cover $\bar{\Pi}\colon \bar{S} \to \overline{P}$, where $\rho\colon \overline{P} \to P$ is a sequence of blowups at the singular points of $R$. Thus one has a fibration $\bar{f} = \varphi \circ \rho \circ \bar{\Pi}\colon \bar{S} \to C$. For convenience, the relatively minimal model $f\colon S \to C$ of $\bar{f}$ is said to be a *covering over $\varphi$ with respect to $\Pi$*:

$$
\begin{array}{ccccc}
\bar{S} & = & \bar{S} & \xrightarrow{\sigma} & S_0 \\
 & & \downarrow{\bar{\Pi}} & & \downarrow{\Pi} \\
 & & \overline{P} & \xrightarrow{\rho} & P \\
 & & & & \downarrow{\varphi} \\
S & \xrightarrow{f} & C & = & C.
\end{array}
$$

After a base change $\pi\colon \widetilde{C} \to C$, we have the pullback fibrations $\tilde{f}\colon \tilde{S} \to \widetilde{C}$ (relatively minimal) and $\tilde{\varphi}\colon \widetilde{P} \to \widetilde{C}$. In fact, $\tilde{f}$ is a covering of $\tilde{\varphi}$ with respect to the covering $\widetilde{\Pi}\colon \widetilde{P} \times_P S_0 \to \widetilde{P}$ whose branch locus is $\tilde{\pi}^*R$. For convenience, we say that $(\tilde{f}, \tilde{\varphi}, \widetilde{\Pi})$ is the pullback of $(f, \varphi, \Pi)$ by the base change $\pi$:

$$
\begin{array}{ccc}
\tilde{S} \dashrightarrow^{1:1} \widetilde{P} \times_P S_0 & \dashrightarrow & S_0 \\
\downarrow{\widetilde{\Pi}} & & \downarrow{\Pi} \\
\widetilde{P} \dashrightarrow^{\tilde{\pi}} & \dashrightarrow & P \\
\downarrow{\tilde{\varphi}} & & \downarrow{\varphi} \\
\widetilde{C} & \xrightarrow{\pi} & C.
\end{array}
$$

**Lemma 3.1.** *Assume that $\Pi$ is a cyclic cover of degree $n$ with branch locus $R \equiv nL$ for some divisor $L$ on $P$. Let $A$ be a very ample divisor on $C$ of degree $e$ and $q_1 + \cdots + q_{ne}$ be a sufficiently general element of $|nA|$. Then we can get a cyclic cover $\Pi' \colon S_0' \to P$ whose branch locus is*

$$R' = R + \varphi^*(q_1 + \cdots + q_{ne})$$

*and a relatively minimal fibration $f' \colon S' \to C$ with respect to $\Pi'$. In particular, $\Pi$, $\Pi'$ coincide outside $\bigcup_{i=1}^{ne} \varphi^{-1}(q_i)$.*

*Furthermore, after a base change $\pi \colon \widetilde{C} \to C$ of degree $n$ which is totally ramified over $q_1, \ldots, q_{ne}$, the pullback fibrations of $f$, $f'$ coincide.*

*Proof.* The existence of such a cyclic cover derives from the obvious fact that $R' \equiv n(L + \varphi^*(A))$.

In order to prove the second part, it is enough to show that the pullback fibrations of $f$, $f'$ come from the same cover. In fact, the branch locus of $\widetilde{\Pi}'$ is

$$\tilde{\pi}^* R' = \tilde{\pi}^* R + \sum_{i=1}^{ne} \tilde{\varphi}^* \pi^*(q_i) = \tilde{\pi}^* R + n \sum_{i=1}^{ne} \tilde{\varphi}^*(\tilde{q}_i),$$

where $\tilde{q}_i = \pi^{-1}(q_i)$. So these two covers coincide by a normalization. $\square$

**Lemma 3.2.** *Under the assumptions and notation in Lemma 3.1, we have that*

(1) *the modular invariants of $f$ and $f'$ coincide respectively, i.e.,*

$$\kappa(f') = \kappa(f), \quad \lambda(f') = \lambda(f), \quad \delta(f') = \delta(f);$$

(2) *the Chern numbers of the $F_i$ coincide with each other respectively, where $F_i = f'^*(q_i)$ $(i = 1, \ldots, ne)$, i.e.,*

$$c_1^2(F_i) = c_1^2(F_j), \quad c_2(F_i) = c_2(F_j), \quad \chi_{F_i} = \chi_{F_j} \quad (i \neq j).$$

*For convenience, let $F$ be one of the singular fibers $F_i$. Therefore, the slope of $f'$ is*

$$(3.1) \qquad \qquad \lambda_{f'} = \frac{\kappa(f) + ne \cdot c_1^2(F)}{\lambda(f) + ne \cdot \chi_F}.$$

*Proof.* Let $\pi \colon \widetilde{C} \to C$ be the cyclic cover of degree $n$ totally ramified over $q_1, \ldots, q_{ne}$. Denote by $\tilde{f} \colon \widetilde{S} \to \widetilde{C}$ (resp., $\tilde{f}' \colon \widetilde{S}' \to \widetilde{C}$ or $\tilde{\varphi} \colon \widetilde{P} \to \widetilde{C}$) the pullback fibration of $f$ (resp., $f'$ or $\varphi$) with respect to $\pi$. We assume that they are relatively minimal.

(1) By Lemma 3.1, the two fibrations $\tilde{f}$ and $\tilde{f}'$ are the same. So

$$\kappa(\tilde{f}) = \kappa(\tilde{f}'), \quad \lambda(\tilde{f}) = \lambda(\tilde{f}'), \quad \delta(\tilde{f}) = \delta(\tilde{f}'),$$

and thus, by (2.4),

$$\kappa(f') = \kappa(f), \quad \lambda(f') = \lambda(f), \quad \delta(f') = \delta(f).$$

(2) Given a general point $q \in C$. The fiber $\Gamma = \varphi^*(q)$ of $\varphi$ is exactly the one in Example 2.7 (resp., Example 2.8 or Example 2.9) for $n = 2$ (resp., $n = 3$ or $n > 3$). Hence the Chern numbers of the fiber $F = f'^*(q)$ are determined uniquely by (2.2).

Combining (1), (2), and (2.5), we obtain (3.1). □

**Remark 3.3.** In the proof of Lemma 3.2(2), if $(n, g_0, m) = (2, 0, 2(g + 2))$ (resp., $(n, g_0, m) = (2, 2, 2(g - 3))$), then the dual graph of $F_i$ is just Figure 1 (resp., Figure 2). Similarly, if $(n, g_0, r, m) = (3, 0, 4, 5)$, then the dual graph of $F_i$ is Figure 3(d).

**Corollary 3.4.** *Under the notation and assumptions in Lemma 3.1, denote by $F$ one of the singular fibers of $f'$ over the $q_i$. If $f$ is semistable of genus $g$ and $\lambda_F < \lambda_f$, then for each rational number $r \in (\lambda_F, \lambda_f]$, there is a fibration of genus $g$ with slope $r$.*

*Proof.* Take a base change $\pi \colon \widetilde{C} \to C$ of degree $d$ ramified over noncritical points of $f$. Let $(\tilde{f}, \tilde{\varphi}, \widetilde{\Pi})$ be the pullback of $(f, \varphi, \Pi)$ with respect to $\pi$. Note that $\tilde{f}$ is semistable and $\widetilde{\Pi}$ is also a cyclic cover of degree $n$.

By (2.4), one has

$$(3.2) \qquad \frac{\kappa(\tilde{f})}{\kappa(f)} = \frac{\lambda(\tilde{f})}{\lambda(f)} = d.$$

So $\lambda_{\tilde{f}} = \lambda_f$.

Applying Lemma 3.2 and (3.2) to $(\tilde{f}, \tilde{\varphi}, \widetilde{\Pi})$, one gets a new triple $(f', \tilde{\varphi}, \Pi')$. The slope of $f'$ is

$$(3.3) \qquad \lambda_{f'} = \frac{d \cdot \kappa(f) + ne \cdot c_1^2(F)}{d \cdot \lambda(f) + ne \cdot \chi_F}.$$

For $r \in (\lambda_F, \lambda_f)$, one can find $d$, $e$ such that

$$\frac{d}{ne} = \frac{\chi_F}{\chi_f} \cdot \frac{r - \lambda_f}{\lambda_f - r}.$$

Thus the slope of $f'$ in (3.3) is $r$. □

## §4. The proof of Theorem 1.3

*Proof of Theorem* 1.3. We prove the theorem in two steps.

*Step* 1. Firstly, we consider the case of hyperelliptic fibrations of genus $g \geq 2$. That is, for each $g \geq 2$ and $r \in \mathbb{Q} \cap [\lambda_m(g), \lambda_M^h(g)]$, we will show that there exists a hyperelliptic fibration of genus $g$ with slope $r$.

By [LT13], we can find a semistable hyperelliptic fibration $f \colon S \to C$ of genus $g$ with slope $\lambda_f = \lambda_M^h(g)$. The fibration $f$ is a covering over a ruled surface $\varphi \colon P \to C$ with respect to a double cover $\Pi \colon S_0 \to P$. Thus we may assume that $r \in \mathbb{Q} \cap (\lambda_m(g), \lambda_M^h(g))$.

Let $F_h$ be the new singular fiber as in Lemma 3.2(2). The dual graph of $F_h$ is Figure 1 by Remark 3.3. From Example 2.4, one gets $\lambda_{F_h} = \lambda_m(g) < \lambda_M^h(g)$. Therefore, the result for hyperelliptic fibrations is obtained by Corollary 3.4.

*Step* 2. Since each fibration of genus 2 is hyperelliptic, we only need to consider $g \geq 4$. In this case, [GD-H91] gives a Kodaira fibration $f \colon S \to C$ of genus $g$ which is a covering over a fibration $\varphi \colon P \to C$ of genus 2 with respect to a double cover $\Pi \colon S_0 \to P$ whose branch locus consists of $2(g - 3)$ sections of $\varphi$.

Let $F_g$ be the new singular fiber in Lemma 3.2(2) in this case. The dual graph of $F_g$ is Figure 2; see Remark 3.3. Similarly to above, there exists a fibration of genus $g$ with slope $r$, for each $r \in \mathbb{Q} \cap (\lambda_{F_g}, \lambda_M(g))$.

From Example 2.4, one has

$$\lambda_m(g) < \lambda_{F_g} = \frac{4g - 4}{g - 2} < \lambda_M^h(g).$$

Thus

$$[\lambda_m(g), \lambda_M^h(g)] \cup (\lambda_{F_g}, \lambda_M(g)] = [\lambda_m(g), \lambda_M(g)].$$

Now we finish the proof by combining with Step 1. □

## §5. The geography of slopes for $g = 3$

The construction of Kodaira fibration in [GD-H91] does not work for $g = 3$. Zaal [Za95] gave another construction: the obtained Kodaira fibration $f \colon S \to C$ of genus 3 satisfies that $S$ is a quadruple cover over a ruled surface. Unfortunately, this quadruple cover is not cyclic.

The result we obtain in this case is Theorem 1.4, whose proof is as follows.

*Proof of Theorem* 1.4. We take a semistable fibration $f$ of genus 3 with slope $\lambda_f = 9$ as in Example 2.6. Let $F$ be the new singular fiber in Lemma 3.2(2) in this case. The dual graph of $F$ is Figure 3(d) by Remark 3.3. Since $f$ derives from

a cyclic covering over a ruled surface, by Corollary 3.4, there exists a fibration of genus 3 with slope $r$ for each $r \in \mathbb{Q} \cap (\lambda_F, 9)$.

From Example 2.5, one can see that

$$\lambda_m(3) = \frac{8}{3} < \lambda_F = 3 < \lambda_M^h(3) = 8.5.$$

Then we know that

$$[\lambda_m(3), \lambda_M^h(g)] \cup (\lambda_F, 9] = [\lambda_m(3), 9].$$

Now we finish the proof by combining with Step 1 in the proof of Theorem 1.3.  □

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