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Regularization of Relative Holonomic $\mathcal{D}$-Modules

by

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Abstract

Let X and S be complex analytic manifolds where S plays the role of a parameter space. Using the sheaf $\mathcal{D}^\infty_{X\times S/S}$ of relative differential operators of infinite order, we construct functorially the regular holonomic $\mathcal{D}_{X\times S/S}$-module $\mathcal{M}_{\mathrm{reg}}$ associated to a relative holonomic $\mathcal{D}_{X\times S/S}$-module $\mathcal{M}$, extending to the relative case classical theorems by Kashiwara–Kawai: denoting by $\mathcal{M}^\infty$ the tensor product of $\mathcal{M}$ by $\mathcal{D}^\infty_{X\times S/S}$ we make $\mathcal{M}^\infty$ explicit in terms of the sheaf of holomorphic solutions of $\mathcal{M}$. As a consequence of the relative Riemann–Hilbert correspondence we conclude that $\mathcal{M}^\infty$ and $\mathcal{M}^\infty_{\mathrm{reg}}$ are isomorphic.

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§1. Introduction

The relative framework we deal with is associated to a projection

$$p\colon X\times S\to S,$$

where X and S are complex manifolds. Throughout this work we identify the relative cotangent bundle $T^*(X\times S/S)$ to $T^*X\times S$ and d_X and d_S will denote respectively the complex dimension of X and of S. Let $\mathcal{D}_{X\times S/S}$ be the subsheaf of $\mathcal{D}_{X\times S}$ of operators commuting with $p^{-1}\mathcal{O}_S$ and let $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_{X\times S/S})$ be the abelian category of coherent $\mathcal{D}_{X\times S/S}$-modules. A $\mathcal{D}_{X\times S/S}$ -holonomic module is a coherent $\mathcal{D}_{X\times S/S}$-module whose characteristic variety is contained in a product $\Lambda\times S$ where Λ is $\mathbb{C}^*$-conic analytic lagrangian in T^*X (cf. [22, 20, 15]). The datum of a strict (i.e. a $p^{-1}\mathcal{O}_S$-flat) holonomic $\mathcal{D}_{X\times S/S}$-module is equivalent to

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the datum of a flat family of holonomic $\mathcal{D}_X$-modules with characteristic variety contained in Λ.

Let $\mathcal{D}^\infty_{X\times S/S}$ denote the subsheaf of $\mathcal{D}^\infty_{X\times S}$ of operators commuting with $p^{-1}\mathcal{O}_S$. As pointed out in [21, Rem. 2, p. 406], the sheaf of rings $\mathcal{D}^\infty_{X\times S/S}$ is faithfully flat over $\mathcal{D}_{X\times S/S}$. Indeed, the method of the proof of [21, Thm. 3.4.1] which concerns the relative microdifferential case out of the zero section of $T^*(X\times S/S)$ adapts to the sheaves $\mathcal{D}_{X\times S/S}$ and $\mathcal{D}^\infty_{X\times S/S}$.

The relative setting means here that $\mathcal{D}_X$ and $\mathcal{D}^\infty_X$ are replaced respectively by $\mathcal{D}_{X\times S/S}$ and $\mathcal{D}^\infty_{X\times S/S}$ and that we consider relative holonomic modules. Our main result is Theorem 8, which proves a relative version of the following Kashiwara–Kawai theorem ([9, Thm. 1.4.9]): Let $\mathcal{D}^\infty_X$ denote the sheaf of linear differential operators on X with possibly infinite order. To any holonomic $\mathcal{D}_X$-module one associates $\mathcal{M}^\infty := \mathcal{D}^\infty_X \otimes_{\mathcal{D}_X} \mathcal{M}$ and, if $F = \mathrm{Sol}\,\mathcal{M}$, then $\mathcal{M}^\infty \simeq \mathrm{R}\mathcal{H}\mathrm{om}(F,\mathcal{O}_X)$.

The same authors introduce in [9] a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}_{\mathrm{reg}}$ contained in $\mathcal{M}^\infty$ and prove in [9, Thm. 5.2.1] a $\mathcal{D}^\infty_X$-isomorphism:

$$\mathcal{M}^\infty \simeq \mathcal{M}^\infty_{\mathrm{reg}}. \tag{1}$$

In (b) of Theorem 12 we extend this result to the relative setting. The proof is based on the relative Riemann–Hilbert correspondence obtained in [3] and [4], since one previous step is to prove that $(\bullet)_{\mathrm{reg}} \simeq \mathrm{RH}^S(\mathrm{Sol}\,\bullet)[-d_X]$. The latter isomorphism is a contribution to the understanding of the functor RH^S.

The task is not trivial, although we dispose of a good notion of regularity recalled below, as well as of the inspiration provided by the techniques in [9]. Let us explain why:

One big difference from the absolute to the relative case is that the triangulated category of $\mathcal{D}_{X\times S/S}$-complexes having bounded holonomic cohomologies ($\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$) is not stable under the inverse image functor by morphisms $f\times \mathrm{Id}\colon X'\times S\to X\times S$. Such constraint entails a loss of several functorial properties (for instance localization, algebraic supports cohomology).

The notions of S-$\mathbb{R}$- and S-$\mathbb{C}$-constructibility were introduced in [15] for objects in $\mathsf{D}^{\mathrm{b}}(p^{-1}\mathcal{O}_S)$, as well as a natural duality and a middle perversity t-structure on the triangulated category $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ whose objects have S-$\mathbb{C}$-constructible cohomologies. A perverse object with perverse dual is then equivalent to the datum of a flat family of perverse sheaves on X.

The lack of functorialities in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ prevents us from stating an irregular relative Riemann–Hilbert correspondence by simply adapting the strategies used in the absolute case as treated by D'Agnolo–Kashiwara (cf. [1]). For a satisfactory functorial behavior, regularity is necessary as proved in [3, 4].

Recall that a regular holonomic $\mathcal{D}_{X\times S/S}$-module is a holonomic $\mathcal{D}_{X\times S/S}$-module satisfying the following condition: the (derived) holomorphic restriction to each fiber of p is a regular holonomic complex on X. We also consider the associated triangulated category $(\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S}))$ of complexes having bounded regular holonomic cohomologies.

It is then natural to ask what kind of "regularity" can be associated to any holonomic $\mathcal{D}_{X\times S/S}$-module.

Recall that the relative Riemann–Hilbert equivalence was first proved in [3] assuming that $d_S = 1$:

The functor ${}^{\mathrm{p}}\mathrm{Sol}\colon \mathcal{M} \mapsto \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{O}_{X\times S})[d_X]$ from $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ to $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ admits a right and left adjoint denoted by RH^S and thus ${}^{\mathrm{p}}\mathrm{Sol}$ is an equivalence of categories.

In [4], the same authors proved that this equivalence holds true for arbitrary d_S.

In the absolute case (meaning that $S = \mathrm{pt}$) we recover Kashiwara's regular Riemann–Hilbert correspondence, and, if $X = \mathrm{pt}$, we get the natural duality on the bounded derived category of complexes with $\mathcal{O}_S$-coherent cohomologies.

We now make our results precise:

If $\mathcal{M}$ is a holonomic $\mathcal{D}_{X\times S/S}$-module, we define

$$\mathcal{M}^\infty := \mathcal{D}^\infty_{X\times S/S} \otimes_{\mathcal{D}_{X\times S/S}} \mathcal{M}$$

and we generalize this definition by flatness to $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$.

In our main result (Theorem 8) we prove that if $\mathcal{M}$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ and $F = {}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M}$ then $\mathcal{M}^\infty \simeq \mathrm{R}\mathcal{H}\mathrm{om}_{p^{-1}\mathcal{O}_S}(F, \mathcal{O}_{X\times S})$ (to compare with [9, Thm. 1.4.9]). As a consequence one concludes in Theorem 12 that if $\mathcal{M}$ is a holonomic $\mathcal{D}_{X\times S/S}$-module then $\mathcal{M}_{\mathrm{reg}} \simeq \mathrm{RH}^S({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M})$ and so (1) holds true in this setting.

The simplest example is the following: for a submanifold Z of X, one has

$$\begin{aligned}&\mathrm{RH}^S(\mathbb{C}_{Z\times S} \otimes p^{-1}\mathcal{O}_S)^\infty[-d_X]\\ &\quad\simeq \mathrm{T}\mathcal{H}\mathrm{om}(\mathbb{C}_{Z\times S}, \mathcal{O}_{X\times S})^\infty \simeq B^\infty_{Z\times S|X\times S}[-d]\\ &\quad\simeq \mathrm{R}\mathcal{H}\mathrm{om}(\mathbb{C}_{Z\times S}, \mathcal{O}_{X\times S}),\end{aligned}$$

where d is the codimension of Z and $\mathrm{T}\mathcal{H}\mathrm{om}$ was introduced in [11].

Another example is provided by [9, p. 814], replacing $a \in \mathbb{C}$ by a holomorphic function $a(s)$ without zeros on some open $S := \Omega \subset \mathbb{C}$. For $X = \mathbb{C}$, we consider the $\mathcal{D}_{X\times S/S}$-module (holonomic, non-regular) defined by

$$(x^2\partial_x - a(s))u(x,s) = 0.$$

We then obtain (cf. [9, p. 815]) an equivalent system substituting the generator u by $u_0 = u$ and introducing $u_1 = -x\partial_x u$,

$$(2)\qquad \begin{cases} x\partial_x u_0 + u_1 = 0, \\ -a(s)u_0 - xu_1 = 0. \end{cases}$$

After multiplication by matrices in $\mathcal{D}^\infty_{X\times S/S}$ (the matrices provided by [9] which now depend on the parameter s), one concludes a $\mathcal{D}^\infty_{X\times S/S}$-isomorphism from the $\mathcal{D}^\infty_{X\times S/S}$-module extension of (2) to the $\mathcal{D}^\infty_{X\times S/S}$-module (with generators w_0, w_1) extension of the regular holonomic $\mathcal{D}_{X\times S/S}$-module

$$(3)\qquad \begin{cases} xw_0 - a(s)w_1 = 0, \\ x\partial_x w_1 = 0. \end{cases}$$

We remark that [9] uses microlocal techniques for the proof of the regularity of $\mathcal{M}_{\mathrm{reg}}$. With the more recent notion of microsupport ([10]) and the results in [22], the necessary tools in the relative framework (see Section 3 on technical lemmas) are easier to prove. Together with the relative Riemann–Hilbert correspondence, our task is much simplified; in particular, we no longer need to microlocalize.

§2. A short reminder on the relative Riemann–Hilbert correspondence

Below we summarize the background from [15, 16, 3, 4] that we shall need in the sequel.

§2.1. Holonomic and regular holonomic $\mathcal{D}_{X\times S/S}$-modules

(a) We say that a $p^{-1}\mathcal{O}_S$-module is strict if it is flat over $p^{-1}\mathcal{O}_S$.

(b) We recall that $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_{X\times S/S})$ is holonomic if the characteristic variety $\mathrm{Char}(\mathcal{M})$ is contained in $\Lambda \times S$, where Λ is analytic $\mathbb{C}^*$-conic lagrangian subset of T^*X; we denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ the associated triangulated category whose objects are the bounded complexes with holonomic cohomologies.

(c) There is a well-defined duality functor

$$\boldsymbol{D}\colon\ \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})^{\mathrm{op}}$$

given by

$$\boldsymbol{D}\mathcal{M} \coloneqq \mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{D}_{X\times S/S} \otimes_{\mathcal{O}_{X\times S}} \Omega^{\otimes^{-1}}_{X\times S/S})[d_X],$$

where $\Omega_{X\times S/S}$ denotes the sheaf of relative differential forms of maximal degree.

(d) $\boldsymbol{D}$ is an involution, i.e. $\boldsymbol{D}\boldsymbol{D} = \mathrm{Id}$.

(e) We recall a tool introduced in [15], the holomorphic restriction to each fiber of p:

$$\forall\, s \in S, \quad \mathrm{Li}_s^*(\bullet) := \bullet \overset{L}{\otimes}_{p^{-1}\mathcal{O}_S} p^{-1}(\mathcal{O}_S/\mathcal{J}_s),$$

where $\mathcal{J}_s$ is the maximal ideal of functions vanishing in s.

(f) A Nakayama lemma variation: Let $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ and assume that $Li^*_{s_o}\mathcal{M} = 0$ for each $s_o \in S$. Then $\mathcal{M} = 0$.

(g) Let $\mathcal{M}$ be an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$. Then $\boldsymbol{D}\mathcal{M}$ is concentrated in degree zero and $\mathcal{H}^0\boldsymbol{D}\mathcal{M}$ is strict if and only if $\mathcal{M}$ is itself concentrated in degree zero and $\mathcal{H}^0\mathcal{M}$ is a strict $\mathcal{D}_{X\times S/S}$-module.

(h) We say that $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_{X\times S/S})$ is regular holonomic if it is holonomic and for all $s \in S$, $\mathrm{Li}_s^*\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_X)$; we denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ the associated triangulated full subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$.

(i) $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ is stable by duality.

(j) $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ and $\mathrm{Mod}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ are closed under taking extensions in $\mathrm{Mod}(\mathcal{D}_{X\times S/S})$ and subquotients in $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_{X\times S/S})$.

§2.2. *S*-constructibility

We say that a sheaf L of $p^{-1}\mathcal{O}_S$-modules is S-locally constant coherent if, locally on $X \times S$, L is isomorphic to $p^{-1}G$, where G is an $\mathcal{O}_S$-coherent module. Such an L is also called an S-local system. We recall the following full triangulated subcategories of $\mathsf{D}^{\mathrm{b}}(p^{-1}\mathcal{O}_S)$:

- An object $F \in \mathsf{D}^{\mathrm{b}}(p^{-1}\mathcal{O}_S)$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ if there exists a $\mathbb{C}$-analytic stratification $(X_\alpha)_{\alpha\in A}$ of X, such that for all $j \in \mathbb{Z}$, for all $\alpha \in A$, $\mathcal{H}^j F|_{X_\alpha\times S}$ is S-locally constant coherent. We say for short that F is S-$\mathbb{C}$-constructible.
- Replacing $\mathbb{C}$-analyticity by subanalyticity with respect to the real analytic manifold $X_{\mathbb{R}}$ underlying X, we obtain the notion of S-$\mathbb{R}$-constructibility and the corresponding triangulated category $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p^{-1}\mathcal{O}_S)$. The category $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ is a full subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p^{-1}\mathcal{O}_S)$.
- If $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p^{-1}\mathcal{O}_S)$ then for each $x \in X$, $F|_{\{x\}\times S}$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S)$.
- There is a natural duality functor $\boldsymbol{D}\colon \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p^{-1}\mathcal{O}_S) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p^{-1}\mathcal{O}_S)^{\mathrm{op}}$ which is an involution given by

$$\boldsymbol{D}F = \mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(F, p^{-1}\mathcal{O}_S)[2d_X].$$

- $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ is stable by duality.

§2.3. A middle perversity t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$

Here we consider the two subcategories ${}^{\mathrm{p}}\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ and ${}^{\mathrm{p}}\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ defined as follows:

We have $F \in {}^{\mathrm{p}}\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ (resp. $F \in {}^{\mathrm{p}}\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$) if for an adapted μ-stratification $(X_\alpha)_{\alpha\in A}$, noting $i_\alpha\colon X_\alpha \hookrightarrow X$,

$$\forall\,\alpha \text{ and } \forall\, j > -\dim(X_\alpha), \quad \mathcal{H}^j(i_\alpha^{-1}F) = 0$$
$$(\text{resp. } \forall\,\alpha \text{ and } \forall\, j < -\dim(X_\alpha), \quad \mathcal{H}^j(i_\alpha^{!}F) = 0.)$$

We say that F of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ is *perverse* if $F \in {}^{\mathrm{p}}\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ and $F \in {}^{\mathrm{p}}\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$, that is, F belongs to the heart of the t-structure defined above.

Remark 1. Note that $\boldsymbol{D}$ is not t-exact for this t-structure; in particular, it does not preserve perversity.

Theorem 2 ([16]). *For a given object $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$, F and $\boldsymbol{D}F$ are perverse if and only if for all $s_o \in S$, $\mathrm{Li}^*_{s_o}(F)$ is perverse in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$.*

§2.4. Link with holonomicity

We have the following link with holonomic $\mathcal{D}_{X\times S/S}$-modules. Let us note ${}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M} = \mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{O}_{X\times S})[d_X]$ and ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} = \mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{O}_{X\times S}, \mathcal{M})[d_X]$.

Then (cf. [2, 15, 16]) we have the following properties:

- $\mathrm{Sol}, \mathrm{DR}\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ take values in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ and $\boldsymbol{D}\,{}^{\mathrm{p}}\mathrm{Sol} = {}^{\mathrm{p}}\mathrm{DR} = {}^{\mathrm{p}}\mathrm{Sol}\,\boldsymbol{D}$.
- If $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ then ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is perverse (cf. [2, Thm. 4.1]).
- If F is such that $\boldsymbol{D}F$ is perverse then $\mathrm{RH}^S(F)$ is concentrated in degree zero (cf. [2, Thm. 4.1]). In particular, for any holonomic $\mathcal{D}_{X\times S/S}$-module, $\mathrm{RH}^S({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M})$ is concentrated in degree zero.
- Given $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$, $\mathcal{M}$ and $\boldsymbol{D}\mathcal{M}$ are strict $\mathcal{D}_{X\times S/S}$-modules if and only if ${}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M}$ and ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} = \boldsymbol{D}\,{}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{M}$ are perverse.

§2.5. The functor RH^S

With the subanalytic tools developed in [13, 14], the functor RH^S was first introduced in [16], followed by [3] (case $d_S = 1$) and by [4] (general case). Kashiwara's functor RH (cf. [7]) is recovered with $d_S = 0$. Below we give a short reminder of its construction and main results:

Let $\rho_S\colon X\times S \to X_{\mathrm{sa}}\times S$ be the natural morphism of sites introduced in [14]. The functor ρ_S^{-1} admits a left adjoint $\rho_{S!}$ which is exact. We note $\mathcal{O}^{t,S}_{X\times S}$ the relative subanalytic sheaf on $X_{\mathrm{sa}} \times S$ associated in [14] to the subanalytic sheaf $\mathcal{O}^t_{X\times S}$ on $(X \times S)_{\mathrm{sa}}$ (introduced in [12]; see also [18]).

The functor RH^S on $\mathsf{D}^b(p^{-1}\mathcal{O}_S)^{\mathrm{op}}$ is given by

$$\mathrm{RH}^S(\bullet) \coloneqq \rho_S^{-1}\,\mathrm{R}\mathcal{H}\mathrm{om}_{\rho_{S*}p^{-1}\mathcal{O}_S}(R\rho_{S*}(\bullet), \mathcal{O}^{t,S}_{X\times S})[d_X].$$

Theorem 3 ([16, 3, 4]). *We have the following properties:*

(a) RH^S *induces an equivalence of categories:* $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ *compatible with duality.*

(b) F *is perverse with a perverse dual if and only if* $\mathrm{RH}^S(F)$ *is strict and concentrated in degree zero.*

(c) *For* $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ *and* $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$, *we have a natural isomorphism in* $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$,

$$\begin{aligned}&\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathrm{RH}^S(F)[-d_X])\\ &\qquad \xrightarrow{\sim} \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathrm{R}\mathcal{H}\mathrm{om}_{p^{-1}\mathcal{O}_S}(F, \mathcal{O}_{X\times S})).\end{aligned}$$

§2.6. Topological aspects of $\mathcal{O}_S$

The sheaf $\mathcal{O}_S$ is made up of complete bornological algebras (multiplicatively convex sheaf of Fréchet algebras over S). In the category of sheaves of complete bornological modules over $\mathcal{O}_S$ (denoted by $\mathrm{Born}(\mathcal{O}_S)$), Houzel (cf. [6]) introduced a notion of tensor product $\bullet\,\widehat{\otimes}_{\mathcal{O}_S}\bullet$. To the latter one associates a family of functors $\bullet\,\widehat{\otimes}\,\mathcal{M}$ on the category of bornological vector spaces, depending functorialy on $\mathcal{M} \in \mathrm{Born}(\mathcal{O}_S)$ (cf. [22, Sect. 3.4]). We have

$$\mathcal{O}_{X\times S}|_{\{x\}\times S} \simeq \mathcal{O}_{X,x}\,\widehat{\otimes}\,\mathcal{O}_S. \tag{4}$$

Then (4) shows that $\mathcal{O}_{X\times S}|_{\{x\}\times S}$ is a so-called FN-free as well as a DFN-free $\mathcal{O}_S$-module (cf. [22, p. 25] for the definition and also [19]).

In particular, given another complex manifold Y, we have

$$\mathcal{O}_{X\times Y\times S}|_{\{(x,y)\}\times S} \simeq (\mathcal{O}_{X\times S}|_{\{x\}\times S})\,\widehat{\otimes}_{\mathcal{O}_S}\,(\mathcal{O}_{Y\times S}|_{\{y\}\times S}). \tag{5}$$

§3. Technical lemmas

In order to prove the main theorem we shall need the following results:

§3.1. Complements on S-$\mathbb{R}$-constructible sheaves

We refer to [10, Chap. VIII] for the background on constructibility.

Notation 4. For short we shall keep the notation p as well as $p^{-1}\mathcal{O}_S$ without referring to the manifold X, whenever there is no risk of ambiguity.

Let X and Y be complex manifolds. Let $q_1\colon X\times Y\times S\to X\times S$ be the first projection and $q_2\colon X\times Y\times S\to Y\times S$ be the second projection, which is illustrated by the following commutative diagram:

$$\begin{array}{ccc} X\times Y\times S & \xrightarrow{q_1} & X\times S \\ {\scriptstyle q_2}\downarrow & \searrow^{p} & \downarrow{\scriptstyle p} \\ Y\times S & \xrightarrow{p} & S. \end{array} \tag{6}$$

Lemma 5. *For any $F\in\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p^{-1}\mathcal{O}_S)$ on $X\times S$ and any object $\mathcal{G}$ of $\mathsf{D}^{\mathrm{b}}(p^{-1}\mathcal{O}_S)$ on $Y\times S$, the functorial morphism*

$$\begin{aligned} T(F) &:= q_1^{-1}\,\mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(F,p^{-1}\mathcal{O}_S)\otimes^{L}_{p^{-1}\mathcal{O}_S} q_2^{-1}\mathcal{G} \\ &\to T'(F) := \mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(q_1^{-1}F,q_2^{-1}\mathcal{G}) \end{aligned} \tag{7}$$

is an isomorphism.

Proof. The proof is now simpler than that of [9, Lem. B.3] since we dispose of the notion of microsupport and of its properties (cf. [10, Chap. V]). It is sufficient to check the isomorphism locally. Furthermore, arguing by induction on the length of F, we may assume that F is in degree zero, that is, F is an S-$\mathbb{R}$-constructible sheaf.

We recall the following result (cf. Lemma A.9 in [5], which is the complete version of [4]).

Lemma 6. *Let F be an S-$\mathbb{R}$-constructible sheaf on $X\times S$. Then there exist*

- *a locally finite covering $(U(\sigma))_{\sigma\in\Delta}$ of X by open subanalytic relatively compact subsets of X,*
- *for each $\sigma\in\Delta$ a coherent $\mathcal{O}_S$-module $G_\sigma(F)$ on S,*
- *and an epimorphism $\bigoplus_{\sigma\in\Delta}\mathbb{C}_{U(\sigma)}\boxtimes G_\sigma(F)\to F$.*

Let us assume for a moment that $F=\mathbb{C}_U\boxtimes G$ for some open relatively compact subanalytic subset U of X and for some coherent $\mathcal{O}_S$-module G. In that case, the proof of Lemma 5 is as follows. Regarding the left-hand term of (7) we have a chain of isomorphisms:

$$\begin{aligned} &q_1^{-1}\,\mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(\mathbb{C}_U\boxtimes G,p^{-1}\mathcal{O}_S)\otimes^{L}_{p^{-1}\mathcal{O}_S} q_2^{-1}\mathcal{G} \\ &\quad\simeq q_1^{-1}\,\mathrm{R}\mathcal{H}om(\mathbb{C}_{U\times S},\mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(p^{-1}G,p^{-1}\mathcal{O}_S))\otimes^{L}_{p^{-1}\mathcal{O}_S} q_2^{-1}\mathcal{G} \\ &\quad\underset{\text{(a)}}{\simeq} q_1^{-1}\,\mathrm{R}\mathcal{H}om(\mathbb{C}_{U\times S},\mathbb{C}_{X\times S})\otimes q_1^{-1}\,\mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(p^{-1}G,p^{-1}\mathcal{O}_S)\otimes^{L}_{p^{-1}\mathcal{O}_S} q_2^{-1}\mathcal{G} \\ &\quad\underset{\text{(b)}}{\simeq} q_1^{-1}\,\mathrm{R}\mathcal{H}om(\mathbb{C}_{U\times S},\mathbb{C}_{X\times S})\otimes \mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(p^{-1}G,q_2^{-1}\mathcal{G}). \end{aligned}$$

Isomorphism (a) follows by [10, Prop. 5.4.14(ii)] and isomorphism (b) follows by the coherence of G.

Similarly, the right-hand term of (7) becomes isomorphic to

$$\begin{aligned}&\mathrm{R}\mathcal{H}\mathrm{om}(q_1^{-1}\mathbb{C}_{U\times S}, \mathrm{R}\mathcal{H}\mathrm{om}_{p^{-1}\mathcal{O}_S}(p^{-1}G, q_2^{-1}\mathcal{G}))\\&\quad\simeq \mathrm{R}\mathcal{H}\mathrm{om}(\mathbb{C}_{U\times Y\times S}, \mathrm{R}\mathcal{H}\mathrm{om}_{p^{-1}\mathcal{O}_S}(p^{-1}G, q_2^{-1}\mathcal{G})).\end{aligned}$$

We have

$$q_1^{-1}\,\mathrm{R}\mathcal{H}\mathrm{om}(\mathbb{C}_{U\times S}, \mathbb{C}_{X\times S}) \simeq \mathrm{R}\mathcal{H}\mathrm{om}(\mathbb{C}_{U\times Y\times S}, \mathbb{C}_{X\times Y\times S}).$$

Thus, for $F = \mathbb{C}_U \boxtimes G$, Lemma 5 follows by [10, Prop. 5.4.14(ii)].

As a consequence, Lemma 5 holds true for sheaves of the form

$$(*)\qquad \bigoplus_{\sigma\in\Delta} \mathbb{C}_{U(\sigma)} \boxtimes G_\sigma(F).$$

We shall now prove the general case ($F \in \mathrm{Mod}_{\mathbb{R}-c}(p^{-1}\mathcal{O}_S)$) by a standard argument. The epimorphism of Lemma 6 induces the following exact sequence:

$$0 \to F' \to K \to F \to 0,$$

where K has the form $(*)$, thus K and F' belong to $\mathrm{Mod}_{\mathbb{R}-c}(p^{-1}\mathcal{O}_S)$. We consider the associated distinguished triangles

$$\begin{aligned}T(F) \to T(K) \to T(F') \xrightarrow{+1},\\ T'(F) \to T'(K) \to T'(F') \xrightarrow{+1}.\end{aligned}$$

Thus (7) reads $T(K) \simeq T'(K)$ in $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}-c}(p^{-1}\mathcal{O}_S)$. There exist integers $N < M$ only depending on T, T' and $\mathcal{G}$ such that the j-cohomology groups of $T(\bullet)$, $T'(\bullet)$, with $\bullet$ replaced by F, F', K (see (7)), vanish for $j \notin [N, M]$. We have $\mathcal{H}^N T(K) \simeq \mathcal{H}^N T'(K)$, thus $\mathcal{H}^N T(F) \to \mathcal{H}^N T'(F)$ is injective (since $\mathcal{H}^{N-1}T(F') = 0 = \mathcal{H}^{N-1}T'(F')$). As F is arbitrary, the same holds true for F replaced by F'. By the five lemma it follows that $\mathcal{H}^N T(F) \simeq \mathcal{H}^N T'(F)$ and so $\mathcal{H}^N T(F') \simeq \mathcal{H}^N T'(F')$ again because F is arbitrary. We then pursue this argument recursively, which ends after a finite number of steps. □

§3.2. A complement on relative holonomic modules

Let X and Y be complex manifolds and let us consider diagram (6).

Lemma 7. *Let $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ Then we have a natural isomorphism*

$$\begin{aligned}&q_1^{-1}\,\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{O}_{X\times S}) \otimes_{p^{-1}\mathcal{O}_S} q_2^{-1}\mathcal{O}_{Y\times S}\\&\quad\xrightarrow{\sim} \mathrm{R}\mathcal{H}\mathrm{om}_{q_1^{-1}\mathcal{D}_{X\times S/S}}(q_1^{-1}\mathcal{M}, \mathcal{O}_{X\times Y\times S}).\end{aligned}$$

Proof. We adapt [9, Prop. 1.4.3]. Since the morphism is well defined, it is enough to prove that, for any $x \in X$, $y \in Y$, it induces an isomorphism

$$\begin{aligned}&\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{O}_{X\times S})|_{\{x\}\times S} \otimes_{\mathcal{O}_S} \mathcal{O}_{Y\times S}|_{\{y\}\times S}\\ &\quad\simeq \mathrm{R}\mathcal{H}\mathrm{om}_{q_1^{-1}\mathcal{D}_{X\times S/S}}(q_1^{-1}\mathcal{M}, \mathcal{O}_{X\times Y\times S})|_{\{(x,y)\}\times S}.\end{aligned}$$

For any $s \in S$, in a neighborhood of (x, s), we now replace $\mathcal{M}$ by a bounded locally free $\mathcal{D}_{X\times S/S}$-resolution $(\mathcal{D}^k_{X\times S/S}, p_k)_{k\in\mathbb{Z}} \xrightarrow[\mathrm{QIS}]{} \mathcal{M}$. Then we may assume that $\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{O}_{X\times S})|_{\{x\}\times S}$ is quasi-isomorphic to the complex $((\mathcal{O}^k_{X\times S})|_{\{x\}\times S}, p_k^\top)$ and that $\mathrm{R}\mathcal{H}\mathrm{om}_{q_1^{-1}\mathcal{D}_{X\times S/S}}(q_1^{-1}\mathcal{M}, \mathcal{O}_{X\times Y\times S})|_{\{(x,y)\}\times S}$ is quasi-isomorphic to the complex $((\mathcal{O}^k_{X\times Y\times S})|_{\{(x,y)\}\times S}, p_k^\top)$.

We have $\mathcal{O}^k_{X\times S}|_{\{x\}\times S} \simeq \mathcal{O}^k_{X,x}\widehat{\otimes}\mathcal{O}_S$ and $\mathcal{O}^k_{X\times Y\times S}|_{\{(x,y)\}\times S} \simeq \mathcal{O}_{X\times Y,(x,y)}\widehat{\otimes}\mathcal{O}_S$ and so $\mathcal{O}^k_{X\times S}|_{\{x\}\times S}$ as well as $\mathcal{O}^k_{X\times Y\times S}|_{\{(x,y)\}\times S}$ are FN-free $\mathcal{O}_S$-modules in the sense of [19].

Since $\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{O}_{X\times S})|_{\{x\}\times S}$ has $\mathcal{O}_S$-coherent cohomologies we are in condition to apply [22, Prop. 3.13], and, in view of (5), to conclude quasi-isomorphisms

$$\begin{aligned}(\mathcal{O}^k_{X\times Y\times S}, p_k^\top)|_{\{(x,y)\}\times S} &\simeq (\mathcal{O}^k_{X\times S}, p_k^\top)|_{\{x\}\times S}\,\widehat{\otimes}_{\mathcal{O}_S}\,(\mathcal{O}_{Y\times S})|_{\{y\}\times S}\\ &\simeq \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{O}_{X\times S})|_{\{x\}\times S} \otimes_{\mathcal{O}_S} \mathcal{O}_{Y\times S}|_{\{y\}\times S}. \quad\square\end{aligned}$$

§4. Main result

§4.1. Statement and proof of the main result

Let Δ denote the diagonal of $X \times X$.

The canonical section of

$$i^{-1}_{\Delta\times S}\mathcal{H}^{d_X}_{\Delta\times S}(\mathcal{O}_{X\times X\times S}) \otimes_{\mathcal{O}_{X\times S}} \Omega_{X\times S/S} = i^{-1}_{\Delta\times S}B^\infty_{X\times S|X\times X\times S} \otimes_{\mathcal{O}_{X\times S}} \Omega_{X\times S/S}$$

corresponding to the global section 1 of $\mathcal{D}^\infty_{X\times S/S}$ allows us to define an isomorphism of sheaves of rings

$$\begin{aligned}\mathcal{D}^\infty_{X\times S/S} &\simeq i^{-1}_{\Delta\times S}B^\infty_{X\times S|X\times X\times S} \otimes_{\mathcal{O}_{X\times S}} \Omega_{X\times S/S}\\ &\simeq i^{-1}_{\Delta\times S}R\Gamma_{\Delta\times S}(\mathcal{O}_{X\times X\times S}) \otimes_{\mathcal{O}_{X\times S}} \Omega_{X\times S/S}[d_X].\end{aligned} \tag{8}$$

Theorem 8. *Let* $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$.

Let $F = \mathrm{Sol}\,\mathcal{M} = \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{O}_{X\times S})$. *Then we have a natural isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathcal{D}^\infty_{X\times S/S})$,

$$\mathcal{M}^\infty \simeq \mathrm{R}\mathcal{H}\mathrm{om}_{p^{-1}\mathcal{O}_S}(F, \mathcal{O}_{X\times S}).$$

Proof. In view of (8), we have isomorphisms

$$\begin{aligned}
&\mathcal{D}^{\infty}_{X\times S/S}\otimes_{\mathcal{D}_{X\times S/S}}\mathcal{M}\\
&\quad\simeq i^{-1}_{\Delta\times S}R\Gamma_{\Delta\times S}(\mathcal{O}_{X\times X\times S})\otimes_{\mathcal{O}_{X\times S}}\Omega_{X\times S/S}[d_X]\otimes_{\mathcal{D}_{X\times S/S}}\boldsymbol{D}\boldsymbol{D}\mathcal{M}\\
&\quad\simeq i^{-1}_{\Delta\times S}R\Gamma_{\Delta\times S}(\mathrm{R}\mathcal{H}om_{q_1^{-1}\mathcal{D}_{X\times S/S}}(q_1^{-1}\boldsymbol{D}\mathcal{M},\mathcal{O}_{X\times X\times S}))[2d_X].
\end{aligned}$$

According to Lemma 7, we have

$$\begin{aligned}
&\mathrm{R}\mathcal{H}om_{q_1^{-1}\mathcal{D}_{X\times S/S}}(q_1^{-1}\boldsymbol{D}\mathcal{M},\mathcal{O}_{X\times X\times S})\\
&\quad\simeq q_1^{-1}\,\mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\boldsymbol{D}\mathcal{M},\mathcal{O}_{X\times S})\otimes_{p^{-1}\mathcal{O}_S}q_2^{-1}\mathcal{O}_{X\times S}.
\end{aligned}$$

On the other hand, the holonomicity of $\mathcal{M}$ implies the isomorphism

$$\begin{aligned}
&\mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\boldsymbol{D}\mathcal{M},\mathcal{O}_{X\times S})\\
&\quad\simeq \mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(\mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M},\mathcal{O}_{X\times S}),p^{-1}\mathcal{O}_S).
\end{aligned}$$

According to Lemma 5 with $X=Y$, $F=\mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M},\mathcal{O}_{X\times S})$ and $\mathcal{G}=\mathcal{O}_{X\times S}$, we conclude a natural isomorphism

$$\begin{aligned}
&\mathrm{R}\mathcal{H}om_{q_1^{-1}\mathcal{D}_{X\times S/S}}(q_1^{-1}\boldsymbol{D}\mathcal{M},\mathcal{O}_{X\times X\times S})\\
&\quad\simeq \mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(q_1^{-1}\,\mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M},\mathcal{O}_{X\times S}),q_2^{-1}\mathcal{O}_{X\times S}).
\end{aligned}$$

Applying $i^{-1}_{\Delta\times S}R\Gamma_{\Delta\times S}$ and the shift $[2d_X]$ to both terms, we finally deduce a natural isomorphism

$$\mathcal{M}^{\infty}\simeq \mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(\mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M},\mathcal{O}_{X\times S}),\mathcal{O}_{X\times S}),$$

which follows by the sequence of isomorphisms

$$\begin{aligned}
&i^{-1}_{\Delta\times S}R\Gamma_{\Delta\times S}(q_2^{-1}\mathcal{O}_{X\times S})\\
&\quad\simeq i^{-1}_{\Delta\times S}\,\mathrm{R}\mathcal{H}om(\mathbb{C}_{\Delta\times S},q_2^{-1}\mathcal{O}_{X\times S})\\
&\quad\underset{(\mathrm{a}')}{\simeq} i^{-1}_{\Delta\times S}(\mathrm{R}\mathcal{H}om(\mathbb{C}_{\Delta\times S},\mathbb{C}_{X\times X\times S})\otimes q_2^{-1}\mathcal{O}_{X\times S})\\
&\quad\underset{(\mathrm{b}')}{\simeq} \mathcal{O}_{X\times S}[-2d_X],
\end{aligned}\tag{9}$$

where (a′) follows by [10, Prop. 5.4.14(ii)] and (b′) by the commutation of the functors $\otimes$ and $i^{-1}_{\Delta\times S}$. □

Corollary 9. *The following properties hold true:*

(a) *We have an isomorphism of functors on* $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$*:*

$$\mathrm{RH}^S(\bullet)^{\infty}[-d_X]\simeq \mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(\bullet,\mathcal{O}_{X\times S}).$$

(b) *Let* $\mathcal{M},\mathcal{N}\in \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$*. We have a natural isomorphism in* $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$*:*

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times S/S}}(\mathcal{M},\mathcal{N})\simeq \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{X\times S/S}}(\mathcal{M},\mathcal{N}^{\infty}).$$

Proof. (a) It is an immediate consequence of Theorem 8 since $\mathrm{Sol}[d_X]\circ \mathrm{RH}^S=\mathrm{Id}$.

(b) We set $\mathcal{N}=\mathrm{RH}^S(G)$ with $G={}^{\mathrm{p}}\mathrm{Sol}(\mathcal{N})\in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$; then $\mathcal{N}^{\infty}\simeq \mathrm{R}\mathcal{H}\mathrm{om}_{p^{-1}\mathcal{O}_S}(G,\mathcal{O}_{X\times S})[d_X]$ and the result follows by Theorem 3(c). □

Definition 10. If $\mathcal{M}$ is a holonomic $\mathcal{D}_{X\times S/S}$-module, we denote by $\mathcal{M}_{\mathrm{reg}}$ the subsheaf of $\mathcal{M}^{\infty}$ of local sections u satisfying the following condition: there exists a coherent ideal $\mathcal{J}$ in $\mathcal{D}_{X\times S/S}$ such that $\mathcal{J}u=0$ and $\mathcal{D}_{X\times S/S}/\mathcal{J}$ is regular holonomic.

Lemma 11. $\mathcal{M}_{\mathrm{reg}}$ *is a* $\mathcal{D}_{X\times S/S}$*-module.*

Proof. The proof is similar to that in [9, Prop. 1.1.20]. If u is a local section of $\mathcal{M}_{\mathrm{reg}}$, let $\mathcal{J}$ be a left ideal of $\mathcal{D}_{X\times S/S}$ as in Definition 10, and let $P\in\mathcal{D}_{X\times S/S}$; then the left ideal $\mathcal{J}'$ of operators Q such that $QP\in\mathcal{J}$ is coherent and $\mathcal{D}_{X\times S/S}/\mathcal{J}'$ is isomorphic to a coherent $\mathcal{D}_{X\times S/S}$-submodule of $\mathcal{D}_{X\times S/S}/\mathcal{J}$; hence, in view of (j) of Section 2.1, it is regular holonomic so that the conditions of Definition 10 are satisfied by Pu. □

Clearly, the correspondence

$$\mathcal{M}\in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})\mapsto \mathcal{M}_{\mathrm{reg}}\in \mathrm{Mod}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$$

defines a left exact functor.

Theorem 12. *We have the following properties:*

(a) *Let* $\mathcal{N}\in \mathrm{Mod}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$*. Then* $\mathcal{N}=\mathcal{N}_{\mathrm{reg}}$*.*

(b) *Let* $\mathcal{N}\in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$*. Then* $\mathcal{N}_{\mathrm{reg}}$ *is a regular holonomic* $\mathcal{D}_{X\times S/S}$*-module isomorphic to* $\mathrm{RH}^S({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{N})$*. In particular,* $\mathcal{N}^{\infty}\simeq\mathcal{N}^{\infty}_{\mathrm{reg}}$*.*

Proof. (a) By the assumption of regularity, we derive a natural inclusion $\mathcal{N}\subset\mathcal{N}_{\mathrm{reg}}$. Let us now prove the inclusion $\mathcal{N}_{\mathrm{reg}}\subset\mathcal{N}$. Let u be a local section of $\mathcal{N}_{\mathrm{reg}}$ and let $\mathcal{J}$ be a left coherent ideal of $\mathcal{D}_{X\times S/S}$ such that $\mathcal{J}u=0$ and such that $\mathcal{D}_{X\times S/S}/\mathcal{J}$ is regular holonomic. We thus deduce a natural morphism $\phi\colon \mathcal{D}_{X\times S/S}/\mathcal{J}\to\mathcal{N}^{\infty}$ as the composition of $\mathcal{D}_{X\times S/S}/\mathcal{J}\twoheadrightarrow \mathcal{D}_{X\times S/S}u\hookrightarrow\mathcal{N}^{\infty}$. Applying Corollary 9(b) to the cohomologies of degree zero with $\mathcal{M}=\mathcal{D}_{X\times S/S}/\mathcal{J}$, ϕ factors through $\mathcal{N}$, thus $\mathcal{D}_{X\times S/S}u\subset\mathcal{N}$.

(b) According to Theorem 8 we have a $\mathcal{D}_{X\times S/S}$-linear isomorphism

$$\Phi\colon \mathcal{N}^{\infty}\simeq \mathrm{RH}^S({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{N})^{\infty}.$$

In view of (a), we conclude by a similar argument that $\Phi(\mathcal{N}_{\mathrm{reg}})$ is contained in $\mathrm{RH}^S({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{N})$. Similarly, using Φ^{-1}, we conclude that $\mathcal{N}_{\mathrm{reg}}$ contains $\Phi^{-1}(\mathrm{RH}^S({}^{\mathrm{p}}\mathrm{Sol}\,\mathcal{N}))$. Thus Φ provides the desired $\mathcal{D}_{X\times S/S}$-isomorphism. □

§4.2. Example

We shall assume that $d_S = 1$. Our goal is to make $(\bullet)^\infty$ explicit in the case of the relative hermitian duality(cf. [17]) by proving the relative variant of [8, Rem. 2.1].

We denote by $\mathcal{D}\mathrm{b}_{X\times S}$ the sheaf of distributions on the real analytic manifold $X_{\mathbb{R}} \times S_{\mathbb{R}}$ underlying $X \times S$ and by $\mathcal{D}\mathrm{b}_{X\times S/S}$ the subsheaf of $\mathcal{D}\mathrm{b}_{X\times S}$ of germs of distributions holomorphic along S. We call $\mathcal{D}\mathrm{b}_{X\times S/S}$ the sheaf of relative distributions. We denote by $\overline{X}$ the complex conjugate manifold of the manifold X. We recall the main results [17, (Thm. 2)]:

(a) The relative Hermitian duality functor

$$\mathrm{C}^S_{X,\overline{X}}(\bullet) \coloneqq \mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\bullet, \mathcal{D}\mathrm{b}_{X\times S/S})$$

induces an equivalence

$$\mathrm{C}^S_{X,\overline{X}} : \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S}) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{\overline{X}\times S/S})^{\mathrm{op}}.$$

(b) $\mathrm{C}^S_{\overline{X},X} \circ \mathrm{C}^S_{X,\overline{X}} \simeq \mathrm{Id}$.

(c) Moreover, the relative conjugation functor

$$\mathrm{c}^S_{X,\overline{X}} \coloneqq \mathrm{C}^S_{X,\overline{X}} \circ \boldsymbol{D}$$

induces an equivalence

$$\mathrm{c}^S_{X,\overline{X}} \colon \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S}) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{\overline{X}\times S/S}),$$

and there is an isomorphism of functors

$${}^{\mathrm{p}}\mathrm{Sol}_{\overline{X}} \circ \mathrm{c}^S_{X,\overline{X}} \simeq {}^{\mathrm{p}}\mathrm{Sol}_X \colon \mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S}) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S).$$

Let $B_{X_{\mathbb{R}}\times S_{\mathbb{R}}}$ be the sheaf of Sato hyperfunctions on $X_{\mathbb{R}} \times S_{\mathbb{R}}$ which we regard as an oriented manifold. Let $B_{X\times S/S}$ denote the subsheaf of $B_{X_{\mathbb{R}}\times S_{\mathbb{R}}}$ of germs of hyperfunctions which are holomorphic along the parameter manifold S.

Proposition 13. *Let $\mathcal{M}$ be an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$. Then we have*

$$\mathrm{C}^S_{X,\overline{X}}(\mathcal{M})^\infty \simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, B_{X\times S/S}).$$

Proof. We note that, by definition of $B_{X_{\mathbb{R}}\times S_{\mathbb{R}}}$, we have, for each orientation on $X_{\mathbb{R}} \times S_{\mathbb{R}}$, an isomorphism of $\mathcal{D}^\infty_{X\times S/S}$-modules

$$B_{X\times S/S} \simeq R\Gamma_{X_{\mathbb{R}}\times S}(\mathcal{O}_{X\times\overline{X}\times S})[2d_X],$$

where, as usual, we regard $X \times \overline{X}$ as a complexification of $X_{\mathbb{R}}$. Hence

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, B_{X\times S/S}) \simeq R\Gamma_{X_{\mathbb{R}}\times S}(\mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{O}_{X\times\overline{X}\times S}))[2d_X].$$

Let q_1 denote the projection $X \times \overline{X} \times S \to X \times S$ and let q_2 denote the projection $X \times \overline{X} \times S \to \overline{X} \times S$. We have

$$\begin{aligned}
&R\Gamma_{X_{\mathbb{R}}\times S}\,\mathrm{R}\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{O}_{X\times\overline{X}\times S})[2d_X] \\
&\quad\underset{(\mathrm{a}')}{\simeq} R\Gamma_{X_{\mathbb{R}}\times S}(q_1^{-1}\,\mathrm{Sol}\,\mathcal{M}\otimes_{p^{-1}\mathcal{O}_S} q_2^{-1}\mathcal{O}_{\overline{X}\times S})[2d_X] \\
&\quad\simeq R\Gamma_{X_{\mathbb{R}}\times S}(\mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(q_1^{-1}\,\mathrm{DR}\,\mathcal{M}, p^{-1}\mathcal{O}_S)\otimes_{p^{-1}\mathcal{O}_S} q_2^{-1}\mathcal{O}_{\overline{X}\times S})[2d_X] \\
&\quad\underset{(\mathrm{b}')}{\simeq} R\Gamma_{X_{\mathbb{R}}\times S}\,\mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(q_1^{-1}\,\mathrm{DR}\,\mathcal{M}, q_2^{-1}\mathcal{O}_{\overline{X}\times S})[2d_X] \\
&\quad\underset{(\mathrm{c}')}{\simeq} \mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(\mathrm{DR}\,\mathcal{M}, \mathcal{O}_{\overline{X}\times S}) \\
&\quad\underset{(\mathrm{d}')}{\simeq} \mathrm{R}\mathcal{H}om_{p^{-1}\mathcal{O}_S}(\mathrm{Sol}\,\mathrm{C}^S_{X,\overline{X}}(\mathcal{M}), \mathcal{O}_{\overline{X}\times S}) \\
&\quad\underset{(\mathrm{e}')}{\simeq} \mathrm{C}^S_{X,\overline{X}}(\mathcal{M})^{\infty},
\end{aligned}$$

where (a′) follows by Lemma 7, (b′) follows by Lemma 5, (c′) follows by a similar argument to (9), (d′) follows by (c) and Section 2.4 and (e′) follows by Theorem 8. □

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