New Fueter-Type Variables Associated to the Global Operator in the Quaternionic Case

by

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Abstract

The purpose of this paper is to develop a new theory of three non-commuting quaternionic variables and its related Schur analysis theory for a modified version of the quaternionic global operator.

§1. Introduction

Quaternionic analysis, or more generally Clifford analysis, allows one to extend the classical theory of holomorphic functions of one complex variable into function theories of hypercomplex variables. A well-developed theory in this non-commutative setting is the theory of Cauchy–Fueter regular functions (or monogenic functions); see [13, 33] and the references therein. In 2007 a new theory of regular functions of a quaternionic variable was introduced in [29]. These are the so-called *slice hyperholomorphic* functions on quaternions. For more details on this topic we refer the reader to the books [21, 20, 28] and the references therein. This theory generated several interesting applications in different areas of mathematics and physics due

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to the discovery of the notion of the S-spectrum for quaternionic operators intro-
duced in [17]; see also [16, 15, 20, 30] and the references therein. In particular, this
function theory allowed one to develop the quaternionic counterpart of operator
theory, Schur analysis, and quantum mechanics; see [4, 16, 15, 20].

Let us recall that Blaschke products are building blocks of classical function
theory on the disk as in [37] and, more generally, in Schur analysis, and they also
constitute essential tools of the Beurling–Lax theorem, which characterizes shift-
invariant subspaces of the Hardy space. A fundamental technique that we use in
this paper is based on the research developed in [19], where the authors discovered
a new approach to the theory of slice hyperholomorphic functions, using a special
operator with non-constant coefficients called the *quaternionic global operator*. The
authors of [19] proved that under suitable conditions on the domain, the space of
slice hyperholomorphic functions is strictly contained in the kernel of this global
operator with non-constant coefficients. This quaternionic global operator was used
in [5] to prove a counterpart of the well-known Fueter–Sce mapping theorem in
the case of quaternionic polyanalytic functions. It was also used to study poly-slice
monogenic functions introduced in [2, 5].

The purpose of this paper is to develop a theory of three non-commuting
quaternionic variables and a related Schur analysis for the quaternionic global
operator. We first set the framework for our work, and define the skew field of
quaternions (denoted by $\mathbb{H}$) as the space of elements of the form

$$(1.1) \qquad q = x_0 + x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3,$$

where x_0, x_1, x_2, x_3 are real numbers and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ satisfy the Cayley multipli-
cation table. A number of the form q in (1.1) is called a quaternion, and

$$\vec{q} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3$$

is called its vector part (more information on quaternions is provided in Section 2).

As in Definition 2.4, the global operator with non-constant coefficients, which
was introduced for the first time in [19], was initially written as

$$(1.2) \qquad G_q(f) := |\vec{q}|^2 \partial_{x_0} f(q) + \vec{q} \sum_{l=1}^{3} x_l \partial_{x_l} f(q).$$

The operator G_q acts on functions of a quaternionic variable which are dif-
ferentiable with respect to the real variables and it was first used to develop a
Cauchy formula in the case of slice hyperholomorphic functions; see [19]. For a
distributional approach to the Cauchy problem in this context see [22]. In par-
ticular, in [19], it was proved that slice-hyperholomorphic functions are strictly

included in the kernel of the global operator G_q. However, the global operator we work with, denoted by V_q, is the normalized form of G_q, $V_q = \frac{G_q}{|\vec{q}|^2}$, namely

$$(1.3) \qquad V_q = \frac{\partial}{\partial x_0} - \frac{1}{\vec{q}} \sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u}.$$

The global operator V_q was used in [5] to develop the Fueter mapping theorem for poly slice monogenic functions. In [3] the global operator was used to study infinite-order differential operators acting on entire hyperholomorphic functions and it was used also in [31] to study global differential equations for slice regular functions. Recent extensions and new results on the global operator and related topics can be found in [14, 32, 35].

Definition 1.1. A quaternionic-valued function f which is at least C^1 in the real variables in a domain $\Omega_R \subset \mathbb{R}^4$ is called V_q-regular if $V_q f = 0$.

In this paper we will focus on a family of V_q-regular functions which are real-analytic and we develop a Cauchy–Kovalevskaia product (denoted CK-product) for this space. We consider the version of Gleason problem associated to V_q, which will allow us to introduce a new type of Fueter-like variables and develop a Schur analysis and rational functions for this operator. For consistency we now introduce the notation that will be used throughout the paper.

Definition 1.2. Given $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ such that $x_u \neq 0$ for some $u = 1, 2, 3$, we define the V_q-Fueter variables

$$(1.4) \qquad \mu_u(x) = x_u \left(1 + \frac{x_0}{\vec{q}}\right), \quad u = 1, 2, 3,$$

and, for $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$, we write, with a slight abuse of notation,

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3},$$

where x^α does not depend on x_0.

The functions μ_u, $u = 1, 2, 3$, are the counterpart of the classical Fueter variables (see Definition 2.1 for the latter) in the present setting. These new variables allow us to define and study a counterpart of Schur analysis in this setting. In Theorem 3.6 we prove a more general form of the following.

Theorem 1.3. *The functions*

$$(1.5) \qquad \mu_u(x) = x_u \left(1 + \frac{x_0}{\vec{q}}\right), \quad u = 1, 2, 3,$$

are V_q-regular on $\mathbb{H}^* := \{q = x_0 + \vec{q}, \ \vec{q} \neq 0\}$.

Throughout the paper we will exchange the quadruple $x = (x_0, x_1, x_2, x_3)$ with $x_u \neq 0$ for some $u = 1, 2, 3$ with the quaternionic variable $q = x_0 + \vec{q} \in \mathbb{H}^*$ so that we can consider μ_u as functions of both. For a given $x \in \mathbb{H}^*$, $\mu_u(x)$ commute pairwise and we recover Fueter-like monomials as in Definition 3.3:

$$\mu^\alpha(x) = \mu_1^{\alpha_1}(x)\mu_2^{\alpha_2}(x)\mu_3^{\alpha_3}(x), \tag{1.6}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$.

The functions μ^α are V_q-regular on $\mathbb{H}^*$ (see Section 3) and we will see that they can be rewritten as

$$\mu^\alpha(x) = x^\alpha \left(1 + \frac{x_0}{\vec{q}}\right)^{|\alpha|}. \tag{1.7}$$

As explained in the following remarks, and throughout the paper, there are important and fundamental differences between the cases considered in [7, 9, 11] (and in particular the case of Fueter variables in [9]) and the present setting.

Remark 1.4. In the setting of Fueter variables, the Cauchy–Fueter operator $D = \partial_{x_0} + \mathbf{e}_1\partial_{x_1} + \mathbf{e}_2\partial_{x_2} + \mathbf{e}_3\partial_{x_3}$ and the operators of partial differentiation commute for smooth functions. This is not the case for the operator V_q, which does not commute with differentiations with respect to x_u, $u = 1, 2, 3$. So, for instance, $\frac{\partial f}{\partial x_1}$ need not be V_q-regular when f is V_q-regular. Since V_q and $\frac{\partial f}{\partial x_0}$ commute, $\frac{\partial f}{\partial x_0}$ is V_q-regular when f is V_q-regular. As an illustration, the case $f(x) = \mu_1(x)$ is considered in Example 3.9. The function $\frac{\partial \mu_1}{\partial x_2}$ is not V_q-regular, while the function $\frac{\partial \mu_1}{\partial x_0}$ is V_q-regular but cannot be written as a convergent Fueter-like series of the μ^α monomials.

To present these ideas we adopt the following structure: In Section 2 we review basic notation and definitions of quaternions, slice hyperholomorphic functions, and the quaternionic global operator with non-constant coefficients. In Section 3 we introduce the Fueter-like variables and prove a Gleason-type theorem in this setting. Section 4 is devoted to the study of a Cauchy–Kovalevskaia-type extension and CK-type product corresponding to these Fueter-like variables. In Sections 5 and 6 we study some examples of reproducing kernel Hilbert spaces generated by the Fueter-like variables, including a counterpart of Arveson space and Schur multipliers. Finally, in Section 7 we obtain Blaschke factors in this framework, followed by building a rational function theory in Section 8. We conclude the paper with a general description of the next steps we will undertake using these new Fueter-like variables.

§2. Preliminary results

We recall that the non-commutative field of quaternions is defined and denoted by

$$\mathbb{H} = \{q = x_0 + x_1 i + x_2 j + x_3 k : x_0, x_1, x_2, x_3 \in \mathbb{R}\},$$

where the imaginary units satisfy the multiplication rules

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = -ji = k, \; jk = -kj = i, \; ki = -ik = j.$$

On $\mathbb{H}$ the conjugate and the modulus of q are defined respectively by

$$\bar{q} = x_0 - \vec{q}, \quad \vec{q} = x_1 i + x_2 j + x_3 k,$$

and

$$|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

Throughout the paper we use the notation $\mathbf{e}_0 = 1$, $\mathbf{e}_1 = i$, $\mathbf{e}_2 = j$, and $\mathbf{e}_3 = k$ interchangeably for the imaginary units. It is important to note that the conjugation satisfies the property $\overline{pq} = \bar{q}\bar{p}$ for any $p, q \in \mathbb{H}$.

An important tool in the quaternionic case is the symmetric product of n quaternionic numbers $q_1, q_2, \ldots, q_n$:

$$(2.1) \qquad q_1 \times q_2 \times \cdots \times q_n = \frac{1}{n!} \sum_{\sigma \in S_n} q_{\sigma(1)} q_{\sigma(2)} \cdots q_{\sigma(n)},$$

where the sum is over the set S_n of all permutations on n indices.

In the classical case, Fueter [26, 27] used the following variables to describe the kernel of the Cauchy–Fueter operator.

Definition 2.1. The classical Fueter variables are $\zeta_l = x_l - x_0 \mathbf{e}_l$ and we denote by ζ^α the symmetric product $\zeta_1^{\times \alpha_1} \times \zeta_2^{\times \alpha_2} \times \zeta_3^{\times \alpha_3}$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

These variables are Fueter-regular with respect to the classical Cauchy–Fueter operator.

We now define the setting for slice hyperholomorphic functions. The unit sphere

$$\left\{\vec{q} = x_1 i + x_2 j + x_3 k : x_1^2 + x_2^2 + x_3^2 = 1\right\}$$

coincides with the set of all imaginary units given by

$$\mathbb{S} = \left\{q \in \mathbb{H} : q^2 = -1\right\}.$$

Definition 2.2. Let $f: \Omega \to \mathbb{H}$ be a $\mathcal{C}^1$ function on a given domain $\Omega \subset \mathbb{H}$. Then f is said to be (left) slice hyperholomorphic function if, for every $I \in \mathbb{S}$, the restriction f_I to $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$, with variable $q = x + Iy$, is holomorphic on

$\Omega_I \coloneqq \Omega \cap \mathbb{C}_I$, that is, it has continuous partial derivatives with respect to x and y and the function $\overline{\partial_I} f \colon \Omega_I \to \mathbb{H}$ defined by

$$\overline{\partial_I} f(x + Iy) \coloneqq \frac{1}{2} \Big(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \Big) f_I(x + yI)$$

vanishes identically on Ω_I. The set of slice hyperholomorphic functions will be denoted by $\mathcal{SR}(\Omega)$.

The right quaternion vector space $\mathcal{SR}(\Omega)$ is endowed with the natural topology of uniform convergence on compact sets. The characterization of such functions on a ball centered at the origin is given by the following theorem.

Theorem 2.3 (Series expansion [29]). *An $\mathbb{H}$-valued function f is slice hyperholomorphic on $B(0, R)$ if and only if it has a series expansion of the form*

$$f(q) = \sum_{n=0}^{+\infty} q^n a_n,$$

converging on $B(0, R) = \{q \in \mathbb{H}; |q| < R\}$.

Another interesting approach to defining slice hyperholomorphic functions is to consider them as solutions of a special global operator with non-constant coefficients that was introduced and studied in [19, 22, 31]. This leads to the following definition.

Definition 2.4. Let Ω be an open set in $\mathbb{H}$ and $f \colon \Omega \to \mathbb{H}$ a function of class $\mathcal{C}^1$. We define the global operator $G_q(f)$ by

$$G_q(f) \coloneqq |\vec{q}|^2 \partial_{x_0} f(q) + \vec{q} \sum_{l=1}^{3} x_l \partial_{x_l} f(q),$$

for any $q = x_0 + \vec{q} \in \Omega$.

It was proved in [19] that any slice hyperholomorphic function is in the kernel of G_q on axially symmetric slice domains. We briefly recall the definition of such a domain

Definition 2.5. A domain $\Omega \subset \mathbb{H}$ is said to be a slice domain (or just s-domain) if $\Omega \cap \mathbb{R}$ is non-empty and for all $I \in \mathbb{S}$, the set $\Omega_I \coloneqq \Omega \cap \mathbb{C}_I$ is a domain of the complex plane $\mathbb{C}$. If, moreover, for every $q = x + Iy \in \Omega$, the whole sphere $x + y\mathbb{S} \coloneqq \{x + Jy; \ J \in \mathbb{S}\}$ is contained in Ω, we say that Ω is an axially symmetric slice domain.

There are many other interesting properties of the global operator G_q that have been studied in the literature, in particular in [18]. We recall some here.

Proposition 2.6. *Let Ω be an open set in $\mathbb{H}$ and $f, g \colon \Omega \to \mathbb{H}$ two functions of class $\mathcal{C}^1$. Then, for $q = x_0 + \vec{q} \in \Omega$, we have*

(1) $G(fg) = G(f)g + fG(g) + (\vec{q}f - f\vec{q})\sum_{l=1}^{3} x_l \partial_{x_l} g$.

In particular, it holds that

(2) $G(f\lambda + g) = G(f)\lambda + G(g)$ *for all $\lambda \in \mathbb{H}$,*
(3) $G(x_0 f) = |\vec{q}|^2 f + x_0 G(f)$ *and $G(\vec{q}f) = -|\vec{q}|^2 f + \vec{q}G(f)$,*
(4) $G(q^k f) = q^k G(f)$ *for all $k \in \mathbb{N}$.*

§3. The V_q-Fueter variables, a new type of Fueter variable

In this section we introduce the Gleason setting that will yield the new V_q-Fueter variables in the quaternionic case, in a natural way. Through this process, we find these variables, which correspond to the operator V_q, when we use the same technique the authors applied in the real ternary case [11], in the split quaternionic case [7], and in the regular Fueter quaternionic case [9]. For completion, in the case of Grassmann variables, see also [8].

It is also worth pointing out that our work here is the first to find a counterpart to the classical Fueter variables through the treatment of this case of a differential operator with non-constant coefficients. All the applications mentioned before used this method for operators with constant coefficients.

The strategy involves a clever application of the chain rule, and, for a V_q-regular function f, we compute

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(tx)\, \mathrm{d}t.$$

In the previous examples of ternary and split quaternions, we can take $a = 0$ and $b = 1$. In the present case a cannot be chosen to be 0 because of the singularity at the origin.

We will see that we can apply the same technique to obtain the Fueter-like variables μ_u in the proof of the following theorem.

Theorem 3.1. *Let Ω be an open domain of $\mathbb{H}^*$. For a V_q-regular function $f \in C^1(\Omega)$, let $a, b \in \Omega$ be such that $[a, b] = \{(1-t)a + tb,\ 0 \le t \le 1\} \subset \Omega$. Then*

$$(3.1) \qquad f(b) - f(a) = \sum_{u=1}^{3} \mu_u(b-a) R_u^{ab} f,$$

where

(3.2)
$$R_u^{ab} f = \int_a^b \frac{\partial f}{\partial x_u}(a + t(b - a)) \, \mathrm{d}t.$$

Proof. The chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}t} f(a + tx) = x_0 \frac{\partial f}{\partial x_0}(a + tx) + \sum_{u=1}^{3} x_u \frac{\partial f}{\partial x_u}(a + tx)$$

$$= \frac{x_0}{\vec{q}} \sum_{u=1}^{3} x_u \frac{\partial f}{\partial x_u}(a + tx) + \sum_{u=1}^{3} x_u \frac{\partial f}{\partial x_u}(a + tx)$$

$$= \sum_{u=1}^{3} \mu_u(x) \frac{\partial f}{\partial x_u}(a + tx),$$

and the result follows by integrating back and setting $x = b - a$. The reader should note that in the second step of the equality we use the fact that f is in the kernel of V_q. $\square$

Remark 3.2. Note that

(3.3)
$$\mu_u(b - a) \neq \mu_u(b) - \mu_u(a).$$

Furthermore, the operators R_u^{ab} do not commute with V_q.

For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ we use the multi-index notation and set

(3.4)
$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \quad \text{and} \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

Definition 3.3. For the V_q-Fueter variables μ_u and $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$, we define the product

(3.5)
$$\mu^\alpha(x) = \mu_1^{\alpha_1}(x) \mu_2^{\alpha_2}(x) \mu_3^{\alpha_3}(x),$$

for every $x \in \mathbb{H}^*$.

Remark 3.4. Since the V_q-Fueter variables μ_u commute, in the above definition we do not need to use the symmetric product and, in fact, we can rewrite the product as

(3.6)
$$\mu^\alpha(x) = x^\alpha \left(1 + \frac{x_0}{\vec{q}}\right)^{|\alpha|}.$$

Proposition 3.5. *Let Ω be an open domain in $\mathbb{H}^*$. For every $x \in \Omega \subset \mathbb{H}^*$ it holds that*

(3.7)
$$|\mu^\alpha|^2 = |x|^{2\alpha} \left(1 + \frac{x_0^2}{x_1^2 + x_2^2 + x_3^2}\right)^{|\alpha|},$$

and, in particular, we have

(3.8)
$$|\mu_u(x)|^2 \le |\zeta_u(x)|^2, \quad u = 1, 2, 3.$$

Proof. This follows from the fact that $\frac{x_0}{\vec{q}}$ has no real part on Ω and

(3.9)
$$|\mu_u(x)|^2 = x_u^2 + \frac{x_u^2 x_0^2}{x_1^2 + x_2^2 + x_3^2} \le x_u^2 + x_0^2 = |\zeta_u(x)|^2.$$

$\square$

We can now prove that the V_q-Fueter products μ^α are in the kernel of V_q on $\mathbb{H}^*$.

Theorem 3.6. *It holds that μ^α are in the kernel of the operator V_q on any open domain $\Omega \subset \mathbb{H}^*$. Moreover, we have*

(3.10)
$$V_q \mu^\alpha(x) = 0,$$

for every $x \in \Omega$.

Proof. We divide the verification into a number of steps.

Step 1: It holds that

(3.11)
$$\frac{\partial}{\partial x_u} \frac{1}{\vec{q}} = \frac{\mathbf{e}_u}{\vec{q}^2} + \frac{2x_u}{\vec{q}^3}, \quad u = 1, 2, 3.$$

Indeed, we have
$$\frac{1}{\vec{q}} = \frac{\bar{\vec{q}}}{\vec{q}^2} = -\frac{\vec{q}}{x_1^2 + x_2^2 + x_3^2}.$$

Hence
$$\frac{\partial}{\partial x_u} \frac{1}{\vec{q}} = \frac{-\mathbf{e}_u(x_1^2 + x_2^2 + x_3^2) + 2x_u\vec{q}}{(x_1^2 + x_2^2 + x_3^2)^2}$$
$$= \frac{\mathbf{e}_u\vec{q}^2}{\vec{q}^4} + \frac{2x_u\vec{q}}{\vec{q}^3}$$

and hence the result.

Step 2: It holds that

(3.12)
$$\frac{\partial}{\partial x_u}\left(1 + \frac{x_0}{\vec{q}}\right)^{|\alpha|} = \sum_{\substack{t,s\in\mathbb{N}_0 \\ t+s=|\alpha|}} \left(1 + \frac{x_0}{\vec{q}}\right)^t \left(\frac{x_0\mathbf{e}_u}{\vec{q}^2} + \frac{2x_ux_0}{\vec{q}^3}\right)\left(1 + \frac{x_0}{\vec{q}}\right)^s,$$

where $u = 1, 2, 3$.

Equation (3.12) is a direct consequence of (3.11) and of the formula for the derivative of f^n when f is a matrix-valued function (and in particular quaternionic valued) of (say) a real variable w:

$$(3.13) \qquad \frac{\mathrm{d}f^n}{\mathrm{d}w} = \sum_{\substack{t,s \in \mathbb{N}_0 \\ t+s=|\alpha|}} f^t f' f^s.$$

Step 3: We have

$$(3.14) \qquad \frac{\partial}{\partial x_0}\left(1+\frac{x_0}{\vec{q}}\right)^{|\alpha|} = \frac{|\alpha|}{\vec{q}}\left(1+\frac{x_0}{\vec{q}}\right)^{|\alpha|-1}.$$

This is because $1+\frac{x_0}{\vec{q}}$ commutes with its derivative with respect to x_0, and formula (3.13) reduces then to the classical formula.

Step 4: We now calculate

$$\frac{1}{\vec{q}}\sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u}\mu^\alpha.$$

We have

$$\frac{1}{\vec{q}}\sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u} x^\alpha\left(1+\frac{x_0}{\vec{q}}\right)^{|\alpha|}$$

$$= \frac{1}{\vec{q}}\left[\sum_{u=1}^{3} x_u\left(\alpha_u x^{\alpha-e_u}\left(1+\frac{x_0}{\vec{q}}\right)^{|\alpha|}\right.\right.$$

$$\left.\left. + x^\alpha \sum_{\substack{t,s\in\mathbb{N}_0 \\ t+s=|\alpha|-1}}\left(1+\frac{x_0}{\vec{q}}\right)^t\left(\frac{x_0\mathbf{e}_u}{\vec{q}^2}+\frac{2x_u x_0}{\vec{q}^3}\right)\left(1+\frac{x_0}{\vec{q}}\right)^s\right)\right]$$

$$= \frac{1}{\vec{q}}\left[|\alpha|x^\alpha\left(1+\frac{x_0}{\vec{q}}\right)^{|\alpha|}\right.$$

$$\left. + x^\alpha \sum_{\substack{t,s\in\mathbb{N}_0 \\ t+s=|\alpha|-1}}\left(1+\frac{x_0}{\vec{q}}\right)^t\left(\sum_{u=1}^{3}\frac{x_u x_0\mathbf{e}_u}{\vec{q}^2}+\frac{2x_u^2 x_0}{\vec{q}^3}\right)\left(1+\frac{x_0}{\vec{q}}\right)^s\right]$$

$$= \frac{x^\alpha}{\vec{q}}\left[|\alpha|\left(1+\frac{x_0}{\vec{q}}\right)^{|\alpha|}\right.$$

$$\left. + \sum_{\substack{t,s\in\mathbb{N}_0 \\ t+s=|\alpha|-1}}\left(1+\frac{x_0}{\vec{q}}\right)^t\left(\frac{x_0\vec{q}}{\vec{q}^2}-\frac{2\vec{q}^2 x_0}{\vec{q}^3}\right)\left(1+\frac{x_0}{\vec{q}}\right)^s\right]$$

$$= \frac{x^\alpha}{\vec{q}} \left[|\alpha| \left(1 + \frac{x_0}{\vec{q}} \right)^{|\alpha|} - \sum_{\substack{t,s \in \mathbb{N}_0 \\ t+s=|\alpha|-1}} \left(1 + \frac{x_0}{\vec{q}} \right)^t \frac{x_0}{\vec{q}} \left(1 + \frac{x_0}{\vec{q}} \right)^s \right]$$

$$= \frac{x^\alpha}{\vec{q}} \left[|\alpha| \left(1 + \frac{x_0}{\vec{q}} \right)^{|\alpha|} - |\alpha| \left(1 + \frac{x_0}{\vec{q}} \right)^{|\alpha|-1} \frac{x_0}{\vec{q}} \right].$$

Step 5: We can now compute $V_q \mu^\alpha$. Using (3.14) and the previous step we have

$$V_q \mu^\alpha = x^\alpha \frac{|\alpha|}{\vec{q}} \left(1 + \frac{x_0}{\vec{q}} \right)^{|\alpha|-1} - \frac{x^\alpha}{\vec{q}} \left[|\alpha| \left(1 + \frac{x_0}{\vec{q}} \right)^{|\alpha|} - |\alpha| \left(1 + \frac{x_0}{\vec{q}} \right)^{|\alpha|-1} \frac{x_0}{\vec{q}} \right]$$

$$= \frac{|\alpha| x^\alpha}{\vec{q}} \left(1 + \frac{x_0}{\vec{q}} \right)^{|\alpha|-1} \left(1 - \left(1 + \frac{x_0}{\vec{q}} \right) + \frac{x_0}{\vec{q}} \right)$$

$$= 0.$$

This ends the proof. $\square$

Remark 3.7. We observe that both functions μ^α and ζ^α coincide with x^α when $x_0 = 0$. It is important to note that these are two different extensions of the same real function x^α leading to two different regular function theories. In fact, μ^α is the V_q-regular extension of x^α, while ζ^α gives the classical Fueter extension. However, the classical Fueter variables ζ^α extend x^α to the whole space of quaternions while μ^α extend x^α to domains of $\mathbb{H}^*$.

Proposition 3.8. *For any $n \in \mathbb{N}$, the function q^n is in $\ker V_q$ and, moreover,*

$$q^n = \sum_{|\alpha|=n} \mu^\alpha c_{\alpha,n}$$

where, with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,

(3.15) $$c_{\alpha,n} = \frac{n!}{\alpha!} \mathbf{e}_1^{\times \alpha_1} \times \mathbf{e}_2^{\times \alpha_2} \times \mathbf{e}_3^{\times \alpha_3},$$

where the symmetric product is taken among all the products of the units $\mathbf{e_u}$.

Proof. In $\mathbb{H}^*$ we have

$$q^n = (x_0 + \vec{q})^n$$
$$= \left(1 + \frac{x_0}{\vec{q}} \right)^n (\vec{q})^n$$

$$= \left(1 + \frac{x_0}{\vec{q}}\right)^n \left(\sum_{|\alpha|=n} x^\alpha c_{\alpha,n} \right)$$

$$= \sum_{|\alpha|=n} x^\alpha \left(1 + \frac{x_0}{\vec{q}}\right)^n c_{\alpha,n},$$

for some $c_{\alpha,n} \in \mathbb{H}$ which can be expressed in terms of symmetrized products as in (3.15) by known formulas. $\qquad\square$

We note that (3.15) does not take into account the Cayley table of multiplication for the quaternions.

Example 3.9. Let us examine $\mu_1(x) = x_1(1 + \frac{x_0}{\vec{q}}) = x_1 + x_0\frac{x_1}{\vec{q}}$. The function $\frac{\partial \mu_1}{\partial x_2} = x_1 x_0(\frac{\mathbf{e}_2}{\vec{q}^2} + \frac{2x_2}{\vec{q}^3})$ is not V_q-regular, while the function $\frac{\partial \mu_1}{\partial x_0} = \frac{x_1}{\vec{q}}$ is V_q-regular but cannot be written as a convergent Fueter-like series of the μ^α.

Discussion of Example 3.9. We have

$$\frac{\partial \mu_1}{\partial x_2}(x) = x_1 x_0 \left(\frac{\mathbf{e}_2}{\vec{q}^2} + \frac{2x_2}{\vec{q}^3}\right),$$

and so

$$\frac{\partial^2 \mu_1}{\partial x_0 \partial x_2}(x) = x_1 \left(\frac{\mathbf{e}_2}{\vec{q}^2} + \frac{2x_2}{\vec{q}^3}\right)$$

is independent of x_0. On the other hand,

$$\sum_{u=1}^3 x_u \frac{\partial}{\partial x_u}\left(\frac{\partial \mu_1}{\partial x_2}\right) = \sum_{u=1}^3 x_u \frac{\partial}{\partial x_u}\left(x_1 x_0 \left(\frac{\mathbf{e}_2}{\vec{q}^2} + \frac{2x_2}{\vec{q}^3}\right)\right)$$

$$= x_0\left[\sum_{u=1}^3 x_u \frac{\partial}{\partial x_u}\left(x_1\left(\frac{\mathbf{e}_2}{\vec{q}^2} + \frac{2x_2}{\vec{q}^3}\right)\right)\right].$$

Hence

$$\left(V_q \frac{\partial \mu_1}{\partial x_2}\right)(x) = x_1\left(\frac{\mathbf{e}_2}{\vec{q}^2} + \frac{2x_2}{\vec{q}^3}\right) - x_0\left[\sum_{u=1}^3 x_u \frac{\partial}{\partial x_u}\left(x_1\left(\frac{\mathbf{e}_2}{\vec{q}^2} + \frac{2x_2}{\vec{q}^3}\right)\right)\right].$$

Setting $x_0 = 0$ shows that $V_q \frac{\partial \mu_1}{\partial x_2} \not\equiv 0$.

We now turn to $\frac{\partial \mu_1}{\partial x_0}$, which is V_q-regular since differentiation with respect to x_0 commutes with V_q. Assume now that

$$\frac{\partial \mu_1}{\partial x_0} = \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha c_\alpha$$

for some quaternionic coefficients c_α, and where the convergence is assumed in some set $\Omega \subset \mathbb{H}^*$. In other words,

$$(3.16) \qquad \frac{x_1}{\vec{q}} = \sum_{\alpha \in \mathbb{N}_0^3} x^\alpha \left(1 + \frac{x_0}{\vec{q}}\right)^{|\alpha|} c_\alpha.$$

Setting $x_0 = 0$ in this equation, and multiplying both sides by $\vec{q}$ leads to

$$(3.17) \qquad \begin{aligned} x_1 = &\sum_{\alpha \in \mathbb{N}_0^3} x_1^{\alpha_1+1} x_2^{\alpha_2} x_3^{\alpha_3} \mathbf{e}_1 c_\alpha + \sum_{\alpha \in \mathbb{N}_0^3} x_1^{\alpha_1} x_2^{\alpha_2+1} x_3^{\alpha_3} \mathbf{e}_2 c_\alpha \\ &+ \sum_{\alpha \in \mathbb{N}_0^3} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3+1} \mathbf{e}_3 c_\alpha, \end{aligned}$$

where this equality is valid a priori only for (x_1, x_2, x_3) such that $x_u \neq 0$ for some $u = 1, 2, 3$. By classical results on series and summable families, (3.17) can be extended to $x_u = 0$. Comparing the linear terms on both sides of (3.17) leads to

$$x_1 = x_1 \mathbf{e}_1 c_{0,0,0} + x_2 \mathbf{e}_2 c_{0,0,0} + x_3 \mathbf{e}_3 c_{0,0,0},$$

which is impossible, leading to a contradiction. $\qquad \square$

Just as in the classical case, let us now define the conjugate operator of V_q, denoted by $\overline{V_q}$, to be the operator defined by

$$(3.18) \qquad \overline{V_q} := \frac{\partial}{\partial x_0} + \frac{1}{\vec{q}} \sum_{u=1}^{3} \mathbf{e}_u \frac{\partial}{\partial x_u}.$$

We can prove the following proposition.

Proposition 3.10. *Let Ω be an open domain in $\mathbb{H}^*$. If f is V_q-regular on Ω, then $\overline{V_q}(f)$ is also V_q-regular on Ω. Moreover, in this case we have*

$$(3.19) \qquad \frac{1}{2}\overline{V_q}f(q) = \frac{\partial}{\partial x_0}f(q), \quad \forall q = x_0 + \vec{q}.$$

Proof. We first observe that

$$(V_q + \overline{V_q})f(q) = 2\frac{\partial}{\partial x_0}f(q).$$

However, since f is V_q-regular, then it belongs to $\ker(V_q)$ so that $V_q(f) = 0$, leading to

$$\overline{V_q}f(q) = 2\frac{\partial}{\partial x_0}f(q).$$

Thus, applying the operator V_q and, using the fact that it commutes with $\frac{\partial}{\partial x_0}$, we obtain

$$V_q \overline{V_q} f(q) = 2V_q \frac{\partial}{\partial x_0} f(q)$$

$$= 2 \frac{\partial}{\partial x_0} V_q f(q)$$

$$= 0.$$

So, the function $\overline{V_q}(f)$ is V_q-regular, which ends the proof. $\qquad\square$

Proposition 3.11. *It holds that for every $q \in \mathbb{H}^*$ we have*

$$\frac{1}{2} \overline{V_q} \mu^\alpha(q) = \frac{|\alpha|}{q} \mu^\alpha(q).$$

Proof. We use the previous computations of $\frac{\partial}{\partial x_0} \mu^\alpha$ and $\frac{1}{q} \sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u} \mu^\alpha$ to obtain

$$\overline{V_q} \mu^\alpha(q) = 2|\alpha| \frac{x^\alpha}{\overline{q}} \left(1 + \frac{x_0}{\overline{q}}\right)^{|\alpha|-1}.$$

Hence, we get

$$\overline{V_q} \mu^\alpha(q) = 2|\alpha| q^{-1} \mu^\alpha(x),$$

which is the desired equality. $\qquad\square$

Remark 3.12. It is easy to see that a consequence of this proposition yields the counterpart of the properties of the number operators and we have

$$M_q \overline{V_q}(\mu^\alpha) = 2|\alpha| \mu^\alpha,$$

where M_q represents the left quaternionic multiplication.

§4. Cauchy–Kovalevskaia product

In this section we build a Cauchy–Kovalevskaia product using the new variables μ^α. We start by building a CK-extension of a function in the kernel of V_q, $f = f_0 + \mathbf{e}_1 f_1 + \mathbf{e}_2 f_2 + \mathbf{e}_3 f_3$, where f_0, f_1, f_2, f_3 are real-valued differentiable functions on an open subset $\Omega \subset \mathbb{H}^*$. The equation $V_q f = 0$ can be rewritten as

$$(4.1) \qquad\qquad \frac{\partial f_j}{\partial x_0} = G_j, \quad j = 0, 1, 2, 3,$$

where G_j is real analytic in the variables x_0, x_1, x_2, x_3. The right-hand side can be written in terms of the partial derivatives with respect to the other three variables,

i.e. $\frac{\partial f_j}{\partial x_k}$, $j = 0, 1, 2, 3$, $k = 1, 2, 3$, and they are given by the following system on an open subset $\Omega \subset \mathbb{H}^*$:

$$
\begin{aligned}
\frac{\partial f_0}{\partial x_0} &= \frac{1}{x_1^2 + x_2^2 + x_3^2}(x_1 E f_1 + x_2 E f_2 + x_3 E f_3), \\
\frac{\partial f_1}{\partial x_0} &= -\frac{1}{x_1^2 + x_2^2 + x_3^2}(-x_1 E f_0 - x_2 E f_3 + x_3 E f_2), \\
\frac{\partial f_2}{\partial x_0} &= -\frac{1}{x_1^2 + x_2^2 + x_3^2}(-x_2 E f_0 - x_1 E f_3 + x_3 E f_1), \\
\frac{\partial f_3}{\partial x_0} &= -\frac{1}{x_1^2 + x_2^2 + x_3^2}(-x_3 E f_0 - x_1 E f_2 + x_2 E f_1),
\end{aligned}
$$

(4.2)

where E denotes the Euler operator

$$
E = \sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u}.
$$

We can therefore apply the Cauchy–Kovalevskaia theorem (see e.g. [23, §7, p. 39], which asserts that the system (4.1) has a unique solution near a real point possibly different from the origin) for given initial real analytic values $f_j(0, x_1, x_2, x_3)$ on an open domain $\widetilde{\Omega} \subset (\mathbb{H}^* \cap \{x_0 = 0\})$. This solution, F, is defined on an open set $\Omega \subset \mathbb{H}^*$, where $\widetilde{\Omega} = \Omega \cap \{x_0 = 0\}$, and we call it the *CK-extension of f with respect to V_q*.

Definition 4.1. The CK-extension to Ω with respect to the operator V_q found above is denoted by $F = \mathrm{CK}_{V_q}(f)$.

Remark 4.2. The choices

$$
f_0(0, x_1, x_2, x_3) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}
$$

and

$$
f_1(0, x_1, x_2, x_3) = f_2(0, x_1, x_2, x_3) = f_3(0, x_1, x_2, x_3) = 0
$$

lead, in the case of the variables x^α to

(4.3) $\mu^\alpha = \mathrm{CK}_{V_q}(x^\alpha)$.

Definition 4.3. We can now define the CK-product of two functions f, g in the kernel of V_q to be

(4.4) $f \star_{V_q} g = \mathrm{CK}_{V_q}(f(0, x_1, x_2, x_3) g(0, x_1, x_2, x_3))$,

for every $x \in \Omega$.

It is easy to see that, for any two quaternions c, d, we have that the CK-product of two powers of these Fueter variables behaves in a very nice way:

Proposition 4.4. *On the entire domain $\mathbb{H}^*$ we have that*

$$(4.5) \qquad \mu^\alpha c \star_{V_q} \mu^\beta d = \mu^{\alpha+\beta} cd.$$

Proof. The proof is left to the reader, with the hint that the main argument is the uniqueness of the CK extension of x^α, x^β, and $x^{\alpha+\beta}$, respectively. $\square$

Theorem 4.5. *For every $x \in \mathbb{H}^*$ it holds that*

$$(4.6) \qquad \left(\exp\left(x_0 \frac{1}{\vec{q}} E\right)\right)(x^\alpha) = \mu^\alpha.$$

Proof. We proceed in a number of steps.

Step 1: It holds that

$$(4.7) \qquad E\left(\frac{1}{\vec{q}}\right) = -\frac{1}{\vec{q}}.$$

Indeed, using (3.11) we can write

$$
\begin{aligned}
E\left(\frac{1}{\vec{q}}\right) &= \sum_{u=1}^{3} x_u \left(\frac{\mathbf{e}_u}{\vec{q}^2} + \frac{2x_u}{\vec{q}^3}\right) \\
&= \frac{\vec{q}}{\vec{q}^2} + 2\frac{x_1^2 + x_2^2 + x_3^2}{\vec{q}^3} \\
&= \frac{\vec{q}}{\vec{q}^2} - 2\frac{\vec{q}^2}{\vec{q}^3} \\
&= -\frac{1}{\vec{q}}.
\end{aligned}
$$

Step 2: For non-commuting functions of a real variable it holds that

$$(4.8) \qquad (fg)' = f'g + fg'.$$

This follows from

$$f(t)g(t) - f(t_0)g(t_0) = (f(t) - f(t_0))g(t) + f(t_0)(g(t) - g(t_0)).$$

Note that f and g may commute at a joint value t but we do not assume that $f(t)g(s) = g(s)f(t)$ for $t \neq s$.

Step 3: It holds that

$$(4.9) \qquad E\left(\frac{x^\alpha}{\bar{q}^n}\right) = \begin{cases} (|\alpha| - n)\frac{x^\alpha}{\bar{q}^n}, & 0 \le n \le |\alpha|, \\ 0 & \text{otherwise.} \end{cases}$$

We proceed by induction. The case $n = 0$ corresponds to the formula

$$E(x^\alpha) = |\alpha| x^\alpha.$$

We then write

$$\frac{x^\alpha}{\bar{q}^{n+1}} = \frac{x^\alpha}{\bar{q}^n}\frac{1}{\bar{q}}$$

and apply (4.8) with $f(x) = \frac{x^\alpha}{\bar{q}^n}$ and $g(x) = \frac{1}{\bar{q}}$ to get

$$
\begin{aligned}
E\left(\frac{x^\alpha}{\bar{q}^{n+1}}\right) &= \sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u}\left(\frac{x^\alpha}{\bar{q}^{n+1}}\right) \\
&= \left(\sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u}\left(\frac{x^\alpha}{\bar{q}^n}\right)\right)\frac{1}{\bar{q}} + \frac{x^\alpha}{\bar{q}^n}\left(\sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u}\frac{1}{\bar{q}}\right) \\
&= \underbrace{(|\alpha| - n)\frac{x^\alpha}{\bar{q}^n}\ \frac{1}{\bar{q}}}_{\text{induction at rank } n} + \frac{x^\alpha}{\bar{q}^n}\underbrace{\frac{-1}{\bar{q}}}_{\text{by (4.7)}} \\
&= (|\alpha| - n - 1)\frac{x^\alpha}{\bar{q}^{n+1}}.
\end{aligned}
$$

Step 4: We prove

$$(4.10) \qquad \left(E\frac{1}{\bar{q}}\right)^n(x^\alpha) = (|\alpha| - 1)(|\alpha| - 2)\cdots(|\alpha| - n + 1)\frac{x^\alpha}{\bar{q}^n}, \quad n = 1, 2, \dots.$$

We proceed by induction. The case $n = 1$ corresponds to the previous step. Then

$$
\begin{aligned}
\left(E\frac{1}{\bar{q}}\right)^{n+1}(x^\alpha) &= E\frac{1}{\bar{q}}\left(\left(E\frac{1}{\bar{q}}\right)^n(x^\alpha)\right) \\
&= E\frac{1}{\bar{q}}\left((|\alpha| - 1)(|\alpha| - 2)\cdots(|\alpha| - n + 1)\frac{x^\alpha}{\bar{q}^n}\right) \\
&= (|\alpha| - 1)(|\alpha| - 2)\cdots(|\alpha| - n + 1)E\left(\frac{x^\alpha}{\bar{q}^n}\right)\frac{1}{\bar{q}} \\
&= (|\alpha| - 1)(|\alpha| - 2)\cdots(|\alpha| - n + 1)(|\alpha| - n)\frac{x^\alpha}{\bar{q}^{n+1}},
\end{aligned}
$$

where we have used the induction hypothesis to go from the first to the second line, and formula (4.9) to go from the third line to the fourth.

Step 5: We prove (4.6). We first note that, in view of (4.10),

$$\left(E\frac{1}{\vec{q}}\right)^n(x^\alpha) = 0$$

for $n \geq |\alpha|$. Furthermore, since

$$\left(\frac{1}{\vec{q}}E\right)^n = \frac{1}{\vec{q}}\left(E\frac{1}{\vec{q}}\right)^{n-1}E, \quad n = 1, 2, \ldots,$$

we have that the series

$$\left(\exp\left(x_0\frac{1}{\vec{q}}E\right)\right)(x^\alpha) = x^\alpha + \sum_{n=1}^\infty \frac{x_0^n}{n!}\left(\frac{1}{\vec{q}}E\right)^n(x^\alpha)$$

$$= x^\alpha + \sum_{n=1}^\infty \frac{x_0^n}{n!}\frac{1}{\vec{q}}\left(E\frac{1}{\vec{q}}\right)^{n-1}E(x^\alpha)$$

has only a finite number of non-zero terms, and is therefore equal to

$$\left(\exp\left(x_0\frac{1}{\vec{q}}E\right)\right)(x^\alpha) = x^\alpha + \sum_{n=1}^\infty \frac{x_0^n}{n!}\frac{1}{\vec{q}}\left(E\frac{1}{\vec{q}}\right)^{n-1}E(x^\alpha)$$

$$= x^\alpha\left(\sum_{n=0}^{|\alpha|-1}\frac{|\alpha|(|\alpha|-1)\cdots(|\alpha|-n+1)}{n!}\left(\frac{x_0}{\vec{q}}\right)^n\right)$$

$$= x^\alpha\left(1 + \frac{x_0}{\vec{q}}\right)^{|\alpha|}.$$

The theorem is now proven.   □

Let b and c be two quaternions, and writing

$$q^n b = \sum_{\alpha\in\mathbb{N}_0^3} \mu^\alpha c_{\alpha,n}b \quad \text{and} \quad q^m c = \sum_{\alpha\in\mathbb{N}_0^3} \mu^\alpha c_{\alpha,m}c,$$

we have the $\star_{V_q}$ product of the two:

$$q^n b \star_{V_q} q^m c = \sum_{\alpha,\beta\in\mathbb{N}_0^3} \mu^{\alpha+\beta}c_{\alpha,n}bc_{\beta,m}c.$$

In the expression above, since b and c are quaternions they do not commute with $c_{\alpha,n}$ and $c_{\beta,m}$; however, when $b \in \mathbb{R}$ this expression reduces to

$$q^n b \star_{V_q} q^m c = \sum_{\alpha,\beta\in\mathbb{N}_0^3} \mu^{\alpha+\beta}c_{\alpha,n}c_{\beta,m}bc$$

(4.11)

$$= \sum_{\gamma\in\mathbb{N}_0^3}\left(\sum_{\substack{\alpha+\beta=\gamma\\\alpha,\beta\in\mathbb{N}_0^3}} c_{\alpha,n}c_{\beta,m}\right)bc, \quad b \in \mathbb{R}, \ c \in \mathbb{H}.$$

When $x_0 = 0$ we have $q = \vec{q}$ and the $\star_{V_q}$ product reduces to the pointwise product. We have

$$\vec{q}^n = \sum_{\alpha \in \mathbb{N}_0} x^\alpha c_{\alpha, n}$$

and

$$\vec{q}^n \vec{q}^m = \vec{q}^{n+m},$$

so that

$$c_{\gamma, n+m} = \sum_{\substack{\alpha + \beta = \gamma \\ \alpha, \beta \in \mathbb{N}_0^3}} c_{\alpha, n} c_{\beta, m},$$

$$q^n b \star_{\text{slice}} q^m c = q^{n+m} bc$$

(4.12)
$$= \sum_{\gamma \in \mathbb{N}_0^3} \mu^\gamma c_{\gamma, n+m} bc.$$

We then have the following proposition.

Proposition 4.6. *Let $b \in \mathbb{R}$ and $c \in \mathbb{H}$. On the entire domain $\mathbb{H}^*$ it holds that*

(4.13)
$$q^n b \star_{V_q} q^m c = q^n b \star_{\text{slice}} q^m c = q^{n+m} bc$$

and, more generally, for f intrinsic,

$$f \star_{V_q} g = f \star_{\text{slice}} g = fg.$$

The term *intrinsic* used in slice quaternionic analysis means that the function preserves all slices. This equality does not hold for more general functions in the kernel of V_q.

§5. Reproducing kernel Hilbert spaces

We now set the stage for the Schur analysis theory in this case and we begin by defining a reproducing kernel for the Hilbert space of power series in μ^α.

For r, R, and ρ strictly positive, let us first define the domain:

(5.1) $\qquad \Omega_{r, R, \rho} = \big\{ x \in \mathbb{R}^4 : r < |x_u| < R, \ u = 1, 2, 3, \text{ and } |x_0| < \rho \big\}.$

First we see that for any $x \in \Omega_{r, R, \rho} \subset \mathbb{H}^*$, we have

(5.2) $\qquad\qquad\qquad\qquad |\mu^\alpha(x)| \leq L_{r, R, \rho}^{|\alpha|},$

where $L_{r, R, \rho} = R(1 + \frac{\rho}{\sqrt{3}r})$.

Let $\Omega_{r, R, \rho}$ and L be defined as above. From standard positivity arguments, as in [7, 11] the next proposition follows.

Proposition 5.1. *Let c_α be a family of positive numbers for α in a subset S of $\mathbb{N}_0^3$. Assume that*

$$\sum_{\alpha \in S} \frac{L_{r_1,R_1,\rho_1}^{2|\alpha|}}{c_\alpha} < \infty$$

for all r_1, R_1, ρ_1 such that

(5.3) $r < r_1 < R_1 < R \quad and \quad 0 < \rho_0 < \rho.$

Then the function

(5.4) $K_{\mathbf{c}}(x,y) = \sum_{\alpha \in S} \frac{\mu^\alpha(x)\overline{\mu^\alpha(y)}}{c_\alpha}$

is positive definite in $\Omega_{r,R,\rho}$ and the associated reproducing kernel Hilbert space consists of the Fueter-like series with quaternionic coefficients f_α:

(5.5) $\mathfrak{H}(K_{\mathbf{c}}) = \left\{ f = \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha f_\alpha : f \text{ abs. conv. in } \Omega_{r,R,\rho}, \ \sum_{\alpha \in \mathbb{N}_0^3} c_\alpha |f_\alpha|^2 < \infty \right\}.$

Proof. The proof follows from the usual arguments as in [7, 11].   □

Proposition 5.2. *Elements of $\mathfrak{H}(K_{\mathbf{c}})$ are V_q-regular on*

(5.6) $\Omega(\mathfrak{H}(K_{\mathbf{c}})) = \left\{ x \in \mathbb{H}^* : \sum_{\alpha \in \mathbb{N}_0^3} \frac{|\mu^\alpha(x)|^2}{c_\alpha} < \infty \right\}.$

Proof. We proceed in a number of steps.

Step 1: Let $a \in (0,1)$. Then

$$\sum_{\alpha \in \mathbb{N}_0^3} a^{2|\alpha|} |\alpha|^2 < \infty,$$

and in particular $a^{|\alpha|}|\alpha|$ are uniformly bounded. By the Cauchy–Schwarz inequality,

$$(\alpha_1 + \alpha_2 + \alpha_3)^2 \le 3(\alpha_1^2 + \alpha_2^2 + \alpha_3^2).$$

Hence

$$\sum_{\alpha \in \mathbb{N}_0^3} a^{2|\alpha|}|\alpha|^2 \le 3 \sum_{\alpha \in \mathbb{N}_0^3} a^{2|\alpha|}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)$$

$$= 3 \sum_{u=1}^{3} \left(\sum_{\alpha \in \mathbb{N}_0^3} a^{2\alpha_1} a^{2\alpha_2} a^{2\alpha_3} \alpha_u^2 \right).$$

But

$$\left(\sum_{\alpha \in \mathbb{N}_0^3} a^{2\alpha_1} a^{2\alpha_2} a^{2\alpha_3} \alpha_1^2 \right) = \left(\sum_{\alpha_1=0}^{\infty} a^{2\alpha_1} \alpha_1^2 \right) \left(\sum_{\alpha_2=0}^{\infty} a^{2\alpha_2} \right) \left(\sum_{\alpha_3=0}^{\infty} a^{2\alpha_3} \right) < \infty,$$

and similarly for $u = 2$ and $u = 3$.

Step 2: For r_1 and R_1 as in (5.3),

$$\text{(5.7)} \qquad \sum_{\alpha \in \mathbb{N}_0^3} \frac{|\frac{\partial \mu^\alpha}{\partial x_1}|^2}{c_\alpha} < \infty$$

for $r_1 \leq |x_1| \leq R_1$, and similarly for x_2 and x_3.

We have

$$\frac{\partial \mu^\alpha}{\partial x_1} = \alpha_1 x^{\alpha - e_1} \left(1 + \frac{x_0}{\vec{q}}\right)^{|\alpha|}$$

$$+ x^\alpha \left[\sum_{\substack{t,s \in \mathbb{N}_0 \\ t+s=|\alpha|-1}} \left(1 + \frac{x_0}{\vec{q}}\right)^t \left(\frac{x_0 \mathbf{e}_u}{\vec{q}^2} + \frac{2 x_u x_0}{\vec{q}^3}\right) \left(1 + \frac{x_0}{\vec{q}}\right)^s \right]$$

and so

$$\left|\frac{\partial \mu^\alpha}{\partial x_1}\right| \leq \alpha_1 R_1^{|\alpha|-1} + R_1^{|\alpha|} |\alpha| \left(1 + \frac{\rho_1}{\sqrt{3}r_1}\right)^{|\alpha|-1} \left(\frac{\rho_1}{3r_1^2} + \frac{2 R_1 \rho_1}{3^{3/2} r_1^{3/2}}\right)$$

$$\leq R_1^\alpha \left(1 + \frac{\rho_1}{\sqrt{3}r_1}\right)^{|\alpha|} |\alpha| \underbrace{\left[1 + \frac{1}{1 + \frac{\rho_1}{\sqrt{3}r_1}} \left(\frac{\rho_1}{3r_1^2} + \frac{2 R_1 \rho_1}{3^{3/2} r_1^{3/2}}\right)\right]}_{= M, \text{ independent of } \alpha}.$$

Let R_2 be such that $R_1 < R_2 < R$. We can write

$$\sum_{\alpha \in \mathbb{N}_0^3} \frac{|\frac{\partial \mu^\alpha}{\partial x_1}|^2}{c_\alpha} \leq M \sum_{\alpha \in \mathbb{N}_0^3} \frac{R_1^{2|\alpha|} (1 + \frac{\rho_1}{\sqrt{3}r_1})^{2|\alpha|} |\alpha|^2}{c_\alpha}$$

$$= M \sum_{\alpha \in \mathbb{N}_0^3} \frac{R_2^{2|\alpha|} (1 + \frac{\rho_1}{\sqrt{3}r_1})^{2|\alpha|} |\alpha|}{c_\alpha} \left[\left(\frac{R_1}{R_2}\right)^{2|\alpha|} |\alpha|^2\right]$$

$$\leq M \sum_{\alpha \in \mathbb{N}_0^3} \frac{R_2^{2|\alpha|} (1 + \frac{\rho_1}{\sqrt{3}r_1})^{2|\alpha|} |\alpha|^2}{c_\alpha}$$

$$< \infty,$$

using Step 1, for some constant M.

Step 3: For r_1 and R_1 as in (5.3) we have

$$\text{(5.8)} \qquad \sum_{\alpha \in \mathbb{N}_0^3} \frac{|\frac{\partial \mu^\alpha}{\partial x_0}|^2}{c_\alpha} < \infty$$

for $|x_0| \leq \rho_1$.

This follows from

$$\frac{\partial}{\partial x_0}\mu^\alpha = x^\alpha\left[\sum_{\substack{t,s\in\mathbb{N}_0 \\ t+s=|\alpha|-1}}\left(1+\frac{x_0}{\vec{q}}\right)^t\left(\frac{1}{\vec{q}}\right)\left(1+\frac{x_0}{\vec{q}}\right)^s\right]$$

and the corresponding bound

$$\left|\frac{\partial}{\partial x_0}\mu^\alpha\right| \le |\alpha|R_1^{|\alpha|}\left(1+\frac{\rho_1}{\sqrt{3}r_1}\right)^{|\alpha|-1}\frac{1}{\sqrt{3}r_1}.$$

This step can be proven directly using the Appell-type property, i.e. $V_{\bar{q}}$ acts on powers of μ the same as $\frac{\partial}{\partial x_0}$.

Step 4: We prove that, pointwise, for $f = \sum_{\alpha\in\mathbb{N}_0^3}\mu^\alpha f_\alpha \in \mathfrak{H}(K_\mathbf{c})$,

$$\frac{\partial}{\partial x_u}\sum_{\alpha\in\mathbb{N}_0^3}\mu^\alpha f_\alpha = \sum_{\alpha\in\mathbb{N}_0^3}\frac{\partial}{\partial x_u}\mu^\alpha f_\alpha.$$

Using the Cauchy–Schwarz inequality and the previous lemma we see that the series of derivatives

$$\sum_{\alpha\in\mathbb{N}_0^3}\frac{\partial}{\partial x_u}\mu^\alpha f_\alpha = \sum_{\alpha\in\mathbb{N}_0^3}\frac{\frac{\partial}{\partial x_u}\mu^\alpha}{\sqrt{c_\alpha}}\sqrt{c_\alpha}f_\alpha$$

converges uniformly in intervals $[r_1, R_1]$. Since the series $\sum_{\alpha\in\mathbb{N}_0^3}\mu^\alpha f_\alpha$ converges in $\mathfrak{H}(K_\mathbf{c})$ then it converges pointwise and a classical calculus theorem allows us to conclude that we have convergence in Step 4. This classical theorem speaks of sequences and not of summable families, but we can reduce the latter to the case of sequences by identifying $\mathbb{N}_0^3$ and $\mathbb{N}_0$ via a bijection.

This solves the cases of $u = 1, 2, 3$. The case $u = 0$ is treated in a similar way and we leave the details to the reader.

We now apply Step 4 four times in the definition of V_q. More precisely,

$$V_q\left(\sum_{\alpha\in\mathbb{N}_0^3}\mu_\alpha f_\alpha\right) = \frac{\partial}{\partial x_0}\left(\sum_{\alpha\in\mathbb{N}_0^3}\mu_\alpha f_\alpha\right) - \frac{1}{\vec{q}}\sum_{u=1}^3 x_u\frac{\partial}{\partial x_u}\left(\sum_{\alpha\in\mathbb{N}_0^3}\mu_\alpha f_\alpha\right)$$

$$= \left(\sum_{\alpha\in\mathbb{N}_0^3}\frac{\partial}{\partial x_0}\mu_\alpha f_\alpha\right) - \frac{1}{\vec{q}}\sum_{u=1}^3 x_u\left(\sum_{\alpha\in\mathbb{N}_0^3}\frac{\partial}{\partial x_u}\mu_\alpha f_\alpha\right)$$

$$= \sum_{\alpha\in\mathbb{N}_0^3}V_q\mu^\alpha f_\alpha$$

$$= 0.$$

This concludes the proof. $\qquad\qquad\square$

Let us now turn to a definition of a backward-shift and multiplication operator in this case and we set

$$(5.9) \qquad e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad \text{and} \quad e_3 = (0,0,1).$$

Definition 5.3. The multiplication and backward shift operators on $\mathbb{H}^*$ are

$$(5.10) \qquad M_u \mu^\alpha = \mu^{\alpha+e_u}$$

$$(5.11) \qquad B_u \mu^\alpha = \frac{\alpha_u}{|\alpha|} \mu^{\alpha-e_u}, \quad u = 1,2,3$$

with the understanding that $B_u \mu^\alpha = 0$ if $\alpha_u = 0$.

Note that

$$(5.12) \qquad \left(\sum_{u=1}^{3} B_u M_u \right) \mu^\alpha = \mu^\alpha.$$

As expected, we have the following result:

Theorem 5.4. *Assume that the domain of definition of M_u in $\mathfrak{H}(K_{\mathbf{c}})$ contains the linear span of the Fueter polynomials μ^α. Then*

$$(5.13) \qquad M_u^*(\mu^\alpha) = \frac{c_\alpha}{c_{\alpha-e_u}} \mu^{\alpha-e_u},$$

with the understanding that the right-hand side of (5.13) *equals 0 if $\alpha_u = 0$.*

Proof. It is easy to check that

$$\begin{aligned}
\langle M_u \mu^\alpha, \mu^\beta \rangle &= \langle \mu^\alpha, \mu^{\beta+e_u} \rangle \\
&= c_\alpha \delta_{\alpha, \beta+e_u} \\
&= c_\alpha \delta_{\alpha-e_u, \beta} \\
&= \frac{c_\alpha}{c_{\alpha-e_u}} c_{\alpha-e_u} \delta_{\alpha-e_u, \beta} \\
&= \frac{c_\alpha}{c_{\alpha-e_u}} \langle \mu^{\alpha-e_u}, \mu^\beta \rangle.
\end{aligned}$$

This concludes the proof. $\qquad \square$

§6. Arveson space and Schur multipliers

For properties of the classical Arveson space, as well as a motivation for its definition, the reader can turn to [12, 24]. We now turn to the definition of an Arveson space in our case, which we will denote by $\mathfrak{A}$, in the present setting. We start with the definition of the counterpart of the *unit ball* in our case:

$$(6.1) \qquad \Omega_A = \Omega(\mathfrak{A}) = \Big\{ x \in \mathbb{H}^*, \ \sum_{\alpha \in \mathbb{N}_0^3} |\mu^\alpha(x)|^2 \frac{|\alpha|!}{\alpha!} < \infty \Big\}.$$

Definition 6.1. The V_q-Arveson space is the Hilbert space of absolutely convergent V_q-series on the unit ball

$$(6.2) \qquad \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha(x) c_\alpha,$$

with

$$(6.3) \qquad \langle f, f \rangle = \sum_{\alpha \in \mathbb{N}_0^3} \frac{\alpha!}{|\alpha|!} |c_\alpha|^2 < \infty,$$

that is,

$$(6.4) \quad \mathfrak{A} = \Big\{ f(x) = \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha(x) c_\alpha : f \text{ abs. conv. with } \langle f, f \rangle = \sum_{\alpha \in \mathbb{N}_0^3} \frac{\alpha!}{|\alpha|!} |c_\alpha|^2$$
$$< \infty \Big\}.$$

Here, the notation is that of Definition 3.3.

Definition 6.2. The reproducing kernel for this Arveson space is

$$(6.5) \qquad K_{\mathfrak{A}}(x, y) = \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha(x) \overline{\mu^\alpha(y)} \frac{|\alpha|!}{\alpha!}.$$

The domain of this kernel is defined by the following.

Proposition 6.3. *Elements of the Arveson space are V_q-regular in*

$$(6.6) \qquad \Omega_A = \Omega(\mathfrak{A}) = \Big\{ x \in \mathbb{H}^*, \ \sum_{\alpha \in \mathbb{N}_0^3} |\mu^\alpha(x)|^2 \frac{|\alpha|!}{\alpha!} < \infty \Big\}.$$

Proof. This follows from the reproducing kernel property, using the same type of arguments as in Section 5. $\qquad \square$

Evaluation at the origin does not exist; however, we can now define its counterpart by setting, for $f \in \mathfrak{A}$:

$$C(f) = c_{0,0,0}.$$

Theorem 6.4. *The Arveson space is the unique reproducing kernel Hilbert space (up to a multiplicative constant) of V_q power series such that M_1, M_2 and M_3 are bounded operators there and satisfy*

$$(6.7) \qquad I - \sum_{u=1}^{3} M_u M_u^* = CC^*.$$

Furthermore,

$$(6.8) \qquad M_u^* = B_u, \quad u = 1, 2, 3,$$

in the Arveson space.

Proof. We first assume that (6.7) (sometimes called the structural identity) is in force and compute the associated inner product. We proceed in a number of steps.

Step 1: We have that

$$C^*1 = 1_{\mathbb{H}}$$

(the constant function equal identically to 1).

Indeed,

$$
\begin{aligned}
(C^*1)(y) &= \langle C^*1, K_{\mathfrak{A}}(\cdot, y)\rangle_{\mathfrak{A}} \\
&= \langle 1_{\mathbb{H}}, CK_{\mathfrak{A}}(\cdot, y)\rangle_{\mathbb{H}} \\
&= 1.
\end{aligned}
$$

Here we used the fact that $CK_{\mathfrak{A}}(\cdot, y) = 1_{\mathbb{H}}$.

Step 2: $M_u^*1 = 0$. This is (5.13) with $\alpha = (0, 0, 0)$.

Step 3: Let α and β be different from $(0, 0, 0)$, and assume that (6.7) is in force. We have

(6.9)
$$1 = \sum_{u=1}^{3} \frac{c_\alpha}{c_{\alpha-e_u}}, \quad \alpha \in \mathbb{N}^3.$$

Indeed,

$$\langle \mu^\alpha, \mu^\beta \rangle = \sum_{u=1}^{3} \langle M_u^* \mu^\alpha, M_u^* \mu^\beta \rangle,$$

that is,

$$c_\alpha \delta_{\alpha,\beta} = \sum_{u=1}^{3} \frac{c_\alpha c_\beta}{c_{\alpha-e_u} c_{\beta-e_u}} \langle \mu^{\alpha-e_u} \mu^{\beta-e_u} \rangle$$

or, equivalently, (6.9) holds.

Step 4: The inner product is that of the Arveson space. From (6.9) we have that the sequence $d_\alpha = \frac{1}{c_\alpha}$ satisfies

$$d_\alpha = \sum_{u=1}^{3} d_{\alpha-e_u}, \quad \alpha \in \mathbb{N}^3.$$

Together with $d_{0,0,0} = 1$ we get $c_\alpha = c_{0,0,0} \frac{\alpha!}{|\alpha|!}$.

The converse is proven by reading these arguments backwards, with $c_\alpha = \frac{\alpha!}{|\alpha|!}$. $\qquad\square$

Proposition 6.5. *Assume that S is an $\mathbb{H}^{n \times m}$-valued multiplier defined on Ω_A. Then*

$$(6.10) \qquad \big(M_S^*(K_{\mathfrak{A}}(\cdot, y))\xi\big)(x) = \sum_{\alpha \in \mathbb{N}_0^3} \frac{|\alpha|!}{\alpha!} \mu^\alpha(x)(S \star_{V_q} \mu^\alpha)(y)^* \xi, \quad \forall \xi \in \mathbb{H}^n.$$

Proof. Since M_S^* is continuous we obtain

$$
\begin{aligned}
\langle M_S^*(K_{\mathfrak{A}}(\cdot, y)\xi), K_{\mathfrak{A}}(\cdot, x)\eta \rangle_{\mathfrak{A}} &= \langle K_{\mathfrak{A}}(\cdot, y)\xi, S \star_{V_q} K_{\mathfrak{A}}(\cdot, x)\eta \rangle_{\mathfrak{A}} \\
&= \sum_{\alpha \in \mathbb{N}_0^3} \frac{|\alpha|!}{\alpha!} \mu^\alpha(x) \langle K_{\mathfrak{A}}(\cdot, y)\xi, S\eta \star_{V_q} \mu^\alpha \rangle_{\mathfrak{A}} \\
&= \sum_{\alpha \in \mathbb{N}_0^3} \frac{|\alpha|!}{\alpha!} \mu^\alpha(x) \overline{\xi^* S\eta \star_{V_q} \mu^\alpha(y)} \\
&= \eta^* \sum_{\alpha \in \mathbb{N}_0^3} \frac{|\alpha|!}{\alpha!} \mu^\alpha(x) (S\xi \star_{V_q} \mu^\alpha(y))^* \\
&= \eta^* \sum_{\alpha \in \mathbb{N}_0^3} \frac{|\alpha|!}{\alpha!} \mu^\alpha(x) (S \star_{V_q} \mu^\alpha)(y)^* \xi. \qquad \square
\end{aligned}
$$

Theorem 6.6. *An $\mathbb{H}^{n \times m}$-valued function S is a contractive multiplier (i.e. a Schur multiplier) from $\mathfrak{A}^m$ into $\mathfrak{A}^n$ if and only if the $\mathbb{H}^{n \times n}$-valued kernel*

$$(6.11) \qquad K(x, y) = \sum_{\alpha \in \mathbb{N}_0^3} \frac{|\alpha|!}{\alpha!} \big\{ \mu^\alpha(x) \overline{\mu^\alpha(y)} I_n - (\mu^\alpha \star_{V_q} S)(x)(\mu^\alpha \star_{V_q} S)(y)^* \big\}$$

is positive definite in $\Omega_A = \Omega(\mathfrak{A})$.

Proof. To simplify notation we assume $n = m = 1$. Assume first that M_S is a contraction. Then $I_{\mathfrak{A}} - M_S M_S^*$ is a positive operator from $\mathfrak{A}$ into itself. The positivity of the kernel (6.11) follows then from the formula

$$
\begin{aligned}
&\langle (I_{\mathfrak{A}} - M_S M_S^*)K_{\mathfrak{A}}(\cdot, y), K_{\mathfrak{A}}(\cdot, x) \rangle_{\mathfrak{A}} \\
&= \sum_{\alpha \in \mathbb{N}_0^3} \frac{|\alpha|!}{\alpha!} \big\{ \mu^\alpha(x) \overline{\mu^\alpha(y)} I_n - (\mu^\alpha \star_{V_q} S)(x)(\mu^\alpha \star_{V_q} S)(y)^* \big\},
\end{aligned}
$$

which in turn is obtained from (6.10).

Conversely, if the kernel (6.11) is positive definite in $\Omega(\mathfrak{A})$, the right linear span of the pairs of functions

$$\left((K_{\mathfrak{A}}(\cdot, y))(x), \sum_{\alpha \in \mathbb{N}_0^3} \frac{|\alpha|!}{\alpha!} (\mu^\alpha(x)(S \star_{V_q} \mu^\alpha)(y)^*) \right) \in \mathfrak{A} \times \mathfrak{A}$$

defines a densely defined contractive relation, which extends therefore to the graph of an everywhere defined contraction; the adjoint of this contraction is M_S.   □

In view of the structural identity (6.9), the Fueter-like variables are Schur multipliers. In the next section we present another important Schur multiplier, which is the counterpart here of an elementary Blaschke factor.

§7. V_q Blaschke-type factors

As mentioned in the introduction, Blaschke products are building blocks of classical function theory on the disk as in [37] and, more generally, in Schur analysis, and they also constitute essential tools of the Beurling–Lax theorem, which characterizes shift-invariant subspaces of the Hardy space.

We denote by $\Omega_1(\mathfrak{A})$ the set of $x \in \mathbb{H}^*$ such that

$$\sum_{u=1}^{3} |\mu_u(x)|^2 < 1.$$

By inequalities (3.8) we have

$$\left\{ x \in \mathbb{H}^*; \ 3x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1 \right\} \subset \Omega_1(\mathfrak{K}).$$

Remark 7.1. It is essential to consider elements of $\mathbb{H}^*$, for example the element $x = (\frac{1}{3}, 0, 0, 0) \in \{x \in \mathbb{R}^4; \ 3x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1\}$. However, $\sum_{u=1}^{3} |\mu_u(x)|^2$ is not defined.

It is also interesting to note that the above ellipsoid also appears in [9].

Theorem 7.2. *Let $a \in \Omega_1(\mathfrak{A})$ and set $\mu(a) = (\mu_1(a)\ \mu_2(a)\ \mu_3(a))$; with this notation we have that $\mu(a)\mu(a)^* = \sum_{u=1}^{3} |\mu_u(a)|^2 < 1$. Then the multiplication operator by $\mu(a)$ on the left is a strict contraction from $\mathfrak{A}^3$ into $\mathfrak{A}$ and the map*

(7.1)
$$\begin{aligned} B_a(x) &= (1 - \mu(a)\mu(a)^*)^{1/2}(1 - \mu(x)\mu(a)^*)^{-\star_{V_q}} \\ &\quad \star_{V_q} (\mu(x) - \mu(a))(I_3 - \mu(a)^*\mu(a))^{-1/2} \end{aligned}$$

is a Schur multiplier from $\mathfrak{A}^3$ into $\mathfrak{A}$.

Proof. The proof follows the proofs in [6, Prop. 4.1, p. 11] and [9, Thm. 4.7, p. 146], and is briefly outlined. In the complex setting we also refer to [36], where a different, but equivalent expression is given for B_a. We set

$$J = \begin{pmatrix} I_{\mathfrak{A}} & 0 \\ 0 & -I_{\mathfrak{A}^3} \end{pmatrix}.$$

Step 1: The operator-matrix

$$H(a) = \begin{pmatrix} (I_{\mathfrak{A}} - M_{\mu(a)}M_{\mu(a)}^*)^{-1/2} & -M_{\mu(a)}(I_{\mathfrak{A}^3} - M_{\mu(a)}^*M_{\mu(a)})^{-1/2} \\ -M_{\mu(a)}^*(I_{\mathfrak{A}} - M_{\mu(a)}M_{\mu(a)}^*)^{-1/2} & (I_{\mathfrak{A}^3} - M_{\mu(a)}^*M_{\mu(a)})^{-1/2} \end{pmatrix}$$

is J-unitary, i.e.

$$H(a)JH(a)^* = H(a)^*JH(a) = J.$$

See [25] for the general case where $M_{\mu(a)}$ is replaced by an arbitrary strict contraction between Hilbert spaces. The operator-matrix $H(a)$ is called the Halmos extension of $M_{\mu(a)}$.

Step 2: The operator

$$\begin{pmatrix} I_{\mathfrak{A}} & M_\mu \end{pmatrix} H(a)JH(a)^* \begin{pmatrix} I_{\mathfrak{A}} \\ M_\mu^* \end{pmatrix}$$

is non-negative.

Indeed, $H(a)JH(a)^* = J$ by the previous step, and $I_{\mathfrak{A}} - M_\mu M_\mu^* \geq$ by (6.9).

Step 3: B_a is a Schur multiplier. It suffices to write

$$\begin{pmatrix} I_{\mathfrak{A}} & M_\mu \end{pmatrix} H(a)$$
$$= \left((I_{\mathfrak{A}} - M_\mu M_{\mu(a)}^*)(I_{\mathfrak{A}} - M_{\mu(a)}M_{\mu(a)}^*)^{-1/2} \ (M_\mu - M_{\mu(a)})(I_{\mathfrak{A}^3} - M_{\mu(a)}^*M_{\mu(a)})^{-1/2} \right)$$
$$= \left(\sqrt{1 - \mu(a)^*\mu(a)} \right)^{1/2} (I_{\mathfrak{A}} - M_\mu M_{\mu(a)}^*) \begin{pmatrix} I_{\mathfrak{A}} & M_{B_a} \end{pmatrix}.$$

This concludes the proof. □

§8. V_q rational functions

The notion of a rational function is important in Schur analysis, and leads to state space representations of linear systems. The study of hypercomplex rational functions using the Cauchy–Kovalevskaia extension theorem originates with the work of Laville; see [34].

Definition 8.1. The $\mathbb{H}^{n \times m}$-valued V_q-regular function on an open domain $\Omega \subset \mathbb{H}^*$ is called V_q-rational if its restriction to $x_0 = 0$ can be written as

$$(8.1) \qquad R(0, x_1, x_2, x_3) = D + C\left(I_N - \sum_{u=1}^3 x_u A_u \right)^{-1} \left(\sum_{u=1}^3 x_u B_u \right),$$

where $D \in \mathbb{H}^{n \times m}$, $C \in \mathbb{H}^{n \times N}$, $A_1, A_2, A_3 \in \mathbb{H}^{N \times N}$, and $B_1, \ldots, B_3 \in \mathbb{H}^{N \times m}$.

Equivalently, taking the V_q-extension, we can write

$$(8.2) \qquad R(x) = D + C(I - \mu(x)A)^{-\star_{V_q}} \star_{V_q} \mu(x)B,$$

where

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}.$$

Proposition 8.2. *The Blaschke factor* (7.1) *is a V_q-rational function on $\Omega_1(\mathfrak{A})$.*

Proof. We follow the proof of [6, Prop. 4.1, p. 12] and first recall that for a contraction $K \in \mathbb{H}^{s \times r}$ it holds that

$$K(I_r - K^*K)^{1/2} = (I_s - KK^*)^{1/2}K \quad \text{and} \quad K^*(I_s - KK^*)^{1/2} = (I_r - K^*K)^{1/2}K^*.$$

These equalities are used in the computations below.

To see that B_a is V_q-rational we write (with $\mu(x) = (x_1\ x_2\ x_3)$)

$$
\begin{aligned}
& B_a(0, x_1, x_2, x_3) \\
& \quad = B_a(0,0,0,0) + B_a(0, x_1, x_2, x_3) - B_a(0,0,0,0) \\
& \quad = -\underbrace{(1 - \mu(a)\mu(a)^*)^{1/2}\mu(a)(I_3 - \mu(a)^*\mu(a))^{-1/2}}_{\mu(a)} \\
& \qquad + (1 - \mu(a)\mu(a)^*)^{1/2}(1 - \mu(x)\mu(a)^*)^{-1}(\mu(x) - \mu(a))(I_3 - \mu(a)^*\mu(a))^{-1/2} \\
& \qquad + (1 - \mu(a)\mu(a)^*)^{1/2}\mu(a)(I_3 - \mu(a)^*\mu(a))^{-1/2} \\
& \quad = -\mu(a) \\
& \qquad + (1 - \mu(a)\mu(a)^*)^{1/2}(1 - \mu(x)\mu(a)^*)^{-1} \\
& \qquad \quad \times \left\{ \mu(x) - \mu(a) + (1 - \mu(x)\mu(a)^*)\mu(a) \right\}(I_3 - \mu(a)^*\mu(a))^{-1/2} \\
& \quad = -\mu(a) + (1 - \mu(a)\mu(a)^*)^{1/2}(1 - \mu(x)\mu(a)^*)^{-1}\mu(x)(I_3 - \mu(a)^*\mu(a))^{1/2},
\end{aligned}
$$

which is of the form (8.1) with

$$(8.3) \qquad T = \begin{pmatrix} A_1\ B_1 \\ A_2\ B_2 \\ A_3\ B_3 \\ C\ D \end{pmatrix} = \begin{pmatrix} \mu(a)^* & (I_3 - \mu(a)^*\mu(a))^{1/2} \\ (1 - \mu(a)\mu(a)^*)^{1/2} & -\mu(a) \end{pmatrix}.$$

$\square$

Remark 8.3. We note that (8.3) is co-isometric. Existence of a co-isometric realization is a general property of Schur multipliers, and will be considered elsewhere. Here, T is in fact unitary.

Rational functions were defined in the setting of the Cauchy operator in [9]. When restricted to $x_0 = 0$, one obtains the same class of functions, namely functions of the form (8.1). The V_q-rational functions are stable under sum, V_q-product, and V_q-inversion when sizes are compatible. The arguments are the same as in [9], to which we refer the reader.

§9. Conclusions and future endeavors

We are now extending this work to analyze a theory of Schur–Agler functions (see [1] and see [10] for the Fueter case) and Schur multipliers, as well as to expand the rational function theory in this context.

Consider a system of linear ordinary differential equations with variable coefficients of the form

$$V_j(f) = \frac{\partial f_j}{\partial x_0} - \sum_{u=1}^{3} \sum_{k=0}^{3} a_{k,u}(x) \frac{\partial f_k}{\partial x_u} = 0, \quad j = 0, 1, 2, 3,$$

where the $a_{k,u}$ are real analytic on some open subset of the real line, and the system of equations is denoted by V. Thanks to the Cauchy–Kovalevskaia theorem one can define V-Fueter variables as the CK extensions of the functions x_u, $u = 1, 2, 3$, and monomials as CK-extensions of the monomials x^α. It seems difficult though to develop a general theory which goes beyond some trivial facts. The cases considered in [9] (for the Cauchy–Fueter operator, factorizing the $\mathbb{R}^4$ Laplacian) and in [11] associated to the operator

$$\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

studied in the setting of split quaternion, together with the present analysis for the V_q-operator, exhibit how different each specific case can be.

Acknowledgments

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