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A priori bounds for geodesic diameter. Part I. Integral chains with coefficients in a complete normed commutative group

Ulrich Menne and Christian Scharrer

Abstract. As service to the community, we provide – for Euclidean space – a basic treatment of locally rectifiable chains and of the complex of locally integral chains. In this setting, we may beneficially develop the idea of a complete normed commutative group bundle over the Grassmann manifold whose fibre is the coefficient group of the chains. Our exposition also sheds new light on some algebraic aspects of the theory. Finally, we indicate an extension to a geometric approach to locally flat chains centring on locally rectifiable chains rather than completion procedures.

1. Introduction

Throughout the introduction, *m is a nonnegative integer, n and d are positive integers, U is an open subset of $\mathbf{R}^n$, and G is a complete normed commutative group.* Our notation is based on H. Federer's treatise [11]; see Section 1.4. In particular, $\mathbf{I}_m^{\mathrm{loc}}(U)$, $\mathscr{R}_m^{\mathrm{loc}}(U)$, and $\mathscr{F}_m^{\mathrm{loc}}(U)$ denote the commutative groups of those m-dimensional currents in U which locally are integral, rectifiable, and flat, respectively.

1.1. Overview

The primary goal of this first paper of our series is to provide a self-contained exposition with complete proofs of all basic facts for locally rectifiable chains and locally integral chains in U with coefficients in G to be employed in the third and final paper (see [21]). The Euclidean setting allows us to beneficially employ the concept of m-dimensional approximate tangent planes in $\mathbf{R}^n$ through the usage of rectifiable varifolds and the study of the *complete normed commutative group bundle* $\mathbf{G}(n, m, G)$ over the Grassmann manifold $\mathbf{G}(n, m)$ with fibre G; the latter is an idea originating from F. Almgren Section 2.4 and Subsection 2.6 (d) in [3]) and T. De Pauw and R. Hardt (Section 3.6 in [9]) which is not developed in those works.

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In comparison to W. Fleming [13] and T. De Pauw and R. Hardt [9, 10], five distinctive features of our approach may be summarised as follows. Firstly, all our classes of chains are based on *local chains*.[1] Secondly, we freely use algebraic properties of commutative groups and topological properties of normed commutative groups following N. Bourbaki. Thirdly, the *closure theorem* – or the boundary rectifiability theorem, in the terminology of L. Simon – in the context of integer coefficients (i.e., for rectifiable currents) plays a central role in our construction of integral chains with coefficients in a general complete normed commutative group. Fourthly, we construct the chain complex of *simple locally integral G chains* as starting point for a closure procedure – based on pairs of locally rectifiable G chains – leading to locally integral G chains. By a simple locally integral G chain, we mean a locally rectifiable G chain which is expressible as finite sum of products of locally integral $\mathbf{Z}$ chains (isomorphically, locally integral currents) with elements of G. Our choice is motivated by the favourable closedness properties of this chain complex under restriction, push forward, and slicing; traditionally, polyhedral chains or Lipschitz chains serve as starting point to construct flat G chains by means of completion. Fifthly, we indicate how the concepts of locally integral G chain and locally rectifiable G chain can be used to construct the chain complex of locally flat G chains by taking a suitable quotient – again based on pairs of locally rectifiable G chains; previous approaches firstly define the chain complex of flat G chains and obtain integral G chains as a subcomplex. Thus, in our proposed treatment, locally rectifiable G chains are central for both constructions: that of locally integral G chains, and that of locally flat G chains.

The group of rectifiable G chains, as defined by T. De Pauw and R. Hardt in Section 3.6 of [9], is isomorphic to ours of locally rectifiable G chain with finite mass, see 3.5. For $G = \mathbf{R}$, our concepts are isomorphic to those of H. Federer [12]; see 5.1 and 7.3. For $G = \mathbf{Z}/d\mathbf{Z}$, our integral G chains are isomorphic to the corresponding subgroup, $\mathbf{I}_m^d(U)$, of flat chains modulo d as defined by H. Federer in 4.2.26 of [11]; see 5.2.

1.2. Outline by section

Preliminaries. In this section, we gather six strings of preparations: Firstly, we summarise basic properties of normed commutative groups in 2.1–2.5; in particular, we recall that it may happen that G is isomorphic to $\mathbf{Z}$ as commutative group but not so as normed commutative group, see 2.3. Secondly, we make some measure-theoretic preparations in 2.6–2.12. Thirdly, we construct two auxiliary functions of class ∞ in 2.14–2.16. The second string and the third string will be employed throughout this series of papers, see 2.13 and 2.17. Fourthly, in analogy to

$$\mathscr{F}_m(U) = \{Q + \partial R : Q \in \mathscr{R}_m(U), R \in \mathscr{R}_{m+1}(U)\},$$

we represent $\mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n)$ as quotient vector space in 2.18–2.20. Fifthly, in 2.21–2.24, we similarly exhibit $\mathscr{F}_m^{\mathrm{loc}}(U)$ as quotient commutative group. Sixthly, we study weights of rectifiable m varifolds in 2.25–2.37; this includes the area formula – including a version for G valued functions – in 2.27 and 2.31, the coarea formula in 2.33, and the Cartesian product in 2.36.

[1] B. White indicated the extension of key elements of the theory of [13] to the local setting in the Appendix of [26].

Rectifiable chains. We begin by defining and studying the complete normed commutative group bundle $\mathbf{G}(n, m, G)$ over $\mathbf{G}(n, m)$ with fibre G in 3.1–3.4. This allows to introduce the necessary operations for locally rectifiable G chains – addition, right-multiplication on $\mathbf{G}(n, m, \mathbf{Z})$ with members of G, push forward, slicing, and Cartesian product – firstly on the level of the bundle. Then, the complete normed commutative group

$$\mathscr{R}_m^{\mathrm{loc}}(U, G)$$

of m-dimensional *locally rectifiable G chains* in U is defined using equivalence classes of certain $\mathscr{H}^m$ measurable $\mathbf{G}(n, m, G)$ valued functions in 3.5. To each S in $\mathscr{R}_m^{\mathrm{loc}}(U, G)$ correspond the weight $\|S\|$ of an m-dimensional rectifiable varifold in U and a representing function (i.e., a member of the equivalence class S)

$$\vec{S};$$

the role of the latter is analogous to the product $\boldsymbol{\Theta}^m(\|Q\|, \cdot)\vec{Q}$ for $Q \in \mathscr{R}_m^{\mathrm{loc}}(U)$. Whenever $\|S\|$ is absolutely continuous with respect to the weight ϕ of some m-dimensional rectifiable varifold in U, there exists a representing function of S which is *adapted* to ϕ, see 3.5; for instance, $\vec{S}$ is adapted to $\|S\|$. This concept allows to combine the results on the bundle $\mathbf{G}(n, m, G)$ with those on rectifiable varifolds to study the afore-mentioned operations on $\mathscr{R}_m^{\mathrm{loc}}(U, G)$ in 3.6–3.8.

Integral chains. We construct the complete normed commutative group $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ of m-dimensional locally integral G chains in U and the corresponding boundary operator ∂_G in six steps.

Step 1 (integer coefficients). To define the subgroup $\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z})$ of $\mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z})$ and the boundary operator $\partial_{\mathbf{Z}}$ corresponding to $\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z})$, we employ the canonical isomorphism of commutative groups $\mathscr{R}_m^{\mathrm{loc}}(U) \simeq \mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z})$ in 4.1.

Step 2 (algebra lemma). For homomorphisms $i: A \to B$ of commutative groups, we establish an equivalent condition to $i \otimes \mathbf{1}_H$ being univalent (i.e., injective) for every commutative group H in 4.2; this is accomplished by expressing H as inductive limit of its finitely generated subgroups and employing the structure theorem for finitely generated commutative groups. We also recall that, contrary to the category of vector spaces, univalentness of i does not imply the same for the homomorphism $i \otimes \mathbf{1}_H$, see 4.3.

Step 3 (application of the closure theorem for rectifiable currents). In 4.4, we verify the condition obtained in Step 2 for the inclusion map i of $\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z})$ into $\mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z})$ by means of the *closure theorem* for rectifiable currents (see Theorem 4.2.16 (2) in [11] or Theorem 30.3 in [23]). This amounts to verifying, for every positive integer d, we have $Q \in \mathbf{I}_m^{\mathrm{loc}}(U)$ whenever $Q \in \mathscr{R}_m^{\mathrm{loc}}(U)$ and $dQ \in \mathbf{I}_m^{\mathrm{loc}}(U)$.

Step 4 (simple locally rectifiable G chains). In 4.5, we use the canonical multiplication

$$\cdot : \mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z}) \times G \to \mathscr{R}_m^{\mathrm{loc}}(U, G),$$

to obtain the induced homomorphism

$$\rho_{U,m,G} : \mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G \to \mathscr{R}_m^{\mathrm{loc}}(U, G),$$

whose image is dense in $\mathscr{R}_m^{\mathrm{loc}}(U, G)$. Based on the structure theorem for finitely generated commutative groups, we next prove that

$$\rho_{U,m,G} \text{ is univalent}$$

in 4.6. In case G is finite, we deduce

$$\mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G \simeq \mathscr{R}_m^{\mathrm{loc}}(U, G),$$

see 4.8.

Step 5 (simple locally integral G chains). By Steps 3 and 4, the composition of homomorphisms

$$\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G \xrightarrow{\quad i \otimes \mathbf{1}_G \quad} \mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G \xrightarrow{\quad \rho_{U,m,G} \quad} \mathscr{R}_m^{\mathrm{loc}}(U, G),$$

where $i \colon \mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \to \mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z})$ is the inclusion, is univalent; its image consists, by definition, of all m-dimensional *simple locally integral G chains* in U. By univalentness, the boundary operator ∂_G of this chain complex may be defined by

$$\partial_G(S \cdot g) = (\partial_{\mathbf{Z}} S) \cdot g, \quad \text{for } S \in \mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \text{ and } g \in G,$$

in 4.9; this is in accordance with the previous definition in case $G = \mathbf{Z}$.

Step 6 (closure operation). We let

$$\mathbf{I}_0^{\mathrm{loc}}(U, G) = \mathscr{R}_0^{\mathrm{loc}}(U, G).$$

Whenever $m \geq 1$, the complete normed commutative group $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ is defined in 4.11 as closure of the subgroup of

$$\mathscr{R}_m^{\mathrm{loc}}(U, G) \times \mathscr{R}_{m-1}^{\mathrm{loc}}(U, G)$$

consisting of all pairs $(S, \partial_G S)$ corresponding to m-dimensional simple locally integral G chains S in U; the boundary operator

$$\partial_G : \mathbf{I}_m^{\mathrm{loc}}(U, G) \to \mathbf{I}_{m-1}^{\mathrm{loc}}(U, G)$$

is then induced by the shift operator mapping $(S, T) \in \mathscr{R}_m^{\mathrm{loc}}(U, G) \times \mathscr{R}_{m-1}^{\mathrm{loc}}(U, G)$ onto $(T, 0)$ if $m \geq 2$ and onto T if $m = 1$. Clearly, ∂_G is continuous. We then show in 4.13 that the canonical projection of $\mathscr{R}_m^{\mathrm{loc}}(U, G) \times \mathscr{R}_{m-1}^{\mathrm{loc}}(U, G)$ onto its first factor, restricted to $\mathbf{I}_m^{\mathrm{loc}}(U, G)$, is univalent; this allows us to subsequently

> identify $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ with a dense subgroup of $\mathscr{R}_m^{\mathrm{loc}}(U, G)$ so that ∂_G extends the boundary operator on simple locally integral G chains.

In this process, establishing that we have $T = 0$ whenever $(0, T) \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$ is ultimately reduced to the case that $U = \mathbf{R}^n$ and that $\operatorname{spt} \|T\|$ is a compact subset of an $(m-1)$-dimensional vector subspace.

During Steps 1–6, we keep track in 4.1, 4.5, and 4.10 of how the operations push forward, Cartesian product, and slicing on rectifiable G chains interact with the intermediately constructed boundary operators. This is crucial: firstly, in the identification of $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ with a subgroup of $\mathscr{R}_m^{\mathrm{loc}}(U, G)$ carried out in Step 6, whence the properties of these operations for $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ in 4.13 and the homotopy formula in 4.16 follow, and secondly, in proving in 4.18 that the restriction operators $r_m\colon \mathscr{R}_m^{\mathrm{loc}}(V, G) \to \mathscr{R}_m^{\mathrm{loc}}(U, G)$ satisfy

$$r_m[\mathbf{I}_m^{\mathrm{loc}}(V, G)] \subset \mathbf{I}_m^{\mathrm{loc}}(U, G)$$

and commute with ∂_G, whenever $U \subset V \subset \mathbf{R}^n$ and V is open.

To compare with classical examples, we define (see 3.5 and 4.17) the subgroups

$$\mathscr{R}_m(U, G) = \mathscr{R}_m^{\mathrm{loc}}(U, G) \cap \{S : \operatorname{spt}\|S\| \text{ is compact}\},$$
$$\mathbf{I}_m(U, G) = \mathbf{I}_m^{\mathrm{loc}}(U, G) \cap \{S : \operatorname{spt}\|S\| \text{ is compact}\}.$$

Moreover, Steps 1, 2, and 4 allow us to define the subgroup $\mathscr{P}_m(U, G)$ of $\mathscr{R}_m^{\mathrm{loc}}(U, G)$ consisting of all m-dimensional *polyhedral* G *chains* in U, see 4.5.

Classical coefficient groups. We compare our treatment of rectifiable and integral G chains with that of the classical cases $G = \mathbf{R}$ in [12] and $G = \mathbf{Z}/d\mathbf{Z}$ in 4.2.26 of [11]. In 5.1, we provide canonical isomorphisms – commuting with the boundary operators, restriction, push forward, Cartesian product, and slicing – showing that

$$\mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}) \simeq \mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n) \cap \{Q : Q \text{ has positive densities}\}.$$

This isomorphism maps $\mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}) \cap \{S : S \text{ is simple}\}$ onto the

$$\text{real linear span of } \mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^n) \text{ in } \mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n).$$

For $m \geq 1$, we determine the closure of these groups to obtain

$$\mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}) \simeq \mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n) \cap \{Q : Q \text{ and } \partial Q \text{ have positive densities}\};$$

this is based on the deformation theorem for members of the group on the right from [12] and the resulting approximation theorem by push forwards of m-dimensional real polyhedral chains in $\mathbf{R}^n$ by diffeomorphisms of class 1 of $\mathbf{R}^n$.

Similarly, relying on the approximation theorem for members of $\mathbf{I}_m^d(\mathbf{R}^n)$ from 4.2.26 in [11], we construct canonical isomorphisms

$$\mathscr{R}_m(\mathbf{R}^n, \mathbf{Z}/d\mathbf{Z}) \simeq \mathscr{R}_m^d(\mathbf{R}^n), \quad \mathbf{I}_m(\mathbf{R}^n, \mathbf{Z}/d\mathbf{Z}) \simeq \mathbf{I}_m^d(\mathbf{R}^n),$$
$$\mathbf{I}_m(\mathbf{R}^n, \mathbf{Z}/d\mathbf{Z}) \cap \{S : S \text{ is simple}\} \simeq \big\{(Q)^d : Q \in \mathbf{I}_m(\mathbf{R}^n)\big\}$$

in 5.2; in particular, R. Young's structural result

$$\mathbf{I}_m^d(\mathbf{R}^n) = \{(Q)^d : Q \in \mathbf{I}_m(\mathbf{R}^n)\}$$

in Corollary 1.5 of [27] may be restated by saying that every $S \in \mathbf{I}_m(\mathbf{R}^n, \mathbf{Z}/d\mathbf{Z})$ is simple. To clarify the literature regarding the impossibility of an analogous structural result for $\mathbf{I}_{m,K}^d(\mathbf{R}^n)$ for general compact subsets K of $\mathbf{R}^n$, we include in 5.3 an unpublished correction listed by H. Federer.

Constancy theorem. To construct an example of a one-dimensional indecomposable integral G chain whose associated rectifiable varifold is decomposable in the second paper of our series (see Example 6.8 in [20]), we provide a constancy theorem for m-dimensional locally integral G chains whose boundary lies outside of an m-dimensional connected orientable submanifold M of class 1 of U in 6.1. In the model case that $m = n$ and

$$M = \mathbf{R}^m \cap \{x : a_i < x_i < b_i \text{ for } i = 1, \dots, m\},$$

where $-\infty < a_i < b_i < \infty$ for $i = 1, \dots, m$, the constancy theorem yields that T in $\mathbf{I}_m(\mathbf{R}^m, G)$, satisfying $\operatorname{spt}\|\partial_G T\| \subset \operatorname{Bdry} M$, equals $Q \cdot g$, for some $g \in G$, where Q in $\mathbf{I}_m(\mathbf{R}^m, \mathbf{Z})$ corresponds to $(\mathscr{L}^m \llcorner M) \wedge \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_m \in \mathbf{I}_m(\mathbf{R}^m)$. We prove the general case and the model case by simultaneous induction on m. This is mostly based on H. Federer's arguments for the classical coefficient groups in 4.1.31 (2) and 4.2.3 of [11] – see also the end of 4.2.26 on p. 432 of [11] –, which we adapt and merge by means of our restriction operators r_m, see 6.2 and 6.3.

Flat chains. To conclude the development of the present paper, we indicate in 7.1 how to extend our approach to include a chain complex of locally flat G chains. Namely, we construct complete normed commutative groups $\mathscr{F}_m^{\mathrm{loc}}(U, G)$ together with continuous boundary operators ∂_G such that $\mathscr{P}_m(U, G)$ is dense in $\mathscr{F}_m^{\mathrm{loc}}(U, G)$ and

$$\mathscr{F}_m^{\mathrm{loc}}(U, G) = \big\{S + \partial_G T : S \in \mathscr{R}_m^{\mathrm{loc}}(U, G), T \in \mathscr{R}_{m+1}^{\mathrm{loc}}(U, G)\big\}.$$

In fact, $\mathscr{F}_m^{\mathrm{loc}}(U, G)$ is defined to be the quotient

$$\big(\mathscr{R}_m^{\mathrm{loc}}(U, G) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U, G)\big) \big/ H_m,$$

where the *closed* subgroup H_m of $\mathscr{R}_m^{\mathrm{loc}}(U, G) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U, G)$ is given by

$$H_m = \big(\mathbf{I}_m^{\mathrm{loc}}(U, G) \times \mathbf{I}_{m+1}^{\mathrm{loc}}(U, G)\big) \cap \{(S, T) : S + \partial_G T = 0\}.$$

Finally, we obtain (in 7.2 and 7.3) canonical isomorphisms

$$\mathscr{F}_m^{\mathrm{loc}}(U, \mathbf{Z}) \simeq \mathscr{F}_m^{\mathrm{loc}}(U) \quad \text{and} \quad \mathscr{F}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}) \simeq \mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n);$$

these are based on the representations of $\mathscr{F}_m^{\mathrm{loc}}(U)$ and $\mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n)$ recorded earlier.

1.3. Remarks

Constancy theorem. The case that M, for some cubical subdivision of $\mathbf{R}^n$, equals the m-skeleton minus the $(m-1)$-skeleton (e.g., $M = \mathbf{W}'_m \sim \mathbf{W}'_{m-1}$) is a basic ingredient for deformation theorems. For general G and such M, a constancy theorem was first formulated by W. Fleming in Lemma 7.2 of [13] for compactly supported normal G chains; however, similar to H. Federer for classical coefficient groups in 4.2.3 of [11] and T. De Pauw and R. Hardt for general G in Theorem 6.3 of [10], we avoid unspecific references to topology in our argument.

Possible continuation of these notes. Besides extending the various operations studied for $\mathscr{R}_m^{\mathrm{loc}}(U,G)$ to $\mathscr{F}_m^{\mathrm{loc}}(U,G)$, one would surely intend to add notions for a Borel regular measure $\|S\|$ and for the support of S associated with S in $\mathscr{F}_m^{\mathrm{loc}}(U,G)$ and to obtain suitable deformation theorems. In this regard, we would expect the representation

$$\mathscr{F}_m^{\mathrm{loc}}(U,G) = \big\{S + \partial_G T : S \in \mathscr{R}_m^{\mathrm{loc}}(U,G), T \in \mathscr{R}_{m+1}^{\mathrm{loc}}(U,G)\big\}$$

to be particularly expedient. For $G = \mathbf{Z}$ or $G = \mathbf{R}$, the deformation theorems in 4.2.9 of [11] and in Section 4 of [12], respectively, form the key ingredients in proving the above isomorphisms for $\mathscr{F}_m^{\mathrm{loc}}(\mathbf{R}^n, G)$ which, for these G, yield an affirmative answer to the following question. Assuming $m \geq 1$,

is $\mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^n,G)$ equal to $\mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n,G) \cap \big\{S : \partial_G S \in \mathscr{R}_{m-1}^{\mathrm{loc}}(\mathbf{R}^n,G)\big\}$?

If successful, these extensions would yield a geometric approach to $\mathscr{F}_m^{\mathrm{loc}}(U,G)$ with $\mathscr{R}_m^{\mathrm{loc}}(U,G)$ and the complete normed commutative group bundle $\mathbf{G}(n,m,G)$ taking the centre stage instead of functional analytic completion procedures.

Background. A notion of flat G chains in Euclidean space was first introduced by W. Fleming in [13]; for the special case $G = \mathbf{Z}/d\mathbf{Z}$, H. Federer provided an alternative approach to W. Fleming's theory in 4.2.26 of [11]. Returning to general G, the first six sections of [13] form the foundation for B. White's improved deformation theorem and his subsequent rectifiability theorem of flat chains in [24] and [25]. These developments are comprised in [9], where T. De Pauw and R. Hardt extended them to general metric spaces.

Development of these notes. Originally, we intended to draw from the most general and self-contained account [9] of T. De Pauw and R. Hardt for our applications in the third paper of our series (see [21]). The present notes then grew out of an attempt to provide to the reader – with the due simplifications entailed by the Euclidean setting – the relevant definitions from [9]. Focusing on local chains and employing the bundle $\mathbf{G}(n,m,G)$ appeared to be natural choices in approaching rectifiable chains in this context. Defining the relevant operations on $\mathscr{R}_m^{\mathrm{loc}}(U,G)$ then also entailed the inclusion of some seemingly well-known but hard-to-cite properties of rectifiable varifolds. Next, the ambition to provide a direct route to locally integral G chains – without prior construction of $\mathscr{F}_m^{\mathrm{loc}}(U,G)$ by completion – raised the question whether (see Steps 2–5) certain canonical homomorphisms were univalent and whether (see Step 6) the group $\mathbf{I}_m^{\mathrm{loc}}(U,G)$ constructed could in fact be identified with a subgroup of $\mathscr{R}_m^{\mathrm{loc}}(U,G)$. The proof of consistency with previous work on chains with classical coefficient groups and the related correction of 4.2.26 in [11] dutifully followed. The simplification of an example in the second paper of our series (see Example 6.8 in [20]) then gave rise to adding the constancy theorem in the submanifold setting; thereby, the adaptation of the existing proof strategies to our context led to the study of the restriction operators. Finally – with our primary goal obtained –, we realised that our approach could be extended to yield a viable definition for $\mathscr{F}_m^{\mathrm{loc}}(U,G)$. Thus, we decided to indicate this direction which entailed documenting the seemingly well-known but hard-to-cite representations of $\mathscr{F}_m^{\mathrm{loc}}(U)$ and $\mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n)$.

W. Fleming's approach. As a possible alternative to constructing the afore-mentioned direct route, we also considered to simply draw from W. Fleming's original theory in [13], which is formulated in the Euclidean setting. However, studying the first four sections thereof, we found that some parts of the treatment required to be formalised, expanded, and (at times) corrected to become entirely satisfactory. We accordingly deemed it advisable to avoid just referring the reader to [13] for proofs. Nonetheless, we have been inspired by W. Fleming's work – for instance, regarding how to identify $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ with a subgroup of $\mathscr{R}_m^{\mathrm{loc}}(U, G)$, see 4.14 –, and we believe that our notes could in fact be partially of assistance to readers intending to study the paper [13].

H. Federer's approach. H. Federer's treatment of the case $G = \mathbf{Z}/d\mathbf{Z}$ and ours of general G share the essential role of the case of integer coefficients. Noting

$$d\,\mathbf{I}_m(\mathbf{R}^n) \subset d\,\mathscr{F}_m(\mathbf{R}^n) \subset \mathscr{F}_m(\mathbf{R}^n) \cap \{T : T \equiv 0 \mod d\},$$

his quotient approach to flat chains modulo d leads to the following commutative diagram:

$$\begin{array}{ccccc}
\mathbf{I}_m(\mathbf{R}^n)/d\,\mathbf{I}_m(\mathbf{R}^n) & \longrightarrow & \mathscr{R}_m(\mathbf{R}^n)/d\mathscr{R}_m(\mathbf{R}^n) & \longrightarrow & \mathscr{F}_m(\mathbf{R}^n)/d\mathscr{F}_m(\mathbf{R}^n) \\
{\scriptstyle\text{univalent}}\big\downarrow & & \big\downarrow{\scriptstyle\simeq} & & \big\downarrow{\scriptstyle\text{onto}} \\
\mathbf{I}_m^d(\mathbf{R}^n) & \xrightarrow{\ \subset\ } & \mathscr{R}_m^d(\mathbf{R}^n) & \xrightarrow{\ \subset\ } & \mathscr{F}_m^d(\mathbf{R}^n).
\end{array}$$

The two horizontal arrows in the top row are univalent by the closure theorem (see Theorem 4.2.16 (2) (3) in [11]); the two horizontal arrows in the bottom row are inclusions; the middle vertical arrow is an isomorphism by [11], p. 430, which corresponds[2] to our isomorphism $\rho_{\mathbf{R}^n, m, \mathbf{Z}/d\mathbf{Z}}$; hence, the left vertical arrow is univalent; and the right vertical arrow is onto by definition of $\mathscr{F}_m^d(\mathbf{R}^n)$.

R. Young's structural results. In Corollary 1.6 of [27], R. Young then established that the left and right vertical arrows in the preceding commutative diagram are isomorphisms. As the isomorphisms $A \otimes (\mathbf{Z}/d\mathbf{Z}) \simeq A/dA$, corresponding to commutative groups A, form a natural transformation, the following commutative diagram – in which all horizontal arrows are univalent – results:

$$\begin{array}{ccccc}
\mathbf{I}_m(\mathbf{R}^n) \otimes (\mathbf{Z}/d\mathbf{Z}) & \longrightarrow & \mathscr{R}_m(\mathbf{R}^n) \otimes (\mathbf{Z}/d\mathbf{Z}) & \longrightarrow & \mathscr{F}_m(\mathbf{R}^n) \otimes (\mathbf{Z}/d\mathbf{Z}) \\
\big\downarrow{\scriptstyle\simeq} & & \big\downarrow{\scriptstyle\simeq} & & \big\downarrow{\scriptstyle\simeq} \\
\mathbf{I}_m^d(\mathbf{R}^n) & \xrightarrow{\ \subset\ } & \mathscr{R}_m^d(\mathbf{R}^n) & \xrightarrow{\ \subset\ } & \mathscr{F}_m^d(\mathbf{R}^n).
\end{array}$$

[2] Correspondence refers to the commutative diagram below, where $i : \mathscr{R}_m(\mathbf{R}^n) \to \mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n)$ is the inclusion, and $\iota_{\mathbf{R}^n, m}$ and $\mu_{\mathbf{R}^n, m, d}$ are the isomorphisms of 4.1 and 5.2, respectively:

$$\begin{array}{ccccc}
\mathscr{R}_m(\mathbf{R}^n)/d\mathscr{R}_m(\mathbf{R}^n) & \xrightarrow[\simeq]{} & \mathscr{R}_m(\mathbf{R}^n) \otimes (\mathbf{Z}/d\mathbf{Z}) & \xrightarrow[\text{univalent}]{i \otimes 1_{\mathbf{Z}/d\mathbf{Z}}} & \mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n) \otimes (\mathbf{Z}/d\mathbf{Z}) \\
\big\downarrow & & & & \simeq\big\downarrow{\scriptstyle \iota_{\mathbf{R}^n, m} \otimes 1_{\mathbf{Z}/d\mathbf{Z}}} \\
{\scriptstyle\simeq} & & & & \mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{Z}) \otimes (\mathbf{Z}/d\mathbf{Z}) \\
\big\downarrow & & & & \simeq\big\downarrow{\scriptstyle \rho_{\mathbf{R}^n, m, \mathbf{Z}/d\mathbf{Z}}} \\
\mathscr{R}_m^d(\mathbf{R}^n) & \xrightarrow[\simeq]{\mu_{\mathbf{R}^n, m, d}} & \mathscr{R}_m(\mathbf{R}^n, \mathbf{Z}/d\mathbf{Z}) & \xrightarrow{\ \subset\ } & \mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{Z}/d\mathbf{Z}).
\end{array}$$

1.4. Notation

Our notation follows [19]; thus, we are largely consistent with H. Federer's terminology in geometric measure theory (see [11], pp. 669–676) and W. Allard's notation for varifolds (see [2]). We mention two exceptions: whenever f is a relation, we employ $f[A]$ to mean $\{y : (x, y) \in f \text{ for some } x \in A\}$ and, whenever T is an m-dimensional vector subspace of $\mathbf{R}^n$, the canonical projection of $\mathbf{R}^n$ onto T is denoted by $T_\natural$. Additionally, following H. Federer, see p. 414 of [12], we say an m-dimensional locally flat chain Q in $\mathbf{R}^n$ has *positive densities* if and only if Q is representable by integration and $\boldsymbol{\Theta}^{*m}(\|Q\|, x) > 0$ for $\|Q\|$ almost all x; by Section 1 in [12], this concept yields the analogue for real coefficients to that of m-dimensional locally rectifiable currents in $\mathbf{R}^n$ for integer coefficients.

2. Preliminaries

2.1 Definition. Suppose G is a commutative group.

Then, a function $\sigma\colon G \to \{r : 0 \le r < \infty\}$ is termed a *group norm* on G if and only if $\sigma^{-1}[\{0\}] = \{0\}$ and, whenever $g, h \in G$, we have

$$\sigma(g) = \sigma(-g) \quad \text{and} \quad \sigma(g+h) \le \sigma(g) + \sigma(h).$$

We associate with such σ the metric $\rho\colon G \times G \to \mathbf{R}$ on G, defined by $\rho(g,h) = \sigma(g-h)$ for $g, h \in G$, and often write $|g|$ instead of $\sigma(g)$.

2.2 Remark. Defining $s\colon G \times G \to G$ by $s(g,h) = g - h$ for $g, h \in G$, we see that $\operatorname{Lip} s \le 1$ with respect to the metric on $G \times G$ with value

$$\rho(g, g') + \rho(h, h') \quad \text{at } ((g,h),(g',h')) \in (G \times G)^2.$$

In particular, G is a topological group; it is complete as uniform space if and only if ρ is complete. If H is a closed subgroup of G and $p\colon G \to G/H$ is the quotient map, then $\operatorname{dist}(\cdot, H) \circ p^{-1}$ constitutes a group norm on G/H which induces the quotient topology and p is an open map; if G is complete, so is G/H, and we have $\operatorname{Lip} f = \operatorname{Lip}(f \circ p)$ whenever f maps G/H into some metric space. Whenever H is another normed commutative group, we endow $G \times H$ with the group norm whose value at $(g,h) \in G \times H$ equals $|g| + |h| \in \mathbf{R}$. Taking the standard group norm on $\mathbf{Z}$, the canonical bilinear map from $\mathbf{Z} \times G$ into G, mapping $(d, g) \in \mathbf{Z} \times G$ onto $d \cdot g \in G$, is Lipschitzian on bounded sets.

2.3 Example. If we have $G = \mathbf{R}/\mathbf{Z}$, $r \in \mathbf{R} \sim \mathbf{Q}$, and H is the subgroup of G generated by $\{r + d : d \in \mathbf{Z}\}$, then G is a complete normed group by 2.2, and H is infinite and therefore dense in G by Corollary to Proposition 11 in Section 1.5, Chapter 7, of [5] in conjunction with Proposition 1 in Section 2.1, Chapter 3, of [4]; hence, H is isomorphic to $\mathbf{Z}$ as commutative group but not so as normed commutative group, because H has no isolated points.

2.4 Definition. Suppose G is a complete normed commutative group, f is a function whose domain contains a set A with values in G, and $\sum_{a \in A} |f(a)| < \infty$.

Then, extending finite summation, we define the sum $\sum_A f$, also denoted by $\sum_{a\in A} f(a)$, in G by requiring that, for whenever $\varepsilon > 0$, there exists a finite subset C of A such that $\left|\sum_A f - \sum_B f\right| \le \varepsilon$ for every finite subset B of A containing C.

2.5 Remark. If $h \colon A \to Y$, then $\sum_A f = \sum_{y \in Y} \sum_{h^{-1}[\{y\}]} f$.

2.6 Definition. Whenever ϕ measures X and f is a $\{y : 0 \leq y \leq \infty\}$ valued function whose domain contains ϕ almost all of X, we define the measure $\phi \llcorner f$ over X by

$$(\phi \llcorner f)(A) = \int_A^* f \,\mathrm{d}\phi \quad \text{for } A \subset X.$$

2.7 Remark. Basic properties of this measure are listed in 2.4.10 of [11]. Moreover, if f is ϕ measurable, then

$$(\phi \llcorner f)(A) = \inf\big\{(\phi \llcorner f)(B) : A \subset B,\, B \text{ is } \phi \text{ measurable}\big\} \quad \text{for } A \subset X;$$

if additionally X is a topological space, ϕ is Borel regular, and $\{x : f(x) > 0\}$ is ϕ almost equal to a Borel set, then $\phi \llcorner f$ is Borel regular.

2.8 Remark. If X is a locally compact Hausdorff space, ϕ is a Radon measure over X, and $0 \leq f \in \mathbf{L}_1^{\mathrm{loc}}(\phi)$, then $\phi \llcorner f$ is a Radon measure over X, provided X is the union of a countable family of compact subsets of X. The supplementary hypothesis "provided ... of X" may not be omitted; in fact, one may take f to be the characteristic function of the set constructed in 9.41 (e) of [14] .

2.9 Definition. Whenever ϕ measures X, Y is a topological space, and f is a Y valued function with $\operatorname{dmn} f \subset X$, we define the measure $f_\# \phi$ over Y by

$$f_\# \phi(B) = \phi(f^{-1}[B]) \quad \text{for } B \subset Y.$$

2.10 Remark. This slightly extends 2.1.2 in [11], where $\operatorname{dmn} f = X$ is required.

2.11 Lemma. *Suppose ϕ is a Radon measure over a locally compact Hausdorff space X, Y is a separable metric space, f is a ϕ measurable Y valued function, and X is ϕ almost equal to the union of a countable family of compact subsets of X.*

Then, $f_\# \phi$ is a Borel regular measure over Y.

Proof. Clearly, all closed subsets of Y are $f_\# \phi$ measurable by 2.1.2 in [11]. To prove the Borel regularity, we employ Lusin's theorem in 2.3.5 of [11] to reduce the problem to the case $C = \operatorname{spt} \phi$ is compact and $f | C$ is continuous. Then, supposing $B \subset Y$ and $\varepsilon > 0$, we employ 2.2.5 in [11] to choose an open subset U of X with $f^{-1}[B] \subset U$ and $\phi(U) \leq \varepsilon + f_\# \phi(B)$, define an open subset V of Y by $V = Y \sim f[C \sim U]$, and verify that

$$B \subset V, \quad f^{-1}[V] \subset U \cup (X \sim C),$$

whence it follows $f_\# \phi(V) \leq \varepsilon + f_\# \phi(B)$. ∎

2.12 Remark. In the context of Radon measures and proper maps, a related statement is available from 2.2.17 in [11].

2.13 Remark. Apart of 2.27 and 2.36 below, 2.7, 2.8, or 2.11 will also be employed in 3.5 and in 4.5, 7.11, 7.13, 9.11, 9.13, and 9.16 of [20].

2.14 Theorem. *Suppose A is a closed subset of $\mathbf{R}^n$.*

Then, there exists a nonnegative function $f : \mathbf{R}^n \to \mathbf{R}$ of class ∞ such that

$$A = \{x : f(x) = 0\}, \quad \mathrm{D}^i f(x) = 0 \text{ whenever } x \in A \text{ and } i \text{ a positive integer},$$
$$\text{and } \{x : f(x) \geq y\} \text{ is compact for } 0 < y < \infty.$$

Proof. We abbreviate $U = \mathbf{R}^n \sim A$, assume $U \neq \varnothing$, apply the construction in 3.1.13 of [11] with $\Phi = \{\mathbf{R}^n \sim A\}$, arrange the elements of the resulting set S in a univalent sequence $s_1, s_2, s_3, \ldots$, and, taking $\varepsilon_i = \inf\{2^{-i}, \exp(-3/h(s_i))\}$, define $g : U \to \mathbf{R}$ by

$$g(x) = \sum_{i=1}^{\infty} \varepsilon_i v_{s_i}(x) \quad \text{for } x \in U.$$

For every positive integer j, we then estimate

$$\|\mathrm{D}^j g(x)\| \leq (129)^m V_j h(x)^{-j} \exp(-1/h(x)) \quad \text{for } x \in U,$$
$$\{x : g(x) \geq 2^{-j}\} \subset \textstyle\bigcup_{i=1}^{j} \mathbf{B}(s_i, 10h(s_i)).$$

Therefore, we may take f to be the extension of g to $\mathbf{R}^n$ by 0. ∎

2.15 Remark. A special case of the preceding theorem is employed in 8.1 (2) of [2] to demonstrate the sharpness of the regularity theorem in Section 8 of [2].

2.16 Corollary. *Suppose U is an open subset of $\mathbf{R}^n$ and E_0 and E_1 are disjoint relatively closed subsets of U.*

Then, there exists $f \in \mathscr{E}(U, \mathbf{R})$ satisfying $E_i \subset \operatorname{Int}\{x : f(x) = i\}$ for $i \in \{0, 1\}$ and $0 \leq f \leq 1$.

Proof. We choose, for $i \in \{0, 1\}$, disjoint relatively closed sets A_i with $E_i \subset \operatorname{Int} A_i$ and, by 2.14, applied with A replaced by $\mathbf{R}^n \sim (U \sim A_i)$, also $g_i \in \mathscr{E}(U, \mathbf{R})$ satisfying $g_i \geq 0$ and $\{x : g_i(x) = 0\} = A_i$, and take $f = g_0/(g_0 + g_1)$. ∎

2.17 Remark. Apart of 2.21 below, 2.14 or 2.16 will also be employed in 4.5, 4.13, and 4.18, as well as in 6.15 and 7.7 of [20] and in 7.3 of [21].

2.18 Theorem. *Suppose m is a nonnegative integer, n is a positive integer, $Z \in \mathbf{F}_m(\mathbf{R}^n)$, K is a compact subset of $\mathbf{R}^n$, and $\operatorname{spt} Z \subset \operatorname{Int} K$.*

Then, there exist $Q \in \mathbf{F}_{m,K}(\mathbf{R}^n)$ and $R \in \mathbf{F}_{m+1,K}(\mathbf{R}^n)$, both with positive densities, such that $Z = Q + \partial R$.

Proof. We choose compact subsets B and C of $\mathbf{R}^n$ with $\operatorname{spt} Z \subset \operatorname{Int} B$, $B \subset \operatorname{Int} C$, and $C \subset \operatorname{Int} K$, and notice that $Z \in \mathbf{F}_{m,B}(\mathbf{R}^n)$ by 4.1.12 in [11]. Employing Theorem 4.1.23 in [11] with K replaced by B, we construct $P_i \in \mathbf{P}_{m,C}(\mathbf{R}^n)$ satisfying

$$\sum_{i=1}^{\infty} \mathbf{F}_C(P_i) < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} P_i = Z.$$

Using Lemma 4.2.23 in [11] with $V = \operatorname{Int} K$ and $X = P_i$, we pick $R_i \in \mathbf{P}_{m+1}(\mathbf{R}^n)$ with $\operatorname{spt} R_i \subset K$ and

$$\sum_{i=1}^{\infty} \big(\mathbf{M}(P_i - \partial R_i) + \mathbf{M}(R_i)\big) < \infty.$$

We define

$$Q = \sum_{i=1}^{\infty} (P_i - \partial R_i) \in \mathbf{F}_{m,K}(\mathbf{R}^n),$$

as well as

$$R = \sum_{i=1}^{\infty} R_i \in \mathbf{F}_{m+1,K}(\mathbf{R}^n),$$

with $Z = Q + \partial R$ by means of $\mathbf{F}_K$ convergent series and 4.1.12 in [11]. Finally, noting that $\mathbf{R}^n$ is countably $(\|Q\|, m)$ rectifiable and countably $(\|R\|, m+1)$ rectifiable, Q and R have positive densities by condition (V) in Section 1 of [12]. ∎

2.19 Corollary. *Suppose that m is a nonnegative integer, n is a positive integer, and $Z \in \mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n)$.*

Then, there exist $Q \in \mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n)$ and $R \in \mathbf{F}_{m+1}^{\mathrm{loc}}(\mathbf{R}^n)$, both with positive densities, such that $Z = Q + \partial R$.

Proof. In view of 4.1.12 in [11], this follows from 2.18 employing a suitable partition of unity; for instance, one may apply the construction in 3.1.13 of [11] with $\Phi = \{\mathbf{R}^n\}$ and Lemma 3.1.12 in [11] with $U = \mathbf{R}^n$, $h(x) = 1/20$ for $x \in \mathbf{R}^n$, $\lambda = 0$, and $\alpha = \beta = 20$. ∎

2.20 Remark. Defining the linear map L from $F_{n,m}$ onto $\mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n)$, where

$$F_{n,m} = \big(\mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n) \times \mathbf{F}_{m+1}^{\mathrm{loc}}(\mathbf{R}^n)\big) \cap \{(Q, R) : Q \text{ and } R \text{ have positive densities}\},$$

by $L(Q, R) = Q + \partial R$ for $(Q, R) \in F_{n,m}$, we obtain a vector space isomorphism

$$F_{n,m}/\ker L \simeq \mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n).$$

2.21 Lemma. *Suppose m is a nonnegative integer, n is a positive integer, U is an open subset of $\mathbf{R}^n$, A is a relatively closed subset of U, S is an open subset of U, $Z \in \mathscr{F}_m^{\mathrm{loc}}(U)$, $W \in \mathscr{F}_m(U)$, and*

$$A \cap \operatorname{spt} Z \subset S, \quad A \cap \operatorname{spt}(Z - W) = \varnothing.$$

Then, there exist $Q \in \mathscr{R}_m(U)$ and $R \in \mathscr{R}_{m+1}(U)$ such that

$$\operatorname{spt} Q \cup \operatorname{spt} R \subset S \quad \textit{and} \quad A \cap \operatorname{spt}(Z - Q - \partial R) = \varnothing.$$

Proof. We pick $X \in \mathscr{R}_m(U)$ and $Y \in \mathscr{R}_{m+1}(U)$ with $W = X + \partial Y$, observe that we may choose an open subset T of U with

$$A \subset T, \quad (\operatorname{Clos} T) \cap \operatorname{spt} Z \subset S, \quad T \cap \operatorname{spt}(Z - W) = \varnothing,$$

and obtain, from 2.16, a locally Lipschitzian function $f : U \to \mathbf{R}$ with

$$f(x) \le 0 \ \text{ for } x \in T \cap \operatorname{spt} Z \quad \text{and} \quad f(x) \ge 1 \ \text{ for } x \in U \sim S.$$

Noting $T \cap \operatorname{spt}(X + \partial Y) \subset \operatorname{spt} Z$, we employ 4.2.1, 4.3.1, 4.3.4 and 4.3.6 in [11] to select $0 < y < 1$ with $\langle Y, f, y \rangle \in \mathscr{R}_m(U)$ and

$$T \cap \operatorname{spt}\big(X \llcorner \{x : f(x) > y\} + \partial(Y \llcorner \{x : f(x) > y\}) - \langle Y, f, y \rangle\big) \subset \operatorname{spt} Z.$$

Thus, we may take $Q = X \llcorner \{x : f(x) \leq y\} + \langle Y, f, y \rangle$ and $R = Y \llcorner \{x : f(x) \leq y\}$ because $T \cap \operatorname{spt}(X + \partial Y - Q - \partial R) \subset T \cap \{x : f(x) \geq y\} \cap \operatorname{spt} Z = \varnothing$. ∎

2.22 Theorem. *Suppose m and n are integers, $m \geq 0$, $n \geq 1$, U is an open subset of $\mathbf{R}^n$, and $Z \in \mathscr{F}_m^{\mathrm{loc}}(U)$.*

Then, there exist $Q \in \mathscr{R}_m^{\mathrm{loc}}(U)$ and $R \in \mathscr{R}_{m+1}^{\mathrm{loc}}(U)$ such that $Z = Q + \partial R$.

Proof. Assume $U \neq \varnothing$. Let Φ denote the class of all open subsets T of U such that, for some $W \in \mathscr{F}_m(U)$, we have $T \cap \operatorname{spt}(Z - W) = \varnothing$; hence $\bigcup \Phi = U$. We define the function $h \colon U \to \{r : 0 < r < \infty\}$ by

$$h(x) = \frac{1}{20} \sup\{\inf\{1, \operatorname{dist}(x, \mathbf{R}^n \sim T)\} : T \in \Phi\} \quad \text{for } x \in U.$$

Applying the construction in 3.1.13 of [11], we obtain a set S such that, arranging its elements into an univalent sequence $s_1, s_2, s_3, \ldots$ in U, we have $U = \bigcup_{i=1}^{\infty} \mathbf{B}(s_i, 5h(s_i))$ and

$$\operatorname{card}\{i : \mathbf{B}(x, 10h(x)) \cap \mathbf{B}(s_i, 10h(s_i)) \neq \varnothing\} \leq (129)^n \quad \text{for } x \in U.$$

Next, we construct $R_1, R_2, R_3, \ldots$ in $\mathscr{R}_m(U)$ and $Q_1, Q_2, Q_3, \ldots$ in $\mathscr{R}_{m+1}(U)$ satisfying $\operatorname{spt} Q_i \cup \operatorname{spt} R_i \subset \mathbf{B}(s_i, 10h(s_i))$ for every positive integer i and

$$K_i \cap \operatorname{spt} Z_i = \varnothing \quad \text{for every nonnegative integer } i,$$

where we abbreviated

$$K_i = \bigcup_{j=1}^{i} \mathbf{B}(s_j, 5h(s_j)) \quad \text{and} \quad Z_i = Z - \sum_{j=1}^{i} (Q_j + \partial R_j);$$

in fact, this is trivial for $i = 0$ and, if $R_1, \ldots, R_{i-1}$ and $Q_1, \ldots, Q_{i-1}$ with these properties have been constructed for some positive integer i, then, noting that

$$K_i \cap \operatorname{spt} Z_{i-1} \subset \mathbf{B}(s_i, 5h(s_i)),$$

we may take Q_i and R_i to be the currents furnished by applying 2.21 with A, S, and Z replaced by $K_i \cap \mathbf{B}(s_i, 10h(s_i))$, $\mathbf{U}(s_i, 10h(s_i))$, and Z_{i-1} because

$$(K_i \sim \mathbf{B}(s_i, 10h(s_i))) \cap \operatorname{spt} Z_i \subset K_{i-1} \cap \operatorname{spt} Z_{i-1} = \varnothing.$$

Let $Q = \sum_{i=1}^{\infty} Q_i$ and $R = \sum_{i=1}^{\infty} R_i$. If T is open and $\operatorname{Clos} T$ is a compact subset of U, then $T \subset K_i$ for some i, hence $T \cap \operatorname{spt}(Z - Q - \partial R) = \varnothing$. ∎

2.23 Remark. Defining the homomorphism η from $\mathscr{R}_m^{\mathrm{loc}}(U) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U)$ onto $\mathscr{F}_m^{\mathrm{loc}}(U)$ by $\eta(Q, R) = Q + \partial R$ for $(Q, R) \in \mathscr{R}_m^{\mathrm{loc}}(U) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U)$, we obtain an isomorphism

$$\big(\mathscr{R}_m^{\mathrm{loc}}(U) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U)\big) \big/ \ker \eta \simeq \mathscr{F}_m^{\mathrm{loc}}(U).$$

2.24 Remark. For the flat G chains of [13], a representation analogous to those of 2.18, 2.19, and 2.22 was obtained in Proposition 2.1 of [1].

2.25 Theorem. *Suppose ϕ is a Radon measure over an open subset X of $\mathbf{R}^n$, m is a nonnegative integer, X is countably (ϕ, m) rectifiable, and*

$$\boldsymbol{\Theta}^{*m}(\phi, a) < \infty \quad \text{for } \phi \text{ almost all } a.$$

Then, ϕ is the weight of some member of $\mathbf{RV}_m(X)$ and, whenever R is a compact m-rectifiable subset of X and f maps a subset of X into $\mathbf{R}^\nu$, we have that, for $\mathscr{H}^m$ almost all $a \in R$ with $\boldsymbol{\Theta}^m(\phi, a) > 0$,

$$\operatorname{Tan}^m(\phi, a) = \operatorname{Tan}^m(\mathscr{H}^m \llcorner R, a) \text{ is an } m\text{-dimensional vector space},$$
$$(\phi, m) \operatorname{ap} \mathrm{D} f(a) = (\mathscr{H}^m \llcorner R, m) \operatorname{ap} \mathrm{D} f(a).$$

Proof. Whenever R is a compact m-rectifiable subset of X, noting $\mathscr{H}^m(R) < \infty$, we infer $\phi(R \cap \{a : \boldsymbol{\Theta}^m(\phi, a) = 0\}) = 0$ from 2.10.19 (1) in [11]. It follows that

$$\boldsymbol{\Theta}^{*m}(\phi, a) > 0 \quad \text{for } \phi \text{ almost all } a.$$

Next, suppose f maps a subset of X into $\mathbf{R}^\nu$, R is a compact m-rectifiable subset of X, and the Borel sets R_i are defined by

$$R_i = R \cap \{a : 1/i < \boldsymbol{\Theta}^{*m}(\phi, a) < i\}$$

whenever i is a positive integer; hence, there holds

$$\mathscr{H}^m \llcorner R_i \leq \phi \llcorner R_i \leq 2^m i \, \mathscr{H}^m \llcorner R_i$$

by 2.10.19 (1) (3) in [11], and

$$\boldsymbol{\Theta}^m(\phi \llcorner X \sim R_i, a) = 0, \quad \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner R \sim R_i, a) = 0,$$

for $\mathscr{H}^m$ almost all $a \in R_i$ by 2.10.19 (4) in [11], whence we infer

$$\begin{aligned} \operatorname{Tan}^m(\phi, a) &= \operatorname{Tan}^m(\phi \llcorner R_i, a) \\ &= \operatorname{Tan}^m(\mathscr{H}^m \llcorner R_i, a) = \operatorname{Tan}^m(\mathscr{H}^m \llcorner R, a) \in \mathbf{G}(n, m), \\ (\phi, m) \operatorname{ap} \mathrm{D} f(a) &= (\phi \llcorner R_i, m) \operatorname{ap} \mathrm{D} f(a) \\ &= (\mathscr{H}^m \llcorner R_i, m) \operatorname{ap} \mathrm{D} f(a) = (\mathscr{H}^m \llcorner R, m) \operatorname{ap} \mathrm{D} f(a) \end{aligned}$$

for $\mathscr{H}^m$ almost all $a \in R_i$ by Theorem 3.2.19 in [11]. The function mapping ϕ almost all $a \in R_i$ onto $\operatorname{Tan}^m(\phi, a) \in \mathbf{G}(n, m)$ therefore is $\phi \llcorner R_i$ measurable by Lemma 3.2.25 and 3.2.28 (2) (4) in [11]. Thus, defining $V \in \mathbf{V}_m(X)$ with $\|V\| = \phi$ by

$$V(k) = \int k(x, \operatorname{Tan}^m(\phi, x)) \, \mathrm{d}\phi \, x \quad \text{for } k \in \mathscr{K}(X \times \mathbf{G}(n, m)),$$

we notice that, for ϕ almost all a, we have

$$V^{(a)}(\beta) = \beta(\operatorname{Tan}^m(\phi, a)) \quad \text{for } \beta \in \mathscr{K}(\mathbf{G}(n, m))$$

by Theorems 2.8.18 and 2.9.13 in [11]. Since $\phi = \mathscr{H}^m \llcorner \boldsymbol{\Theta}^m(\phi, \cdot)$ by 2.8 (5) in [2], we have $V \in \mathbf{RV}_m(X)$ by Theorem 3.5 (1a) in [2], and the conclusion follows. ∎

2.26 Remark. Recalling Theorem 3.5 (1b) in [2], we infer the following assertion: if also χ satisfies the hypotheses of 2.25 with ϕ replaced by χ, then, we have that, for $\mathscr{H}^m$ almost all a with $\boldsymbol{\Theta}^m(\phi, a) > 0$ and $\boldsymbol{\Theta}^m(\chi, a) > 0$,

$$\operatorname{Tan}^m(\phi, a) = \operatorname{Tan}^m(\chi, a) \text{ is an } m\text{-dimensional vector space}$$
$$\text{and } (\phi, m) \operatorname{ap} \mathrm{D} f(a) = (\chi, m) \operatorname{ap} \mathrm{D} f(a).$$

2.27 Theorem. *Suppose X and Y are open subsets of $\mathbf{R}^n$ and $\mathbf{R}^\nu$, respectively, $f : X \to Y$ is locally Lipschitzian, m is a nonnegative integer with $m \leq n$ and $m \leq \nu$, $V \in \mathbf{RV}_m(X)$, $f \,|\, \operatorname{spt}\|V\|$ is proper, and $g : Y \to \mathbf{R}^\mu$ is a locally Lipschitzian map.*

Then, a member W of $\mathbf{RV}_m(Y)$ may be defined by

$$W(k) = \int_A k(f(x), \operatorname{im}(\|V\|, m) \operatorname{ap} \mathrm{D} f(x)) \, (\|V\|, m) \operatorname{ap} \mathrm{J}_m f(x) \, \mathrm{d}\|V\| \, x$$

whenever $k \in \mathscr{K}(Y \times \mathbf{G}(\nu, m))$, where $A = \{x : (\|V\|, m) \operatorname{ap} \mathrm{J}_m f(x) > 0\}$ and

$$(\|V\|, m) \operatorname{ap} \mathrm{J}_m f = \big\| \textstyle\bigwedge_m (\|V\|, m) \operatorname{ap} \mathrm{D} f \big\|;$$

in particular, we have $\|W\| = f_\#(\|V\| \llcorner (\|V\|, m) \operatorname{ap} \mathrm{J}_m f)$. Moreover, for $\mathscr{H}^m$ almost all $y \in Y$, there holds

$$\boldsymbol{\Theta}^m(\|W\|, y) = \sum_{x \in f^{-1}[\{y\}]} \boldsymbol{\Theta}^m(\|V\|, x),$$

and, if $f(x) = y$ and $\boldsymbol{\Theta}^m(\|V\|, x) > 0$, then

$$\operatorname{im}(\|V\|, m) \operatorname{ap} \mathrm{D} f(x) = \operatorname{Tan}^m(\|W\|, y) \text{ is an } m\text{-dimensional vector space},$$

g is $(\|W\|, m)$ approximately differentiable at y, and

$$(\|V\|, m) \operatorname{ap} \mathrm{D}(g \circ f)(x) = (\|W\|, m) \operatorname{ap} \mathrm{D} g(y) \circ (\|V\|, m) \operatorname{ap} \mathrm{D} f(x).$$

Proof. As $\|\bigwedge_m L\| = \|\bigwedge_m (L \circ T_\natural)\|$ for $T \in \mathbf{G}(n, m)$ and $L \in \operatorname{Hom}(T, \mathbf{R}^\nu)$, the legitimacy of the definition of W and the equation for $\|W\|$ follow from Lemma 4.5 (1) (2) in [18], 2.8, 2.11, and 2.2.3 and 3.2.28 (4) in [11]. To prove the remaining conclusions, we firstly consider the special case that V is associated with a compact subset K of X which is contained in an m-dimensional submanifold of class 1 of $\mathbf{R}^n$, and

$$\text{either } (\|V\|, m) \operatorname{ap} \mathrm{J}_m f(x) = 0 \text{ for } \mathscr{H}^m \text{ almost all } x \in K,$$
$$\text{or } f|K \text{ is univalent and } (f|K)^{-1} \text{ is Lipschitzian.}$$

If the second alternative holds, then, noting that $(\|V\|, m) \operatorname{ap} \mathrm{J}_m f(x) > 0$ for $\mathscr{H}^m$ almost all $x \in K$ by Corollaries 2.10.11 and 3.2.20 in [11], we infer from Lemma 3.2.17 in [11], in conjunction with 2.10.19 (4), Rademacher's theorem in 3.1.6, and 3.1.19 (4) in [11], that

$$\operatorname{Tan}^m(\mathscr{H}^m \llcorner f[K], f(x)) = \operatorname{im}(\mathscr{H}^m \llcorner K, m) \operatorname{ap} \mathrm{D} f(x) \in \mathbf{G}(n, m),$$
$$(\mathscr{H}^m \llcorner K, m) \operatorname{ap} \mathrm{D}(g \circ f)(x)$$
$$= (\mathscr{H}^m \llcorner f[K], m) \operatorname{ap} \mathrm{D} g(f(x)) \circ (\mathscr{H}^m \llcorner K, m) \operatorname{ap} \mathrm{D} f(x)$$

for $\mathscr{H}^m$ almost all $x \in K$. Therefore, for both alternatives, W is the rectifiable varifold associated with $f[K]$ by Corollary 3.2.20 in [11]. As the characteristic functions of K and $f[K]$ are $\mathscr{H}^m$ almost equal to $\boldsymbol{\Theta}^m(\|V\|, \cdot)$ and $\boldsymbol{\Theta}^m(\|W\|, \cdot)$, respectively, by 2.10.19 (4) and Theorem 3.2.19 in [11], the conclusion in the special case now follows by Corollary 2.10.11 in [11] and Theorem 3.5 (1b) in [2]. In the general case, we use 3.5 (1) in [15], 2.2.3 and Lemma 3.2.2 in [11], and 2.26 to express $V = \sum_{i=1}^{\infty} c_i V_i$ for some $0 < c_i < \infty$ and some varifolds V_i satisfying the conditions of the special case, whence we deduce the conclusion by Theorem 3.5 (2) in [2] in conjunction with Corollary 2.10.11 in [11], Theorem 3.5 (1b) in [2], and 2.26. ∎

2.28 Remark. If f is of class ∞, then $W = f_\# V$ as defined in 3.2 of [2]; thus, we extend the definition of $f_\# V$ to encompass the presently considered maps f.

2.29 Remark. If additionally $\mu \geq m$, Z is an open subset $\mathbf{R}^\mu$, $g\colon Y \to Z$ is locally Lipschitzian, and $g \circ f | \operatorname{spt}\|V\|$ is proper, then

$$(g \circ f)_\# V = g_\#(f_\# V) \in \mathbf{RV}_m(Z).$$

Allowing for the case $m > \nu$, we leave the term $f_\# V$ undefined, but we are still assured that $(g \circ f)_\# V = 0$, because $\mathscr{H}^m(\operatorname{im} f) = 0$ implies $\mathscr{H}^m(\operatorname{im}(g \circ f)) = 0$.

2.30 Corollary. *If additionally u is $\|V\| \llcorner (\|V\|, m) \operatorname{ap} \mathrm{J}_m f$ integrable, then*

$$\big(\|V\| \llcorner (\|V\|, m) \operatorname{ap} \mathrm{J}_m f\big)(u) = \int \sum_{M \cap f^{-1}[\{y\}]} \boldsymbol{\Theta}^m(\|V\|, \cdot) u \, \mathrm{d}\mathscr{H}^m\, y,$$

where $M = \{x : \boldsymbol{\Theta}^m(\|V\|, x) > 0\}$.

Proof. If u is the characteristic function of a $\|V\|$ measurable set B over X, then

$$\boldsymbol{\Theta}^m(\|V\| \llcorner B, x) = \boldsymbol{\Theta}^m(\|V\|, x) u(x) \quad \text{for } \mathscr{H}^m \text{ almost all } x \in X$$

by Theorems 2.8.18 and 2.9.11 in [11] and Theorem 3.5 (1b) in [2]; whence, recalling Corollary 2.10.11 in [11] and 2.26, we infer the conclusion by applying 2.27 with V replaced by $V \llcorner B \times \mathbf{G}(n, m)$. Next, considering the subcase $A = B$, the assertion extends to $\|V\| \llcorner (\|V\|, m) \operatorname{ap} \mathrm{J}_m f$ measurable sets B. Therefore, the general case follows by means of the usual approximation procedure, see 2.1.1 (6) (10), Theorems 2.3.3 and 2.4.4 (6), and Corollary 2.4.8 in [11]. ∎

2.31 Corollary. *Suppose additionally G is a complete separable normed commutative group, v is a G valued $\|V\| \llcorner (\|V\|, m) \operatorname{ap} \mathrm{J}_m f$ measurable function,*

$$|v(x)| \leq \boldsymbol{\Theta}^m(\|V\|, x) \quad \textit{for } \|V\| \llcorner (\|V\|, m) \operatorname{ap} \mathrm{J}_m f \textit{ almost all } x,$$

and $M = \{x : \boldsymbol{\Theta}^m(\|V\|, x) > 0\}$.

Then, an $\mathscr{H}^m$ measurable function ξ may be defined by

$$\xi(y) = \sum_{x \in M \cap f^{-1}[\{y\}]} v(x) \in G \quad \textit{whenever } y \in Y,$$

where $\mathscr{H}^m$ refers to the m-dimensional Hausdorff measure over Y.

Proof. We recall Corollary 2.10.11 in [11] and Theorem 3.5 (1b) in [2]. As M is countably $(\mathscr{H}^m, m)$ rectifiable, the function $N(f|B \cap M, \cdot)$ is $\mathscr{H}^m$ measurable whenever B is a Borel subset of X by Corollary 3.2.20 in [11]. This implies the conclusion in the special case that $v\colon M \to G$ is a Borel function with finite image and $|v(x)| \leq \boldsymbol{\Theta}^m(\|V\|, x)$ for $x \in M$. In the general case, there exists a sequence $w_1, w_2, w_3, \ldots$ of functions satisfying the conditions of the special case and

$$\lim_{i \to \infty} w_i(x) = v(x) \quad \text{for } \|V\| \llcorner (\|V\|, m) \operatorname{ap} \mathrm{J}_m f \text{ almost all } x.$$

Since $\mathscr{H}^m\big(f[M \cap \{x : (\|V\|, m) \operatorname{ap} \mathrm{J}_m f(x) = 0\}]\big) = 0$, the conclusion follows. ■

2.32. If $T \in \mathbf{G}(n, m)$ is associated with the simple m vector $\zeta \in \bigwedge_m \mathbf{R}^n$ and $h \in \operatorname{Hom}(T, \mathbf{R}^\kappa)$ satisfies $\bigwedge_\kappa h \neq 0$, then, cf. Theorem 4.3.8 (3) in [11], $\ker h$ is associated with

$$\zeta \llcorner \textstyle\bigwedge^\kappa (h \circ T_\natural)(\omega) \quad \text{whenever } 0 \neq \omega \in \bigwedge^\kappa \mathbf{R}^\kappa.$$

2.33 Theorem. *Suppose X is an open subset of $\mathbf{R}^n$, $f : X \to \mathbf{R}^\kappa$ is locally Lipschitzian, m is an integer, $\kappa \leq m \leq n$, $V \in \mathbf{RV}_m(X)$, the prefix* ap *denotes $(\|V\|, m)$ approximate differentiation, and $\operatorname{ap} \mathrm{J}_\kappa f = \|\bigwedge_\kappa \operatorname{ap} \mathrm{D} f\|$. Moreover, suppose W is a function such that $y \in \operatorname{dmn} W$ if and only if $y \in \mathbf{R}^\kappa$ and*

$$\big(\mathscr{H}^{m-\kappa} \llcorner f^{-1}[\{y\}]\big) \llcorner \boldsymbol{\Theta}^m(\|V\|, \cdot)$$

is the weight of some $Z \in \mathbf{RV}_{m-\kappa}(X)$ and such that in this case $W(y) = Z$.

Then, the following two statements hold.

(1) *The function W is $\mathscr{L}^\kappa$ measurable and, for $\mathscr{L}^\kappa$ almost all y, we have*

$$\operatorname{Tan}^{m-\kappa}(\|W(y)\|, x) = \ker \operatorname{ap} \mathrm{D} f(x) \quad \text{for } \|W(y)\| \text{ almost all } x.$$

(2) *If g is $\|V\| \llcorner \operatorname{ap} \mathrm{J}_\kappa f$ integrable, then*

$$\int g \,\mathrm{d}(\|V\| \llcorner \operatorname{ap} \mathrm{J}_\kappa f) = \iint g \,\mathrm{d}\|W(y)\| \,\mathrm{d}\mathscr{L}^\kappa y.$$

Proof. Denoting the statement that results from replacing $\|W(y)\|$ in (2) by

$$\phi(y) = \big(\mathscr{H}^{m-\kappa} \llcorner f^{-1}[\{y\}]\big) \llcorner \boldsymbol{\Theta}^m(\|V\|, \cdot)$$

by (2)′ and recalling Theorem 2.10.25 in [11] as well as Theorem 3.5 (1b) in [2], we readily deduce (2)′ from 3.5 (2) in [15], whence, in conjunction with 2.10.19 (3) and Theorem 3.2.22 (2) in [11], we infer that $\phi(y) = \|W(y)\|$ for $\mathscr{L}^\kappa$ almost all y by 2.25. Using Lemma 3.2.25 and 3.2.28 (2) (4) in [11] and 2.32, we deduce that the function mapping $a \in A$ onto $\ker \operatorname{ap} \mathrm{D} f(a) \in \mathbf{G}(n, m-\kappa)$ is $\|V\| \llcorner A$ measurable from Lemma 4.5 (1) (2) in [18], where $A = \{a : \operatorname{ap} \mathrm{J}_\kappa f(a) > 0\}$. In view of Example 2.23 in [19] and (2)′, we may thus define an $\mathscr{L}^\kappa$ measurable $\mathbf{V}_{m-\kappa}(X)$ valued function Z such that, for $\mathscr{L}^\kappa$ almost all y,

$$Z(y)(k) = \int k(x, \ker \operatorname{ap} \mathrm{D} f(x)) \,\mathrm{d}\phi(y)\, x \quad \text{whenever } k \in \mathscr{K}(X \times \mathbf{G}(n, m-\kappa))$$

and it remains to show that $Z(y)$ is rectifiable for $\mathscr{L}^\kappa$ almost all y. Recalling 2.26, this follows from 3.5 (1) in [15] in the special case that f is of class 1 to which the general case may be reduced by means of Lemma 11.1 in [19]. ■

2.34 Remark. If C is an $\mathscr{L}^\kappa$ Vitali relation, then, for $\mathscr{L}^\kappa$ almost all y,

$$W(y)(k) = (C) \lim_{S \to y} \mathscr{L}^\kappa(S)^{-1} \int_{f^{-1}[S]} k(x, \ker \operatorname{ap} \mathrm{D} f(x)) \,\mathrm{d}(\|V\| \llcorner \operatorname{ap} \mathrm{J}_\kappa f)\, x$$

whenever $k \in \mathscr{K}(U \times \mathbf{G}(n, m-\kappa))$ by Theorem 2.9.8 in [11] in conjunction with Theorem 3.5 (1b) in [2] and Example 2.23 in [19]. Thus, in case $\kappa = 1$ and $\|\delta V\|$ is a Radon measure, we recall Theorem 2.8.17 in [11] and Remark 8.5, Example 8.7, Lemma 8.29, and Theorem 12.1 in [19] to similarly conclude

$$\|V \partial \{x : f(x) > y\}\| = \big(\mathscr{H}^{m-1} \llcorner f^{-1}[\{y\}]\big) \llcorner \boldsymbol{\Theta}^m(\|V\|, \cdot) \quad \text{for } \mathscr{L}^1 \text{ almost all } y.$$

2.35 Remark. For $X = \mathbf{R}^n$, the statement of 2.33 is analogous to those for flat chains with positive densities in Theorems 4.3.2 (2) and 4.3.8 (2) (3) in [11] with $U = \mathbf{R}^n$.

2.36 Theorem. *Suppose X and Y are open subsets of $\mathbf{R}^n$ and $\mathbf{R}^\nu$, respectively, m and μ are nonnegative integers, $m \leq n$, $\mu \leq \nu$, ϕ and χ are the weights of some members of $\mathbf{RV}_m(X)$ and $\mathbf{RV}_\mu(Y)$, respectively, and*

$$M = \{x : \boldsymbol{\Theta}^m(\phi, x) > 0\}, \quad N = \{y : \boldsymbol{\Theta}^\mu(\chi, y) > 0\}.$$

Then, there holds

$$(\mathscr{H}^m \llcorner M) \times (\mathscr{H}^\mu \llcorner N) = \mathscr{H}^{m+\mu} \llcorner (M \times N)$$

and $M \times N$ is $\mathscr{H}^{m+\mu}$ almost equal to $\{z : \boldsymbol{\Theta}^{m+\mu}(\phi \times \chi, z) > 0\}$.

Proof. We let $Z = \{z : \boldsymbol{\Theta}^{m+\mu}(\phi \times \chi, z) > 0\}$ and recall 2.4.10 in [11] and Theorem 3.5 (1b) in [2]. By Theorem 3.6 (1) (2) in [15], we have $\phi \times \chi = \mathscr{H}^{m+\mu} \llcorner \boldsymbol{\Theta}^{m+\mu}(\phi \times \chi, \cdot)$ and

$$\boldsymbol{\Theta}^{m+\mu}(\phi \times \chi, (x, y)) = \boldsymbol{\Theta}^m(\phi, x) \boldsymbol{\Theta}^\mu(\chi, y) \quad \text{for } \phi \times \chi \text{ almost all } (x, y);$$

in particular, we have $\mathscr{H}^{m+\mu}(Z \sim (M \times N)) = 0$ and, since $\boldsymbol{\Theta}^{m+\mu}_*(\phi \times \chi, (x, y)) > 0$ for $(x, y) \in M \times N$ by Fubini's theorem, see 2.6.2 (2) in [11], also $\mathscr{H}^{m+\mu}((M \times N) \sim Z) = 0$. Finally,

$$\big(\phi \llcorner \boldsymbol{\Theta}^m(\phi, \cdot)^{-1}\big) \times \big(\chi \llcorner \boldsymbol{\Theta}^\mu(\chi, \cdot)^{-1}\big) = \big((\phi \times \chi) \llcorner \boldsymbol{\Theta}^{m+\mu}(\phi \times \chi, \cdot)^{-1}\big)$$

may be verified by means of Fubini's theorem, see 2.6.2 (4) in [11], and 2.7. ∎

2.37 Remark. By 3.2.24 in [11], for general Borel subsets M of $\mathbf{R}^n$ and N of $\mathbf{R}^\nu$ which are countably $(\mathscr{H}^m, m)$, respectively $(\mathscr{H}^\mu, \mu)$, rectifiable, it may happen that

$$(\mathscr{H}^m \llcorner M) \times (\mathscr{H}^\mu \llcorner N) \neq \mathscr{H}^{m+\mu} \llcorner (M \times N).$$

3. Rectifiable chains

3.1. Suppose m and n are nonnegative integers, $n \geq 1$, G is a normed commutative group, the function $\beta\colon \bigwedge_m \mathbf{R}^n \to \bigodot_2 \bigwedge_m \mathbf{R}^n$ satisfies $\beta(\zeta) = \zeta \odot \zeta / 2$ for $\zeta \in \bigwedge_m \mathbf{R}^n$, and the Grassmann manifolds $\mathbf{G}_0(n, m)$ and $\mathbf{G}(n, m)$ are canonically isometrically identified with subsets of $\bigwedge_m \mathbf{R}^n$ and $\bigodot_2 \bigwedge_m \mathbf{R}^n$, respectively, as in 3.2.28 (1) (4) of [11]. Abbreviating

$\alpha = \beta|\mathbf{G}_o(n,m)$, we recall from 3.2.28 (3) (4) in [11] that α is an open map from $\mathbf{G}_o(n,m)$ onto $\mathbf{G}(n,m)$ and

$$|\alpha(\zeta) - \alpha(\zeta')|^2 = |\zeta - \zeta'|^2(1 + \zeta \bullet \zeta')/2 \quad \text{for } \zeta, \zeta' \in \mathbf{G}_o(n,m);$$

hence, $\operatorname{Lip}\alpha = 1$ if $m \leq n$, $\operatorname{Lip}\alpha = 0$ if $m > n$, and $\operatorname{Lip}\big((\alpha|U)^{-1}\big) < \infty$ whenever $U \subset \mathbf{G}_o(n,m)$ with $\operatorname{diam} U < 2$.

We endow the set $B = \mathbf{G}_o(n,m) \times G$ with the metric R such that

$$R\big((\zeta, g), (\zeta', g')\big) = |\zeta - \zeta'| + |g - g'| \quad \text{for } (\zeta, g), (\zeta', g') \in B,$$

and study the quotient Γ of B induced by the action of the two-element subgroup $\{\mathbf{1}_B, -\mathbf{1}_B\}$ of isometries of B. Taking $p\colon B \to \Gamma$ to be the canonical projection of B onto Γ, we define maps $P\colon \Gamma \to \mathbf{G}(n,m)$, $N\colon \Gamma \to \mathbf{R}$, and

$$+ : (\Gamma \times \Gamma) \cap \{(\gamma, \gamma') : P(\gamma) = P(\gamma')\} \to \Gamma$$

by requiring that, whenever $(\zeta, g), (\zeta, g') \in B$, we have

$$P(p(\zeta, g)) = \alpha(\zeta), \quad N(p(\zeta, g)) = |g| \quad \text{and} \quad p(\zeta, g) + p(\zeta, g') = p(\zeta, g + g').$$

For $T \in \mathbf{G}(n,m)$, the set $H = P^{-1}[\{T\}]$ endowed with addition $+|(H \times H)$ and group norm $N|H$ forms a normed commutative group. Next, noting that $\inf R[\gamma \times \gamma'] = \operatorname{dist}(b, \gamma')$ for $b \in \gamma$, we define a metric $\rho\colon \Gamma \times \Gamma \to \mathbf{R}$ on Γ by

$$\rho(\gamma, \gamma') = \inf R[\gamma \times \gamma'] \quad \text{for } \gamma, \gamma' \in \Gamma.$$

Clearly, $\operatorname{Lip} p \leq 1$, the metric ρ induces the quotient topology on Γ, and p is an open map. Moreover, whenever f maps Γ into some metric space, there holds $\operatorname{Lip} f = \operatorname{Lip}(f \circ p)$, hence

$$\operatorname{Lip} p = 1 = \operatorname{Lip} P \ \text{ if } m \leq n, \quad \operatorname{Lip} p = 0 = \operatorname{Lip} P \ \text{ if } m > n.$$

Whenever $(\zeta, g), (\zeta', g') \in B$, we notice that

$$\begin{aligned}\rho\big(p(\zeta, g), p(\zeta', g')\big) &= \inf\{|\zeta - \zeta'| + |g - g'|, |\zeta + \zeta'| + |g + g'|\} \\ &\geq \inf\{|\zeta - \zeta'| + |g - g'|, 2 - |\zeta - \zeta'|\}.\end{aligned}$$

Thus, for $U \subset \mathbf{G}_o(n,m)$ with $\operatorname{diam} U < 2$, the map $(p|(U \times G))^{-1}$, whose domain equals $P^{-1}[\alpha[U]]$, is locally Lipschitzian, whence we infer that the Lipschitzian map

$$\psi_U = p \circ \big((\alpha|U)^{-1} \times \mathbf{1}_G\big) \text{ from } \alpha[U] \times G \text{ onto } P^{-1}[\alpha[U]]$$

possesses a locally Lipschitzian inverse; moreover, we have

$$P(\psi_U(T, g)) = T, \quad (N \circ \psi_U)(T, g) = |g|, \quad \psi_U(T, g) + \psi_U(T, g') = \psi_U(T, g + g')$$

for $T \in \alpha[U]$ and $g, g' \in G$. Therefore, the maps N and

$$+ : (\Gamma \times \Gamma) \cap \{(\gamma, \gamma') : P(\gamma) = P(\gamma')\} \to \Gamma$$

are locally Lipschitzian; also, if G is complete, so are the domains of these maps.

Whenever σ is a Γ valued function, we let $|\sigma| = N \circ \sigma$. If also τ is a Γ valued function, $\sigma + \tau$ shall be the function with domain

$$(\operatorname{dmn} \sigma) \cap (\operatorname{dmn} \tau) \cap \{x : P(\sigma(x)) = P(\tau(x))\}$$

and value $\sigma(x) + \tau(x)$ at x in its domain. Henceforward, we will denote the metric space Γ by $\mathbf{G}(n, m, G)$, P by $\boldsymbol{\pi}_{\mathbf{G}(n,m,G)}$, and the function N by $|\cdot|$.

3.2. Suppose m and n are nonnegative integers, $n \geq 1$, G is a complete normed commutative group, and, for such G, the map $p_G \colon \mathbf{G}_{\mathrm{o}}(n, m) \times G \to \mathbf{G}(n, m, G)$ denotes the quotient map and the Cartesian product $\mathbf{G}(n, m, \mathbf{Z}) \times G$ is endowed with the metric whose value at $((\delta, g), (\delta', g')) \in (\mathbf{G}(n, m, \mathbf{Z}) \times G)^2$ equals

$$\rho(\delta, \delta') + |g - g'|, \quad \text{where } \rho \text{ is the metric on } \mathbf{G}(n, m, \mathbf{Z}).$$

Then, we define a map $\lambda \colon \mathbf{G}(n, m, \mathbf{Z}) \times G \to \mathbf{G}(n, m, G)$ by requiring

$$\lambda(p_{\mathbf{Z}}(\zeta, d), g) = p_G(\zeta, d \cdot g) \quad \text{for } \zeta \in \mathbf{G}_{\mathrm{o}}(n, m),\ d \in \mathbf{Z}, \text{ and } g \in G.$$

Using the maps ψ_U of 3.1 with G replaced by $\mathbf{Z}$, we verify that λ is locally Lipschitzian and that $\lambda | \big(\boldsymbol{\pi}_{\mathbf{G}(n,m,\mathbf{Z})}^{-1}[\{T\}] \times G\big) \to \boldsymbol{\pi}_{\mathbf{G}(n,m,G)}^{-1}[\{T\}]$ is bilinear for $T \in \mathbf{G}(n, m)$. Henceforward, we will denote $\lambda(\delta, g)$ by $\delta \cdot g$ and, whenever σ is a $\mathbf{G}(n, m, \mathbf{Z})$ valued function and $g \in G$, we will designate by $\sigma \cdot g$ the function with the same domain as σ and value $\sigma(x) \cdot g$ at $x \in \operatorname{dmn} \sigma$. We also note that

$$|\delta \cdot g| \leq |\delta||g| \quad \text{for } \delta \in \mathbf{G}(n, m, \mathbf{Z}) \text{ and } g \in G \text{ with equality if } |\delta| \leq 1.$$

3.3. Suppose m is a nonnegative integer, n is a positive integer, G is a normed commutative group, and Y is a normed space. Then, we endow

$$\mathbf{G}(n, m, G, Y) = \{(\gamma, h) : \gamma \in \mathbf{G}(n, m, G), h \in \operatorname{Hom}(\boldsymbol{\pi}_{\mathbf{G}(n,m,G)}(\gamma), Y)\}$$

with a metric whose value at $((\gamma, h), (\gamma', h')) \in \mathbf{G}(n, m, G, Y)^2$ equals

$$\rho(\gamma, \gamma') + \|h \circ (\operatorname{dmn} h)_\natural - h' \circ (\operatorname{dmn} h')_\natural\|,$$

where ρ is the metric on $\mathbf{G}(n, m, G)$. If G and Y are complete, so is $\mathbf{G}(n, m, G, Y)$.

Whenever ν is a positive integer, we employ the quotient maps

$$p : \mathbf{G}_{\mathrm{o}}(n, m) \times G \to \mathbf{G}(n, m, G) \quad \text{and} \quad q : \mathbf{G}_{\mathrm{o}}(\nu, m) \times G \to \mathbf{G}(\nu, m, G)$$

to define the map

$$P : \mathbf{G}(n, m, G, \mathbf{R}^\nu) \cap \left\{(\gamma, h) : \textstyle\bigwedge_m h \neq 0\right\} \to \mathbf{G}(\nu, m, G)$$

so that, whenever $T \in \mathbf{G}(n, m)$, $g \in G$, $h \in \operatorname{Hom}(T, \mathbf{R}^\nu)$, $\bigwedge_m h \neq 0$, and T is associated with $\zeta \in \mathbf{G}_{\mathrm{o}}(n, m)$, we have

$$P(p(\zeta, g), h) = q\left(\frac{\bigwedge_m (h \circ (\operatorname{dmn} h)_\natural)(\zeta)}{\left|\bigwedge_m (h \circ (\operatorname{dmn} h)_\natural)(\zeta)\right|}, g \right).$$

Clearly, dmn P is an open subset of $\mathbf{G}(n,m,G,\mathbf{R}^\nu)$. Employing the maps ψ_U of 3.1, we readily verify that P is a locally Lipschitzian function. Henceforth, we will denote $P(\gamma,h)$ by $h_\# \gamma$. We also note that, whenever $(\gamma,h),(\gamma',h) \in \mathbf{G}(n,m,G,\mathbf{R}^\nu)$ satisfy $\bigwedge_m h \neq 0$ and $\boldsymbol{\pi}_{\mathbf{G}(n,m,G)}(\gamma) = \boldsymbol{\pi}_{\mathbf{G}(n,m,G)}(\gamma')$, we have

$$|h_\#\gamma| = |\gamma|, \quad \boldsymbol{\pi}_{\mathbf{G}(\nu,m,G)}(h_\#\gamma) = \operatorname{im} h, \quad h_\#(\gamma+\gamma') = h_\#\gamma + h_\#\gamma',$$
$$(i\circ h)_\#\gamma = i_\#(h_\#\gamma) \quad \text{whenever } i \in \operatorname{Hom}(\operatorname{im} h, \mathbf{R}^\mu) \text{ and } \textstyle\bigwedge_m i \neq 0,$$

where μ is a positive integer, and

$$h_\#(\delta\cdot g) = (h_\#\delta)\cdot g \quad \text{whenever } (\delta,h) \in \mathbf{G}(n,m,\mathbf{Z},\mathbf{R}^\nu),\ \textstyle\bigwedge_m h \neq 0, \text{ and } g \in G.$$

Similarly, whenever κ is a positive integer and $\kappa \le m$, we employ the quotient maps $p{:}\,\mathbf{G}_{\mathrm{o}}(n,m)\times G \to \mathbf{G}(n,m,G)$ and $r{:}\,\mathbf{G}_{\mathrm{o}}(n,m-\kappa)\times G \to \mathbf{G}(n,m-\kappa,G)$ and recall 2.32 to define

$$Q : \mathbf{G}(n,m,G,\mathbf{R}^\kappa) \cap \big\{(\gamma,h) : \textstyle\bigwedge^\kappa h \neq 0\big\} \to \mathbf{G}(n,m-\kappa,G)$$

so that, whenever $T \in \mathbf{G}(n,m)$, $g \in G$, $h \in \operatorname{Hom}(T,\mathbf{R}^\kappa)$, $\bigwedge^\kappa h \neq 0$, and T is associated with $\zeta \in \mathbf{G}_{\mathrm{o}}(n,m)$, we have

$$Q(p(\zeta,g),h) = r\left(\frac{\zeta \llcorner \bigwedge^\kappa (h\circ(\operatorname{dmn} h)_\natural)(\omega)}{\left|\zeta \llcorner \bigwedge^\kappa (h\circ(\operatorname{dmn} h)_\natural)(\omega)\right|}, g\right) \quad \text{whenever } 0 \neq \omega \in \textstyle\bigwedge^\kappa \mathbf{R}^\kappa.$$

Clearly, dmn Q is an open subset of $\mathbf{G}(n,m,G,\mathbf{R}^\kappa)$. Employing the maps ψ_U of 3.1, we readily verify that Q is a locally Lipschitzian function. Henceforth, we will denote $Q(\gamma,h)$ by $\gamma \llcorner h$. We finally note that, whenever $(\gamma,h),(\gamma',h) \in \mathbf{G}(n,m,G,\mathbf{R}^\kappa)$ satisfy $\bigwedge^\kappa h \neq 0$ and $\boldsymbol{\pi}_{\mathbf{G}(n,m,G)}(\gamma) = \boldsymbol{\pi}_{\mathbf{G}(n,m,G)}(\gamma')$, we have

$$|\gamma \llcorner h| = |\gamma|, \quad \boldsymbol{\pi}_{\mathbf{G}(n,m-\kappa,G)}(\gamma\llcorner h) = \ker h, \quad (\gamma+\gamma')\llcorner h = \gamma\llcorner h + \gamma'\llcorner h$$

and that, whenever $(\delta,h) \in \mathbf{G}(n,m,\mathbf{Z},\mathbf{R}^\kappa)$, $\bigwedge^\kappa h \neq 0$, and $g \in G$, we have

$$(\delta\cdot g)\llcorner h = (\delta\llcorner h)\cdot g.$$

3.4. Suppose m and μ are nonnegative integers, n and ν are positive integers,

$$p : \mathbf{G}_{\mathrm{o}}(n,m)\times\mathbf{Z} \to \mathbf{G}(n,m,\mathbf{Z}), \quad q : \mathbf{G}_{\mathrm{o}}(\nu,\mu)\times G \to \mathbf{G}(\nu,\mu,G),$$

and $r{:}\,\mathbf{G}_{\mathrm{o}}(n+\nu,m+\mu)\times G \to \mathbf{G}(n+\nu,m+\mu,G)$ are the quotient maps,

$$P : \mathbf{R}^n \to \mathbf{R}^n\times\mathbf{R}^\nu, \quad Q : \mathbf{R}^\nu \to \mathbf{R}^n\times\mathbf{R}^\nu,$$
$$P(x) = (x,0) \quad \text{and} \quad Q(y) = (0,y) \quad \text{for } (x,y) \in \mathbf{R}^n\times\mathbf{R}^\nu,$$

and $\mathbf{R}^n\times\mathbf{R}^\nu \simeq \mathbf{R}^{n+\nu}$. Then, we define a map

$$\kappa : \mathbf{G}(n,m,\mathbf{Z})\times\mathbf{G}(\nu,\mu,G) \to \mathbf{G}(n+\nu,m+\mu,G)$$

so that, whenever $\zeta \in \mathbf{G}_{\mathrm{o}}(n,m)$, $\eta \in \mathbf{G}_{\mathrm{o}}(\nu,\mu)$, $d \in \mathbf{Z}$, and $g \in G$, we have

$$\kappa(p(\zeta,d),q(\eta,g)) = r\big(\textstyle\bigwedge_m P(\zeta)\wedge\bigwedge_\mu Q(\eta), d\cdot g\big).$$

Employing the maps ψ_U of 3.1 with (n, m, G) replaced by $(n, m, \mathbf{Z})$ and (ν, μ, G), respectively, we readily verify that κ is locally Lipschitzian and that

$$\kappa \big| \big(\boldsymbol{\pi}_{\mathbf{G}(n,m,\mathbf{Z})}^{-1}[\{S\}] \times \boldsymbol{\pi}_{\mathbf{G}(\nu,\mu,G)}^{-1}[\{T\}] \big) \to \boldsymbol{\pi}_{\mathbf{G}(n+\nu,m+\mu,G)}^{-1}[\{S \times T\}]$$

is bilinear whenever $S \in \mathbf{G}(n, m)$ and $T \in \mathbf{G}(\nu, \mu)$. Henceforward, we will denote $\kappa(\delta, \gamma)$ by $\delta \times \gamma$ and, whenever σ and τ are $\mathbf{G}(n, m, \mathbf{Z})$ and $\mathbf{G}(\nu, \mu, G)$ valued functions, respectively, we will designate by $\sigma \times \tau$ the function with domain $\operatorname{dmn} \sigma \times \operatorname{dmn} \tau$ and value $\sigma(x) \times \tau(y)$ at $(x, y) \in \operatorname{dmn} \sigma \times \operatorname{dmn} \tau$. We finally note

$$|\delta \times \gamma| \le |\delta||\gamma| \text{ with equality if } G = \mathbf{Z} \text{ or } G = \mathbf{R} \text{ or } |\delta| \le 1,$$
$$\delta \times (\delta' \cdot g) = (\delta \times \delta') \cdot g$$

for $\delta \in \mathbf{G}(n, m, \mathbf{Z})$, $\delta' \in \mathbf{G}(\nu, \mu, \mathbf{Z})$, $\gamma \in \mathbf{G}(\nu, \mu, G)$, and $g \in G$. (Defining $\mathbf{G}(0, 0, G) \simeq G$ and allowing for $\nu = 0$, the $\cdot$ operation of 3.2 could be considered a special case of the present $\times$ operation with $\mu = \nu = 0$.)

3.5. Suppose m and n are integers, $m \ge 0$, $n \ge 1$, U is an open subset of $\mathbf{R}^n$, $\mathscr{H}^m$ refers to the m-dimensional Hausdorff measure over U, and G is a complete normed commutative group. Then, we let $L(U, m, G)$ denote the set of all $\mathbf{G}(n, m, G)$ valued functions σ such that the following conditions are satisfied: $\operatorname{dmn} \sigma \subset U$, $M = \{x : |\sigma(x)| > 0\}$ is $\mathscr{H}^m$ measurable, σ is $\mathscr{H}^m \llcorner M$ measurable, for some separable subset Z of $\mathbf{G}(n, m, G)$, we have $\sigma(x) \in Z$ for $\mathscr{H}^m$ almost all $x \in M$,

$$\int_{K \cap M} |\sigma| \, \mathrm{d}\mathscr{H}^m < \infty \quad \text{whenever } K \text{ is a compact subset of } U,$$

U is countably (ϕ, m) rectifiable, and $\operatorname{Tan}^m(\phi, x) = \boldsymbol{\pi}_{\mathbf{G}(n,m,G)}(\sigma(x))$ for ϕ almost all x, where we abbreviated the measure $(\mathscr{H}^m \llcorner M) \llcorner |\sigma|$ over U by ϕ; hence, σ is a ϕ measurable function and ϕ is the weight of some member of $\mathbf{RV}_m(U)$ by 2.7, 2.2.3 in [11], and 2.26. Elements σ and τ of $L(U, m, G)$ are termed *equivalent* if and only if the functions $\sigma|\{x : |\sigma(x)| > 0\}$ and $\tau|\{x : |\tau(x)| > 0\}$ are $\mathscr{H}^m$ almost equal; the resulting set of equivalence classes is denoted by

$$\mathscr{R}_m^{\mathrm{loc}}(U, G)$$

and its members are called m-dimensional *locally rectifiable G chains* in U. Whenever $S \in \mathscr{R}_m^{\mathrm{loc}}(U, G)$, we denote by $\|S\|$ the Radon measure over U, which is equal to

$$(\mathscr{H}^m \llcorner \{x : |\sigma(x)| > 0\}) \llcorner |\sigma| \quad \text{for } \sigma \in S,$$

and we employ Theorems 2.8.18 and 2.9.13 in [11] to define the function

$$\vec{S}$$

to be the member of S characterised by requiring that, for $\sigma \in S$ and $a \in U$, we have $a \in \operatorname{dmn} \vec{S}$ if and only if

$$\boldsymbol{\Theta}^m(\|S\|, a) > 0 \quad \text{and} \quad (\|S\|, V) \operatorname*{ap\,lim}_{x \to a} \sigma(x) \in \mathbf{G}(n, m, G)$$

and in this case $\vec{S}(a)$ equals that approximate limit, where V is the $\|S\|$ Vitali relation given by $V = \{(x, \mathbf{B}(x, r)) : x \in U, 0 < r < \operatorname{dist}(x, \mathbf{R}^n \sim U)\}$.

In case $m > n$, we have $L(U, m, G) = \{\varnothing\}$, hence $\mathscr{R}_m^{\mathrm{loc}}(U, G)$ contains a single element, and, if $S \in \mathscr{R}_m^{\mathrm{loc}}(U, G)$, then $\|S\| = 0$ and $\vec{S} = \varnothing$.

The considerations of this paragraph rely on 2.26 and on Theorem 3.5 (1b) in [2]. We have

$$\|S\| = \mathscr{H}^m \llcorner \boldsymbol{\Theta}^m(\|S\|, \cdot) \quad \text{for } S \in \mathscr{R}_m^{\mathrm{loc}}(U, G).$$

For $\sigma \in L(U, m, G)$, we say that σ *is adapted to* ϕ if and only if ϕ is the weight of some member of $\mathbf{RV}_m(U)$ if $m \le n$, ϕ is the zero measure over U if $m > n$, the domain of σ is $\mathscr{H}^m$ almost equal to $\{x : \boldsymbol{\Theta}^m(\phi, x) > 0\}$, and

$$\boldsymbol{\pi}_{\mathbf{G}(n,m,G)}(\sigma(x)) = \operatorname{Tan}^m(\phi, x) \quad \text{for } \phi \text{ almost all } x;$$

for instance, $\vec{S}$ is adapted to $\|S\|$ for $S \in \mathscr{R}_m^{\mathrm{loc}}(U, G)$. If $S \in \mathscr{R}_m^{\mathrm{loc}}(U, G)$ and A is $\|S\|$ measurable, then we define $S \llcorner A \in \mathscr{R}_m^{\mathrm{loc}}(U, G)$ by requiring $\sigma | A \in S \llcorner A$ whenever $\sigma \in S$; hence, $\|S \llcorner A\| = \|S\| \llcorner A$. Next, we define the sum

$$S + T \in \mathscr{R}_m^{\mathrm{loc}}(U, G)$$

of S and T in $\mathscr{R}_m^{\mathrm{loc}}(U, G)$ by requiring

$$\rho = (\vec{S} + \vec{T}) \cup \big(\vec{S} | (U \sim \operatorname{dmn} \vec{T})\big) \cup \big(\vec{T} | (U \sim \operatorname{dmn} \vec{S})\big) \in S + T.$$

We infer

$$\|S + T\| = (\|S\| + \|T\|) \llcorner |\rho| / \Theta \le \|S\| + \|T\|,$$

where $\Theta = \boldsymbol{\Theta}^m(\|S\|, \cdot) + \boldsymbol{\Theta}^m(\|T\|, \cdot)$, and note $\sigma + \tau$ belongs to $S + T$ and is adapted to ϕ whenever σ in S and τ in T are both adapted to ϕ; for instance, to $\phi = \|S\| + \|T\|$. It follows that

$$(S + T) \llcorner A = S \llcorner A + T \llcorner A \quad \text{whenever } A \text{ is } \|S\| + \|T\| \text{ measurable},$$

and that $\mathscr{R}_m^{\mathrm{loc}}(U, G)$ is a commutative group which is a complete topological group when endowed with the group norm with value

$$\sum_{i=1}^{\infty} \inf\{2^{-i}, \|S\|(K_i)\} \quad \text{at } S \in \mathscr{R}_m^{\mathrm{loc}}(U, G),$$

where $K_i = U \cap \{x : |x| \le i, \operatorname{dist}(x, \mathbf{R}^n \sim U) \ge 1/i\}$; hence, if $A_1, A_2, A_3, \dots$ is a disjoint sequence of $\|S\|$ measurable sets, then $S \llcorner \bigcup_{i=1}^{\infty} A_i = \sum_{i=1}^{\infty} S \llcorner A_i$. We also let

$$\mathscr{R}_m(U, G) = \mathscr{R}_m^{\mathrm{loc}}(U, G) \cap \{S : \operatorname{spt}\|S\| \text{ is compact}\}.$$

Finally, with respect to the group norm whose value at S equals $\|S\|(\mathbf{R}^n)$,

$$\mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n, G) \cap \{S : \|S\|(\mathbf{R}^n) < \infty\}$$

is a complete normed commutative group; its members correspond to the m-dimensional rectifiable G chains of Section 3.6 in [9]. We also notice that if G is equal to a finite direct sum of cyclic groups with their standard group norm, then

$$\mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n, G) \cap \big\{S : \text{both } \|S\| \text{ and } (\vec{S})_\# \|S\| \text{ have compact support}\big\}$$

is a subgroup thereof; if $m \le n$, then its members correspond to the G varifolds of dimension m in $\mathbf{R}^n$, as defined in Section 2.4 and Subsection 2.6 (d) of [3].

3.6. Suppose that m is a nonnegative integer, n and ν are positive integers, U and V are open subsets of $\mathbf{R}^n$ and $\mathbf{R}^\nu$, respectively, G is a complete normed commutative group, $S \in \mathscr{R}_m^{\mathrm{loc}}(U, G)$, $f : U \to V$ is locally Lipschitzian, $f | \operatorname{spt} \|S\|$ is proper,

$$\chi = f_\# \big(\|S\| \llcorner \| \textstyle\bigwedge_m (\|S\|, m) \operatorname{ap} \mathrm{D} f \| \big),$$

and $Y = \{ y : \operatorname{Tan}^m(\chi, y) \in \mathbf{G}(\nu, m) \}$.

Then, using 2.26, 2.27, 2.31, and the maps ψ_U of 3.1 with n replaced by ν, we obtain an $\mathscr{H}^m \llcorner Y$ measurable function τ, defined on a subset of Y by

$$\tau(y) = \sum_{x \in (\operatorname{dmn} \vec{S}) \cap f^{-1}[\{y\}]} \big((\|S\|, m) \operatorname{ap} \mathrm{D} f(x) \big)_\# \big(\vec{S}(x) \big) \quad \text{whenever } y \in Y,$$

where the summation is understood to be computed in the complete normed commutative group $\boldsymbol{\pi}^{-1}_{\mathbf{G}(\nu,m,G)}[\{\operatorname{Tan}^m(\chi, y)\}]$, we define $f_\# S$ in $\mathscr{R}_m^{\mathrm{loc}}(V, G)$ by requiring that τ belongs to $f_\# S$ and we have

$$\| f_\# S \| = \chi \llcorner |\tau| / \boldsymbol{\Theta}^m(\chi, \cdot) \leq \chi, \quad \operatorname{spt} \| f_\# S \| \subset f[\operatorname{spt} \|S\|].$$

Applying 2.30 to the characteristic function u of $U \cap \{ x : f(x) \neq g(x) \}$, we see that if also $g : U \to V$ is locally Lipschitzian, $g | \operatorname{spt} \|S\|$ is proper, and $g(x) = f(x)$ for $\|S\|$ almost all x, then $f_\# S = g_\# S$. Employing 2.26 and 2.27, we verify that, if σ in S is adapted to ϕ, $f | \operatorname{spt} \phi$ is proper, $\psi = f_\# (\phi \llcorner \| \bigwedge_m (\phi, m) \operatorname{ap} \mathrm{D} f \|)$, and $\Upsilon = \{ \upsilon : \operatorname{Tan}^m(\psi, \upsilon) \in \mathbf{G}(\nu, m) \}$, then Y is $\mathscr{H}^m$ almost contained in Υ and the function ρ, defined on a subset of Υ by

$$\rho(\upsilon) = \sum_{x \in (\operatorname{dmn} \sigma) \cap f^{-1}[\{\upsilon\}]} \big((\phi, m) \operatorname{ap} \mathrm{D} f(x) \big)_\# (\sigma(x)) \quad \text{whenever } \upsilon \in \Upsilon,$$

where the sum is computed in $\boldsymbol{\pi}^{-1}_{\mathbf{G}(\nu,m,G)}[\{\operatorname{Tan}^m(\psi, \upsilon)\}]$, belongs to $f_\# S$ and is adapted to ψ; in particular, τ is adapted to χ. Therefore, we firstly obtain

$$(f_\# S) \llcorner B = f_\# (S \llcorner f^{-1}[B]) \quad \text{whenever } B \text{ is } f_\# \|S\| \text{ measurable},$$

as in this case $f^{-1}[B]$ is $\|S\|$ measurable by 2.1.5 (1) (4) in [11], secondly, in view of 3.3 and 3.5,

$$f_\# (S + T) = f_\# S + f_\# T$$

whenever also $T \in \mathscr{R}_m^{\mathrm{loc}}(U, G)$ and $f | \operatorname{spt} \|T\|$ is proper, and thirdly, using 2.5, 2.27, 2.29, and 3.3,

$$(g \circ f)_\# S = g_\# (f_\# S)$$

whenever also μ is a positive integer, W is an open subset of $\mathbf{R}^\mu$, $g : V \to W$ is locally Lipschitzian, and $g \circ f | \operatorname{spt} \|S\|$ is proper. Finally, the homomorphism

$$f_\# : \mathscr{R}_m^{\mathrm{loc}}(U, G) \cap \{ T : \operatorname{spt} \|T\| \subset C \} \to \mathscr{R}_m^{\mathrm{loc}}(V, G)$$

is continuous whenever C is a relatively closed subset of U such that $f | C$ is proper.

3.7. Suppose m and μ are nonnegative integers, n and ν are positive integers, U and V are open subsets of $\mathbf{R}^n$ and $\mathbf{R}^\nu$, respectively, G is a complete normed commutative group, $S \in \mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z})$, and $T \in \mathscr{R}_\mu^{\mathrm{loc}}(V, G)$. Then, in view of Theorem 3.5 (1b) in [2], Theorem 3.6 (1) (2) in [15], 2.26, 2.36, and 3.4, we define the Cartesian product

$$S \times T \in \mathscr{R}_{m+\mu}^{\mathrm{loc}}(U \times V, G)$$

by requiring it to contain

$$\vec{S} \times \vec{T},$$

we have $\|S \times T\| \leq \|S\| \times \|T\|$ with equality if $G = \mathbf{Z}$ or $G = \mathbf{R}$ or $\boldsymbol{\Theta}^m(\|S\|, x) \leq 1$ for $\mathscr{H}^m$ almost all $x \in U$, and we notice that, if σ in S is adapted to ϕ, τ in T is adapted to χ,

$$M = \{x : \boldsymbol{\Theta}^m(\phi, x) > 0\}, \quad \text{and} \quad N = \{y : \boldsymbol{\Theta}^\mu(\chi, y) > 0\},$$

then, $\operatorname{dmn}(\vec{S} \times \vec{T})$ is $\mathscr{H}^{m+\mu}$ almost contained in $M \times N$ and $(\sigma|M) \times (\tau|N)$ belongs to $S \times T$ and is adapted to $\phi \times \chi$; hence, $\vec{S} \times \vec{T}$ is adapted to $\|S\| \times \|T\|$. (Examples with $\mathscr{H}^{m+\mu}(\operatorname{dmn}(\sigma \times \tau) \sim (M \times N)) > 0$ are readily constructed by means of 3.2.24 in [11].) Therefore, we may verify that

$$\times : \mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z}) \times \mathscr{R}_\mu^{\mathrm{loc}}(V, G) \to \mathscr{R}_{m+\mu}^{\mathrm{loc}}(U \times V, G)$$

is a continuous bilinear operation using 3.4 and 3.5. Finally, if $q \colon U \times V \to V$ satisfies $q(x, y) = y$ for $(x, y) \in U \times V$ and $\operatorname{spt}\|S\|$ is compact, then

$$q_\#(S \times T) = 0 \ \text{ if } m > 0 \quad \text{and} \quad q_\#(S \times T) = \left(\textstyle\sum \vec{S}\right) \cdot T \ \text{ if } m = 0,$$

where the isomorphism $\mathbf{G}(n, 0, \mathbf{Z}) \simeq \mathbf{Z}$ induced by the map $\psi_{\{1\}}$ of 3.1 is used.

3.8. Suppose κ, m, and n are positive integers satisfying $\kappa \leq m$, U is an open subset of $\mathbf{R}^n$, G is a complete normed commutative group, $S \in \mathscr{R}_m^{\mathrm{loc}}(U, G)$, and $f \colon U \to \mathbf{R}^\kappa$ is locally Lipschitzian. Then, defining a $\mathbf{G}(n, m-\kappa, G)$ valued function τ on a subset of U by

$$\tau(x) = \vec{S}(x) \llcorner (\|S\|, m) \operatorname{ap} \mathrm{D} f(x) \quad \text{whenever } x \in U$$

and employing the notation of 3.5, we infer

$$\tau | f^{-1}[\{y\}] \in L(U, m-\kappa, G) \quad \text{for } \mathscr{L}^\kappa \text{ almost all } y$$

from 2.33; for such y, we may define $\langle S, f, y\rangle$ to be the unique member of $\mathscr{R}_{m-\kappa}^{\mathrm{loc}}(U, G)$ containing $\tau | f^{-1}[\{y\}]$, so that in particular,

$$\operatorname{Tan}^{m-\kappa}(\|\langle S, f, y\rangle\|, x) = \ker (\|S\|, m) \operatorname{ap} \mathrm{D} f(x) \quad \text{for } \|\langle S, f, y\rangle\| \text{ almost all } x.$$

Moreover, for $\mathscr{L}^\kappa$ almost all y, we additionally have

$$\|\langle S, f, y\rangle\| = \left(\mathscr{H}^{m-\kappa} \llcorner f^{-1}[\{y\}]\right) \llcorner \boldsymbol{\Theta}^m(\|S\|, \cdot).$$

Using 2.26 and 2.33, we verify that, if σ in S is adapted to ϕ, $X = \{x : \boldsymbol{\Theta}^m(\phi, x) > 0\}$, ρ is a $\mathbf{G}(n, m-\kappa, G)$ valued function, $\operatorname{dmn} \rho \subset X$, and

$$\rho(x) = \sigma(x) \llcorner (\phi, m) \operatorname{ap} \mathrm{D} f(x) \quad \text{for } \mathscr{H}^m \text{ almost all } x \in X,$$

then, for $\mathscr{L}^\kappa$ almost all y, the function $\rho | f^{-1}[\{y\}]$ belongs to $\langle S, f, y\rangle$ and is adapted to $(\mathscr{H}^{m-\kappa} \llcorner f^{-1}[\{y\}]) \llcorner \boldsymbol{\Theta}^m(\phi, \cdot)$; in particular, the function $\tau | f^{-1}[\{y\}]$ in $\langle S, f, y\rangle$ is adapted to $\|\langle S, f, y\rangle\|$ for $\mathscr{L}^\kappa$ almost all y. In view of 2.33, 3.3, and 3.5, we deduce that, if A is $\|S\|$ measurable and $T \in \mathscr{R}_m^{\mathrm{loc}}(U, G)$, then, for $\mathscr{L}^\kappa$ almost all y,

$$\langle S \llcorner A, f, y\rangle = \langle S, f, y\rangle \llcorner A \quad \text{and} \quad \langle S + T, f, y\rangle = \langle S, f, y\rangle + \langle T, f, y\rangle.$$

Finally, taking $K_i = U \cap \{x : |x| \le i, \operatorname{dist}(x, \mathbf{R}^n \sim U) \ge 1/i\}$, we will show that, whenever

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \inf\{2^{-i}, \|S_j\|(K_i)\} < \infty$$

for some $S_j \in \mathscr{R}_m^{\mathrm{loc}}(U, G)$, there holds

$$\lim_{j\to\infty} \langle S_j, f, y\rangle = 0 \quad \text{for } \mathscr{L}^\kappa \text{ almost all } y;$$

in fact, whenever $\varepsilon_i > 0$ satisfy $\varepsilon_i \operatorname{Lip}(f|K_i)^\kappa \le 1$ and $\Phi\colon \mathbf{R}^\kappa \to \{t : 0 < t \le 1\}$ satisfies $\int \Phi \, \mathrm{d}\mathscr{L}^\kappa = 1$, applying 2.33 (2) yields

$$\int \inf\{2^{-i}, \varepsilon_i \|\langle S_j, f, y\rangle\|(K_i)\}\Phi(y) \, \mathrm{d}\mathscr{L}^\kappa \, y \le \inf\{2^{-i}, \|S_j\|(K_i)\}.$$

4. Integral chains

4.1. Whenever m and n are integers, $m \ge 0$, $n \ge 1$, and U is an open subset of $\mathbf{R}^n$, we employ the quotient map $p\colon \mathbf{G}_{\mathrm{o}}(n, m) \times \mathbf{Z} \to \mathbf{G}(n, m, \mathbf{Z})$ and Theorem 4.1.28 in [11] to define

$$\iota_{U,m} : \mathscr{R}_m^{\mathrm{loc}}(U) \to \mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z})$$

by letting $\iota_{U,m}(Q) \in \mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z})$ contain $\tau\colon X \to \mathbf{G}(n, m, \mathbf{Z})$ given by

$$\tau(x) = p\big(\vec{Q}(x), \boldsymbol{\Theta}^m(\|Q\|, x)\big) \quad \text{for } x \in X,$$

where $X = \{x : 0 < \boldsymbol{\Theta}^m(\|Q\|, x) \in \mathbf{Z} \text{ and } \vec{Q}(x) \in \mathbf{G}_{\mathrm{o}}(n, m)\}$, whenever $Q \in \mathscr{R}_m^{\mathrm{loc}}(U)$; hence, $\|\iota_{U,m}(Q)\| = \|Q\|$, τ is adapted to $\|Q\|$, and $\iota_{U,m}$ yields an isomorphism of commutative groups. For such m and U, these isomorphisms have the following four properties whenever $Q \in \mathscr{R}_m^{\mathrm{loc}}(U)$: Firstly, if A is $\|Q\|$ measurable, then

$$(\iota_{U,m}(Q)) \llcorner A = \iota_{U,m}(Q \llcorner A)$$

by 4.1.7 in [11] and 3.5; secondly, if ν is a positive integer, V is an open subset of $\mathbf{R}^\nu$, $f\colon U \to V$ is locally Lipschitzian, and $f | \operatorname{spt} Q$ is proper, then

$$f_\#(\iota_{U,m}(Q)) = \iota_{V,m}(f_\# Q)$$

by the first property, 3.6, and 4.1.7, 4.1.14, and 4.1.30 in [11]; thirdly, if μ and ν are nonnegative integers, $\nu \ge 1$, V is an open subset of $\mathbf{R}^\nu$, and $R \in \mathscr{R}_\mu^{\mathrm{loc}}(V)$, then

$$\iota_{U,m}(Q) \times \iota_{V,\mu}(R) = \iota_{U\times V, m+\mu}(Q \times R)$$

by 4.1.8 in [11] and 3.7; and, finally, if κ is a positive integer, $\kappa \le m$, and $f : U \to \mathbf{R}^\kappa$ is locally Lipschitzian, then

$$\langle \iota_{U,m}(Q), f, y\rangle = \iota_{U,m-\kappa}(\langle Q, f, y\rangle) \quad \text{for } \mathscr{L}^\kappa \text{ almost all } y$$

by the first property, 3.8, and 4.1.7, 4.3.1, and Theorems 4.3.2 (2) and 4.3.8 in [11].

Next, whenever m and U are as before, we let

$$\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) = \iota_{U,m}[\mathbf{I}_m^{\mathrm{loc}}(U)], \quad \mathscr{P}_m(U, \mathbf{Z}) = \iota_{U,m}[\mathscr{P}_m(U)]$$

and, if $m \ge 1$, we employ $\partial : \mathbf{I}_m^{\mathrm{loc}}(U) \to \mathbf{I}_{m-1}^{\mathrm{loc}}(U)$ to define the homomorphism

$$\partial_{\mathbf{Z}} : \mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \to \mathbf{I}_{m-1}^{\mathrm{loc}}(U, \mathbf{Z})$$

such that

$$\partial_{\mathbf{Z}} \circ \iota_{U,m} = \iota_{U,m-1} \circ \partial.$$

Requiring the monomorphisms mapping $S \in \mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z})$ onto

$$(S, \partial_{\mathbf{Z}} S) \in \mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z}) \times \mathscr{R}_{m-1}^{\mathrm{loc}}(U, \mathbf{Z}) \text{ if } m \ge 1, \quad S \in \mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z}) \text{ if } m = 0$$

to be isometric, the groups $\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z})$ are endowed with the structure of complete normed commutative groups. For $S \in \mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z})$ with $m \ge 1$, we have

$$\partial_{\mathbf{Z}}(\partial_{\mathbf{Z}} S) = 0 \text{ if } m \ge 2, \quad \operatorname{spt} \|\partial_{\mathbf{Z}} S\| \subset \operatorname{spt} \|S\|,$$

if ν is a positive integer, V is an open subset of $\mathbf{R}^\nu$, $f : U \to V$ is a locally Lipschitzian map, and $f \mid \operatorname{spt}\|S\|$ is proper, then we have $f_\# S \in \mathbf{I}_m^{\mathrm{loc}}(V, \mathbf{Z})$ with $\partial_{\mathbf{Z}}(f_\# S) = f_\#(\partial_{\mathbf{Z}} S)$ by 4.1.7 in [11], and, if $f : U \to \mathbf{R}$ is locally Lipschitzian, then there holds

$$S \llcorner \{x : f(x) > y\} \in \mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}),$$
$$\partial_{\mathbf{Z}}(S \llcorner \{x : f(x) > y\}) = \langle S, f, y\rangle + (\partial_{\mathbf{Z}} S) \llcorner \{x : f(x) > y\}$$

for $\mathscr{L}^1$ almost all y by 4.2.1, 4.3.1, and 4.3.4 in [11]. From 4.1.8 in [11], we infer that, if additionally μ is a nonnegative integer, ν is a positive integer, and V is an open subset of $\mathbf{R}^\nu$, we have $S \times T \in \mathbf{I}_{m+\mu}^{\mathrm{loc}}(U \times V, \mathbf{Z})$ and

$$\partial_{\mathbf{Z}}(S \times T) = (\partial_{\mathbf{Z}} S) \times T + (-1)^m \cdot (S \times \partial_{\mathbf{Z}} T) \text{ if } m > 0 < \mu,$$
$$\partial_{\mathbf{Z}}(S \times T) = (\partial_{\mathbf{Z}} S) \times T \text{ if } m > \mu = 0, \quad \partial_{\mathbf{Z}}(S \times T) = S \times \partial_{\mathbf{Z}} T \text{ if } m = 0 < \mu$$

whenever $S \in \mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z})$ and $T \in \mathbf{I}_\mu^{\mathrm{loc}}(V, \mathbf{Z})$. Finally, we let

$$\mathbf{I}_m(U, \mathbf{Z}) = \mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \cap \{S : \operatorname{spt}\|S\| \text{ is compact}\}.$$

4.2 Lemma. *Suppose $f : B \to A$ is a homomorphism of commutative groups.*

Then, the following two conditions are equivalent.

(1) *Whenever d is a nonnegative integer, the homomorphism*

$$f_d : B/dB \to A/dA,$$

induced by f, is univalent.

(2) *Whenever G is a commutative group, the homomorphism $f \otimes \mathbf{1}_G$ is univalent.*

Proof. The homomorphisms f_d correspond to $f \otimes \mathbf{1}_{\mathbf{Z}/d\mathbf{Z}}$ via the canonical isomorphisms $B/dB \simeq B \otimes (\mathbf{Z}/d\mathbf{Z})$ and $A/dA \simeq A \otimes (\mathbf{Z}/d\mathbf{Z})$ noted in Corollary 2 to Proposition 6 in Section 3.6, Chapter II, of [6]. If these homomorphisms are univalent, then so are the homomorphisms $f \otimes \mathbf{1}_G$ whenever G is a finitely generated commutative group by Proposition 7 in Section 3.7, Chapter II, of [6] and Theorem 2 on p. 19 of Chapter VII in [8], whence we deduce the validity of (2) by Corollary 4 to Proposition 7 in Section 6.3, Chapter II, of [6]. ∎

4.3 Remark. The conditions imply that f is univalent but the converse does not hold; in fact, one may take $A = \mathbf{Z}$, $B = 2\mathbf{Z}$, f the inclusion map, and $G = \mathbf{Z}/2\mathbf{Z}$ by Remark to Corollary to Proposition 5 in Section 3.6, Chapter II, of [6].

4.4 Example. If f is the inclusion map of a pair (A, B) and the conditions of 4.2 hold, then we shall identify $B \otimes G$ with the subset $(f \otimes \mathbf{1}_G)[B \otimes G]$ of $A \otimes G$. We will prove that, whenever m is a nonnegative integer, $C \subset U \subset \mathbf{R}^n$, U is open, C is closed relative to U, we may take the pair (A, B) to equal

$$(\mathbf{I}_m(U,\mathbf{Z}), \mathscr{P}_m(U,\mathbf{Z})), \quad \big(\mathbf{I}_m^{\mathrm{loc}}(U,\mathbf{Z}), \mathbf{I}_m(U,\mathbf{Z})\big), \quad \big(\mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z}), \mathscr{R}_m(U,\mathbf{Z})\big),$$
$$\big(\mathbf{I}_m^{\mathrm{loc}}(U,\mathbf{Z}), \mathbf{I}_m^{\mathrm{loc}}(U,\mathbf{Z}) \cap \{S : \operatorname{spt}\|S\| \subset C\}\big),$$
$$(\mathscr{R}_m(U,\mathbf{Z}), \mathbf{I}_m(U,\mathbf{Z})), \quad \text{or} \quad \big(\mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z}), \mathbf{I}_m^{\mathrm{loc}}(U,\mathbf{Z})\big).$$

We recall 4.1. Then, concerning the first pair, we verify that, if d is a positive integer, $Q \in \mathbf{I}_m(U)$, and $dQ \in \mathscr{P}_m(U)$, then $Q \in \mathscr{P}_m(U)$, by employing the representation of the image of dQ in $\mathscr{P}_m(\mathbf{R}^n)$ by oriented convex cells obtained in 4.1.32 of [11]; concerning the last two pairs, localising by means of slicing in the case of the last pair, we similarly make use of the closure theorem, see 4.2.16 (2) in [11]; and the remaining pairs trivially satisfy the conditions.

4.5. Suppose m is a nonnegative integer, n is a positive integer, U is an open subset of $\mathbf{R}^n$, and G is a complete normed commutative group. Then, we may define a bilinear operation from $\mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z}) \times G$ into $\mathscr{R}_m^{\mathrm{loc}}(U,G)$ by requiring that

$$S \cdot g \in \mathscr{R}_m^{\mathrm{loc}}(U,G)$$

contains $\sigma \cdot g$ whenever σ in $S \in \mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z})$ and $g \in G$; hence, $\|S \cdot g\| \le |g| \|S\|$ with equality if $\boldsymbol{\Theta}^m(\|S\|, x) = 1$ for $\|S\|$ almost all x. For $S \in \mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z})$ and $g \in G$, we notice the following four properties. If A is $\|S\|$ measurable, then $(S \cdot g) \llcorner A = (S \llcorner A) \cdot g$; if ν is a positive integer, V is an open subset of $\mathbf{R}^\nu$, $f : U \to V$ is locally Lipschitzian, and $f \,|\, \operatorname{spt}\|S\|$ is proper, then we have $f_\#(S \cdot g) = (f_\# S) \cdot g$; if μ is a nonnegative integer, ν is a positive integer, V is an open subset of $\mathbf{R}^\nu$, and $T \in \mathscr{R}_\mu^{\mathrm{loc}}(V,\mathbf{Z})$, then $(S \times T) \cdot g = S \times (T \cdot g)$; and, if κ is a positive integer, $\kappa \le m$, and $f : U \to \mathbf{R}^\kappa$ is locally Lipschitzian, then we have $\langle S \cdot g, f, y \rangle = \langle S, f, y \rangle \cdot g$ for $\mathscr{L}^\kappa$ almost all y.

Next, recalling 4.4, we will study the homomorphism

$$\rho_{U,m,G} : \mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z}) \otimes G \to \mathscr{R}_m^{\mathrm{loc}}(U,G),$$

characterised by

$$\rho_{U,m,G}(S \otimes g) = S \cdot g \quad \text{for } S \in \mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z}) \text{ and } g \in G.$$

Clearly, $\rho_{U,m,\mathbf{Z}}$ is the canonical isomorphism $\mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z}) \otimes \mathbf{Z} \simeq \mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z})$,

$$\rho_{U,m,G}[\mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z}) \otimes G] = \mathscr{R}_m^{\mathrm{loc}}(U,G) \quad \text{if } G \text{ is finite,}$$

and

$$\rho_{U,0,G}[\mathscr{R}_0(U,\mathbf{Z}) \otimes G] = \mathscr{R}_0(U,G).$$

In general, noting $\rho_{U,m,G}[\mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z}) \otimes G]$ is dense in $\mathscr{R}_m^{\mathrm{loc}}(U,G)$ and that $\mathbf{I}_m(U,\mathbf{Z})$ is dense in $\mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z})$, we obtain that

$$\rho_{U,m,G}[\mathbf{I}_m(U,\mathbf{Z}) \otimes G] \text{ is dense in } \mathscr{R}_m^{\mathrm{loc}}(U,G).$$

Whenever $S \in \rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U,\mathbf{Z}) \otimes G]$ and C is a relatively closed neighbourhood of $\operatorname{spt}\|S\|$ in U, there holds $S \in \rho_{U,m,G}[(\mathbf{I}_m^{\mathrm{loc}}(U,\mathbf{Z}) \cap \{T : \operatorname{spt}\|T\| \subset C\}) \otimes G]$; in fact, this is readily verified using 2.16 and 4.1. Finally, we let

$$\mathscr{P}_m(U,G) = \rho_{U,m,G}[\mathscr{P}_m(U,\mathbf{Z}) \otimes G].$$

4.6 Theorem. *Suppose m is a nonnegative integer, n is a positive integer, U is an open subset of $\mathbf{R}^n$, G is a complete normed commutative group, and $\rho_{U,m,G}$ is as in 4.5.*

Then, $\rho_{U,m,G}$ is a monomorphism.

Proof. Suppose $\xi \in \ker \rho_{U,m,G}$. Then, for some finitely generated subgroup H of G, we have $\xi \in \operatorname{im}(\mathbf{1}_{\mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z})} \otimes i)$, where $i \colon H \to G$ is the inclusion map. In view of Theorem 2 on p. 19 of Chapter VII in [8], there exist integers r and s with $0 \le r \le s$ and integers d_t with $d_t \ge 2$ such that

$$A = \bigoplus_{t=1}^{r} (\mathbf{Z}/d_t\mathbf{Z}) \oplus \bigoplus_{t=r+1}^{s} \mathbf{Z} \simeq H \quad \text{(as commutative groups).}$$

Choosing $h_t \in H$ corresponding to a generator of the t-th summand of A under this isomorphism for $t = 1, \ldots, s$, we express

$$\xi = \sum_{t=1}^{s} S_t \otimes i(h_t) \quad \text{for some } S_t \in \mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z}).$$

Selecting σ_t in S_t adapted to $\phi = \sum_{t=1}^{s} \|S_t\|$, we infer that $\sum_{t=1}^{s} \sigma_t \cdot i(h_t)$, which belongs to $0 \in \mathscr{R}_m^{\mathrm{loc}}(U,G)$, is adapted to ϕ by 3.5. The preceding isomorphism then yields

$$\sigma_t(x) \cdot i(h_t) = 0 \in \boldsymbol{\pi}_{\mathbf{G}(n,m,G)}^{-1}[\{\operatorname{Tan}^m(\phi,x)\}] \quad \text{for } t = 1, \ldots, s$$

for ϕ almost all x; hence, we have

$$\sigma_t(x) \in d_t \boldsymbol{\pi}_{\mathbf{G}(n,m,\mathbf{Z})}^{-1}[\{\operatorname{Tan}^m(\phi,x)\}] \quad \text{for } t = 1, \ldots, r,$$
$$\sigma_t(x) = 0 \in \boldsymbol{\pi}_{\mathbf{G}(n,m,\mathbf{Z})}^{-1}[\{\operatorname{Tan}^m(\phi,x)\}] \quad \text{for } t = r+1, \ldots, s$$

for such x. Finally, for $t = 1, \ldots, r$, we infer $S_t \otimes i(h_t) = 0$, as $S_t \in d_t \mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z})$ and $d_t \cdot h_t = 0$, whereas, for $t = r+1, \ldots, s$, we clearly have $S_t = 0$. ∎

4.7 Remark. Since $S = 0$ or $d = 0$ whenever $S \in \mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z})$, $d \in \mathbf{Z}$, and $d \cdot S = 0$, we infer from Proposition 1 in Section 2.3, Chapter I, of [7] that $\mathbf{1}_{\mathscr{R}_m^{\mathrm{loc}}(U,\mathbf{Z})} \otimes i$ is a monomorphism. In view of 2.3, we also note that H may fail to be isomorphic to A as normed group, when A is endowed with the standard group norm.

4.8 Remark. If G is finite, then $\rho_{n,m,G}$ accordingly induces isomorphisms

$$\mathscr{R}_m(U, \mathbf{Z}) \otimes G \simeq \mathscr{R}_m(U, G) \quad \text{and} \quad \mathscr{P}_m(U, \mathbf{Z}) \otimes G \simeq \mathscr{P}_m(U, G)$$

by 3.5, 4.4, and 4.5.

4.9 Corollary. *Whenever m is a positive integer, there exists a unique homomorphism* (*see* 4.1 *and* 4.5)

$$\partial_G : \rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G] \to \rho_{U,m-1,G}[\mathbf{I}_{m-1}^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G]$$

such that $\partial_G(S \cdot g) = (\partial_{\mathbf{Z}} S) \cdot g$ for $S \in \mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z})$ and $g \in G$. Moreover, in the case $G = \mathbf{Z}$, this homomorphism agrees with

$$\partial_{\mathbf{Z}} : \mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \to \mathbf{I}_{m-1}^{\mathrm{loc}}(U, \mathbf{Z}).$$

Proof. In view of 4.6, it is sufficient for the principal conclusion to recall from 4.4 that the canonical homomorphism from $\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G$ into $\mathscr{R}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G$ is univalent. The postscript is then readily verified by means of 4.5. ■

4.10 Remark. Employing 4.1 and 4.5 with suitable relatively closed neighbourhoods C of $\operatorname{spt}\|S\|$ in U, we verify that, whenever $S \in \rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G]$, we have $\partial_G(\partial_G S) = 0$ if $m \geq 2$, $\operatorname{spt}\|\partial_G S\| \subset \operatorname{spt}\|S\|$, and

$$f_\# S \in \rho_{V,m,G}[\mathbf{I}_m^{\mathrm{loc}}(V, \mathbf{Z}) \otimes G], \quad \text{with} \quad \partial_G(f_\# S) = f_\#(\partial_G S),$$

whenever ν is a positive integer, V is an open subset of $\mathbf{R}^\nu$, and $f : U \to V$ is a locally Lipschitzian map such that $f | \operatorname{spt}\|S\|$ is proper; in fact, for the last equation, suitability of C amounts to properness of $f|C$. Finally, from 4.1 and 4.5, we infer that, if μ is a nonnegative integer, ν is a positive integer, V is an open subset of $\mathbf{R}^\nu$, then

$$S \times T \in \rho_{U \times V, m+\mu, G}[\mathbf{I}_{m+\mu}^{\mathrm{loc}}(U \times V, \mathbf{Z}) \otimes G],$$
$$\partial_G(S \times T) = (\partial_{\mathbf{Z}} S) \times T + (-1)^m \cdot (S \times \partial_G T) \text{ if } m > 0 < \mu,$$
$$\partial_G(S \times T) = (\partial_{\mathbf{Z}} S) \times T \text{ if } m > \mu = 0, \quad \partial_G(S \times T) = S \times \partial_G T \text{ if } m = 0 < \mu$$

for $S \in \mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z})$ and $T \in \rho_{V,\mu,G}[\mathbf{I}_\mu^{\mathrm{loc}}(V, \mathbf{Z}) \otimes G]$, and, recalling 3.5 and 3.8, that, if $f : U \to \mathbf{R}$ is locally Lipschitzian and $S \in \rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G]$, then

$$S \llcorner \{x : f(x) > y\} \in \rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G],$$
$$\partial_G(S \llcorner \{x : f(x) > y\}) = \langle S, f, y \rangle + (\partial_G S) \llcorner \{x : f(x) > y\}$$

for $\mathscr{L}^1$ almost all y.

4.11 Definition. Suppose m is a nonnegative integer, n is a positive integer, U is an open subset of $\mathbf{R}^n$, and G is a complete normed commutative group.

Then, we define the complete normed commutative group $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ of m-dimensional *locally integral G chains* in U to be the subgroup (see 4.5)

$$\operatorname{Clos}\left\{(S, \partial_G S) : S \in \rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G]\right\}$$

of $\mathscr{R}_m^{\mathrm{loc}}(U, G) \times \mathscr{R}_{m-1}^{\mathrm{loc}}(U, G)$, if $m \geq 1$, and $\mathbf{I}_m^{\mathrm{loc}}(U, G) = \mathscr{R}_m^{\mathrm{loc}}(U, G)$, if $m = 0$. In the case $m \geq 1$, we recall 4.9 and 4.10 to define the continuous homomorphism

$$\partial_G : \mathbf{I}_m^{\mathrm{loc}}(U, G) \to \mathbf{I}_{m-1}^{\mathrm{loc}}(U, G)$$

by $\partial_G(S, T) = (T, 0)$ if $m \geq 2$ and $\partial_G(S, T) = T$ if $m = 1$ for $(S, T) \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$.

4.12 Remark. Defining the monomorphisms (see 4.9)

$$\kappa_{U,m,G} : \rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G] \to \mathbf{I}_m^{\mathrm{loc}}(U, G),$$
$$\kappa_{U,m,G}(S) = (S, \partial_G S) \text{ if } m \geq 1, \quad \kappa_{U,m,G}(S) = S \text{ if } m = 0,$$

whenever $S \in \rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G]$, we employ 4.10 to verify that, if $m \geq 1$, then

$$\partial_G\, \kappa_{U,m,G}(S) = \kappa_{U,m-1,G}(\partial_G S) \quad \text{for } S \in \rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G].$$

In the case $G = \mathbf{Z}$, the map $\kappa_{U,m,G}$ is an isometry between $\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z})$, as defined in 4.1, and $\mathbf{I}_m^{\mathrm{loc}}(U, G)$, as defined in 4.11, by 4.9. In the general case, the next theorem will allow us to subsequently identify $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ with a dense subgroup of $\mathscr{R}_m^{\mathrm{loc}}(U, G)$ by showing that the homomorphism mapping $(S, T) \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$ onto $S \in \mathscr{R}_m^{\mathrm{loc}}(U, G)$ is univalent; the isometric isomorphism $\kappa_{U,m,\mathbf{Z}}$ will then correspond to the identity map on $\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z})$ justifying our notation for $G = \mathbf{Z}$.

4.13 Theorem. *Suppose m and n are positive integers, U is an open subset of $\mathbf{R}^n$, and G is a complete normed commutative group.*

Then, the following eight statements hold.

(1) *If $(0, T) \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$, then $T = 0$.*

Henceforward, we will identify $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ with the image of the monomorphism mapping $(S, T) \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$ onto $S \in \mathscr{R}_m^{\mathrm{loc}}(U, G)$.

(2) *The subgroup $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ is dense in $\mathscr{R}_m^{\mathrm{loc}}(U, G)$.*

(3) *The subgroup $\rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G]$, see 4.5, is dense in $\mathbf{I}_m^{\mathrm{loc}}(U, G)$, and the boundary operator $\partial_G : \mathbf{I}_m^{\mathrm{loc}}(U, G) \to \mathbf{I}_{m-1}^{\mathrm{loc}}(U, G)$ yields the unique continuous extension of*

$$\partial_G : \rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G] \to \rho_{U,m-1,G}[\mathbf{I}_{m-1}^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G],$$

see 4.9.

(4) *If $m \geq 2$, then $\partial_G(\partial_G S) = 0$ for $S \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$.*

(5) *For $S \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$, we have $\operatorname{spt}\|\partial_G S\| \subset \operatorname{spt}\|S\|$.*

(6) *If ν is a positive integer, V is an open subset of $\mathbf{R}^\nu$, $f : U \to V$ is locally Lipschitzian, $S \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$, and $f \mid \operatorname{spt}\|S\|$ is proper, then*

$$f_\# S \in \mathbf{I}_m^{\mathrm{loc}}(V, G) \quad \textit{and} \quad \partial_G(f_\# S) = f_\#(\partial_G S).$$

(7) *If* $S \in \mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z})$*,* μ *is a nonnegative integer,* ν *is a positive integer,* V *is an open subset of* $\mathbf{R}^\nu$*, and* $T \in \mathbf{I}_\mu^{\mathrm{loc}}(V, G)$*, then* $S \times T \in \mathbf{I}_{m+\mu}^{\mathrm{loc}}(U \times V, G)$ *and*

$$\begin{aligned} \partial_G(S \times T) &= (\partial_{\mathbf{Z}} S) \times T + (-1)^m \cdot (S \times \partial_G T) && \text{if } m > 0 < \mu, \\ \partial_G(S \times T) &= (\partial_{\mathbf{Z}} S) \times T && \text{if } m > \mu = 0, \\ \partial_G(S \times T) &= S \times \partial_G T && \text{if } m = 0 < \mu. \end{aligned}$$

(8) *If* $S \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$ *and* $f : U \to \mathbf{R}$ *is locally Lipschitzian, then there holds*

$$\begin{gathered} S \llcorner \{x : f(x) > y\} \in \mathbf{I}_m^{\mathrm{loc}}(U, G), \\ \partial_G(S \llcorner \{x : f(x) > y\}) = \langle S, f, y\rangle + (\partial_G S) \llcorner \{x : f(x) > y\} \end{gathered}$$

for $\mathscr{L}^1$ *almost all* y*.*

Proof. The statement (4) is trivial. We will prove the following three assertions.

(9) If $(S, T) \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$ and $f : U \to \mathbf{R}$ is locally Lipschitzian, then

$$(S \llcorner \{x : f(x) > y\}, \langle S, f, y\rangle + T \llcorner \{x : f(x) > y\}) \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$$

for $\mathscr{L}^1$ almost all y.

(10) If $(S, T) \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$ and C is a neighbourhood of $\operatorname{spt}(\|S\| + \|T\|)$ which is relatively closed in U, then

$$(S, T) \in \operatorname{Clos}\big\{(R, \partial_G R) : R \in \rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G], \operatorname{spt}\|R\| \subset C\big\}.$$

(11) If $(S, T) \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$, ν is a positive integer, V is an open subset of $\mathbf{R}^\nu$, $f : U \to V$ is locally Lipschitzian, and $f | \operatorname{spt}(\|S\| + \|T\|)$ is proper, then $(f_\# S, f_\# T) \in \mathbf{I}_m^{\mathrm{loc}}(V, G)$.

For (9) and (11), the special case that $S \in \rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G]$ and $T = \partial_G S$ was noted in 4.10. The general case of (9) then follows by approximation by means of 3.5 and 3.8. Applying 2.16 with $E_0 = U \sim \operatorname{Int} C$ and $E_1 = \operatorname{spt}(\|S\| + \|T\|)$, we deduce (10) from 4.10, again using 3.5 and 3.8. Selecting a relatively closed neighbourhood C of $\operatorname{spt}(\|S\| + \|T\|)$ in U such that $f | C$ is proper, the general case of (11) follows from the special case by approximation based on 3.6, 4.10, and (10).

To prove (1) in the case $m > n$, it suffices to note that $\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z})$ and thus also $\operatorname{im} \kappa_{U,m,G}$ and its closure $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ are trivial groups.

To prove (1) in the case $m \le n$, we suppose there were $(0, T) \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$ such that $T \neq 0$. Six further properties could be assumed. Firstly, compactness of $\operatorname{spt}\|T\|$ by (9); secondly, $U = \mathbf{R}^n$ by (11); thirdly, $\|T\|(M) > 0$, where $M = \{x + g(x) : x \in P\}$ for some $P \in \mathbf{G}(n, m-1)$ and some $g \colon P \to P^\perp$ with $\operatorname{Lip} g < \infty$, see Kirszbraun's theorem 2.10.43, 3.1.19 (5), and Theorem 3.2.29 in [11]; fourthly, $M = P$ by (11); fifthly, $\|T\|(M) > \|T\|(\mathbf{R}^n \sim M)$ by applying (9) with $f(x)$ replaced by $|x - a|$ for some $a \in \mathbf{R}^n$ with

$$\lim_{r \to 0+} \frac{\|T\|(\mathbf{B}(a, r) \sim M)}{\|T\| \, \mathbf{B}(a, r)} = 0,$$

see Theorems 2.8.18 and 2.9.11 in [11]; and, sixthly, $\operatorname{spt}\|T\| \subset M$ by applying (11) with f replaced by $M_\natural$, as $(M_\natural)_\# T \neq 0$ would be ensured by the facts

$$(M_\natural)_\#(T \llcorner M) = T \llcorner M \quad \text{and} \quad \|(M_\natural)_\#(T \llcorner \mathbf{R}^n \sim M)\|(\mathbf{R}^n) \le \|T\|(\mathbf{R}^n \sim M),$$

see 3.5 and 3.6. Then, taking a compact neighbourhood C of $\operatorname{spt}\|T\|$, we could employ (10) to secure $R_i \in \rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{Z}) \otimes G]$, with $\operatorname{spt}\|R_i\| \subset C$ for every positive integer i such that

$$\lim_{i\to\infty}(R_i, \partial_G R_i) = (0, T) \quad \text{in } \mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n, G) \times \mathscr{R}_{m-1}^{\mathrm{loc}}(\mathbf{R}^n, G)$$

and, recalling 3.6, additionally require that $\operatorname{spt}\|R_i\| \subset M$ by applying 3.6 and 4.10 with f replaced by $M_{\natural}$. Since $M \in \mathbf{G}(n, m-1)$, this would imply $R_i = 0$ and thus $\partial_G R_i = 0$ for every positive integer i, in contradiction to $T \neq 0$.

Having established (1), the remaining conclusions pose no difficulty: (2) is implied by 4.5; 4.12 yields (3); (5) may be inferred from (9); (6) follows from (5) and (11); (7) is deduced from 3.7, 4.10, and (3); finally, (8) reduces to (9). ∎

4.14 Remark. The proof of (1) was inspired by [13], p. 163.

4.15 Remark. If $S \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$ and C is a neighbourhood of $\operatorname{spt}\|S\|$ which is relatively closed in U, then S belongs to the closure of $\rho_{U,m,G}[\mathbf{I}_m^{\mathrm{loc}}(U, \mathbf{Z}) \otimes G] \cap \{T : \operatorname{spt}\|T\| \subset C\}$ in $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ by (5) and (10).

4.16 Corollary. *Suppose additionally that ν is a positive integer, V is an open subset of $\mathbf{R}^\nu$, A is an open subinterval of $\mathbf{R}$, $0 \in A$, $t \in A$, $h\colon A \times U \to V$ is locally Lipschitzian, $f\colon U \to V$ and $g\colon U \to V$ satisfy*

$$f(x) = h(0, x) \quad \text{and} \quad g(x) = h(t, x) \quad \text{for } x \in U,$$

$S \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$, and $h|(\operatorname{spt}[0, t] \times \operatorname{spt}\|S\|)$ is proper.

Then, there holds $h_\#([0, t] \times S) \in \mathbf{I}_{m+1}^{\mathrm{loc}}(V, G)$ and

$$g_\# S - f_\# S = \partial_G h_\#([0, t] \times S) + h_\#([0, t] \times \partial_G S), \qquad \text{if } m \geq 1,$$
$$g_\# S - f_\# S = \partial_G h_\#([0, t] \times S), \qquad \text{if } m = 0;$$

here, $[0, t]$ in $\mathbf{I}_1^{\mathrm{loc}}(A)$ is identified with $\iota_{A,1}([0, t])$ in $\mathbf{I}_1^{\mathrm{loc}}(A, \mathbf{Z})$, see 4.1.

Proof. Recalling 3.6 and 3.7, it suffices to compute the boundary of the chain

$$h_\#([0, t] \times S) \in \mathbf{I}_{m+1}^{\mathrm{loc}}(V, G)$$

by means of 4.13 (5) (6) (7). ∎

4.17 Definition. Suppose m and n are integers, $m \geq 0$, $n \geq 1$, U is an open subset of $\mathbf{R}^n$, and G is a complete normed commutative group.

Then, we let $\mathbf{I}_m(U, G) = \mathbf{I}_m^{\mathrm{loc}}(U, G) \cap \{S : \operatorname{spt}\|S\| \text{ is compact}\}$.

4.18 Theorem. *Suppose n is a positive integer, $U \subset V \subset \mathbf{R}^n$, $i\colon U \to V$ is the inclusion map, U and V are open, G is a complete normed commutative group, and*

$$r_m : \mathscr{R}_m^{\mathrm{loc}}(V, G) \to \mathscr{R}_m^{\mathrm{loc}}(U, G), \qquad \text{whenever } m \text{ is a nonnegative integer},$$

are characterised by $\vec{T}|U \in r_m(T)$ for $T \in \mathscr{R}_m^{\mathrm{loc}}(V, G)$.

Then, there holds

$$r_m[\mathbf{I}_m^{\mathrm{loc}}(V, G)] \subset \mathbf{I}_m^{\mathrm{loc}}(U, G)$$

and, in case $m \geq 1$, also $\partial_G r_m(T) = r_{m-1}(\partial_G T)$ whenever $T \in \mathbf{I}_m^{\mathrm{loc}}(V, G)$. In particular, we have $i_\#[\mathbf{I}_m(U, G)] = \mathbf{I}_m(V, G) \cap \{T : \operatorname{spt}\|T\| \subset U\}$.

Proof. Clearly, r_m are continuous homomorphisms. Assuming $m \geq 1$, we thus define the closed subgroup H of $\mathbf{I}_m^{\mathrm{loc}}(V, G)$ to consist of those $T \in \mathbf{I}_m^{\mathrm{loc}}(V, G)$ with

$$(r_m(T), r_{m-1}(\partial_G T)) \in \operatorname{Clos}\{(S, \partial_G S) : S \in \mathbf{I}_m(U, G)\},$$

where the closure is taken in $\mathscr{R}_m^{\mathrm{loc}}(U, G) \times \mathscr{R}_{m-1}^{\mathrm{loc}}(U, G)$. Employing 4.13 (6), we readily verify $i_\#[\mathbf{I}_m(U, G)] \subset H$. Recalling 4.1, we infer

$$\big(\rho_{V,m,G}[\mathbf{I}_m^{\mathrm{loc}}(V, \mathbf{Z}) \otimes G]\big) \cap \{T : \operatorname{spt}\|T\| \text{ is a compact subset of } U\} \subset i_\#[\mathbf{I}_m(U, G)]$$

from 4.5; hence, $\mathbf{I}_m(V, G) \cap \{T : \operatorname{spt}\|T\| \subset U\} \subset H$ by 4.15.

To prove $H = \mathbf{I}_m^{\mathrm{loc}}(V, G)$, we next suppose $T \in \mathbf{I}_m^{\mathrm{loc}}(V, G)$ and obtain a locally Lipschitzian map $f : V \to \mathbf{R}$ such that $U = \{x : f(x) > 0\}$ and the set $\{x : f(x) \geq y\}$ is compact for $y > 0$ from 2.14. By 4.13 (8), $\mathscr{L}^1$ almost all y belong to the set Y of $y \in \mathbf{R}$ with $T \llcorner \{x : f(x) > y\} \in \mathbf{I}_m^{\mathrm{loc}}(V, G)$ and

$$\partial_G(T \llcorner \{x : f(x) > y\}) = \langle T, f, y\rangle + (\partial_G T) \llcorner \{x : f(x) > y\};$$

therefore, we conclude $T \in H$ because $0 \in \operatorname{Clos}(Y \cap \{y : y > 0\})$. As $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ is isometric to the subgroup $\{(S, \partial_G S) : S \in \mathbf{I}_m^{\mathrm{loc}}(U, G)\}$ of $\mathscr{R}_m^{\mathrm{loc}}(U, G) \times \mathscr{R}_{m-1}^{\mathrm{loc}}(U, G)$ and $\mathbf{I}_m^{\mathrm{loc}}(U, G)$ is complete, the conclusion follows. ■

5. Classical coefficient groups

5.1 Example. Whenever m is a nonnegative integer and n is a positive integer, we employ the quotient map $p : \mathbf{G}_{\mathrm{o}}(n, m) \times \mathbf{R} \to \mathbf{G}(n, m, \mathbf{R})$ and Section 1 of [12] to define isomorphisms (of commutative groups)

$$\lambda_{n,m} : \mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n) \cap \{Q : Q \text{ has positive densities}\} \to \mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R})$$

by letting $\lambda_{n,m}(Q) \in \mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R})$ contain $\tau : X \to \mathbf{G}(n, m, \mathbf{R})$ given by

$$\tau(x) = p\big(\vec{Q}(x), \boldsymbol{\Theta}^m(\|Q\|, x)\big) \quad \text{for } x \in X,$$

where $X = \{x : 0 < \boldsymbol{\Theta}^m(\|Q\|, x) < \infty \text{ and } \vec{Q}(x) \in \mathbf{G}_{\mathrm{o}}(n, m)\}$, whenever Q is an m-dimensional locally flat chain in $\mathbf{R}^n$ with positive densities. The formal properties of $\iota_{\mathbf{R}^n,m}$ listed in the first paragraph of 4.1 are shared by $\lambda_{n,m}$ and have the same proof; moreover, we have $\lambda_{n,m}(rQ) = \iota_{\mathbf{R}^n,m}(Q) \cdot r$ for $Q \in \mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^n)$ and $r \in \mathbf{R}$. Recalling 4.5 and 4.9, we infer that

$$\mathscr{P}_m(\mathbf{R}^n, \mathbf{R}) = \lambda_{n,m}[\mathbf{P}_m(\mathbf{R}^n)],$$
$$\rho_{\mathbf{R}^n,m,\mathbf{R}}[\mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{Z}) \otimes \mathbf{R}] = \lambda_{n,m}[D_{n,m}],$$
$$\text{where } D_{n,m} \text{ is the real linear span of } \mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^n) \text{ in } \mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n),$$

and, if $m \geq 1$, that

$$\partial_{\mathbf{R}}\, \lambda_{n,m}(Q) = \lambda_{n,m-1}(\partial Q) \quad \text{for } Q \in D_{n,m}.$$

Next, we define commutative groups by

$$I_{n,m} = \mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n) \cap \{Q : Q \text{ and } \partial Q \text{ have positive densities}\} \quad \text{if } m \geq 1,$$
$$I_{n,0} = \mathbf{F}_0^{\mathrm{loc}}(\mathbf{R}^n) \cap \{Q : Q \text{ has positive densities}\}.$$

Clearly, we have $\mathbf{I}_0^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}) = \lambda_{n,0}[I_{n,0}]$. For $m \geq 1$, we will establish that

$$\mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}) = \lambda_{n,m}[I_{n,m}] \quad \text{with} \quad \partial_{\mathbf{R}} \lambda_{n,m}(Q) = \lambda_{n,m-1}(\partial Q) \text{ for } Q \in I_{n,m}.$$

Defining the group norm σ on $I_{n,m}$ by

$$\sigma(Q) = \sum_{i=1}^{\infty} \big(\inf\{2^{-i}, \|Q\| \, \mathbf{B}(0,i)\} + \inf\{2^{-i}, \|\partial Q\| \, \mathbf{B}(0,i)\} \big) \quad \text{for } Q \in I_{n,m},$$

we note that $\lambda_{n,m}$ maps $D_{n,m}$ isometrically onto $\rho_{\mathbf{R}^n,m,\mathbf{R}}[\mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{Z}) \otimes \mathbf{R}]$. Recalling that $\rho_{\mathbf{R}^n,m,\mathbf{R}}[\mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{Z}) \otimes \mathbf{R}]$ is dense in $\mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R})$ and $\mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R})$ is complete, it therefore suffices to prove that $D_{n,m}$ is σ dense in $I_{n,m}$, since $I_{n,m}$ is σ complete. Observing that $\mathbf{F}_m(\mathbf{R}^n) \cap I_{n,m}$ is σ dense in $I_{n,m}$ by 4.2.1, 4.3.1, and Theorem 4.3.8 in [11], this is a consequence of the deformation theorem obtained in Section 4 of [12]; in fact, employing Section 4 in [12] instead of the deformation theorem 4.2.9 in [11] in its derivation, one readily verifies that one may replace $\mathbf{I}_m(\mathbf{R}^n)$ and $\mathscr{P}_m(\mathbf{R}^n)$ by $\mathbf{F}_m(\mathbf{R}^n) \cap I_{n,m}$ and $\mathbf{P}_m(\mathbf{R}^n)$, respectively, in the approximation theorem 4.2.20 in [11], and clearly we have $g_\# P \in D_{n,m}$ whenever $P \in \mathbf{P}_m(\mathbf{R}^n)$ and $g \colon \mathbf{R}^n \to \mathbf{R}^n$ is Lipschitzian.

5.2 Example. Here, we relate the present treatment to 4.2.26 in [11]. Whenever m is a nonnegative integer, n and d are positive integers, and U is an open subset of $\mathbf{R}^n$, we let $p \colon \mathbf{G}_{\mathrm{o}}(n,m) \times (\mathbf{Z}/d\mathbf{Z}) \to \mathbf{G}(n,m,\mathbf{Z}/d\mathbf{Z})$ denote the quotient map and define $\mu_{U,m,d}$ to be the composition of isomorphisms

$$\begin{aligned} \mathscr{R}_m^d(U) &\simeq \mathscr{R}_m(U)/d\mathscr{R}_m(U) \\ &\simeq \mathscr{R}_m(U) \otimes (\mathbf{Z}/d\mathbf{Z}) \simeq \mathscr{R}_m(U,\mathbf{Z}) \otimes (\mathbf{Z}/d\mathbf{Z}) \simeq \mathscr{R}_m(U,\mathbf{Z}/d\mathbf{Z}), \end{aligned}$$

where the inverse of the first isomorphism thereof is induced by the homomorphism mapping $Q \in \mathscr{R}_m(U)$ onto $(Q)^d \in \mathscr{R}_m^d(U)$, the second is canonical, the third is $\iota_{U,m} \otimes \mathbf{1}_{\mathbf{Z}/d\mathbf{Z}}$, and the fourth is induced by $\rho_{U,m,\mathbf{Z}/d\mathbf{Z}}$; see pp. 426 and 430 of [11], Corollary 2 to Proposition 6 in Section 3.6, Chapter II, of [6], 4.1, and 4.8, respectively. If $Q \in \mathscr{R}_m(U)$ and $S = \mu_{U,m,d}\big((Q)^d\big)$, then the function $\sigma \colon X \to \mathbf{G}(n,m,\mathbf{Z}/d\mathbf{Z})$, defined by

$$\sigma(x) = p\big(\vec{Q}(x), \boldsymbol{\Theta}^m(\|Q\|, x) \cdot 1\big) \quad \text{for } x \in X,$$

where $X = \{x : \vec{Q}(x) \in \mathbf{G}_{\mathrm{o}}(n,m) \text{ and } 0 < \boldsymbol{\Theta}^m(\|Q\|, x) \in \mathbf{Z}\}$ and $1 \in \mathbf{Z}/d\mathbf{Z}$, belongs to S. We readily verify the following three properties for $Q \in \mathscr{R}_m(U)$ with $q = (Q)^d$. Firstly, $\|\mu_{U,m,d}(q)\| = \|Q\|^d$; secondly,

$$(\mu_{U,m,d}(q)) \llcorner A = \mu_{U,m,d}(q \llcorner A) \quad \text{whenever } A \text{ is } \|Q\|^d \text{ measurable};$$

and, thirdly,

$$f_\#(\mu_{U,m,d}(q)) = \mu_{U,m,d}(f_\# q)$$

whenever ν is a positive integer, V is an open subset of $\mathbf{R}^\nu$, and $f : U \to V$ is locally Lipschitzian. Recalling 4.5 and 4.9, we obtain that

$$\mathscr{P}_m(U, \mathbf{Z}/d\mathbf{Z}) = \mu_{U,m,d}[\mathscr{P}_m^d(U)],$$
$$\rho_{U,m,\mathbf{Z}/d\mathbf{Z}}[\mathbf{I}_m(U,\mathbf{Z}) \otimes (\mathbf{Z}/d\mathbf{Z})] = \mu_{U,m,d}[\{(Q)^d : Q \in \mathbf{I}_m(U)\}],$$
$$\partial_{\mathbf{Z}/d\mathbf{Z}}\, \mu_{U,m,d}(q) = \mu_{U,m-1,d}(\partial q) \quad \text{whenever } q = (Q)^d,\ Q \in \mathbf{I}_m(U), \text{ and } m \geq 1.$$

Clearly, $\mathbf{I}_0(U, \mathbf{Z}/d\mathbf{Z}) = \mu_{U,0,d}[\mathbf{I}_0^d(U)]$. For $m \geq 1$, we will establish that

$$\mathbf{I}_m(U, \mathbf{Z}/d\mathbf{Z}) = \mu_{U,m,d}[\mathbf{I}_m^d(U)],$$
$$\partial_{\mathbf{Z}/d\mathbf{Z}}\, \mu_{U,m,d}(q) = \mu_{U,m-1,d}(\partial q) \quad \text{for } q \in \mathbf{I}_m^d(U).$$

We abbreviate

$$D = \{(Q)^d : Q \in \mathbf{I}_m(U)\} \quad \text{and} \quad E = \rho_{U,m,\mathbf{Z}/d\mathbf{Z}}[\mathbf{I}_m(U,\mathbf{Z}) \otimes (\mathbf{Z}/d\mathbf{Z})],$$

define the group norm τ on $\mathbf{I}_m(U, \mathbf{Z}/d\mathbf{Z})$ by

$$\tau(S) = (\|S\| + \|\partial_{\mathbf{Z}/d\mathbf{Z}}\, S\|)(U) \quad \text{for } S \in \mathbf{I}_m(U, \mathbf{Z}/d\mathbf{Z}),$$

and notice that $\mu_{U,m,d}|D$ is an isometry (onto E) with respect to $\mathbf{N}^d$ and τ. One may readily infer the conclusion by combining the following four assertions, to be shown for every compact subset K of U. Firstly, $\mathbf{I}_m^d(U) \cap \{r : \operatorname{spt}^d r \subset K\}$ is $\mathbf{N}^d$ complete;[3] secondly, $\mathbf{I}_m(U, \mathbf{Z}/d\mathbf{Z}) \cap \{S : \operatorname{spt}\|S\| \subset K\}$ is τ complete; thirdly, each $q \in \mathbf{I}_m^d(U)$ with $\operatorname{spt}^d q \subset \operatorname{Int} K$ belongs to the $\mathbf{N}^d$ closure of $D \cap \{r : \operatorname{spt}^d r \subset K\}$; and, fourthly, each $S \in \mathbf{I}_m(U, \mathbf{Z}/d\mathbf{Z})$ with $\operatorname{spt}\|S\| \subset \operatorname{Int} K$ belongs to the τ closure of $E \cap \{T : \operatorname{spt}\|T\| \subset K\}$. The first two are elementary, the fourth follows from 4.15, and the special case $U = \mathbf{R}^n$ of the third is a consequence of the approximation theorem $(4.2.20)^d$ on p. 432 of [11]. Observing that $i_\#$ induced by the inclusion map $i : U \to \mathbf{R}^n$ maps maps D and $\mathbf{I}_m^d(U)$ onto $\{(Q)^d : Q \in \mathbf{I}_m(\mathbf{R}^n), \operatorname{spt}^d Q \subset U\}$ and $\mathbf{I}_m^d(U) \cap \{r : \operatorname{spt}^d r \subset U\}$, respectively, the general case of the third reduces to the special case thereof.

Finally, we record the structural theorem

$$\mathbf{I}_m^d(U) = \{(Q)^d : Q \in \mathbf{I}_m(U)\}$$

from Corollary 1.5 in [27]; in fact, the cited source treats the main case $U = \mathbf{R}^n$ to which the case $U \neq \mathbf{R}^n$ is readily reduced (for instance, by means of 4.5).

5.3 Remark. In line with the list of corrections to [11] that H. Federer maintained and distributed, we mention that, in contrast with the structural theorem, it had been known that

$$\{(Q)_K^d : Q \in \mathbf{I}_{m,K}(U)\}$$

is in general a proper subset of $\mathbf{I}_{m,K}^d(U)$; in fact, we will show that, taking S and f as on p. 426 of [11], $m = d = 2$, $U = \mathbf{R}^6$, and $K = \operatorname{spt} S$, an example is furnished by $(S)_K^2$.

[3] As is implicit in 4.2.26 in [11], we define $\operatorname{spt}^d t$ for $t \in \mathscr{F}_m^d(U)$ by requiring $\operatorname{spt}^d (T)^d = \operatorname{spt}^d T$ for $T \in \mathscr{F}_m(U)$.

Clearly, $S \in \mathscr{R}_{2,K}(\mathbf{R}^6)$ and $\mathscr{F}_K^2(\partial S) = 0$, so that we have $(S)_K^2 \in \mathbf{I}_{2,K}^2(\mathbf{R}^6)$. Moreover, if

$$Q \in \mathscr{F}_{2,K}(\mathbf{R}^6) \quad \text{and} \quad (Q)_K^2 = (S)_K^2,$$

then, since $\mathscr{R}_{3,K}(\mathbf{R}^6) = \{0\}$ so that $\mathscr{F}_{2,K}(\mathbf{R}^6) = \mathscr{R}_{2,K}(\mathbf{R}^6)$, that $\mathscr{F}_K(R) = \mathbf{M}(R)$ whenever $R \in \mathscr{F}_{2,K}(\mathbf{R}^6)$, and that $2\mathscr{F}_{2,K}(\mathbf{R}^6)$ is $\mathscr{F}_K$ closed in $\mathscr{F}_{2,K}(\mathbf{R}^6)$, we conclude

$$Q - S \in 2\mathscr{R}_{2,K}(\mathbf{R}^6),$$

whence we infer that

$$\boldsymbol{\Theta}^2(\|Q\|, x) \text{ is an odd integer for } \mathscr{H}^2 \text{ almost all } x \in K.$$

Using that $B = f[\mathbf{S}^2]$ is a nonorientable connected compact two-dimensional submanifold of class ∞ of $\mathbf{R}^6$, we construct $c > 0$ such that $\mathbf{M}(\partial(Q \llcorner C_j)) \geq c j^{-1}$ for every positive integer j, where $C_j = f[\mathbf{R}^3 \cap \{x : |x| = j^{-1/2}\}]$, so that

$$\mathbf{M}(\partial Q) \geq \sum_{j=1}^{\infty} \mathbf{M}(\partial(Q \llcorner C_j)) = \infty;$$

in fact, since $\mathscr{F}_{3,B}(\mathbf{R}^6) \subset \mathbf{F}_{3,B}(\mathbf{R}^6) = \{0\}$ by 4.1.15 in [11], we notice $\mathbf{F}_B(R) = \mathbf{M}(R)$ for $R \in \mathbf{F}_{2,B}(\mathbf{R}^6)$ by 4.1.24 in [11] and $\mathscr{F}_B(R) = \mathbf{M}(R)$ for $R \in \mathscr{F}_{2,B}(\mathbf{R}^6)$, whence we firstly deduce that

$$\inf\{\mathbf{M}(\partial R) : R \in \mathbf{F}_{2,B}(\mathbf{R}^6), \mathbf{M}(R) = 1\} > 0$$

by 4.1.31 (2) and the compactness theorem 4.2.17 (1) in [11], and then that we may take c to be the infimum of the set of numbers $\mathbf{M}(\partial R)$ corresponding to all $R \in \mathscr{R}_{2,B}(\mathbf{R}^6)$ such that $\boldsymbol{\Theta}^2(\|R\|, x)$ is an odd integer for $\mathscr{H}^2$ almost all $x \in B$, because $c > 0$ by the closure theorem 4.2.16 (2) and the compactness theorem 4.2.17 (2) in [11].

5.4 Remark. The preceding remark shall complete the literature as H. Federer's list of corrections – which does not contain proofs of the above statements – remains unpublished, and the corrections in Remark 2.5 in [22] and Section 4 of [16] are not entirely satisfactory. Regarding [16], a revised version will appear in [17].

6. Constancy theorem

6.1 Theorem. *Suppose m and n are positive integers, U is an open subset of $\mathbf{R}^n$, M is a connected m-dimensional submanifold of class 1 of U, ζ is an m vector field orienting M, G is a complete normed commutative group, $S \in \mathbf{I}_m^{\mathrm{loc}}(U, G)$,*

$$M \cap \operatorname{spt}\|S\| \neq \varnothing, \quad \operatorname{spt}\|\partial_G S\| \subset U \sim M,$$

and $(\operatorname{spt}\|S\|) \sim M$ is closed relative to U.

Then, for some nonzero member g of G, there holds (see 4.1)

$$\operatorname{spt}\|S - \iota_{U,m}((\mathscr{H}^m \llcorner M) \wedge \zeta) \cdot g\| \subset U \sim M.$$

Proof. We firstly establish that if either $m = 1$ or $m > 1$ and the statement of the theorem holds with m replaced by $m - 1$, then the following assertion holds: *if* $-\infty < a_i < b_i < \infty$ *for* $i = 1, \ldots, m$,

$$C = \mathbf{R}^m \cap \{x : a_i \leq x_i \leq b_i \text{ for } i = 1, \ldots, m\},$$

and $T \in \mathbf{I}_m(\mathbf{R}^m, G)$ *satisfies* $\operatorname{spt}\|\partial_G T\| \subset \operatorname{Bdry} C$, *then there exists* g *in* G *with*

$$T = \iota_{\mathbf{R}^m,m}([a_1, b_1] \times \cdots \times [a_m, b_m]) \cdot g.$$

We abbreviate

$$B = \mathbf{R}^m \cap \{x : x_1 = b_1\} \quad \text{and} \quad I = \iota_{\mathbf{R},1}([0, 1]),$$

and define $f\colon \mathbf{R}^m \to \mathbf{R}^m$ and $h\colon \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m$ by

$$\begin{aligned} f(x) &= (a_1, x_2, \ldots, x_m), \\ h(t, x) &= (1 - t) f(x) + tx = ((1 - t)a_1 + tx_1, x_2, \ldots, x_m) \end{aligned}$$

for $(t, x) \in \mathbf{R} \times \mathbf{R}^m$. We see from 4.16 that

$$T = h_\#(I \times ((\partial_G T) \llcorner B));$$

in fact, in view of 3.6, we conclude that $f_\# T = 0$, that $h_\#(I \times T) = 0$, and likewise, since $\mathscr{H}^m(h[\mathbf{R} \times ((\operatorname{Bdry} C) \sim B)]) = 0$, that $h_\#\big(I \times ((\partial_G T) \llcorner (\mathbf{R}^m \sim B))\big) = 0$. If $m = 1$, then $B = \{b_1\}$ and there exists $g \in G$ with $(\partial_G T) \llcorner B = \iota_{\mathbf{R},0}([b_1]) \cdot g$ and

$$T = \iota_{\mathbf{R},1}\big(h_\#([0, 1] \times [b_1])\big) \cdot g = \iota_{\mathbf{R},1}([a_1, b_1]) \cdot g,$$

both by 4.1 and 4.5. If $m > 1$, then, assuming $T \neq 0$, denoting the standard basis of $\mathbf{R}^m$ by $e_1, \ldots, e_m$, and taking $n = m$,

$$\begin{gathered} U = \mathbf{R}^m \cap \{x : x_1 > a_1, a_i < x_i < b_i \text{ for } i = 2, \ldots, m\}, \quad M = B \cap U, \\ \zeta(y) = e_2 \wedge \cdots \wedge e_m \text{ for } y \in M, \quad S = r_{m-1}(\partial_G T) \in \mathbf{I}^{\mathrm{loc}}_{m-1}(U, G), \text{ see 4.18}, \end{gathered}$$

hence $B \cap \operatorname{Bdry} C$ and $M = U \cap \operatorname{Bdry} C$ are $\mathscr{H}^{m-1}$ almost equal, $\partial_G S = 0$ by 4.13 (4), and $\varnothing \neq \operatorname{spt}\|S\| \subset M$ since $T \neq 0$, we apply the statement of the theorem with m replaced by $m - 1$ to infer, for some $g \in G$, that

$$S = \iota_{U,m-1}\big((\mathscr{H}^{m-1} \llcorner M) \wedge \zeta\big) \cdot g;$$

thus,

$$\begin{aligned} (\partial_G T) \llcorner B &= \iota_{\mathbf{R}^m,m-1}\big((\mathscr{H}^{m-1} \llcorner M) \wedge \zeta\big) \cdot g \\ &= \iota_{\mathbf{R}^m,m-1}\big(\xi_\#([a_2, b_2] \times \cdots \times [a_m, b_m])\big) \cdot g, \end{aligned}$$

where $\xi\colon \mathbf{R}^{m-1} \to \mathbf{R}^m$ is defined by $\xi(y) = (b_1, y_1, \ldots, y_{m-1})$ for $y \in \mathbf{R}^{m-1}$, so that 4.1 and 4.5 yield

$$\begin{aligned} T &= \iota_{\mathbf{R}^m,m}\big(h_\#([0, 1] \times \xi_\#([a_2, b_2] \times \cdots \times [a_m, b_m]))\big) \cdot g \\ &= \iota_{\mathbf{R}^m,m}([a_1, b_1] \times \cdots \times [a_m, b_m]) \cdot g, \end{aligned}$$

because $h(t, \xi(y)) = ((1-t)a_1 + tb_1, y_1, \dots, y_{m-1})$ for $(t, y) \in \mathbf{R} \times \mathbf{R}^{m-1}$.

Secondly, we will show that the special case $n = m$ and $U = M = \mathbf{R}^m$ of the statement of the theorem holds for m whenever the assertion of the first paragraph holds for this m. Abbreviating

$$C_r = \mathbf{R}^m \cap \{x : |x_i| \leq r \text{ for } i = 1, \dots, m\},$$

we see that $\mathscr{L}^1$ almost all positive real numbers belong to the set A of $0 < r < \infty$ satisfying $S \llcorner C_r \in \mathbf{I}_m(\mathbf{R}^n, G)$ and $\operatorname{spt}\|\partial_G(S \llcorner C_r)\| \subset \operatorname{Bdry} C_r$ by 4.13 (8). The assertion of the first paragraph then yields $\alpha\colon A \to G$ with

$$S \llcorner C_r = \iota_{\mathbf{R}^m,m}\big((\mathscr{L}^m \llcorner C_r) \wedge \zeta\big) \cdot \alpha(r) \quad \text{for } r \in A.$$

Recalling 3.5 and 4.5, we compute

$$\begin{aligned}\iota_{\mathbf{R}^m,m}&\big((\mathscr{L}^m \llcorner C_r) \wedge \zeta\big) \cdot (\alpha(s) - \alpha(r)) \\ &= S \llcorner (C_s \sim C_r) - \iota_{\mathbf{R}^m,m}\big((\mathscr{L}^m \llcorner (C_s \sim C_r)) \wedge \zeta\big) \cdot \alpha(s)\end{aligned}$$

whenever $r, s \in A$ and $r < s$ and conclude that $\operatorname{im} \alpha$ consists of a single point $g \in G$, that $S = \iota_{\mathbf{R}^m,m}(\mathscr{L}^m \wedge \zeta) \cdot g$, and that $g \neq 0$.

It remains to prove that the statement of the theorem holds for m whenever the assertion of the first paragraph holds for this m. For this purpose, we consider the class Ω of all (V, h) such that V is an open subset U, $h \in G$, and

$$\begin{gathered} M \cap V \neq \varnothing, \quad V \cap \operatorname{spt}\|S\| \subset M, \quad (\mathscr{H}^m \llcorner M \cap V) \wedge \zeta \in \mathscr{R}_m^{\mathrm{loc}}(U), \\ V \cap \operatorname{spt}\big\|S - \iota_{U,m}\big((\mathscr{H}^m \llcorner M \cap V) \wedge \zeta\big) \cdot h\big\| = \varnothing. \end{gathered}$$

To establish $M = \bigcup\{M \cap V : (V, h) \in \Omega\}$, we suppose $a \in M$, recall that nontrivial open balls in $\mathbf{R}^m$ are diffeomorphic to $\mathbf{R}^m$, and employ 3.1.19 (4) in [11] in constructing an open subset V of U with $a \in V$, $V \cap \operatorname{spt}\|S\| \subset M$, and $\mathscr{H}^m(M \cap V) < \infty$ as well as maps $\phi\colon V \to \mathbf{R}^m$ and $\psi\colon \mathbf{R}^m \to V$ of class 1 such that

$$\phi \circ \psi = \mathbf{1}_{\mathbf{R}^m}, \quad M \cap V = \operatorname{im} \psi.$$

We observe that ψ and $\phi | \operatorname{im} \psi$ are proper, and choose an orientation η of $\mathbf{R}^m$ such that

$$\psi_\#(\mathscr{L}^m \wedge \eta) = (\mathscr{H}^m \llcorner M \cap V) \wedge \zeta \in \mathscr{R}_m^{\mathrm{loc}}(V)$$

by means of 4.1.31 in [11] and 3.6. As $V \cap \operatorname{spt}\|\partial_G S\| = \varnothing$ by our hypothesis and 4.13 (4), we may apply 4.18 (with the roles of U and V exchanged) to obtain $S' = r_m(S) \in \mathbf{I}_m^{\mathrm{loc}}(V, G)$ with $\partial_G S' = 0$. Defining $T = \phi_\# S' \in \mathbf{I}_m^{\mathrm{loc}}(\mathbf{R}^m, G)$ with $\partial_G T = 0$ by 4.13 (6), the special case yields $h \in G$ with

$$T = \iota_{\mathbf{R}^m,m}(\mathscr{L}^m \wedge \eta) \cdot h.$$

Noting $\psi \circ \phi | \operatorname{im} \psi = \mathbf{1}_{\operatorname{im} \psi}$, we conclude

$$S' = \psi_\# T = \iota_{V,m}\big((\mathscr{H}^m \llcorner M \cap V) \wedge \zeta\big) \cdot h$$

by 3.6 and 4.5, whence we infer $(V, h) \in \Omega$.

If (V_1, h_1) and (V_2, h_2) belong to Ω and $M \cap V_1 \cap V_2 \neq \varnothing$, then $h_1 = h_2$; in fact, we have $\mathscr{H}^m(M \cap V_1 \cap V_2) > 0$ and, by 3.5 and 4.5, we have

$$\begin{aligned} & V_1 \cap V_2 \cap \operatorname{spt} \big\| \iota_{U,m}\big((\mathscr{H}^m \llcorner M \cap V_1 \cap V_2) \wedge \zeta\big) \cdot (h_1 - h_2) \big\| \\ & \qquad \subset \operatorname{spt} \big\| \iota_{U,m}\big((\mathscr{H}^m \llcorner M \cap V_1) \wedge \zeta\big) \cdot h_1 - \iota_{U,m}\big((\mathscr{H}^m \llcorner M \cap V_2) \wedge \zeta\big) \cdot h_2 \big\| \end{aligned}$$

with the latter set not meeting $V_1 \cap V_2$. Since $M \cap \operatorname{spt}\|S\| \neq \varnothing$, we can select a nonzero $g \in G$ with

$$\Upsilon = \Omega \cap \{(V, h) : h = g\} \neq \varnothing$$

and infer that $\bigcup \{M \cap V : (V, g) \in \Omega\}$ is a nonempty, relatively open and relatively closed subset of M, hence equals M. We conclude $\Upsilon = \Omega$. Since $g \neq 0$ and

$$\|S\| \llcorner V = (\mathscr{H}^m \llcorner M \cap V)|g| \quad \text{for } (V, g) \in \Omega$$

by 4.5, we also have $(\mathscr{H}^m \llcorner M) \wedge \zeta \in \mathscr{R}_m^{\mathrm{loc}}(U)$ and the conclusion follows. ∎

6.2 Remark. The preceding theorem is modelled on 4.1.31 (2) in [11] where the case $G = \mathbf{R}$ is treated. In the present case, orientability of M must be part of the hypotheses (rather than the conclusion) by p. 432 in 4.2.26 of [11] and 5.2.

6.3 Remark. The special case $U = \mathbf{R}^n$ and M an m-dimensional cube is a basic ingredient of deformation theorems. For $G = \mathbf{Z}$ or $G = \mathbf{R}$, this follows from the constancy theorem derived in 4.1.4 and 4.1.7 of [11]. An alternative approach for $G = \mathbf{R}$, designed to be extendable to $G = \mathbf{Z}/d\mathbf{Z}$, is given in 4.2.3 of [11]. However, the flat chain $X \llcorner \mathbf{R}^m \sim H$ constructed in that proof does not belong to the domain of $\psi_\#$ as claimed; this is easily circumvented for $G = \mathbf{R}$ but requires some further arguments for $G = \mathbf{Z}/d\mathbf{Z}$. To avoid this difficulty and as our boundary operator ∂_G is not yet defined on all of $\mathscr{R}_m^{\mathrm{loc}}(U, G)$, the present proof (applicable to locally integral chains only) merges the extensions of 4.1.31 (2) and 4.2.3 in [11] to general G in a simultaneous inductive argument by means of the restriction operators r_m constructed in 4.18. Finally, in the context of the flat G chains of [9], a different approach to the special case is chosen in Theorem 6.3 in [10] on which a constancy theorem for chains in *Lipschitz submanifolds* of complete separable metric spaces (see Theorem 7.6 in [10]) is based.

7. Flat chains

7.1. Suppose n is a positive integer, U is an open subset of $\mathbf{R}^n$, and G is a complete normed commutative group. Whenever m is a nonnegative integer, we note

$$H_m = \big(\mathbf{I}_m^{\mathrm{loc}}(U, G) \times \mathbf{I}_{m+1}^{\mathrm{loc}}(U, G)\big) \cap \{(S, T) : S + \partial_G T = 0\}$$

is a closed subgroup of $\mathscr{R}_m^{\mathrm{loc}}(U, G) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U, G)$ by 4.13 (1) (4) and recall 2.2 to define the complete normed commutative group $\mathscr{F}_m^{\mathrm{loc}}(U, G)$ of m-dimensional *locally flat G chains* in U to be the quotient

$$\big(\mathscr{R}_m^{\mathrm{loc}}(U, G) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U, G)\big) \big/ H_m.$$

Since the composition of canonical continuous homomorphisms

$$\mathscr{R}_m^{\mathrm{loc}}(U,G) \to \mathscr{R}_m^{\mathrm{loc}}(U,G) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U,G) \to \mathscr{F}_m^{\mathrm{loc}}(U,G)$$

is univalent, *we will henceforth identify $\mathscr{R}_m^{\mathrm{loc}}(U,G)$ with its image in $\mathscr{F}_m^{\mathrm{loc}}(U,G)$*. Whenever m is a positive integer, noting that the continuous homomorphisms

$$b_m : \mathscr{R}_m^{\mathrm{loc}}(U,G) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U,G) \to \mathscr{R}_{m-1}^{\mathrm{loc}}(U,G) \times \mathscr{R}_m^{\mathrm{loc}}(U,G),$$

defined by $b_m(S,T) = (0,S)$ for $(S,T) \in \mathscr{R}_m^{\mathrm{loc}}(U,G) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U,G)$, satisfy the conditions $b_m[H_m] \subset H_{m-1}$ by 4.13(4), $b_m \circ b_{m+1} = 0$, and $(\partial_G S, -S) \in H_{m-1}$ for $S \in \mathbf{I}_m^{\mathrm{loc}}(U,G)$, they induce continuous quotient homomorphisms

$$\partial_G : \mathscr{F}_m^{\mathrm{loc}}(U,G) \to \mathscr{F}_{m-1}^{\mathrm{loc}}(U,G),$$

with $\partial_G(\partial_G S) = 0$ for $S \in \mathscr{F}_{m+1}^{\mathrm{loc}}(U,G)$, such that the diagram

$$\begin{array}{ccccc}
\mathbf{I}_m^{\mathrm{loc}}(U,G) & \longrightarrow & \mathscr{R}_m^{\mathrm{loc}}(U,G) & \longrightarrow & \mathscr{F}_m^{\mathrm{loc}}(U,G) \\
\downarrow \partial_G & & & & \downarrow \partial_G \\
\mathbf{I}_{m-1}^{\mathrm{loc}}(U,G) & \longrightarrow & \mathscr{R}_{m-1}^{\mathrm{loc}}(U,G) & \longrightarrow & \mathscr{F}_{m-1}^{\mathrm{loc}}(U,G)
\end{array}$$

commutes. Whenever m is a nonnegative integer, we notice that the quotient homomorphism of $\mathscr{R}_m^{\mathrm{loc}}(U,G) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U,G)$ onto $\mathscr{F}_m^{\mathrm{loc}}(U,G)$ maps

$$(S,T) \in \mathscr{R}_m^{\mathrm{loc}}(U,G) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U,G) \quad \text{onto} \quad S + \partial_G T \in \mathscr{F}_m^{\mathrm{loc}}(U,G);$$

in particular,

$$\mathscr{F}_m^{\mathrm{loc}}(U,G) = \big\{ S + \partial_G T : S \in \mathscr{R}_m^{\mathrm{loc}}(U,G), T \in \mathscr{R}_{m+1}^{\mathrm{loc}}(U,G) \big\},$$

and we record that

$$\mathscr{P}_m(U,G) \text{ is dense in } \mathscr{F}_m^{\mathrm{loc}}(U,G);$$

in fact, to prove the second assertion, recalling 4.1, 4.5, and 4.9, we notice that the subgroup $\rho_{U,m,G}[\mathbf{I}_m(U,\mathbf{Z}) \otimes G]$ is dense in $\mathscr{F}_m^{\mathrm{loc}}(U,G)$ and observe that $\mathbf{R}^n$ may be replaced by U in Corollary 4.2.21 of [11].

7.2 Example. Suppose m is a nonnegative integer, n is a positive integer, and U is an open subset of $\mathbf{R}^n$. Recalling 2.23 and 4.1, the commutative groups $\mathscr{F}_m^{\mathrm{loc}}(U)$ and $\mathscr{F}_m^{\mathrm{loc}}(U,\mathbf{Z})$ are isomorphic via

$$\mathscr{F}_m^{\mathrm{loc}}(U) \simeq \big(\mathscr{R}_m^{\mathrm{loc}}(U) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U)\big) \big/ \ker \eta \simeq \mathscr{F}_m^{\mathrm{loc}}(U,\mathbf{Z}),$$

where the second isomorphism is induced by $\iota_{U,m} \times \iota_{U,m+1}$; in fact, $\iota_{U,m} \times \iota_{U,m+1}$ maps

$$\ker \eta = \big(\mathbf{I}_m^{\mathrm{loc}}(U) \times \mathbf{I}_{m+1}^{\mathrm{loc}}(U)\big) \cap \{(Q,R) : Q + \partial R = 0\}$$

onto

$$\big(\mathbf{I}_m^{\mathrm{loc}}(U,\mathbf{Z}) \times \mathbf{I}_{m+1}^{\mathrm{loc}}(U,\mathbf{Z})\big) \cap \{(S,T) : S + \partial_{\mathbf{Z}} T = 0\}.$$

The preceding isomorphisms

$$\mathscr{F}_m^{\mathrm{loc}}(U) \simeq \mathscr{F}_m^{\mathrm{loc}}(U, \mathbf{Z})$$

commute with the boundary operators ∂ and $\partial_{\mathbf{Z}}$. Finally, topologising the commutative group $\mathscr{F}_m^{\mathrm{loc}}(U)$ as in 4.3.16 of [11], they also are homeomorphisms because 2.16 and 2.22 allow us to employ slicing to verify that basic neighbourhoods of 0 in $\mathscr{F}_m^{\mathrm{loc}}(U)$ are given by the family of sets

$$\eta\big[(\mathscr{R}_m^{\mathrm{loc}}(U) \times \mathscr{R}_{m+1}^{\mathrm{loc}}(U)) \cap \{(Q, R) : (\|Q\| + \|R\|)(W) < \delta\}\big]$$

corresponding to all pairs (W, δ) such that W is open, Clos W is a compact subset of U, and $\delta > 0$.

7.3 Example. Proceeding as in 7.2, with 2.23 and 4.1 replaced by 2.20 and 5.1, we obtain an isomorphism of chain complexes with

$$\mathbf{F}_m^{\mathrm{loc}}(\mathbf{R}^n) \simeq \mathscr{F}_m^{\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}).$$

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Ulrich Menne
Department of Mathematics, National Taiwan Normal University
No. 88, Sec. 4, Tingzhou Rd., 116059 Wenshan Dist., Taipei City, Taiwan (R. O. C.);
ulrich.menne@math.ntnu.edu.tw

Christian Scharrer
Institute for Applied Mathematics, University of Bonn
Endenicher Allee 60, 53115 Bonn, Germany;
scharrer@iam.uni-bonn.de