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Arithmetic Algebraic Geometry. – *On Brauer–Manin obstruction to the Hasse principle and weak approximation for homogeneous spaces under connected reductive groups over global fields*, by Nguyen Quoc Thang, communicated on 8 November 2024.

Abstract. – We give some new formulas via some exact sequences for computing an obstruction to the weak approximation on non-abelian cohomology sets and homogeneous spaces over global fields, with stabilizers belonging to some class of non-connected subgroups. As a consequence, we show that the Brauer–Manin obstruction to the weak approximation for such spaces is the only one. Along the way, we show that the Brauer–Manin obstructions to the Hasse principle and weak approximation for homogeneous spaces under connected reductive groups over global function fields with stabilizers belonging to a certain class of non-necessarily connected groups are the only ones, extending some of Borovoi's results obtained for number fields in this regard.

Keywords. – Brauer groups, weak approximation, Galois cohomology, Tate–Shafarevich kernel, local and global field, reductive group.

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Contents

1. Introduction

Let k be a field, k_s a separable closure of k in an algebraic closure $\bar{k}$ of k, and $\Gamma := \mathrm{Gal}(k_s/k)$ the absolute Galois group of k. Denote by V the set of all places (that is, the set of all equivalent classes of valuations) of k and let k_v be the completion of k at $v \in V$.

Let X be a smooth, geometrically integral k-variety. We say that X satisfies *the Hasse principle with respect to* V if $X(k) \neq \emptyset$ once we have $\prod_{v\in V} X(k_v) \neq \emptyset$. We say that X has *the weak approximation property with respect to a finite subset* $S \subset V$ if the closure $\overline{X(k)}^S$ of $X(k)$ is dense in the product topology via the diagonal embedding in $\prod_{v\in S} X(k_v)$:

$$\overline{X(k)}^S = \prod_{v\in S} X(k_v),$$

and that X has *the weak approximation property over* k if the above holds for any finite set $S \subset V$, or equivalently, $\overline{X(k)} = \prod_v X(k_v)$.

For an affine algebraic group scheme G defined over a field k (cf. e.g. [35]), let $\mathrm{H}^i_{\mathrm{fppf}}(k, G) := \mathrm{H}^i_{\mathrm{fppf}}(\bar{k}/k, G(\bar{k}))$ be the flat cohomology in degree i (≤ 1 if G is non-commutative) of G (which is isomorphic to Galois cohomology $\mathrm{H}^i(k, G) := \mathrm{H}^i(\Gamma, G(k_s))$ in degree i if G is smooth) and let

$$Ш^i(G) := \mathrm{Ker}\left(\mathrm{H}^i_{\mathrm{fppf}}(k, G) \longrightarrow \prod_{v\in V} \mathrm{H}^i_{\mathrm{fppf}}(k_v, G)\right)$$

be the *Tate–Shafarevich kernel* in degree i of G, and for a subset $S \subset V$, we define

$$Ш^i_S(G) := \mathrm{Ker}\left(\mathrm{H}^i_{\mathrm{fppf}}(k, G) \longrightarrow \prod_{v\in V\setminus S} \mathrm{H}^i_{\mathrm{fppf}}(k_v, G)\right),$$

whenever it makes sense (cf. [34]).

If X is a smooth variety over k, let $\mathrm{Br}(X) := \mathrm{H}^2_{et}(X, \mathbf{G}_m)$ denote the cohomological Brauer group of X. For a field extension K/k, we denote $X \times_k K$ the base change of X from k to K. In the case $K = k_s$ (resp. $K = k_v$), we denote $X_s = X \times_k k_s$ (resp. $X_v = X \times_k k_v$) for short. Then, we have natural homomorphisms

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(X_s),$$

where the image of the former lies in the kernel of the latter. Following [16, 17, 38, 40], we set

$$\mathrm{Br}_1(X) := \mathrm{Ker}\left(\mathrm{Br}(X) \longrightarrow \mathrm{Br}(X_s)\right),$$
$$\mathrm{Br}_0(X) := \mathrm{Im}\left(\mathrm{Br}(k) \longrightarrow \mathrm{Br}(X)\right),$$

$$\mathrm{Br}_a(X) := \mathrm{Br}_1(X)/\operatorname{Br}_0(X),$$
$$Б_\omega(X) := \bigl\{x \in \mathrm{Br}_a(X) \mid x_v = 0 \text{ for almost all } v \in V\bigr\},$$

and finally

$$Б(X) := \bigl\{x \in \mathrm{Br}_a(X) \mid x_v = 0 \text{ for all } v \in V\bigr\},$$

where for $x \in \mathrm{Br}_a(X)$, we denote by x_v the image of x in $\mathrm{Br}_a(X_v)$. For a subset of places $S \subset V$, we denote

$$Б_S(X) := \mathrm{Ker}\Bigl(\mathrm{Br}_a(X) \longrightarrow \prod_{v \notin S} \mathrm{Br}_a(X_v)\Bigr),$$

then $Б_\omega(X) = \varinjlim_S Б_S(X) = \bigcup_S Б_S(X)$, and notice that $Б(X) = Б_\emptyset(X)$. Denote by $Y^D = \mathrm{Hom}(Y, \mathbf{Q}/\mathbf{Z})$ the Pontrjagin dual of a torsion abelian group Y.

Now let X be an irreducible, smooth, geometrically integral variety defined over a global field k and assume that $\prod_{v \in V} X(k_v) \neq \emptyset$.

Consider the following natural (Brauer–Manin) pairings:

$$\prod_v X(k_v) \times Б_\omega(X) \longrightarrow \mathbf{Q}/\mathbf{Z} \quad \Bigl(\text{resp. } \prod_{v \in S} X(k_v) \times Б_S(X) \longrightarrow \mathbf{Q}/\mathbf{Z}\Bigr),$$
$$\bigl\langle (x_v), b\bigr\rangle := \sum_v \mathrm{inv}_v\bigl(b_v(x_v)\bigr)$$

(cf. [38, Lem. 6.2]) and write $(x_v) \perp b$ if $\langle (x_v), b\rangle = 0$. We define the Brauer–Manin sets:

$$\Bigl(\prod_v X(k_v)\Bigr)^{Б_\omega(X)} := \Bigl\{(x_v) \in \prod_v X(k_v) \mid (x_v) \perp Б_\omega(X)\Bigr\}$$

and

$$\Bigl(\prod_{v \in S} X(k_v)\Bigr)^{Б_S(X)} := \Bigl\{(x_v) \in \prod_{v \in S} X(k_v) \mid (x_v) \perp Б_S(X)\Bigr\},$$

respectively. It is well known that

$$X(k) \neq \emptyset \implies \Bigl(\prod_v X(k_v)\Bigr)^{Б_\omega(X)} \neq \emptyset.$$

If the converse implication holds, then we say that *the Brauer–Manin obstruction to the Hasse principle is the only one*.

Now assume that $X(k) \neq \emptyset$. Then, by continuity, for any finite set S of places of k, we have the following inclusions:

$$\overline{X(k)}^S \subseteq \Bigl(\prod_{v \in S} X(k_v)\Bigr)^{Б_S(X)}, \quad \overline{X(k)} \subseteq \Bigl(\prod_v X(k_v)\Bigr)^{Б_\omega(X)}.$$

If it happens that

$$\overline{X(k)}^S = \Big(\prod_{v\in S} X(k_v)\Big)^{\text{Б}_S(X)} \quad \Big(\text{resp. } \overline{X(k)} = \Big(\prod_{v} X(k_v)\Big)^{\text{Б}_\omega(X)}\Big),$$

then we say that *the Brauer–Manin obstruction to the weak approximation in S (resp. over k) is the only one*.

One of the important tools to study the arithmetic of algebraic varieties (say via Hasse principle and the weak approximation) is the study of Brauer–Manin pairing (resp. obstruction). In [5], M. Borovoi has proved a general result for *homogeneous spaces under connected affine algebraic groups*, which basically reduces to the following. If H is a smooth affine algebraic group, let H° be the connected component of H, $R_u(H)$ the unipotent radical of H, $H^{\text{red}} := H^\circ/R_u(H)$ the largest reductive quotient, $H^{\text{ss}} := [H^{\text{red}}, H^{\text{red}}]$ the semisimple part of H^{red}, $H^{\text{tor}} := H^{\text{red}}/H^{\text{ss}}$ the maximal torus quotient of H, and $H^{\text{ssu}} := \text{Ker}(H^\circ \to H^{\text{tor}})$. This last subgroup is normal in both H and H° and one denotes the quotient $H^{(\text{m})} := H/H^{\text{ssu}}$. Let $X^*(H) := \text{Hom}(H, \mathbf{G}_m)$. Then, we have the exact sequence

$$1 \longrightarrow H^{\text{tor}} \longrightarrow H^{(\text{m})} \longrightarrow \pi_0(H) \longrightarrow 1;$$

i.e., $H^{(\text{m})}$ is an extension of a torus by the finite étale group scheme $\pi_0(H)$ (the group of connected components of H).

Theorem 1.1 (Cf. [5, Sec. 2]). *Let k be a number field, G a connected affine k-group, H a k-subgroup of G, and $X := G/H$, such that G^{red} has simply connected semisimple part and H/H^{ssu} is commutative, thus of multiplicative type. Then, the Brauer–Manin obstructions to the Hasse principle and the weak approximation for X are the only one.* ∎

If G is an algebraic k-group, one denotes by

$$\text{A}(G) := \prod_{v\in V} G(k_v)/\overline{G(k)} \quad \Big(\text{resp. } \text{A}(S,G) := \prod_{v\in S} G(k_v)/\overline{G(k)}^S\Big)$$

the *defect* (or *obstruction*) *to the weak approximation property of G over k* (resp. *obstruction to weak approximation at S*), where $\overline{G(k)}$ denotes the closure of $G(k)$ in the product of $G(k_v)$. We have the exact sequences of pointed sets

$$1 \longrightarrow \overline{G(k)}^S \longrightarrow \prod_{v\in S} G(k_v) \longrightarrow \text{A}(S,G) \longrightarrow 1, \tag{1.1}$$

$$1 \longrightarrow \overline{G(k)} \longrightarrow \prod_{v} G(k_v) \longrightarrow \text{A}(G) \longrightarrow 1, \tag{1.2}$$

where the distinguished element of $\text{A}(S,G)$ (resp. $\text{A}(G)$) is the equivalence class of the identity element of the product.

It is now a well-known and classical result that over a number field (resp. global function field) k, one can compute effectively the obstruction to weak approximation $\mathrm{A}(S,G)$ and $\mathrm{A}(G)$ for any connected (resp. connected and reductive) k-group G via various cohomological invariants of the group G (see e.g. [14, 38, 42]).

There is a natural question as how to compute the obstruction to weak approximation for other class of varieties, such as homogeneous spaces under connected algebraic groups. One way is to extend the results mentioned above and also the exact sequences (1.2) and (1.2) to the case of homogeneous spaces under connected linear algebraic groups. The first extensions of these exact sequences to the case of homogeneous spaces under connected algebraic groups were obtained by Borovoi ([7, Thm. 1.3] for connected reductive stabilizers). Then, it was extended further by Borovoi and Schlank ([10, Thm. 5.1] for non-necessary connected stabilizers) for k being a number field, though in the later case, no short exact sequences were given.

Theorem 1.2. *Let k be a number field.*

(1) (Cf. [7, Thm. 1.3]) *Let G be a connected affine k-group, such that $\mathrm{A}(G)=1$, $Ш(G)=1$, H a connected k-subgroup of G, and let $X := G/H$. Then, there is an exact sequence of pointed sets*

$$1 \longrightarrow \overline{X(k)}^S \longrightarrow \prod_{v\in S} X(k_v) \longrightarrow C_S(X) \longrightarrow 1,$$

where $C_S(X)$ is a certain cohomological invariant of X (see Section 3).

(2) (Cf. [10, Lem. 3.4, Thm. 5.1 (ii)]) *Let G be a quasi-trivial k-group, H a smooth (non-necessarily connected) k-subgroup of G. Then, there is an exact sequence of pointed sets*

$$1 \longrightarrow \overline{X(k)}^S \longrightarrow \prod_{v\in S} X(k_v) \longrightarrow Ч^1_S(H^{(\mathrm{m})}).$$ ∎

The aim of the present paper is to extend Theorem 1.1 to the case of global function fields, extend Theorem 1.2 (1) to any global field, where H is quasi-connected (thus not necessarily connected), and extend the exact sequence in Theorem 1.2 (2) further to the right for quasi-connected groups H. Due to the complexity of the behavior of unipotent radicals in characteristic $p>0$, we have to restrict our attention to connected reductive groups only.

Our approach is as follows. First we express an obstruction to the weak approximation in the cohomology groups (see Section 3) of the stabilizer. Our first main result in the paper is the following theorem (Section 2 for unexplained notion and notation), which computes *the obstruction to weak approximation in cohomology*.

Theorem 1.3 (See Theorem 3.4). *Let k be a global field, H a quasi-connected reductive k-group, and let S be a finite set of places of k. Then, the following hold.*

(1) *We have the exact sequences of pointed sets*

$$\mathrm{H}^1(k,H) \xrightarrow{\gamma_S} \prod_{v\in S} \mathrm{H}^1(k_v,H) \xrightarrow{f_S} Ч^1_S(H^{(\mathrm{m})}) \longrightarrow 1,$$

$$\mathrm{H}^1_{\mathrm{ab}}(k,H) \xrightarrow{\gamma_S} \prod_{v\in S} \mathrm{H}^1_{\mathrm{ab}}(k_v,H) \xrightarrow{f_{\mathrm{ab},S}} Ч^1_S(H^{(\mathrm{m})}) \longrightarrow 1.$$

(2) *There are natural isomorphisms of finite abelian groups*

$$Ч^1_{\mathrm{ab},S}(H) \xrightarrow{\simeq} Ч^1_S(H^{(\mathrm{m})}), \quad Ч^1_{\mathrm{ab},\omega}(H) \xrightarrow{\simeq} Ч^1_\omega(H^{(\mathrm{m})}).$$

(3) *We have the following exact sequences of pointed sets, functorial in H:*

$$1 \longrightarrow \overline{\gamma_V(\mathrm{H}^1(k,H))} \xrightarrow{i_V} \prod_{v\in V} \mathrm{H}^1(k_v,H) \xrightarrow{f_V} Ч^1_\omega(H^{(\mathrm{m})}) \longrightarrow 1,$$

$$1 \longrightarrow \overline{\gamma_{\mathrm{ab},V}(\mathrm{H}^1_{\mathrm{ab}}(k,H))} \xrightarrow{i_{\mathrm{ab},V}} \prod_{v\in V} \mathrm{H}^1_{\mathrm{ab}}(k_v,H) \xrightarrow{f_{\mathrm{ab},V}} Ч^1_\omega(H^{(\mathrm{m})}) \longrightarrow 1,$$

where the closure is taken in the respective products.

Then, by using Theorem 1.3 as a main tool, we derive the following extension of Borovoi's and Borovoi–Schlank's results mentioned above to the case of homogeneous spaces under connected reductive groups with quasi-connected stabilizers. The following theorem resumes the main results of Section 4.

Theorem 1.4 (See Theorems 4.1, 4.8, 4.11). *Let k be a global field, S a finite set of places of k, G a quasi-trivial connected reductive k-group, H a smooth quasi-connected reductive k-subgroup of G, and $X = G/H$.*

Then, we have the following commutative diagrams, where in each of them, the columns are exact, the third rows represent such a pairing with the right kernel being trivial and the left kernel being $\overline{X(k)}^S$ (resp. $\overline{X(k)}$), and the fourth rows represent perfect pairings of finite abelian groups

$$\begin{tikzcd}
1 \arrow[d] & & & \\
\overline{X(k)}^S \arrow[d] & & 1 & \\
\prod_{v\in S} X(k_v) \arrow[d] & \times & Б_S(X)/Б(X) \arrow[u] \arrow[r] & \mathbf{Q}/\mathbf{Z} \\
Ч^1_S(H^{(\mathrm{m})}) \arrow[d] & \times & Ш^1_S(\widehat{H})/Ш^1(\widehat{H}) \arrow[u, "\simeq"] \arrow[r] & \mathbf{Q}/\mathbf{Z} \arrow[u, leftrightarrow, "="'] \\
1 & & 1 \arrow[u] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[d] & & & \\
\overline{X(k)} \arrow[d] & & 1 & \\
\prod_v X(k_v) \arrow[d] & \times & \text{Б}_\omega(X)/\text{Б}(X) \arrow[u] \arrow[r] & \mathbf{Q}/\mathbf{Z} \\
\text{Ч}^1_\omega(H^{(\mathrm{m})}) \arrow[d] & \times & \text{Ш}^1_\omega(\hat{H})/\text{Ш}(\hat{H}) \arrow[u, "\simeq"'] \arrow[r] & \mathbf{Q}/\mathbf{Z} \arrow[u, leftrightarrow, "="'] \\
1 & & 1. \arrow[u] &
\end{tikzcd}$$

Here, the closure $\overline{X(k)}^S$ *and* $\overline{X(k)}$ *are taken in the product* $\prod_{v\in S} X(k_v)$ *and the product* $\prod_v X(k_v)$*, respectively. In particular, the Brauer–Manin obstruction to the weak approximation in* S *and over* k *for* X *is the only one.*

Then, we apply them to study the Brauer–Manin obstruction to the weak approximation for a class of homogeneous spaces with *quasi-connected reductive stabilizers* over any global field.

Finally, we extend some of Borovoi's main results [5, Thms. 2.2 and 2.4] to the global function field case.

THEOREM 1.5 (See Theorems 5.2, 5.3). *Let* k *be a field with no real places,* G *a connected reductive* k*-group with simply connected semisimple part, and* X *a* k*-homogeneous space under* G *with smooth stabilizer* H *such that* $H^{(\mathrm{m})}$ *is commutative. Then, the Brauer–Manin obstructions to the Hasse and weak approximation for* X *are the only ones.*

Note that over global function fields, the case where the stabilizer is a *connected reductive group* has been considered before (cf. [44, 45] and independently, also in [20, 21]). This partially verifies an analog for global function fields of a conjecture due to Colliot-Thélène (see [13] for the original number field statement) for homogeneous spaces. We remark also that the case of non-connected stabilizer is in general hard to deal with (cf. [47, Sec. 3.2.1]) and the general case is widely open. An announcement of some of the main results of the present paper was given in [44]. Here is the outline of the paper.

In Section 2, we recall some basic facts needed in the sequel.

In Section 3, we discuss some new formulas computing an obstruction to weak approximation in non-abelian cohomology of quasi-connected reductive groups, where the main result is Theorem 3.4.

Then, in Section 4, we apply these new formulas to compute an obstruction to weak approximation in homogeneous spaces under connected reductive groups with quasi-connected stabilizers over global fields and prove some of our main results (Theorems 4.1, 4.8, and 4.11). In particular, these results also show that over any global field k, the Brauer–Manin obstruction to the weak approximation in homogeneous spaces under connected reductive groups with quasi-connected reductive k-stabilizers is the only one.

Finally, in Section 5, we apply our obtained results and the method given by Borovoi to show that over *global fields without real places*, the Brauer–Manin obstruction to the Hasse principle (resp. to the weak approximation) in homogeneous spaces under connected reductive groups with simply connected semisimple part and stabilizers H such that $H^{(\mathrm{m})}$ is commutative is the only one (see Theorems 5.2, 5.3 and Proposition 5.4 for more precise statements).

All the results obtained here in the case of quasi-connected stabilizers and the extension of Borovoi's main results of [5] to global function fields are new. In the particular case of *connected stabilizers*, in the case of global function fields, some of the results were announced in [44], which was submitted in November 2020, and then in [45]. It turns out that some parts of the results were also obtained independently in [20] (online July 20, 2021), [21] (submitted July 29, 2021), with quite a different method of proof.

2. Preliminaries. Kottwitz group and related reciprocity sequence

2.1. *Kottwitz group and Kottwitz exact sequence*

We first need some function field analogs of several important results due to Kottwitz, upon which the results of Borovoi are based. Then, by following the approach taken by Borovoi [7], we extend Borovoi's formulas to the function field case.

For a connected reductive group G defined over a local or global field k, Kottwitz introduced (cf. [31, Secs. 1 and 2]) the following group:

$$\mathcal{A}(G) := \pi_0\big(Z(\widehat{G})^\Gamma\big)^D$$

(which is isomorphic to the dual $\mathrm{Pic}(G)^D$ of the Picard group of G by [30, Sec. 2.4.1]). Here, $\widehat{G}$ denotes the Langlands dual of G and $Z(\cdot)$ denotes the center of $(\cdot)$. If we denote by $X_*(T)$ the co-character group of a torus T, then Borovoi in [6, Sec. 1.4] defined the algebraic fundamental group of G by

$$\pi_1^B(G) := \mathrm{Coker}\big(X_*(T^{\mathrm{sc}}) \longrightarrow X_*(T)\big),$$

where T^{sc} is a maximal k-torus of G^{ss} mapped upon a maximal k-torus T of G. (In loc. cit., the field k is assumed of characteristic 0, but Borovoi's definition works well in any other characteristic, and can be extended to reductive group schemes over any base scheme; see [9, Def. 3.11].) Next, Borovoi considers the following group (see [7, Sec. 1.1]):

$$B(G) := \left(\pi_1^B(G)_\Gamma\right)_{\mathrm{tors}}.$$

Finally, in [14, Sec. 6.1], Colliot-Thélène introduced the algebraic fundamental group of G by setting

$$\pi_1^C(G) := \mathrm{Coker}(F_* \to P_*),$$

where

$$1 \longrightarrow F \longrightarrow H \longrightarrow G \longrightarrow 1$$

is a flasque resolution of G and $P = H^{\mathrm{tor}}$. Then, by [14, Prop. A.2], there is a natural isomorphism $\pi_1^B(G) \simeq \pi_1^C(G)$ over any field k. We need the following results in the sequel.

Theorem 2.1 (Cf. [7, Prop. 3.2] (char. $k=0$), [14, Props. 6.3, A.2], [43, Thm. 1.1]). *Let k be any field, and let G be a connected reductive k-group. Then, there are isomorphisms of functors*

$$\mathcal{A}(G) \xrightarrow{\simeq} B(G) \xrightarrow{\simeq} \mathrm{Pic}(G)^D. \qquad \blacksquare$$

In the following theorem, the last term $Ш^2(G^{\mathrm{tor}})$ was added in [43, Thm. 4.7] and the proof was given there for any global field.

Theorem 2.2. *Let k be a field and let G be a connected reductive k-group.*

(1) (See [31, Thm. 1.2], [14, Thm. 9.1] for $\mathfrak{p}$-adic fields and [43, Thm. 2.5] for all non-archimedean local fields) *With the above notation, if k is a non-archimedean local field, then there is a canonical isomorphism* $\mathrm{H}^1(k, G) \simeq \mathcal{A}(G)$.

(2) (See [31, Prop. 2.6], [6, Thm. 5.15], [14, Thm. 9.4] for number fields (without the term $Ш^2(G)$) and [43, Thm. 4.7] for all global fields) *With the above notation, if k is a global field, then there is the exact sequence*

$$1 \longrightarrow Ш^1(G) \longrightarrow \mathrm{H}^1(k,G) \longrightarrow \bigoplus_v \mathrm{H}^1(k_v,G) \longrightarrow \mathcal{A}(G) \longrightarrow Ш^2(G^{\mathrm{tor}}) \longrightarrow 1,$$

and the above exact sequence is functorial in G. $\blacksquare$

2.2. *Quasi-trivial and quasi-connected reductive groups*

Recall that (cf. [14, Def. 2.1]) a smooth connected affine group G defined over a field k (supposed reductive, if char. $k > 0$) is *quasi-trivial* if $k_s[G]^*/k_s^*$ is a permutation $\mathrm{Gal}(k_s/k)$-module and $\mathrm{Pic}(G_s) = 0$.

Equivalently (cf. [14, Prop. 2.2]), G is quasi-trivial if G^{tor} is a quasi-trivial k-torus and G^{ss} is semisimple simply connected.

A smooth affine k-group G is called *quasi-connected reductive* (cf. [33, Sec. 1.3.1]) if there is an exact sequence

$$1 \longrightarrow G \longrightarrow G_1 \longrightarrow T \longrightarrow 1,$$

where G_1 is a connected reductive k-group and T is a k-torus.

For a quasi-connected reductive k-group G, we may define after Labesse [33] the group $\mathrm{H}^i_{\mathrm{ab}}(\cdot, G)$ which is the abelianized (fppf) cohomology of G (see [33, Sec. 1.6], [43, Sec. 1.4]). If k is a global field and S is a finite set of places of k, denote

$$Ч^1_{\mathrm{ab},S}(G) := \mathrm{Coker}\left(\mathrm{H}^1_{\mathrm{ab}}(k, G) \to \prod_{v\in S} \mathrm{H}^1_{\mathrm{ab}}(k_v, G)\right),$$

$$Ч^1_{\mathrm{ab},\omega}(G) := \mathrm{Coker}\left(\mathrm{H}^1_{\mathrm{ab}}(k, G) \to \prod_{v} \mathrm{H}^1_{\mathrm{ab}}(k_v, G)\right).$$

3. Some formulas for obstructions to the weak approximation in cohomology

Let G be a connected reductive group defined over a global field k, H a smooth k-subgroup of G, and X the homogeneous k-space G/H under G.

In [3–7], for the case of number fields, Borovoi gave various formulas to compute an obstruction to the weak approximation for X in terms of a certain quotient of some auxiliary groups that are related to the groups $\mathcal{A}(G)$, $B(G)$ and the Brauer group of G.

Here, following Borovoi, we give some extensions to the case of global fields with no real places (in particular global function fields) and then apply them to Brauer–Manin obstruction to the weak approximation of homogeneous spaces. To do this, we need some formulas for an obstruction to the weak approximation in cohomology groups of the stabilizer with respect to some natural topology on the group cohomology.

3.1. *Some formulas and notation*

First we recall some notations that will be used in the sequel. Let $\pi_1(G)$ be the algebraic fundamental group defined by Borovoi (cf. [6]; see also some related information in Section 2). For $B(G) := (\pi_1(G)_\Gamma)_{\mathrm{tors}}$ (or the same, $B(G) = \mathrm{Pic}(G)^D$), for each $v \in V$, each finite set S of places of k, we denote $B_v(G) = B(G \times k_v)$, $\lambda_v : B_v(G) \to B(G)$ the natural map, and denote by

$$B^S(G) := \left\langle \mathrm{Im}(\lambda_v) \mid v \notin S \right\rangle$$

the subgroup of $B(G)$ generated by the images of λ_v for $v \notin S$, $B'(G) := B^{\emptyset}(G)$ the subgroup of $B(G)$ generated by the images of λ_v for all v, $B^\omega(G) := \bigcap_S B^S(G)$, where S runs over all finite subsets of places of k,

$$C_S(G) := B'(G)/B^S(G), \quad C_\omega(G) := B'(G)/B^\omega(G);$$

thus, we have $C_\omega(G) = \varprojlim_S C_S(G)$.

Let $B(H,G) := \mathrm{Ker}(B(H) \to B(G))$ which corresponds to the k-morphism of connected reductive k-groups $H \to G$ and for each place v, let

$$B_v(H,G) := \mathrm{Ker}\big(B_v(H) \to B_v(G)\big).$$

We obtain for each v a homomorphism $\lambda_v : B_v(H,G) \to B(H,G)$, and

$$B^S(H,G) := \big\langle \lambda_v\big(B_v(H,G)\big) \mid v \notin S \big\rangle$$

the subgroup of $B(H,G)$ generated by the corresponding images for all $v \notin S$. Finally, we set $B'(H,G) := B^\emptyset(H,G)$ when $S = \emptyset$; thus, we have

$$B^\omega(H,G) \subseteq B^S(H,G) \subseteq B'(H,G) \subseteq B(H,G)$$

and set

$$C_S(X) := B'(H,G)/B^S(H,G), \quad C_\omega(X) := B'(H,G)/B^\omega(H,G),$$

so we have $C_\omega(X) = \varprojlim_S C_S(X)$. Notice that a priori, the quantity $C_S(X)$ depends on each of the following inputs S, G, H and we have

$$C_\omega(X) = 0 \iff B'(H,G) = B^S(H,G) \text{ for all } S \iff C_S(X) = 0 \text{ for all } S.$$

In the particular case when $\mathrm{Pic}(G) = \mathrm{Pic}(G \times k_s) = 0$, we have

$$\begin{aligned} B(H,G) &= B(H) = \mathrm{Pic}(H)^D, \\ B(H_v, G_v) &= B(H_v) = \mathrm{Pic}(H_v)^D, \\ B(H \times k_s, G \times k_s) &= B(H \times k_s) = \mathrm{Pic}(H \times k_s)^D \end{aligned}$$

and since $B(H \times k_{v,s}) = B(H \times k_s)$, for the map $\overline{\lambda} : B(H \times k_s) \to B(H)$, we have

$$\mathrm{Im}(\overline{\lambda}) = \mathrm{Im}(\overline{\lambda}_v) \subseteq \mathrm{Im}(\lambda_v) \tag{3.1}$$

for all v. Hence, we also have

$$\mathrm{Im}(\overline{\lambda}) \subseteq \mathrm{Im}(\oplus_{v \notin S} \lambda_v);$$

i.e.,

$$\mathrm{Im}(\overline{\lambda}) \subseteq B^S(H) \subseteq B'(H) \tag{3.2}$$

for all S. We need the following formula for the duality of torsion abelian groups. Namely, for a homomorphism $f : A \to B$ of such groups, we have

$$[\mathrm{Im}(f)]^D \xrightarrow{\simeq} B^D/(\mathrm{Coker}(f))^D \simeq B^D/\mathrm{Ker}(f^D : B^D \to A^D), \tag{3.3}$$

$$[\mathrm{Ker}(f)]^D \xrightarrow{\simeq} \mathrm{Coker}(f^D : B^D \to A^D). \tag{3.4}$$

Then, by setting

$$\mathrm{Pic}_1(H) := \mathrm{Ker}\left(\mathrm{Pic}(H) \to \mathrm{Pic}(H \times k_s)\right),$$

we have

$$\begin{aligned} \mathrm{Pic}_1(H)^D &= \left[\mathrm{Ker}\left(\mathrm{Pic}(H) \to \mathrm{Pic}(H \times k_s)\right)\right]^D \\ &\xrightarrow{\simeq} \mathrm{Coker}\left[\mathrm{Pic}(H \times k_s)^D \to \mathrm{Pic}(H)^D\right] \\ &\xrightarrow{\simeq} \mathrm{Coker}\left[B(H \times k_s) \to B(H)\right] \\ &\xrightarrow{\simeq} B(H)/\mathrm{Im}(\bar{\lambda}), \end{aligned}$$

so we have

$$\mathrm{Pic}_1(H)^D \xrightarrow{\simeq} B(H)/\mathrm{Im}(\bar{\lambda}) \tag{3.5}$$

and similarly

$$\begin{aligned} \mathrm{Pic}_1(H_v)^D &= \left[\mathrm{Ker}\left(\mathrm{Pic}(H) \to \mathrm{Pic}(H \times \bar{k}_v)\right)\right]^D \\ &\xrightarrow{\simeq} B(H_v)/\mathrm{Im}(\bar{\lambda}). \end{aligned} \tag{3.6}$$

Therefore, by Theorem 2.2, we have natural inclusions

$$B^S(H)/\mathrm{Im}(\bar{\lambda}) \subseteq B'(H)/\mathrm{Im}(\bar{\lambda}). \tag{3.7}$$

Notice that we have

$$\begin{aligned} B^S(H)/\mathrm{Im}(\bar{\lambda}) &= \left\langle \mathrm{Im}(\lambda_v)/\mathrm{Im}(\bar{\lambda}) \mid v \notin S \right\rangle \\ &= \left\langle \mathrm{Im}\left[B_v(H)/\mathrm{Im}(\bar{\lambda}) \to B(H)/\mathrm{Im}(\bar{\lambda})\right] \mid v \notin S \right\rangle \\ &= \left\langle \mathrm{Im}\left[\mathrm{Pic}_1(H_v)^D \to \mathrm{Pic}_1(H)^D\right] \mid v \notin S \right\rangle, \end{aligned} \tag{3.8}$$

$$\begin{aligned} B'(H)/\mathrm{Im}(\bar{\lambda}) &= \left\langle \mathrm{Im}(\lambda_v)/\mathrm{Im}(\bar{\lambda}) \mid v \in V \right\rangle \\ &= \left\langle \mathrm{Im}\left[B_v(H)/\mathrm{Im}(\bar{\lambda}) \to B(H)/\mathrm{Im}(\bar{\lambda})\right] \mid v \in V \right\rangle \\ &= \left\langle \mathrm{Im}\left[\mathrm{Pic}_1(H_v)^D \to \mathrm{Pic}_1(H)^D\right] \mid v \in V \right\rangle, \end{aligned} \tag{3.9}$$

so we have

$$C_S(H) = B'(H)/B^S(H) = \left[B'(H)/\operatorname{Im}(\bar{\lambda})\right]/\left[B^S(H)/\operatorname{Im}(\bar{\lambda})\right] = \frac{\left\langle \operatorname{Im}\left[\operatorname{Pic}_1(H_v)^D \to \operatorname{Pic}_1(H)^D\right] \mid v \in V\right\rangle}{\left\langle \operatorname{Im}\left[\operatorname{Pic}_1(H_v)^D \to \operatorname{Pic}_1(H)^D\right] \mid v \notin S\right\rangle}. \tag{3.10}$$

We need the following generalization of a lemma due to Borovoi, where we restrict ourselves to the case of reductive groups to avoid complications related with unipotent radicals. The part (1) of the following lemma was proved in [15, Lem. 1.5] in char. 0, where the stabilizers are connected.

LEMMA 3.1. *Let k be a field, and let G' be a connected reductive k-group.*

(1) *If H' is a connected k- (resp. k_s-) subgroup of G', then one can represent the k-homogeneous space $X := G'/H'$ as $X = G/H$, where H is a connected k- (resp. k_s-) subgroup of G and G is a quasi-trivial reductive k-group. If H' is smooth, H can be taken smooth, too.*

(2) *If H is a quasi-connected reductive k-group, then $H^{(\mathrm{m})}$ is also a smooth k-group of multiplicative type.*

PROOF. (1) The same proof of [15, Lem. 1.5] for the case char. $k = 0$ also holds in the case char. $k > 0$.

(2) First proof. Let H be quasi-connected reductive and consider the exact sequence

$$1 \longrightarrow H \longrightarrow H_1 \longrightarrow T \longrightarrow 1,$$

where H_1 is connected reductive and T is a torus. Since H^{ssu} is a characteristic subgroup of H° and H is normal in H_1, H^{ssu} is also normal in H_1 and we may consider the quotient $Q := H_1/H^{\mathrm{ssu}}$. Then, we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & H^{\mathrm{ssu}} \arrow[r, "\lambda'"] \arrow[d, "\simeq"] & H \arrow[r, "\gamma'"] \arrow[d, "\alpha"] & H^{(\mathrm{m})} \arrow[r] \arrow[d, "\gamma"] & 1 \\
1 \arrow[r] & H^{\mathrm{ssu}} \arrow[r, "\beta'"] & H_1 \arrow[r, "\alpha'"] \arrow[d, "\beta"] & Q \arrow[r] \arrow[d, "\lambda"] & 1 \\
 & & T \arrow[r, "\simeq"] \arrow[d] & T \arrow[d] & \\
 & & 1 & 1. &
\end{tikzcd}$$

Since $H^{\mathrm{ss}} \subseteq H^{\mathrm{ssu}}$, Q is a torus. Therefore, $H^{(\mathrm{m})}$ is a subgroup of a torus, thus of multiplicative type.

Second proof. (I am thankful to the referee for this short proof.) Note that in general if H is just a kernel of a morphism from a connected affine group onto a torus, then $H^{(\mathrm{m})}$ needs not be a group of multiplicative type. Let H be a quasi-connected reductive group over a field k. Let H° be its connected components and $H^{\mathrm{ss}} = H^{\mathrm{der}} = [H^\circ, H^\circ]$ its semisimple part (derived subgroup). Let us prove that $H^{(\mathrm{m})} = H/H^{\mathrm{der}}$ is of multiplicative type.

We have a short exact sequence:

$$1 \longrightarrow H \longrightarrow H_1 \longrightarrow T \longrightarrow 1, \tag{3.11}$$

where T is a torus and H_1 is reductive and connected. Then, $H_1^{\mathrm{der}} = [H_1, H_1]$ maps to the identity in T as T is commutative; hence, $H_1^{\mathrm{der}} \subseteq H$. Since H_1^{der} is connected, we also get $H_1^{\mathrm{der}} \subseteq H^\circ$. Since H_1 is reductive, H_1^{der} is semisimple, and thus $H_1^{\mathrm{der}} = (H_1^{\mathrm{der}})^{\mathrm{der}} = [H_1^{\mathrm{der}}, H_1^{\mathrm{der}}]$. It follows that

$$H^{\mathrm{der}} \subseteq H_1^{\mathrm{der}} = (H_1^{\mathrm{der}})^{\mathrm{der}} \subseteq H^{\mathrm{der}},$$

and thus $H^{\mathrm{der}} = H_1^{\mathrm{der}}$. Hence, by quotienting the first two terms in (3.11), we get an exact sequence:

$$1 \longrightarrow H^{(\mathrm{m})} = H/H^{\mathrm{der}} \longrightarrow H_1/H_1^{\mathrm{der}} \longrightarrow T \longrightarrow 1.$$

Since H_1 is connected, $H_1/H_1^{\mathrm{der}} = H_1^{(\mathrm{m})}$ is a torus, and any subgroup of it is of multiplicative type as wished. ∎

We need in the sequel the following description of the closure of k-points in the product of local points. The first statement generalizes to homogeneous spaces a well-known result for connected reductive groups (see [26, Satz 1], [38, Proof of Thm. 3.3] (for number fields) and [42, Thm. 1.1] (for global function fields)).

Lemma 3.2. *Let k be a global field and let G be a connected reductive k-group.*

(1) *Let H be a smooth k-subgroup of G. Then, for the k-homogeneous $X = G/H$, and for any finite set S of places of k, the closure $\overline{X(k)}^S$ of $X(k)$ in $\prod_{v\in S} X(k_v)$ is an open subset.*

(2) *Let H be a connected reductive k-subgroup of G. Assume that*

  (a) $\mathrm{A}(S, G) = 1$, $\mathrm{Ker}(\mathrm{H}^1(k, G) \to \prod_{v\in S} \mathrm{H}^1(k_v, G)) = 1$, *and*

  (b) *the localization map* $\mathrm{H}^1(k, H) \to \prod_{v\in S} \mathrm{H}^1(k_v, H)$ *is surjective.*

  Then, the k-variety $X := G/H$ has weak approximation in S.

(3) (See [14, Prop. 9.2] for number fields, and [42, Prop. 2.2 (3)] for global function fields) *Let k be any global field and let G be a quasi-trivial reductive k-group. Then, G has the weak approximation property over k and the Hasse principle for* H^1 *holds for G. Moreover, if k has no real places, then* $\mathrm{H}^1(k,G)$ *and* $\mathrm{H}^1(k_v,G)$ *(for all v) are trivial.*

(4) (See [7, Cor. 1.7] for number fields) *Let k be a number field, H a connected reductive k-group. Then, the k-variety $X = G/H$ has the weak approximation property in ∞.*

(5) *If G is quasi-trivial, then we have $B(G) = 0$, $B'(G) = 0$, $B^S(G) = 0$.*

Proof. (1) We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
G(k) & \xrightarrow{\pi} & X(k) & \xrightarrow{\delta} & \mathrm{H}^1(k,H) & \xrightarrow{i^*} & \mathrm{H}^1(k,G) \\
{\scriptstyle i}\downarrow & & {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle \gamma_S} & & \downarrow{\scriptstyle \beta_S} \\
\prod_{v\in S} G(k_v) & \xrightarrow{\pi_S} & \prod_{v\in S} X(k_v) & \xrightarrow{\delta_S} & \prod_{v\in S}\mathrm{H}^1(k_v,H) & \xrightarrow{i_S^*} & \prod_{v\in S}\mathrm{H}^1(k_v,G).
\end{array}$$

Since H is a smooth k-group, π_S is an open mapping by Implicit Function Theorem, so $\mathrm{Im}(\pi_S)$ is an open subset in $\prod_{v\in S} X(k_v)$. On the one hand, by [26, Satz 1], [38, Proof of Thm. 3.3] (for number fields) and [42, Thm. 1.1] (for global function fields), $\overline{G(k)}^S$ is an open (and normal) subgroup of $\prod_{v\in S} G(k_v)$. On the other hand, the action mapping

$$G(k) \times X(k) \longrightarrow X(k) \subset \overline{X(k)}^S$$

extends by continuity to the action mapping

$$G(k) \times \overline{X(k)}^S \longrightarrow \overline{X(k)}^S,$$

and thus also the action mapping

$$\overline{G(k)}^S \times \overline{X(k)}^S \longrightarrow \overline{X(k)}^S.$$

This implies that if $x \in \overline{X(k)}^S$, then $\overline{G(k)}^S \cdot x \subset \overline{X(k)}^S$, and therefore $\overline{X(k)}^S$ is open in $\prod_{v\in S} X(k_v)$.

(2) Follows by standard diagram chase (see e.g. [29], [38, Sec. 3]). Since the argument is short, we present it here. Given $x_S \in \prod_{v\in S} X(k_v)$, since γ_S is surjective, from the diagram above, we have $\delta_S(x_S) = \gamma_S(h)$, for some $h \in \mathrm{H}^1(k,H)$, so

$$i_S^*(\gamma_S(h)) = \beta_S(i^*(h)) = 1.$$

This implies that $i^*(h) = 1$; i.e., $h = \delta(x)$, $x \in X(k)$; thus, $\delta_S(x) = \delta_S(x_S)$. Therefore,

$$x_S \in \pi_S\Big(\prod_{v\in S} G(k_v)\Big) \cdot x.$$

By assumption, $G(k)$ is dense in $\prod_{v\in S} G(k_v)$, so this implies that

$$x_S \in \pi_S(\overline{G(k)}^S)\cdot x \subset \overline{X(k)}^S.$$

(3) If k is a number field, then the weak approximation property and the Hasse principle for H^1 for G were proved in [14, Prop. 9.2].

If k is a global function field, then the weak approximation for G was proved in [42, Prop. 2.2]. By [14, Prop. 2.2], there is an exact sequence

$$1 \longrightarrow G^{\mathrm{ss}} \longrightarrow G \longrightarrow T \longrightarrow 1,$$

where G^{ss} is a semisimple simply connected k-group, and T is a quasi-trivial k-torus. By Harder's theorem [26, Satz A] (resp. Bruhat–Tits' theorem [12, Thm. 4.7]), we have $\mathrm{H}^1(k, G^{\mathrm{ss}}) = 1$ (resp. $\mathrm{H}^1(k_v, G^{\mathrm{ss}}) = 1$), and also since T is an induced k- torus, $\mathrm{H}^1(k, T) = 1$ (resp. $\mathrm{H}^1(k_v, T) = 1$). This implies that $\mathrm{H}^1(k, G) = 1$.

Finally, if k is a totally imaginary number field, then as above, $\mathrm{H}^1(k_v, G^{\mathrm{ss}}) = 1$ by Bruhat–Tits' theorem quoted above, and the Hasse principle for G^{ss} (cf. [36, Ch. VI]) implies that $\mathrm{H}^1(k, G) = 1$.

(4) Let $1 \to F \to G_1 \xrightarrow{\pi} G \to 1$ be a flasque resolution of G, where G_1 is a quasi-trivial reductive k-group. Then, for $H_1 := \pi^{-1}(H)$, we have $X = G/H \simeq G_1/H_1$, so we may assume from the beginning that G is a quasi-trivial reductive k-group. It is well known that any connected k-group has weak approximation with respect to ∞ (see [38, Cor. 3.5]) and any quasi-trivial k-group G satisfies the Hasse principle and weak approximation over k (see (3)), so the condition (a) of (2) holds. Also, according to a result by Kneser–Harder (see [38, Lem. 1.12]), the localization map $\mathrm{H}^1(k, H) \to \prod_{v\in\infty} \mathrm{H}^1(k_v, H)$ is surjective. Hence, the condition (b) of (2) holds and now (4) follows from (2).

(5) If G is a quasi-trivial reductive group, then the exact sequence

$$1 \longrightarrow G^{\mathrm{ss}} \longrightarrow G \longrightarrow T \longrightarrow 1$$

induces the following exact sequence (cf. [38, Cor. 6.11]):

$$\mathrm{Pic}(T) \longrightarrow \mathrm{Pic}(G) \longrightarrow \mathrm{Pic}(G^{\mathrm{ss}}).$$

Since G^{ss} is simply connected and T is an induced k-torus, they have trivial Picard groups, so this sequence shows that $\mathrm{Pic}(G) = 0$. Therefore, $B(G) = 0$, $B^S(G) = 0$. ■

3.2. Approximation in cohomology

Next, we consider an application to the approximation problem in the cohomology groups (or sets) in the framework as in [42, Sec. 4] (see also [24]).

(a) Recall that for a field k with a place v, let k_v be the completion of k with respect to v. We may endow the cohomology sets $\mathrm{H}^i_{\mathrm{fppf}}(k_v, H)$, $i \geq 0$ for a group scheme H of finite type over k_v with the natural topology as in [1], [42, Sec. 4], or [46] (which is called canonical or special topology there). Recall that for smooth k_v-groups, the corresponding topology is *discrete*, and if H is commutative, this topology agrees with the one defined in [34]. We also equip $\mathrm{H}^i_{\mathrm{fppf}}(k, H)$ with the weakest topology such that for any subset W of places, all the connecting maps and the natural maps

$$\gamma_W : \mathrm{H}^i_{\mathrm{fppf}}(k, H) \longrightarrow \prod_{v \in W} \mathrm{H}^i_{\mathrm{fppf}}(k_v, H)$$

are continuous with respect to the topology of the product of $\prod_{v \in W} \mathrm{H}^i_{\mathrm{fppf}}(k_v, H)$.

(b) Similarly, for a short complex of diagonalizable k-groups (resp. k_v-groups), we endow the hypercohomology sets

$$\mathcal{H}^i_{\mathrm{fppf}}\big(k, [P \to Q]\big) \quad \big(\text{resp. } \mathcal{H}^i_{\mathrm{fppf}}\big(k_v, [P \to Q]\big)\big)$$

as in [32, p. 138, p. 146].

(c) We may drop the subscript “fppf” in the case the groups under consideration are smooth. We may also define a natural topology on the abelian groups $\mathrm{H}^i_{\mathrm{ab}}(k, G)$ (resp. $\mathrm{H}^i_{\mathrm{ab}}(k_v, G)$) for quasi-connected reductive k-groups (resp. k_v-groups) G in the same way as in [32, p. 138, p. 146], so that all the connecting maps and the maps

$$\begin{aligned} \mathrm{ab}^0_G &: G(k_v) = \mathrm{H}^0(k_v, G) \longrightarrow \mathrm{H}^0_{\mathrm{ab}}(k_v, G), \\ \mathrm{ab}^1_G &: \mathrm{H}^1(k_v, G) \longrightarrow \mathrm{H}^1_{\mathrm{ab}}(k_v, G) \end{aligned}$$

are continuous.

More precisely, for a quasi-connected reductive group H defined over a field k by the exact sequence

$$1 \longrightarrow G \longrightarrow G_1 \longrightarrow T \longrightarrow 1,$$

let G_1^{sc} be the simply connected covering of the semisimple part G_1^{ss} of G_1. Then, we have $G_1^{\mathrm{ss}} \subset G$, so we have a short complex

$$G_{\mathrm{ab}} := [G_1^{sc} \to G],$$

which is quasi-isomorphic to the short complex $[Z_1^{sc} \to Z]$, where Z_1^{sc}, Z denote the center of G_1^{sc} and G, respectively. Let

$$\rho : G_1^{sc} \longrightarrow G^{\mathrm{ss}} \longrightarrow G$$

be the composite mapping and set $K := \mathrm{Ker}(\rho)$. Recall that (see [6, Sec. 3.10], [11, Sec. 2.16.1], [23, Prop. 3.10]) we have the long exact sequence

$$\begin{aligned} 1 &\longrightarrow K(k) \longrightarrow G_1^{sc}(k) \xrightarrow{\rho} G(k) \xrightarrow{\mathrm{ab}^0} \mathrm{H}^0_{\mathrm{ab}}(k,G) \\ &\longrightarrow \mathrm{H}^1(k,G_1^{sc}) \xrightarrow{\rho_*} \mathrm{H}^1(k,G) \xrightarrow{\mathrm{ab}^1} \mathrm{H}^1_{\mathrm{ab}}(k,G). \end{aligned}$$

Here, the maps ab^0, ab^1 are defined from the natural morphism of complexes

$$[1 \to G] \longrightarrow [G_1^{sc} \to G];$$

thus,

$$\begin{aligned} \mathrm{ab}^0 &: G(k) = \mathcal{H}^0\big(k,[1 \to G]\big) \longrightarrow \mathcal{H}^0\big(k,[G_1^{sc} \to G]\big), \\ \mathrm{ab}^1 &: \mathrm{H}^1(k,G) = \mathcal{H}^1\big(k,[1 \to G]\big) \longrightarrow \mathcal{H}^1\big(k,[G_1^{sc} \to G]\big). \end{aligned}$$

(d) Assume that k_v is the completion of k with respect to a place v and the notation is as above. Then, the natural topology on k_v-points of algebraic groups is just induced from that of k_v. Further, the topology on $\mathrm{H}^0_{\mathrm{ab}}(k_v,G)$ is the weakest such that ab^0 is continuous (hence also open). Since for smooth k-groups H, the canonical topology on $\mathrm{H}^1(k_v,G)$ is discrete (see [1, Cor. 5.1.3]), it is natural to endow the group $\mathrm{H}^1_{\mathrm{ab,fppf}}(k_v,G)$ with the discrete topology.

(e) The definition of *weak approximation, almost weak approximation* in cohomology sets can be naturally extended to the case of fppf cohomology of a group scheme H defined over a field k, and we use the same definition as in [42, Sec. 4] for *non-commutative group schemes*. (Recall that a k-variety X has *almost weak approximation over k* if for a finite set S of places of k, X has the weak approximation property with respect to $V \setminus S$.) For unipotent group schemes, the weak approximation in degree 1 cohomology sets always holds (see [42, Prop. 4.3]). Further, the obstruction is given there for the case of *commutative group schemes* over global fields. It is natural to ask what the obstruction to the weak approximation in such cohomology sets can be in the *non-commutative case* (see [42, Rem. 4.5]).

Here, we apply the above results to extend the results of [42, Sec. 4] to study the obstruction to weak approximation in cohomology sets in the case of quasi-connected reductive k-groups over global fields with no real places.

(f) For a quasi-connected reductive k-group H, and its related group $H^{(\mathrm{m})}$ and any subset $S \subset V$, we denote by

$$\gamma_S : \mathrm{H}^1(k,H) \longrightarrow \prod_{v \in S} \mathrm{H}^1(k_v,H) =: \Pi_S,$$

$$\gamma_V : \mathrm{H}^1(k,H) \longrightarrow \prod_{v} \mathrm{H}^1(k_v,H) =: \Pi_V,$$

$$\gamma_{\mathrm{ab},S} : \mathrm{H}^1_{\mathrm{ab}}(k, H) \longrightarrow \prod_{v \in S} \mathrm{H}^1_{\mathrm{ab}}(k_v, H) =: \Pi_S,$$

$$\gamma_{\mathrm{ab},V} : \mathrm{H}^1_{\mathrm{ab,fppf}}(k, H) \longrightarrow \prod_{v} \mathrm{H}^1_{\mathrm{ab}}(k_v, H) =: \Pi_V,$$

$$\mu_S : \mathrm{H}^1(k, H^{(\mathrm{m})}) \longrightarrow \prod_{v \in S} \mathrm{H}^1(k_v, H^{(\mathrm{m})}),$$

$$\mu_V : \mathrm{H}^1(k, H^{(\mathrm{m})}) \longrightarrow \prod_{v \in V} \mathrm{H}^1(k_v, H^{(\mathrm{m})}),$$

$$\mu_{\mathrm{ab},S} : \mathrm{H}^1_{\mathrm{ab}}(k, H^{(\mathrm{m})}) \longrightarrow \prod_{v \in S} \mathrm{H}^1_{\mathrm{ab}}(k_v, H^{(\mathrm{m})}),$$

$$\mu_{\mathrm{ab},V} : \mathrm{H}^1_{\mathrm{ab}}(k, H^{(\mathrm{m})}) \longrightarrow \prod_{v \in V} \mathrm{H}^1_{\mathrm{ab}}(k_v, H^{(\mathrm{m})})$$

the localization maps. Further, denote by

$$\pi_S := \prod_{v \in S} \pi_v : \prod_{v \in S} \mathrm{H}^1(k_v, H) \longrightarrow \prod_{v \in S} \mathrm{H}^1(k_v, H^{(\mathrm{m})}),$$

$$\pi_{\mathrm{ab},S} := \prod_{v \in S} \pi_{\mathrm{ab},v} : \prod_{v \in S} \mathrm{H}^1_{\mathrm{ab}}(k_v, H) \longrightarrow \prod_{v \in S} \mathrm{H}^1_{\mathrm{ab}}(k_v, H^{(\mathrm{m})})$$

the maps induced from $H \to H^{(\mathrm{m})}$,

$$\mathrm{ab}_S : \prod_{v \in S} \mathrm{H}^1(k_v, H^{(\mathrm{m})}) \longrightarrow \prod_{v \in S} \mathrm{H}^1_{\mathrm{ab},v}(k_v, H^{(\mathrm{m})})$$

the product of the maps $\mathrm{ab}_v : \mathrm{H}^1(k_v, H^{(\mathrm{m})}) \to \mathrm{H}^1_{\mathrm{ab},v}(k_v, H^{(\mathrm{m})})$, and

$$p_S : \prod_{v \in S} \mathrm{H}^1_{\mathrm{ab},v}(k_v, H^{(\mathrm{m})}) \longrightarrow \text{Ч}^1_{\mathrm{ab},S}(H^{(\mathrm{m})}) := \mathrm{Coker}(\mu_{\mathrm{ab},S}).$$

These maps give rise to a composite mapping $f_S := p_S \circ \mathrm{ab}_S \circ \pi_S$

$$f_S : \prod_{v \in S} \mathrm{H}^1(k_v, H) \xrightarrow{\pi_S} \prod_{v \in S} \mathrm{H}^1(k_v, H^{(\mathrm{m})}) \xrightarrow{\mathrm{ab}_S} \prod_{v \in S} \mathrm{H}^1_{\mathrm{ab}}(k_v, H^{(\mathrm{m})}) \xrightarrow{p_S} \text{Ч}^1_{\mathrm{ab},S}(H^{(\mathrm{m})})$$

and the composite mapping $g_S := f_S \circ \delta_S$

$$g_S : \prod_{v \in S} X(k_v) \xrightarrow{\delta_S} \prod_{v \in S} \mathrm{H}^1(k_v, H) \xrightarrow{f_S} \text{Ч}^1_{\mathrm{ab},S}(H^{(\mathrm{m})}).$$

REMARK 1. (1) Since for the quasi-connected reductive group H, the group $H^{(\mathrm{m})}$ is commutative, so we may identify $\mathrm{H}^i_{\mathrm{ab}}(k, H^{(\mathrm{m})})$ and $\mathrm{H}^i(k, H^{(\mathrm{m})})$.

(2) Let $A_S := \mathrm{Im}(\gamma_S)$. For each pair S, T of sets of places of k, $S \subseteq T$, let $p_S^T : \Pi_T \to \Pi_S$ be the projection onto the S-component of $\Pi_T = \Pi_S \times \Pi_{T\setminus S}$ and let q_S^T be the restriction of p_S^T to A_T. Then, it is clear that the systems (A_S, q_S^T) and (Π_S, p_S^T) form surjective projective systems (i.e., the projection maps are surjective). Let

$$\widehat{\gamma_V\big(\mathrm{H}^1(k,H)\big)} := \varprojlim_S (A_S, q_S^T), \quad \Pi_V := \varprojlim_S (\Pi_S, p_S^T).$$

We note that Π_V can be naturally identified with the product $\prod_v \mathrm{H}^1(k_v, H)$. Next, assume that all the sets $\mathrm{H}^1(k,H)$, $\mathrm{H}^1(k_v,H)$ are equipped with some natural topology (cf. [1]). Then, we equip the products Π_S, Π_V with the product topology and A_S with the topology induced from that of Π_S, and we equip the projective limits with the usual topology of the limit.

The following proposition extends to global fields without real places some results obtained earlier by Borovoi in the case of number fields.

THEOREM 3.3. *Let k be a global field without real places, H a quasi-connected reductive group, and S a finite set of places of k. Then, with the above notation, the following hold.*

(1) *There are natural isomorphisms of finite abelian groups*

$$\text{(3.12)} \qquad Ч^1_{\mathrm{ab},S}(H) \xrightarrow{\simeq} Ч^1_S(H^{(\mathrm{m})}), \quad Ч^1_{\mathrm{ab},\omega}(H) \xrightarrow{\simeq} Ч^1_\omega(H^{(\mathrm{m})}).$$

(2) *We have the following exact sequences of pointed sets, functorial in H:*

$$\text{(3.13)} \qquad 1 \longrightarrow \gamma_S\big(\mathrm{H}^1(k,H)\big) \xrightarrow{i_S} \prod_{v\in S} \mathrm{H}^1(k_v,H) \xrightarrow{f_S} Ч^1_S(H^{(\mathrm{m})}) \longrightarrow 1,$$

$$\text{(3.14)} \qquad 1 \longrightarrow \overline{\gamma_V\big(\mathrm{H}^1(k,H)\big)} \xrightarrow{i_V} \prod_{v\in V} \mathrm{H}^1(k_v,H) \xrightarrow{f_V} Ч^1_\omega(H^{(\mathrm{m})}) \longrightarrow 1.$$

Here i_S, i_V are natural inclusions.

(3) *If H is connected reductive, then we have natural isomorphisms of finite abelian groups*

$$c_S : Ч^1_S(H^{(\mathrm{m})}) = Ч^1_S(H^{(\mathrm{tor})}) \xrightarrow{\simeq} C_S(H),$$
$$c_\omega : Ч^1_\omega(H^{(\mathrm{m})}) = Ч^1_\omega(H^{(\mathrm{tor})}) \xrightarrow{\simeq} C_\omega(H),$$

and thus also the following exact sequences of pointed sets, functorial in H:

$$\text{(3.15)} \qquad 1 \longrightarrow \gamma_S\big(\mathrm{H}^1(k,H)\big) \xrightarrow{i_S} \prod_{v\in S} \mathrm{H}^1(k_v,H) \xrightarrow{c_S\circ f_S} C_S(H) \longrightarrow 1,$$

$$\text{(3.16)} \qquad 1 \longrightarrow \overline{\gamma_V\big(\mathrm{H}^1(k,H)\big)} \xrightarrow{i_V} \prod_{v\in V} \mathrm{H}^1(k_v,H) \xrightarrow{c_V\circ f_V} C_\omega(H) \longrightarrow 1.$$

Remark 2. If k is a number field, then it was proved that (see [3, Thm. 1.2]) if H is a *connected reductive group* over k, then there is the following exact sequence, closely related to (3.13):

$$\text{(3.13}'\text{)} \qquad \mathrm{H}^1(k,H) \xrightarrow{j_S} \prod_{v\in S} \mathrm{H}^1(k_v,H) \longrightarrow C_S(H) \longrightarrow 1,$$

that (see [3, Thm. 1.4] or [8, Prop. 3.7]) there is an isomorphism

$$C_S(H) \simeq C_S(H^{\mathrm{tor}}),$$

and that (see [8, Prop. 3.9])

$$C_S(H^{\mathrm{tor}}) \simeq Ш^1_S(H^{\mathrm{tor}}).$$

Moreover (see [7, Prop. 1.9]), if $H \subset G$ is a connected k-subgroup of a simply connected k-group G, and $X = G/H$, then there is an isomorphism

$$C_S(X) \simeq Ш^1_S(H^{\mathrm{tor}}),$$

and (see [8, Lem. 3.4]) when G is quasi-trivial, we have

$$C_S(X) \simeq C_S(H).$$

Thus, (3.13) is a function field analog (in fact, an extension) of (3.13′) in the case of quasi-connected groups (and when H is connected and reductive, we have $H^{(\mathrm{m})} = H^{\mathrm{tor}}$, so one gets the exact analog). In Theorem 3.4 below, (3.13) will be extended to any quasi-connected k-group H over any global field k.

Proof of Theorem 3.3. We embed H as a subgroup into a quasi-trivial reductive k-group G, and let $X := G/H$. Assume that H is defined via the exact sequence $1 \to H \to G_1 \to T \to 1$, where G_1 is connected reductive and T is a torus. Note that since H, G_1, T are smooth, by [1, Cor. 5.1.3], the natural topologies on $\mathrm{H}^1(k_v,H)$, $\mathrm{H}^1(k_v,G_1)$, $\mathrm{H}^1(k_v,T)$ are discrete. Therefore, the (finite) product topology is also discrete and the image of $\mathrm{H}^1(k,H)$ in $\prod_{v\in S}\mathrm{H}^1(k_v,H)$ is also a closed subset there. Thus,

$$\overline{\gamma_S\big(\mathrm{H}^1(k,H)\big)} = \gamma_S\big(\mathrm{H}^1(k,H)\big).$$

Recall that if k is a global function field, then we have

$$\mathrm{H}^1(k,H) \xrightarrow{\simeq} \mathrm{H}^1_{\mathrm{ab}}(k,H),\ \mathrm{H}^1(k_v,H) \xrightarrow{\simeq} \mathrm{H}^1_{\mathrm{ab}}(k_v,H)$$

(cf. [43, Prop. 1.14]). One can use the same proof of [43, Prop. 1.14], which was given for the case of *global function fields* to extend this result to the present case (*k has no* *real places*). Thus, further, we may regard $\mathrm{H}^1(k,H), \mathrm{H}^1(k_v,H)$ as *abelian topological groups*.

(1) Since H is quasi-connected reductive, we have $H^{\mathrm{ssu}} = H^{\mathrm{ss}}$ and the exact sequence

$$1 \longrightarrow H^{\mathrm{ss}} \longrightarrow H \xrightarrow{\pi} H^{(\mathrm{m})} \longrightarrow 1. \tag{3.17}$$

We derive from this the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
\mathrm{H}^1(k,H^{\mathrm{ss}}) \arrow[r, "\alpha_S"] \arrow[d] & \prod_{v\in S}\mathrm{H}^1(k_v,H^{\mathrm{ss}}) \arrow[d] \\
\mathrm{H}^1(k,H) \arrow[r, "\gamma_S"] \arrow[d, "\pi_*"] & \prod_{v\in S}\mathrm{H}^1(k_v,H) \arrow[d, "\pi_S"] \\
\mathrm{H}^1(k,H^{(\mathrm{m})}) \arrow[r, "\mu_S"] & \prod_{v\in S}\mathrm{H}^1(k_v,H^{(\mathrm{m})}).
\end{tikzcd}$$

Since H^{ss} is semisimple, we know by [2, Thm. 1.7] (for number fields) and [46, Thm. 3.8.1, p. 4301] (for global function fields) that the localization map

$$\alpha_S : \mathrm{H}^1(k,H^{\mathrm{ss}}) \longrightarrow \prod_{v\in S}\mathrm{H}^1(k_v,H^{\mathrm{ss}})$$

is surjective. Now we pass to abelianized cohomology (see [6, 9, 11, 33]) in the flat setting. Denote by

$$\mathrm{ab}^i_H : \mathrm{H}^i(k,H) \longrightarrow \mathrm{H}^i_{\mathrm{ab}}(k,H)$$

the natural map of cohomology and let

$$\mathrm{ab}^i_{H,S} : \prod_{v\in S}\mathrm{H}^i(k_v,H) \longrightarrow \prod_{v\in S}\mathrm{H}^i_{\mathrm{ab}}(k_v,H)$$

be the product of the maps $\mathrm{ab}^i_{H,v}$. The exact sequence (3.17) defines the following commutative diagram with exact rows and the first two exact columns:

$$\begin{tikzcd}
\mathrm{H}^1_{\mathrm{ab}}(k,H^{\mathrm{ss}}) \arrow[r, "\alpha_{\mathrm{ab},S}"] \arrow[d, "\nu"] & \prod_{v\in S}\mathrm{H}^1_{\mathrm{ab}}(k_v,H^{\mathrm{ss}}) \arrow[r, "\alpha'_{\mathrm{ab}}"] \arrow[d, "\nu_S"] & \text{Ч}^1_{\mathrm{ab},S}(H^{\mathrm{ss}}) \arrow[d] \\
\mathrm{H}^1_{\mathrm{ab}}(k,H) \arrow[r, "\gamma_{\mathrm{ab},S}"] \arrow[d, "\pi_{\mathrm{ab}}"] & \prod_{v\in S}\mathrm{H}^1_{\mathrm{ab}}(k_v,H) \arrow[r, "\gamma'_{\mathrm{ab}}"] \arrow[d, "\pi_{\mathrm{ab},S}"] & \text{Ч}^1_{\mathrm{ab},S}(H) \arrow[d, "\varphi_{\mathrm{ab},S}"] \\
\mathrm{H}^1_{\mathrm{ab}}(k,H^{(\mathrm{m})}) \arrow[r, "\mu_{\mathrm{ab},S}"] \arrow[d] & \prod_{v\in S}\mathrm{H}^1_{\mathrm{ab,fppf}}(k_v,H^{(\mathrm{m})}) \arrow[r, "\mu'_{\mathrm{ab}}"] \arrow[d] & \text{Ч}^1_{\mathrm{ab},S}(H^{(\mathrm{m})}) \arrow[d] \\
\mathrm{H}^2_{\mathrm{ab}}(k,H^{\mathrm{ss}}) \arrow[r, "\alpha^1_{\mathrm{ab},S}"] & \prod_{v\in S}\mathrm{H}^2_{\mathrm{ab}}(k_v,H^{\mathrm{ss}}) \arrow[r, "\alpha^{1\prime}_{\mathrm{ab}}"] & \text{Ч}^2_{\mathrm{ab},S}(H^{\mathrm{ss}}),
\end{tikzcd}$$

where $\varphi_{\mathrm{ab},S}$ is induced from the pair $\pi_{\mathrm{ab}}, \pi_{\mathrm{ab},S}$. Let F be the fundamental group of H^{ss}. Then, we know by [2, Thm. 1.7] (for number fields) and [46, Thm. 3.8.1, p. 4301] (for global function fields) that the localization map α_S is surjective; hence, so is $\alpha_{\mathrm{ab},S}$ due to the commutativity of the following diagram:

$$\begin{tikzcd} \mathrm{H}^1(k, H^{\mathrm{ss}}) \arrow[r, "\alpha_S"] \arrow[d, "\mathrm{ab}_{H^{\mathrm{ss}}}"] & \prod_{v\in S} \mathrm{H}^1(k_v, H^{\mathrm{ss}}) \arrow[d, "\mathrm{ab}_{H^{\mathrm{ss}},S}"] \\ \mathrm{H}^1_{\mathrm{ab}}(k, H^{\mathrm{ss}}) \arrow[r, "\alpha_{\mathrm{ab},S}"] & \prod_{v\in S} \mathrm{H}^1_{\mathrm{ab,fppf}}(k_v, H^{\mathrm{ss}}) \end{tikzcd}$$

and the surjectivity of $\mathrm{ab}_{H^{\mathrm{ss}},S}$. Therefore, $Ч^1_{\mathrm{ab},S}(H^{\mathrm{ss}}) = 1$. Next, we show that there are isomorphisms

$$\mathrm{H}^2_{\mathrm{ab}}(k, H^{\mathrm{ss}}) \xrightarrow{\simeq} \mathrm{H}^3(k, F),$$
$$\mathrm{H}^2_{\mathrm{ab,fppf}}(k_v, H^{\mathrm{ss}}) \xrightarrow{\simeq} \mathrm{H}^3(k_v, F).$$

Indeed, let $\widetilde{H}$ denote the simply connected covering of H^{ss} with canonical projection $\widetilde{H} \xrightarrow{\partial} H^{\mathrm{ss}}$. Then, the short complex $[\widetilde{H} \xrightarrow{\partial} H^{\mathrm{ss}}]$ forms a quasi-abelian crossed module in the terminology of [23] (see [23, Ex. 3.3]). Then, it follows from [23, (3.5)], applied to the quasi-abelian crossed module $[\widetilde{H} \xrightarrow{\partial} H^{\mathrm{ss}}]$, that we have an exact sequence, for $i \geq 0$:

$$\cdots \longrightarrow \mathrm{H}^{i-1}_{\mathrm{fppf}}\big(k, \mathrm{Coker}(\partial)\big) \longrightarrow \mathrm{H}^{i+1}_{\mathrm{fppf}}\big(k, \mathrm{Ker}(\partial)\big) \longrightarrow \mathrm{H}^{i}_{\mathrm{ab,fppf}}\big(k, [\widetilde{H} \xrightarrow{\partial} H^{\mathrm{ss}}]\big)$$
$$\longrightarrow \mathrm{H}^{i}_{\mathrm{fppf}}\big(k, \mathrm{Coker}(\partial)\big) \longrightarrow \cdots.$$

Since $\partial : \widetilde{H} \to H^{\mathrm{ss}}$ is surjective, we have $\mathrm{H}^{i-1}_{\mathrm{fppf}}(k, \mathrm{Coker}(\partial)) = \mathrm{H}^{i}_{\mathrm{fppf}}(k, \mathrm{Coker}(\partial)) = 1$, so we get an isomorphism

$$\mathrm{H}^2_{\mathrm{ab,fppf}}(k, H^{\mathrm{ss}}) \xrightarrow{\simeq} \mathrm{H}^3_{\mathrm{fppf}}(k, F),$$

and similarly,

$$\mathrm{H}^2_{\mathrm{ab,fppf}}(k_v, H^{\mathrm{ss}}) \xrightarrow{\simeq} \mathrm{H}^3_{\mathrm{fppf}}(k_v, F).$$

Then, since S contains no real places, and since $\mathrm{cd}(k_v) = 2$, we have

$$\mathrm{H}^3_{\mathrm{fppf}}(k, F) = 1, \quad \mathrm{H}^3_{\mathrm{fppf}}(k_v, F) = 1.$$

Hence, $\pi_{\mathrm{ab},S}$ and $\varphi_{\mathrm{ab},S}$ are surjective. We show next that $\varphi_{\mathrm{ab},S}$ is injective, thus an isomorphism. Let $z \in \prod_{v\in S} \mathrm{H}^1_{\mathrm{ab,fppf}}(k_v, H)$ such that $\varphi_{\mathrm{ab},S}(\gamma'_{\mathrm{ab}}(z)) = 1$. Then, $1 = \mu'_{\mathrm{ab}}(\pi_{\mathrm{ab},S}(z))$, so

$$\pi_{\mathrm{ab},S}(z) = \mu_{\mathrm{ab},S}(t), \quad t \in \mathrm{H}^1_{\mathrm{ab,fppf}}(k, H^{(\mathrm{m})}).$$

Since $\mathrm{H}^2_{\mathrm{ab,fppf}}(k_v, H^{\mathrm{ss}}) = 1$, we have $t = \pi_{\mathrm{ab}}(s)$, for some $s \in \mathrm{H}^1_{\mathrm{ab,fppf}}(k, H)$. Thus, $\pi_{\mathrm{ab},S}(z) = \pi_{\mathrm{ab},S}(\gamma_{\mathrm{ab},S}(s))$, so

$$z \equiv \gamma_{\mathrm{ab},S}(s) \quad \big(\operatorname{mod} \operatorname{Ker}(\pi_{\mathrm{ab},S})\big);$$

i.e.,

$$z \equiv \gamma_{\mathrm{ab},S}(s) \quad \big(\operatorname{mod} \operatorname{Im}(\nu_S)\big).$$

Since $\alpha_{\mathrm{ab},S}$ is surjective, this implies that

$$z = \gamma_{\mathrm{ab},S}(s) + \nu_S(u), \quad u \in \prod_{v \in S} \mathrm{H}^1_{\mathrm{ab,fppf}}(k_v, H^{\mathrm{ss}}).$$

Therefore, $z = \gamma_{\mathrm{ab},S}(s + \nu(u))$, so $\gamma'_{\mathrm{ab}}(z) = 1$, and $\varphi_{\mathrm{ab},S}$ is a bijection; thus, we have

$$Ч^1_{\mathrm{ab},S}(H) \xrightarrow{\simeq} Ч^1_{\mathrm{ab},S}(H^{(\mathrm{m})}) \xrightarrow{\simeq} Ч^1_S(H^{(\mathrm{m})})$$

as asserted. The isomorphism $Ч^1_{\mathrm{ab},\omega}(H) \simeq Ч^1_\omega(H^{(\mathrm{m})})$ is proved in a similar way. That they are finite abelian groups follows from [45, Prop. 1.4.1].

(2) The functoriality of the sequences can be verified without any difficulty. The proof of (3.13) follows from the fact that $\varphi_{\mathrm{ab},S}$ is an isomorphism.

Further, we proceed to prove (3.14). The idea is that we pass to the inverse limit of $\gamma_S(\mathrm{H}^1(k, H))$ and show that the limit and the closure are the same. We need the following well-known "folklore" statement.

Statement. (a) *If* $1 \to A_\alpha \to B_\alpha \to C_\alpha \to 1$ *is a projective system of exact sequences of abelian groups, then for their projective limit, we have an exact sequence*

$$1 \longrightarrow \varprojlim_\alpha A_\alpha \longrightarrow \varprojlim_\alpha B_\alpha \longrightarrow \varprojlim_\alpha C_\alpha$$

and if all the projection maps are surjective, then also an exact sequence

$$1 \longrightarrow \varprojlim_\alpha A \longrightarrow \varprojlim_\alpha B \longrightarrow \varprojlim_\alpha C \longrightarrow 1.$$

(b) *Let* $(B_\alpha, f_\alpha^\beta)$, $\alpha \in I$ *be a projective system of topological spaces, where for* $\alpha \le \beta$, *the transition map* $f_\alpha^\beta : B_\beta \to B_\alpha$ *is a surjective and closed mapping. Let for each* α, A_α *be a closed non-empty subset of* B_α, *such that with* f_α^β *restricted to* (A_α), $(A_\alpha, f_\alpha^\beta)$ *is also a projective system. Then,* $\varprojlim_\alpha A_\alpha$ *is a closed subset of* $\varprojlim_\alpha B_\alpha$. ■

Now by (1), we have the following exact sequences of abelian groups (which hold due to (3.12), (3.13)) (recall that $A_S = \gamma_S(\mathrm{H}^1(k, H))$):

$$1 \longrightarrow A_S \longrightarrow \prod_{v \in S} \mathrm{H}^1(k_v, H) \longrightarrow Ч^1_S(H^{(\mathrm{m})}) \longrightarrow 1,$$

and by Statement (a), we have the exact sequence of abelian groups

$$1 \longrightarrow \widehat{\gamma_V\big(\mathrm{H}^1(k,H)\big)} \longrightarrow \prod_{v\in V} \mathrm{H}^1(k_v,H) \longrightarrow \text{Ш}^1_\omega(H^{(\mathrm{m})}) \longrightarrow 1. \tag{3.18}$$

We need to show that we have

$$\overline{\gamma_V\big(\mathrm{H}^1(k,H)\big)} = \widehat{\gamma_V\big(\mathrm{H}^1(k,H)\big)}.$$

For each finite set S, $\gamma_S(\mathrm{H}^1(k,H))$ is a closed Hausdorff topological subgroup of the product $\prod_{v\in S}\mathrm{H}^1(k_v,H)$ (which is equipped with the discrete topology, see above). Since G is quasi-trivial, it satisfies the weak approximation property over k and we have the following commutative diagram with the exact second column, where we show that the first column of which is also exact:

$$\begin{tikzcd}
 & \overline{G(k)} \arrow[r, "="] \arrow[d, "\bar\pi"] & \prod_v G(k_v) \arrow[d, "\pi'"] & & \\
1 \arrow[r] & \overline{X(k)} \arrow[r] \arrow[d, "\delta'_V"] & \prod_v X(k_v) \arrow[r] \arrow[d, "\delta_V"] & \text{Ш}^1_\omega(H^{(\mathrm{m})}) \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \widehat{\gamma_V\big(\mathrm{H}^1(k,H)\big)} \arrow[r, "i"] \arrow[d] & \prod_v \mathrm{H}^1(k_v,H) \arrow[r] \arrow[d] & \text{Ш}^1_\omega(H^{(\mathrm{m})}) \arrow[r] & 1 \\
 & 1 & 1. & &
\end{tikzcd} \tag{3.19}$$

By assumption, k has no real places, so over such a field k, by Lemma 3.2 (3), we have $\mathrm{H}^1(k,G) = \mathrm{H}^1(k_v,G) = 1$; hence, the map δ'_V is surjective. As in the above diagram, since δ'_V is continuous, we have

$$\widehat{\gamma_V\big(\mathrm{H}^1(k,H)\big)} = \delta'_V\big(\overline{X(k)}\big) \subseteq \overline{\delta'_V\big(X(k)\big)} = \overline{\gamma_V\big(\mathrm{H}^1(k,H)\big)};$$

thus,

$$\widehat{\gamma_V\big(\mathrm{H}^1(k,H)\big)} \subseteq \overline{\gamma_V\big(\mathrm{H}^1(k,H)\big)}. \tag{3.20}$$

By Statement (b) or [27, Ch. 2, Thm. 6.14], the projective limit

$$\widehat{\gamma_V\big(\mathrm{H}^1(k,H)\big)} = \varprojlim_S \big(\gamma_S\big(\mathrm{H}^1(k,H)\big)\big)$$

is a closed Hausdorff subgroup of the product $\prod_v \mathrm{H}^1(k_v,H)$, which clearly contains $\gamma_V(\mathrm{H}^1(k,H))$, so we have

$$\overline{\gamma_V\big(\mathrm{H}^1(k,H)\big)} \subseteq \widehat{\gamma_V\big(\mathrm{H}^1(k,H)\big)}. \tag{3.21}$$

From the inclusions (3.20) and (3.21), we deduce that the two groups $\overline{\gamma_V(\mathrm{H}^1(k,H))}$ and $\widehat{\gamma_V(\mathrm{H}^1(k,H))}$ are equal and (3.14) holds.

(3) If H is connected and reductive, we have $H^{(\mathrm{m})} = H^{\mathrm{tor}}$. If k is a number field, the assertion follows from [7, Prop. 1.9], so we assume that k is a global function field.

First we claim that there is the exact sequence

$$\mathrm{H}^1_{\mathrm{ab}}(k,H) \xrightarrow{\gamma_{\mathrm{ab},S}} \prod_{v\in S} \mathrm{H}^1_{\mathrm{ab}}(k_v,H) \xrightarrow{\beta''_{\mathrm{ab}}} C_S(H) \longrightarrow 1. \tag{3.22}$$

By Theorems 2.1 and 2.2, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{H}^1(k,H) & \xrightarrow{\gamma_V} & \bigoplus_v \mathrm{H}^1(k_v,H) & \xrightarrow{\xi} & B'(H) & \longrightarrow & 1 \\
\simeq\downarrow & & \alpha\downarrow & & \downarrow\beta & & \\
\mathrm{H}^1(k,H) & \xrightarrow{\gamma_S} & \bigoplus_{v\in S} \mathrm{H}^1(k_v,H) & \xrightarrow{\xi_S} & \mathrm{Coker}(\gamma_S) & \longrightarrow & 1,
\end{array}$$

where $\xi = \oplus_v \xi_v$, and each ξ_v is the composition

$$\xi_v : \mathrm{H}^1(k_v,H) \xrightarrow{\varphi_v} B_v(H) \xrightarrow{\lambda_v} B'(H),$$

where φ_v is a bijection by [43, Thm. 2.5], γ_V, γ_S are localization maps, and

$$\alpha : (x_v)_{v\in V} \mapsto (x_v)_{v\in S}.$$

It is clear that β is surjective and a chase on the diagram shows that $\mathrm{Ker}(\beta) = B^S(H)$. In fact, on the one hand, let $x \in \mathrm{Ker}(\beta)$,

$$x = \sum_{w\in W} \xi_w(x_w), \quad x_w \in \mathrm{H}^1(k_w,H).$$

Then, $x = \xi(\sum_w x_w) = \xi(y)$, where $y = \sum_w x_w$, so $\xi_S(\alpha(y)) = 0$; thus,

$$\alpha(y) = \gamma_S(h) = \alpha\big(\gamma(h)\big), \quad h \in \mathrm{H}^1(k,H).$$

Hence,

$$y - \gamma(h) \in \mathrm{Ker}(\alpha) = \bigoplus_{v\notin S} \mathrm{H}^1(k_v,H),$$

so $x \in B^S(H)$. On the other hand, it is clear that $B^S(H) \subseteq \mathrm{Ker}(\beta)$, so

$$\mathrm{Coker}(\gamma_S) \simeq B'(H)/B^S(H) = C_S(H); \tag{3.23}$$

hence, the sequence (3.22) is exact, where we take $\beta''_{\mathrm{ab}} := \xi_S$.

From above and the claim, we derive the following diagram, where the first row is exact due to (3.23):

$$\begin{tikzcd} \mathrm{H}^1_{\mathrm{ab}}(k,H) \arrow[r, "\gamma_{\mathrm{ab},S}"] \arrow[d, "="] & \bigoplus_{v\in S}\mathrm{H}^1_{\mathrm{ab}}(k_v,H) \arrow[r, "\beta''_{\mathrm{ab}}"] \arrow[d, "="] & C_S(H) \arrow[r] \arrow[d, "\wp_S"] & 1 \\ \mathrm{H}^1_{\mathrm{ab}}(k,H) \arrow[r, "\gamma_{\mathrm{ab},S}"] & \bigoplus_{v\in S}\mathrm{H}^1_{\mathrm{ab}}(k_v,H) \arrow[r, "\gamma''_{\mathrm{ab}}"] & Ч^1_{\mathrm{ab},S}(H^{(\mathrm{m})}) \arrow[r] & 1, \end{tikzcd}$$

where $\gamma''_{\mathrm{ab}} := \mu'_{\mathrm{ab}} \circ \pi_{\mathrm{ab},S}$ and $\wp_S$ is a map defined as follows. If $x \in C_S(H)$, then we lift x to an element $y \in \prod_{v\in S}\mathrm{H}^1_{\mathrm{ab}}(k_v, H)$, and then let $\wp_S(x) := \gamma'_{\mathrm{ab}}(y)$. It is clear that if x lifts to y', then $y' - y \in \mathrm{Ker}(\gamma''_{\mathrm{ab}}) = \mathrm{Im}(\gamma_{\mathrm{ab}})$, so

$$y' = y + \gamma_{\mathrm{ab}}(z), \quad z \in \mathrm{H}^1_{\mathrm{ab}}(k, H).$$

Thus, $\gamma'_{\mathrm{ab}}(y) = \gamma'_{\mathrm{ab}}(y')$, so the map $\wp_S$ is well defined. Thus, $\gamma''_{\mathrm{ab}} = \wp_S \circ \beta''_{\mathrm{ab}}$ and this last diagram is commutative with exact rows.

One checks that $\wp_S$ is a bijection, thus an isomorphism. From this, we see that $C_S(H) \simeq Ч^1_S(H^{(\mathrm{m})})$ and similarly, $C_\omega(H) \simeq Ч^1_\omega(H^{(\mathrm{m})})$. From all the above, we see that (3) holds.

Theorem 3.3 is proven. ∎

Now we are able prove the general case of Theorem 3.3 (1), (2).

Theorem 3.4. *Let k be a global field, H a quasi-connected reductive k-group, and S a finite set of places of k. Then,*

(1) *we have the exact sequence of pointed sets*

$$1 \longrightarrow \gamma_S\big(\mathrm{H}^1(k,H)\big) \xrightarrow{i_S} \prod_{v\in S}\mathrm{H}^1(k_v,H) \xrightarrow{f_S} Ч^1_S(H^{(\mathrm{m})}) \longrightarrow 1,$$

(2) *we have the exact sequence of pointed sets*

$$1 \longrightarrow \gamma_{\mathrm{ab},S}\big(\mathrm{H}^1_{\mathrm{ab}}(k,H)\big) \xrightarrow{i_{\mathrm{ab},S}} \prod_{v\in S}\mathrm{H}^1_{\mathrm{ab}}(k_v,H) \xrightarrow{f_{\mathrm{ab},S}} Ч^1_S\big(H^{(\mathrm{m})}\big) \longrightarrow 1, \tag{3.24}$$

(3) *there are natural isomorphisms of finite abelian groups:*

$$Ч^1_{\mathrm{ab},S}(H) \xrightarrow{\simeq} Ч^1_S(H^{(\mathrm{m})}), \quad Ч^1_{\mathrm{ab},\omega}(H) \xrightarrow{\simeq} Ч^1_\omega(H^{(\mathrm{m})}),$$

(4) *we have the following exact sequences of pointed sets, functorial in H:*

$$1 \longrightarrow \overline{\gamma_V\big(\mathrm{H}^1(k,H)\big)} \xrightarrow{i_V} \prod_{v\in V}\mathrm{H}^1(k_v,H) \xrightarrow{f_V} Ч^1_\omega(H^{(\mathrm{m})}) \longrightarrow 1, \tag{3.25}$$

$$1 \longrightarrow \overline{\gamma_{\mathrm{ab},V}(\mathrm{H}^1_{\mathrm{ab}}(k,H))} \xrightarrow{i_{\mathrm{ab},V}} \prod_{v\in V}\mathrm{H}^1_{\mathrm{ab}}(k_v,H) \xrightarrow{f_{\mathrm{ab},V}} Ч^1_\omega(H^{(\mathrm{m})}) \longrightarrow 1. \tag{3.26}$$

Before proving the theorem, we need the following lemma. For simplicity, we consider only the case of Galois cohomology (which will be used later on).

LEMMA 3.5. *Let k be a global field, $1 \to A \to B \xrightarrow{\pi} C \to 1$ an exact sequence of smooth k-group schemes, and k_v a completion of k.*

(1) *In the following commutative diagram of cohomology, where δ is the coboundary map:*

$$\begin{array}{ccc} C(k) & \xrightarrow{\delta} & \mathrm{H}^1(k, A) \\ {\scriptstyle i_v}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_v} \\ C(k_v) & \xrightarrow{\delta_v} & \mathrm{H}^1(k_v, A) \end{array}$$

for any element $c \in C(k)$, $y \in \mathrm{H}^1(k, A)$, we have

$$\mathrm{res}_v(c \cdot y) = c \cdot \mathrm{res}_v(y),$$

where i_v is the natural embedding and $\cdot$ denotes the action [39, Ch. I, Sec. 5.5] *of $C(k)$ on $\mathrm{H}^1(k, A)$ (resp. $C(k_v)$ on $\mathrm{H}^1(k_v, A)$).*

(2) *If we equip $C(k_v)$, $C(k)$ with the v-adic topology and $\mathrm{H}^1(k_v, A)$ with the discrete topology and the weakest one on $\mathrm{H}^1(k, A)$ such that all the maps are continuous, then the action $\cdot$ of $C(k_v)$ on $\mathrm{H}^1(k_v, A)$ is continuous.*

PROOF. Let $\Gamma := \mathrm{Gal}(k_s/k)$, and let $\Gamma_v := \mathrm{Gal}(k_{v,s}/k_v)$, considered as the decomposition group of v, thus as a closed subgroup of Γ (cf. [39, Ch. II, Sec. 6]: the treatment given there was for number fields, but it is also valid for any global field).

(1) The map res_v is just the restriction map

$$\mathrm{res}_v : (a_s)_{s\in\Gamma} \mapsto (a_s)_{s\in\Gamma_v}.$$

Recall that if y is the class of the 1-cocycle $(a_s)_{s\in\Gamma}$, and c has a lifting $b \in B(k_s)$, $\pi(b) = c$, then we have $\pi(b) = \pi(b)^s$, $\forall s \in \Gamma$, so $b^{-1}\,{}^s b \in \mathrm{Ker}(\pi) = A(k_s)$. Then, $c \cdot y$ is the class of the 1-cocycle $s \mapsto (b^{-1}a_s^s b)$, $s \in \Gamma$, and this class does not depend on the choice of lifting. (In fact, if there is another lift $b_1 \in B(k_s)$ for c, then we have $b_1 = ba$, $a \in A(k_s)$, so $b_1^{-1}a_s^s b_1 = a^{-1}(b^{-1}a_s^s b)^s a$ is equivalent to $(b^{-1}a_s^s b)$ as a 1-cocycle with values in A.)

By definition, $\mathrm{res}_v(c \cdot y)$ is the class of the 1-cocycle $t \mapsto (b^{-1}a_t^t b)$, $t \in \Gamma_v$. Let $b_1 \in B(k_{v,s})$ be a lifting of c (considered as an element of $C(k_v)$). Then, we have, on the one hand, $\pi(b_1) = c = \pi(b)$, so $b_1 = ba$, $a \in A(k_{v,s})$, and on the other hand, $c \cdot \mathrm{res}_v(y)$ is the class of the 1-cocycle $t \mapsto (b_1^{-1}\,a_t^t b_1)$, $t \in \Gamma_v$, which is thus $t \mapsto (a^{-1}b^{-1}a_t^t b^t a)$, $t \in \Gamma_v$ which is equivalent to the 1-cocycle $t \mapsto (b^{-1}a_t^t b)$, $t \in \Gamma_v$. Hence, their classes in $\mathrm{H}^1(k_v, A)$ coincide.

(2) To show that the map

$$C(k_v) \times \mathrm{H}^1(k_v, A) \xrightarrow{f} \mathrm{H}^1(k_v, A), \quad (c, x) \mapsto c \cdot x$$

is continuous, we need only to show that for any $x \in \mathrm{H}^1(k_v, A)$, $f^{-1}(x)$ is open. By definition,

$$f^{-1}(x) = \{z = (c, y) \in C(k_v) \times \mathrm{H}^1(k_v, A) \mid c \cdot y = x\},$$

so this simply means that y belongs to the $C(k_v)$-orbit of x, so we have

$$f^{-1}(x) = \{z = (c, c^{-1} \cdot x) \in C(k_v) \times \mathrm{H}^1(k_v, A) \mid c \in C(k_v)\}.$$

Let $\mathrm{Stab}_C(x)$ be the stabilizer of x in $C(k_v)$ and let

$$C(k_v) = \bigcup_{j \in J} \mathrm{Stab}_C(x) c_j, \quad j \in J,$$

be a partition of $C(k_v)$ into right cosets of $\mathrm{Stab}_C(x)$. Then, we have

$$f^{-1}(x) = \bigcup_{j \in J,\, c \in \mathrm{Stab}_C(x)} (cc_j, c_j^{-1} \cdot x) = \bigcup_j \big(\mathrm{Stab}_C(x) c_j \times (c_j^{-1} \cdot x)\big).$$

Since A is smooth, the map π is separable, so $\pi(B(k_v))$ is an open subgroup of $C(k_v)$, and it is true for the twist ${}_aB$ (instead of B), for any $a \in \mathrm{H}^1(k, A)$. By [39, Ch. I, Prop. 39 (iii)], we have $\mathrm{Stab}_C(x) = \pi({}_xB(k_v))$. Therefore, $\mathrm{Stab}_C(x)$ is an open subset of $C(k_v)$, and so is $\mathrm{Stab}_C(x)c_j$ for all j. The topology on $\mathrm{H}^1(k_v, A)$ is discrete, so each point $c_j^{-1} \cdot x$ is open in $\mathrm{H}^1(k_v, A)$. It implies that $f^{-1}(x)$ is the union of open subsets

$$\big(\mathrm{Stab}_C(x) c_j \times (c_j^{-1} \cdot x)\big)$$

of $C(k_v) \times \mathrm{H}^1(k_v, A)$, so is also open as required. ∎

Remark 3. One can state the above lemma in a more general form in the framework of [22, Prop. 3.3.3, Ch. III].

Proof of Theorem 3.4. (1) Since H is quasi-connected, $H^{(\mathrm{m})}$ is a smooth k-group of multiplicative type by Lemma 3.1. Let H be defined from the exact sequence

$$1 \longrightarrow H \longrightarrow H_1 \longrightarrow T \longrightarrow 1,$$

where H_1 is a connected reductive k-group and T is a k-torus. Let

$$1 \longrightarrow F \longrightarrow Q \longrightarrow T \longrightarrow 1$$

be a flasque resolution of T, where F is a flasque k-torus and Q is an induced k-torus. Then, we have the following commutative diagram with exact rows and columns:

$$\text{(I)}\qquad \begin{tikzcd} & & 1 \arrow[d] & 1 \arrow[d] & \\ & & F \arrow[r, "\simeq"] \arrow[d, "\alpha"] & F \arrow[d, "\gamma"] & \\ 1 \arrow[r] & H \arrow[r, "h"] \arrow[d, "\simeq"] & H_2 \arrow[r, "j"] \arrow[d, "\beta"] & Q \arrow[r] \arrow[d, "\lambda"] & 1 \\ 1 \arrow[r] & H \arrow[r, "\beta\circ h"] & H_1 \arrow[r, "j'"] \arrow[d] & T \arrow[r] \arrow[d] & 1 \\ & & 1 & 1. & \end{tikzcd}$$

Here, $H_2 = H_1 \times_T Q$. By considering the quotients by the derived subgroup of H_2 and H_1, we derive the following analogous commutative diagram with exact rows and columns:

$$\text{(II)}\qquad \begin{tikzcd} & & 1 \arrow[d] & 1 \arrow[d] & \\ & & F \arrow[r, "\simeq"] \arrow[d, "\alpha_m"] & F \arrow[d, "\gamma_m"] & \\ 1 \arrow[r] & H^{(\mathrm{m})} \arrow[r, "h_m"] \arrow[d, "\simeq"] & H_2^{\mathrm{tor}} \arrow[r, "j_m"] \arrow[d, "\beta_m"] & Q \arrow[r] \arrow[d, "\lambda_m"] & 1 \\ 1 \arrow[r] & H^{(\mathrm{m})} \arrow[r, "\beta_m\circ h_m"] & H_1^{\mathrm{tor}} \arrow[r, "j'_m"] \arrow[d] & T \arrow[r] \arrow[d] & 1 \\ & & 1 & 1. & \end{tikzcd}$$

We then derive the following commutative diagram with exact rows:

$$\begin{tikzcd} Q(k) \arrow[r, "\delta"] \arrow[d, "q_S"'] & \mathrm{H}^1(k, H) \arrow[r, "h"] \arrow[d, "\gamma_S"'] & \mathrm{H}^1(k, H_2) \arrow[r, "j_*"] \arrow[d, "\theta_S"] & \mathrm{H}^1(k, Q) \arrow[d] \\ \prod_{v\in S} Q(k_v) \arrow[r, "\delta_S"] & \prod_{v\in S} \mathrm{H}^1(k, H) \arrow[r, "h_S"] & \prod_{v\in S} \mathrm{H}^1(k, H_2) \arrow[r, "j_{*,S}"] & \prod_{v\in S} \mathrm{H}^1(k_v, Q), \end{tikzcd}$$

where q_S is the diagonal map and γ_S, θ_S are the localization maps. Since Q is an induced k-torus, its first cohomology is trivial, so the maps h, h_S are surjective. Then, by considering the induced diagrams of Galois cohomology, we have the following commutative diagram, where the first two rows are exact:

$$
\text{(III)}\qquad
\begin{tikzcd}
 & Q(k) \arrow[r, "\delta"] \arrow[d, "q_S"] & \mathrm{H}^1(k,H) \arrow[r, "h"] \arrow[d, "\gamma_S"] & \mathrm{H}^1(k,H_2) \arrow[r] \arrow[d, "\theta_S"] & 1 \\
 & \prod_{v\in S} Q(k_v) \arrow[r, "\delta_S"] \arrow[d, "q'_S"] & \prod_{v\in S}\mathrm{H}^1(k_v,H) \arrow[r, "h_S"] \arrow[d, "f_S"] & \prod_{v\in S}\mathrm{H}^1(k_v,H_2) \arrow[r] \arrow[d, "\theta'_S"] & 1 \\
1 \arrow[r] & \mathrm{A}(S,Q) \arrow[r, "\delta'_S"] \arrow[d] & Ч^1_S(H^{(\mathrm{m})}) \arrow[r, "h'_S"] \arrow[d] & Ч^1_S(H_2^{\mathrm{tor}}) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1,
\end{tikzcd}
$$

where f_S is the composite map

$$\prod_{v\in S}\mathrm{H}^1(k_v,H)\longrightarrow\prod_{v\in S}\mathrm{H}^1_{\mathrm{ab}}(k_v,H)\longrightarrow\prod_{v\in S}\mathrm{H}^1_{\mathrm{ab}}(k_v,H^{(\mathrm{m})})\longrightarrow Ч^1_S(H^{(\mathrm{m})}),$$

and θ'_S is the composite map

$$\prod_{v\in S}\mathrm{H}^1(k_v,H_2)\longrightarrow\prod_{v\in S}\mathrm{H}^1_{\mathrm{ab}}(k_v,H_2)\longrightarrow\prod_{v\in S}\mathrm{H}^1_{\mathrm{ab}}(k_v,H_2^{\mathrm{tor}})\longrightarrow Ч^1_S(H_2^{\mathrm{tor}}).$$

Similarly, we have also the following commutative diagram with the first two exact rows:

$$
\text{(IV)}\qquad
\begin{tikzcd}
 & Q(k) \arrow[r, "\delta_m"] \arrow[d, "q_S"] & \mathrm{H}^1(k,H^{(\mathrm{m})}) \arrow[r, "h_m"] \arrow[d, "\gamma_{m,S}"] & \mathrm{H}^1(k,H_2^{\mathrm{tor}}) \arrow[d, "\theta_{m,S}"] & \\
 & \prod_{v\in S} Q(k_v) \arrow[r, "\delta_{m,S}"] \arrow[d, "q'_S"] & \prod_{v\in S}\mathrm{H}^1(k_v,H^{(\mathrm{m})}) \arrow[r, "h_{m,S}"] \arrow[d, "f_{m,S}"] & \prod_{v\in S}\mathrm{H}^1(k_v,H_2^{\mathrm{tor}}) \arrow[d, "\theta'_{m,S}"] & \\
1 \arrow[r] & \mathrm{A}(S,Q) \arrow[r, "\delta'_{m,S}"] \arrow[d] & Ч^1_S(H^{(\mathrm{m})}) \arrow[r, "h'_S"] \arrow[d] & Ч^1_S(H_2^{\mathrm{tor}}) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1.
\end{tikzcd}
$$

Claim 1. *The third row of diagram* (IV) *is exact. More precisely, it boils down to an isomorphism of finite abelian groups* $Ч^1_S(H^{(\mathrm{m})})\simeq Ч^1_S(H_2^{\mathrm{tor}})$.

Indeed, it is clear that h'_S is surjective. Assume that $h'_S(x)=1$, $x\in Ч^1_{\mathrm{ab},S}(H^{(\mathrm{m})})$. Let

$$x=f_{m,S}(y),\qquad y\in\prod_{v\in S}\mathrm{H}^1(k_v,H^{(\mathrm{m})}).$$

Then, $\theta'_{m,S}(h_{m,S}(y)) = 1$, so

$$h_{m,S}(y) = \theta_{m,S}(z), \quad z \in \mathrm{H}^1(k, H_2^{\mathrm{tor}}).$$

Take $t \in \mathrm{H}^1(k, H^{(\mathrm{m})})$ such that $h_m(t) = z$, so we have $h_{m,S}(\gamma_{m,S}(t)) = h_{m,S}(y)$. Therefore,

$$y = g_S \cdot \gamma_{m,S}(t), \quad g_S \in \prod_{v \in S} Q(k_v),$$

where $\cdot$ is the action of $\prod_{v\in S} Q(k_v)$ on $\prod_{v\in S} \mathrm{H}^1(k_v, H^{(\mathrm{m})})$ (cf. [39, Ch. I, Sec. 5.5]). We equip $\prod_{v\in S} Q(k_v)$ with the product of v-adic topologies, $v \in S$, and equip the cohomology sets with the natural topology (cf. [34, Ch. III], [1]) such that all the maps appearing in this diagram are continuous. Also, since H, H_2 are smooth, the natural topology on $\prod_{v\in S} \mathrm{H}^1(k_v, H^{(\mathrm{m})})$ and $\prod_{v\in S} \mathrm{H}^1(k_v, H_2^{\mathrm{tor}})$ are discrete ones (cf. [1]). Since Q is an induced k-torus, q_S has dense image, so we may write $g_S = \lim_n g_n$, $g_n \in Q(k)$, and due to the continuity of the action of $\prod_{v\in S} Q(k_v)$ on $\prod_{v\in S} \mathrm{H}^1(k_v, H^{(\mathrm{m})})$ (see above lemma), we have

$$y = g_S \cdot \gamma_{m,S}(t) = \lim_n \gamma_{m,S}(g_n) \cdot \gamma_{m,S}(t) = \lim_n \gamma_{m,S}(g_n \cdot t),$$

where $g_n \cdot t$ denotes the action of g_n on t coming from the action of $Q(k)$ on $\mathrm{H}^1(k, H^{(\mathrm{m})})$. Since $\prod_{v\in S} \mathrm{H}^1(k_v, H^{(\mathrm{m})})$ is finite, discrete, it implies that starting from certain n, we have $y = \gamma_{m,S}(g_n \cdot t)$. It shows that $x = 1$, and we obtain an isomorphism

$$h'_S : Ч^1_S(H^{(\mathrm{m})}) \simeq Ч^1_S(H_2^{\mathrm{tor}}) \tag{3.27}$$

and the claim is proven.

Claim 2. (1) *holds; i.e., the second column of diagram* (III) *is exact.*

The idea of the proof is similar, but for completeness, we give the proof in detail. It is clear that $f_S \circ \gamma_S = 1$. If $x \in \mathrm{Ker}(f_S)$, then we have $1 = h'_S(f_S(x)) = \theta'_S(h_S(x))$. Since H_2 is a connected reductive k-group, we know by Borovoi's result [3, Thms. 1.2 and 1.4] that the third column is exact in the case of number fields. In the case of global function fields, it follows from Theorem 3.3 (2) proved above.

Thus, $h_S(x) = \theta_S(y)$, $y \in \mathrm{H}^1(k, H_2)$. Since h is surjective, we have $y = h(z)$, $z \in \mathrm{H}^1(k, H)$; thus, we have $h_S(\gamma_S(z)) = h_S(x)$. Therefore, there is an element $g_S \in \prod_{v\in S} Q(k_v)$ such that $g_S \cdot \gamma_S(z) = x$, where $\cdot$ denotes the usual action of $\prod_{v\in S} Q(k_v)$ on $\prod_{v\in S} \mathrm{H}^1(k_v, H)$ (cf. [39, Ch. I, Sec. 5.5]). We may equip $\prod_{v\in S} Q(k_v)$ with the product of v-adic topologies, $v \in S$, and equip $\prod_{v\in S} \mathrm{H}^1(k_v, H)$, and $\prod_{v\in S} \mathrm{H}^1(k_v, H_2)$ with the natural topology (cf. [1]) such that all the maps appearing in this diagram are continuous. Also, since H, H'_1 are smooth, the natural topology on $\prod_{v\in S} \mathrm{H}^1(k_v, H)$ and

$\prod_{v\in S} \mathrm{H}^1(k_v, H_2)$ are discrete ones (cf. [1]). As above, we may write $g_S = \lim_n \gamma_S(g_n)$, $g_n \in Q(k)$, and we have

$$x = g_S \cdot \gamma_S(z) = \lim_n \gamma_S(g_n) \cdot \gamma_S(z) = \lim_n \gamma_S(g_n \cdot z),$$

where $g_n \cdot z$ denotes the action of g_n on z (coming from the action of $Q(k)$ on $\mathrm{H}^1(k, H)$). Since $\prod_{v\in S} \mathrm{H}^1(k_v, H)$ is finite, discrete, it implies that starting from certain n, we have $x = \gamma_S(g_n \cdot z)$. It shows that $\mathrm{Ker}(f_S) = \mathrm{Im}(\gamma_S)$. Finally, f_S is surjective since we have shown that h'_S is an isomorphism and since h_S, θ'_S are surjective. Hence, the claim is proven.

(2) We have the commutative diagram

$$\text{(V)} \qquad \begin{tikzcd}
 & \mathrm{H}^1(k,H) \arrow[r, "h"] \arrow[d, "\gamma_S"] & \mathrm{H}^1(k,H_2) \arrow[r, "u"] \arrow[d, "\theta_S"] & \mathrm{H}^1_{\mathrm{ab}}(k,H_2) \arrow[r] \arrow[d, "\theta_{\mathrm{ab},S}"] & 1 \\
 & \prod_{v\in S} \mathrm{H}^1(k_v,H) \arrow[r, "h_S"] \arrow[d, "f_S"] & \prod_{v\in S} \mathrm{H}^1(k_v,H_2) \arrow[r, "u_S"] \arrow[d, "\theta'_S"] & \prod_{v\in S} \mathrm{H}^1_{\mathrm{ab}}(k_v,H_2) \arrow[r] \arrow[d, "\theta'_{\mathrm{ab},S}"] & 1 \\
1 \arrow[r] & \text{Ш}^1_S(H^{(\mathrm{m})}) \arrow[r, "h'_S"] \arrow[d] & \text{Ш}^1_S(H_2^{\mathrm{tor}}) \arrow[r, "\simeq"] \arrow[d] & \text{Ш}^1_S(H_2^{\mathrm{tor}}) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1,
\end{tikzcd}$$

where the first two columns are exact (the first by what we have proved and the second by a result of Borovoi for number fields [3, Thms. 1.2, 1.4] and by what we have proved above for global fields without real places), and the map $\theta'_{\mathrm{ab},S}$ is the composite map

$$\prod_{v\in S} \mathrm{H}^1_{\mathrm{ab}}(k_v, H_2) \longrightarrow \prod_{v\in S} \mathrm{H}^1_{\mathrm{ab}}(k_v, H_2^{\mathrm{tor}}) \longrightarrow \text{Ш}^1_S(H_2^{\mathrm{tor}}).$$

Also, we know that the maps u, u_S are surjective, by [6, Sec. 5] for char. $k = 0$ case and [23, Thm. 5.8] (or [43, 1.12–1.14]) for char. $k > 0$ case. Since θ'_S is surjective, we derive that so is $\theta'_{\mathrm{ab},S}$, and that $\theta'_{\mathrm{ab},S} \circ \theta_{\mathrm{ab},S}$ is a trivial map. Let $\theta'_{\mathrm{ab},S}(x) = 1$. Choose $y \in \prod_{v\in S} \mathrm{H}^1(k_v, H_2)$, such that $u_S(y) = x$. Then, we have

$$1 = \theta'_{\mathrm{ab},S}(x) = \theta'_{\mathrm{ab},S}\big(u_S(y)\big) = \theta'_S(y),$$

so $y = \theta_S(z)$, $z \in \mathrm{H}^1(k, H_2)$; thus,

$$x = u_S\big(\theta_S(z)\big) = \theta_{\mathrm{ab},S}\big(u(z)\big);$$

i.e., $x \in \mathrm{Im}(\theta_{\mathrm{ab},S})$, and the third column is exact.

By the way, we obtain also the following isomorphisms of finite abelian groups:

$$\text{(3.28)} \qquad \text{Ш}^1_{\mathrm{ab},S}(H_2) \simeq \text{Ш}^1_S(H_2^{\mathrm{tor}}).$$

(3) From the exact sequence

$$1 \longrightarrow H \longrightarrow H_2 \longrightarrow Q \longrightarrow 1,$$

by passing to abelianized cohomology, we obtain the commutative diagram

$$\text{(VI)} \qquad \begin{tikzcd}
 & Q(k) \arrow[r, "d"] \arrow[d, "q_S"] & \mathrm{H}^1_{\mathrm{ab}}(k, H) \arrow[r, "h_{\mathrm{ab}}"] \arrow[d, "\gamma_{\mathrm{ab},S}"] & \mathrm{H}^1_{\mathrm{ab}}(k, H_2) \arrow[r] \arrow[d, "\theta_{\mathrm{ab},S}"] & 1 \\
 & \prod_{v \in S} Q(k_v) \arrow[r, "d_S"] \arrow[d, "q'_S"] & \prod_{v \in S} \mathrm{H}^1_{\mathrm{ab}}(k_v, H) \arrow[r, "h_S"] \arrow[d, "f_{\mathrm{ab},S}"] & \prod_{v \in S} \mathrm{H}^1_{\mathrm{ab}}(k_v, H_2) \arrow[r] \arrow[d, "\theta'_{\mathrm{ab},S}"] & 1 \\
1 \arrow[r, "="] & \mathrm{A}(S, Q) \arrow[r, "d'_S"] \arrow[d] & \Sha^1_{\mathrm{ab},S}(H) \arrow[r, "h'_S"] \arrow[d] & \Sha^1_{\mathrm{ab},S}(H_2) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1,
\end{tikzcd}$$

where the second and the third columns and the first and the second rows are exact. Again, by a similar, but simpler, chase on the diagram involving only abelian groups as in the proof of (2), we obtain the following isomorphism:

$$\Sha^1_{\mathrm{ab},S}(H) \xrightarrow{\simeq} \Sha^1_{\mathrm{ab},S}(H_2). \tag{3.29}$$

By (2), we have

$$\Sha^1_S(H^{(\mathrm{m})}) \xrightarrow{\simeq} \Sha^1_S(H_2^{\mathrm{tor}}) \simeq \Sha^1_{\mathrm{ab},S}(H_2). \tag{3.30}$$

Thus, from the above isomorphism and the natural morphism $H \to H^{(\mathrm{m})}$, which induces the following homomorphisms of abelian groups:

$$\mathrm{H}^1_{\mathrm{ab}}(k, H) \longrightarrow \mathrm{H}^1_{\mathrm{ab}}(k, H^{(\mathrm{m})}), \quad \prod_{v \in S} \mathrm{H}^1_{\mathrm{ab}}(k_v, H) \longrightarrow \prod_{v \in S} \mathrm{H}^1_{\mathrm{ab}}(k_v, H^{(\mathrm{m})}),$$

we get finally a natural projection:

$$\Sha^1_{\mathrm{ab},S}(H) \xrightarrow{\simeq} \Sha^1_S(H^{(\mathrm{m})})$$

which is an isomorphism of finite abelian groups due to [45, Prop. 1.4.1], (3.29) and (3.30).

The isomorphism

$$\Sha^1_{\mathrm{ab},\omega}(H) \xrightarrow{\simeq} \Sha^1_{\omega}(H^{(\mathrm{m})})$$

is proved in a similar way, so we omit the details: we show as in Claim 1 that we have an isomorphism

$$\Sha^1_{\omega}(H^{(\mathrm{m})}) \xrightarrow{\simeq} \Sha^1_{\omega}(H_2^{\mathrm{tor}}), \tag{3.27$'$}$$

and then

$$(3.28') \qquad Ч^1_{\mathrm{ab},\omega}(H_2) \xrightarrow{\simeq} Ч^1_\omega(H_2^{tor}).$$

Then, we show

$$(3.29') \qquad Ч^1_{\mathrm{ab},\omega}(H) \xrightarrow{\simeq} Ч^1_{\mathrm{ab},\omega}(H_2),$$

and finally

$$(3.30') \qquad Ч^1_\omega(H^{(\mathrm{m})}) \xrightarrow{\simeq} Ч^1_\omega(H_2^{\mathrm{tor}}) \xrightarrow{\simeq} Ч^1_{\mathrm{ab},\omega}(H_2).$$

(4) The proof of (4) is quite similar to that of Theorem 3.3 (2), by using (1), (2) proved above, so we omit the details.

Theorem 3.4 is thus proven. ∎

We derive the following consequences.

Corollary 3.6. *Let the notation be as in Theorem 3.4. The localization maps*

$$\gamma_\infty : \mathrm{H}^1(k, H) \longrightarrow \prod_{v\in\infty} \mathrm{H}^1(k_v, H),$$

$$\gamma_{\mathrm{ab},\infty} : \mathrm{H}^1_{\mathrm{ab}}(k, H) \longrightarrow \prod_{v\in\infty} \mathrm{H}^1_{\mathrm{ab}}(k_v, H)$$

are surjective.

Proof. By looking at the diagram (III) in the proof of Theorem 3.4, where $S = \infty$, and by using the well-known facts that the localization map for H_2,

$$\theta_\infty : \mathrm{H}^1(k, H_2) \longrightarrow \prod_{v\in\infty} \mathrm{H}^1(k_v, H_2),$$

is surjective and the image of q_∞ is dense in $\prod_{v\in\infty} Q(k_v)$ and applying a suitable twisting argument, we deduce that γ_∞ is surjective.

Since the abeliniazation maps

$$\mathrm{ab}^1_H : \mathrm{H}^1(k, H) \to \mathrm{H}^1_{\mathrm{ab}}(k, H), \quad \mathrm{ab}^1_{v,H} : \mathrm{H}^1(k_v, H) \to \mathrm{H}^1_{\mathrm{ab}}(k_v, H)$$

are surjective for any local and global field (see [33, Prop. 1.6.7] for char. 0 case and [43, Thm. 1.12 (b)] for char. $p > 0$ case) from the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(k, H) \arrow[r, "\mathrm{ab}^1_H"] \arrow[d, "\gamma_\infty"'] & \mathrm{H}^1_{\mathrm{ab}}(k, H) \arrow[d, "\gamma_{\mathrm{ab},\infty}"'] \\
\prod_{v\in\infty} \mathrm{H}^1(k_v, H) \arrow[r, "\mathrm{ab}^1_{\infty,H}"] & \prod_{v\in\infty} \mathrm{H}^1_{\mathrm{ab}}(k_v, H)
\end{tikzcd}$$

and from above, we deduce that $\gamma_{\mathrm{ab},\infty}$ is also surjective. ∎

Corollary 3.7. *Let H be a quasi-connected reductive group defined over a global field k, S a finite set of places of k, and*

$$\gamma_S : \mathrm{H}^1(k, H) \longrightarrow \prod_{v\in S} \mathrm{H}^1(k_v, H),$$
$$\gamma_V : \mathrm{H}^1(k, H) \longrightarrow \prod_{v} \mathrm{H}^1(k_v, H)$$

the localization maps. Then, the following hold.

(1) *The obstructions to weak approximation in S and over k in the first Galois cohomology of H are finite abelian groups $Ч^1_S(H^{(\mathrm{m})})$ and $Ч^1_\omega(H^{(\mathrm{m})})$.*

(2) *The almost weak approximation holds: there is a finite set S_0 of places of k such that for any finite set S of places of k with $S \cap S_0 = \emptyset$, the localization map γ_S is surjective.*

(3) *The obstruction is finite if the following holds: $(|\pi_0(H^{(\mathrm{m})})|, p) = 1$, where $p =$ char. k. In particular, if $H^{(\mathrm{m})} = \{1\}$ (e.g., when H is semisimple), then the weak approximation holds for $\mathrm{H}^1(k, H)$.*

Proof. (1) follows from Theorem 3.3 (2).

(2) We indicate two proofs.

First proof. As in the proof of [42, Prop. 4.4.2 (b)], we may embed $H^{(\mathrm{m})}$ into an induced k-torus P and set $T := P/H^{(\mathrm{m})}$, which is a k-torus. Let S_0 be the set of all places v of k, where v has a non-cyclic decomposition group in a fixed splitting field L for P, thus also for $H^{(\mathrm{m})}$. It then follows from [42, Prop. 4.4.2] that we have

$$Ч^1_S(H^{(\mathrm{m})}) = Ч^1_{S\cap S_0}(H^{(\mathrm{m})})$$

for any finite set of places S of k and for S_0. So if $S \cap S_0 = \emptyset$, then

$$Ч^1_S(H^{(\mathrm{m})}) = Ч^1_{S\cap S_0}(H^{(\mathrm{m})}) = \{1\},$$

and γ_S is surjective. Thus, the image of $\mathrm{H}^1(k, H)$ via $\gamma_{V\setminus S_0}$ is dense in the product $\prod_{v\notin S_0} \mathrm{H}^1(k_v, H)$.

Second proof. We may use the same argument given in the proof of Lemma 4.2 (b) below.

(3) In our case, we note that if $(|\pi_0(H^{(\mathrm{m})})|, p) = 1$, then $\mathrm{H}^1(k_v, H^{(\mathrm{m})})$ is finite for all places v [39, Ch. III]. Since the almost weak approximation holds, it follows that the obstruction to weak approximation is finite. ■

REMARK 4. (1) It is interesting to compare the obstructions obtained above to the one studied in [42, Sec. 4].

(2) Simple examples show that in general, (3.12) does not have an analog in the case $S = V$; i.e., $\gamma_V(\mathrm{H}^1(k, H))$ *may not be closed* in $\prod_v \mathrm{H}^1(k_v, H)$ for non-connected groups H. The reason is that for non-connected H, the closure of $\gamma(\mathrm{H}^1(k, H))$ in product topology on $\prod_v \mathrm{H}^1(k_v, H)$ is very big in general. For example, if H is a connected reductive k-group, then it is well known that the image of the localization map $\gamma_V : \mathrm{H}^1(k, H) \to \prod_v \mathrm{H}^1(k_v, H)$ can land only in the direct sum $\coprod_v \mathrm{H}^1(k_v, H)$.

(3) In [43, p. 86], there was introduced the notion of pseudo-connected groups, the class of which a priori contains that of quasi-connected ones. We show that they are in fact the same. Recall that a smooth affine k-group H is called 0-*quasi-connected reductive* if it is quasi-connected reductive, 1-*quasi-connected reductive* if H is the kernel of a surjective k-morphism from a (0-)quasi-connected reductive G onto a k-torus. By induction, an affine, smooth k-group H is an n-*quasi-connected reductive group* if it is the kernel of a surjective k-morphism from an $(n-1)$-quasi-connected reductive onto a k-torus. One may ask if the class of n-quasi-connected groups strictly contains the quasi-connected ones. We show that any n-quasi-connected group is quasi-connected by induction on n. If $n = 0$, there is nothing to prove and we assume the assertion holds for $n - 1$. Consider the exact sequence

$$1 \longrightarrow H \longrightarrow H_1 \longrightarrow T \longrightarrow 1,$$

where H_1 is $(n-1)$-quasi-connected and T is a k-torus. By the induction hypothesis, H_1 is quasi-connected, say via the exact sequence $1 \to H_1 \to G_1 \to S \to 1$, where G_1 is connected and reductive, and S is a torus. Since S is a torus, thus commutative, we have $DG_1 \subseteq H_1$. Since DG_1 is semisimple, from this inclusion, we derive $DG_1 = [DG_1, DG_1] \subseteq DH_1$. Since T is a torus, thus commutative, we have $DH_1 \subseteq H$; thus, $DG_1 \subseteq H$, so we have a surjective morphism from the torus G_1/DG_1 onto G_1/H; hence, G_1/H is also a torus and we are done.

4. FORMULAS FOR OBSTRUCTION TO WEAK APPROXIMATION IN HOMOGENEOUS SPACES

In this section, we propose an extension of an exact sequence computing an obstruction to the weak approximation for homogeneous spaces due to Borovoi [7] and Borovoi and Schlank [10].

As applications of results proved in Section 3, we derive a generalization of the formula $\mathrm{A}(S, G) \simeq (\text{Б}_S(G)/\text{Б}(G))^D$ or the same, of the Sansuc exact sequence (S') (see [44, Thm. 2.1] and [45, Thm. 2.2.1 (2)]) and also that of the isomorphisms of [44, Prop. 1.3] (or [45, Prop. 1.4.1]), to the case of homogeneous spaces.

The following theorem extends well-known results of Borovoi [7, Thms. 1.3, 1.11] obtained in the number field case to the global function field case and to the case of homogeneous spaces over global fields with quasi-connected reductive stabilizers.

THEOREM 4.1. *Let k be a global field, S a finite set of places of k, and $X = G/H$ a homogeneous space under a quasi-trivial connected reductive k-group G with a quasi-connected reductive k-stabilizer H.*

(1) *There is an exact sequence of pointed sets*

$$1 \longrightarrow \overline{X(k)}^S \longrightarrow \prod_{v\in S} X(k_v) \xrightarrow{g_S} Ч^1_S(H^{(\mathrm{m})}) \longrightarrow 1, \tag{4.1a}$$

and also an exact sequence of pointed sets

$$1 \longrightarrow \overline{X(k)}^S \longrightarrow \prod_{v\in S} X(k_v) \xrightarrow{\xi_S} \big(Б_S(X)/Б(X)\big)^D \longrightarrow 1, \tag{4.1b}$$

or the same,

$$1 \longrightarrow \overline{X(k)}^S \longrightarrow \prod_{v\in S} X(k_v) \longrightarrow \big(Б_S(X)\big)^D \longrightarrow \big(Б(X)\big)^D \longrightarrow 1, \tag{4.1c}$$

where the closure is taken in the finite product $\prod_{v\in S} X(k_v)$.

(2) *The Brauer–Manin obstruction to the weak approximation for $X = G/H$ in S is the only one.*

(3) *If H is connected and reductive, then there are isomorphisms*

$$C_S(X) \simeq \big(Б_S(X)/Б(X)\big)^D, \quad C_\omega(X) \simeq \big(Б_\omega(X)/Б(X)\big)^D. \tag{4.1d}$$

REMARK 5. (1) If k is a number field, G (resp. H) a connected affine k-group (resp. connected k-subgroup of G), such that $\mathrm{A}(G) = 1$, $Ш(G) = 1$, then (4.1a) has been given in [7, Thm. 1.3], (4.1b) and (4.1c) in [7, Thm. 1.11], and (4.1d) in [7, Cor. 1.12].

(2) If k is a global field with no real places, and G is either connected (and H is connected) reductive or quasi-trivial (and H is quasi-connected) reductive k-group, then (4.1a) has been given in [44, Thm. 4.1 (1)] and (4.1b) and (4.1c) in [44, Thm. 4.1 (2)]. Around the same time, (4.1b) and (4.1c) have been given (with a different method of proof) in [20, Thm. 4.2] (where k is a global function field, and G, H are connected reductive k-groups).

PROOF OF THEOREM 4.1. Before proving the theorem, we need some more preparation. First we need the following lemma.

LEMMA 4.2. *Let k be a global field, and let M be a smooth k-group of multiplicative type, embedded as a k-subgroup of an induced k-torus P.*

(a) *If the set of archimedean places $\infty_k \neq \emptyset$, then the localization map*

$$\mathrm{H}^1(k, M) \to \prod_{v \in \infty} \mathrm{H}^1(k_v, M)$$

is surjective.

(b) *If $S = S_0 \amalg S_c$ is the disjoint union, where S_c is the set of all places in S, the decomposition groups of which in some fixed splitting field of P are cyclic, then we have an isomorphism*

$$Ш^1_S(M) \simeq Ш^1_{S_0}(M).$$

(c) *Under the assumption of Theorem 4.1 (1), if for $S = S_0 \amalg S_c$ (notation as in (b)) the following sequence*

$$1 \longrightarrow \overline{X(k)}^S \longrightarrow \prod_{v \in S} X(k_v) \xrightarrow{g_S} Ш^1_S(H^{(\mathrm{m})}) \longrightarrow 1$$

is exact, then the same also holds for S_0; i.e., the following "truncated" sequence is exact:

$$1 \longrightarrow \overline{X(k)}^{S_0} \longrightarrow \prod_{v \in S_0} X(k_v) \xrightarrow{g_{S_0}} Ш^1_{S_0}(H^{(\mathrm{m})}) \longrightarrow 1.$$

In particular, if it holds for $S = S_0 \amalg S_\infty$, where S_0 (resp. S_∞) is the subset of all non-archimedean (resp. archimedean) places of S, then it also holds for S_0.

To prove Lemma 4.2, we start first with (a).

Let $T := P/M$, which is a k-torus. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
P(k) \arrow[r] \arrow[d] & T(k) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^1(k, M) \arrow[r] \arrow[d, "\gamma_S"] & 1 \\
\prod_{v \in \infty} P(k_v) \arrow[r] & \prod_{v \in \infty} T(k_v) \arrow[r, "\delta_S"] & \prod_{v \in \infty} \mathrm{H}^1(k_v, M) \arrow[r] & 1.
\end{tikzcd}$$

Since $T(k)$ is dense in $\prod_{v \in \infty} T(k_v)$ [38, Cor. 3.5], and since the cohomology set $\prod_{v \in \infty} \mathrm{H}^1(k_v, M)$ is discrete and finite, it follows that γ_S is surjective.

For (b), by replacing ∞ by any finite set S of places of k, from the above diagram, one can derive that

$$\mathrm{A}(S, T) \simeq Ш^1_S(M), \quad \text{for any finite } S. \tag{4.2}$$

(See [45, Prop. 1.4.1] for a more general statement.) Let $1 \to F \to Q \to T \to 1$ be a flasque resolution of the above torus $T(= P/M)$, where F is a flasque k-torus and Q is an induced k-torus. Then, the same proof of [42, Thm. 2.3, First proof of (2) on p. 1294], where H (resp. G) is replaced by Q (resp. T), shows that there are the following isomorphisms:

$$\mathrm{A}(S,T) \simeq \mathrm{Coker}(\gamma_S : \mathrm{H}^1(k,F) \longrightarrow \prod_{v\in S} \mathrm{H}^1(k_v,F)), \tag{4.3}$$

$$\mathrm{A}(S,T) \simeq \mathrm{A}(S_0,T), \text{ in particular, } \mathrm{A}(S,T) \simeq \mathrm{A}(S\setminus\infty,T). \tag{4.4}$$

From (4.2)–(4.4), we derive that

$$Ч^1_S(M) \simeq Ч^1_{S_0}(M)$$

as required.

Regarding (c), by functoriality, the following diagram of pointed sets is commutative:

$$\begin{tikzcd}
1 \arrow[r] & \overline{X(k)}^S \arrow[r, "i_S"] \arrow[d, "p_{S_0}"] & X_{S_0}\times X_{S_c} \arrow[r, "g_S"] \arrow[d, "pr_1"] & Ч^1_S(H^{(\mathrm{m})}) \arrow[r] \arrow[d, "p"] & 1 \\
1 \arrow[r] & \overline{X(k)}^{S_0} \arrow[r, "i_{S_0}"] & X_{S_0} \arrow[r, "g_{S_0}"] & Ч^1_{S_0}(H^{(\mathrm{m})}) \arrow[r] & 1,
\end{tikzcd}$$

where we set $X_U := \prod_{v\in U} X(k_v)$ and the first row is exact by assumption. Due to the commutativity, since g_S and p are surjective, so is g_{S_0}. Regarding the exactness of the second row, it is clear that the product $g_{S_0}\circ i_{S_0} = 1$ (trivial mapping). To show that $\mathrm{Ker}(g_{S_0}) = \mathrm{Im}(i_{S_0})$, notice that for the number field case, by part (b), we have $Ч^1_S(M) \simeq Ч^1_{S_0}(M)$. Hence, from the diagram and from this isomorphism, it implies that the second row is also exact. Hence, Lemma 4.2 is proven.

To continue the proof of Theorem 4.1, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
G(k) \arrow[r, "\pi"] \arrow[d, "i"] & X(k) \arrow[r, "\delta"] \arrow[d, "i'"] & \mathrm{H}^1(k,H) \arrow[r, "h"] \arrow[d, "\gamma_S"] & \mathrm{H}^1(k,G) \arrow[d, "\beta_S"] \\
\prod_{v\in S} G(k_v) \arrow[r, "\pi_S"] & \prod_{v\in S} X(k_v) \arrow[r, "\delta_S"] \arrow[d, "g_S"] & \prod_{v\in S}\mathrm{H}^1(k_v,H) \arrow[r, "h_S"] \arrow[d, "f_S"] & \prod_{v\in S}\mathrm{H}^1(k_v,G) \\
& Ч^1_S(H^{(\mathrm{m})}) \arrow[r, "="] \arrow[d] & Ч^1_S(H^{(\mathrm{m})}) \arrow[d] & \\
& 1 & 1, &
\end{tikzcd}$$

where $g_S := f_S \circ \delta_S$. By Theorems 3.3 and 3.4 (with the notation used in the proof there), the following sequence

$$\mathrm{H}^1(k,H) \xrightarrow{\gamma_S} \prod_{v\in S} \mathrm{H}^1(k_v,H) \xrightarrow{\varphi_{\mathrm{ab},S}\circ\gamma'_{\mathrm{ab}}} Ч^1_S(H^{(\mathrm{m})}) \longrightarrow 1$$

is exact; i.e., the second column in the diagram above is exact.

(1)(a) Assume first that k is a number field. Then, by Lemma 4.2 (c), to prove Theorem 4.1 (1), we may enlarge S so as to contain the finite set of all archimedean places of k. Given that $\infty \subset S$, then by Lemma 3.2 (3), we see that the map β_S is a bijection (the Hasse principle for quasi-trivial groups).

Let $K := \mathrm{Im}(\delta)$, $K_S := \mathrm{Im}(\delta_S)$, and $L := \mathrm{Im}(h)$, $L_S := \mathrm{Im}(h_S)$. Then, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
X(k) \arrow[r, "\delta"] \arrow[d, "\gamma"] & \mathrm{H}^1(k,H) \arrow[r, "h"] \arrow[d, "\gamma_S"] & L \arrow[r] \arrow[d, "\theta_S"] & 1 \\
\prod_{v\in S} X(k_v) \arrow[r, "\delta_S"] \arrow[d, "g_S"] & \prod_{v\in S} \mathrm{H}^1(k_v,H) \arrow[r, "h_S"] \arrow[d, "f_S"] & L_S \arrow[r] & 1 \\
Ч^1_{\mathrm{ab},S}(H^{(\mathrm{m})}) \arrow[r, "\simeq"] \arrow[d] & Ч^1_{\mathrm{ab},S}(H^{(\mathrm{m})}) \arrow[d] & & \\
1 & 1, & &
\end{tikzcd} \tag{4.5}$$

where $g_S := f_S \circ \delta_S$.

First we show that the map θ_S is bijective. Note that θ_S is injective since β_S is so. Since H is quasi-connected, by 3.6, the localization map

$$\mathrm{H}^1(k,H) \longrightarrow \prod_{v\in\infty} \mathrm{H}^1(k_v,H)$$

is surjective. To show that θ_S is surjective, we pick an element $z_S = (z_\infty, z_f) \in L_S$, where

$$z_\infty \in \prod_{v\in\infty} \mathrm{H}^1(k_v,G), \quad z_f \in \prod_{v\in S\setminus\infty} \mathrm{H}^1(k_v,G).$$

Since G is quasi-trivial, we know that $\mathrm{H}^1(k_v,G) = 1$ if $v \notin \infty$, so $z_f = 1$. Therefore, $L_S = L_\infty \times (1_f)$ and $z_S = z_\infty \times (1_f)$, where

$$1_f = (1,\ldots,1) \in \prod_{v\in S\setminus\infty} \mathrm{H}^1(k_v,G).$$

Let

$$h_S(y_S) = z_S, \quad y_S = (y_\infty, y_f) \in \prod_{v \in S} \mathrm{H}^1(k_v, H),$$

where $h_S(y_f) = z_f, h_S(y_\infty) = z_\infty$. Then, we have $y_\infty \in \mathrm{Im}(\gamma_S)$ by the surjectivity of γ_∞, so $y_\infty = \gamma_\infty(y)$, $y \in \mathrm{H}^1(k, H)$, and $z_S = h_S(y_\infty, 1) = h_S(\gamma_S(y)) = \theta_S(h(y))$ and $z_S \in \mathrm{Im}(\gamma_S)$. Therefore, θ_S is also surjective, thus bijective as asserted.

(1)(b) If k is a global function field, then we know that $\mathrm{H}^1(k, G) = 1, \mathrm{H}^1(k_v, G) = 1$, so $L = \{1\}$, $L_S = \{1_S\}$; thus, θ_S is trivially bijective. Hence, in all cases, the map θ_S is bijective.

Further, we endow $\mathrm{H}^1(k, H)$ and $\mathrm{H}^1(k_v, H)$ with the natural topology (see Section 3.2) such that all the connecting maps and the localization maps appearing in the diagrams above are continuous, where $G(k)$, $X(k)$ have the usual topology, induced from the product topology on $\prod_{v \in S} G(k_v)$, $\prod_{v \in S} X(k_v)$, respectively.

Now we show that the sequence (4.1a) is exact. Recall that $g_S := f_S \circ \delta_S$. Due to the continuity, it is clear that $\overline{X(k)}^S \subseteq \mathrm{Ker}(g_S)$. Conversely, let $x_S \in \mathrm{Ker}(g_S)$. Then, $\delta_S(x_S) = \gamma_S(y)$, $y \in \mathrm{H}^1(k, H)$. Since

$$1 = h_S\big(\delta_S(x_S)\big) = h_S\big(\gamma_S(y)\big) = \theta_S\big(h(y)\big),$$

and since θ_S is bijective, we have $y = \delta(x)$, for some $x \in X(k)$. So we have $\delta_S(\gamma(x)) = \delta_S(x_S)$ and it implies that $x_S \in \prod_{v \in S} G(k_v) \cdot x$, the orbit of x under $\prod_{v \in S} G(k_v)$. Since G has weak approximation property over k (see Lemma 3.2 (3)), it implies that $x_S \in \overline{X(k)}^S$ as well.

Next, we show that g_S is surjective. We consider the diagram

(4.6)
$$\begin{tikzcd}
X(k) \arrow[r, "\delta"] \arrow[d, "\gamma"] & \mathrm{H}^1(k, H) \arrow[r, "h"] \arrow[d, "\gamma_S"] & L \arrow[r] \arrow[d, "\theta_S"] & 1 \\
\prod_{v \in S} X(k_v) \arrow[r, "\delta_S"] \arrow[d, "g_S"] & \prod_{v \in S} \mathrm{H}^1(k_v, H) \arrow[r, "h_S"] \arrow[d, "f_S"] & L_S \arrow[r] & 1 \\
\mathrm{Im}(f_S \circ \delta_S) \arrow[r, hook] \arrow[d] & \text{Ч}^1_{\mathrm{ab},S}(H^{(\mathrm{m})}) \arrow[d] & & \\
1 & 1 & &
\end{tikzcd}$$

and the maps

$$\gamma_\infty : \mathrm{H}^1(k, H) \longrightarrow \prod_{v \in S \cap \infty} \mathrm{H}^1(k_v, H),$$

$$\gamma_f : \mathrm{H}^1(k, H) \longrightarrow \prod_{v \in S \setminus \infty} \mathrm{H}^1(k_v, H),$$

and for $h_S \in \prod_{v\in S} \mathrm{H}^1(k_v, H)$, let $h_S = (h_\infty, h_f)$, with $h_\infty \in \prod_{v\in S\cap\infty} \mathrm{H}^1(k_v, H)$, $h_f \in \prod_{v\in S\setminus\infty} \mathrm{H}^1(k_v, H)$. Let $\bar{h}_\infty, \bar{h}_f$ be the image of h_∞, h_f in $\prod_{v\in S\cap\infty} \mathrm{H}^1_{\mathrm{ab}}(k_v, H)$ and $\prod_{v\in S\setminus\infty} \mathrm{H}^1_{\mathrm{ab}}(k_v, H)$, via the maps $\mathrm{ab}_{S\cap\infty}$ and $\mathrm{ab}_{S\setminus\infty}$, respectively. Finally, let

$$\bar{\bar{h}}_S = \overline{(\bar{h}_\infty, \bar{h}_f)}$$

be the image of $\bar{h}_S = (\bar{h}_\infty, \bar{h}_f)$ in the group $Ч^1_{\mathrm{ab},S}(H)$.

We have

$$\bar{\bar{h}}_S = \overline{(\bar{h}_\infty, \bar{h}_f)} = \overline{(\bar{h}_\infty, \bar{1})} \cdot \overline{(\bar{1}, \bar{h}_f)}. \tag{4.7}$$

Since $\mathrm{H}^1(k_v, G) = 1$ for $v \notin \infty$, the map $\delta_f : \prod_{v\in S\setminus\infty} X(k_v) \to \prod_{v\in S\setminus\infty} \mathrm{H}^1(k_v, H)$ is surjective; there exists $x_f \in \prod_{v\in S\setminus\infty} X(k_v)$ such that $\delta_f(x_f) = h_f$, so

$$\bar{\bar{h}}_S = \overline{(\bar{h}_\infty, \bar{1})} \cdot \overline{\left(\bar{1}, \bar{\delta}_f(x_f)\right)}. \tag{4.8}$$

Further, since the map

$$\gamma_\infty : \mathrm{H}^1(k, H) \to \prod_{v\in\infty} \mathrm{H}^1(k_v, H)$$

is surjective by Corollary 3.6, there exists $h \in \mathrm{H}^1(k, H)$, such that $\gamma_\infty(h) = h_\infty$. Let $\gamma_S(h) = (\gamma_\infty(h), \gamma_f(h))$. There is $x'_f \in \prod_{v\in S\setminus\infty} X(k_v)$ such that $\delta_f(x'_f) = \gamma_f(h)$. Then, from (4.7), we have

$$\begin{aligned} \overline{\overline{\gamma_S(h)}} &= \overline{\left(\bar{\gamma}_\infty(h), \bar{1}\right)} \cdot \overline{\left(\bar{1}, \bar{\gamma}_f(h)\right)} = \overline{(\bar{h}_\infty, \bar{1})} \cdot \overline{\left(\bar{1}, \bar{\gamma}_f(h)\right)} \\ &= \overline{(\bar{h}_\infty, \bar{1})} \cdot \overline{\left(\bar{1}, \bar{\delta}_f(x'_f)\right)}. \end{aligned} \tag{4.9}$$

From (4.8), it follows that for any element $h_S \in \prod_{v\in S} \mathrm{H}^1(k_v, H)$, we have

$$\bar{\bar{h}}_S \equiv \overline{(\bar{h}_\infty, \bar{1})} \quad \left(\mathrm{mod}\, \overline{\overline{\mathrm{Im}(\delta_f)}}\right),$$

and by (4.9), we have

$$\overline{(\bar{h}_\infty, \bar{1})} \equiv \bar{1} \quad \left(\mathrm{mod}\, \overline{\overline{\mathrm{Im}(\delta_f)}}\right).$$

Since

$$\overline{\overline{\mathrm{Im}(\delta_f)}} \subseteq \mathrm{Im}(g_S),$$

it implies that the injective map

$$\mathrm{Im}(f_S \circ \delta_S) \overset{\hookrightarrow}{\longrightarrow} Ч^1_{\mathrm{ab},S}(H^{(\mathrm{m})})$$

is also sujrective, hence an isomorphism.

Thus, we have proved that there is the following exact sequence, which is (4.1a):

$$1 \longrightarrow \overline{X(k)}^S \longrightarrow \prod_{v\in S} X(k_v) \xrightarrow{g_S} Ч^1_{\mathrm{ab},S}(H^{(\mathrm{m})}) \longrightarrow 1.$$

Since $H^{(\mathrm{m})}$ is of multiplicative type (Lemma 3.1), we have

$$Ч^1_{\mathrm{ab},S}(H^{(\mathrm{m})}) = Ч^1_S(H^{(\mathrm{m})})$$

and by [44, Thm. 1.5] (or [45, Thms. 2.1.1, 2.1.2]), we have

$$Ч^1_S(H^{(\mathrm{m})}) \overset{j_S}{\simeq} (Б_S(X)/Б(X))^D.$$

Hence, from (4.1a), we obtain the exact sequence (4.1b):

$$1 \longrightarrow \overline{X(k)}^S \longrightarrow \prod_{v\in S} X(k_v) \xrightarrow{\zeta_S} (Б_S(X)/Б(X))^D \longrightarrow 1$$

where $\zeta_S := j_S \circ g_S$. Hence, the sequences in Theorem 4.1 (1) are proved to be exact.

(2) Now we show that the Brauer–Manin obstruction to weak approximation in S is the only one; i.e., we have

$$\overline{X(k)}^S = \Big(\prod_{v\in S} X(k_v)\Big)^{Б_S(X)}.$$

We need the following.

Lemma 4.3. *Let k be a global field, S a finite set of places of k, and $X = G/H$ a homogeneous space under a quasi-trivial connected reductive k-group G with a quasi-connected reductive k-stabilizer H. With the above notation, we have the following natural commutative diagram:*

(a)
$$\begin{array}{ccccc}
\prod_{v\in S} X(k_v) & \times & Б_S(X)/Б(X) & \longrightarrow & \mathbf{Q}/\mathbf{Z} \\
\big\downarrow{\scriptstyle g_S} & & {\scriptstyle g'_S}\big\uparrow{\scriptstyle \simeq} & & \big\updownarrow{\scriptstyle =} \\
Ч^1_S(H^{(\mathrm{m})}) & \times & Ш^1_S(\widehat{H^{(\mathrm{m})}})/Ш^1(\widehat{H^{(\mathrm{m})}}) & \longrightarrow & \mathbf{Q}/\mathbf{Z},
\end{array}$$

(b)
$$\begin{array}{ccccc}
\prod_{v\in V} X(k_v) & \times & Б_\omega(X)/Б(X) & \longrightarrow & \mathbf{Q}/\mathbf{Z} \\
\big\downarrow{\scriptstyle g_V} & & {\scriptstyle g'_V}\big\uparrow{\scriptstyle \simeq} & & \big\updownarrow{\scriptstyle =} \\
Ч^1_\omega(H^{(\mathrm{m})}) & \times & Ш^1_\omega(\widehat{H^{(\mathrm{m})}})/Ш^1(\widehat{H^{(\mathrm{m})}}) & \longrightarrow & \mathbf{Q}/\mathbf{Z}.
\end{array}$$

Remark 6. The diagram (a) was given in [10, Thm. 5.1 (ii)] for any k-subgroup H (not necessarily quasi-connected) of G and for any number field k. The proof of

[10, Thm. 5.1 (ii)] given for number fields does not seem to extend to the case of global function fields: The treatment of *arbitrary* k-subgroup H of G is fairly complicated. However, in the case k is a global function field, H is quasi-connected reductive; then, by using [45, Thm. 2.1.2] instead of [10, Thm. 4.1], and using [45, Thm. 2.1.1] instead of [10, Thm. 4.2], with the other arguments unchanged, the proof given there can be extended to cover the case char. $k = p > 0$.

Below we give another approach, which is natural in some sense, which is based on a result of [18, Prop. 2.7]. Note that our map g_S is the map $(-c_S)$ in [10, Sec. 3.3].

Proof of Lemma 4.3. The proof consists of two steps.

Step 1. We will show that there is such a commutative diagram as stated.

First we recall the following result. Given an algebraic k-group H, $\mathrm{Ext}_{k-\mathrm{gr}}(H, \mathbf{G}_m)$ denotes the group of isomorphic extensions of H by $\mathbf{G}_m$, $\mathrm{Ext}^c_{k-\mathrm{gr}}(H, \mathbf{G}_m)$ the abelian group of isomorphic central extensions of H by $\mathbf{G}_m$, and $\mathrm{Ext}_{k-\mathrm{abgr}}(H, \mathbf{G}_m)$ the group of abelian central extensions of H by $\mathbf{G}_m$.

Let $p : Y \to X$ be an H-torsor. Then, it gives rise to a class $\xi \in \mathrm{H}^1(X, H)$ and a (boundary) map

$$\mathrm{ev}_Y : X(k) \to \mathrm{H}^1(k, H).$$

Every central extension

$$1 \longrightarrow \mathbf{G}_m \longrightarrow H_1 \longrightarrow H \longrightarrow 1$$

with its class θ gives rise to an exact sequence of pointed sets

$$\mathrm{H}^1(X, H_1) \longrightarrow \mathrm{H}^1(X, H) \xrightarrow{p_\theta} \mathrm{H}^2_{\mathrm{fppf}}(X, \mathbf{G}_m) = \mathrm{Br}(X);$$

thus, one gets a natural pairing

$$\mathrm{H}^1(X, H) \times \mathrm{Ext}^c_{k-\mathrm{gr}}(H, \mathbf{G}_m) \longrightarrow \mathrm{Br}(X); \quad (\xi, \theta) \mapsto p_\theta(\xi),$$

and also a homomorphism

$$\rho_{\mathrm{tors}}(Y) : \mathrm{Ext}^c_{k-\mathrm{gr}}(H, \mathbf{G}_m) \to \mathrm{Br}(X).$$

Then, the key result we need is the following.

Proposition 4.4 ([18, Prop. 2.7]). *Let H be a (not necessarily connected) linear algebraic group over k. Let $M = H^{(\mathrm{m})}$ denote the maximal quotient of H which is a group of multiplicative type. Let $Y \to X$ be an H-torsor. With notation as above,* *there is the commutative diagram*

$$\begin{array}{ccccc}
X(k) & \times & \mathrm{Br}(X) & \longrightarrow & \mathrm{Br}(k) \\
\Big\downarrow{\scriptstyle \mathrm{ev}_Y} & & \Big\uparrow{\scriptstyle \rho_{\mathrm{tors}}(Y)} & & \Big\updownarrow{\scriptstyle =} \\
\mathrm{H}^1(k,H) & \times & \mathrm{Ext}^c_{k-\mathrm{gr}}(H,\mathbf{G}_m) & \longrightarrow & \mathrm{Br}(k) \\
\Big\downarrow & & \Big\uparrow & & \Big\updownarrow{\scriptstyle =} \\
\mathrm{H}^1(k,H^{(\mathrm{m})}) & \times & \mathrm{Ext}^c_{k-\mathrm{gr}}(H^{(\mathrm{m})},\mathbf{G}_m) & \longrightarrow & \mathrm{Br}(k) \\
\Big\downarrow{\scriptstyle =} & & \Big\uparrow & & \Big\updownarrow{\scriptstyle =} \\
\mathrm{H}^1(k,H^{(\mathrm{m})}) & \times & \mathrm{Ext}_{k-\mathrm{abgr}}(H^{(\mathrm{m})},\mathbf{G}_m) & \longrightarrow & \mathrm{Br}(k) \\
\Big\downarrow{\scriptstyle =} & & \Big\uparrow{\scriptstyle \simeq} & & \Big\updownarrow{\scriptstyle =} \\
\mathrm{H}^1(k,H^{(\mathrm{m})}) & \times & \mathrm{H}^1(k,\widehat{H^{(\mathrm{m})}}) & \longrightarrow & \mathrm{Br}(k).
\end{array}$$

Further, by the discussion preceding [18, Prop. 2.8], the image of the resulting map from $\mathrm{H}^1(k,\widehat{H^{(\mathrm{m})}})$ to $\mathrm{Br}(X)$ actually lands in the subgroup $\mathrm{Br}_1(X)$, so we obtain the commutative diagram

$$\begin{array}{ccccc}
X(k) & \times & \mathrm{Br}_1(X) & \longrightarrow & \mathrm{Br}(k) \\
\Big\downarrow & & \Big\uparrow & & \Big\updownarrow{\scriptstyle =} \\
\mathrm{H}^1(k,H^{(\mathrm{m})}) & \times & \mathrm{H}^1(k,\widehat{H^{(\mathrm{m})}}) & \longrightarrow & \mathrm{Br}(k),
\end{array}$$

thus also a commutative diagram

$$\begin{array}{ccccc}
X(k) & \times & \mathrm{Br}_a(X) & \longrightarrow & \mathrm{Br}(k) \\
\Big\downarrow{\scriptstyle =} & & \Big\uparrow & & \Big\updownarrow{\scriptstyle =} \\
X(k) & \times & \mathrm{Br}_1(X) & \longrightarrow & \mathrm{Br}(k) \\
\Big\downarrow & & \Big\uparrow & & \Big\updownarrow{\scriptstyle =} \\
\mathrm{H}^1(k,H^{(\mathrm{m})}) & \times & \mathrm{H}^1(k,\widehat{H^{(\mathrm{m})}}) & \longrightarrow & \mathrm{Br}(k).
\end{array}$$

By considering a finite set S of places of k, then we obtain the commutative diagram

$$\begin{array}{ccccc}
\prod_{v\in S} X(k_v) & \times & Б_S(X) & \longrightarrow & \mathbf{Q}/\mathbf{Z} \\
\Big\downarrow & & \Big\uparrow & & \Big\updownarrow{\scriptstyle =} \\
\prod_{v\in S}\mathrm{H}^1(k_v,H^{(\mathrm{m})}) & \times & \prod_{v\in S}\mathrm{H}^1(k_v,\widehat{H^{(\mathrm{m})}}) & \longrightarrow & \mathbf{Q}/\mathbf{Z} \\
\Big\downarrow & & \Big\uparrow & & \Big\updownarrow{\scriptstyle =} \\
Ч^1_S(H^{(\mathrm{m})}) & \times & Ш^1_S(\widehat{H^{(\mathrm{m})}}) & \longrightarrow & \mathbf{Q}/\mathbf{Z},
\end{array}$$

or the commutative diagram

$$\begin{tikzcd}
\prod_{v\in S} X(k_v) \arrow[d] & \times & \text{Б}_S(X)/\text{Б}(X) \arrow[r] & \mathbf{Q}/\mathbf{Z} \\
\prod_{v\in S} \mathrm{H}^1(k_v, H^{(\mathrm{m})}) \arrow[d] & \times & \prod_{v\in S} \mathrm{H}^1(k_v, \widehat{H^{(\mathrm{m})}}) \arrow[u] \arrow[r] & \mathbf{Q}/\mathbf{Z} \arrow[u, leftrightarrow, "="'] \\
\text{Ч}^1_S(H^{(\mathrm{m})}) & \times & \text{Ш}^1_S(\widehat{H^{(\mathrm{m})}})/\text{Ш}^1_S(\widehat{H^{(\mathrm{m})}}) \arrow[u] \arrow[r] & \mathbf{Q}/\mathbf{Z}, \arrow[u, leftrightarrow, "="']
\end{tikzcd}$$

and finally the commutative diagram

$$\begin{tikzcd}
\prod_{v\in S} X(k_v) \arrow[d, "g_S"] & \times & \text{Б}_S(X)/\text{Б}(X) \arrow[r] & \mathbf{Q}/\mathbf{Z} \\
\text{Ч}^1_S(H^{(\mathrm{m})}) & \times & \text{Ш}^1_S(\widehat{H^{(\mathrm{m})}})/\text{Ш}^1(\widehat{H^{(\mathrm{m})}}) \arrow[u, "g'_S"] \arrow[r] & \mathbf{Q}/\mathbf{Z}. \arrow[u, leftrightarrow, "="']
\end{tikzcd} \tag{4.10}$$

STEP 2. We will show that the right up arrow g'_S is an isomorphism.

Remark that the bottom row is a perfect pairing due to [10, Thm. 5.1 (ii)] (for any k-group H and k a number field) and [45, Thm. 2.1.1 (2)] for any global field k. We show first that g'_S is injective. Let $h \in \text{Ш}^1_S(\widehat{H^{(\mathrm{m})}})/\text{Ш}^1(\widehat{H^{(\mathrm{m})}})$, such that $g'_S(h) = 1$. Then, since (4.10) is commutative, for any $x_S \in \prod_{v\in S} X(k_v)$, we have the pairing

$$\big(g_S(x_S), h\big) = \big(x_S, g'_S(h)\big) = 1.$$

Since we have shown above that the map g_S is surjective, it follows that h is orthogonal to the whole group $\text{Ч}^1_S(H^{(\mathrm{m})})$ with respect to the given pairing. Since the second pairing is a perfect pairing, it follows that $h = 1$. Therefore, on the one hand, π is a monomorphism. On the other hand, by [10, Thm. 5.1 (i)] for k a number field, and [45, Thm. 2.1.2 (i)], for any global field k, there exists an isomorphism of finite abelian groups

$$\text{Ч}^1_S(H^{(\mathrm{m})})^D \xrightarrow{\simeq} \text{Ш}^1_S(\widehat{H^{(\mathrm{m})}})/\text{Ш}^1(\widehat{H^{(\mathrm{m})}}) \xrightarrow{\simeq} \text{Б}_S(X)/\text{Б}(X).$$

Since $H^{(\mathrm{m})}$ is smooth (by assumption) and it is of multiplicative type, then it is well known that $\prod_{v\in S} \mathrm{H}^1(k_v, H^{(\mathrm{m})})$ is finite ([39, Ch. III] and [45, Prop. 1.4.1]); hence, so is $\text{Ч}^1_S(H^{(\mathrm{m})})^D$, so g'_S is also surjective, hence an isomorphism.

The case S is replaced by V is treated in a similar way and by using [45, Thm. 2.1.1] and [45, Prop. 1.4.1] regarding the finitude of $\text{Ч}^1_\omega(H^{(\mathrm{m})})$. Hence, Lemma 4.3 is proved. ∎

Notice that the surjective map g_S is canonical and the isomorphism

$$j_S : Ч^1_S(H^{(\mathrm{m})}) \xrightarrow{\simeq} \big(Б_S(X)/Б(X)\big)^D$$

(from [44, Thm. 1.5] or [45, Thm. 2.1.2]) comes from a perfect duality of [44, Thm. 1.5] (or [45, Thm. 2.1.2 (1)(ii)]), so from Lemma 4.3, it follows that the composite map

$$\zeta_S : \prod_{v\in S} X(k_v) \xrightarrow{g_S} Ч^1_S(H^{(\mathrm{m})}) \xrightarrow{j_S} \big(Б_S(X)/Б(X)\big)^D$$

thus also comes from this same duality, which is nothing than the map coming from the pairing

$$\prod_{v\in S} X(k_v) \times \big(Б_S(X)/Б(X)\big) \longrightarrow \mathbf{Q}/\mathbf{Z}.$$

Therefore, if $x_S \in (\prod_{v\in S} X(k_v))^{Б_S(X)}$, then we have

$$\zeta_S(x_S) = j_S\big(g_S(x_S)\big) = 1.$$

From the exact sequence (4.1a), (4.1b), we have $g_S(x_S) = 1$; i.e., $x_S \in \overline{X(k)}^S$ as required. In particular, it implies that the Brauer–Manin obstruction to the weak approximation in S is the only one. Hence, part (2) of Theorem 4.1 is proven.

(3) Since H is connected and reductive, by Lemma 3.1, we may represent the homogeneous space $X = G/H$ in the form of a quotient G_0/H_0, where G_0 is quasi-trivial reductive k-group and H_0 is a connected reductive k-subgroup of G_0, so we may assume that G is quasi-trivial. By Theorem 3.3 (3), we have the natural isomorphisms of finite abelian groups

$$c_S : Ч^1_S(H^{(\mathrm{tor})}) \xrightarrow{\simeq} C_S(H),$$
$$c_\omega : Ч^1_\omega(H^{(\mathrm{tor})}) \xrightarrow{\simeq} C_\omega(H),$$

and by the proof of Theorem 4.1 (1), we have isomorphisms of finite abelian groups

$$c'_S : Ч^1_S(H^{(\mathrm{tor})}) \xrightarrow{\simeq} \big(Б_S(X)/Б(X)\big)^D,$$
$$c'_\omega : Ч^1_\omega(H^{(\mathrm{tor})}) \xrightarrow{\simeq} \big(Б_\omega(X)/Б(X)\big)^D.$$

Since G is quasi-trivial, by Lemma 3.2 (5), we have $B'(G) = 0$, $B^S(G) = 0$, so

$$B(H,G) = B(H), \quad B(H_v,G_v) = B(H_v),$$
$$B'(H,G) = B'(H), \quad B^S(H,G) = B^S(H);$$

thus, we have

$$C_S(H) = B'(H)/B^S(H) = B'(H,G)/B^S(H,G) = C_S(X), \tag{4.11}$$

so finally

$$C_S(X) \xrightarrow{\simeq} \left(Б_S(X)/Б(X)\right)^D,$$

and the isomorphism

$$C_\omega(X) \xrightarrow{\simeq} \left(Б_\omega(X)/Б(X)\right)^D$$

follows from the fact that $Б_\omega(X) = \varinjlim_S Б_S(X)$ and $C_\omega(X) = \varprojlim_S (C_S(X))$.

Theorem 4.1 is thus proven. ■

REMARK 7. Another way to prove this last assertion (namely, the isomorphisms (4.1d)) in the case H is connected is as follows (cf. also [7]). As above, we may assume that G is quasi-trivial. It follows from [38, Prop. 6.10] that we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Pic}_1(H) \arrow[r, "\gamma_1"] \arrow[d, "p"'] & \mathrm{Br}_1(X) \arrow[r, "\delta_1"] \arrow[d, "\alpha"'] & \mathrm{Br}_1(G) \arrow[d, "\beta"'] \\
0 \arrow[r] & \mathrm{Pic}(H) \arrow[r, "\gamma'"] \arrow[d, "p'"'] & \mathrm{Br}(X) \arrow[r, "\delta'"] \arrow[d, "\alpha'"'] & \mathrm{Br}(G) \arrow[d, "\beta'"'] \\
0 \arrow[r] & \mathrm{Pic}(H \times k_s) \arrow[r, "\gamma''"] & \mathrm{Br}(X \times k_s) \arrow[r, "\delta''"] & \mathrm{Br}(G \times k_s),
\end{tikzcd}$$

where $\mathrm{Pic}_1(H) := \mathrm{Ker}(p')$, hence also the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}_1(H) \arrow[r, "\gamma_1"] \arrow[d, "="'] & \mathrm{Br}_1(X) \arrow[r, "\delta_1"] \arrow[d, "\alpha_1"'] & \mathrm{Br}_1(G) \arrow[d, "\beta_1"'] \\
0 \arrow[r] & \mathrm{Pic}_1(H) \arrow[r, "\gamma_1'"] & \mathrm{Br}_a(X) \arrow[r, "\delta_1'"] & \mathrm{Br}_a(G)
\end{tikzcd}$$

with $\mathrm{Pic}_1(H) = \mathrm{Ker}(p') = \mathrm{Ker}(\delta_1)$ $(= \mathrm{Ker}(\delta_1'))$. Therefore, we also have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Pic}_{1,S}(H) \arrow[r, "\gamma_2"] \arrow[d, "p_1"'] & Б_S(X) \arrow[r, "\delta_2"] \arrow[d, "\alpha"'] & Б_S(G) \arrow[d, "\beta"'] \\
0 \arrow[r] & \mathrm{Pic}_1(H) \arrow[r, "\gamma_1'"] \arrow[d, "p_1'"'] & \mathrm{Br}_a(X) \arrow[r, "\delta_1'"] \arrow[d, "\alpha'"'] & \mathrm{Br}_a(G) \arrow[d, "\beta'"'] \\
0 \arrow[r] & \prod_{v \notin S} \mathrm{Pic}_1(H_v) \arrow[r, "\gamma_S"] & \prod_{v \notin S} \mathrm{Br}_a(X_v) \arrow[r, "\delta_{1,S}'"] & \prod_{v \notin S} \mathrm{Br}_a(G_v),
\end{tikzcd}$$

where $\mathrm{Pic}_{1,S}(H) := \mathrm{Ker}(p_1)$. Since $Б_S(G) = 0$ and $Б(G) = 0$, we have

$$\mathrm{Pic}_{1,S}(H) \xrightarrow{\simeq} Б_S(X), \quad \mathrm{Pic}_{1,\emptyset}(H) = \mathrm{Pic}_1(H) = Б(X).$$

Applying the formulas (3.3)–(3.4) (at the beginning of Section 3) to the groups

$$A := \bigoplus_{v \notin S} \mathrm{Pic}_1(H_v)^D, \quad B := \mathrm{Pic}_1(H)^D$$

with a natural map $f_S : A \to B$,

$$f_S := \bigoplus_{v \notin S} f_v,$$

where

$$f_v : \mathrm{Pic}_1(H_v)^D \longrightarrow \mathrm{Pic}_1(H)^D, \quad v \in V.$$

Then, we have $\mathrm{Im}(f_S) = \langle \mathrm{Im}(f_v) \mid v \notin S \rangle$, $\mathrm{Im}(f_\emptyset) = \langle \mathrm{Im}(f_v) \mid v \in V \rangle$. Then,

$$f_S^D : \mathrm{Pic}_1(H) \longrightarrow \prod_{v \notin S} \mathrm{Pic}_1(H_v),$$

so $\mathrm{Ker}(f_S^D) = \mathrm{Pic}_{1,S}(H) = Б_S(X)$ (see above) and we have

$$A^D = \prod_{v \notin S} \mathrm{Pic}_1(H_v), \quad B^D = \mathrm{Pic}_1(H),$$

and thus one obtains the isomorphisms

$$\begin{aligned}\big(\mathrm{Im}(f_S)\big)^D &\xrightarrow{\simeq} B^D / \mathrm{Ker}(f_S^D : B^D \to A^D) \\ &\xrightarrow{\simeq} \mathrm{Pic}_1(H)/\mathrm{Ker}(f_S^D) \xrightarrow{\simeq} \mathrm{Pic}_1(H)/Б_S(X),\end{aligned}$$

so $\mathrm{Im}(f_S) = (\mathrm{Pic}_1(H)/Б_S(X))^D$, and applying this to $S = \emptyset$, we get

$$\mathrm{Im}(f_\emptyset) = \big(\mathrm{Pic}_1(H)/Б_\emptyset(X)\big)^D = \big(\mathrm{Pic}_1(H)/Б(X)\big)^D.$$

Hence,

$$\begin{aligned}\mathrm{Im}(f_\emptyset)/\mathrm{Im}(f_S) &\xrightarrow{\simeq} \big[\big(\mathrm{Pic}_1(H)/Б(X)\big)^D\big]/\big[\big(\mathrm{Pic}_1(H)/Б_S(X)\big)^D\big] \\ &\xrightarrow{\simeq} \big[Б_S(X)/Б(X)\big]^D.\end{aligned}$$

But by (3.10), we also have $\mathrm{Im}(f_\emptyset)/\mathrm{Im}(f_S) \simeq C_S(H)$; hence, combined with (4.11), we have

$$C_S(X) = C_S(H) \xrightarrow{\simeq} \big[Б_S(X)/Б(X)\big]^D. \tag{4.12}$$

Thus, from (4.12), we derive (4.1d). (The case $C_\omega(X) = C_\omega(H) \simeq [Б_\omega(X)/Б(X)]^D$ is similar.)

We derive some consequences from Theorem 4.1 and its proof.

COROLLARY 4.5. *Let k be a global field, S a finite set of places of k, and H a quasi-connected reductive k-group. Then, for any quasi-trivial reductive k-group G containing H as a closed k-subgroup and $X = G/H$, we have the exact sequences*

$$\mathrm{H}^1(k,H) \xrightarrow{\gamma_S} \prod_{v\in S} \mathrm{H}^1(k_v,H) \longrightarrow \big(Б_S(X)/Б(X)\big)^D \longrightarrow 1,$$

$$1 \longrightarrow \widehat{\gamma_V(\mathrm{H}^1(k,H))} \longrightarrow \prod_{v} \mathrm{H}^1(k_v,H) \longrightarrow \big(Б_\omega(X)/Б(X)\big)^D \longrightarrow 1.$$

PROOF. We divide the proof into two cases.

Case 1. H is connected and reductive. By Theorem 3.3, we have to prove that

$$C_S(H) \xrightarrow{\simeq} \big(Б_S(X)/Б(X)\big)^D. \tag{4.13}$$

By (4.11), we have $C_S(H) \simeq C_S(X)$, and by (4.12), we have

$$C_S(X) \simeq \big(Б_S(X)/Б(X)\big)^D;$$

thus, we have

$$C_S(H) \simeq \big(Б_S(X)/Б(X)\big)^D.$$

Case 2. H is quasi-connected reductive. Since $H^{(\mathrm{m})}$ is commutative, by Theorem 3.3, we have the exact sequence

$$\mathrm{H}^1(k,H) \longrightarrow \prod_{v\in S} \mathrm{H}^1(k_v,H) \longrightarrow Ч^1_S(H^{(\mathrm{m})}) \longrightarrow 1. \tag{4.14}$$

By [44, Thm. 1.5] (or [45, Thm. 2.1.2 (2)]), since $H^{(\mathrm{m})}$ is of multiplicative type, we have an isomorphism of finite abelian groups

$$Ч^1_S(H^{(\mathrm{m})}) \xrightarrow{\simeq} \big(Ш^1_S(\widehat{H^{(\mathrm{m})}})/Ш^1(\widehat{H^{(\mathrm{m})}})\big)^D.$$

Since $\widehat{H^{(\mathrm{m})}} \simeq \hat{H}$, we also have

$$Ч^1_S(H^{(\mathrm{m})}) \xrightarrow{\simeq} \big(Ш^1_S(\hat{H})/Ш^1(\hat{H})\big)^D.$$

Again, by [44, Thm. 1.5] (or [45, Thm. 2.1.2]), we have

$$Ш^1_S(\hat{H}) \xrightarrow{\simeq} Б_S(X), \quad Ш^1(\hat{H}) \xrightarrow{\simeq} Б(X);$$

hence, also

$$Ч^1_S(H^{(\mathrm{m})}) \xrightarrow{\simeq} \big(Б_S(X)/Б(X)\big)^D.$$

Hence, from (4.14), we obtain an exact sequence

$$\mathrm{H}^1(k,H) \xrightarrow{\gamma_S} \prod_{v\in S} \mathrm{H}^1(k_v,H) \longrightarrow \big(Б_S(X)/Б(X)\big)^D \longrightarrow 1$$

as required. The other exact sequence is treated in the same way. ■

We derive some other immediate consequences from Theorems 3.3 and 4.1. The strategy of most of the proofs follows the same proofs given in [7], so we give here only a sketch of the proofs.

Corollary 4.6. *Let k be a global field, S a finite set of places of k, and G a connected reductive k-group.*

(1) (See [7, Cor. 1.5] for number field case) *Let H be a connected reductive k-subgroup of G, and let $X := G/H$. Then, X has weak approximation in S (resp. over k) if and only if $C_S(X)$ (resp. $C_\omega(X)$) is trivial.*

(2) *Let G be a quasi-trivial group, H a quasi-connected reductive k-subgroup of G, and $X := G/H$. Then, X has weak approximation in S (resp. over k) if and only if $Ч^1_S(H^{(\mathrm{m})})$ (resp. $Ч^1_\omega(H^{(\mathrm{m})})$) is trivial.*

(3) (See [7, Cor. 1.6] for number field case) *Let G, H, X be as in* (1). *Let L/k be a finite Galois extension in k_s which splits H^{tor} (i.e.,* $\mathrm{Gal}(L/k)$ *acts trivially on $X^*(H^{\mathrm{tor}})$) and let S_0 be the set of all places of k which have non-cyclic decomposition groups in L. Then, we have $C_S(X) = C_{S\cap S_0}(X)$. In particular, if $S \cap S_0 = \emptyset$, then X has weak approximation in S.*

(4) *Let k, G, H, X be as in* (2). *Let L/k be a finite Galois extension in k_s which splits $H^{(\mathrm{m})}$, and let S_0 be the set of all places of k which have non-cyclic decomposition groups in L. Then, we have $Ч^1_S(H^{(\mathrm{m})}) \simeq Ч^1_{S\cap S_0}(H^{(\mathrm{m})})$. In particular, if $S\cap S_0 = \emptyset$, then X has weak approximation in S.*

Proof. (1) and (2). It follows from Theorems 3.3 and 4.1.

(3) and (4). From Theorems 3.3 and 4.1, we deduce that it suffices to investigate the triviality of the groups $Ч^1_{\mathrm{ab},S}(H^{(\mathrm{m})})$ and $Ч^1_{\mathrm{ab},\omega}(H^{(\mathrm{m})})$. In case (3) (resp. case (4)), $H^{(\mathrm{m})}$ is a k-torus (resp. a group of multiplicative type), which is split over L. Then, the assertion follows from standard statements regarding weak approximation for the first Galois cohomology of tori (resp. groups of multiplicative type) in this case (see [42, Thm. 4.4, Prop. 4.4.2], where the arguments of the proof given there also hold in the case of number fields). ■

COROLLARY 4.7 (See [3, Thm. 1.4] for number field case). *Keep the notation as in Corollary 4.6.*

(1) *With the assumption as in Corollary 4.6 (1), (3), we have canonical isomorphisms of finite abelian groups*

$$C_S(H) \xrightarrow{\simeq} C_S(H^{\mathrm{tor}}) \xrightarrow{\simeq} C_{S\cap S_0}(H^{\mathrm{tor}}),$$

where S_0 stands for the set of all places of k, ramified in a fixed splitting field of H^{tor} with non-cyclic decomposition groups.

(2) *With the assumption as in Corollary 4.6 (2), (4), we have canonical isomorphisms of finite abelian groups*

$$Ч^1_{\mathrm{ab},S}(H) \xrightarrow{\simeq} Ч^1_S(H^{(\mathrm{m})}) \xrightarrow{\simeq} Ч^1_{S\cap S_0}(H^{(\mathrm{m})}),$$

where S_0 stands for the set of all places of k, ramified in a fixed splitting field of $H^{(\mathrm{m})}$ with non-cyclic decomposition groups.

PROOF. (1) If in Theorem 3.3 (3), we let $T = H^{\mathrm{tor}}$ be a torus, then we obtain an isomorphism

$$C_S(T) \simeq Ч^1_S(T)$$

and

$$C_S(H) \simeq Ч^1_S(H^{\mathrm{tor}}) = Ч^1_S(T) \simeq C_S(T) = C_S(H^{\mathrm{tor}}). \tag{4.15}$$

To prove the second isomorphism, we may assume that G is a quasi-trivial reductive group. Then, by Corollary 4.6 (4), we have

$$Ч^1_S(T) \simeq Ч^1_{S\cap S_0}(T),$$

and again by (4.15), we have

$$C_{S\cap S_0}(H^{\mathrm{tor}}) = C_{S\cap S_0}(T) \simeq Ч^1_{S\cap S_0}(T) \simeq C_{S\cap S_0}(H).$$

As yet another argument to prove the first isomorphism, we let $X = G/H$, where G is a quasi-trivial reductive k-group containing H. Then, by Corollary 4.5 and (4.13), we have

$$C_S(H) \xrightarrow{\simeq} \big(Б_S(X)/Б(X)\big)^D. \tag{4.16}$$

From the proof of [45, Thm. 2.1.1], the case B is connected, and it follows that we have the isomorphism of abelian groups

$$Ш^1_S(\widehat{H})/Ш^1(\widehat{H}) \xrightarrow{\simeq} Б_S(X)/Б(X). \tag{4.17}$$

By Theorem 3.3 (3), we have an isomorphism $Ч^1_S(H^{\mathrm{tor}}) \simeq C_S(H^{\mathrm{tor}})$, and by [44, Thm. 1.5] (or [45, Thm. 2.1.1]), the isomorphisms

$$Ч^1_S(H^{\mathrm{tor}}) \xrightarrow{\simeq} C_S(H^{\mathrm{tor}}) \xrightarrow{\simeq} \left(Ш^1_S(\widehat{H^{\mathrm{tor}}})/Ш^1(\widehat{H^{\mathrm{tor}}})\right)^D. \tag{4.18}$$

Since $\widehat{H} = \widehat{H^{\mathrm{tor}}}$ from (4.16)–(4.18), we derive that $C_S(H) \simeq C_S(H^{\mathrm{tor}})$.

To show that $C_S(H^{\mathrm{tor}}) \simeq C_{S\cap S_0}(H^{\mathrm{tor}})$, it suffices to note that we have

$$Ч^1_S(H^{\mathrm{tor}}) \xrightarrow{\simeq} C_S(H^{\mathrm{tor}})$$

for any S and that

$$Ч^1_S(H^{\mathrm{tor}}) \xrightarrow{\simeq} Ч^1_{S\cap S_0}(H^{\mathrm{tor}}).$$

(2) Similar proof. ∎

Remark 8. It follows from all the results proved above that over any global field k, any homogeneous k-space X under a connected reductive k-group G with a quasi-connected k-stabilizer H has *almost weak approximation over* k; i.e., X has weak approximation outside a finite set of places.

From the following isomorphism

$$\mathrm{A}(G) \xrightarrow{\simeq} \left(Б_\omega(G)/Б(G)\right)^D$$

(see [44, Thm. 2.1] (or [45, Thm. 2.2.1])) for connected reductive groups, we derive a generalization of this isomorphism for obstruction to weak approximation in homogeneous spaces over k as follows. In fact, it can be seen as a global version of Theorem 4.1 and also of an analog of the Sansuc exact sequence (S) (see [44, Thm. 2.1] (or [45, Thm. 2.2.1])) in the case of homogeneous spaces.

Theorem 4.8. *Let k be a global field, G a quasi-trivial reductive k-group, H a quasi-connected reductive k-subgroup of G, and $X = G/H$.*

(1) *We have the exact sequence of pointed sets*

$$1 \longrightarrow \overline{X(k)} \longrightarrow \prod_v X(k_v) \longrightarrow Ч^1_\omega(H^{(\mathrm{m})}) \longrightarrow 1. \tag{4.19}$$

(2) *There is the exact sequence of pointed sets*

$$1 \longrightarrow \overline{X(k)} \longrightarrow \prod_v X(k_v) \longrightarrow \left(Б_\omega(X)/Б(X)\right)^D \longrightarrow 1 \tag{4.20}$$

(or the same, $1 \to \overline{X(k)} \to \prod_v X(k_v) \to (Б_\omega(X))^D \to (Б(X))^D \to 1$).

(3) *The Brauer–Manin obstruction to the weak approximation of X over k is the only one.*

Proof. (1) We have the following commutative diagram, similar to the one given in the proof of Theorem 4.1, with exact rows

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & \\
G(k) \arrow[r, "\pi"] \arrow[d, "i"] & X(k) \arrow[r, "\delta'"] \arrow[d, "i'"] & \widehat{\gamma_V\big(\mathrm{H}^1(k,H)\big)} \arrow[r, "h"] \arrow[d, "j_V"] & \mathrm{H}^1(k,G) \arrow[d, "j_V"] \\
\prod_v G(k_v) \arrow[r, "\pi_V"] & \prod_v X(k_v) \arrow[r, "\delta_V"] \arrow[d, "g_V"] & \prod_v \mathrm{H}^1(k_v,H) \arrow[r, "h_V"] \arrow[d, "f_V"] & \prod_v \mathrm{H}^1(k_v,G) \\
 & \text{Ч}^1_\omega(H^{(\mathrm{m})}) \arrow[r, "="] \arrow[d] & \text{Ч}^1_\omega(H^{(\mathrm{m})}) \arrow[d] & \\
 & 1 & 1, &
\end{tikzcd}$$

where j_V is just an inclusion, $g_V := f_V \circ \delta_V$, and δ' is the composite map

$$\delta' : X(k) \xrightarrow{\delta} \mathrm{H}^1(k,H) \xrightarrow{j_V} \widehat{\gamma_V\big(\mathrm{H}^1(k,H)\big)} \xrightarrow{i_V} \prod_v \mathrm{H}^1(k_v,H).$$

We know by Lemma 3.2 (3) that β_V is a bijection (Hasse principle), and by Theorem 3.3, the second column is exact. Further, we endow $\mathrm{H}^1(k,H)$ and $\mathrm{H}^1(k_v,H)$ with natural topology (see Section 3.2 after the proof of Lemma 3.2) such that all the maps appearing in the diagrams above are continuous. Then, by the proof of Theorem 3.3, we have a natural identification

$$\widehat{\gamma_V\big(\mathrm{H}^1(k,H)\big)} \xrightarrow{\simeq} \overline{\gamma_V\big(\mathrm{H}^1(k,H)\big)}.$$

It follows from a similar argument in the proof of Theorem 4.1 that g_V is surjective. Due to the continuity, we have $\overline{X(k)} \subseteq \mathrm{Ker}(g_V)$. Conversely, let $x \in \mathrm{Ker}(g_V)$. Then,

$$\delta_V(x) = j_V(y), \tag{4.21}$$

where

$$y \in \overline{\gamma_V\big(\mathrm{H}^1(k,H)\big)}, \quad y = (y_v), \quad y_v \in \mathrm{H}^1(k_v,H).$$

Take any finite set S of places of k and consider the diagram (4.5). Let $x_S := p_S(x)$ be the projection of x via

$$p_S : \prod_v X(k_v) \longrightarrow \prod_{v\in S} X(k_v)$$

and take any open neighborhood U of y in $\prod_{v\in V} \mathrm{H}^1(k_v,H)$ in the form

$$U = \prod_{v\notin S} \mathrm{H}^1(k_v,H) \times \prod_{v\in S} U_v,$$

where U_v is an open neighborhood of y_v in $\mathrm{H}^1(k_v, H)$, for all $v \in S$. Then, it is clear that

$$y_S := (y_v)_{v\in S} \in \overline{\gamma_S\big(\mathrm{H}^1(k, H)\big)}^S$$

(the closure is taken in $\prod_{v\in S} \mathrm{H}^1(k_v, H)$). Due to the discreteness of the topology on the product $\prod_{v\in S} \mathrm{H}^1(k_v, H)$ (cf. [1, Sec. 5]), we have

$$\overline{\gamma_S\big(\mathrm{H}^1(k, H)\big)}^S = \gamma_S\big(\mathrm{H}^1(k, H)\big),$$

so this implies that $y_S \in \gamma_S(\mathrm{H}^1(k, H))$, and we have, for some $z \in \mathrm{H}^1(k, H)$, $\gamma_S(z) = y_S$. So from (4.21), we derive $\delta_S(x_S) = \gamma_S(z)$. As in the proof of Theorem 3.4, it implies that

$$x_S \in \pi_S\Big(\prod_{v\in S} \big(G(k_v)\big)\Big) \cdot x,$$

for some $x \in X(k)$. Since G has weak approximation property in S over k (see Lemma 3.2 (3)), this means that $x_S \in \overline{X(k)}^S$. Since this is true for any S, this implies that $x \in \overline{X(k)}$ since G has weak approximation property over k. Thus, we have proved that there is the exact sequence

$$1 \longrightarrow \overline{X(k)} \longrightarrow \prod_v X(k_v) \xrightarrow{g_V} Ч^1_\omega(H^{(\mathrm{m})}) \longrightarrow 1. \tag{4.22}$$

(2) From the isomorphism $\hat{H} \simeq \widehat{H^{(\mathrm{m})}}$ and from [44, Thm. 1.5] (or [45, Thm. 2.1.2]), we derive that

$$\begin{aligned} Ч^1_\omega(H^{(\mathrm{m})}) &\xrightarrow{\simeq} \big(Ш^1_\omega(\widehat{H^{(\mathrm{m})}})/Ш^1(\widehat{H^{(\mathrm{m})}})\big)^D \\ &\xrightarrow{\simeq} \big(Ш^1_\omega(\hat{H})/Ш^1(\hat{H})\big)^D \\ &\xrightarrow{\simeq} \big(Б_\omega(X)/Б(X)\big)^D. \end{aligned} \tag{4.23}$$

Thus, from (4.22) and (4.23), we obtain the exact sequence (4.20).

(3) Finally, to prove that the Brauer–Manin obstruction to the weak approximation over k for X is the only one, we need to show that

$$\overline{X(k)} = \Big(\prod_v X(k_v)\Big)^{Б_\omega(X)}. \tag{4.24}$$

For any point $x = (x_v) \in (\prod_v X(k_v))^{Б_\omega(X)}$, let U be any open neighborhood of x. By choosing U sufficiently small, we may assume that U is in the form (I am thankful to the referee for this observation):

$$U = U_{v_1} \times \cdots \times U_{v_t} \times \prod_{v\notin T} X(k_v),$$

where $T := \{v_1, \dots, v_t\}$ is a finite set of places and U_{v_i} is an open neighborhood of x_{v_i} in X_{v_i}. Since

$$x = (x_v) \in \Big(\prod_v X(k_v)\Big)^{\text{Б}_\omega(X)},$$

we have

$$x_T := (x_{v_1}, \dots, x_{v_t}) \in \Big(\prod_{v \in T} X(k_v)\Big)^{\text{Б}_T(X)}.$$

We have proved that (see the proof of Theorem 4.1)

$$\overline{X(k)}^S = \Big(\prod_{v \in S} X(k_v)\Big)^{\text{Б}_S(X)}$$

for any finite set S of places, so by taking $S = T$, there exists

$$x' \in X(k) \cap (U_{v_1} \times \cdots \times U_{v_t});$$

hence, we have $x' \in X(k) \cap U$ and it implies that $x \in \overline{X(k)}$ and (4.24) holds. ■

Remark 9. In the case k is a number field, in [19, Main Theorem, Cor. 6.3], there was proved a more general exact sequence (in the same spirit) for the homogeneous spaces $X = G/H$, G is a connected linear algebraic k-group, and H is a *connected k-subgroup* of G.

Corollary 4.9. *If G is quasi-trivial and H is quasi-connected reductive, then we have the exact sequence*

$$\overline{G(k)} \longrightarrow \overline{X(k)} \xrightarrow{\delta_V} \gamma\big(\widehat{\mathrm{H}^1(k, H)}\big), \tag{4.25}$$

where δ_V is surjective if k has no real places.

Proof. From the diagram drawn in the proof of Theorem 4.8, we derive the commutative diagram

$$\begin{tikzcd}
 & \overline{G(k)} \arrow[r, "="] \arrow[d, "\bar{\pi}"] & \prod_v G(k_v) \arrow[d, "\pi'"] & & \\
1 \arrow[r] & \overline{X(k)} \arrow[r] \arrow[d, "\delta'_V"] & \prod_v X(k_v) \arrow[r] \arrow[d, "\delta_V"] & \text{Ч}^1_{\mathrm{ab},\omega}(H^{(\mathrm{m})}) \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \gamma\big(\widehat{\mathrm{H}^1(k, H)}\big) \arrow[r, "i"] & \prod_v \mathrm{H}^1(k_v, H) \arrow[r] & \text{Ч}^1_{\mathrm{ab},\omega}(H^{(\mathrm{m})}) \arrow[r] & 1.
\end{tikzcd} \tag{4.26}$$

Here, $\overline{\pi}$ is induced from π and δ'_V is induced from δ. Since δ_V is surjective, from the diagram, it implies that δ'_V is also surjective. A chase on the diagram shows that $\mathrm{Ker}(\delta'_V) = \overline{G(k)}$ (since G has weak approximation over k). Hence, the left column is exact and we obtain the exact sequence (4.25). If, moreover, k has no real places, then $\mathrm{H}^1(k, G) = 1$, $\mathrm{H}^1(k_v, G) = 1$, for all v, so δ'_V is surjective as asserted. ■

REMARK 10. (1) In the next section (see Theorem 5.3), we will prove that over global fields k with no real places, the Brauer–Manin obstruction for weak approximation is the only one for another (more general) class of homogeneous spaces, namely, for those $X = G/H$ where G is a connected reductive k-group with simply connected semisimple part and with the stabilizer H such that $H^{(\mathrm{m})}$ is commutative.

(2) Locally, at truncated places, we have the following analogous commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & \overline{G(k)}^S \arrow[r, "="] \arrow[d, "\overline{\pi}_S"] & \prod_{v\in S} G(k_v) \arrow[d, "\pi_S"] & & \\
1 \arrow[r] & \overline{X(k)}^S \arrow[r] \arrow[d, "\delta'_S"] & \prod_{v\in S} X(k_v) \arrow[r, "g_S"] \arrow[d, "\delta_S"] & Ч^1_{\mathrm{ab},S}(H^{(\mathrm{m})}) \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \gamma_S\big(\mathrm{H}^1(k, H)\big) \arrow[r, "i_S"] & \prod_{v\in S} \mathrm{H}^1(k_v, H) \arrow[r] & Ч^1_{\mathrm{ab},S}(H^{(\mathrm{m})}) \arrow[r] & 1.
\end{tikzcd}$$

Recall that a Galois extension L/k is called *metacyclic* if its Galois group is metacyclic; i.e., all of its Sylow subgroups are cyclic.

COROLLARY 4.10 (See [8, Thm 4.2] for number field case). *With notation and assumption as in Corollary 4.6, if H^{tor} (resp. $H^{(\mathrm{m})}$) is split over a metacyclic extension of k (e.g. if $H^{(\mathrm{m})} = 1$), then $X = G/H$ has weak approximation in S.*

PROOF. It follows from Corollary 4.6. ■

From Theorems 4.1 and 4.8 and their proofs, we derive the following.

THEOREM 4.11 (See [38, proof of Thm. 8.12] for connected linear algebraic groups and [10, Thm. 5.1] for homogeneous spaces over number fields). *Let k be a global field, S a finite set of places of k, G a connected reductive k-group, H a k-subgroup of G, and $X = G/H$. Assume that G is quasi-trivial and H is quasi-connected reductive. Then, we have the following commutative diagrams, where in each of the diagrams, the columns are exact, the third rows represent such a pairing with the right kernel being trivial and the left kernel being $\overline{X(k)}^S$ (resp. $\overline{X(k)}$), and the fourth rows represent* *perfect pairings of finite abelian groups:*

$$\begin{tikzcd}
1 \arrow[d] & & & \\
\overline{X(k)}^{S} \arrow[d] & & 1 & \\
\prod_{v\in S} X(k_v) \arrow[d] & \times & \text{Б}_S(X)/\text{Б}(X) \arrow[u] \arrow[r] & \mathbf{Q}/\mathbf{Z} \\
\text{Ч}^1_S(H^{(\mathrm{m})}) \arrow[d] & \times & \text{Ш}^1_S(\hat{H})/\text{Ш}^1(\hat{H}) \arrow[u, "\simeq"'] \arrow[r] & \mathbf{Q}/\mathbf{Z} \arrow[u, leftrightarrow, "="'] \\
1 & & 1 \arrow[u] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[d] & & & \\
\overline{X(k)} \arrow[d] & & 1 & \\
\prod_{v} X(k_v) \arrow[d] & \times & \big(\text{Б}_\omega(X)/\text{Б}(X)\big) \arrow[u] \arrow[r] & \mathbf{Q}/\mathbf{Z} \\
\text{Ч}^1_\omega(H^{(\mathrm{m})}) \arrow[d] & \times & (\text{Ш}^1_\omega(\hat{H})/\text{Ш}(\hat{H})) \arrow[u, "\simeq"'] \arrow[r] & \mathbf{Q}/\mathbf{Z} \arrow[u, leftrightarrow, "="'] \\
1 & & 1. \arrow[u] &
\end{tikzcd}$$

Here, the closure $\overline{X(k)}^{S}$ and $\overline{X(k)}$ are taken in the product $\prod_{v\in S} X(k_v)$ and the product $\prod_v X(k_v)$, respectively. In particular, the Brauer–Manin obstruction to the weak approximation for X in S and over k is the only one. ■

5. Brauer–Manin obstruction to the Hasse principle and weak approximation for homogeneous spaces: Stabilizers as extensions of groups of multiplicative type

In this section, for the global fields with no real places, we investigate the Brauer–Manin obstruction to the Hasse principle and weak approximation for homogeneous spaces under connected reductive groups with stabilizers H such that $H^{(\mathrm{m})}$ is commutative and prove Theorems 5.2 and 5.3, which are an extension of Borovoi's main result to the global function field case. First we need the following.

LEMMA 5.1 (Cf. [5, Sec. 4.1]). *Let k be a field, X a homogeneous space under a smooth affine algebraic connected k-group G, and H the stabilizer of a point $x \in X(k_s)$, which is smooth. Then, there is a k-group $H^{\langle \mathrm{m} \rangle}$, which is a k-form of $H^{(\mathrm{m})} = H/H^{\mathrm{ssu}}$, such that if H_1 is the stabilizer of another point $x_1 \in X(k_s)$ and $\lambda_g : H \to H_1, h \mapsto g^{-1}hg$ the conjugation H to H_1, then λ_g induces a k-isomorphism $\overline{\lambda}_g : H^{\langle \mathrm{m} \rangle} \to H_1^{\langle \mathrm{m} \rangle}$. In particular, $H^{\langle \mathrm{m} \rangle}$ is uniquely determined and depends only on G and X.*

PROOF. The proof given in [5, Sec. 4.1, pp. 189–190] (under the assumption that $\mathrm{char.}\, k = 0$ (p. 183)) still holds in the case $\mathrm{char.}\, k = p > 0$. Since H/H^{ssu} is an extension of the finite group $\pi_0(H)$ by the torus H^{tor}, the same also holds for $H^{\langle \mathrm{m} \rangle}$. ∎

Notice that the k-group $H^{\langle \mathrm{m} \rangle}$ is an extension of a finite k-group by a k-torus (see Section 1). Hence, if it is of multiplicative type, then it is commutative, but not the converse. In the case $\mathrm{char.}\, k = 0$, the finite group scheme $\pi_0(H)$ contains no unipotent elements, so if $H^{\langle \mathrm{m} \rangle}$ is commutative, then $H^{\langle \mathrm{m} \rangle}$ is of multiplicative type. In the case $\mathrm{char.}\, k = p > 0$, then $H^{\langle \mathrm{m} \rangle}$ may contain non-trivial finite unipotent subgroups, in particular, non-trivial unipotent elements.

We have the following extension of a result of Borovoi [5, Cor. 2.5] to the case of global fields with no real places, in particular, global function fields. The proof follows the same method (with suitable modifications) which uses the embedding trick as in Borovoi [5] given for number fields. One should remark that in the case of number fields, the condition $H^{(\mathrm{m})}$ being commutative implies that it is of multiplicative type (see [5]), but in the case of global function field, $H^{(\mathrm{m})}$ may contain a non-trivial unipotent element, so it may not be of multiplicative type. Therefore, our arguments need be modified so as to overcome this difficulty.

THEOREM 5.2 (Cf. [5, Cor. 2.5] for number field case). *Let k be a global field with no real places, S a finite set of places of k, and X a homogeneous k-space under a connected reductive group G with simply connected semisimple part and with a smooth stabilizer H such that $H^{(\mathrm{m})}$ is commutative. Then, the Brauer–Manin obstruction to the Hasse principle for X is the only one.*

PROOF. Assuming that $(\prod_v X(k_v))^{\text{Б}_\omega(X)} \neq \emptyset$, we will show that $X(k) \neq \emptyset$. By Lemma 5.1, there exists a k-form $H^{\langle \mathrm{m} \rangle}$ of the group of multiplicative type $H^{(\mathrm{m})}$. We apply the fibration method as in the proof of [5] and the similar proof of [5, Cor. 2.5] ($\mathrm{char.}\, k = 0$) and [45, Prop. 4.3.3 and Thm. 4.3.6] ($\mathrm{char.}\, k = p > 0$) and the embedding trick due to Borovoi.

Case 1. The inclusion $H \hookrightarrow G$ induces an injection $H^{(\mathrm{m})} \to G^{\mathrm{tor}}$. In our case, this means that $H \cap G^{\mathrm{ss}} = \{1\}$. Consider the quotient $\pi : X \to Y := X/G^{\mathrm{ss}}$. Then, Y is a k-homogeneous space under the k-torus $G/G^{\mathrm{ss}} = G^{\mathrm{tor}}$, and we have a map

$$\pi_V : \prod_v X(k_v) \longrightarrow \prod_v Y(k_v).$$

Since G^{ss} is simply connected, we have $\mathrm{Pic}(G^{\mathrm{ss}}) = 0$ by [38, Lem. 6.9 (iv)], so from [38, Prop. 6.10], it follows that we have the exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Br}_a(Y) \longrightarrow \mathrm{Br}_a(X) \longrightarrow \mathrm{Br}_a(G^{\mathrm{ss}}) = 0, \tag{5.1}$$

so $\mathrm{Br}_a(Y) \simeq \mathrm{Br}_a(X)$ and also for all $v \in V$, $\mathrm{Br}_a(Y_v) \simeq \mathrm{Br}_a(X_v)$; hence,

$$Б_S(Y) \simeq Б_S(X), \quad \text{for any finite set } S \subset V, \tag{5.2}$$

$$Б_\omega(Y) \simeq Б_\omega(X). \tag{5.3}$$

Consider the commutative diagram

$$\begin{array}{ccccc}
\prod_v X(k_v) & \times & Б_\omega(X) & \longrightarrow & \mathbf{Q}/\mathbf{Z} \\
\Big\downarrow \pi_V & & \simeq \Big\uparrow \pi'_V & & \Big\updownarrow = \\
\prod_v Y(k_v) & \times & Б_\omega(Y) & \longrightarrow & \mathbf{Q}/\mathbf{Z},
\end{array}$$

where the pairings are the Brauer–Manin pairing. For any $x \in \prod_v X(k_v)$ and any $b \in Б_\omega(X)$, we have

$$\langle x, b\rangle = \big\langle x, \pi'_V(b')\big\rangle = \big\langle \pi_V(x), b'\big\rangle$$

for some $b' \in Б_\omega(Y)$. Hence, given that $(\prod_v X(k_v))^{Б_\omega(X)} \neq \emptyset$, we have

$$\Big(\prod_v Y(k_v)\Big)^{Б_\omega(Y)} \neq \emptyset.$$

Now as in the proof of [5, Prop. 3.5] (or similar proof of [45, Prop. 4.3.3 (i)]), Y is a torsor under a k-torus, which has locally everywhere rational points, and we know that in this case, the Brauer–Manin to the Hasse principle is the only one. Therefore, one concludes that $Y(k) \neq \emptyset$ by [44, Thm. 3.3] (or [45, Prop. 3.2.1]).

For a k-point $y \in Y(k)$, $X_y := \pi^{-1}(y)$ is a homogeneous k-space under G^{ss} and with a stabilizer $H_1 := H \cap G^{\mathrm{ss}}$, which is trivial by our assumption. This means that X_y is a G^{ss}-torsor, which is a trivial torsor since by assumption G^{ss} is simply connected and we have $\mathrm{H}^1(k, G^{\mathrm{ss}}) = 1$ by Harder (see [25, Hauptsatz], [26, Satz A] and [36, Ch. 6]). Thus, $X_y(k) \neq \emptyset$ and hence so is $X(k)$.

Case 2. General case. Here, we proceed in few steps.

Step 1. Construction of some embeddings. We let $\varphi : H \to H^{(\mathrm{m})} \simeq_{k_s} H^{\langle \mathrm{m}\rangle}$ be the composition of the projection from H onto $H^{(\mathrm{m})}$ and followed by any k_s-isomorphism $H^{(\mathrm{m})} \simeq H^{\langle \mathrm{m}\rangle}$. Since $H^{\langle \mathrm{m}\rangle}$ is defined over k and is commutative, then by [42, Proof of Prop. 4.4.1], we may take a k-embedding j of $H^{\langle \mathrm{m}\rangle}$ into a smooth special commutative k-group (i.e., a smooth affine k-group such that its first Galois cohomology over L is trivial, for any field extension L/k)

$$T = T^s \times T^u, \tag{5.4}$$

with T^s being an induced k-torus and T^u a k-split unipotent group.

Next, we consider the embedding (over k_s) with respect to the maps $H \to G$ (embedding) and the map $j' = j \circ \varphi : H \to T$,

$$H \hookrightarrow F := G \times T, \quad h \mapsto \big(h, j'(h)\big) \in G \times T,$$

all of which are defined over k_s. We have $F^{\mathrm{ss}} = G^{\mathrm{ss}}$, so $F/F^{\mathrm{ss}} = G^{\mathrm{tor}} \times T$. It is clear that the inclusion $H \hookrightarrow F$ induces an *injection* $H^{\langle \mathrm{m}\rangle} \to F/F^{\mathrm{ss}}$. For a k-variety U, we denote $U_s = U \times_k k_s$. Consider the quotient $Z := F/H$. Here, F acts naturally on X where T acts on X by trivial action. Therefore, X becomes naturally also an F-homogeneous space. Now, $X = G/H$ being defined over k, it is clear that $Z := F/H$ is also *defined over* k.

With the natural action of F_s on X_s (with trivial action of T_s on X_s), there is an F_s-equivariant morphism φ_s from the resulting F_s-homogeneous space $Z_s := F_s/H_s$ into $X_s = F_s/(H_s \times T_s) = G_s/H_s$. Explicitly for $z = H_s(g,t) \in Z_s$, set $\varphi_s(z) = H_s g$ and one can check as in [5, p. 190] that $\varphi_s : Z_s \to X_s$ is a T_s-torsor, which is defined over k_s, but may be not over k. However, if L/k is an extension and $X(L) \neq \emptyset$, then the T_L-torsor $\varphi : Z_L \to X_L$ is defined over L since then we may assume that H is defined over L.

Step 2. Construction of some T-torsor with the same arithmetic Brauer groups as X.

Claim. *Under the assumption that $\prod_v X(k_v) \neq \emptyset$, there exists a k-form (Z_0, φ_0) of (Z_s, φ_s) such that φ_0 induces the isomorphisms*

$$\text{Б}_S(X) \xrightarrow{\simeq} \text{Б}_S(Z_0), \quad \text{Б}_\omega(X) \xrightarrow{\simeq} \text{Б}_\omega(Z_0), \quad \text{Б}(X) \xrightarrow{\simeq} \text{Б}(Z_0).$$

(a) *The existence of such a k-form* (Z_0, φ_0). In fact, this can be proceeded just as in the proof of [45, Thm. 4.3.6], using [45, Prop. 4.3.5] as follows. For the sake of the convenience of the reader, we give it here.

Since $X(k_v) \neq \emptyset$, the same argument as in [5, p. 192] implies that the T-torsor $\varphi_v : Z_v \to X_v$ is defined over k_v. Then, the exact sequence

$$Z(k_v) \xrightarrow{\varphi_v} X(k_v) \longrightarrow \mathrm{H}^1(k_v, T) = 1$$

(T being a k-special commutative group) shows that φ_v is surjective; hence, from $X(k_v) \neq \emptyset$, it implies that $Z(k_v) \neq \emptyset$, and it is true for all $v \in V$. In particular, the projection $\prod_v Z(k_v) \to \prod_v X(k_v)$ is surjective.

Consider the gerbe $\mathcal{Z}$ associated with the k-homogeneous space Z corresponding to the pair (Z, F) and its band $\mathrm{lien}(\mathcal{Z})$ as in [22, Ch. IV, Sec. 5]. Since $Z(k_v) \neq \emptyset$, by [22, Ch. IV, Prop. 5.1.4 (ii)], the corresponding class $c(\mathcal{Z}_v)$ of the gerbe $\mathcal{Z}_v$ in $\mathrm{H}^2_{\mathrm{fppf}}(k_v, \mathrm{lien}(\mathcal{Z}))$ is neutral. By [22, Ch. IV, Prop. 5.1.4 (iii), p. 295], cf. also [5, p. 192], the automorphism of the T-torsor Z is isomorphic to T and we have $\mathrm{lien}(\mathcal{Z}) \simeq \mathrm{lien}(T)$. Therefore,

$$\mathrm{H}^2_{\mathrm{fppf}}\big(k_v, \mathrm{lien}(\mathcal{Z})\big) \xrightarrow{\simeq} \mathrm{H}^2_{\mathrm{fppf}}(k_v, T).$$

Then, the corresponding cohomological class $\eta(\mathcal{Z})_v \in \mathrm{H}^2(k_v, T)$ is also trivial and this is true for all places v of k. Hence, $\eta(\mathcal{Z}) \in Ш^2(T)$. Since T^u is commutative, we have $\mathrm{H}^2_{\mathrm{fppf}}(L, T^u) = 1$ for any field extension L/k. By the Hasse principle for the Brauer group of global fields, we have $Ш^2(T) = 0$; thus, $\eta(\mathcal{Z}) = 0$ in $\mathrm{H}^2(k, T)$. Therefore, the class $c(\mathcal{Z})$ in $\mathrm{H}^2_{\mathrm{fppf}}(k, \mathrm{lien}(\mathcal{Z}))$ is neutral, so again by [22, Ch. IV, Prop. 5.1.4 (iii)], there exists a T-torsor Z_0, defined over k, which dominates Z; i.e., there is a T-equivariant morphism $\pi_0 : Z_0 \to Z$. Since Z_s is also a T_s-torsor, it means that Z_0 is a k-form (Z, φ) of (Z_s, φ_s); hence, (a) is proved.

(b) *The isomorphisms of Brauer groups.* The assertion of the claim regarding the isomorphisms of Brauer groups and their quotients follows by using the same idea in the proof of [5, Lemma 4.4, pp. 190–191]. For the T-torsor $\varphi : Z_0 \to X$ under a smooth special commutative k-group $T = T^s \times T^u$ (as in (5.4)), there are associated with it another two torsors, namely the T^u-torsor $Z_0 \to Z_0/T^u$ and the T^s-torsor $Z_1 := Z_0/T^u \to Z_0/T = X$, thus also two fibrations:

$$1 \longrightarrow T^s \longrightarrow Z_1 \longrightarrow X \longrightarrow 1,$$
$$1 \longrightarrow T^u \longrightarrow Z_0 \longrightarrow Z_1 \longrightarrow 1.$$

Then, related to the fibration $T^s \to Z_1 \to X$, we have the exact sequence

$$0 = \mathrm{Pic}(T^s) \longrightarrow \mathrm{Br}_1(X) \longrightarrow \mathrm{Br}_1(Z_1) \longrightarrow \mathrm{Br}_a(T^s),$$

and from this we derive the exact sequence

$$0 \longrightarrow \mathrm{Br}_a(X) \longrightarrow \mathrm{Br}_a(Z_1) \longrightarrow \mathrm{Br}_a(T^s)$$

over k and similar ones over all k_v; thus, also

$$0 \longrightarrow \text{Б}_S(X) \longrightarrow \text{Б}_S(Z_1) \longrightarrow \text{Б}_S(T^s).$$

Since $\text{Б}_\omega(T^s) = 0$ by [38, Prop. 6.9 (v)], it follows that for all S we have

$$\text{Б}_S(X) \simeq \text{Б}_S(Z_1), \quad \text{Б}_\omega(X) \simeq \text{Б}_\omega(Z_1), \quad \text{and also} \quad \text{Б}(X) \simeq \text{Б}(Z_1). \tag{5.5}$$

By Proposition A.1 in the appendix, applied to our case, we have the following exact sequence (T^u being k-split unipotent group):

$$0 = \mathrm{Pic}(T^u) \longrightarrow \mathrm{Br}_1(Z_1) \longrightarrow \mathrm{Br}_1(Z_0) \longrightarrow \mathrm{Br}_a(T^u),$$

and similarly as above, an exact sequence

$$0 \longrightarrow \mathrm{Br}_a(Z_1) \longrightarrow \mathrm{Br}_a(Z_0) \longrightarrow \mathrm{Br}_a(T^u),$$

and finally we have

(5.6) $Б_\omega(Z_0) \simeq Б_\omega(Z_1), \quad Б_S(Z_0) \simeq Б_S(Z_1), \ \forall S,$ and also $Б(Z_0) \simeq Б(Z_1).$

From these isomorphisms, the claim follows.

Now for Z_1 as above, we have surjective maps

$$\psi_V : \prod_v Z_0(k_v) \longrightarrow \prod_v Z_1(k_v),$$
$$\psi'_V : \prod_v Z_1(k_v) \longrightarrow \prod_v X(k_v);$$

hence, we have

$$\psi_V\Big(\Big(\prod_v Z_0(k_v)\Big)^{Б_\omega(Z_0)}\Big) = \Big(\prod_v Z_1(k_v)\Big)^{Б_\omega(Z_1)},$$
$$\psi'_V\Big(\Big(\prod_v Z_1(k_v)\Big)^{Б_\omega(Z_1)}\Big) = \Big(\prod_v X(k_v)\Big)^{Б_\omega(X)}$$

since the induced homomorphisms

$$\psi_* : Б_\omega(Z_1) \longrightarrow Б_\omega(Z_0) \quad \big(\text{resp. } \psi'_* : Б_\omega(X) \longrightarrow Б_\omega(Z_1)\big)$$

are isomorphisms by our assumption (resp. by the claim (5.5) above). Given that $(\prod_v X(k_v))^{Б_\omega(X)} \neq \emptyset$, clearly this implies that

$$\Big(\prod_v Z_0(k_v)\Big)^{Б_\omega(Z_0)} \neq \emptyset, \tag{5.7}$$

$$\Big(\prod_v Z_1(k_v)\Big)^{Б_\omega(Z_1)} \neq \emptyset. \tag{5.8}$$

Step 3. The rest of the proof. Now the rest of the proof follows the arguments given as in the Case 1. Namely, consider the quotient

$$\phi : Z_0 \to Y_0 := Z_0/F^{\mathrm{ss}}.$$

Then, naturally, Y_0 is a k-homogeneous space under the k-group $F/F^{\mathrm{ss}} = G^{\mathrm{tor}} \times T$. Due to injection $H^{\langle m\rangle} \hookrightarrow T$, we have $H \cap F^{\mathrm{ss}} = \{1\}$. By the same argument as in Case 1 above, we have isomorphisms

$$\text{(5.9)} \qquad Б_S(Z_0) \simeq Б_S(Y_0), \quad \forall \text{ finite set } S,$$

$$\text{(5.10)} \qquad Б_\omega(Z_0) \simeq Б_\omega(Y_0).$$

By (5.7), we have $(\prod_v Z_0(k_v))^{Б_\omega(Z_0)} \neq \emptyset$, so this implies that $(\prod_v Y_0(k_v))^{Б_\omega(Y_0)} \neq \emptyset$, too, due to (5.10). The k-variety Y_0 is a k-homogeneous space under the connected commutative k-group $G^{\mathrm{tor}} \times T$; hence, it is a torsor under a quotient k-group N of $G^{\mathrm{tor}} \times T$. N is a connected smooth commutative k-group, with the unipotent part N^u of N being the image of T^u, hence also k-split. Therefore, N^u has trivial Galois cohomology in degree 1; thus, Y_0 can be considered also a k-torsor under N^s, the k-toric part of N. Since the Brauer–Manin obstruction for the Hasse principle for torsors under connected reductive groups, in particular tori, is the only one ([38, Cor. 8.7] for number fields, and [45, Prop. 3.2.1 and Thm. 4.3.6], and [20, Thm. 2.5] for global function fields), this implies that $Y_0(k) \neq \emptyset$.

For a k-point $y \in Y_0(k)$, $Z_y := \phi^{-1}(y)$ is a homogeneous k-space under $F^{\mathrm{ss}} = G^{\mathrm{ss}}$ and with a stabilizer $H_1 := H \cap F^{\mathrm{ss}} = \{1\}$ by our assumption. Thus, Z_y is an F^{ss}-torsor, which is a trivial torsor since by assumption $F^{\mathrm{ss}} = G^{\mathrm{ss}}$ is simply connected and $\mathrm{H}^1(k, G^{\mathrm{ss}}) = 1$ by Harder (see [25, Hauptsatz], [26, Satz A], and [36, Ch. 6]). Thus, $Z_y(k) \neq \emptyset$, and hence so is $Z_0(k)$. Since $\varphi(Z_0(k)) \subseteq X(k)$, $X(k) \neq \emptyset$ as required. ∎

The following result extends [5, Cor. 2.5] (proved for number field case) to the case of global fields with no real places, in particular, global function fields. This contains no formulas for the obstruction to weak approximation like in Theorem 4.8, but it is more general than Theorem 4.8, regarding the Brauer–Manin obstruction.

THEOREM 5.3 (Cf. [5, Cor. 2.5] for number field case). *Let k be a global field with no real places, S a finite set of places of k, and X a homogeneous k-space under a connected reductive group G with simply connected semisimple part and with a smooth stabilizer H defined over k, such that $H^{(\mathrm{m})}$ is commutative. Then, the Brauer–Manin obstruction to the weak approximation in S (resp. over k) for X is the only one.*

PROOF. Assume that $X(k) \neq \emptyset$ and consider any finite set S of places. Take any point $x_S \in (\prod_{v\in S} X(k_v))^{Б_S(X)}$ and we will show that $x_S \in \overline{X(k)}$. We distinguish two cases.

Case 1. The inclusion $H \hookrightarrow G$ induces an injection $H^{(\mathrm{m})} \to G^{\mathrm{tor}}$. Then, we are back to Case 1 in the proof of Theorem 5.2, where we have shown that the k-variety

$Y := X/G^{\mathrm{ss}}$ is a k-torsor under G^{tor}, and $Y(k) \neq \emptyset$, since $X(k)$ is so. Then, it was shown by [38, Cors. 8.7, 8.13] for number fields and [45, Prop. 3.2.1] for any global fields that the Brauer–Manin obstruction to the weak approximation for torsors under connected reductive k-groups (in particular under tori) is the only one.

Case 2. General case. As in Case 2 in the proof of Theorem 5.2, there exists a smooth special commutative k-group $T = T^s \times T^u$, where T^s is an induced k-torus and T^u a k-split unipotent group, with a k-embedding $H^{\langle \mathrm{m} \rangle} \hookrightarrow T$. Then, we consider the k-group $F = G \times T$, the k-variety $Z = F/H$. As above, with the group T as in Case 2 in the proof of Theorem 5.2, we have $\mathrm{H}^1(k, T) = 1$, so again from the exact sequence

$$Z_0(k) \xrightarrow{\varphi_0} X(k) \longrightarrow \mathrm{H}^1(k, T)$$

and from $X(k) \neq \emptyset$ and the surjectivity of φ_0, it implies that $Z_0(k) \neq \emptyset$. As in the proof of Theorem 5.2, we have the following commutative diagram with φ_S being surjective:

$$\begin{tikzcd}
\prod_{v\in S} Z_0(k_v) \arrow[d, "\varphi_S"] & \times & \text{Б}_S(Z_0) \arrow[r] & \mathbf{Q}/\mathbf{Z} \arrow[d, leftrightarrow, "="] \\
\prod_{v\in S} X(k_v) & \times & \text{Б}_S(X) \arrow[u, "\varphi'_S", "\simeq"'] \arrow[r] & \mathbf{Q}/\mathbf{Z}.
\end{tikzcd}$$

Hence, we have

$$\varphi_S\Big(\Big(\prod_{v\in S} Z_0(k_v)\Big)^{\text{Б}_S(Z_0)}\Big) = \Big(\prod_{v\in S} X(k_v)\Big)^{\text{Б}_S(X)}.$$

Thus, there is $z_S \in (\prod_{v\in S} Z_0(k_v))^{\text{Б}_S(Z_0)}$ such that $\varphi_S(z_S) = x_S$. Since Z_0 is a homogeneous space under F and by our construction we have $H^{\langle \mathrm{m} \rangle} \hookrightarrow F/F^{\mathrm{ss}}$ (see Case 2 in the proof of Theorem 5.2), we have

$$z_S \in \overline{Z_0(k)}^S,$$

which implies that $x_S \in \overline{X(k)}$. In particular, the Brauer–Manin obstruction to the weak approximation in S is the only one.

The general case of weak approximation over k follows from this (which can be proceeded as at the end of the proof of Theorem 5.2). (We may proceed by using similar arguments just as in the proofs of Case 2 in the proof of Theorem 5.2 (general case) above (or [45, Prop. 4.3.3 and Thm. 4.3.6]). All the statements are valid and proved with the same argument given there, so to shorten the paper, we omit the long verification of the details.) (Alternatively, we may argue directly by taking the limit over S from the case when working with finite set S of places. I am thankful to the referee for this remark.) ∎

Remark 11. In [37, Thm. 1.9], under some conditions on the Tate–Shafarevich group and strong approximation for the k-subgroup H of G, it has been shown that the Brauer–Manin obstruction to the strong approximation for $X := G/H$ is the only one; cf. also [19, Cor. 6.3] and [20] for the case of weak approximation over global function fields, where H is assumed connected.

5.1. Relations with torsion primes

For a connected reductive group G, let $F(G)$ be the fundamental group of G, i.e., the fundamental group of the semisimple part G' of G. Namely, if $\pi : \widetilde{G} \to G'$ is the central isogeny from the semisimple simply connected covering $\widetilde{G}$ of G', then $F(G) := \mathrm{Ker}(\pi)$.

We may write

$$F(G) = F_s(G) \times F_p(G),$$

where $F_s(G)$ and $F_p(G)$ denote the separable (i.e., smooth) and purely inseparable part of $F(G)$, respectively. Thus, if char. $k = p$, then over k_s there are isomorphisms

$$F_s(G) \simeq \prod_i \mu_{m_i}, \quad F_p(G) \simeq \prod_j \mu_{p^{n_j}},$$

where μ_n denotes the standard finite group scheme of n-roots of unity and $(p, m_i) = 1$ for all i.

A prime l is called a *torsion prime for* G (cf. [41, Sec. 2]) if for some regular semisimple subgroup K of G, the fundamental group $F(K)$ of K has l-torsion. Denote by $P(G)$ the set of all torsion primes for G.

Recall that the set $P(G)$ may also be defined as follows. If Φ is a root system for G, $L(\Phi)$ denotes the root lattice for Φ, and Φ^* denotes the dual root system. A prime l is called a *torsion prime for* Φ if $L(\Phi^*)/L(\Phi_1^*)$ has l-torsion for some closed subsystem $\Phi_1 \subseteq \Phi$ and we denote by $P(\Phi)$ the torsion primes for Φ. Then, by [41, Lem. 2.5], we have

$$P(G) = P\big(\Phi(G)\big) \cup P\big(F(G)\big),$$

where $P(F(G))$ denotes the set of torsion primes for $F(G)$.

Assume that H is a subgroup of multiplicative type of G. Consider the decomposition of $H/H^\circ(H \cap Z(G))$ into a product of finite cyclic groups:

$$H/H^\circ\big(H \cap Z(G)\big) = A \times B,$$

where A (resp. B) is the product of all cyclic groups of torsion in common (resp. not in common) with G. Let $a(H)$ be the number of cyclic groups appearing in A.

Proposition 5.4. *Let k be a global field with no real places, S a finite set of places of k, and X a homogeneous k-space under a connected reductive group G with a stabilizer H such that $H^{(\mathrm{m})}$ is commutative. Assume that one of the following holds:*

(1) *There exists a central extension*

$$1 \longrightarrow F \longrightarrow H_1 \xrightarrow{\pi} G \longrightarrow 1$$

of G such that H_1 is a connected reductive group with H_1^{ss} simply connected and $\pi^{-1}(H)^{(\mathrm{m})}$ is commutative;

(2) *H is a group of multiplicative type such that $a(H) \leq 1$;*

(3) *H° is a regular torus in G.*

Then, the Brauer–Manin obstruction to the Hasse principle and the weak approximation in S (resp. over k) for X are the only ones.

Proof. In case (2), by [41, Cor. 2.25], H is contained in a torus T of G; therefore, it is contained in a maximal k-torus T' of G. The preimage $\pi^{-1}(T')$ is then a maximal k-torus of H_1; thus, it satisfies all the assumptions of (1), so (2) implies (1). In case (3), it is known that if H° is a regular torus of G (i.e., if it contains a regular element of G), then $T := Z_G(H^\circ)$ is a maximal torus of G by [28, Sec. 26.2, Cor. A]. In particular, we have $H \subseteq T$ and again (3) implies (1).

To prove the assertion under the assumption (1), for the central extension

$$1 \longrightarrow F \longrightarrow H_1 \xrightarrow{\pi} G \longrightarrow 1,$$

the preimage $\pi^{-1}(H)$ is a subgroup of multiplicative type of H_1. We also have

$$H_1/\pi^{-1}(H) \xrightarrow{\simeq} G/H,$$

so we may reduce everything to the case, where G^{ss} is simply connected and the assertion now follows from Theorems 5.2 and 5.3. ∎

Appendix. Picard and Brauer groups of a fibration under a smooth connected affine group scheme

The purpose of this appendix is to present a statement, which is used in the proof of Theorems 5.2 and 5.3. It is a slight extension (to the case we need) of one important result due to J.-J. Sansuc [38, Prop. 6.10, Cor. 6.11]. For a k-variety X, denote $U(X) := k[X]^*/k^*$.

Proposition A.1 (Cf. [38, Prop. 6.10]). *Let k be a field, G a connected smooth affine k-group scheme, such that $G = G^{\mathrm{red}} \cdot R_{\mathrm{u}}(G)$ is a semi-direct product over k of the reductive part of G with k-split unipotent radical $R_{\mathrm{u}}(G)$. Let $\pi : X \to Y$ be a Y-torsor under G. Then, one has a natural exact sequence of abelian groups*

$$\begin{aligned} 1 &\longrightarrow U(Y) \longrightarrow U(X) \longrightarrow X^*(G)_k \longrightarrow \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\varphi} \mathrm{Pic}(G) \\ &\longrightarrow \mathrm{Br}(Y) \longrightarrow \mathrm{Br}(X). \end{aligned}$$

If π has (locally) a section with respect to the Zariski topology, one has the exact sequence

$$1 \longrightarrow U(Y) \longrightarrow U(X) \longrightarrow X^*(G)_k \longrightarrow \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(G) \longrightarrow 1.$$

If $\varphi_s : \mathrm{Pic}(X_s) \to \mathrm{Pic}(G_s)$ is surjective, e.g., if $\mathrm{Pic}(G_s) = 1$, or $\pi_s : X_s \to Y_s$ locally has a section with respect to the Zariski topology, then one has the exact sequence

$$\begin{aligned} 1 &\longrightarrow U(Y) \longrightarrow U(X) \longrightarrow X^*(G)_k \longrightarrow \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\varphi} \mathrm{Pic}(G) \\ &\longrightarrow \mathrm{Br}_1(Y) \longrightarrow \mathrm{Br}_1(X) \longrightarrow \mathrm{Br}_e(G). \end{aligned}$$

Then, one derives the following.

Corollary A.2 (See [38, Cor. 6.11]). *Let $1 \to G' \to G \to G'' \to 1$ be an exact sequence of smooth connected affine k-groups, where $G' = G'^{\mathrm{red}} R_{\mathrm{u}}(G')$ is a semi-direct product over k of the reductive part of G' with k-split unipotent radical. Then, we have the natural exact sequences*

$$\begin{aligned} 1 &\longrightarrow X^*(G'')_k \longrightarrow X^*(G)_k \longrightarrow X^*(G')_k \longrightarrow \mathrm{Pic}(G'') \longrightarrow \mathrm{Pic}(G) \\ &\longrightarrow \mathrm{Pic}(G') \longrightarrow \mathrm{Br}(G'') \longrightarrow \mathrm{Br}(G) \longrightarrow \mathrm{Br}(G'), \\ 1 &\longrightarrow X^*(G'')_k \longrightarrow X^*(G)_k \longrightarrow X^*(G')_k \longrightarrow \mathrm{Pic}(G'') \longrightarrow \mathrm{Pic}(G) \\ &\longrightarrow \mathrm{Pic}(G') \longrightarrow \mathrm{Br}_a(G'') \longrightarrow \mathrm{Br}_a(G) \longrightarrow \mathrm{Br}_a(G') \longrightarrow \mathrm{Br}_e(G). \end{aligned}$$

Remark A.3. (1) Sansuc's original result in Proposition A.1 was stated under the assumption that if the field k is *non-perfect*, then G should be reductive.

(2) The condition on the unipotent radical of G is automatically fulfilled when $\mathrm{char}.\, k = 0$ by a classical result of G. Mostow.

Proof of Proposition A.1 and Corollary A.2. The same proof given in [38] goes through verbatim. The only place where we have to assume the splitness of the semi-product $G = G^{\mathrm{red}} \cdot R_{\mathrm{u}}(G)$ and that of $R_{\mathrm{u}}(G)$ is where we use [38, Lem. 6.6]. (The pathological behavior of unipotent radicals shows that the assumption on the splitness is perhaps the best one for the proposition to hold.) ∎

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