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Differential Equations. – *Natural annihilators and operators of constant rank over* $\mathbb{C}$, by FRANZ GMEINEDER and STEFAN SCHIFFER, communicated on 8 November 2024.

ABSTRACT. – Even if the Fourier symbols of two constant rank differential operators have the same nullspace for each non-trivial phase space variable, the nullspaces of those differential operators might differ by an infinite-dimensional space. Under the natural condition of constant rank over $\mathbb{C}$, we establish that the equality of nullspaces on the Fourier symbol level already implies the equality of the nullspaces of the differential operators in $\mathcal{D}'$ modulo polynomials of a fixed degree. In particular, this condition allows one to speak of *natural annihilators* within the framework of complexes of differential operators.

KEYWORDS. – linear partial differential operators, constant rank operators, elliptic operators, Poincaré lemma.

MATHEMATICS SUBJECT CLASSIFICATION 2020. – 35G35 (primary); 13N10, 35G05 (secondary).

1. INTRODUCTION

1.1. Aim and scope

Let V, W, X be three real, finite-dimensional inner product spaces and let, for $k, \ell \in \mathbb{N}$,

$$\mathbb{A} := \sum_{|\alpha|=k} \mathbb{A}_\alpha \partial^\alpha, \quad \mathbb{B} := \sum_{|\beta|=\ell} \mathbb{B}_\beta \partial^\beta \tag{1.1}$$

be two constant coefficient differential operators on $\mathbb{R}^n$ from V to W or from W to X, respectively. By this we understand that for each $|\alpha| = k$ and $|\beta| = \ell$, we have $\mathbb{A}_\alpha \in \mathcal{L}(V; W)$ or $\mathbb{B}_\beta \in \mathcal{L}(W; X)$.

For instance, this setting comprises the usual gradient Du for maps $u: \mathbb{R}^n \to \mathbb{R}^N$ or the symmetric gradient $\varepsilon(u) := \frac{1}{2}(Du + Du^\top)$ for maps $u: \mathbb{R}^n \to \mathbb{R}^n$ as frequently employed in nonlinear elasticity; these can be recovered by the particular choices $(V, W) = (\mathbb{R}^N, \mathbb{R}^{N\times n})$ or $(V, W) = (\mathbb{R}^n, \mathbb{R}^{n\times n}_{\mathrm{sym}})$, respectively. To describe the main question of the present paper, note that

$$\mathrm{C}^\infty(\mathbb{R}^n; \mathbb{R}^N) \xrightarrow{D} \mathrm{C}^\infty(\mathbb{R}^n; \mathbb{R}^{N\times n}) \xrightarrow{\mathrm{curl}} \mathrm{C}^\infty(\mathbb{R}^n; \mathbb{R}^{N\times n}),$$

$$\mathrm{C}^\infty(\mathbb{R}^n; \mathbb{R}^n) \xrightarrow{\varepsilon} \mathrm{C}^\infty(\mathbb{R}^n; \mathbb{R}^{n\times n}_{\mathrm{sym}}) \xrightarrow{\mathrm{curlcurl}^\top} \mathrm{C}^\infty(\mathbb{R}^n; \mathbb{R}^d),$$

for suitable $d \in \mathbb{N}$, are sequences that are exact at the corresponding mid point vector spaces. Here, we have set for $u = (u_{jk})_{1\le j\le N,\ 1\le k\le n}$ and $v = (v_{jk})_{1\le j,k\le n}$

$$
\begin{aligned}
\mathrm{curl}(u) &= (\partial_k u_{ji} - \partial_i u_{jk})_{ijk},\\
\mathrm{curlcurl}^\top(v) &= (\partial_{ij} v_{kl} + \partial_{kl} v_{ij} - \partial_{il} v_{kj} - \partial_{kj} v_{il})_{ijkl}.
\end{aligned}
\tag{1.2}
$$

The first example is the usual gradient-curl-complex, whereas the second one is referred to as the Saint–Venant compatibility complex (see, e.g., [5]). In the language of Fourier analysis, this circumstance can be restated by the associated symbol complex

$$V \xrightarrow{\mathbb{A}[\xi]} W \xrightarrow{\mathbb{B}[\xi]} X \quad \text{being exact at } W \text{ for all } \xi \in \mathbb{R}^n \setminus \{0\}$$

for the corresponding choices $(\mathbb{A},\mathbb{B}) = (D,\mathrm{curl})$ or $(\mathbb{A},\mathbb{B}) = (\varepsilon,\mathrm{curlcurl}^\top)$, respectively. This means that for each $\xi \in \mathbb{R}^n \setminus \{0\}$, we have $\mathbb{A}[\xi](V) = \ker(\mathbb{B}[\xi])$, where

$$\mathbb{A}[\xi] = \sum_{|\alpha|=k} \xi^\alpha \mathbb{A}_\alpha, \quad \mathbb{B}[\xi] = \sum_{|\beta|=\ell} \xi^\beta \mathbb{B}_\beta, \quad \xi \in \mathbb{R}^n. \tag{1.3}$$

In this situation, we call $\mathbb{A}$ a *potential* of $\mathbb{B}$, and $\mathbb{B}$ an *annihilator* of $\mathbb{A}$. Annihilators are far from being uniquely determined: For instance, letting Δ be the usual Laplacian, each $\Delta^j\mathbb{B}$ ($j \in \mathbb{N}$) satisfies $\ker(\Delta^j\mathbb{B}[\xi]) = \ker(|\xi|^{2j}\mathbb{B}[\xi]) = \ker(\mathbb{B}[\xi])$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$. Still, in the above examples with $(\mathbb{A},\mathbb{B}) = (D,\mathrm{curl})$ or $(\mathbb{A},\mathbb{B}) = (\varepsilon,\mathrm{curlcurl}^\top)$, the nullspaces of $\mathbb{B}$ and $\Delta^j\mathbb{B}$ differ by an infinite-dimensional vector space. Thus, denoting the class of annihilators of a given differential operator $\mathbb{A}$ by

$$\mathrm{An}(\mathbb{A}) := \left\{ \mathbb{B} : \begin{array}{l} \mathbb{B} \text{ is of the form (1.3) for some vector space } X, \\ \ker\big(\mathbb{B}[\xi]\big) = \mathbb{A}[\xi](V) \text{ for all } \xi \in \mathbb{R}^n \setminus \{0\} \end{array} \right\}, \tag{1.4}$$

it is logical to ask for a subset $\mathscr{C} \subset \mathrm{An}(\mathbb{A})$ with the property that the distributional nullspaces of $\mathbb{B},\mathbb{B}' \in \mathscr{C}$ only differ by a finite-dimensional vector space each, and under which conditions on $\mathbb{A}$ the class $\mathscr{C}$ is non-empty.

1.2. Operators with constant rank over $\mathbb{C}$

We first recall some terminology that is customary in the above context. Following the works of Schulenberger and Wilcox [23] and Murat [19] (also see Fonseca and Müller [10]), operators $\mathbb{A}$ and $\mathbb{B}$ of the form (1.3) are said to be of *constant rank (over* $\mathbb{R}$*)* provided $\dim_{\mathbb{R}}(\mathbb{A}[\xi](V))$ or $\dim_{\mathbb{R}}(\mathbb{B}[\xi](W))$, respectively, are independent of the phase space variable $\xi \in \mathbb{R}^n \setminus \{0\}$. By Raiţă [20] (also see [2]), every constant rank operator possesses a constant rank potential.

Towards the above question from (1.4)ff., a strengthening of the notion of constant rank is required.

DEFINITION 1.1 (Constant rank over $\mathbb{C}$). Let $\mathbb{B}$ be a differential operator as in (1.1). We say that $\mathbb{B}$ has *constant rank over* $\mathbb{C}$ provided

$$\dim_{\mathbb{C}}\big(\mathbb{B}[\xi](V+\mathrm{i}\,V)\big)\quad\text{is independent of } \xi\in\mathbb{C}^n\setminus\{0\}. \tag{1.5}$$

For (1.5), note that whenever a complex phase variable $\xi=\mathrm{Re}(\xi)+\mathrm{i}\,\mathrm{Im}(\xi)$ is inserted into $\mathbb{B}[\xi]$, it consequently gives rise to a linear map $\mathbb{B}[\xi]\colon V+\mathrm{i}\,V\to W+\mathrm{i}\,W$. Similarly, as the constant rank operators generalise the notion of (overdetermined real) elliptic differential operators $\mathbb{A}$ á la Hörmander and Spencer [16, 25], operators of constant rank over $\mathbb{C}$ generalise the concept of $\mathbb{C}$-elliptic operators in the spirit of Smith [24] (also see [4, 13, 17]). Here, an operator $\mathbb{A}$ is called *(real or $\mathbb{R}$-) elliptic* provided $\mathbb{A}[\xi]\colon V\to W$ is injective for all $\xi\in\mathbb{R}^n\setminus\{0\}$, and *$\mathbb{C}$-elliptic* provided $\mathbb{A}[\xi]\colon V+\mathrm{i}\,V\to W+\mathrm{i}\,W$ is injective for all $\xi\in\mathbb{C}^n\setminus\{0\}$. Adopting the terminology of Definition 1.1, the main result of the present paper is as follows.

THEOREM 1.2. *Let $\mathbb{B}$, $\widetilde{\mathbb{B}}$ be two differential operators with constant rank over $\mathbb{C}$. Then, the following are equivalent:*

(a) *For all $\xi\in\mathbb{C}^n\setminus\{0\}$, we have*

$$\ker\big(\mathbb{B}[\xi]\big)=\ker\big(\widetilde{\mathbb{B}}[\xi]\big).$$

(b) *There exist two finite-dimensional vector subspaces $\mathcal{X}_1,\mathcal{X}_2$ of the W-valued polynomials on $\mathbb{R}^n$ such that*

$$\ker(\mathbb{B})+\mathcal{X}_1=\ker(\widetilde{\mathbb{B}})+\mathcal{X}_2, \tag{1.6}$$

where ker *is understood as the nullspace in $\mathcal{D}'(\mathbb{R}^n;W)$, so e.g.*

$$\ker(\mathbb{B})=\big\{T\in\mathcal{D}'(\mathbb{R}^n;W)\colon\ \mathbb{B}T=0\big\}.$$

Let us note that if the Fourier symbols $\mathbb{B}[\xi]$ and $\widetilde{\mathbb{B}}[\xi]$ have the same nullspace for any ξ, then they are both annihilators of some differential operator $\mathbb{A}$ with constant rank in $\mathbb{C}$. Also note that the statement of Theorem 1.2 is false if we drop the assumption that $\mathbb{B}$ and $\widetilde{\mathbb{B}}$ satisfy the constant rank property over $\mathbb{C}$ (cf. Example 4.3).

In the language of algebraic geometry, the proof of Theorem 1.2 relies on a vectorial Nullstellensatz to be stated and established in Section 3 below. Nullstellensatz techniques have been employed in slightly different contexts (see [14, 17, 24]). However, these by now routine applications to differential operators (to be revisited in detail in Section 3) do not prove sufficient to establish Theorem 1.2.

If a differential operator $\mathbb{A}$ has an annihilator $\mathbb{B}$ of constant complex rank, this annihilator is in some sense minimal when being compared with other annihilators (so e.g. $\mathbb{D}\circ\mathbb{B}$ for (real) elliptic operators $\mathbb{D}$ on $\mathbb{R}^n$ from X to some finite-dimensional real vector space Y). Thus, annihilators of constant complex rank – provided existent – are *natural*. Even though the condition of constant rank over $\mathbb{C}$ might seem quite restrictive, it is satisfied for a wealth of operators to be gathered below. Following the discussion in [3, §6] (also see [12, 21]), it is also this class of operators for which one expects truncation theorems that play, e.g., a role in plasticity problems.

We wish to point out that when preparing this note, we became aware that the above theorem can also be established as a consequence of Härkönen, Niklasson and Raita [15, Thm. 1.2]. In particular, [15] makes use of a decomposition

$$\ker_{C^\infty} \mathbb{A} = \ker_{C^\infty} \mathbb{A}_c + \ker_{C^\infty} \mathbb{A}_u;$$

that holds for any operator $\mathbb{A}$. Within our setting of complex constant rank $\mathbb{A}_u$ can be shown to be complex elliptic. To arrive at this decomposition requires some deeper tools from commutative algebra. Instead, our proof only hinges on the Hilbert Nullstellensatz and elementary linear algebra.

1.3. *Organisation of the document*

Apart from this introductory section, the paper is organised as follows. In Section 2, we gather examples of operators arising in applications that verify the constant rank condition over $\mathbb{C}$. Section 3 then is devoted to a suitable variant of a vectorial Nullstellensatz, which displays the pivotal step in the proof of Theorem 1.2 in Section 4.

1.4. *Notation*

For $k \in \mathbb{N}$, we denote by $\mathcal{P}_k(\mathbb{R}^n;\mathbb{R}^d)$ the $\mathbb{R}^d$-valued polynomials on $\mathbb{R}^n$ of degree at most k; the space of $\mathbb{R}^d$-valued polynomials p on $\mathbb{R}^n$ which are homogeneous of degree k, so satisfying $p(\lambda x) = \lambda^k p(x)$ for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, is denoted as $\mathcal{P}_k^h(\mathbb{R}^n;\mathbb{R}^d)$. Moreover, given a ring $\mathcal{R}$, we use the convention $I \trianglelefteq \mathcal{R}$ to express that I is an ideal in $\mathcal{R}$.

2. (Non-)Examples of operators of constant rank over $\mathbb{C}$

In this section, we discuss some (non-)examples that satisfy the algebraic condition of constant rank over $\mathbb{C}$ from Definition 1.1 and arise frequently in applications.

Example 2.1 ($\mathbb{C}$-elliptic operators). $\mathbb{C}$-ellipticity of an operator $\mathbb{A}$ of the form (1.3) means that

$$\mathbb{A}[\xi]\colon V + \mathrm{i}\,V \to W + \mathrm{i}\,W$$

is injective for any $\xi \in \mathbb{C}^n \setminus \{0\}$. Such operators have constant rank over $\mathbb{C}$ by definition; trivially, the usual k-th order gradients are $\mathbb{C}$-elliptic. As discussed e.g. in [4, Ex. 2.2], the symmetric gradient $\varepsilon(u) := (\frac{1}{2}(\partial_i u_j + \partial_j u_i))_{ij}$ for maps $u\colon \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbb{C}$-elliptic for $n \geq 2$, and so is the trace-free gradient

$$\varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}\operatorname{div}(u)E_n$$

with the $(n \times n)$-unit matrix E_n provided $n \geq 3$. These operators play a crucial role in elasticity, plasticity or fluid mechanics; see, e.g., [11].

Example 2.2 (The curl- and curlcurl$^\top$-operator). Given $n \geq 2$ and $u\colon \mathbb{R}^n \to \mathbb{R}^{N\times n}$, we define $\operatorname{curl}(u)$ as in the introduction. Note that, for $v \in \mathbb{C}^{N\times n}$, $\operatorname{curl}[\xi](v) = 0$ for $\xi \in \mathbb{C}^n \setminus \{0\}$ if and only if $v = a\xi^\top$ for some $a \in \mathbb{C}^N$, so $\dim_{\mathbb{C}}(\ker(\operatorname{curl}[\xi])) = N$. Similarly, for the Saint–Venant-compatibility complex, one explicitly verifies that $\operatorname{curlcurl}^\top[\xi](v) = 0$ if and only if $v = a \odot \xi = \frac{1}{2}(a\xi^\top + \xi a^\top)$ for some $a \in \mathbb{C}^n$. Thus, $\dim_{\mathbb{C}}(\ker(\operatorname{curlcurl}^\top[\xi])) = n$, and the validity of the constant rank property follows.

Example 2.3 (Divergence-type operators). For $n \geq 2$ and $u = (u_1, \dots, u_n)\colon \mathbb{R}^n \to \mathbb{R}^n$, the divergence $\operatorname{div}(u) = \sum_{i=1}^n \partial_i u_i$ has symbol $\operatorname{div}[\xi](v) := \sum_{i=1}^n \xi_i v_i$. Therefore, with $\xi \in \mathbb{C}^n \setminus \{0\}$, we have $\sum_{i=1}^n \xi_i v_i = 0$ provided $v \in \xi^\perp$, and thus

$$\dim_{\mathbb{C}}\big(\ker\big(\operatorname{div}[\xi]\big)\big) = n - 1.$$

Hence, div is of constant complex rank. An operator that arises in the relaxation of static problems, cf. [6], is the divergence of symmetric matrices; the same argument as above establishes that the divergence of symmetric matrices is of constant complex rank.

Example 2.4 (The Laplacian). The (scalar) Laplacian $\mathbb{B} = \Delta$ does not satisfy the constant rank condition over $\mathbb{C}$. For instance, let $n = 2$. Writing $\xi = (\xi_1, \xi_2)^\top \in \mathbb{C}^2$, the relevant symbol in view of Definition 1.1 is

$$\mathbb{B}[\xi] = \xi^\top \xi = \xi_1^2 + \xi_2^2. \tag{2.1}$$

The polynomial given by (2.1) vanishes if and only if $\xi \in \mathbb{C}(1, \mathrm{i})^\top$ or $\xi \in \mathbb{C}(1, -\mathrm{i})^\top$, and so

$$\ker_{\mathbb{C}}\big(\mathbb{B}[\xi]\big) = \begin{cases} \mathbb{C} & \text{if } \xi = \lambda(1,\mathrm{i})^\top \text{ or } \xi = \lambda(1,-\mathrm{i})^\top,\ \lambda \in \mathbb{C} \\ \{0\} & \text{otherwise,} \end{cases}$$

and so the constant rank condition is violated over $\mathbb{C}$; still, over the base field $\mathbb{R}$, the Laplacian is elliptic and hence of constant rank over $\mathbb{R}$. Likewise, *any* real elliptic operator $\mathbb{B}\colon \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; V)$ cannot be complex elliptic by a projective version of the fundamental theorem of algebra.

3. A Nullstellensatz for operators of constant complex rank

The proof of Theorem 1.2 hinges on a variant of the Hilbert Nullstellensatz from algebraic geometry stated in Theorem 3.2 below. For the reader's convenience, let us first display a classical version of the Hilbert Nullstellensatz as a background tool, which may e.g. be found in [8, §4.1, Thm. 2].

Lemma 3.1 (HNS). *Let $\mathbb{F}$ be an algebraically closed field and $\mathfrak{A} \trianglelefteq \mathbb{F}[X_1, \dots, X_n]$ an ideal. Then, we have $\sqrt{\mathfrak{A}} = \mathcal{I}(V(\mathfrak{A}))$, where*

- $\sqrt{\mathfrak{A}} := \{\mathbf{x} \in \mathbb{F}[X_1, \dots, X_n] \colon \exists m \in \mathbb{N}_0 \colon \mathbf{x}^m \in \mathfrak{A}\}$ *is the* radical of $\mathfrak{A}$,
- $V(\mathfrak{A}) := \{x = (x_1, \dots, x_n) \in \mathbb{F}^n \colon \forall \mathbf{x} \in \mathfrak{A} \colon \mathbf{x}(x) = 0\}$ *is the* set of common zeros of $\mathfrak{A}$, *and*
- $\mathcal{I}(V(\mathfrak{A})) := \{\mathbf{x} \in \mathbb{F}[X_1, \dots, X_n] \colon \forall x \in V(\mathfrak{A}) \colon \mathbf{x}(x) = 0\}$.

The standard use of this result in the context of differential operators (see Remark 3.4 below) does not prove sufficient for Theorem 1.2. Hence, let $d, k, l \in \mathbb{N}$. For $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, l\}$, we consider homogeneous polynomials $p_{ij} \in \mathbb{C}[\xi_1, \dots, \xi_n]$ of order k and the system of equations

$$v^\intercal p(\xi)_j := \sum_{i=1}^{d} p_{ij}(\xi) v_i = 0, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n, \; j \in \{1, \dots, l\}, \tag{3.1}$$

where $v = (v_1, \dots, v_d) \in \mathbb{C}^d$. In accordance with Definition 1.1, we say that the system (3.1) satisfies the *constant rank property over* $\mathbb{C}$ if there exists an $r \in \{0, \dots, d\}$ such that for every $\xi \in \mathbb{C}^n \setminus \{0\}$ the vector space

$$\mathcal{X}_\xi\big((p_{ij})_{ij}\big) := \big\{v = (v_1, \dots, v_d) \in \mathbb{C}^d \colon v^\intercal p(\xi) = 0 \text{ for all } j \in \{1, \dots, l\}\big\}$$

has dimension $(d - r)$ over $\mathbb{C}$. We may now state the main ingredient for the proof of Theorem 1.2, which arises as a generalisation of the usual Hilbert Nullstellensatz.

Theorem 3.2 (Vectorial Nullstellensatz for constant rank operators). *Let $d, k, l \in \mathbb{N}$ and, for $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, l\}$, let $p_{ij} \in \mathbb{C}[\xi_1, \dots, \xi_n]$ be homogeneous polynomials of degree k such that (3.1) satisfies the* constant rank property over $\mathbb{C}$. *Let $b_1, \dots, b_d \in \mathbb{C}[\xi_1, \dots, \xi_n]$, $v = (v_1, \dots, v_d) \in \mathbb{C}^d$ and define*

$$B[\xi](v) := \sum_{i=1}^{d} v_i b_i(\xi) = v^\top b(\xi).$$

Suppose that for any $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n \setminus \{0\}$ and $v = (v_1, \dots, v_d) \in \mathbb{C}^d$ we have that

$$v^\intercal p(\xi) = 0 \implies B[\xi](v) = 0, \tag{3.2}$$

and let $q \in \mathbb{C}[\xi_1, \ldots, \xi_n]$ be a homogeneous polynomial of degree ≥ 1. Then, there exist polynomials $h_j \in \mathbb{C}[\xi_1, \ldots, \xi_n]$, $j \in \{1, \ldots, l\}$, and an $m \in \mathbb{N}$, such that for all $\xi \in \mathbb{C}^n$ and all $v \in \mathbb{C}^d$, there holds

$$q^m(\xi)B[\xi](v) = h^\mathsf{T}(\xi)p(\xi)v. \tag{3.3}$$

REMARK 3.3. This Nullstellensatz is a generalisation of the *elliptic* Nullstellensatz (cf. [24]) in the following way. There, it is assumed that the operator $B = \mathrm{Id}$ is the identity operator on V. Then, assumption (3.2) corresponds to $v^\mathsf{T}p(\xi) = 0 \Rightarrow v = 0$, i.e. to the assumption that p is complex elliptic.

PROOF. Let the polynomials p_{ij} satisfy the constant rank property for some fixed $r \in \{0, \ldots, d\}$. We define sets

$$\mathcal{J} = \{J \subset \{1, \ldots, l\} : |J| = r\}, \quad \mathcal{I} = \{I \subset \{1, \ldots, d\} : |I| = r\}.$$

For a subset $J \in \mathcal{J}$, we write $J = \{j(1), \ldots, j(r)\}$ for $j(1) < \cdots < j(r)$ and likewise for $I \in \mathcal{I}$, $I = \{i(1), \ldots, i(r)\}$ for $i(1) < \cdots < i(r)$. Define the matrix $M_{IJ} \in \mathbb{C}^{r \times r}$ by its entries via

$$(M_{IJ})_{\beta\gamma} := p_{i(\beta),j(\gamma)}.$$

Now consider an arbitrary $(r \times r)$-minor of $P(\xi) = (p_{ij}(\xi))_{ij}$; any such minor arises as $\det(M_{IJ}(\xi))$ for some $I \in \mathcal{I}$, $J \in \mathcal{J}$. If $\xi \in \mathbb{C}^n \setminus \{0\}$ is a common zero of all $q_{IJ} := \det(M_{IJ})$, then

$$\dim_\mathbb{C} \left(\mathcal{X}_\xi\big((p_{ij})_{ij}\big)\right) \neq d - r$$

by virtue of the constant rank property over $\mathbb{C}$. On the other hand, by the homogeneity of the p_{ij}'s, $\xi = 0$ is a common zero of the q_{IJ}'s, and so is the only common zero of the q_{IJ}'s.

On the other hand, $\xi = 0$ is a zero of any homogeneous polynomial $q \in \mathbb{C}[\xi_1, \ldots, \xi_n]$ of degree ≥ 1. Thus, the Hilbert Nullstellensatz from Lemma 3.1 implies the existence of an $m \in \mathbb{N}$ and polynomials $g_{IJ} \in \mathbb{C}[\xi_1, \ldots, \xi_n]$ ($I \in \mathcal{I}$, $J \in \mathcal{J}$) such that

$$q^m = \sum_{J \in \mathcal{J}} \sum_{I \in \mathcal{I}} g_{IJ} \det(M_{IJ}). \tag{3.4}$$

We now come to the definition of h_j as appearing in (3.3). For the matrix M_{IJ} and $\gamma \in \{1, \ldots, r\}$, we define the matrix M_{IJ}^γ as the matrix where the γ-th column vector is replaced by $(b_{i(\beta)})_{\beta=1,\ldots,r}$; i.e.,

$$M_{IJ}^\gamma = \begin{pmatrix} p_{i(1)j(1)} & \cdots & p_{i(1)j(\gamma-1)} & b_{i(1)} & p_{i(1)j(\gamma+1)} & \cdots & p_{i(1)j(r)} \\ \cdots & & \cdots & \cdots & \cdots & & \cdots \\ p_{i(r)j(1)} & \cdots & p_{i(r)j(\gamma-1)} & b_{i(r)} & p_{i(r)j(\gamma+1)} & \cdots & p_{i(r)j(r)} \end{pmatrix}.$$

We then define for $j \in \{1, \ldots, l\}$

$$h_j := \sum_{\gamma=1}^{r} \sum_{I \in \mathcal{I}} \sum_{J \in \mathcal{J} : j(\gamma)=j} g_{IJ} \det(M_{IJ}^{\gamma}) \tag{3.5}$$

and claim that

$$\sum_{\gamma=1}^{r} p_{ij(\gamma)} \det(M_{IJ}^{\gamma}) = b_i \det M_{IJ} \quad \text{for all } i \in \{1, \ldots, d\}, \tag{3.6}$$

$$\sum_{j=1}^{l} h_j \Big(\sum_{i=1}^{d} p_{ij} v_i \Big) = q^m \sum_{i=1}^{d} b_i v_i, \tag{3.7}$$

so that the h_j's will satisfy (3.3). Let us see how (3.7) follows from (3.6): In fact,

$$\begin{aligned} \sum_{j=1}^{l} h_j \Big(\sum_{i=1}^{d} p_{ij} v_i \Big) &\overset{(3.5)}{=} \sum_{j=1}^{l} \sum_{i=1}^{d} \sum_{\gamma=1}^{r} \sum_{I \in \mathcal{I}} \sum_{J \in \mathcal{J} : j(\gamma)=j} g_{IJ} \det(M_{IJ}^{\gamma}) p_{ij} v_i \\ &= \sum_{J \in \mathcal{J}} \sum_{I \in \mathcal{I}} g_{IJ} \Big(\sum_{i=1}^{d} \sum_{\gamma=1}^{r} p_{ij(\gamma)} \det(M_{IJ}^{\gamma}) v_i \Big) \\ &\overset{(3.6)}{=} \sum_{J \in \mathcal{J}} \sum_{I \in \mathcal{I}} g_{IJ} \det(M_{IJ}) \cdot \Big(\sum_{i=1}^{d} b_i v_i \Big) \\ &\overset{(3.4)}{=} q^m \sum_{i=1}^{d} b_i v_i. \end{aligned}$$

Hence, it remains to show (3.6). To this end, for $\beta, \gamma \in \{1, \ldots, r\}$, let us define the matrix $M_{I(\beta)J(\gamma)}$ as the $(r-1) \times (r-1)$ matrix, where the γ-th column of M_{IJ} and the β-th row have been removed. By the Laplace expansion formula and the definition of M_{IJ}^{γ}, we then obtain

$$\det(M_{IJ}^{\gamma}) = \sum_{\beta=1}^{r} (-1)^{\beta+\gamma} b_{i(\beta)} \det(M_{I(\beta)J(\gamma)}).$$

Hence,

$$\sum_{\gamma=1}^{r} p_{ij(\gamma)} \det(M_{IJ}^{\gamma}) = \sum_{\beta,\gamma=1}^{r} (-1)^{\beta+\gamma} b_{i(\beta)} \det(M_{I(\beta)J(\gamma)}) p_{ij(\gamma)}. \tag{3.8}$$

Now consider the $(r+1) \times (r+1)$-matrix M defined by

$$M := \begin{pmatrix} p_{i(1)j(1)} & \cdots & p_{i(1)j(r)} & b_{i(1)} \\ \vdots & \ddots & \vdots & \vdots \\ p_{i(r)j(1)} & \cdots & p_{i(r)j(r)} & b_{i(r)} \\ p_{ij(1)} & \cdots & p_{ij(r)} & b_i \end{pmatrix}.$$

By (3.2), for each $\xi \in \mathbb{C}^n \setminus \{0\}$, the subspace of $v \in \mathbb{C}^d$ such that

$$\sum_{i=1}^{d} p_{ij}(\xi) v_i = 0 \text{ for all } j \in \{1, \ldots, l\}, \qquad \sum_{i=1}^{d} v_i b_i(\xi) = 0$$

is $\mathcal{X}_\xi((p_{ij})_{ij})$ and thus has dimension $(d-r)$. Therefore, all $(r+1) \times (r+1)$ minors of the matrix corresponding to these linear equations vanish. In particular, the determinant of the matrix M is 0. Denote by M^β the $(r \times r)$-submatrix of M, where the last column and the β-th row of M are eliminated. We apply the Laplace expansion formula twice to M (in the last column and then in the last row), to see that

$$\begin{aligned} 0 &= \det(M) \\ &= \left(\sum_{\beta=1}^{r} b_{i(\beta)} (-1)^{r+1+\beta} \det(M^\beta) \right) + b_i \det(M_{IJ}) \\ &= \left(\sum_{\gamma=1}^{r} \sum_{\beta=1}^{r} (-1)^{r+1+\beta} (-1)^{r+\gamma} b_{i(\beta)} p_{ij(\gamma)} \det(M_{I(\beta)J(\gamma)}) \right) + b_i \det(M_{IJ}). \end{aligned}$$

Therefore,

$$b_i \det(M_{IJ}) = \sum_{\gamma=1}^{r} \sum_{\beta=1}^{r} (-1)^{\beta+\gamma} b_{i(\beta)} p_{ij(\gamma)} \det(M_{I(\beta)J(\gamma)}),$$

which establishes (3.6). The proof is complete. ■

REMARK 3.4. We briefly comment on the by now well-understood situation of $\mathbb{C}$-elliptic differential operators $\mathbb{A}$, where the Hilbert Nullstellensatz is typically applied as follows (cf. [24], [17, Lem. 4, Thm. 5], [14, Prop. 3.2], [9, §3]). Let $V \cong \mathbb{R}^N$, $W \cong \mathbb{R}^m$ and let $\mathbb{A}$ be a homogeneous differential operator on $\mathbb{R}^n$ from V to W. Then, $\mathbb{C}$-ellipticity of $\mathbb{A}$ implies by virtue of the Hilbert Nullstellensatz that there exists $k \in \mathbb{N}$ with the following property. There exists a linear, homogeneous differential operator $\mathbb{L}$ on $\mathbb{R}^n$ from W to $V \odot^k \mathbb{R}^n$ of order $(k-1)$ such that $D^k = \mathbb{L}\mathbb{A}$. Inserting this relation into the usual Sobolev integral representation of $u \in \mathrm{C}^\infty(\overline{\mathrm{B}_1(0)}; V)$ (cf. [1, §4] or [18, Thm. 1.1.10.1]) and integrating by parts then yields a polynomial P of order

$(k-1)$ such that

$$u(x) = P(x) + \int_{\mathrm{B}_1(0)} K(x,y)\mathbb{A}u(y)\mathrm{d}y$$

for all $x \in \mathrm{B}_1(0)$ and all $u \in \mathrm{C}^\infty(\overline{\mathrm{B}_1(0)}; V)$; here, the function

$$K\colon \mathrm{B}_1(0) \times \mathrm{B}_1(0) \to \mathcal{L}(W;V)$$

is a suitable integral kernel. This, in particular, implies that $\dim(\ker(\mathbb{A})) < \infty$.

In our situation, an approach via the Sobolev representation formula does not work: The operators $\mathbb{B}, \widetilde{\mathbb{B}}$ from Theorem 1.2 do not have finite-dimensional nullspaces, but their nullspaces differ by finite-dimensional vector spaces. However, we may replace the formula $D^k = \mathbb{L}\mathbb{A}$ by the relation $D^k\mathbb{B} = \mathbb{L}\widetilde{\mathbb{B}}$ and still get some quantitative estimates, also cf. [22].

4. Proof of Theorem 1.2

We assume that $\mathbb{A}\colon \mathrm{C}^\infty(\mathbb{R}^n;\mathbb{R}^d) \to \mathrm{C}^\infty(\mathbb{R}^n;\mathbb{R}^l)$ is a homogeneous differential operator of the form

$$\mathbb{A}u = \sum_{|\alpha|=k} \mathbb{A}_\alpha \partial^\alpha u.$$

In fact, in what follows, one might also assume that $\mathbb{A}_\alpha$ is a complex matrix and that $\mathbb{A}$ is not entirely homogeneous, but only coordinate-wise homogeneous, as Theorem 3.2 does only require these assumptions (also cf. Remark 4.4).

Based on Theorem 3.2, the proof of Theorem 1.2 requires two additional ingredients that we record next.

Lemma 4.1. *Let $\mathbb{A}\colon \mathrm{C}^\infty(\mathbb{R}^n;\mathbb{R}^d) \to \mathrm{C}^\infty(\mathbb{R}^n;\mathbb{R}^l)$ be a homogeneous differential operator of order k. Define the differential operator*

$$\nabla \circ \mathbb{A}\colon \mathrm{C}^\infty(\mathbb{R}^n;\mathbb{R}^d) \to \mathrm{C}^\infty(\mathbb{R}^n;\mathbb{R}^l \times \mathbb{R}^n)$$

componentwisely by

$$\big((\nabla \circ \mathbb{A})u\big)_i = \partial_i \mathbb{A}u, \quad i \in \{1,\dots,n\}.$$

Then, we have

$$\ker(\nabla \circ \mathbb{A}) = \ker(\mathbb{A}) + \mathcal{P}_k(\mathbb{R}^n;\mathbb{R}^d). \tag{4.1}$$

Observe that this result *does not* require the constant rank property.

PROOF. Suppose that $u \in \ker(\nabla \circ \mathbb{A})$. Then, $\mathbb{A}u$ is a constant function. Consider the space $W \subset \mathbb{R}^l$ defined by $W := \operatorname{span}\{\mathbb{A}[\xi](\mathbb{R}^d) : \xi \in \mathbb{R}^n\}$. Note that, on the one hand, $\mathbb{A}u \in W$ pointwisely, and, on the other hand,

$$W = \mathbb{A}\mathcal{P}_k^h(\mathbb{R}^n;\mathbb{R}^d) = \mathbb{A}\mathcal{P}_k(\mathbb{R}^n;\mathbb{R}^d). \tag{4.2}$$

The last line can be seen by considering, for $|\beta| = k$ and $v \in \mathbb{R}^d$, the polynomials $p_\beta(x) := \frac{x^\beta}{\beta!}v$. Then, for any $\xi \in \mathbb{R}^n$,

$$\mathbb{A}\Big(\sum_{|\beta|=k} \xi^\beta p_\beta\Big) = \sum_{|\alpha|=k}\sum_{|\beta|=k} \xi^\beta \mathbb{A}_\alpha \partial^\alpha p_\beta = \sum_{|\alpha|=k} \xi^\alpha \mathbb{A}_\alpha v$$

and so (4.2) follows by the homogeneity of $\mathbb{A}$ of degree k. In particular, for every $u \in \ker(\nabla \circ \mathbb{A})$, we can find a polynomial p of degree k with $\mathbb{A}(u - p) = 0$. Hence,

$$\ker(\nabla \circ \mathbb{A}) \subset \ker(\mathbb{A}) + \mathcal{P}_k(\mathbb{R}^n;\mathbb{R}^d).$$

On the other hand, since $\mathbb{A}$ is homogeneous and of order k, every element of $\ker(\mathbb{A}) + \mathcal{P}_k(\mathbb{R}^n;\mathbb{R}^d)$ belongs to the nullspace of $\nabla \circ \mathbb{A}$. Thus, (4.1) follows and the proof is complete. ■

COROLLARY 4.2 (Kernels of annihilators). *Let $\mathbb{A}^{(1)}$ and $\mathbb{A}^{(2)}$ be two homogeneous differential operators of order $k^{(1)}$ and $k^{(2)}$, which have constant rank over $\mathbb{C}$ and both act on $\mathrm{C}^\infty(\mathbb{R}^n;\mathbb{R}^d)$. Moreover, suppose that their Fourier symbols satisfy*

$$\ker\big(\mathbb{A}^{(2)}[\xi]\big) \subset \ker\big(\mathbb{A}^{(1)}[\xi]\big) \quad \text{for all } \xi \in \mathbb{C}^n. \tag{4.3}$$

Then, the following hold:

(a) *There exists $\tilde{k} \in \mathbb{N}$ and a differential operator $\mathcal{B}$, such that*

$$\nabla^{\tilde{k}} \circ \mathbb{A}^{(2)} = \mathcal{B} \circ \mathbb{A}^{(1)}.$$

(b) *For the nullspace of $\mathbb{A}^{(1)}$, we have*

$$\{u \in \mathrm{L}^1_{\mathrm{loc}} : \mathbb{A}^{(1)}u = 0\} \subset \{u \in \mathrm{L}^1_{\mathrm{loc}} : \mathbb{A}^{(2)}u = 0\} + V,$$

where V is a finite-dimensional vector space (consisting of polynomials).

(c) *If, in addition,*

$$\ker\big(\mathbb{A}^{(1)}[\xi]\big) = \ker\big(\mathbb{A}^{(2)}[\xi]\big),$$

then we may write

$$\{u \in \mathrm{L}^1_{\mathrm{loc}} : \mathbb{A}^{(1)}u = 0\} + V = \{u \in \mathrm{L}^1_{\mathrm{loc}} : \mathbb{A}^{(2)}u = 0\} + W$$

for finite-dimensional vector spaces V and W consisting of polynomials.

PROOF. Ad (a). We aim to apply Theorem 3.2, and we explain the setting first. Assuming that $\mathbb{A}^{(1)}$ is $\mathbb{R}^{l_1}$-valued and $\mathbb{A}^{(2)}$ is $\mathbb{R}^{l_2}$-valued, we may write for $v = (v_1, \dots, v_d) \in \mathbb{C}^d$

$$\mathbb{A}^{(1)}[\xi]v = \Big(\sum_{i=1}^{d} A_{ij}^{(1)}(\xi)v_i\Big)_{j=1,\dots,l_1} \quad \text{and} \quad \mathbb{A}^{(2)}[\xi]v = \big(\mathbb{A}_m^{(2)}(\xi)v\big)_{m=1,\dots,l_2},$$

where every $\mathbb{A}_m^{(2)}(\xi)v$ can be written as

$$\mathbb{A}_m^{(2)}(\xi)v = \sum_{i=1}^{d} v_i b_{im}(\xi).$$

For each $m \in \{1, \dots, l_2\}$, we apply Theorem 3.2 to $p_{ij}[\xi] = A_{ij}^{(1)}[\xi]$ and $B[\xi] = \mathbb{A}_m^{(2)}(\xi)$; note that its applicability is ensured by (4.3).

In consequence, for every component $\mathbb{A}_m^{(2)}$ with $m \in \{1, \dots, l_2\}$ and $a \in \{1, \dots, n\}$, we may find $N(a,m) \in \mathbb{N}$ and polynomials $h_{j,a} \in \mathbb{C}[\xi_1, \dots, \xi_n]$, such that

$$\xi_a^{N(a,m)} \mathbb{A}_m^{(2)}(\xi) = \sum_{j=1}^{l_1} h_{j,a}(\xi) \sum_{i=1}^{d} A_{ij}^{(1)}(\xi)v_i.$$

Therefore, choosing $\tilde{k} := n \max_{m\in\{1,\dots,l_2\},\, a\in\{1,\dots,n\}} N(a,m)$, we obtain that for every $\alpha \in \mathbb{N}^n$ with $|\alpha| = \tilde{k}$ and $m \in \{1, \dots, l_2\}$, there exists $h_{j\alpha}$ such that

$$\xi^\alpha \mathbb{A}_m^{(2)}(\xi) = \sum_{j=1}^{l_1} h_{j\alpha}(\xi) \sum_{i=1}^{d} A_{ij}^{(1)}(\xi)v_i.$$

Defining the differential operator $\mathscr{B}$ according to this Fourier symbol, (a) follows; i.e.,

$$\mathscr{B}[\xi]_{m,\alpha}(w) = \sum_{j=1}^{l_1} h_{j\alpha}(\xi)w_j, \quad m \in \{1, \dots, l_2\}.$$

Ad (b). This directly follows from Lemma 4.1. Indeed, applying Lemma 4.1 $\tilde{k}$-times, there exists a finite-dimensional space $\tilde{V}$ of polynomials such that

$$\{u \in \mathrm{L}^1_{\mathrm{loc}} : \nabla^{\tilde{k}} \mathbb{A}^{(2)}u = 0\} = \{u \in \mathrm{L}^1_{\mathrm{loc}} : \mathbb{A}^{(2)}u = 0\} + \tilde{V}.$$

As $\ker \mathbb{A}^{(1)} \subset \ker \mathscr{B} \circ \mathbb{A}^{(1)} = \ker \nabla^{\tilde{k}} \circ \mathbb{A}^{(2)}$, the result directly follows. Finally, (c) is immediate by applying (b) in both directions. The proof is complete. ∎

We may now turn to the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Direction (a)⇒(b) of Theorem 1.2 is just Corollary 4.2; using convolution, one may first observe this for $\mathrm{L}^1_{\mathrm{loc}}$ functions and then generalise it to $\mathcal{D}'$. On the other hand, direction (b)⇒(a) follows from a routine construction (see e.g. [10, 13, 24]) of plane waves which we outline for the reader's convenience. Suppose towards a contradiction that there exists $\xi \in \mathbb{C}^n \setminus \{0\}$ such that $\ker(\mathbb{B}[\xi]) \neq \ker(\widetilde{\mathbb{B}}[\xi])$. Without loss of generality, we may then assume there exists $v \in \mathbb{C}^l \setminus \{0\}$ such that

$$v \in \ker\left(\mathbb{B}[\xi]\right) \setminus \ker\left(\widetilde{\mathbb{B}}[\xi]\right).$$

Consider the waves $u_h(x) := e^{\mathrm{i}\, x\cdot h\xi} v$ for $h \in \mathbb{Z}$ and sort by real and imaginary parts. Observe that all u_h are linearly independent as functions and, in particular,

$$\mathcal{X} = \left\{ \sum_{h=1}^{H} a_h \operatorname{Re}\left(u_h(x)\right) a_1, \dots, a_h \in \mathbb{R},\ H \in \mathbb{N} \right\}$$

forms an infinite-dimensional vector space. One may now calculate that $\mathbb{B}u = 0$ for any $u \in \mathcal{X}$, but for $u = \sum_{h=1}^{H} a_h u_h(x)$, we have

$$\widetilde{\mathbb{B}}u = \sum_{h=1}^{H} a_h i^k h^k \widetilde{\mathbb{B}}[\xi](v) e^{\mathrm{i}\, x\cdot h\xi} \neq 0$$

due to the linear independence of $e^{\mathrm{i}\, x\cdot h\xi}$ and also $\widetilde{\mathbb{B}}(\operatorname{Re} u) \neq 0$. Thus, we find an infinite-dimensional vector space $\mathcal{X}$ that is in $\ker \mathbb{B}$, but $\mathcal{X} \cap \ker \widetilde{\mathbb{B}} = \{0\}$. ∎

EXAMPLE 4.3. In general, direction (a)⇒(b) in Theorem 1.2 will fail if $\mathbb{B}$ and $\widetilde{\mathbb{B}}$ *do not* satisfy the complex constant rank property. As one readily verifies, if we take $\mathbb{B} = \Delta$ and $\widetilde{\mathbb{B}} = \Delta^2$ to be the Laplacian and the Bi-Laplacian (and so both violate the constant rank condition over $\mathbb{C}$ by Example 2.4) in $n = 2$ dimensions,

$$\ker_{\mathbb{C}}\left(\mathbb{B}[\xi]\right) = \ker_{\mathbb{C}}\left(\widetilde{\mathbb{B}}[\xi]\right) = \begin{cases} \mathbb{C} & \text{if } \xi = \lambda(1,\mathrm{i})^\top \text{ or } \xi = \lambda(1,-\mathrm{i})^\top,\ \lambda \in \mathbb{C}, \\ \{0\} & \text{otherwise.} \end{cases}$$

Denote $\ker(\Delta)$ and $\ker(\Delta^2)$ as the nullspaces of Δ or Δ^2, respectively, in $\mathcal{D}'(\mathbb{R}^n)$. Denoting the homogeneous harmonic polynomials on $\mathbb{R}^n$ by $\mathcal{P}_{\mathrm{ho}}(\mathbb{R}^n)$, we have

$$\ker(\Delta) + \widetilde{\mathcal{P}} \subset \ker(\Delta^2),$$

where $\widetilde{\mathcal{P}} = \{v\colon\ \Delta v = p \text{ for some } p \in \mathcal{P}_{\mathrm{ho}}(\mathbb{R}^n)\}$, and from here one sees that the nullspaces of $\mathbb{B}$ and $\widetilde{\mathbb{B}}$ differ by an infinite-dimensional vector space. As one can see from the proof of Theorem 1.2, however, the implication (b)⇒(a) can be shown by contradiction, even if the complex constant rank property does not hold.

REMARK 4.4. Up to now, we assumed that the polynomials p_{ij} are homogeneous polynomials of order k. This assumption is motivated by the fact that we deal with homogeneous differential operators. However, we can also define the *constant rank* *property* when not all polynomials have the same order. In particular, for polynomials p_{ij} as in (3.1), we may weaken the assumption to p_{ij} having order $k_j \in \mathbb{N}$, and the statement of the vectorial Nullstellensatz still holds true.

For the corresponding differential operator, this includes the following setting. The operator $\mathbb{B} = (\mathbb{B}_0, \dots, \mathbb{B}_k)$ is componentwisely defined via homogeneous differential operators $\mathbb{B}_i \colon \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W_i)$ of order i (for $i = 0$, the operator $\mathbb{B}_0$ is similarly understood to be a linear map). In particular,

$$\mathbb{B} \colon \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W_0 \times \cdots \times W_k).$$

The constant rank property in this setting means that there exists $r \in \mathbb{N}$ such that

$$\bigcap_{i=0}^{k} \ker\big(\mathbb{B}_i[\xi]\big) = r, \qquad \text{for all } \xi \in \mathbb{C}^n \setminus \{0\}.$$

Observe that it is not required at all, that each homogeneous component satisfies the constant rank property itself; e.g., $\mathbb{B}u = (\partial_1 u, \partial_2^2 u)$.

In view of Lemma 4.1, we can however also transform this setting into a fully homogeneous one, while only allowing an additional finite-dimensional nullspace. Indeed, the operator $\widetilde{\mathbb{B}}$ given by

$$\widetilde{\mathbb{B}} = (\nabla^k \circ \mathbb{B}_0, \nabla^{k-1} \circ \mathbb{B}_1, \dots, \mathbb{B}_k)$$

is homogeneous of order k and its nullspace only differs by a finite-dimensional space from the nullspace of $\mathbb{B}$.

4.1. On natural annilators

For now, we have seen that if $\mathbb{B}[\xi]$ and $\widetilde{\mathbb{B}}[\xi]$ have the same nullspace for all $\xi \in \mathbb{C}^n \setminus \{0\}$, then their nullspaces as differential operators only differ by finite-dimensional subspaces. Given the nullspaces $V(\xi) = \ker(\mathbb{B}[\xi])$ for some differential operator $\mathbb{B}$, it is natural to ask for a *minimal* differential operator in the sense of nullspaces; i.e.,

$$\ker\big(\mathbb{B}_0[\xi]\big) = V(\xi)$$

and $\ker(\mathbb{B}_0) \subset \ker(\widetilde{\mathbb{B}})$ for all $\widetilde{\mathbb{B}}$ with $\ker(\widetilde{\mathbb{B}}[\xi]) = V(\xi)$. Indeed, let us assume that we are given a *potential* operator $\mathbb{A} \colon \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; \mathbb{R}^d)$ that obeys

$$\operatorname{Im} \mathbb{A}[\xi] = \ker \mathbb{B}[\xi].$$

Such a potential operator exists due to the constant rank property, cf. [20]. Consider the set $\mathscr{B} \subset (\mathbb{C}[\xi_1, \dots, \xi_n])^d$ as the set of annihilator polynomials that vanish on $\mathbb{A}[\xi]$; i.e.,

$$\mathscr{B} = \Big\{ p \in \mathbb{C}[\xi_1, \dots, \xi_n]^d \colon \sum_{i=1}^{d} p_i(\xi) \circ \mathbb{A}_i(\xi) = 0 \Big\}.$$

Then, $\mathscr{B}$ is a submodule of the free module $\mathbb{C}[\xi_1, \dots, \xi_n]^d$ and is finitely generated. As there exists some $\mathbb{B}$ such that the relation

$$\operatorname{Im} \mathbb{A}[\xi] = \ker \mathbb{B}[\xi]$$

is satisfied for all $\xi \in \mathbb{C}^n \setminus \{0\}$, the same can be said about any generator of $\mathscr{B}$; i.e., if $p_1, \dots, p_l$ is a generator of $\mathscr{B}$, then

$$\mathbb{B}_0 v = \sum_{i=1}^{l} \big(p_i(\xi) v\big) \cdot e_i$$

is an annihilator of $\mathbb{A}$; i.e., $\operatorname{Im} \mathbb{A}[\xi] = \ker \mathbb{B}_0[\xi]$. Moreover, due to its properties as a generator, for any $\mathbb{B}$ that satisfies $\ker \mathbb{B}_0[\xi] = \ker \mathbb{B}[\xi]$, we have some polynomial $\mathbb{L}$ such that

$$\mathbb{B}[\xi] = \mathbb{L} \circ \mathbb{B}_0.$$

If the generator is further minimal, we also might speak of a *natural* annihilator of $\mathbb{B}$ (which is, of course, not unique, as the choice of a basis for a submodule is equally not unique).

Example 4.5. Let us consider a few examples that already appeared in Section 2.

(a) If $\mathbb{A} = \nabla$, then $\mathbb{B}_0 = \operatorname{curl}$ is a natural annihilator.

(b) Likewise, if $\mathbb{A}$ is the operator of exterior differentiation on differential forms, then the exterior derivative itself is the natural annihilator.

(c) If $\mathbb{A} = (\nabla + \nabla^\top)$, then $\mathbb{B}_0 = \operatorname{curl}\operatorname{curl}^\top$ is a natural annihilator.

(d) Consider the k-th order gradient $\nabla^k = (\partial_\alpha)_{|\alpha|=k}$ acting on $C^\infty(\mathbb{R}^n; \mathbb{R})$. A candidate for the annihilator is the operator consisting of entries of the form

$$\partial^\alpha v_\beta - \partial^\beta v_\alpha.$$

This is, however, not a natural annihilator. Indeed, viewing ∇^k as a concatenation of ∇ and ∇^{k-1}, i.e. $\nabla^k \colon C^\infty(\mathbb{R}^n; \mathbb{R}) \to C^\infty(\mathbb{R}^n; \mathbb{R}^n \otimes (V^{k-1}))$, applying the curl to the first-component is a natural annihilator. In particular, the coordinates of this operator are

$$\partial^j v_\beta - \partial^k v_\alpha \quad \text{if the multi-indices obey } e_j + \beta = e_k + \alpha.$$

(e) Even if the operator $\mathbb{A}$ is homogeneous, the natural annihilator does not need to be. A simple example is an operator acting on a pair of functions, e.g.

$$\mathbb{A}(u_1, u_2) = \big(\nabla u_1, (\nabla + \nabla^\top) u_2\big),$$

where a natural annihilator obviously is $\mathbb{B}_0(v_1, v_2) = \operatorname{curl} v_1 \operatorname{curl}\operatorname{curl}^\top v_2$. Considering the natural annihilator of this $\mathbb{B}_0$ again yields an operator that is componentwise homogeneous.

Another, more complex, example is the $\mathbb{C}$-elliptic operator

$$\mathbb{A}\colon \mathrm{C}^\infty(\mathbb{R}^3;\mathbb{R}^3) \to \mathrm{C}^\infty(\mathbb{R}^3;\mathbb{R}^{3\times 3})$$

given by

$$\mathbb{A}(v_1, v_2, v_3) = \begin{pmatrix} \partial_1 v_1 & \partial_2 v_1 + \partial_1 v_2 & \partial_3 v_1 \\ \partial_2 v_1 + \partial_1 v_2 & \partial_2 v_2 & \partial_3 v_2 \\ \partial_1 v_3 & \partial_2 v_3 & \partial_3 v_3 \end{pmatrix}$$

i.e. a mixture of symmetric and full gradient. A natural annihilator $\mathbb{B}_0$ of this is always non-homogeneous (with homogeneous rows), while a natural annihilator of this operator in turn will be fully non-homogeneous (i.e. we cannot even arrange homogeneous rows). Indeed, it does not seem clear what condition some operator $\mathbb{A}$ needs to satisfy to ensure that a natural annihilator is fully homogeneous (which also means that *any* natural annihilator is homogeneous).

4.2. Remarks on a Poincaré-type lemma

The classical Poincaré lemma asserts that a differential form ω is closed, i.e., $\mathrm{d}\omega = 0$, if and only if $\omega = \mathrm{d}u$. This result holds on *all* domains of a certain geometry and regularity. Generalising this result to constant rank operators, one has that if

$$\operatorname{Im}\mathbb{A}[\xi] = \ker\mathbb{B}[\xi] \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

then

$$\mathbb{B}(u) = 0, \quad \int_{\mathrm{T}^n} u = 0 \quad \Longrightarrow u = \mathbb{A}v$$

and sharp estimates on the norm of v may also be given in suitable spaces (e.g. L^p, $\mathrm{W}^{k,p}$ when $1 < p < \infty$). Results like this can, for instance, be proven by using the Fourier transform and naturally also hold on the full space.

The situation is not so clear in the general case of functions defined on open and bounded (and sufficiently regular) subsets $\Omega \subset \mathbb{R}^n$ (with or without prescribed boundary values). In order to get reasonable results for the underlying functions, the complex constant rank condition is a suitable assumption in this setting.

The previous results have the following consequences for the Poincaré lemma. Let $\mathbb{A}$ and $\mathbb{B}$ be given operators of order $k_\mathbb{A}$ and $k_\mathbb{B}$ such that $\operatorname{Im}\mathbb{A}[\xi] = \ker\mathbb{B}[\xi]$ for all $\xi \in \mathbb{C}^n \setminus \{0\}$. In particular, those operators have complex constant rank.

(a) We may reduce to the problem, where $\mathbb{B}$ is of order one: Suppose that we are given a function u that satisfies $\mathbb{B}u = 0$. Then, $\tilde{u} = \nabla^{k_{\mathbb{B}}-1}u$ also satisfies a differential equation, namely, $\widetilde{B}\tilde{u} = 0$ (where $\widetilde{B}$ comes through rewriting $\mathbb{B}$ as a concatenation of $\widetilde{\mathbb{B}}$ and $\nabla^{k_{\mathbb{B}}-1}$), as well as the condition curl $\tilde{u}$ which establishes that $\tilde{u}$ is a gradient of order $k_{\mathbb{B}} - 1$. One may check that $\widetilde{\mathbb{A}} = \nabla^{k_{\mathbb{B}}-1} \circ \mathbb{A}$ is a potential of $\widetilde{\mathbb{B}}$. Instead of searching for a Poincaré lemma for $\mathbb{B}$ and $\mathbb{A}$ we can therefore establish a Poincaré lemma for $\widetilde{\mathbb{B}}$ and $\widetilde{\mathbb{A}}$. In particular, the operation $u \mapsto \tilde{u}$ is reversible up to polynomials of order $k_{\mathbb{B}} - 1$.

(b) We may reduce to the problem, where $\mathbb{A}$ is of order one: As before, we may write $\mathbb{A}$ as a concatenation of $\widetilde{A}$ and $\nabla^{k_{\mathbb{A}}-1}$. Hence, all functions of the form $u = \mathbb{A}v$ can be written as $\widetilde{A}\tilde{v}$ under the additional side-constraint curl $v = 0$. The operator given by

$$\mathbb{A}'v = \begin{pmatrix} \widetilde{A}v \\ \operatorname{curl} v \end{pmatrix}$$

then has an annihilator $\mathbb{B}'$ (of order that might be larger than one). Showing a Poincaré lemma for $\mathbb{A}'$ and $\mathbb{B}'$ and applying it to vectors $(u\ 0)$ (such that the second condition is curl $v = 0$) then also yields a Poincaré lemma for $\mathbb{A}$ and $\mathbb{B}$.

(c) If $k_{\mathbb{A}} = k_{\mathbb{B}} = 1$ and the space dimension is two, the situation is quite special: We can show that $\mathbb{A}\colon \mathrm{C}^\infty(\mathbb{R}^n; U) \to \mathrm{C}^\infty(\mathbb{R}^n; V)$ (after suitable coordinate transformation) can be decomposed into

$$\mathbb{A}u = \begin{pmatrix} \nabla u_1 \\ \operatorname{curl} u_2 \\ 0 \end{pmatrix}$$

if $u = (u_1, u_2, u_3) \in U_1 \oplus U_2 \oplus U_3 = U$. This can be, for instance, seen by setting $U_3 = \ker \mathbb{A}[e_1] \cap \ker \mathbb{A}[e_2]$, $U_1 = (\ker \mathbb{A}[e_1] + \ker \mathbb{A}[e_2])^\perp$ and choosing U_3 accordingly. As $\mathbb{B}$ has order one, we conclude that the elliptic part (on U_1) needs to be the gradient. A similar argument then holds for the non-elliptic part. Therefore, by using classical results for the Poincaré lemma for ∇ (see, for instance, [7]), we get a Poincaré lemma for $\mathbb{A}$ and $\mathbb{B}$ and through the reductions made before also for a wider class of operators in $n = 2$.

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