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Mathematical Physics. – *Self-similar solution to the Stefan problem on surfaces of revolution*, by Benedetta Calusi, Angiolo Farina and Roberto Gianni, communicated on 8 November 2024.

Abstract. – In this paper, we study the one-phase Stefan problem on a surface of revolution, characterizing those surfaces admitting a self-similar solution.

Keywords. – self-similar solution, Stefan problem, surface of revolution.

Mathematics Subject Classification 2020. – 35C06 (primary);
35K99, 53A05 (secondary).

1. Introduction

The Stefan problem is extensively studied because of its simple mathematical formulation and since a wide variety of phenomena and processes can be reduced to a problem of Stefan type (e.g. see [1, 4–7, 11–16, 18, 20, 22, 23, 27, 30, 31] and references cited therein). Indeed, such mathematical problems arise from a wide field of applications since several natural phenomena and industrial processes involve the presence of an evolving interface with diffusion and phase change [8, 26]. The works by Crank [9], by Fasano and Primicerio [17], and by Tarzia [29] are the fundamental review of the problem of phase transition related to a moving front (free boundary).

The ice-liquid water phase transition is the most common example of a Stefan problem. Such a mathematical model consists of solving the heat equations for the solid (ice) and liquid phases and the equation (representing energy balance) for the interface separating ice and liquid (usually named Stefan condition). The reduced one-phase setting consists in assuming that the temperature in one phase is uniform and constant so that one has just to solve a parabolic equation coupled with the evolution equation for the interface [19].

In many physical and biological models, diffusion and phase change processes can be formulated, with a certain degree of approximation, on a surface with particular geometries, i.e., they occur on thin layers usually modelled as curved surfaces. Some examples are diffusional transport of substances in the cell membranes [10], and crystal growth [24]. Literature on this subject is quite vast, e.g. [2, 3, 24, 28, 32].

The formulation of phase change problems on a surface is a limit case in which the process takes place in a material occupying a layer which is thin with respect to the domains in which it is embedded. Further conditions must be fulfilled concerning the mutual heat exchange, for instance thermal insulation, which here will be assumed yielding the absence of source terms in the heat conduction equation. In particular, the surface is assumed to be thermally insulated from the 3D environment in which the surface is embedded. Together with the absence of irradiation, this reflects in the absence of source terms in the heat equation. In the reduced scheme, the free boundary, i.e., the interface where phase change takes place, is identified by a curve on the surface. Examples of a singular front evolution on thin layer are given in [25]. Here, the thin material layers are treated as two-dimensional continua since their thickness is negligible compared to the dimensions of the adjacent bulk materials. The layers are thus mathematically represented as surfaces with their own thermomechanical properties. These surfaces therefore carry mass, momentum, energy, and entropy. A thermodynamic theory of phase boundaries can be therefore developed on the postulation of conservation laws together with constitutive assumptions. We refer the readers also to [21] where more details are provided.

In 1976, Stewartson and Waechter [28] provided the complete asymptotic theory for Stefan problem on a sphere. Recently, an approximate analytical solution for a two-phase Stefan problem has been developed based on asymptotic analysis with fully phase-dependent thermophysical properties to model outward solidification on an annulus [32].

Faraudo [10] studied analytically the diffusional transport on two-dimensional curved surfaces highlighting the influence of the surface local curvature on chemicals diffusion.

In [2,3], new insights on PDEs on evolving spaces are illustrated. The Stefan problem on a moving hypersurface is analyzed by regularization in [2]. Prokhorova in [24] investigates the self-similar regimes in the problem of crystal growing from a pure melt on an isothermal surface. In particular, the author discusses the modeling of the spherical crystal growth by analyzing the two-dimensional crystallization on surfaces of revolution.

The aim of this paper is to study the one-phase Stefan problem on a surface of revolution, looking for those surfaces for which the problem admits self-similar solutions of the form $u(s,t) = f(\frac{g(s)}{h(t)})$, where s is the arc length on the curve generating the surface. We assume that the surface and the environment do not exchange heat so that the thermal energy diffuses only on the surface. In particular, looking for axisymmetric solutions, we focus on the surfaces of revolution about the z-axis and consider problems with azimuthal symmetry. We assume that the initial condition for the interface is $z =$ constant, and because of the required symmetry, $z = b(t)$ represents the free boundary at time t.

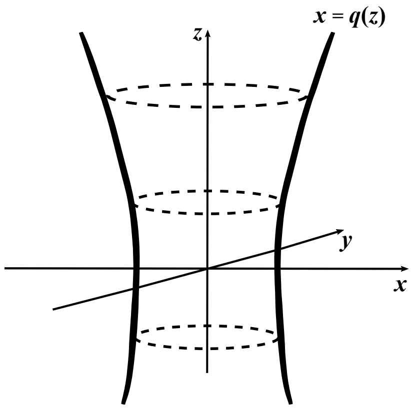

FIGURE 1. Geometry of the problem: surfaces of revolution about the z-axis.

We assume that the temperature for $z > b(t)$ is below the melting temperature, u_{melting} (solid phase), while for $z < b(t)$ the medium is in the liquid state at the constant temperature u_{melting}. Figure 1 shows the geometry of the problem.

We find that not all surfaces admit self-similar solutions. The only surfaces of revolution that admit self-similar solutions depend on a real parameter. We also find the functional form of $g(s)$, $h(t)$.

The paper is organized as follows. In Section 2, we introduce the background and general formulation of the Stefan problem on a surface. In Sections 3 and 4, we look for solutions of self-similar type. In Section 5, we classify the surfaces of revolution admitting similarity solutions. The last section is devoted to discussion and conclusions.

2. Background: The Stefan problem on a regular surface

We briefly report the theory of the Stefan problem on regular surfaces in $\mathbb{R}^3$.

As mentioned in the introduction, diffusion and phase change processes on a thin layer can be modelled, with a certain degree of approximation, replacing the layer with a 2D surface (see, e.g., [21]). Within such an approximation, we consider the Stefan problem on a regular surface, Σ. We denote by $\Sigma_l(t)$ and $\Sigma_s(t)$ the liquid and solid domains, respectively, assuming that they are separated by an a priori unknown curve $\Gamma(t)$. Therefore, $\Sigma = \Sigma_l(t) \cup \Sigma_s(t) \cup \Gamma(t) \subset \mathbb{R}^3$. We denote as u the surface temperature, u is larger than the melting temperature u_{melting} in Σ_l, while $u \leq u_{\text{melting}}$ in Σ_s.

We now assume that there is no heat exchange between the surface and the environment. Denoting by $\mathbf{j}$ the heat flux and by κ the thermal conductivity (constant and uniform) of the material filling the thin layer, the upscaling process leading to the formulation of heat conduction and phase change on a surface leads to conclude that

Fourier's law takes the form

$$\mathbf{j} = -\kappa \nabla_\Sigma u,$$

where

$$\nabla_\Sigma(\cdot) = \frac{\partial(\cdot)}{\partial w^\alpha} \boldsymbol{X}^\alpha = \frac{\partial(\cdot)}{\partial w^\alpha} g^{\alpha\beta} \boldsymbol{X}_\beta$$

is the surface gradient and $g^{\alpha\beta}$ the inverse of the covariant metric tensor $g_{\alpha\beta}, \alpha, \beta = 1, 2$. The heat equation acquires the form

$$\rho c \frac{\partial u}{\partial t} = \kappa \Delta_\Sigma u, \tag{2.1}$$

where ρ and c are the density and specific heat (constant and uniform) and

$$\Delta_\Sigma(\cdot) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial w^\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial\,(\cdot)}{\partial w^\beta} \right)$$

is the Laplace–Beltrami operator with $g = \det(g_{\alpha\beta})$.

We suppose that u is rescaled so that the melting temperature vanishes, i.e., $u_{\text{melting}} = 0$. We assume that the temperature in Σ_s, denoted by u_s, is non-positive, and the temperature in Σ_l, denoted by u_l, is non-negative, while at the interface,

$$u_s|_\Gamma = u_l|_\Gamma = 0. \tag{2.2}$$

We then assume that Γ can be locally represented parametrically in an implicit form as

$$\mathcal{F}(w^1, w^2, t) = 0, \quad t > 0,$$

so that the usual Stefan condition rewrites as [21]

$$g^{\alpha\beta} \left(\kappa_l \frac{\partial u_l}{\partial w^\alpha} - \kappa_s \frac{\partial u_s}{\partial w^\alpha} \right) \frac{\partial \mathcal{F}}{\partial w^\beta} = \lambda \rho \frac{\partial \mathcal{F}}{\partial t}, \tag{2.3}$$

where λ is the melting specific latent heat and κ_l, κ_s are thermal conductivity in the liquid and solid phase, respectively.

3. One-phase Stefan problem on a surface of revolution

Here and in the sequel, we set $(x_1, x_2, x_3) = (x, y, z)$ and $(w^1, w^2) = (\theta, z)$. We then consider a surface of revolution generated by rotating the graph of the function $x = q(z) \geq 0$, $z \in \mathcal{D}$ where $\mathcal{D}$ is the domain of definition of q, around the z-axis, as shown in Figure 2, namely,

$$\begin{cases} x(\theta, z) = q(z) \cos\theta, \\ y(\theta, z) = q(z) \sin\theta, \qquad (\theta, z) \in [0, 2\pi] \times \mathcal{D}. \\ z(\theta, z) = z, \end{cases}$$

The covariant metric tensor $g_{\alpha\beta}$, $\alpha, \beta = 1, 2$, is

$$g_{11} = q(z)^2, \quad g_{12} = 0, \quad g_{22} = 1 + \left(\frac{dq}{dz}\right)^2,$$

and

$$\sqrt{g} = \sqrt{\det(g_{\alpha\beta})} = q(z)\sqrt{1 + q'(z)^2}.$$

We assume that the initial condition for the interface is $z =$ constant, and since we require azimuthal symmetry, $\mathcal{F}(z, t) = z - b(t) = 0$ represents the interface between the liquid and solid phase and Σ_s is the portion $z > b(t)$. In Σ_s, the temperature u evolves according to (2.1), while $u = 0$ for $z \leq b$ (one-phase Stefan problem). By introducing the following scaling

$$t = t_{\text{ref}} t^*, \quad z = Hz^*, \quad q = Hq^*, \quad b = Hb^*,$$

with H characteristic length (e.g., the length of the sample) and

$$t_{\text{ref}} = \frac{\rho c H^2}{\kappa},$$

system (2.1), (2.2), (2.3) takes the dimensionless form

$$\begin{cases} u_t = \frac{1}{q(z)\sqrt{1+q'(z)^2}} \frac{\partial}{\partial z}\left(\frac{q(z)}{\sqrt{1+q'(z)^2}} \frac{\partial u}{\partial z}\right), & 0 < t,\ b(t) < z, \\ u\big(b(t), t\big) = 0, \\ \Lambda \dot{b}(t) = \frac{1}{1+q'(z)^2} \frac{\partial u}{\partial z}\Big|_b, \end{cases} \tag{3.1}$$

with $\Lambda = \lambda/c$ and where we omit writing * to keep the notation as light as possible.

We proceed introducing

$$s = \int_0^z \sqrt{1 + q'(\xi)^2} d\xi, \tag{3.2}$$

where we integrate starting from zero without loss of generality, since we can reduce to this case by means of a vertical translation along the z-axis. We denote the inverse of (3.2) as $z = \hat{z}(s)$. We then set

$$\hat{q}(s) = q\big(\hat{z}(s)\big), \iff q(z) = \hat{q}\big(s(z)\big). \tag{3.3}$$

In particular, we denote by $\sigma(t)$ the free boundary in the s coordinate, namely,

$$\sigma(t) = \int_0^{b(t)} \sqrt{1 + q'(\xi)^2} d\xi,$$
$$\dot{\sigma}(t) = \sqrt{1 + q'\big(b(t)\big)^2}\, \dot{b}(t).$$

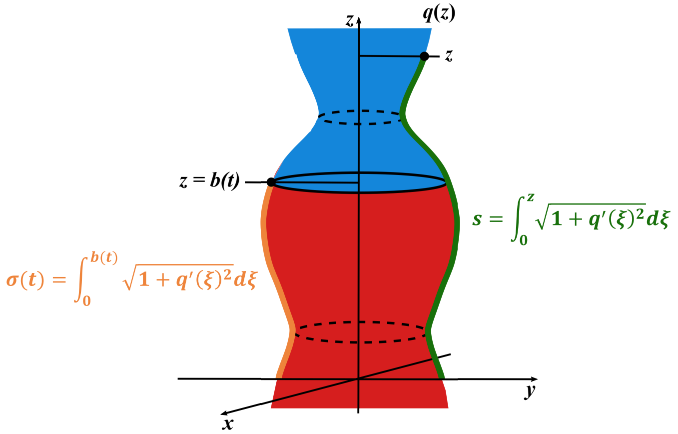

FIGURE 2. Geometry of a surface of revolution generated by rotating the graph of the function $x = q(z)$ around the z-axis. The blue and red regions represent the part of the material in the solid and liquid state, respectively.

System (3.1) becomes

$$\begin{cases} u_t = u_{ss} + B(s)u_s, \\ u\big(\sigma(t), t\big) = 0, \\ \Lambda\dot{\sigma}(t) = \frac{\partial u}{\partial s}\Big|_{\sigma}, \end{cases} \tag{3.4}$$

where

$$B(s) = \frac{\hat{q}'(s)}{\hat{q}(s)}. \tag{3.5}$$

We refer the readers to [13–16] for the well-posedness of problem (3.4).

4. LOOKING FOR SELF-SIMILAR SOLUTIONS

We look for a solution in the form

$$u(s,t) = f(\eta), \tag{4.1}$$

where

$$\eta = \frac{g(s)}{h(t)}, \tag{4.2}$$

assuming here and in the sequel neither g nor h are constant and requiring

$$\frac{g\big(\sigma(t)\big)}{h(t)} = \zeta,$$

with ζ non-vanishing constant and $f(\zeta) = 0$. However, since h and g are defined, at least at this stage, up to multiplicative constants, we select these constants such that $\zeta = 1$. Hence,

$$\frac{g\big(\sigma(t)\big)}{h(t)} = 1, \implies g'\big(\sigma(t)\big)\dot{\sigma}(t) = \dot{h}(t), \tag{4.3}$$

where $g(s)$ is C^3 and $h(t)$ is C^2 in the interior of their respective domains of definition.

It is worth noting that if we look for the solution in the form $u(s,t) = k(t)f(\eta)$, then the function $k(t)$ has to be constant and formula (4.1) is automatically recovered (see the appendix for more details).

Plugging (4.1), (4.2) into system (3.4) and noting that

$$\begin{aligned}\frac{\partial u}{\partial s}\bigg|_\sigma &= \frac{\partial}{\partial s}\Bigg(\underbrace{f\bigg(\frac{g(s)}{h(t)}\bigg)}_{\underset{(4.2)}{=} f(\eta)}\Bigg)\Bigg|_\sigma = \bigg(\frac{g'(s)}{h(t)}f'(1)\bigg)\bigg|_\sigma \\ &= \frac{g'\big(\sigma(t)\big)}{h(t)}f'(1) \underset{(4.3)}{=} \frac{\dot{h}(t)}{\dot{\sigma}(t)h(t)}f'(1),\end{aligned}$$

we have

$$\begin{cases} f''\big(\frac{g'}{h}\big)^2 + \big(\frac{g''}{h} + g\frac{\dot{h}}{h^2} + B\frac{g'}{h}\big)f' = 0, \\ f(1) = 0, \\ \Lambda\dot{\sigma}^2 = \frac{\dot{h}(t)}{h(t)}f'(1), \end{cases} \tag{4.4}$$

where $f'(1) \neq 0$ so that $\sigma \neq$ constant. In the sequel, we assume that the coefficient of f' is non-vanishing.

The key point is to select the functional form of $g(s)$ and $h(t)$ so that equation $(4.4)_1$ can be rewritten in terms of the sole variable $\eta = g(s)/h(t)$. We rewrite $(4.4)_1$ as

$$f'' + \bigg(\frac{h}{g'}\bigg)^2\bigg(\frac{g''}{h} + g\frac{\dot{h}}{h^2} + B\frac{g'}{h}\bigg)f' = 0,$$

deducing that a condition for a self-similar solution to exist is that the coefficient of f' must be a function of η,

$$\begin{aligned}\varphi(\eta) &= \bigg(\frac{h}{g'}\bigg)^2\bigg(\frac{g''}{h} + g\frac{\dot{h}}{h^2} + B\frac{g'}{h}\bigg) \\ &= \frac{g''(s)h(t)}{\big(g'(s)\big)^2} + B(s)\frac{h(t)}{g'(s)} + \frac{g(s)\dot{h}(t)}{\big(g'(s)\big)^2}.\end{aligned} \tag{4.5}$$

By differentiating equation (4.5) with respect to t, we get

$$-\varphi'(\eta)\frac{g(s)}{h^2(t)}\dot{h}(t) = \frac{g''(s)\dot{h}(t)}{\big(g'(s)\big)^2} + B(s)\frac{\dot{h}(t)}{g'(s)} + \frac{g(s)\ddot{h}(t)}{\big(g'(s)\big)^2}. \tag{4.6}$$

We multiply by $(g')^2/g$ equation (4.6) and then differentiate it with respect to s. We thus get an equation no longer containing the derivatives of h

$$\begin{aligned} &-\varphi''(\eta)\frac{\big(g'(s)\big)^3}{h^3(t)} - \varphi'(\eta)\frac{2g'(s)g''(s)}{h^2(t)} \\ &\qquad = \frac{g'''g + B'g'g + B(s)g''g - g''g' - B(g')^2}{g^2} := \frac{F_1(s)}{g^2(s)}. \end{aligned} \tag{4.7}$$

Now, we select $s = s_0$ and set $g(s_0)/h(t) = v$, thus, φ becomes a function of v. Hence, we rewrite (4.7) as

$$\begin{aligned} F_1(s_0) &= g^2(s_0)\Big(-\varphi_{vv}(v)\frac{\big(g'(s_0)\big)^3}{h^3(t)} - \varphi_v(v)\frac{2g'(s_0)g''(s_0)}{h^2(t)}\Big) \\ &= -\varphi_{vv}(v)v^3\frac{\big(g'(s_0)\big)^3}{g(s_0)} - \varphi_v(v)v^2 2g'(s_0)g''(s_0) \\ &:= \varphi_{vv}(v)v^3F_2(s_0) + \varphi_v(v)v^2F_3(s_0), \end{aligned} \tag{4.8}$$

where

$$F_2(s_0) = -\frac{\big(g'(s_0)\big)^3}{g(s_0)} \quad \text{and} \quad F_3(s_0) = -2g'(s_0)g''(s_0). \tag{4.9}$$

We remark that since $g(s)$ cannot be constant, $F_2(s_0) \not\equiv 0$. Therefore, equation (4.8) rewrites in the form

$$\varphi_{vv}(v) + \frac{\varphi_v(v)}{v}\underbrace{\left(\frac{F_3(s_0)}{F_2(s_0)}\right)}_{A_1(s_0)} = \frac{1}{v^3}\underbrace{\left(\frac{F_1(s_0)}{F_2(s_0)}\right)}_{A_2(s_0)},$$

whose solution, for a prescribed s_0, is

$$\begin{cases} \varphi(v) = \frac{A_2(s_0)}{(2-A_1(s_0))v} + K_1\frac{v^{1-A_1(s_0)}}{1-A_1(s_0)} + K_2, & A_1(s_0) \neq 1, 2, \\ \varphi(v) = A_2(s_0)\frac{1}{v} + K_1\log|v| + K_2, & A_1(s_0) = 1, \\ \varphi(v) = \int^v \frac{A_2(s_0)\log|\zeta|}{\zeta^2}d\zeta - K_1\frac{1}{v} + K_2, & A_1(s_0) = 2, \end{cases}$$

where K_1, K_2 are arbitrary constants. We now set

$$A_1(s_0) = r = \text{constant},$$

namely, recalling (4.9),

$$\begin{aligned} A_1(s_0) &= \frac{F_3(s_0)}{F_2(s_0)} = 2g'(s_0)g''(s_0)\frac{g(s_0)}{\big(g'(s_0)\big)^3} \\ &= \frac{2g''(s_0)g(s_0)}{\big(g'(s_0)\big)^2} = r = \text{constant}, \end{aligned}$$

which, setting $g' = w(g)$ so that $g'' = w_g(g)w(g)$, implies that

$$\frac{w_g(g)}{w(g)} = \frac{r}{2g}, \implies \ln|w| = \frac{r}{2}\ln|g| + C,\ C = \text{constant}. \tag{4.10}$$

Therefore, from (4.10), we obtain

$$\begin{cases} g' = w(g) = c_1 g, & \text{if } r = 2, \\ 0 \le \big[(g')^2\big]^{1/r} = \big(w^2(g)\big)^{1/r} = c_1 g, & \text{if } r \neq 2, \end{cases} \quad c_1 \in \mathbb{R},$$

i.e.,

$$\begin{cases} \frac{g'}{g} = c_1, & \text{if } r = 2, \\ g'(gc_1)^{-r/2} = 1, \underset{y = c_1 g \ge 0}{\implies} y'y^{-r/2} = c_1, & \text{if } r \neq 2, \end{cases}$$

which easily provides an expression for g,

$$\begin{cases} g(s) = c_2 e^{c_1 s}, & \text{if } r = 2, \\ g(s) = \frac{1}{c_1}\big[(c_1 s + c_2)\big(-\frac{r}{2} + 1\big)\big]^{\frac{1}{-\frac{r}{2}+1}}, & \text{if } r \neq 2, \end{cases} \quad c_1, c_2 \in \mathbb{R}, \tag{4.11}$$

requiring that, for $r \neq 2$, the constants c_1, c_2 and r are selected so that the term in the squared parenthesis is non-negative.

Now, we recall that we are looking for the solution of the type (4.1), (4.2) to the system (3.4), thus, using (4.11), by redefining in a suitable way the function f, we can rewrite $u(s,t)$ as

$$u(s,t) = f(\eta) = \begin{cases} f\big(\frac{e^{c_1 s}}{h(t)}\big), & \text{with } g(s) = e^{c_1 s}, \\ f\big(\frac{c_1 s + c_2}{h(t)}\big), & \text{with } g(s) = c_1 s + c_2. \end{cases} \tag{4.12}$$

By using (4.3) and (4.12), we have

$$\sigma(t) = \begin{cases} \frac{\log|h(t)|}{c_1}, \\ \frac{h(t) - c_2}{c_1}, \end{cases} \implies \dot\sigma(t) = \begin{cases} \frac{\dot h(t)}{h(t)c_1}, \\ \frac{\dot h(t)}{c_1}. \end{cases} \tag{4.13}$$

Thus, by plugging (4.13) into $(4.4)_3$, we have

$$\Lambda\dot{\sigma}^2 = \begin{cases} \Lambda\left(\frac{\dot{h}(t)}{h(t)c_1}\right)^2 = \frac{\dot{h}(t)}{h(t)} f'(1), \\ \Lambda\left(\frac{\dot{h}(t)}{c_1}\right)^2 = \frac{\dot{h}(t)}{h(t)} f'(1), \end{cases}$$

i.e.,

$$\begin{cases} \Lambda\frac{\dot{h}(t)}{h(t)c_1^2} = f'(1), \implies h(t) = d_1 e^{\frac{c_1^2 f'(1)}{\Lambda}t}, \\ \Lambda\frac{\dot{h}(t)}{c_1^2} = \frac{f'(1)}{h(t)}, \implies h(t)^2 = \frac{2c_1^2 f'(1)}{\Lambda}t + d_1, \end{cases} \qquad d_1 = \text{constant}. \tag{4.14}$$

Consequently, by using (4.14), formula (4.12) becomes

$$u(s,t) = f(\eta) = \begin{cases} f\left(\frac{e^{c_1 s}}{e^{c_3 t}}\right), & \text{with } g(s) = e^{c_1 s}, \\ f\left(\frac{c_1 s + c_2}{\sqrt{c_3 t + c_4}}\right), & \text{with } g(s) = c_1 s + c_2, \end{cases} \tag{4.15}$$

with c_1, c_2, c_3, c_4 constants. Moreover, (4.15) can be also rewritten as

$$u(s,t) = f(\eta) = \begin{cases} f(e^{b_1 s + b_2 t}), \\ f\left(\frac{s+b_1}{\sqrt{t+b_2}}\right), \end{cases} \tag{4.16}$$

with $b_{1,2}$ constants and $t + b_2 > 0$.

Now, plugging (4.16) into $(3.4)_1$, we have that

$$\begin{cases} f'(\eta)b_2 e^{b_1 s + b_2 t} = f''(\eta)b_1^2 e^{2(b_1 s + b_2 t)} + f'(\eta)b_1^2 e^{b_1 s + b_2 t} + f'(\eta)b_1 e^{b_1 s + b_2 t} B(s), \\ - f'(\eta)(s+b_1)\frac{(t+b_2)^{-3/2}}{2} = f''(\eta)(t+b_2)^{-1} + f'(\eta)B(s)(t+b_2)^{-1/2}, \end{cases}$$

i.e.,

$$\begin{cases} f'(\eta)b_2 = f''(\eta)b_1^2\eta + f'(\eta)b_1^2 + f'(\eta)b_1 B(s), \\ -f'(\eta)\frac{\eta}{2} = f''(\eta) + f'(\eta)B(s)(s+b_1)\eta^{-1}, \end{cases} \tag{4.17}$$

from which we deduce the admissible values for B,

$$B(s) = \begin{cases} m, \\ \frac{m}{s+b_1}, \end{cases} \qquad m = \text{constant}. \tag{4.18}$$

Equations appearing in (4.17) are analytically solvable and in the first case we get $f(\eta)$ proportional to a power of η, while in the second case $f(\eta)$ is the hypergeometric function.

Now, we show that if we consider

$$\frac{g''}{h} + g\frac{\dot{h}}{h^2} + B\frac{g'}{h} = 0, \tag{4.19}$$

the previous case is essentially retrieved. Indeed, if (4.19) holds true, formula $(4.4)_1$ entails: (i) $f'' = 0$, or (ii) $\frac{g'}{h} = 0$. The second possibility is trivially excluded because of being incompatible with the hypothesis of similarity solution. Considering the first possibility, i.e., $f'' = 0$, rewriting system (4.4), we obtain

$$\begin{cases} f''(\eta) = 0, \\ (4.19) \Rightarrow \frac{\dot{h}(t)}{h(t)} = -\frac{B(s)g'(s)+g''(s)}{g(s)} = C_1 = \text{constant}, \\ f(1) = 0, \\ \Lambda\dot{\sigma}^2 = \frac{\dot{h}(t)}{h(t)} f'(1). \end{cases} \tag{4.20}$$

By using $(4.20)_{1,3}$, we obtain the expression of f, i.e.,

$$f(\eta) = C_2(\eta - 1), \quad C_2 \in \mathbb{R},$$

from which, by exploiting $(4.20)_4$, we have

$$\Lambda\dot{\sigma}^2 = \frac{\dot{h}(t)}{h(t)} f'(1) = \text{constant}, \implies \sigma(t) = C_3 t + C_4,\ C_{3,4} \in \mathbb{R}.$$

Then, recalling $(4.20)_2$, we obtain the following expression for h:

$$h(t) = e^{C_1 t} C_5, \quad C_5 \in \mathbb{R}, \tag{4.21}$$

and, since (4.3) holds true, we get

$$\frac{g\big(\sigma(t)\big)}{h(t)} = 1, \iff g\big(\sigma(t)\big) = h(t) = e^{C_1 t} C_5, \tag{4.22}$$

i.e., $C_3 = C_1$,

$$\sigma(t) = C_1 t + C_4. \tag{4.23}$$

From (4.22), we deduce the functional form of $g(s)$,

$$g(s) = e^s C_6, \tag{4.24}$$

where C_6 is a suitable constant. Thus, from $(4.20)_2$, we get the admissible values for $B(s)$:

$$-\frac{B(s)g'(s) + g''(s)}{g(s)} = -B(s) - 1 = C_1, \implies B(s) = \text{constant}. \tag{4.25}$$

Therefore, expressions (4.21), (4.24), and (4.25) correspond to solutions $(4.16)_1$ and $(4.18)_1$, i.e., we have retrieved a result obtained assuming that the coefficient of f' in $(4.4)_1$ is non-vanishing.

5. Profiles $q(z)$ for which self-similar solutions may exist

We rewrite (4.18) exploiting (3.5) as

$$\frac{\hat{q}'(s)}{\hat{q}(s)} = B(s) = \begin{cases} m, \\ \frac{m}{s+b_1}, \end{cases} \tag{5.1}$$

i.e.,

$$\hat{q}(s) = \begin{cases} Ae^{ms}, \\ A|s+b_1|^m, \end{cases} \qquad A = \text{constant} \in \mathbb{R}. \tag{5.2}$$

We point out that, at this stage, the problem of finding profiles $q(z)$ which admit self-similar solution is not yet completely solved since (5.2) provides just $\hat{q}(s)$.

In the sequel, we illustrate the procedure for finding $q(z)$ from (5.2).

We first analyze $(5.2)_1$, which coupled with (3.2) implies that

$$q(z) = \hat{q}\big(s(z)\big) = Ae^{m\int_0^z \sqrt{1+q'(\xi)^2}d\xi}. \tag{5.3}$$

We can assume, without loss of generality, $A > 0$. By differentiating (5.3) with respect to z, we get

$$q'(z) = q(z)m\big(1+q'(z)^2\big)^{1/2},$$

from which

$$(q')^2 = \frac{m^2q^2}{1-m^2q^2}, \implies q' = \pm\frac{mq}{\sqrt{1-m^2q^2}}, \tag{5.4}$$

for $|q(z)| \leq 1/|m|$ with $m \neq 0$.

By denoting $v = \sqrt{1-m^2q^2}$, so that $m^2q^2 = 1-v^2$, we rewrite (5.4) as

$$-\frac{v^2}{m^2q^2}v' = \pm m, \quad \text{where } v' = \frac{dv(z)}{dz},$$

thus,

$$-\frac{v^2v'}{1-v^2} = \pm m, \implies v' - \frac{v'}{1-v^2} = \pm m,$$

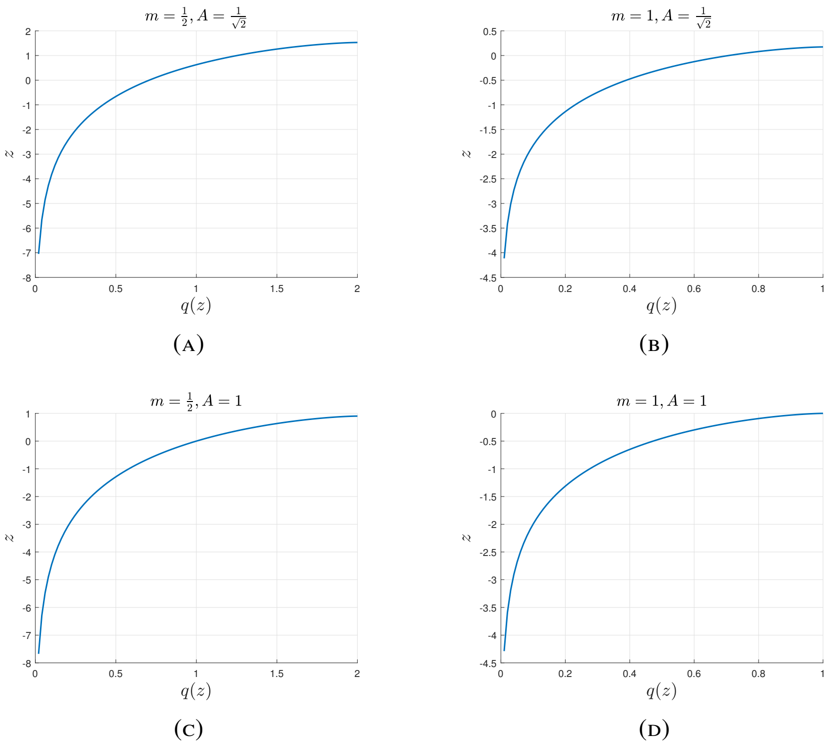

FIGURE 3. Plot of $q(z)$ with a positive sign when $B(s) = m$ with (A,B) $A = 1/\sqrt{2}$ and (C,D) $A = 1$ for different values of m. The surface of revolution is generated by rotating $q(z)$ around the z-axis. The plot of $q(z)$ with a negative sign is obtained by means of a reflection with respect to the horizontal axis.

i.e.,

$$v(z) - v(0) - \operatorname{arctanh}\big(v(z)\big) + \operatorname{arctanh}\big(v(0)\big) = \pm mz,$$

and, recalling that $v = \sqrt{1 - m^2 q^2}$ and $q(z) = x$, we get

$$z = \pm\frac{1}{m}\Big(\sqrt{1 - m^2 x^2} - \sqrt{1 - m^2 A^2} - \operatorname{arctanh}(\sqrt{1 - m^2 x^2}) + \operatorname{arctanh}(\sqrt{1 - m^2 A^2})\Big), \tag{5.5}$$

where $q(0) = A$. Formula (5.5) provides an implicit expression of $q(z)$. The profile of $q(z)$ is displayed in Figure 3 for different values of m and A.

Now, we focus on $(5.2)_2$. Recalling (3.2), we have

$$q(z) = \hat{q}\big(s(z)\big) = A\left|\int_0^z \sqrt{1 + q'(\xi)^2}\,d\xi + b_1\right|^m. \tag{5.6}$$

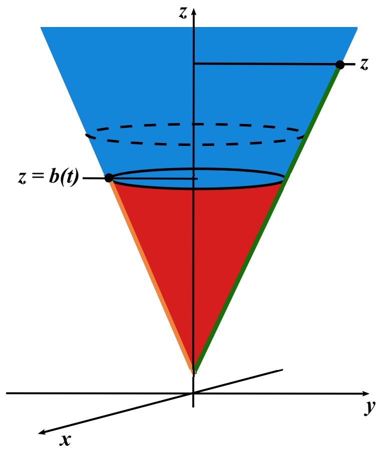

FIGURE 4. Geometry of the surface of revolution given by (5.9).

Proceeding similarly to the previous case, we differentiate (5.6) and then squaring, we get

$$(q')^2 = m^2 A^{2/m} q^{2-2/m}\big[1 + (q')^2\big],$$

i.e., for $|q(z)| \leq 1/|mA^{1/m}|^{\frac{m}{m-1}}$ with $m \neq 0$,

$$\frac{q'\sqrt{1 - m^2 A^{2/m} q^{2-2/m}}}{q^{1-1/m}} = \pm m A^{1/m}, \tag{5.7}$$

$$\implies z = \pm \frac{1}{mA^{1/m}} \int_{q_0}^{q(z)} \frac{\sqrt{1 - m^2 A^{2/m} w^{2-2/m}}}{w^{1-1/m}} dw,$$

where $q_0 = q(0) = A|b_1|^m$. Recalling that $x = q(z)$, we get

$$z = \pm \frac{1}{mA^{1/m}} \int_{x_0}^{x} \frac{\sqrt{1 - m^2 A^{2/m} w^{2-2/m}}}{w^{1-1/m}} dw, \tag{5.8}$$

where $x_0 = q_0 = A|b_1|^m$. If $m = 1$, equation (5.8) reduces to

$$z = Cx + D, \tag{5.9}$$

i.e., the surface of revolution is a cone as depicted in Figure 4.

If $m = 2$, recalling $q(z) = x$, equation (5.7) reduces to

$$z = \pm \frac{1}{2A^{1/2}} \int_{x_0}^{x} \frac{\sqrt{1 - 4Aw}}{w^{1/2}} dw,$$

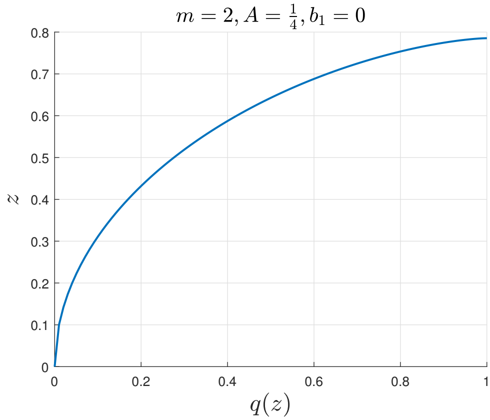

FIGURE 5. Plot of $q(z)$ with a positive sign when $B(s)$ is given by (5.1)$_2$ with $A = 1/4$, $b_1 = 0$, and $m = 2$. The surface of revolution is generated by rotating $q(z)$ around the z-axis. The plot of $q(z)$ with a negative sign is obtained by means of a reflection with respect to the horizontal axis.

which, defining $y = A^{1/2}w^{1/2}$ so that $Aw = y^2$ and $dw = 2y/A\,dy$, becomes

$$\begin{aligned} z &= \pm\frac{1}{A}\int_{A^{1/2}x_0^{1/2}}^{A^{1/2}x^{1/2}} \sqrt{1-(2y)^2}dy \\ &= \pm\frac{1}{4A}\big(2\sqrt{Ax}\sqrt{1-4Ax} + \arcsin(2\sqrt{Ax}) \\ &\qquad\qquad - 2\sqrt{Ax_0}\sqrt{1-4Ax_0} - \arcsin(2\sqrt{Ax_0})\big). \end{aligned}$$

Figure 5 shows the plot of $q(z)$ for $m = 2$.

It is worth noting that (5.5) and (5.8) hold true when $m \neq 0$. If $m = 0$, then (5.2) entails $\hat{q}(s(z)) = A$. Therefore, recalling (3.3), the surface of revolution is a cylinder. In this case, problem (3.4) reduces to the classical 1D Stefan problem with one phase, namely

$$\begin{cases} u_t = u_{ss}, \\ u\big(\sigma(t), t\big) = 0, \\ \Lambda\dot{\sigma}(t) = \frac{\partial u}{\partial s}\big|_{\sigma}, \end{cases}$$

which is known to have the classical $\frac{s}{2\sqrt{t}}$ similarity variable.

We remark that the procedure here developed highlights the existence, besides the cylinder, of other surfaces of revolution for which the similarity variable is of the type $s/\sqrt{t}$, as in formula (4.16). In particular, the set of surfaces admitting $s/\sqrt{t}$ as similarity variable is given by the implicit equation (5.8).

6. Discussion and Conclusion

We investigated the one-phase Stefan problem on revolution surfaces, looking for those ones for which the problem admits a self-similar solution. In particular, we considered the azimuthal symmetry of the thermal field.

If $x = q(z)$ is the function originating the surface of revolution, we have shown that self-similar solutions can exist only for particular choices of $q(z)$.

Moreover, the results show that the problem admits the classical similarity variable of the type $s/\sqrt{t}$ not only when the surface of revolution is a cylinder (as expected) but also when the surface is given by (5.8).

Operating in this way, we have been able to determine and classify the most general forms of the profiles $q(z)$ entailing the existence of similarity solutions.

Possible future developments could involve self-similar solutions on moving surfaces as well as the two-phase Stefan problem.

Appendix

Let us look for solutions in the form

$$u(s,t) = k(t) f(\eta), \tag{A.1}$$

with $k(t)$ not constant, η given by (4.2), and with (4.3) which still holds true. In this case, system (3.4) becomes

$$\begin{cases} f''\left(\frac{g'}{h}\right)^2 + \left(\frac{g''}{h} + g\frac{\dot{h}}{h^2} + B\frac{g'}{h}\right) f' - \frac{\dot{k}}{k} f = 0, \\ f(1) = 0, \\ \Lambda\dot{\sigma}^2 = k(t)\frac{\dot{h}(t)}{h(t)} f'(1), \end{cases} \tag{A.2}$$

since

$$\left.\frac{\partial u}{\partial s}\right|_{\sigma} = k(t)\frac{\dot{h}(t)}{\dot{\sigma}(t)h(t)} f'(1).$$

By multiplying equation $(\mathrm{A.2})_1$ by $\frac{k}{\dot{k}}$ (recall that we can assume $\dot{k} \neq 0$), we have

$$\begin{cases} \left(\frac{g'}{h}\right)^2\frac{k}{\dot{k}} = \phi_1(\eta), \\ \left(\frac{g''}{h} + g\frac{\dot{h}}{h^2} + B\frac{g'}{h}\right)\frac{k}{\dot{k}} = \phi_2(\eta), \end{cases} \tag{A.3}$$

where ϕ_1, ϕ_2 are arbitrary (at this stage) functions of η. Now, setting

$$H(t) = \frac{1}{h(t)}, \underset{(4.2)}{\Longrightarrow} \eta = H(t)g(s), \tag{A.4}$$

$$\frac{k(t)}{\dot{k}(t)} = K(t), \tag{A.5}$$

and

$$\left(\frac{g'}{g}\right)^2 = G(s), \quad G(s) \geq 0, \tag{A.6}$$

equation (A.3)$_1$ entails

$$\phi_1(\eta) = \phi_1\big(g(s)H(t)\big) = \left(\frac{g'}{h}\right)^2 \frac{k}{\dot{k}} = \underbrace{\left(\frac{g'}{g}\right)^2}_{G(s)} \underbrace{\left(\frac{g}{h}\right)^2}_{\eta^2} \underbrace{\frac{k}{\dot{k}}}_{K(t)} = G(s)K(t)\eta^2,$$

namely,

$$G(s)K(t) = \phi_3(\eta), \quad \text{with } \phi_3(\eta) = \phi_3\big(g(s)H(t)\big) = \frac{\phi_1(\eta)}{\eta^2}. \tag{A.7}$$

By differentiating both sides of (A.7) with respect to s and t, we get

$$G'(s)K(t) = \frac{\partial}{\partial s}\big(\phi_3\big(g(s)H(t)\big)\big) = \phi_3'(\eta)g'(s)H(t),$$
$$G(s)\dot{K}(t) = \frac{\partial}{\partial t}\big(\phi_3\big(g(s)H(t)\big)\big) = \phi_3'(\eta)g(s)\dot{H}(t),$$

respectively, whose ratio gives

$$\frac{G'(s)}{G(s)}\frac{K(t)}{\dot{K}(t)} = \frac{g'(s)}{g(s)}\frac{H(t)}{\dot{H}(t)},$$

i.e.,

$$\frac{g'(s)}{g(s)}\frac{G(s)}{G'(s)} = \frac{\dot{H}(t)}{H(t)}\frac{K(t)}{\dot{K}(t)} = \frac{1}{2p} = \text{constant}. \tag{A.8}$$

By using definitions (A.4)–(A.6) and equation (A.8), we have

$$\begin{cases} \frac{g'(s)}{g(s)} = \frac{1}{2p}\frac{G'(s)}{G(s)}, \\ \frac{\dot{H}(t)}{H(t)} = \frac{1}{2p}\frac{\dot{K}(t)}{K(t)}, \end{cases} \Longrightarrow \begin{cases} g(s) = C_1 G^{\frac{1}{2p}}(s) = C_1\left(\frac{g'}{g}\right)^{\frac{1}{p}}, \\ \frac{1}{h(t)} = H(t) = C_2|K|^{\frac{1}{2p}}(t) = C_2\left|\frac{k}{\dot{k}}\right|^{\frac{1}{2p}}, \end{cases}$$

from which we have

$$g'(s) = C_1^{-p}g^{p+1}(s), \Longrightarrow \begin{cases} g(s) = \left[-p(C_1^{-p}s + D_1)\right]^{-\frac{1}{p}}, \\ g''(s) = C_1^{-2p}(p+1)g^{2p+1}(s), \end{cases} \tag{A.9}$$

$$\left|\frac{\dot{k}(t)}{k(t)}\right| = \big(C_2 h(t)\big)^{2p}, \Longrightarrow \frac{\dot{k}(t)}{k(t)} = \pm(C_2)^{2p}h(t)^{2p}, \tag{A.10}$$

with C_1 and D_1 suitable constants such that the term in squared bracket in (A.9)$_1$ is well defined.

Inserting (A.10) in $(\text{A.3})_2$ by assuming $p \neq 0$ and defining $\widehat{C}_2 = \pm(C_2)^{2p}$, we get

$$\left(\frac{g''}{h} + g\frac{\dot{h}}{h^2} + B(s)\frac{g'}{h}\right)\frac{1}{h^{2p}} = \phi_2(\eta)\widehat{C}_2,$$

which implies

$$\frac{g''}{h^{2p+1}} + g\frac{\dot{h}}{h^{2p+2}} + B(s)\frac{g'}{h^{2p+1}} = \phi_2(\eta)\widehat{C}_2 = \phi_3(\eta),$$

i.e.,

$$\frac{g''}{g^{2p+1}} + \frac{g\dot{h}}{g^{2p+1}h} + B(s)\frac{g'}{g^{2p+1}} = \phi_3(\eta)\frac{h^{2p+1}}{g^{2p+1}} = \frac{\phi_3(\eta)}{\eta^{2p+1}} = \phi_4(\eta). \tag{A.11}$$

Differentiating (A.11) with respect to t,

$$\dot{\left(\frac{\dot{h}}{h}\right)}\frac{1}{g^{2p}} = \phi_{4,\eta}\, g\,\dot{\left(\frac{1}{h}\right)},$$

i.e.,

$$\phi_{4,\eta}(\eta) = -\left(\frac{\ddot{h}h - (\dot{h})^2}{\dot{h}}\right)\frac{1}{h^{2p+1}}\frac{1}{\eta^{2p+1}}. \tag{A.12}$$

Let us fix t and integrate (A.12) in η which is still free to move. We get

$$\phi_4(\eta) = K_0\eta^{-2p} + K_1, \qquad K_0 = \frac{\ddot{h}h - (\dot{h})^2}{2p\dot{h}h^{2p+1}}, \qquad K_1 = \text{constant},$$

where K_0 is also a constant since it is a function of the sole variable t, hence,

$$\frac{\ddot{h}h - (\dot{h})^2}{\dot{h}h^{2p+1}} = 2pK_0.$$

Setting $\dot{h}(t) = v(h)$ and recalling that $\dot{h} \neq 0$, we obtain

$$\frac{v_h v h - v^2}{v h^{2p+1}} = 2pK_0,$$

i.e.,

$$\frac{v_h}{h} - \frac{v}{h^2} = 2ph^{2p-1}K_0, \implies \frac{\dot{h}}{h} = \frac{v}{h} = K_0 h^{2p} + K_1. \tag{A.13}$$

On the other hand, from

$$\frac{g\big(\sigma(t)\big)}{h(t)} = 1,$$

and

$$g(s) = \left[-p(C_1^{-p}s + D_1)\right]^{-\frac{1}{p}} = (as+b)^{-\frac{1}{p}}, \quad b = -pD_1 = \text{constant},$$

where $a = -pC_1^{-p} = \text{constant} \neq 0$ so that the solution in (A.1) is not a function of the sole variable t, we have

$$a\sigma(t) + b = \big(h(t)\big)^{-p} \implies \sigma(t) = \frac{1}{a\big(h(t)\big)^p} - \frac{b}{a},$$

and

$$\dot\sigma = -\frac{p}{a}h^{-p-1}\dot h \implies (\dot\sigma)^2 = \frac{p^2}{a^2}h^{-2p-2}(\dot h)^2,$$

thus, from $(\text{A.2})_3$, we obtain

$$\Lambda\frac{p^2}{a^2}\frac{\big(\dot h(t)\big)^2}{h^{2p+2}(t)} = k(t)\frac{\dot h(t)}{h(t)}f'(1).$$

Using (A.13), we get

$$\text{(A.14)} \qquad \Lambda\frac{p^2}{a^2}\frac{K_0h^{2p}+K_1}{h^{2p}} = k(t)f'(1),$$
$$\implies k(t) = \frac{\Lambda}{f'(1)}\frac{p^2}{a^2}\left(K_0 + \frac{K_1}{h^{2p}}\right) = \lambda_0 + \lambda_1 h^{-2p},$$

where $f'(1) \neq 0$ so that $\sigma \neq$ constant, thus,

$$\frac{\dot k(t)}{k(t)} = \frac{\lambda_1(-2p)h^{-2p-1}(t)\dot h(t)}{\lambda_0 h^{2p}+\lambda_1}h^{2p}$$
$$= \frac{-2p\lambda_1 h^{-1}\dot h}{\lambda_0 h^{2p}+\lambda_1} \underset{(\text{A.13})}{=} \frac{-2p\lambda_1(K_0h^{2p}+K_1)}{\lambda_0h^{2p}+\lambda_1},$$

i.e.,

$$\left|\frac{\dot k(t)}{k(t)}\right| = \left|\frac{-2p\lambda_1(K_0h^{2p}+K_1)}{\lambda_0h^{2p}+\lambda_1}\right| \underset{(\text{A.10})}{=} C_2^{2p}h^{2p},$$

which, recalling that $p \neq 0$, is only possible if $\lambda_0 = 0$ and $K_1 = 0$. However, $K_1 = 0$ implies in (A.14) that $k(t) = \text{constant}$ and we go back to the case already treated in the previous section of the paper.

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