Rend. Lincei Mat. Appl. 35 (2024), 311–321
DOI 10.4171/RLM/1041

Partial Differential Equations. – *Higher regularity theory for (s,p)-harmonic functions*, by Verena Bögelein, Frank Duzaar, Naian Liao, Giovanni Molica Bisci and Raffaella Servadei, communicated on 8 November 2024.

Abstract. – In this note, we announce some new regularity estimates concerning higher Sobolev and Hölder regularity for (s,p)-harmonic functions.

Keywords. – fractional p-Laplacian, gradient regularity, Hölder regularity.

Mathematics Subject Classification 2020. – 47G20 (primary); 35B65, 35J70, 35R09 (secondary).

1. Introduction

Recently, nonlocal fractional problems have attracted the interest of the mathematical community both for pure academic research and for their wide applications in many different fields, such as continuum mechanics, phase transition, population dynamics, optimal control, game theory, just to name a few (see, for instance, [2, 3, 9, 19] and the references therein). Motivated by this wide interest we consider the following fractional p-Laplace equation:

$$(-\Delta_p)^s u = 0 \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^N$ with $N \geq 2$, $p > 1$, $s \in (0,1)$ and $(-\Delta_p)^s$ is the fractional p-Laplace operator defined as

$$(-\Delta_p)^s u := \text{p.v.} \int_{\mathbb{R}^N} \frac{2\big|u(x)-u(y)\big|^{p-2}\big(u(x)-u(y)\big)}{|x-y|^{N+sp}}\,\mathrm{d}x.$$

Its solutions are termed the (s,p)-harmonic functions; see Definition 2.1. Equation (1.1) can be seen as the nonlocal counterpart of the p-Laplace equation

$$-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0. \tag{1.2}$$

Classical results state that the gradient of local weak solutions of (1.2) is locally Hölder continuous; see [11, 13, 16, 18] for equations and [1, 15, 17] for systems.

Up to now it is unclear whether an analogue of this result holds true for the fractional p-Laplace equation (1.1). Regularity theory for equations involving the fractional

p-Laplace operator is a challenging open problem and only partial results are present in the current literature. Local boundedness and Hölder regularity with an implicit Hölder exponent were obtained in [10] for every $s \in (0,1)$ and $p \in (1,\infty)$.

When it comes to higher regularity, the situation is more delicate and the picture is less clear. For instance, it is far from trivial to assert that the gradient of an (s,p)-harmonic function u exists in the Sobolev sense. As far as we know, the first decisive results in this direction were proven in [7]. In the regime $p \ge 2$ and $s \in (\frac{p-1}{p}, 1)$, the authors prove that the gradient of u exists in L^p if it is globally bounded. Moreover, the range of s can be improved to $s \in (\frac{p-1}{p+1}, 1)$ when u fulfills a certain boundary condition. Again in the case $p \ge 2$, interior higher Hölder regularity with an explicit Hölder exponent was established in [8]. Provided that $s \in (0, \frac{p-1}{p}]$, the authors proved that (s,p)-harmonic functions are almost $C^{0,\frac{sp}{p-1}}_{\mathrm{loc}}$, while they are almost Lipschitz continuous whenever $s \in (\frac{p-1}{p}, 1)$.

Motivated by the broad interest, we present in this paper substantial improvements of the results from [7, 8]. More precisely, we are able to show that, when $p \ge 2$ and $s \in (\frac{p-2}{p}, 1)$, the gradient of any (s,p)-harmonic function u belongs to the fractional Sobolev space $W^{\beta,q}_{\mathrm{loc}}(\Omega)$ for any $q \ge p$ and any $\beta \in (0, \frac{p}{q}(s - \frac{p-2}{p}))$. In particular, ∇u belongs to $L^q_{\mathrm{loc}}(\Omega)$ for any $q \ge p$. As a meaningful consequence of the above results, we get that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for any $\alpha \in (0,1)$.

On the other hand, whenever $p \in (2,\infty)$ and $s \in (0, \frac{p-2}{p}]$, we are able to show that the fractional differentiability of order s can be improved to any number less than $\frac{sp}{p-2}$, whereas the integrability order q can be any number larger than p; that is, $u \in W^{\gamma,q}_{\mathrm{loc}}(\Omega)$ for any $q \ge p$ and $\gamma \in [s, \frac{sp}{p-2})$. Consequently, as a byproduct of the aforementioned results, we also have that

$$u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega) \quad \text{for any } \gamma \in \Big(0, \frac{sp}{p-2}\Big).$$

All regularity results that we obtain for the case $p \ge 2$ are summarized in Table 1 below. Finally, in the case $p \in (1,2)$ and $s \in (0,1)$, our main result includes that any (s,p)-harmonic function u satisfies $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ for any $q \ge p$. As a consequence of this regularity result and the Morrey-type embedding, we deduce that $u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$ for any $\gamma \in (0,1)$. Prior to us, the known Hölder exponent is any $\gamma \in (0, \min\{1, \frac{sp}{p-1}\})$, obtained in [14]. In addition, we also have acquired some fractional differentiability results for ∇u.

$s \in \big(0, \frac{p-2}{p}\big]$	$s \in \big(\frac{p-2}{p}, 1\big)$
$u \in W^{\gamma,q}_{\mathrm{loc}}(\Omega)$, $\forall \gamma \in \big[s, \frac{sp}{p-2}\big)$	$\nabla u \in W^{\gamma,q}_{\mathrm{loc}}(\Omega)$, $\forall \gamma \in \big(0, \frac{p}{q}\big(s - \frac{p-2}{q}\big)\big)$
$u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$, $\forall \gamma \in \big(0, \frac{sp}{p-2}\big)$	$u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$, $\forall \gamma \in (0,1)$

TABLE 1. Main results for $p \ge 2$.

2. Preliminaries

In this section, we introduce the notation and the definition used along the paper.

- *Hölder space* $C^{0,\alpha}_{\mathrm{loc}}(\Omega)$. For $\alpha \in (0, 1)$, this space consists of locally bounded functions with the finite semi-norm in a ball $B_R(x_o) \Subset \Omega$ given by

$$[w]_{C^{0,\alpha}(B_R(x_o))} := \sup_{x \neq y \in B_R(x_o)} \frac{|w(x) - w(y)|}{|x - y|^\alpha}.$$

- *Fractional Sobolev space* $W^{\gamma,q}_{\mathrm{loc}}(\Omega)$. For $q \in [1, \infty)$ and $\gamma \in (0, 1)$, this space consists of all $L^q_{\mathrm{loc}}(\Omega)$-functions with the finite semi-norm in a ball $B_R(x_o) \Subset \Omega$ given by

$$[w]_{W^{\gamma,q}(B_R(x_o))} := \left[\iint_{B_R(x_o)\times B_R(x_o)} \frac{|w(x) - w(y)|^q}{|x - y|^{N+\gamma q}} \,\mathrm{d}x\mathrm{d}y \right]^{\frac{1}{q}}.$$

- *Tail*. It is defined as follows:

$$\mathrm{Tail}(u; x_o, R) := \left[R^{sp} \int_{\mathbb{R}^N \setminus B_R(x_o)} \frac{|u(x)|^{p-1}}{|x - x_o|^{N+sp}} \,\mathrm{d}x \right]^{\frac{1}{p-1}}.$$

In the above notation, if $x_o = 0$ or if the center point is clear from the context, we omit it. We end this section with the definition of (s, p)-harmonic functions.

Definition 2.1 ((s, p)-harmonic functions). Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $p \in (1, \infty)$ and $s \in (0, 1)$. A function $u \in W^{s,p}_{\mathrm{loc}}(\Omega)$ is called (s, p)-harmonic in Ω if and only if

$$\int_{\mathbb{R}^N} \frac{\big|u(x)\big|^{p-1}}{\big(1 + |x|\big)^{N+sp}} \,\mathrm{d}x < \infty,$$

and

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\big|u(x) - u(y)\big|^{p-2}\big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x - y|^{N+sp}} \,\mathrm{d}x\mathrm{d}y = 0 \tag{2.1}$$

for every $\varphi \in W^{s,p}(\Omega)$ compactly supported in Ω and extended to 0 outside Ω.

3. Regularity of (s, p)-harmonic functions when $p \geq 2$

In this section, we announce the regularity results for (s, p)-harmonic functions, obtained recently in [5]. In order for this, we need to distinguish two cases: when $s \in (0, \frac{p-2}{p}]$ and when $s \in (\frac{p-2}{p}, 1)$. For the sake of simplicity, we formulate all statements for locally bounded (s, p)-harmonic functions, keeping in mind that any (s, p)-harmonic function is locally bounded.

3.1. *Regularity results when $s \in (0, \frac{p-2}{p}]$*

This section is devoted to the regularity of (s, p)-harmonic functions in the case $s \in (0, \frac{p-2}{p}]$, with $p \geq 2$. In this framework, the main results upgrade the solution to a higher fractional Sobolev space.

THEOREM 3.1 (Almost $W^{\frac{sp}{p-2},q}$-regularity). *Let $p \in (2, \infty)$ and $s \in (0, \frac{p-2}{p}]$. Then, for any locally bounded (s, p)-harmonic function in Ω, we have*

$$u \in W^{\gamma,q}_{\mathrm{loc}}(\Omega) \quad \text{for any } q \in [p, \infty) \text{ and } \gamma \in \Big[s, \frac{sp}{p-2}\Big).$$

Moreover, there exists a universal constant $C = C(N, p, s, q, \gamma) \geq 1$ such that for any ball $B_R \equiv B_R(x_o) \Subset \Omega$, we have

$$[u]_{W^{\gamma,q}(B_{\frac{1}{2}R})} \leq \frac{C}{R^\gamma}\Big[R^{s-N(\frac{1}{p}-\frac{1}{q})}[u]_{W^{s,p}(B_R)} + R^{\frac{N}{q}}\big(\|u\|_{L^\infty(B_R)} + \mathrm{Tail}(u; R)\big)\Big].$$

The constant C blows up as $\gamma \uparrow \frac{sp}{p-2}$. ■

For the proof we refer to [5, Theorem 1.1]. Using Theorem 3.1 and the Morrey-type embedding for fractional Sobolev spaces, we immediately obtain the next regularity result.

THEOREM 3.2 (Almost $C^{0,\frac{sp}{p-2}}$-regularity). *Let $p \in (2, \infty)$ and $s \in (0, \frac{p-2}{p}]$. Then, for any locally bounded (s, p)-harmonic function in Ω, we have*

$$u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega) \quad \text{for any } \gamma \in \Big(0, \frac{sp}{p-2}\Big).$$

Moreover, there exists a universal constant $C = C(N, p, s, \gamma) \geq 1$ such that for any ball $B_R \equiv B_R(x_o) \Subset \Omega$, we have

$$[u]_{C^{0,\gamma}(B_{\frac{1}{2}R})} \leq \frac{C}{R^\gamma}\Big[R^{s-\frac{N}{p}}[u]_{W^{s,p}(B_R)} + \|u\|_{L^\infty(B_R)} + \mathrm{Tail}(u; R)\Big].$$

The constant C blows up as $\gamma \uparrow \frac{sp}{p-2}$. ■

For the proof we refer to [5, Theorem 1.2].

3.2. *Regularity results when $s \in (\frac{p-2}{p}, 1)$*

In the case of $p \geq 2$ and $s \in (\frac{p-2}{p}, 1)$, better regularity properties of local weak solutions of (1.1) can be achieved. The first result states that for any (s, p)-harmonic function, the weak gradient ∇u exists and is locally in $L^p(\Omega, \mathbb{R}^N)$.

THEOREM 3.3 (L^p-gradient regularity). *Let $p \in [2,\infty)$ and $s \in (\frac{p-2}{p}, 1)$. Then, for any locally bounded (s,p)-harmonic function in Ω, we have*

$$u \in W^{1,p}_{\mathrm{loc}}(\Omega).$$

Moreover, there exists a universal constant $C = C(N,p,s)$ such that for any ball $B_R \equiv B_R(x_o) \Subset \Omega$, we have

$$\|\nabla u\|_{L^p(B_{\frac{1}{2}R})} \le \frac{C}{R}\big[R^s(1-s)^{\frac{1}{p}}[u]_{W^{s,p}(B_R)} + R^{\frac{N}{p}}\big(\|u\|_{L^\infty(B_R)} + \mathrm{Tail}(u;R)\big)\big].$$

The constant C is stable as $s \uparrow 1$ and blows up as $s \downarrow \frac{p-2}{p}$. ∎

For the proof we refer to [5, Theorem 1.3]. The strategy of the proof relies on the fact that for any (s,p)-harmonic function the difference quotient

$$\frac{u(x+h)-u(x)}{|h|}$$

can be uniformly bounded in L^p. This is proved using an iteration scheme improving the fractional differentiability by increasing the power of the increment $|h|$. This procedure allows us to improve the regularity of u from $W^{s,p}_{\mathrm{loc}}$ to $W^{1,p}_{\mathrm{loc}}$.

Having the L^p-regularity of the gradient at hand, a natural question arises: can this regularity be improved? The answer is affirmative. Indeed, for any $q > p$, the L^q-gradient regularity holds. This higher gradient regularity is the core of the next theorem. The main idea of the proof is to use a Moser-type iteration argument, improving the $W^{1,p}_{\mathrm{loc}}$-regularity to $W^{1,q}_{\mathrm{loc}}$ by a refined estimate for the finite second-order difference of u.

THEOREM 3.4 (L^q-gradient regularity). *Let $p \in [2,\infty)$ and $s \in (\frac{p-2}{p}, 1)$. Then, for any locally bounded (s,p)-harmonic function in Ω, we have*

$$u \in W^{1,q}_{\mathrm{loc}}(\Omega) \quad \text{for any } q \in [p,\infty).$$

Moreover, there exists a universal constant $C = C(N,p,s,q)$ such that for any ball $B_R \equiv B_R(x_o) \Subset \Omega$, the quantitative L^q-gradient estimate

$$\|\nabla u\|_{L^q(B_{\frac{1}{2}R})} \le CR^{\frac{N}{q}-1}\big[R^{s-\frac{N}{p}}(1-s)^{\frac{1}{p}}[u]_{W^{s,p}(B_R)} + \|u\|_{L^\infty(B_R)} + \mathrm{Tail}(u;R)\big]$$

holds true. The constant C is stable in the limit $s \uparrow 1$ and blows up as $s \downarrow \frac{p-2}{p}$.

For the proof we refer to [5, Theorem 1.4]. At this stage, the classical Morrey-type embedding for the Sobolev space $W^{1,q}$ with $q > N$ implies that (s,p)-harmonic functions in the regime $s \in (\frac{p-2}{p}, 1)$ are locally Hölder continuous with exponent $1 - \frac{N}{q}$. Since q can be chosen arbitrarily large according to Theorem 3.4, this means Hölder continuity for each Hölder exponent in $(0, 1)$, as stated in the next theorem.

THEOREM 3.5 (Almost Lipschitz continuity). *Let $p \in [2, \infty)$ and $s \in (\frac{p-2}{p}, 1)$. Then, for any locally bounded (s, p)-harmonic function in Ω, we have*

$$u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega) \quad \textit{for any } \gamma \in (0, 1).$$

Moreover, there exists a universal constant $C = C(N, p, s, \gamma)$ such that for any ball $B_R \equiv B_R(x_o) \Subset \Omega$, we have

$$[u]_{C^{0,\gamma}(B_{\frac{1}{2}R})} \le \frac{C}{R^\gamma}\big[R^{s-\frac{N}{p}}(1-s)^{\frac{1}{p}}[u]_{W^{s,p}(B_R)} + \|u\|_{L^\infty(B_R)} + \mathrm{Tail}(u; R)\big].$$

The constant C is stable as $s \uparrow 1$ and blows up as $s \downarrow \frac{p-2}{p}$. ■

For the proof we refer to [5, Theorem 1.5]. It is a classical result that p-harmonic functions satisfy $|\nabla u|^{\frac{p-2}{2}}\nabla u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ (see [6, 17, 18]). This higher differentiability can be converted into fractional differentiability; namely, $\nabla u \in W^{\beta,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for any $0 < \beta < \frac{2}{p}$. The corresponding result for (s, p)-harmonic functions is contained in the following theorem.

THEOREM 3.6 (Almost $W^{\frac{sp-(p-2)}{q},q}$-gradient regularity). *Let $p \in [2, \infty)$ and $s \in (\frac{p-2}{p}, 1)$. Then, for any locally bounded (s, p)-harmonic function in Ω, we have*

$$\nabla u \in W^{\alpha,q}_{\mathrm{loc}}(\Omega) \quad \textit{for any } q \in [p, \infty) \textit{ and } \alpha \in \big(0, \tfrac{p}{q}\big(s - \tfrac{p-2}{p}\big)\big).$$

Moreover, there exists a universal constant $C = C(N, p, s, q, \alpha)$ such that for any ball $B_R \equiv B_R(x_o) \Subset \Omega$, we have

$$[\nabla u]_{W^{\alpha,q}(B_{\frac{1}{2}R})} \le \frac{CR^{\frac{N}{q}}}{R^{1+\alpha}}\big[R^{s-\frac{N}{p}}(1-s)^{\frac{1}{p}}[u]_{W^{s,p}(B_R)} + \|u\|_{L^\infty(B_R)} + \mathrm{Tail}(u; R)\big].$$

Moreover, the constant C is stable as $s \uparrow 1$ and blows up as $s \downarrow \frac{p-2}{p}$ and $\alpha \uparrow \frac{p}{q}(s - \frac{p-2}{p})$.

For the proof we refer to [5, Theorem 1.6]. In particular, Theorem 3.6 states that when $s \in (\frac{p-2}{p}, 1)$, any (s, p)-harmonic function satisfies

$$\nabla u \in W^{\beta,p}_{\mathrm{loc}}(\Omega) \quad \text{for any } \beta \in \big(0, s - \tfrac{p-2}{p}\big),$$

and thus formally recovers the classical result in the limit $s \uparrow 1$.

We stress that, in this sense, our range of β is sharper than the one obtained previously in [7, Corollaries 1.8 & 1.9]. Moreover, our approach dispenses with any additional assumption on the solution's global behavior and relies solely on the notion of local solution.

4. Regularity of (s, p)-harmonic functions when $p \in (1, 2)$

This section is devoted to the gradient regularity estimate for (s, p)-harmonic functions when $p \in (1, 2)$, obtained in [4].

Theorem 4.1 ($W^{1,q}$-gradient regularity). *Let $p \in (1, 2]$ and $s \in (0, 1)$. Then, for any locally bounded (s, p)-harmonic function in Ω, we have*

$$u \in W^{1,q}_{\mathrm{loc}}(\Omega) \quad \text{for any } q \in [p, \infty).$$

Moreover, there exists a universal constant $C = C(N, p, s, q)$ such that for any ball $B_R \equiv B_R(x_o) \Subset \Omega$, the quantitative L^q-gradient estimate

$$\|\nabla u\|_{L^q(B_{\frac{1}{2}R})} \le C R^{\frac{N}{q}-1} \big[R^{s-\frac{N}{p}} (1-s)^{\frac{1}{p}} [u]_{W^{s,p}(B_R)} + \|u\|_{L^\infty(B_R)} + \mathrm{Tail}(u; R) \big]$$

holds true. The constant C is stable as $s \uparrow 1$

For the proof we refer to [4, Theorem 1.2]. An immediate consequence of the previous theorem, joint with the Morrey-type embedding, is the following result.

Theorem 4.2 (Almost Lipschitz continuity). *Let $p \in (1, 2]$ and $s \in (0, 1)$. Then, for any locally bounded (s, p)-harmonic function in Ω, we have*

$$u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega) \quad \text{for any } \gamma \in (0, 1).$$

Moreover, there exists a universal constant $C = C(N, p, s, \gamma)$ such that for any ball $B_R \equiv B_R(x_o) \Subset \Omega$, we have

$$[u]_{C^{0,\gamma}(B_{\frac{1}{2}R})} \le \frac{C}{R^\gamma} \big[R^{s-\frac{N}{p}} (1-s)^{\frac{1}{p}} [u]_{W^{s,p}(B_R)} + \|u\|_{L^\infty(B_R)} + \mathrm{Tail}(u; R) \big].$$

The constant C is stable as $s \uparrow 1$ and blows up as $s \downarrow \frac{p-2}{p}$. ∎

For the proof we refer to [4, Theorem 1.3].

5. Technical aspects

To bring the novel point to light, let us consider an $(s, 2)$-harmonic function u for simplicity, take φ to be supported in $B_{\frac{2}{3}R}$ and $\varphi = 1$ in $B_{\frac{1}{2}R}$, and split its weak formulation (2.1) into two terms; namely,

$$\iint_{B_R \times B_R} \frac{\big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y$$
$$= -2 \iint_{B_R \times (\mathbb{R}^N \setminus B_R)} \frac{\big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y.$$

The right-hand side can be written as

$$-2\int_{B_R} u(x)\varphi(x)\underbrace{\left[\int_{\mathbb{R}^N\setminus B_R}\frac{\mathrm{d}y}{|x-y|^{N+2s}}\right]}_{\approx\frac{1}{R^{2s}}}\mathrm{d}x$$
$$+2\int_{B_R}\varphi(x)\underbrace{\left[\int_{\mathbb{R}^N\setminus B_R}\frac{u(y)}{|x-y|^{N+2s}}\mathrm{d}x\right]}_{=:\frac{1}{R^{2s}}\mathfrak{t}(x)}\mathrm{d}y.$$

This observation heuristically suggests viewing the problem locally as

$$(-\Delta)^s u(x) \approx \underbrace{-\frac{u(x)}{R^{2s}}+\frac{\mathfrak{t}(x)}{R^{2s}}}_{=:\mathfrak{h}(x)} \quad \text{in } B_{\frac{1}{2}R}. \tag{5.1}$$

Let us call $\mathfrak{h}(x)$ the right-hand side of (5.1). To implement the finite difference scheme to (5.1), we need to control the difference $\mathfrak{h}(x_1)-\mathfrak{h}(x_2)$ for $x_1, x_2 \in B_{\frac{1}{4}R}$. Our key observation is that $\mathfrak{t}(x)$ is a Lipschitz function. More precisely,

$$\big|\mathfrak{t}(x_1)-\mathfrak{t}(x_2)\big| \lesssim \frac{|x_1-x_2|}{R}\cdot \mathrm{Tail}(u; B_{2R}).$$

Therefore, we have

$$\big|\mathfrak{h}(x_1)-\mathfrak{h}(x_2)\big| \lesssim \frac{\big|u(x_1)-u(x_2)\big|}{R^{2s}}+\frac{|x_1-x_2|}{R}\cdot\frac{\mathrm{Tail}(u; B_{2R})}{R^{2s}}.$$

This is a rough explanation of the strategy from the general case of (s, p)-harmonic functions. The details and the technical effort in [5, Lemma 3.1] are much more involved, but this is the underlying idea. Essentially, and as experts know, the analysis of the finite difference of the tail term hinges upon finding the optimal exponent of the increment, as this will be eventually reflected in the fractional differentiability. This is the point that allows us to improve the previous results.

6. New contributions to the higher regularity theory

The topic of higher Sobolev regularity theory for (s, p)-harmonic functions was initiated in [7]. The authors established that $\nabla u \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for certain ranges of s and p, provided some additional condition was assumed for the long-range behavior of u. Later, a similar program about higher Hölder regularity was carried out in [8] and the Hölder exponent could be any number less than $\min\{1, \frac{sp}{p-1}\}$. These works were concerned with the case $p > 2$. Recently, the higher Hölder regularity has been extended to the case $p < 2$ in [14]. In p-s axis, the threshold obtained in [7, 8] is exhibited by the red curve in Figure 1.

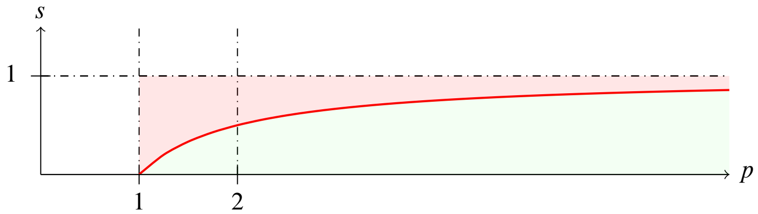

FIGURE 1. Old threshold.

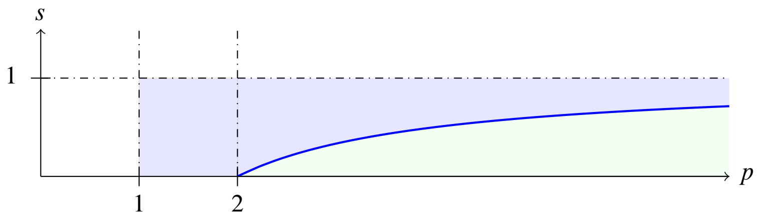

FIGURE 2. New threshold.

Our main contributions are three-fold. First, we establish the existence of ∇u in the Sobolev sense for a wider range of s and p; see Figure 2. Second, we upgrade also the integrability of ∇u to any integrability exponent. Conceivably, this will allow us to establish a Calderón-Zygmund-type theory for ∇u. Third, we improve the Hölder exponent of (s, p)-harmonic functions to any number less than $\min\{1, \frac{sp}{(p-2)_+}\}$.

For simplicity, we have chosen to prove our main results under the assumption of local boundedness and discard an inhomogeneous term on the right-hand side. As the expert knows, once this has been done, we can employ proper freezing and comparison arguments and consider general inhomogeneous terms, and also adding coefficients to the equation.

After the completion of our first work, the interesting preprint [12] appeared. Among other things the authors establish similarly that $\nabla u \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for $s \in (\frac{(p-2)_+}{p}, 1)$.

FUNDING. – N. Liao is supported by the FWF-project P36272-N *On the Stefan type problems*. G. Molica Bisci and R. Servadei have been funded by the European Union - NextGenerationEU within the framework of PNRR Mission 4 - Component 2 - Investment 1.1 under the Italian Ministry of University and Research (MUR) program PRIN 2022 - grant number 2022BCFHN2 - Advanced theoretical aspects in PDEs and their applications - CUP: H53D23001960006 and partially supported by the INdAM-GNAMPA Research Project 2024: Aspetti geometrici e analitici di alcuni problemi locali e non-locali in mancanza di compattezza - CUP E53C23001670001.

References

[1] E. Acerbi – N. Fusco, Regularity for minimizers of nonquadratic functionals: the case $1 < p < 2$. *J. Math. Anal. Appl.* **140** (1989), no. 1, 115–135. Zbl 0686.49004 MR 0997847

[2] D. Applebaum, *Lévy processes and stochastic calculus*. 2nd edn., Cambridge Stud. Adv. Math. 116, Cambridge University Press, Cambridge, 2009. Zbl 1200.60001 MR 2512800

[3] J. Bertoin, *Lévy processes*. Cambridge Tracts in Math. 121, Cambridge University Press, Cambridge, 1996. Zbl 0861.60003 MR 1406564

[4] V. Bögelein – F. Duzaar – N. Liao – G. Molica Bisci – R. Servadei, Gradient regularity for (s, p)-harmonic functions. 2024, arXiv:2409.02012v1.

[5] V. Bögelein – F. Duzaar – N. Liao – G. Molica Bisci – R. Servadei, Regularity for the fractional p-Laplace equation. 2024, arXiv:2406.01568v1.

[6] B. Bojarski – T. Iwaniec, p-harmonic equation and quasiregular mappings. In *Partial differential equations (Warsaw, 1984)*, pp. 25–38, Banach Center Publ. 19, PWN, Warsaw, 1987. Zbl 0659.35035 MR 1055157

[7] L. Brasco – E. Lindgren, Higher Sobolev regularity for the fractional p-Laplace equation in the superquadratic case. *Adv. Math.* **304** (2017), 300–354. Zbl 1364.35055 MR 3558212

[8] L. Brasco – E. Lindgren – A. Schikorra, Higher Hölder regularity for the fractional p-Laplacian in the superquadratic case. *Adv. Math.* **338** (2018), 782–846. Zbl 1400.35049 MR 3861716

[9] R. Cont – P. Tankov, *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004. Zbl 1052.91043 MR 2042661

[10] A. Di Castro – T. Kuusi – G. Palatucci, Local behavior of fractional p-minimizers. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **33** (2016), no. 5, 1279–1299. Zbl 1355.35192 MR 3542614

[11] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Anal.* **7** (1983), no. 8, 827–850. Zbl 0539.35027 MR 0709038

[12] L. Diening – K. Kim – H.-S. Lee – S. Nowak, Higher differentiability for the fractional p-Laplacian. 2024, arXiv:2406.16727v1.

[13] L. C. Evans, A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e. *J. Differential Equations* **45** (1982), no. 3, 356–373. Zbl 0508.35036 MR 0672713

[14] P. Garain – E. Lindgren, Higher Hölder regularity for the fractional p-Laplace equation in the subquadratic case. *Math. Ann.* **390** (2024), no. 4, 5753–5792. Zbl 07940270 MR 4816120

[15] M. Giaquinta – G. Modica, Remarks on the regularity of the minimizers of certain degenerate functionals. *Manuscripta Math.* **57** (1986), no. 1, 55–99. Zbl 0607.49003 MR 0866406

[16] J. L. Lewis, Regularity of the derivatives of solutions to certain degenerate elliptic equations. *Indiana Univ. Math. J.* **32** (1983), no. 6, 849–858. Zbl 0554.35048 MR 0721568

[17] K. Uhlenbeck, Regularity for a class of non-linear elliptic systems. *Acta Math.* **138** (1977), no. 3-4, 219–240. Zbl 0372.35030 MR 0474389

[18] N. N. Ural'ceva, Degenerate quasilinear elliptic systems (in Russian). *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **7** (1968), 184–222. Zbl 0196.12501 MR 0244628

[19] L. Vlahos – H. Isliker – Y. Kominis – K. Hizanidis, Normal and anomalous diffusion: A tutorial. In *Order and chaos. Vol. 10*, edited by T. Bountis, Patras University Press, 2008.

Received 10 July 2024

Verena Bögelein
Fachbereich Mathematik, Universität Salzburg
Hellbrunner Str. 34, 5020 Salzburg, Austria
verena.boegelein@plus.ac.at

Frank Duzaar
Fachbereich Mathematik, Universität Salzburg
Hellbrunner Str. 34, 5020 Salzburg, Austria
frankjohannes.duzaar@plus.ac.at

Naian Liao
Fachbereich Mathematik, Universität Salzburg
Hellbrunner Str. 34, 5020 Salzburg, Austria
naian.liao@plus.ac.at

Giovanni Molica Bisci
Dipartimento di Scienze Pure e Applicate (DiSPeA), University of Urbino Carlo Bo
Piazza della Repubblica, 13, 61029 Urbino, Italy
giovanni.molicabisci@uniurb.it

Raffaella Servadei
Dipartimento di Scienze Pure e Applicate (DiSPeA), University of Urbino Carlo Bo
Piazza della Repubblica, 13, 61029 Urbino, Italy
raffaella.servadei@uniurb.it