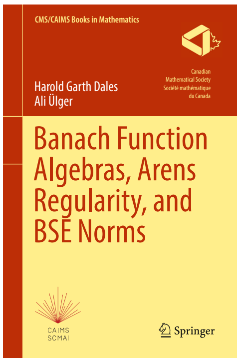
Book review

Banach Function Algebras, Arens Regularity, and BSE Norms
by Harold Garth Dales and Ali Ülger

Reviewed by Betül Tanbay



To the memory of Garth Dales.

Mathematical learning and mathematical research are being severely affected by technology in general, by the internet and artificial intelligence in particular. The pandemic accelerated the change. The mathematical community is discussing all over how knowledge should be created and transmitted, what the speed and the means of sharing should be.

In the middle of this discussion, discretely a book shone after almost a decade of hard work by Garth Dales and Ali Ülger. At an earlier time it would have become immediately a classic. I dare to claim that it is also a classic-to-be for reasons I must explain before I get in the details of the book.

The collective mathematical memory is rather peculiar. The history of mathematics is full of surprises, of problems solved using some seemingly unrelated results in a forgotten topic. Today, many classical fields of mathematics are being treated as obsolete, forgotten regularly at graduate courses, getting less and less research funds. We are at a moment of history where the letters A and I are to appear in any research grant application to achieve acceptance. Young mathematicians are in a huge rush of publishing, bibliometrics seem to have won over insight.

Despite this rush, Garth Dales and Ali Ülger took their time. They started this book when they were both in their seventies, as established researchers in the area. They have worked at distance regularly throughout years, making sure to have a few face to face weeks per year to settle the amassed material. When an open question rose, in the midst of a chapter, they did not avoid it. Real research got in the way. It was sad for the co-author but also for many of us who followed their work, to lose Garth before the book appeared in public.

This "slow research" style resulted in a book that can be used as a course book for a graduate class, but also comprising new unpublished results in the topic and new proofs of known results. After very shortly reminding of the history of the subject, I shall start by tracing how this book can be used as a course book, and mention some of the new results.

The Banach algebras, as a subject, is one of the largest components of functional analysis, with its own problems and techniques. Since its introduction during the second world war, the subject has grown tremendously. Banach algebras are fed essentially by three sources: operator theory (C^*-algebras, von Neumann algebras), harmonic analysis (group algebras, semigroup algebras, measure algebras) and complex analysis (uniform algebras). And there are a multitude other Banach algebras obtained through various constructions such as projective tensor product, direct sums, taking quotients, etc. Although the main emphasis of this very rich book is on commutative semisimple Banach algebras, it also contains many results about non-commutative Banach algebras.

The book consists of six chapters. In the first chapter, the authors fix the notation and the terminology and mention a series of known results in measure theory, bounded linear operators, tensor products of Banach spaces and the geometry of Banach spaces to be used in subsequent chapters.

In Chapter 2, the authors give some information about Banach algebras in general, and C^*-algebras, von Neumann algebras and dual algebras in particular. In this chapter the topic of Arens products and the second dual of Banach algebras equipped with an Arens product are also treated. I mention here two nice results in that chapter that attracted my attention. (1) Theorem 2.2.25, which says that the dual space of a C^*-algebra A has the Schur property iff the sequence space ℓ^2 is not a quotient of A. (2) Corollary 2.2.26, which says that the dual space of the projective tensor product $A \hat{\otimes} B$ of two C^*-algebras A and B is weakly sequentially complete iff A^* or B^* has the Schur property. These results, as far as I know the subject, are new.

In Chapter 3, the authors introduce the notions of Banach function algebras (i.e., commutative semisimple Banach algebras), sequence algebras, projective tensor products of commutative C^*-algebras and uniform algebras. Two important and recent subjects in this chapter are the so-called "separating ball property (SBP)" and the "bounded pointwise approximate identities (BPAI)." A Banach function algebra A is said to have the SBP if giving two distinct characters φ and ψ, there is an a with $\|a\| \leq 1$ such that $\langle a, \varphi \rangle = 1$ and $\langle a, \psi \rangle = 0$. This property and its variants have a multitude of applications for the existence of BPAI's in the maximal ideals of the algebra, the existence of idempotent elements, the existence of topological invariant means for abstract Banach function algebras and Arens regularity.

In Chapter 4, the authors introduce and give the basic properties of three classes of Banach algebras that come from harmonic analysis: first, the group algebra $L^1(G)$ and the measure algebra $M(G)$ of a locally compact group (G), then the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$ of a locally compact group G, introduced by Eymard and finally the Figà–Talamanca–Herz algebra $A_p(G)$ and $PM_p(G)$ of an amenable group G. The authors study various properties of these algebras, including problems concerning when these algebras have bounded approximate identities, when there are ideals in their biduals, and they ask what are their spaces of weakly almost periodic functionals and their multiplier algebras.

In Chapter 5, the authors study the Bochner–Schoenberg–Eberline algebras, the so-called BSE algebras. Let G be a locally compact Abelian group with dual group $\hat{G}$. The BSE theorem gives an inequality that is necessary and sufficient for a continuous bounded function f on $\hat{G}$ to be the Fourier–Stieltjes transform of a measure $\mu \in M(G)$. A BSE-function on the character space of a Banach function algebra A is a continuous function that satisfies the condition in the BSE theorem. The algebra A is said to be a BSE algebra if the Gelfand transform of each multiplier of A satisfies the condition in the BSE theorem and, conversely, if any continuous bounded function on the character space of A satisfying this condition is the Gelfand transform of a multiplier of A. These notions were introduced by Hatori and Takahasi in 1990. Since then quite a few results accumulated about these notions. The authors give a functional analytic treatment of these notions, expose the known results and present several new results, especially about the question when the multiplier algebra of a Banach function algebra is a BSE algebra and when the norm of the algebra A is equivalent to its BSE norm.

Finally, Chapter 6 is about Arens regularity and the topological centres of the second dual of a Banach algebra. In this chapter the authors expose a series of known results in the area together with some new results. The new results are mainly about Arens regularity of the projective tensor product $A \hat{\otimes} B$ of two Banach algebras A and B. To mention just one: Theorem 6.2.13 states that, if both algebras A and B are Arens regular and the algebra A has a bounded approximate identity, then a necessary condition for the algebra $A \hat{\otimes} B$ to be Arens regular is the weak compactness of every bounded linear operator $T\colon A \to B^*$.

The book is well-written; it is very rich in ideas and approaches. The subjects dealt with are clearly explained and illustrated by numerous examples; and the known results are well-referenced: there are 332 references. The index of symbols and the index of terms are quite rich and facilitate the job of the readers. I do hope that the book will help keeping alive careful and deep thinking in pure mathematics.

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