

Magnetic reconnection in magnetohydrodynamics

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Abstract. We provide examples of periodic solutions (in both 2 and 3 dimension) of the magnetohydrodynamics equations such that the topology of the magnetic lines changes during the evolution. This phenomenon, known as magnetic reconnection, is relevant for physicists, in particular in the study of highly conducting plasmas. Although numerical and experimental evidences exist, analytical examples of magnetic reconnection were not known.

1. Introduction

We are interested in the magnetohydrodynamics system, i.e.,

$$(MHD) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla P = \nu \Delta u + (b \cdot \nabla)b, \\ \partial_t b + (u \cdot \nabla)b = (b \cdot \nabla)u + \eta \Delta b, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ u(0, \cdot) = u_0, \quad b(0, \cdot) = b_0, \end{cases}$$

where (for $d = 2, 3$) $b: (0, T) \times \mathbb{T}^d \rightarrow \mathbb{R}^d$ identifies the magnetic field in a resistive incompressible fluid with velocity $u: (0, T) \times \mathbb{T}^d \rightarrow \mathbb{R}^d$. The scalar quantity $P: (0, T) \times \mathbb{T}^d \rightarrow \mathbb{R}$ is the pressure, $\nu \geq 0$ is the viscosity and $\eta \geq 0$ is the resistivity. The system (MHD) describes the behavior of an electrically conducting incompressible fluid, the equations are given by a combination of the Navier–Stokes equations and Maxwell’s equations from electromagnetisms. We point out that the 2D model follows from the 3D model under the assumption that the solution depends only on the first two variables and that u and b have third null component.

In the resistive and viscous case (i.e., $\eta > 0$ and $\nu > 0$), the existence of global weak solutions with finite energy and local strong solutions to (MHD) in two and three dimensions have been proved in [9]. Moreover, for smooth initial data, they proved the smoothness and uniqueness of their global weak solutions in the two-dimensional case. On the other hand, in [39] the authors proved the uniqueness of the local strong solutions in 3D, together with some regularity criteria. See also [31, 40] for existence and uniqueness of

global mild solutions with small initial data. This situation is somewhat reminiscent of the available results for the Navier–Stokes equations. A similar situation arises in the non-viscous case, i.e., when $\nu = 0$: in 2D, the existence of global weak solutions has been proved in [22] for divergence-free initial data in L^2 . The ideal case $\eta = 0$ has attracted the attention of many mathematicians in recent years, and local well-posedness results, at an (essentially) sharp level of Sobolev regularity, are now available [16, 17]. Finally, in the ideal case with no diffusion, i.e., $\nu = \eta = 0$, the local existence of strong solutions for initial data in $H^s(\mathbb{R}^d)$ with $s > d/2 + 1$ has been established in [37, 38]. See also [25, 26, 36, 41], for global existence results of smooth solutions in the ideal case assuming that the initial datum is a small perturbation of a constant steady state.

We are interested in the problem of *magnetic reconnection*. In the non-resistive case ($\eta = 0$), it is known that the integral lines of a sufficiently smooth magnetic field are transported by the fluid (*Alfvén’s theorem*). In particular, the topology of the integral lines of the magnetic field does not change under the evolution. The topological stability of the magnetic structure is related to the conservation of the magnetic helicity, which, in the non-resistive case ($\eta = 0$), becomes a very subtle matter at low regularities, intimately related to anomalous dissipation phenomena. We refer to [2, 14, 15] for some very interesting (positive and negative) results in this direction. On the other hand, in the resistive case ($\eta > 0$), the topology of the magnetic lines may (and it is indeed expected to) change under the fluid evolution, in both 2 and 3 dimensions, even for regular solutions. This phenomenon, known as magnetic reconnection, is of particular relevance for physicists, in particular in the study of highly conducting plasmas. A possible explanation of the phenomenon of the solar flares, large releases of energy from the surface of the sun, involves magnetic reconnection. The energy stored in the magnetic fields over a large period of time is rapidly released during the change of topology of the magnetic lines. It is also worth mentioning that the intensity of the solar flares is of a larger magnitude than the one predicted by the current (MHD) models, suggesting that also some turbulent phenomenon, as cascade of energy, may be involved [35]. Although numerical and experimental evidences exist (see [32, 35] and the references therein), no analytical examples of magnetic reconnection are known. We also refer to [5], where the second author considered the forced (MHD) system and constructed analytical examples of magnetic reconnection in the two-dimensional case. Besides the intrinsic mathematical interest, a better understanding by a rigorous analytic viewpoint may give important insights on the Sweet–Parker model arising in magnetic reconnection theory [35]. Lastly, we remark that an analogous problem exists in the theory of the three-dimensional Navier–Stokes equations, where in the presence of viscosity the topology of the vorticity lines is expected to change under the Navier–Stokes flow. We refer to [6, 7, 10, 20, 33] for an overview on this phenomenon.

Thus, in this work we are concerned with providing analytical examples of this phenomenon. Our main result is the following.

Theorem 1.1. *Consider $d \in \{2, 3\}$. Given any viscosity and resistivity $\nu, \eta > 0$ and any constants $T > 0$ and $M > 0$, there exists a zero-average unique global smooth solution (u, b) of (MHD) on \mathbb{T}^d , with initial datum $(0, b_0)$ and $\|b_0\|_{L^2} = M$, such that the magnetic lines at time $t = 0$ and $t = T$ are not topologically equivalent, meaning that there is no homeomorphism of \mathbb{T}^d into itself mapping the magnetic lines of $b(0, \cdot)$ into those of $b(T, \cdot)$.*

Remark 1.2. Heuristically, one expects a bound for the reconnection time like $T > 1/(\eta N^2)$, where N is a characteristic length scale of the solution. Indeed, the PDE can be interpreted as the Hamiltonian function of the magnetic field being transported by particles following the trajectories of u , $\Delta x \sim u \Delta t$, and also drifting by Brownian motion with η as the diffusion coefficient, $\Delta x \sim (\eta \Delta t)^{1/2}$. Thus, the time it takes a particle to deviate from the trajectory of u a distance Δx equal to the typical scale of variation of the magnetic field (which is what causes mixing and topological change) is $(\Delta x)^2/\eta$ (in this case, $\Delta x = 1/N$). The solutions we are going to construct will have a reconnection time of the order of $T \sim 1/(\eta N^2)$, as explained in Remark 4.1 and Remark 7.1 below.

Note that as M may be very large and the solutions have zero-average, we are considering genuinely large initial magnetic fields (for instance, large in any Sobolev space H^s with $s \geq 0$; see also Remark 1.6 below). It is also possible to consider more general initial velocities $u_0 \neq 0$. In particular, we may consider large initial velocity if we impose some a priori structure (see Remark 4.2); however, we prefer to state the result in the simplest form. Notably, large velocities (without any specific geometrical structure) may be also considered in the 2D case if we work with very large viscosities (see Theorem 1.3).

The proofs of Theorem 1.1 in 2D and in 3D are logically independent. For the sake of readability, we first present the proof of the theorem in 3D, which is an adaptation of the argument from [10]. See also [27] for a simplified construction. The proof of the result in 2D requires new ideas and it is somehow more general, as it should be clear by the following observation: if we have found a 2D solution

$$(b_1(t, x_1, x_2), b_2(t, x_1, x_2)),$$

of (MHD) which exhibits a reconnection, then we have also proved magnetic reconnection for (MHD) in 3D, simply considering the 3D solution

$$(1.1) \quad (b_1(t, x_1, x_2), b_2(t, x_1, x_2), 0);$$

the 2D velocity and pressure must be extended to 3D in the analogous way. On the other hand, an advantage of the genuinely 3D argument will be the possibility to (additionally) prescribe rich topological structures for the magnetic lines, relying upon some deep results about topological richness of Beltrami fields [11–13]. Moreover, the 3D reconnection obtained extending the 2D result to 3D as in (1.1) would not be *structurally stable* in the sense of the Remark 1.4 below. We explain why in Remark 7.4.

It is worth mentioning already that the 2D result and the (genuine) 3D result that we presented are structurally stable indeed; see again Remark 1.4.

As we have noted, the condition $\eta > 0$ is necessary to prove magnetic reconnection for smooth solutions, because for $\eta = 0$, this is forbidden by Alfvén’s theorem. The heuristic behind the phenomenon is that the resistivity allows to break the topological rigidity. In this sense, it is interesting that we can prove magnetic reconnection at arbitrarily small resistivity, namely, for all $\eta > 0$. However, we point out that we are not working in a turbulent regime since the energy of the solutions is concentrated at the scale of dissipation. Indeed, the reconnection time of our solutions is of order $T \gg 1/(N_0^2 \eta)$, so to prove reconnection at a fixed time T and η very small, we need to use fields of length scales much smaller than $\eta^{1/2}$.

Regarding the role played by the viscosity, we mention that Theorem 1.1 might be extended to the case $\nu = 0$, working with growth and stability estimates like the ones used in the Euler equations theory, rather than in the Navier–Stokes one. The price that we must pay to work with viscosity $\nu = 0$ is of course that we will only have local in time results.

However, the viscosity may enter in the argument in an interesting way, since it is valid the principle that large viscosity helps too. We will investigate this in the 2D case, where we can prove a stronger reconnection statement which is valid for any initial velocity in $H^4(\mathbb{T}^2)$ as long as we work with sufficiently large viscosity.

Theorem 1.3. *Consider $d = 2$. Given any resistivity $\eta > 0$ and any constants $T > 0$ and $M, R > 0$, there exists a viscosity $\nu = \nu(M, R) \gg 1$ sufficiently large such that the following holds: for any zero-average u_0 with $\|u_0\|_{H^4} = R$, there exists a zero-average unique global smooth solution (u, b) of (MHD) on \mathbb{T}^2 , with initial datum (u_0, b_0) and $\|b_0\|_{L^2} = M$, such that the magnetic lines at time $t = 0$ and $t = T$ are not topologically equivalent.*

Remark 1.4. The results that we presented above are *structurally stable*, in the sense that if we slightly perturb the initial data (in the appropriate norms) and/or the observation times $t = 0$ and $t = T$, the results are still valid. This will be clear by the proof (see also Remark 4.2). The structural stability of the phenomenon is important from a physical point of view, since it guarantees its observability.

The proofs are based on a perturbative analysis of some particular solutions of the linearized equation, for which one can infer reconnection by a suitable topological argument. More precisely, in the 3D case, we closely follow the idea of an earlier result of the third author and Alberto Enciso and Daniel Peralta-Salas [10], in the context of the Navier–Stokes equation. Thus we consider data for which the magnetic field at time $t = 0$ has the form

$$MB_0 + \delta B_1, \quad \text{for } M > 0 \text{ and } 0 < \delta \ll 1,$$

where B_j are high frequency Beltrami fields (eigenvector of the curl operator) with the following properties:

- (i)_{3D} All the magnetic lines of B_0 wind around a certain direction of the torus, in particular they are all non-contractible. This is a robust topological property, in the sense that it is still valid for all sufficiently small regular perturbations of B_0 .
- (ii)_{3D} The field B_1 has some contractible magnetic line (in fact, in a small ball we can prescribe magnetic lines knotted and linked in complicated ways following the topological results from [10–13]). This is again topologically robust.

The idea is then to choose the relevant parameters, namely δ and the eigenvalues (frequencies) of the Beltrami fields, in such a way that our solution will be sufficiently close (in a regular norm) to MB_0 at time $t = 0$ and to a suitable rescaled version of the field B_1 , at time $T > 0$. Recalling the topological constraint (i)_{3D}, (ii)_{3D} (and their robustness), this will ensure that the magnetic lines of the solution at time $t = 0$ and $t = T$ are not homeomorphic. Thus we must have had magnetic reconnections in the intermediate times and we are also able to quantify the time in which the reconnection takes place, see Remark 4.1.

If we try to use the same strategy in 2D, we encounter the problem that it is not easy to produce a simple high frequency Taylor vector field (which is the 2D analogous of a

Beltrami field) with all non-contractible vortex lines. Moreover, in two dimensions the reconnection is expected to occur at critical points, see [35]. Thus, in the 2D case we use a different topological constraint, that consists in counting the number of stagnation points of the magnetic field. In particular, we define the initial magnetic field as

$$M V_N + \delta V_1, \quad \text{for } M > 0 \text{ and } 0 < \delta \ll 1,$$

where V_N and V_1 are Taylor fields (eigenvectors of the Stokes problem (5.1)) with the following properties:

- (i)_{2D} The field V_N has several stagnation points (namely, $\sim N \gg 1$, half of them are hyperbolic and half of them elliptic). This is robust in the sense that small regular perturbations must have at least as many stagnation points.
- (ii)_{2D} The fields V_1 has exactly four stagnation points, and the topology of the vortex lines is completely prescribed (see figure 2) and robust (V_1 is structurally stable by Theorem 5.3).

Again, one will then choose the relevant parameters in such a way that the solution will share the same topological properties (i)_{2D} and (ii)_{2D} at times $t = 0$ and $t = T > 0$, respectively. This proves magnetic reconnection for intermediate times. Again, we can quantify the time in which the reconnection takes place, see Remark 7.1.

In the last part of the paper, we provide an example of initial magnetic fields which shows instantaneous reconnection under the (MHD) flow. The theorem is the following.

Theorem 1.5. *Consider $M > 0$ and $u_0 \in C^\infty(\mathbb{T}^d)$ with zero-average, where $d \in \{2, 3\}$. There exists a zero-average initial magnetic field b_0 , with $\|b_0\|_{L^2} = M$, such that the following holds: if $d = 3$, there is a zero-average local smooth solution of (MHD) with initial datum (u_0, b_0) which at time $t = 0$ has a tube of magnetic lines that breaks up instantaneously (namely, for any positive time $t > 0$). If $d = 2$, there is a zero-average global smooth solution with initial datum (u_0, b_0) which at time $t = 0$ has an heteroclinic connection that breaks up instantaneously.*

The 3D case of the theorem above again closely follows the idea of [10]: by considering an initial datum which is not structurally stable (again the datum will be a small perturbation of a Beltrami field), one can prove that some magnetic lines sitting on a resonant (embedded 2D) torus rearrange instantaneously their topology. In the 2D case, we exploit the structural instability of the heteroclinic connections of a suitable perturbation of a Taylor field. In particular, we show that an heteroclinic connection is instantly broken. Both results may be proved invoking the Melnikov theory, but in 2D a significant simplification of the argument is available (see Section 9). We are grateful to Daniel Peralta-Salas for this observation.

It is worth mentioning that the idea of looking at the critical points in order to prove bifurcation results for the 2D Navier–Stokes equation was proposed in [23, 24]. In those papers, the authors study topology change of streamlines under short time evolution. They use a Taylor expansion in time and compute the $\mathcal{O}(t)$ correction based on the chosen data to conclude. In particular, they provide different sets of initial data such that the following scenario holds:

- (i) the set of initial critical points bifurcate from one to two, and then to three critical points in finite time;

- (ii) three critical points merge into one critical point in finite time;
- (iii) the bifurcation can be generated by the Navier–Stokes flow and do not require the existence of an initial critical point.

However, the method that we use in the proofs of Theorem 1.1 and Theorem 1.3 to show that the number of critical points changes during the evolution is completely different, and heavily relies on the choice of Taylor fields (and Beltrami fields in 3D) to implement a robust perturbative argument. Also our instantaneous result is different since, rather than control the number of critical points, we actually show that the MHD flow breaks heteroclinic connections between saddle points.

We conclude the introduction with some comments, The first is a remark on the size of our initial data in the 3D part of Theorem 1.1. Let consider for simplicity the case in which $\eta = \nu = 1$.

Remark 1.6. It is well known that in the case of the 3D Navier–Stokes equation, one can use negative order Besov spaces, or even the more general Koch–Tataru space from [21], in order to develop a well-posedness theory for data which are strongly oscillating (as in our case). We recall that the space $\dot{B}_{\infty,\infty}^{-1}$ contains all the critical spaces that in principle would allow a fixed point argument. The relative norm is

$$\|f\|_{\dot{B}_{\infty,\infty}^{-1}} := \sup_{t>0} \sqrt{t} \|e^{t\Delta} f\|_{L^\infty}.$$

The initial data in (the 3D part of) Theorem 1.1 are small with respect to this norm. However, as observed in Remark 4.5 of [10] (in the case of the Navier–Stokes equation), a straightforward modification of our argument would allow handling initial data b_0 which are small perturbations of the Beltrami field

$$B = MN^\alpha(\sin(Nx_3), \cos(Nx_3), 0), \quad \text{for } N \in \mathbb{N} \text{ and } \alpha \leq 1.$$

Now, using the fact that $e^{t\Delta} B = e^{-N^2 t} B$, one can easily show that the $\dot{B}_{\infty,\infty}^{-1}$ norm of B is

$$\frac{1}{\sqrt{2e}} MN^{\alpha-1}.$$

Thus, in the case $\alpha = 1$, we are considering solutions magnetic fields that at the initial time are even large in the $\dot{B}_{\infty,\infty}^{-1}$ norm (and thus large in any critical space). It is interesting to notice that our argument would fail exactly if we take $\alpha > 1$. The same is true for the Navier–Stokes equation case (see again Remark 4.5 in [10]).

The last comment concerns another very important model for analyzing reconnection phenomena is the Hall-MHD system. The latter has an additional term on the left-hand side of the magnetic equation in (MHD), namely, $\text{curl}(\text{curl } b \wedge b)$, which is called the *Hall term*. We refer to [1, 4, 8, 19] and reference therein for an introduction to these equations. It is known that the Hall term alone cannot change the topology of the magnetic field lines; however, it interacts with the magnetic viscosity accelerating the process, see [19]. It is interesting to note that the Hall term is identically zero if b is a Beltrami field. Thus, since a small data theory is available for global smooth solutions of the Hall-MHD [4], we expect that the perturbative argument of Theorem 1.1 can be adapted to the case of the Hall-MHD system as well.

1.1. Organization of the paper

In the rest of the introduction, we describe the notation and some general facts that we will be using throughout the article. In Sections 2 and 5, we recall the concepts of Beltrami and Taylor fields that will be necessary in the proof of the magnetic reconnection for 3D and 2D, respectively. One basic ingredient in our proofs is the stability of *strong* solutions of the MHD system. Sections 3 and 6 are devoted to this matter in the 3D and 2D case, respectively. The magnetic reconnection in the 3D case is proved in Section 4. The magnetic reconnection in the 2D case is proved in Section 7 in the case of small velocities, and in Section 8 in the case of large velocities, but under an additional assumption on the size of the viscosity. Finally, Section 9 is devoted to the instantaneous reconnection.

1.2. Notations and preliminaries

Throughout the paper, we will denote by C a positive constant whose value can change line by line. We will denote by $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ the d -dimensional flat torus equipped with the Lebesgue measure \mathcal{L}^d . We will always work with $d \in \{2, 3\}$. For a given positive integer m and a given d -dimensional vector field $w: \mathbb{T}^d \rightarrow \mathbb{R}^d$, we define

$$|\nabla^m w|^2 := \sum_{|\alpha|=m} |\partial^\alpha w|^2,$$

where $\alpha \in \mathbb{N}^d$ is a multi-index, and

$$\|w\|_{H^r}^2 := \sum_{m=0}^r \int_{\mathbb{T}^d} |\nabla^m w(x)|^2 dx,$$

where r is a given positive integer. We will use p and q to denote real numbers in $[1, +\infty]$. We will adopt the customary notation for Lebesgue spaces $L^p(\mathbb{T}^d)$ and for Sobolev spaces $W^{k,p}(\mathbb{T}^d)$; in particular, $H^k(\mathbb{T}^d) := W^{k,2}(\mathbb{T}^d)$. We will denote with $\|\cdot\|_{L^p}$ (respectively, $\|\cdot\|_{W^{k,p}}$, $\|\cdot\|_{H^k}$) the norms of the aforementioned functional spaces, omitting the domain dependence. Every definition below can be adapted in a standard way to the case of spaces involving time, like, e.g., $L^1([0, T]; L^p(\mathbb{T}^d))$. Moreover, for a time-dependent vector field $w(t, x)$, we define

$$\|w\|_{L^2 W^{r,\infty}}^2 := \sum_{m=0}^r \int_0^\infty \|\nabla^m w(t, \cdot)\|_\infty^2 dt.$$

Working in the 2D case, we will frequently use the interpolation inequality

$$\|f\|_{L^4(\mathbb{T}^2)} \leq C \|f\|_{L^2(\mathbb{T}^2)}^{1/2} \|\nabla f\|_{L^2(\mathbb{T}^2)}^{1/2},$$

valid for zero-average functions, to which we refer as Ladyzenskaya's inequality.

In 3D, a similar role will be played by

$$\|f\|_{L^\infty(\mathbb{T}^3)} \leq C \|f\|_{L^6(\mathbb{T}^3)}^{1/2} \|\nabla f\|_{L^2(\mathbb{T}^3)}^{1/2},$$

again valid for zero-average functions, to which we refer as the Gagliardo–Nirenberg inequality. Finally, we recall the well-known Gronwall lemma.

Lemma 1.7 (Gronwall). *Let f be a nonnegative, absolutely continuous function on $[0, T]$ which satisfies, for a.e. t , the differential inequality*

$$f'(t) \leq \alpha(t)f(t) + \beta(t),$$

where α and β are nonnegative, summable functions on $[0, T]$. Then

$$f(t) \leq e^{A(t)} \left(f(0) + \int_0^t \beta(s) e^{-A(s)} ds \right),$$

for all $t \in [0, T]$, where $A(t) = \int_0^t \alpha(s) ds$.

2. Beltrami fields

In this section, we introduce the main mathematical objects that we need for the 3D magnetic reconnection result. These are the so-called Beltrami fields.

A vector field $B: \mathbb{T}^3 \rightarrow \mathbb{R}^3$ is called a *Beltrami field* with frequency N if it is an eigenfunction of the curl operator with eigenvalue $N \in \mathbb{Z}$, i.e.,

$$(2.1) \quad \operatorname{curl} B = NB.$$

It is in fact easy to check that on \mathbb{T}^3 , the eigenvalues are have the form $\pm|k|$, where $k \in \mathbb{Z}^3$. We will restrict our attention to Beltrami fields of nonzero frequency, which are necessarily divergence-free and have zero mean, i.e.,

$$\operatorname{div} B = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} B \, dx = 0, \quad \text{if } N \neq 0.$$

The general form of a Beltrami field of frequency N is indeed

$$W = \sum_{|k|=|N|} \left(b_k \cos(k \cdot x) + \frac{b_k \times k}{N} \sin(k \cdot x) \right),$$

where $b_k \in \mathbb{R}^3$ are vectors orthogonal to k , that is, $k \cdot b_k = 0$. It is easy to check that B satisfies, additionally,

$$\begin{aligned} (B \cdot \nabla) B &= \nabla \frac{|B|^2}{2}, \\ \Delta B &= -N^2 B. \end{aligned}$$

For our purposes, we consider two topologically non-equivalent Beltrami fields. The first is given explicitly:

$$(2.2) \quad B_0 := (2\pi)^{-3/2} (\sin(N_0 x_3), \cos(N_0 x_3), 0).$$

Note that all the integral lines of the field B_0 (which coincide with those of $\nabla \times B_0$) are either periodic or quasi-periodic (depending on the rationality/irrationality of the ratio $\sin(N_0 x_3) / \cos(N_0 x_3)$) and wind around the 2D tori given by the equation $x_3 = \text{constant}$. In particular, all the integral lines of B_0 are non-contractible. This property is structurally

stable, in the sense that it is still true for small (regular enough) perturbations. More precisely, if

$$(2.3) \quad \|B_0 - B'\|_{C^{3,\alpha}} < \eta,$$

with a small (N_0 -independent) constant, then

- (i) B' does not have any contractible integral line (since the same is true for B_0).

This fact, which is a KAM type theorem, was proved in Lemma 4.2 of [10].

The second Beltrami field is constructed in the following theorem, that is a simplified version of Theorem 2.1 in [10]. It is worth mentioning that the most important part of the proof of this result comes from [11–13].

Theorem 2.1 (see Theorem 2.1 in [10]). *Let S be a finite union of closed curves (disjoint, but possibly knotted and linked) in \mathbb{T}^3 that is contained in the unit ball. For all N_1 large enough and odd, there exists a Beltrami field B_1 with some integral lines diffeomorphic to S (namely, related by a diffeomorphism of \mathbb{T}^3). This set is contained in a ball of radius $1/N_1$ and is structurally stable, namely, there exists η independent of N_1 such that any B'' such that*

$$(2.4) \quad \|B_1 - B''\|_{C^1} < \eta,$$

has a collection of integral lines diffeomorphic to S . Moreover,

$$(2.5) \quad \frac{1}{CN_1} < \|B_1\|_{L^2} < \frac{C}{\sqrt{N_1}}.$$

In particular,

- (ii) B'' has some contractible integral line (since the same is true for B_1).

In fact, the theory developed in [11–13] allows also to prescribe some vortex tubes of arbitrarily complicated topology which can be realized by the vortex lines of the Beltrami field B_1 . In this case, the structural stability requires a stronger norm, namely, $C^{3,\alpha}$. For the purpose of this paper, we will consider the simple scenario from Theorem 2.1, as it is sufficient to prove magnetic reconnection.

The natural numbers N_0 and N_1 will play the role of free parameters in our construction, but eventually we will choose one of them much larger than the other in such a way that our solution at time $t = 0$ will be close to (a rescaled version of) B_0 , while at time $t = T > 0$ will be close to (a rescaled version of) B_1 . Thus, as consequence of (i) and (ii) above, a change of topology of the integral lines must have happened.

3. Stability of regular solutions of the MHD system in 3D

The goal of this section is to provide a stability result for regular solutions of the magnetohydrodynamic system. Later, we will often work with some special reference solutions with zero velocity, but at this stage we prefer to prove a slightly more general perturbative result. The estimates below will be crucial for two reasons: on the one hand, they will quantify the error in the perturbative argument (see the remark below); on the other hand, they will allow us to construct a global solution as a small perturbation of some large global smooth solution (that will indeed be a Beltrami field).

The possibility to run a perturbative argument around strong solutions is of course not a novelty, but the important aspect of the next proposition is that we quantify the error in such a way that it depends only polynomially by the $L^2((0, T); W^{r, \infty}(\mathbb{T}^3))$ norm of the reference solution (u, b) , for $r \geq 1$, while the exponential dependence only involves its $L^2((0, T); L^\infty(\mathbb{T}^3))$ norm; see (3.3). This will be important to handle large initial data in our main theorem.

The initial data will be small perturbations of (u_0, b_0) , namely, we focus on divergence-free vector fields (w_0, m_0) such that

$$\|u_0 - w_0\|_{H^r} + \|b_0 - m_0\|_{H^r} \ll 1.$$

We want to show that there exists a unique global regular solution (w, m) starting from $(0, m_0)$. To do that, we proceed as in [10]: we know that there exists a local solution (w, m) , and we prove that it can be extended globally, under suitable estimates for the Sobolev norms.

Theorem 3.1. *For integer $r \geq 1$ and any $\sigma < \min(\eta, \nu)$, let $(u, b) \in L^2((0, T); W^{r, \infty}(\mathbb{T}^3))$ be a global smooth solution of (MHD) with initial datum (u_0, b_0) of zero mean such that*

$$(3.1) \quad \|u\|_{L^2 W^{k, \infty}} + \|b\|_{L^2 W^{k, \infty}} < C(1 + N^{k-1}),$$

for all integers $0 \leq k \leq r$, where $N \geq 1$. Then, there exists a sufficiently large positive constant c such that, for any divergence-free vector field (w_0, m_0) with zero mean and

$$(3.2) \quad \|u_0 - w_0\|_{H^r} + \|b_0 - m_0\|_{H^r} \leq \frac{1}{c} N^{1-r},$$

the corresponding solution (w, m) to (MHD) is global and satisfies

$$(3.3) \quad \|u(t, \cdot) - w(t, \cdot)\|_{H^k} + \|b(t, \cdot) - m(t, \cdot)\|_{H^k} \\ \leq C(1 + N^{k-1}) e^{C \int_0^t (\|u(\cdot, s)\|_{L^\infty}^2 + \|b(\cdot, s)\|_{L^\infty}^2) ds} (\|u_0 - w_0\|_{H^k} + \|b_0 - m_0\|_{H^k}) e^{-\sigma t},$$

for all $0 \leq k \leq r$ and all $t > 0$, with a σ -dependent constant C .

Remark 3.2. In the next section, we will apply this theorem with the choice $(u, b) = (0, M e^{-\eta N_0^2 t} B_0)$, where B_0 is the Beltrami field defined in (2.2). It is immediate to check that this solves (MHD) with an appropriate choice of the pressure (see next section), and that the assumptions of the theorem are satisfied with $N = N_0$.

Proof. We denote by $P_{u, b}$ and $P_{w, m}$ the pressure function of (u, b) and (w, m) , respectively. We know from [39] that there exists a unique local solution (w, m) of (MHD) starting from (w_0, m_0) . We denote by T^* the local time of existence. We start by proving the bound (3.3), which will be enough to guarantee that the solution is actually global.

Step 1. Preliminaries.

We define $v = w - u$ and $h = m - b$. It is easy to check that (v, h) solves the system

$$(3.4) \quad \begin{cases} \partial_t v + \operatorname{div}(v \otimes v + 2v \otimes u) + \nabla P_{v, h} = \nu \Delta v + \operatorname{div}(h \otimes h + 2h \otimes b), \\ \partial_t h + (v \cdot \nabla) h + (u \cdot \nabla) h + (v \cdot \nabla) b = \eta \Delta h + (h \cdot \nabla) v + (h \cdot \nabla) u + (b \cdot \nabla) v, \\ \operatorname{div} v = \operatorname{div} h = 0, \\ v_0 = w_0 - u_0, \quad h_0 = m_0 - b_0, \end{cases}$$

where $P_{v,h} = P_{w,m} - P_{u,b}$. We recall that, since (u, b) is a global smooth solution, the following energy equalities hold:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} (|u(t, x)|^2 + |b(t, x)|^2) dx &= -\nu \int_{\mathbb{T}^3} |\nabla u(t, x)|^2 dx - \eta \int_{\mathbb{T}^3} |\nabla b(t, x)|^2 dx, \\ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |u(t, x)|^2 dx &= -\nu \int_{\mathbb{T}^3} |\nabla u(t, x)|^2 dx + \int_{\mathbb{T}^3} (b \cdot \nabla) b \cdot u dx, \\ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |b(t, x)|^2 dx &= -\eta \int_{\mathbb{T}^3} |\nabla b(t, x)|^2 dx + \int_{\mathbb{T}^3} (b \cdot \nabla) u \cdot b dx. \end{aligned}$$

Let us now define the time energies e_k as follows:

$$e_k(t) := \sum_{j=0}^k \int_{\mathbb{T}^3} (|\nabla^j v(t, x)|^2 + |\nabla^j h(t, x)|^2) dx.$$

The goal now is to provide bounds on e_k via an induction argument.

Step 2. Estimate on e_0 .

We multiply the first equation in (3.4) by v , and we obtain that

$$\begin{aligned} \partial_t \frac{|v|^2}{2} + (v \cdot \nabla) \frac{|v|^2}{2} + 2(v \cdot \nabla) u \cdot v + \operatorname{div}(P_{v,h} v) \\ = \nu \Delta v \cdot v + (h \cdot \nabla) h \cdot v + (h \cdot \nabla) b \cdot v + (b \cdot \nabla) h \cdot v, \end{aligned}$$

and integrating over \mathbb{T}^3 , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |v|^2 dx + 2 \int_{\mathbb{T}^3} (v \cdot \nabla) u \cdot v dx &= -\nu \int_{\mathbb{T}^3} |\nabla v|^2 dx + \int_{\mathbb{T}^3} (h \cdot \nabla) h \cdot v dx \\ (3.5) \quad &+ \int_{\mathbb{T}^3} (h \cdot \nabla) b \cdot v dx + \int_{\mathbb{T}^3} (b \cdot \nabla) h \cdot v dx. \end{aligned}$$

We multiply the second equation in (3.4) by h and, after integrating over \mathbb{T}^3 , we get that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |h|^2 dx + \int_{\mathbb{T}^3} (v \cdot \nabla) b \cdot h dx &= -\eta \int_{\mathbb{T}^3} |\nabla h|^2 dx + \int_{\mathbb{T}^3} (h \cdot \nabla) v \cdot h dx \\ (3.6) \quad &+ \int_{\mathbb{T}^3} (h \cdot \nabla) u \cdot h dx + \int_{\mathbb{T}^3} (b \cdot \nabla) v \cdot h dx. \end{aligned}$$

We use the identities

$$\begin{aligned} \int_{\mathbb{T}^3} (h \cdot \nabla) h \cdot v dx &= - \int_{\mathbb{T}^3} (h \cdot \nabla) v \cdot h dx, \\ \int_{\mathbb{T}^3} (b \cdot \nabla) h \cdot v dx &= - \int_{\mathbb{T}^3} (b \cdot \nabla) v \cdot h dx, \end{aligned}$$

and summing up (3.5) and (3.6), we get that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} (|v|^2 + |h|^2) dx + 2 \int_{\mathbb{T}^3} (v \cdot \nabla) u \cdot v dx + \int_{\mathbb{T}^3} (v \cdot \nabla) b \cdot h dx \\ (3.7) \quad = -\nu \int_{\mathbb{T}^3} |\nabla v|^2 dx - \eta \int_{\mathbb{T}^3} |\nabla h|^2 dx + \int_{\mathbb{T}^3} (h \cdot \nabla) u \cdot h dx + \int_{\mathbb{T}^3} (h \cdot \nabla) b \cdot v dx. \end{aligned}$$

By using integration by parts and Young's inequality, we obtain the following estimates:

$$\begin{aligned} \left| \int_{\mathbb{T}^3} (v \cdot \nabla) u \cdot v \, dx \right| &= \left| - \int_{\mathbb{T}^3} (v \cdot \nabla) v \cdot u \, dx \right| \\ &\leq \|u\|_{L^\infty} \|v\|_{L^2} \|\nabla v\|_{L^2} \leq C \|u\|_{L^\infty}^2 \|v\|_{L^2}^2 + \varepsilon \|\nabla v\|_{L^2}^2, \end{aligned}$$

where ε is a small constant that will be chosen later, and $C = (4\varepsilon)^{-1}$. Similarly,

$$\begin{aligned} \left| \int_{\mathbb{T}^3} (h \cdot \nabla) u \cdot h \, dx \right| &\leq C \|u\|_{L^\infty}^2 \|h\|_{L^2}^2 + \varepsilon \|\nabla h\|_{L^2}^2, \\ \left| \int_{\mathbb{T}^3} (v \cdot \nabla) b \cdot h \, dx \right| &\leq C \|b\|_{L^\infty}^2 \|v\|_{L^2}^2 + \varepsilon \|\nabla h\|_{L^2}^2, \\ \left| \int_{\mathbb{T}^3} (h \cdot \nabla) b \cdot v \, dx \right| &\leq C \|b\|_{L^\infty}^2 \|h\|_{L^2}^2 + \varepsilon \|\nabla v\|_{L^2}^2. \end{aligned}$$

Then, by substituting in (3.7), we obtain that

$$\begin{aligned} \frac{d}{dt} e_0(t) &\leq C (\|u(t)\|_{L^\infty}^2 + \|b(t)\|_{L^\infty}^2) e_0(t) \\ &\quad - 2(v - 2\varepsilon) \int_{\mathbb{T}^3} |\nabla v|^2 \, dx - 2(\eta - 2\varepsilon) \int_{\mathbb{T}^3} |\nabla h|^2 \, dx. \end{aligned}$$

Finally, since v and h have zero mean for all times in which they are defined, we can use Poincaré's inequality

$$\|f\|_{L^2} \leq \|\nabla f\|_{L^2},$$

and by properly fixing ε , we obtain that

$$\frac{d}{dt} e_0(t) \leq [C (\|u(t)\|_{L^\infty}^2 + \|b(t)\|_{L^\infty}^2) - 2\sigma] e_0(t),$$

where $\sigma \in (0, 1)$ is a fixed quantity which depend on v and η . By Gronwall's lemma, it follows that

$$e_0(t) \leq (\|v_0\|_{L^2}^2 + \|h_0\|_{L^2}^2) \exp\left(C \int_0^t (\|u(\tau)\|_{L^\infty}^2 + \|b(\tau)\|_{L^\infty}^2 \, d\tau - 2\sigma t)\right),$$

which leads to

$$\begin{aligned} &\|u(t, \cdot) - w(t, \cdot)\|_{L^2}^2 + \|b(t, \cdot) - m(t, \cdot)\|_{L^2}^2 \\ &\leq \exp\left(C \int_0^t (\|u(\tau)\|_{L^\infty}^2 + \|b(\tau)\|_{L^\infty}^2 \, d\tau - 2\sigma t)\right) (\|u_0 - w_0\|_{L^2}^2 + \|b_0 - m_0\|_{L^2}^2), \end{aligned}$$

which implies (3.3) for $k = 0$.

Step 3. Inductive step.

We are assuming that, for all $k < r$,

$$\begin{aligned} e_k(t) &\leq C(1 + N^{2k-2}) \exp\left(C \int_0^t (\|u(\tau)\|_{L^\infty}^2 + \|b(\tau)\|_{L^\infty}^2) \, d\tau\right) \\ &\quad \cdot e^{-2\sigma t} (\|v_0\|_{H^k}^2 + \|b_0\|_{H^k}^2). \end{aligned}$$

We will show that the bound holds also for $k = r$. Let $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq r$, and differentiate the equation for the velocity by ∇^α to obtain

$$\begin{aligned} \partial_t \partial^\alpha v - \nu \Delta \partial^\alpha v + \nabla \partial^\alpha P_v + \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta v \cdot \nabla) \partial^{\alpha-\beta} v \\ + \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} [(\partial^\beta u \cdot \nabla) \partial^{\alpha-\beta} v + (\partial^\beta v \cdot \nabla) \partial^{\alpha-\beta} u] \\ = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} [(\partial^\beta h \cdot \nabla) \partial^{\alpha-\beta} h + (\partial^\beta b \cdot \nabla) \partial^{\alpha-\beta} h + (\partial^\beta h \cdot \nabla) \partial^{\alpha-\beta} b]. \end{aligned}$$

Multiply the above equation by $\partial^\alpha v$ and integrating in space, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\partial^\alpha v|^2 dx + \nu \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 dx + \int_{\mathbb{T}^3} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta v \cdot \nabla) \partial^{\alpha-\beta} v \partial^\alpha v dx \\ + \int_{\mathbb{T}^3} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} [(\partial^\beta u \cdot \nabla) \partial^{\alpha-\beta} v \partial^\alpha v + (\partial^\beta v \cdot \nabla) \partial^{\alpha-\beta} u \partial^\alpha v] dx \\ = \int_{\mathbb{T}^3} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} [(\partial^\beta h \cdot \nabla) \partial^{\alpha-\beta} h \partial^\alpha v + (\partial^\beta b \cdot \nabla) \partial^{\alpha-\beta} h \partial^\alpha v + (\partial^\beta h \cdot \nabla) \partial^{\alpha-\beta} b \partial^\alpha v] dx. \end{aligned}$$

We use the divergence-free condition and, by integration by parts and Young's inequality, we can estimate the terms above as follows:

$$\begin{aligned} \left| \int_{\mathbb{T}^3} (\partial^\beta v \cdot \nabla) \partial^{\alpha-\beta} v \partial^\alpha v dx \right| &\leq \frac{\varepsilon}{6} \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 dx + C \int_{\mathbb{T}^3} |\partial^\beta v|^2 |\partial^{\alpha-\beta} v|^2 dx, \\ \left| \int_{\mathbb{T}^3} (\partial^\beta u \cdot \nabla) \partial^{\alpha-\beta} v \partial^\alpha v dx \right| &\leq \frac{\varepsilon}{6} \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 dx + C \int_{\mathbb{T}^3} |\partial^\beta u|^2 |\partial^{\alpha-\beta} v|^2 dx, \\ \left| \int_{\mathbb{T}^3} (\partial^\beta v \cdot \nabla) \partial^{\alpha-\beta} u \partial^\alpha v dx \right| &\leq \frac{\varepsilon}{6} \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 dx + C \int_{\mathbb{T}^3} |\partial^\beta v|^2 |\partial^{\alpha-\beta} u|^2 dx, \\ \left| \int_{\mathbb{T}^3} (\partial^\beta h \cdot \nabla) \partial^{\alpha-\beta} h \partial^\alpha v dx \right| &\leq \frac{\varepsilon}{6} \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 dx + C \int_{\mathbb{T}^3} |\partial^\beta h|^2 |\partial^{\alpha-\beta} h|^2 dx, \\ \left| \int_{\mathbb{T}^3} (\partial^\beta b \cdot \nabla) \partial^{\alpha-\beta} h \partial^\alpha v dx \right| &\leq \frac{\varepsilon}{6} \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 dx + C \int_{\mathbb{T}^3} |\partial^\beta b|^2 |\partial^{\alpha-\beta} h|^2 dx, \\ \left| \int_{\mathbb{T}^3} (\partial^\beta h \cdot \nabla) \partial^{\alpha-\beta} b \partial^\alpha v dx \right| &\leq \frac{\varepsilon}{6} \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 dx + C \int_{\mathbb{T}^3} |\partial^\beta h|^2 |\partial^{\alpha-\beta} b|^2 dx. \end{aligned}$$

By summing up over β the above inequalities, we get that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\partial^\alpha v|^2 dx + \nu \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 dx \leq \varepsilon \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 dx + C \|u(t)\|_{L^\infty}^2 \int_{\mathbb{T}^3} |\partial^\alpha v|^2 dx \\ + C \sum_{k=0}^{r-1} \int_{\mathbb{T}^3} |\nabla^k v|^2 |\nabla^{r-k} v|^2 dx + C \sum_{k=1}^r \int_{\mathbb{T}^3} |\nabla^k u|^2 |\nabla^{r-k} v|^2 dx \\ + C \|b(t)\|_{L^\infty}^2 \int_{\mathbb{T}^3} |\partial^\alpha v|^2 dx + C \sum_{k=0}^{r-1} \int_{\mathbb{T}^3} |\nabla^k h|^2 |\nabla^{r-k} h|^2 dx + C \sum_{k=1}^r \int_{\mathbb{T}^3} |\nabla^k b|^2 |\nabla^{r-k} h|^2 dx. \end{aligned}$$

We use Gagliardo–Nirenberg’s inequality to estimate

$$\|\nabla^k v\|_{L^\infty} \leq C \|\nabla^{k+1} v\|_{L^2}^{1/2} \|\nabla^k v\|_{L^6}^{1/2} \leq C \|\nabla v\|_{H^{k+1}}^{1/2} \|v\|_{H^{k+1}}^{1/2},$$

where the Sobolev embedding is used to get the second inequality. Thus, from the definition of e_r , we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\partial^\alpha v|^2 dx + \nu \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 dx \\ & \leq \varepsilon \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 dx + C(\|u(t)\|_{L^\infty}^2 + \|b(t)\|_{L^\infty}^2) e_r(t) \\ & \quad + C(\|\nabla v(t)\|_{H^r} + \|\nabla h(t)\|_{H^r}) e_r(t)^{3/2} + C \sum_{k=1}^r (\|\nabla^k u(t)\|_{L^\infty}^2 + \|\nabla^k b(t)\|_{L^\infty}^2) e_{r-k}(t) \\ & \leq \varepsilon \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 dx + C(\|u(t)\|_{L^\infty}^2 + \|b(t)\|_{L^\infty}^2) e_r(t) + \varepsilon(\|\nabla v(t)\|_{H^r}^2 + \|\nabla h(t)\|_{H^r}^2) \\ & \quad + C e_r(t)^3 + C \sum_{k=1}^r (\|\nabla^k u(t)\|_{L^\infty}^2 + \|\nabla^k b(t)\|_{L^\infty}^2) e_{r-k}(t), \end{aligned}$$

where in the last computation we applied Young’s inequality. Finally, by using Poincaré’s inequality and summing over α , we end up to

$$\begin{aligned} \frac{d}{dt} \sum_{|\alpha| \leq r} \int_{\mathbb{T}^3} |\partial^\alpha v|^2 dx & \leq -2(\nu - \varepsilon) \sum_{|\alpha| \leq r} \int_{\mathbb{T}^3} |\nabla \partial^\alpha v|^2 dx + C e_r(t)^3 \\ & \quad + C(\|u(t)\|_{L^\infty}^2 + \|b(t)\|_{L^\infty}^2) e_r(t) \\ & \quad + C \sum_{k=0}^{r-1} (\|\nabla^{r-k} u(t)\|_{L^\infty}^2 + \|\nabla^{r-k} b(t)\|_{L^\infty}^2) e_k(t). \end{aligned}$$

We now consider the equation for the magnetic field: applying ∂^α to the equation, we obtain

$$\begin{aligned} \partial_t \partial^\alpha h - \eta \Delta \partial^\alpha h + \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} ((\partial^\beta v \cdot \nabla) \partial^{\alpha-\beta} h + (\partial^\beta u \cdot \nabla) \partial^{\alpha-\beta} h + (\partial^\beta v \cdot \nabla) \partial^{\alpha-\beta} b) \\ = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} ((\partial^\beta h \cdot \nabla) \partial^{\alpha-\beta} v + (\partial^\beta h \cdot \nabla) \partial^{\alpha-\beta} u + (\partial^\beta b \cdot \nabla) \partial^{\alpha-\beta} v). \end{aligned}$$

By multiplying by $\partial^\alpha h$, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} |\partial^\alpha h|^2 dx + \eta \int_{\mathbb{T}^3} |\nabla \partial^\alpha h|^2 dx \\ & \quad + \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} ((\partial^\beta v \cdot \nabla) \partial^{\alpha-\beta} h + (\partial^\beta u \cdot \nabla) \partial^{\alpha-\beta} h + (\partial^\beta v \cdot \nabla) \partial^{\alpha-\beta} b) \cdot \partial^\alpha h dx \\ & \quad = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} ((\partial^\beta h \cdot \nabla) \partial^{\alpha-\beta} v + (\partial^\beta h \cdot \nabla) \partial^{\alpha-\beta} u + (\partial^\beta b \cdot \nabla) \partial^{\alpha-\beta} v) \cdot \partial^\alpha h dx. \end{aligned}$$

We now use estimates similar to those above, where $\nabla^\alpha h$ plays the role of $\nabla^\alpha v$, obtaining

$$\begin{aligned} \frac{d}{dt} \sum_{|\alpha| \leq r} \int_{\mathbb{T}^3} |\partial^\alpha h|^2 dx &\leq -2(\eta - \varepsilon) \sum_{|\alpha| \leq r} \int_{\mathbb{T}^3} |\nabla \partial^\alpha h|^2 dx \\ &+ C (\|u(t)\|_{L^\infty}^2 + \|b(t)\|_{L^\infty}^2) e_r(t) + C e_r(t)^3 \\ &+ \sum_{k=0}^{r-1} (\|\nabla^{r-k} u(t)\|_{L^\infty}^2 + \|\nabla^{r-k} b(t)\|_{L^\infty}^2) e_k(t). \end{aligned}$$

By summing the inequalities, we get

$$(3.8) \quad \begin{aligned} \frac{d}{dt} e_r(t) &\leq -2(v + \eta - 2\varepsilon) e_r(t) + C (\|u(t)\|_{L^\infty}^2 + \|b(t)\|_{L^\infty}^2) e_r(t) + C e_r(t)^3 \\ &+ 2 \sum_{k=0}^{r-1} (\|\nabla^{r-k} u(t)\|_{L^\infty}^2 + \|\nabla^{r-k} b(t)\|_{L^\infty}^2) e_k(t) \end{aligned}$$

Let us assume that $e_r(t)^2$ is small enough that

$$-2(v + \eta - 2\varepsilon) e_r + C e_r^3 \leq -2\sigma e_r.$$

Then, by substituting in (3.8) and using the inductive step on e_k , we obtain (recall that here $k \geq 1$)

$$\begin{aligned} \frac{d}{dt} e_r(t) &\leq -2\sigma e_r + C (\|u(t)\|_{L^\infty}^2 + \|b(t)\|_{L^\infty}^2) e_r \\ &+ C N^{2k-2} \exp\left(-2\sigma t + C \int_0^t (\|u(\tau)\|_{L^\infty}^2 + \|b(\tau)\|_{L^\infty}^2) d\tau\right) \\ &\times \sum_{k=0}^{r-1} (\|\nabla^{r-k} u(t)\|_{L^\infty}^2 + \|\nabla^{r-k} b(t)\|_{L^\infty}^2) Q_k(t) e_k(0), \end{aligned}$$

and hence, by Gronwall's lemma,

$$(3.9) \quad \begin{aligned} e_r(t) &\leq C \exp\left(-2\sigma t + C (\|u\|_{L^2 L^\infty}^2 + \|b\|_{L^2 L^\infty}^2)\right) e_r(0) \\ &\times \left[1 + \sum_{k=0}^{r-1} N^{2k-2} (\|\nabla^{r-k} u\|_{L^2 L^\infty}^2 + \|\nabla^{r-k} b\|_{L^2 L^\infty}^2)\right] \\ &\leq C(1 + N^{2r-2}) \exp\left(-2\sigma t + C (\|u\|_{L^2 L^\infty}^2 + \|b\|_{L^2 L^\infty}^2)\right) e_r(0), \end{aligned}$$

where in the second inequality we used assumption (3.1).

Now, the assumption we made on $e_r(t)$ is satisfied at time $t = 0$ (and thus for short times) by the smallness hypothesis (3.2). Moreover, this can be extended to any further time using the inequality (3.9) and again (3.2). This completes the proof. \blacksquare

4. Magnetic reconnection in 3D

In this section, we provide an example of magnetic reconnection in the three-dimensional case. The idea is to construct a global solution of (MHD) via a perturbative argument in such a way that we can ‘‘control’’ the topology of its magnetic lines at $t = 0$ and $t = T$.

Proof of Theorem 1.1, case $d = 3$. We divide the proof in several steps.

Step 1. Construction of the global smooth solution.

Let B_0 and B_1 be the Beltrami fields defined in (2.2) and in Theorem 2.1, respectively, and consider the couple $(u_0, b_0) = (0, MB_0)$ as an initial datum for (MHD). Then, a global smooth solution of (MHD) is given by

$$(u(t, x), b(t, x)) = (0, Me^{-\eta N_0^2 t} B_0(x)),$$

with pressure

$$P_{u,b}(t, x) = -\frac{1}{2} M^2 e^{-2\eta N_0^2 t} |B_0(x)|^2.$$

This may be easily verified recalling that B_0 is a Beltrami field. Now, consider the couple $(0, m_0)$ as initial datum of (MHD), where m_0 is given by

$$m_0 := MB_0 + \delta B_1.$$

Looking at the definition of (2.2), we immediately see that

$$\|MB_0\|_{H^r} = MN_0^r.$$

We want to construct a global smooth solution (w, m) starting from such an initial datum. First of all, we define

$$h_0 := m_0 - b_0 = \delta B_1.$$

We compute the H^r norm of δB_1 using (2.5), the fact that B_1 is Beltrami and the equivalence between the H^1 norm of B_1 and the L^2 norm of its curl, so that we arrive to

$$CN_1^{r-1} < \|B_1\|_{L^2} < CN_1^{r-1/2}.$$

Thus in particular, we have that

$$\|h_0\|_{H^k} \leq C\delta N_1^k.$$

Since we want to apply Theorem 3.1, we require that

$$(4.1) \quad \delta N_1^r \ll N_0^{1-r}.$$

Thus, recalling also Remark 3.2, we know that there exists a unique global solution (w, m) starting from $(0, m_0)$ and that the difference $h(t, \cdot) = m(t, \cdot) - b(t, \cdot)$ satisfies

$$\|w(t, \cdot)\|_{H^k} + \|h(t, \cdot)\|_{H^k} \leq C(N_0^{k-1} + 1) e^{-\sigma t} \|h_0\|_{H^k}$$

for all $0 \leq k \leq r$.

Step 2. Further estimates on the global solution.

We need more estimates on the difference h in order to control the behavior of the fluid at time $t = T$. Recall that, since in our reference solution $u = 0$, we have that $w = v$, and then h satisfies the following identity:

$$(4.2) \quad \begin{aligned} h(t, \cdot) &= e^{\eta t \Delta} h_0 \\ &+ \int_0^t e^{\eta(t-s)\Delta} \operatorname{div} (h(s) \otimes v(s) - v(s) \otimes h(s) + b(s) \otimes v(s) - v(s) \otimes b(s)) \, ds \end{aligned}$$

For simplicity, we define

$$(4.3) \quad L_h(t, \cdot) := \int_0^t e^{\eta(t-s)\Delta} \operatorname{div} (h(s) \otimes v(s) - v(s) \otimes h(s)) \, ds,$$

$$(4.4) \quad L_b(t, \cdot) := \int_0^t e^{\eta(t-s)\Delta} \operatorname{div} (b(s) \otimes v(s) - v(s) \otimes b(s)) \, ds.$$

First of all, note that by Theorem 3.1 and the estimate (4.1) of the previous step, we get that

$$\|v(t, \cdot)\|_{H^r} + \|h(t, \cdot)\|_{H^r} \leq C N_0^{r-1} e^{-\sigma t} \|h_0\|_{H^r} \leq \delta N_0^{r-1} N_1^r;$$

recall that here we have chosen $v_0 = 0$. Then, by using the above formula, we estimate the tensorial products in L_h and L_b as follows:

$$\begin{aligned} \|h(s) \otimes v(s)\|_{H^{r+1}} &\leq \|h(s)\|_{L^\infty} \|v(s)\|_{H^{r+1}} + \|v(s)\|_{L^\infty} \|h(s)\|_{H^{r+1}} \\ &\leq \|h(s)\|_{H^2} \|v(s)\|_{H^{r+1}} + \|v(s)\|_{H^2} \|h(s)\|_{H^{r+1}} \\ &\leq C \delta^2 N_0^{r+1} N_1^{r+3} e^{-\sigma s}, \\ \|b(s) \otimes v(s)\|_{L^2} &\leq \|b(s)\|_{L^\infty} \|v(s)\|_{L^2} \leq C \delta e^{-\eta N_0^2 s}, \\ \|b(s) \otimes v(s)\|_{H^{r+1}} &\leq C \|v(s)\|_{L^\infty} \|b(s)\|_{H^{r+1}} + C \|v(s)\|_{H^{r+1}} \|b(s)\|_{L^\infty} \\ &\leq C \|v(s)\|_{H^2} \|b(s)\|_{H^{r+1}} + C \|v(s)\|_{H^{r+1}} \|b(s)\|_{L^\infty} \\ &\leq C e^{-\eta N_0^2 s} (\delta N_0^{r+2} N_1^2 + \delta N_0^r N_1^{r+1}) \\ &\leq C e^{-\eta N_0^2 s} \delta N_0^{r+2} N_1^2, \end{aligned}$$

where in the last inequality we assumed that

$$(4.5) \quad N_0^2 \gg N_1^{r-1}.$$

Using (see Lemma 4.3 in [10])

$$(4.6) \quad \|e^{s\Delta} f\|_{H^k} \leq e^{-s} \|f\|_{H^k}, \quad \text{for } f \text{ with zero-average,}$$

we estimate L_h as follows:

$$\begin{aligned} \|L_h(t, \cdot)\|_{H^r} &\leq C \int_0^t \|e^{\eta(t-s)\Delta} (h(s) \otimes v(s))\|_{H^{r+1}} \, ds \\ &\leq C \int_0^t e^{-\eta(t-s)} \|h(s) \otimes v(s)\|_{H^{r+1}} \, ds \\ &\leq C \delta^2 N_0^{r+1} N_1^{r+3} \int_0^t e^{-\eta(t-s)} e^{-\eta\sigma s} \, ds \leq C \delta^2 N_0^{r+1} N_1^{r+3}. \end{aligned}$$

On the other hand, using (see Lemma 4.3 in [10])

$$(4.7) \quad \|e^{s\Delta} f\|_{H^k} \leq C s^{-k/2} \|f\|_{L^2}, \quad \text{for } f \text{ with zero-average,}$$

we estimate

$$\begin{aligned}
& \|L_b(t, \cdot)\|_{H^r} \\
& \leq C \int_0^{t/2} \|e^{\eta(t-s)\Delta} (b(s) \otimes v(s))\|_{H^{r+1}} ds + C \int_{t/2}^t \|e^{\eta(t-s)\Delta} (b(s) \otimes v(s))\|_{H^{r+1}} ds \\
& \leq C \int_0^{t/2} (t-s)^{-(r+1)/2} \|b(s) \otimes v(s)\|_{L^2} ds + C \int_{t/2}^t e^{-\eta(t-s)} \|b(s) \otimes v(s)\|_{H^{r+1}} ds \\
& \leq C\delta \int_0^{t/2} (t-s)^{-(r+1)/2} e^{-\eta N_0^2 s} ds + C\delta N_0^{r+2} N_1^2 \int_{t/2}^t e^{-\eta(t-s)} e^{-\eta N_0^2 s} ds \\
& \leq C\delta N_0^{-2} + C\delta N_0^{r+2} N_1^2 e^{-\eta N_0^2 t/2}.
\end{aligned}$$

Step 3. Choice of the parameters.

In this step, we fix the parameters N_0 , N_1 and δ . First of all, we know that at time $t = 0$, the magnetic field m_0 satisfies

$$m_0 = MB_0 + \delta B_1.$$

Then, let us consider the rescaled vector field

$$M^{-1}m_0,$$

that satisfies

$$\|M^{-1}m_0 - B_0\|_{H^r} \leq C\delta N_1^r \ll N_0^{1-r}.$$

Thus, recalling (2.3) and property (i) of Section 2, using the Sobolev embedding and taking r sufficiently large, we see that the integral lines of $M^{-1}m_0$ are all non-contractible. Furthermore, since the rescaling does not change the topology of its integral lines, all the integral lines of m_0 are non-contractible.

We now consider the behavior of the fluid at time $t = T$. We rescale the magnetic field as

$$\delta^{-1} e^{\eta N_1^2 T} m(T, \cdot),$$

and then, since $m = b + h$ and by formula (4.2), we get

$$\begin{aligned}
\delta^{-1} e^{\eta N_1^2 T} m(T, \cdot) &= B_1 + \frac{M}{\delta} e^{-\eta(N_0^2 - N_1^2)T} B_0(x) \\
&+ \delta^{-1} e^{\eta N_1^2 T} \int_0^T e^{\eta(T-s)\Delta} \operatorname{div} (h(s) \otimes v(s) + b(s) \otimes v(s) - v(s) \otimes h(s) - v(s) \otimes b(s)) ds.
\end{aligned}$$

Our goal is to choose the constants δ , N_0 and N_1 such that

$$\|\delta^{-1} e^{\eta N_1^2 T} m(T, \cdot) - B_1\|_{H^r} \ll 1,$$

for a sufficiently large r . Thus, since $\delta^{-1} e^{\eta N_1^2 T} m(T, \cdot)$ is simply a rescaling of $m(T, \cdot)$, the integral lines of $m(T, \cdot)$ will be locally diffeomorphic to the set \mathcal{S} defined in Section 2, as consequence of (2.4), property (ii) and Sobolev embedding. In particular, $m(T, \cdot)$ will

possess some contractible integral lines, thus the set of its integral lines will be not homotopically equivalent to that of B_1 , proving that magnetic reconnection must have happened between the times $t = 0$ and $t = T$.

It remains to show that \tilde{b}_1 satisfies (2.4). By the estimates proved in the Step 2, we have that

$$(4.8) \quad \begin{aligned} \|M\delta^{-1} e^{-\eta(N_0^2 - N_1^2)T} B_0\|_{H^r} &\leq M\delta^{-1} e^{-\eta(N_0^2 - N_1^2)T} \|B_0\|_{H^r} \\ &\leq M\delta^{-1} e^{-\eta(N_0^2 - N_1^2)T} N_0^r, \end{aligned}$$

$$(4.9) \quad \begin{aligned} \|\delta^{-1} e^{\eta N_1^2 T} L_h(T, \cdot)\|_{H^r} &\leq C\delta^{-1} e^{\eta N_1^2 T} \|L_h(T, \cdot)\|_{H^r} \\ &\leq C\delta^{-1} e^{\eta N_1^2 T} \delta^2 N_0^{r+1} N_1^{r+3} = C\delta N_0^{r+1} N_1^{r+3} e^{\eta N_1^2 T}, \end{aligned}$$

$$(4.10) \quad \begin{aligned} \|\delta^{-1} e^{\eta N_1^2 T} L_b(T, \cdot)\|_{H^r} &\leq \delta^{-1} e^{\eta N_1^2 T} \|L_b(T, \cdot)\|_{H^r} \\ &\leq \delta^{-1} e^{\eta N_1^2 T} \delta N_0^{-2} = e^{\eta N_1^2 T} N_0^{-2}. \end{aligned}$$

In order to make the above quantities small, we define δ as

$$(4.11) \quad \delta := N_0^{-(r+1)} N_1^{-(r+3)} e^{-2\eta N_1^2 T}.$$

With this choice, we can easily verify that (4.1) holds. Moreover, by the choice of δ as above, we have that (4.8) is small if

$$(4.12) \quad e^{-\eta N_0^2 T} \ll N_0^{-2r-1} N_1^{-r-3} M^{-1} e^{-3\eta N_1^2 T}.$$

Note that if N_0 is chosen sufficiently larger than N_1 , (4.5) and (4.12) are satisfied, and the quantities in (4.9) and (4.10) can be made arbitrarily small.

Step 4. Rescaling of the initial datum.

To complete the proof of the theorem, we need to rescale the norm of m_0 in the above construction. This can be done by replacing it with the initial condition $Mm_0/\|m_0\|_{L^2}$, since the rescaling factor does not change anything in the above arguments. ■

Some remarks on the proof above are in order.

Remark 4.1. The reconnection scenario presented in Theorem 1.1 is not instantaneous. We can in fact prove that for a strictly positive time interval, the magnetic field m does not have any contractible integral line. To show this, we define the rescaled vector fields

$$\tilde{m}(t, \cdot) := \frac{m(t, \cdot)}{MN_0^r} \quad \text{and} \quad \tilde{B}_0 := N_0^{-r} B_0,$$

and we obtain that

$$\begin{aligned} \|\tilde{m}(t, \cdot) - \tilde{B}_0\|_{H^r} &\lesssim (1 - e^{-\eta N_0^2 t}) + \frac{\delta}{MN_0^r} e^{-\eta N_1^2 t} N_1^r + \frac{\delta^2 N_0^{r+1} N_1^{r+3}}{N_0^r} \\ &\quad + \frac{\delta}{N_0^{r+2}} + \frac{\delta N_0^{r+2} N_1^2}{N_0^r}. \end{aligned}$$

Thus, given the choice of δ as in (4.11) and the relation $N_1 \ll N_0$, we have that for any $t \in (0, \gamma/(\eta N_0^2))$, it holds

$$\|\tilde{m}(t, \cdot) - \tilde{B}_0\|_{H^r} \lesssim \gamma + \frac{1}{N_0^\beta},$$

for some positive β and γ small enough. So, from Lemma 4.2 of [10], it follows that the solution does not have any contractible integral line at least until time of order $1/(\eta N_0^2)$. On the other hand, in Theorem 1.1 we have shown that, for reconnection to occur, we must have T and N_0 satisfy the relationship (4.12), and then

$$T \gg \frac{1}{\eta N_0^2}.$$

In conclusion, the reconnection takes place in a resistive time scale, i.e., at a time of the order $1/(\eta N_0^2)$.

Remark 4.2. The choice $u_0 = 0$ simplifies the proof, but we may easily generalize the argument to small velocities, namely taking $\|u_0\|_{H^r} = \varepsilon$, where the size of the small parameter ε depends on all the relevant parameters we introduced in the proof. Moreover, we may consider large data u_0 (and ε perturbation of them) if we add an appropriate structure. For example, if we choose $u_0 = MB_0$, it is easy to see that

$$(Me^{-\nu N_0^2 t} B_0(x), Me^{-\eta N_0^2 t} B_0(x))$$

is a smooth solution of (MHD) with initial datum $(u_0, b_0) = (MB_0, MB_0)$, choosing the pressure

$$P = \frac{M^2}{2} (e^{-2\nu N_0^2 t} - e^{-2\eta N_0^2 t}) |B_0|^2.$$

Then, when we estimate the H^r -norm of (4.2), there are additional terms to be computed, i.e.,

$$\int_0^t e^{\eta(t-s)\Delta} \operatorname{div} (h(s) \otimes u(s) - u(s) \otimes h(s)) \, ds,$$

which can be analyzed in the same way we did for L_b to close the argument.

Remark 4.3. In the computations above, it is crucial that $\eta > 0$. Indeed, the choice of the frequency $N_0 = N_0(\eta)$ depends on η , and $N_0(\eta) \rightarrow \infty$ as $\eta \rightarrow 0$, preventing to promote the magnetic reconnection scenario to the ideal case (in fact, we choose N_0 proportional to $\eta^{-1/2}$).

Remark 4.4. Differently to η , the estimates above do not blow up in the vanishing viscosity limit $\nu \rightarrow 0$. It is a reasonable guess that a similar result may hold in the inviscid case for local-in-time solutions. In light of Remark 4.1, in order to prove such a result, one must guarantee that the time of local existence is greater than the time at which reconnection is expected to occur. This imposes an additional smallness condition on the parameter δ . Standard energy estimates imply that δ must be exponentially small in N_0 , which does not allow us to make small the term in (4.8). We leave this for a future work.

5. Taylor fields

After fixing the notations and recalling some definitions and structural stability results, we introduce the main mathematical objects which are needed in our 2D magnetic reconnection proof. These are the so-called Taylor fields, which may be viewed as a 2D counterpart of the Beltrami fields from Section 2.

We say that $v: \mathbb{T}^2 \rightarrow \mathbb{R}^2$ is a *Hamiltonian* vector field if it can be written as the orthogonal gradient of a scalar function ψ , i.e.,

$$v = \nabla^\perp \psi := (\partial_{x_2} \psi, -\partial_{x_1} \psi).$$

Hamiltonian vector fields are by definition divergence-free, and we denote by

$$D_H^r(\mathbb{T}^2) = \{v \in C^r(\mathbb{T}^2) : v \text{ is a Hamiltonian vector field}\}.$$

We recall that a singular point x_0 of a vector field $v \in C^r(\mathbb{T}^2)$ is said to be *non-degenerate* if $\nabla v(x_0)$ is an invertible matrix. It is worth to note that if v is a smooth divergence-free vector field, then a non-degenerate singular point of v must be either a saddle or a center.

Remark 5.1. By the Helmholtz decomposition, an incompressible vector field on \mathbb{T}^2 is either Hamiltonian or a constant vector field. Since we will always work with zero-average vector fields, there will be no difference between being incompressible and Hamiltonian.

We now give the definition of structural stability.

Definition 5.2. A vector field v on \mathbb{T}^d is structurally stable if there is a neighborhood \mathcal{U} of v in $C^1(\mathbb{T}^d)$ such that whenever $v' \in \mathcal{U}$, there is a homeomorphism of \mathbb{T}^d onto itself transforming trajectories of v onto trajectories of v' .

Note that, as a consequence of the classical result of Peixoto [34], no 2D divergence-free vector field with some critical point that is a center is structurally stable under general C^r perturbations. This is because centers might be destroyed adding small sink or sources, which in the incompressible setting are however forbidden. Thus, in the divergence-free case a stronger result holds, see [29]. Moreover, the Peixoto stability result does not allow any kind of connection between saddle points, while in the incompressible case, saddle self-connections are allowed. On the other hand, heteroclinic self connections are expected to give rise to bifurcation phenomena also in the incompressible case (see Theorem 1.5).

Theorem 5.3 (Ma–Wang [29]). *A divergence-free Hamiltonian vector field $v \in D_H^r(\mathbb{T}^2)$ with $r \geq 1$ is structurally stable under Hamiltonian vector field perturbations if and only if*

- *v is regular, i.e., all singular points of v are not degenerate;*
- *all saddle points of v are self-connected.*

Moreover, the set of all C^r structurally stable Hamiltonian vector fields is open and dense in $D_H^r(\mathbb{T}^2)$.

5.1. Building blocks in 2D: Taylor vortices

For the two-dimensional case, we need to introduce a class of vector fields called *Taylor vortices*. We refer to [28, 30] for more details on the following. We consider the eigenvec-

tors of the following Stokes problem:

$$(5.1) \quad \begin{cases} -\Delta \mathcal{V} = \lambda \mathcal{V}, \\ \mathcal{V} = \nabla^\perp \psi. \end{cases}$$

Provided that $\lambda \in \mathbb{N}$, for any couple of integers $n \geq 1, m \geq 0$, we can easily construct a solution of (5.1) with $\lambda = n^2 + m^2$ in the following way:

$$\begin{aligned} \mathcal{V}_{nm}^1 &= (m \sin nx_1 \cos mx_2, -n \cos nx_1 \sin mx_2), \\ \mathcal{V}_{nm}^2 &= (m \cos nx_1 \sin mx_2, -n \sin nx_1 \cos mx_2), \\ \mathcal{V}_{nm}^3 &= (m \cos nx_1 \cos mx_2, n \sin nx_1 \sin mx_2), \\ \mathcal{V}_{nm}^4 &= (m \sin nx_1 \sin mx_2, n \cos nx_1 \cos mx_2), \\ \mathcal{V}_n^1 &= (\sin nx_2, 0), \quad \mathcal{V}_n^2 = (\cos nx_2, 0), \quad \mathcal{V}_n^3 = (0, \sin nx_1), \quad \mathcal{V}_n^4 = (0, \cos nx_1). \end{aligned}$$

Denote by $\mathbb{V}_{nm} = \text{span}\{\mathcal{V}_{nm}^1, \mathcal{V}_{nm}^2, \mathcal{V}_{nm}^3, \mathcal{V}_{nm}^4\}$ and $\mathbb{V}_n = \text{span}\{\mathcal{V}_n^1, \mathcal{V}_n^2, \mathcal{V}_n^3, \mathcal{V}_n^4\}$. Then, by varying n, m , these spaces generate all the (zero-average) solutions of (5.1).

It is important to note that the vector fields \mathcal{V}_{nm} (and \mathcal{V}_n , respectively) are stationary solutions of the Euler equations for a suitable choice of the pressure. For instance, we have

$$\begin{cases} (\mathcal{V}_{nm}^1 \cdot \nabla) \mathcal{V}_{nm}^1 = \nabla P_{nm}^1, \\ \text{div } \mathcal{V}_{nm}^1 = 0, \end{cases}$$

with pressure given by

$$P_{nm}^1 = m^2 (\sin(nx_1))^2 + n^2 (\sin(mx_2))^2.$$

Crucially, all vector fields in \mathbb{V}_{nm} are not Hamiltonian structurally stable, because they are not saddle self-connected, see Figure 1. However, they may have two kind of topological structure, see Theorem 4.5.3 in [30].

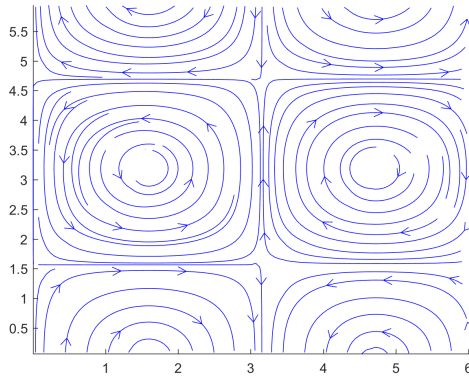


Figure 1. Phase diagram of the vector field \mathcal{V}_{11}^4 .

Let us recall that the set of Hamiltonian structurally stable vector fields is open and dense in the set of Hamiltonian vector fields. This means that, for any given \mathcal{V}_{nm} , one can

find a neighborhood \mathcal{U} , with respect to the topology in C^1 , such that if $\mathcal{V} \in \mathcal{U}$, then \mathcal{V} is a Hamiltonian structurally stable vector field, see Theorem 3.3.2 in [30]. This in particular means that by perturbing \mathcal{V}_{nm} , it is possible to break heteroclinic saddle connections. This is suggested by the fact that for Hamiltonian perturbations, the Melnikov function associated to an heteroclinic saddle connection may be nonzero, while it has to be zero in the case of homoclinic saddle connection (since the perturbation is autonomous, the Melnikov function is indeed a constant in this cases). A rigorous mechanism to break heteroclinic saddle connections has been proposed in [29]. The same holds for the space \mathbb{V}_n with $n > 1$, while in \mathbb{V}_1 , there are vector fields which are structurally stable. An example of a stable Taylor field is given by

$$(5.2) \quad V_1 = \left(\sin x_2, \frac{1}{2} \sin x_1 \right),$$

whose phase diagram can be found in Figure 2. In particular, note that the structural stability follows from the absence of heteroclinic connections. The field V_1 has indeed two saddle points, which are not connected by any integral line. It is also important for us that V_1 solves the stationary Euler $(V_1 \cdot \nabla)V_1 = \nabla P$ equation, with pressure $P = \frac{1}{2} \cos x_1 \cos x_2$.

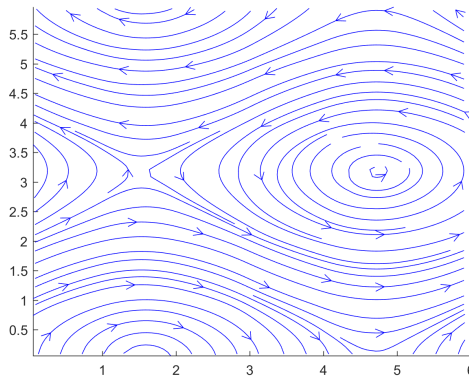


Figure 2. Phase diagram of the vector field V_1 .

Consider $n, m \geq 1$. We set $N^2 := n^2 + m^2$ (note that here N may be not integer). We denote by $V_N = \mathcal{V}_{nm}^1$. In the rest of the paper, we will focus, to fix ideas, on this particular subfamily of Taylor fields with eigenvalue N^2 , but we could similarly consider any other Taylor fields with eigenvalue N^2 with straightforward modifications of the arguments. The critical points of V_N are given by

$$(5.3) \quad x^* := \left(\frac{k_1 \pi}{n}, \frac{k_2 \pi}{m} \right) \quad \text{and} \quad \bar{x} := \left(\frac{\pi}{2n} + \frac{k_1 \pi}{n}, \frac{\pi}{2m} + \frac{k_2 \pi}{m} \right),$$

with $k_1 = 1, \dots, 2n$ and $k_2 = 1, \dots, 2m$ (with a small abuse of notation, we omit the dependence of x^* and \bar{x} on k_1 and k_2 , that will be however irrelevant). By computing the gradient of V_N , we obtain that

$$\nabla V_N = \begin{pmatrix} nm \cos(nx_1) \cos(mx_2) & -m^2 \sin(nx_1) \sin(mx_2) \\ n^2 \sin(nx_1) \sin(mx_2) & -nm \cos(nx_1) \cos(mx_2) \end{pmatrix}.$$

Thus, at the critical points we have

$$(5.4) \quad \nabla V_N(x^*) = \pm \begin{pmatrix} nm & 0 \\ 0 & -nm \end{pmatrix} \quad \text{or} \quad \nabla V_N(\bar{x}) = \pm \begin{pmatrix} 0 & -m^2 \\ n^2 & 0 \end{pmatrix},$$

and ∇V_N is always invertible in the critical points with determinant $\pm n^2 m^2$. Note that the x^* are hyperbolic critical points, while the \bar{x} are elliptic ones.

It is also worth to note that the distance between critical points can be controlled by below with C/N (this is clear by (5.3)). Moreover, a straightforward computation on the norms gives

$$\|V_N\|_{C^k} = c_k N^{k+1}.$$

We conclude this section with the following lemma, which investigates the stability of the critical points of our Taylor fields under small perturbations. If we do not take into account the quantitative bound (5.5) for $\delta_0(N)$, the content of the lemma is an immediate consequence of the implicit function theorem, as the linearization of V_N is invertible at the critical points. However, it will be crucial for us to quantify the size of the perturbative parameter δ , since in our applications it will not be possible to have δ too small (compared to a certain function of the frequency N). For instance, if in the next lemma we would allow the choice $\delta_0(N) = e^{-CN^2}$ with $C \gg 1$, we could not use it anymore in the proof of Theorem 1.1. This is why, in the proof below, we run again the implicit function theorem machinery for this particular example.

Lemma 5.4. *Let V_N be the Taylor field with eigenvalue $N^2 = n^2 + m^2$ defined above, and let $W \in C^1$. For all N sufficiently large (depending only on $\|W\|_{C^1}$), there exists $\delta_0 = \delta_0(N)$ such that the vector field $\tilde{V}: \mathbb{T}^2 \rightarrow \mathbb{R}^2$ defined as*

$$\tilde{V}(x) := V_N(x) + \delta W(x),$$

has at least $8nm$ regular critical points for every $|\delta| < \delta_0(N)$. We may choose

$$(5.5) \quad \delta_0(N) = N^{-L},$$

where L is a fixed large integer.

Proof. First of all, note that the critical points of \tilde{V} are zeros of the vector-valued function $F: \mathbb{T}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$ defined as

$$(5.6) \quad F(x, \delta) = V_N(x) + \delta W(x).$$

At the hyperbolic critical points x^* , we have (see 5.4) that

$$F(x^*, 0) = 0 \quad \text{and} \quad \nabla_x F(x^*, 0) \text{ is invertible,}$$

and the same is true at the elliptic critical points. We can thus apply the implicit function theorem to the function F in a neighborhood of x^* (or \bar{x}): this implies that here exist $\delta_0 = \delta_0(N) > 0$ and a (smooth) curve $x: [-\delta_0, \delta_0] \rightarrow \mathbb{T}^2$ with $x(0) = x^*$ (or $x(0) = \bar{x}$) such that $F(x(\delta), \delta) = 0$ for all $|\delta| < \delta_0$. We must now quantify the value of δ_0 to be as in (5.5). This will imply, in particular, that the new critical points $x(\delta)$, obtained with different choices of x^* and \bar{x} , do not collapse into each others.

We first consider the hyperbolic points. Note that

$$(5.7) \quad \nabla_x F(x^*, \delta) = \pm \begin{pmatrix} nm & 0 \\ 0 & -nm \end{pmatrix} + \delta \nabla W(x^*),$$

thus

$$(5.8) \quad \det \nabla_x F(x^*, \delta) = -n^2 m^2 + \mathcal{O}(\delta nm),$$

and so

$$|\det \nabla_x F(x^*, \delta)| > \frac{N^2}{2},$$

for $|\delta| < \delta_0$, with δ_0 sufficiently small. Here we used that $N \lesssim nm \lesssim N^2$. Hereafter, all the \mathcal{O} may also depend on $\|W\|_{C^1}$. For instance, in (5.8) we should have written $\mathcal{O}(\delta nm \|W\|_{C^1})$. However, we will always omit the dependence to simplify the notations.

This allows us to define a suitable map Φ such that the curve $x(\delta)$ will be constructed as the fixed point of this map. Let denote with $\Gamma_{\rho_0, \delta_0}(x^*, 0)$ the set of all the curves

$$\delta \in [-\delta_0, \delta_0] \rightarrow \gamma(\delta) \in B_{\rho_0}(x^*),$$

where $B_{\rho_0}(x^*)$ is the ball centered at x^* with radius $\rho_0 > 0$ (to be fixed). We endow $\Gamma_{\rho_0, \delta_0}(x^*, 0)$ with the distance

$$\text{dist}(\gamma, \gamma') = \sup_{|\delta| < \delta_0} |\gamma(\delta) - \gamma'(\delta)|.$$

This will give us a continuous curve as fixed point. We could consider a distance which takes into account also the derivative of γ to show that the fixed point is a C^1 curve (in fact, is C^k if $W \in C^k$), but this will be unessential for our purposes. We define

$$\Phi : \gamma(\delta) \in \Gamma_{\rho_0, \delta_0}(x^*, 0) \rightarrow \Phi(\gamma(\delta)) := \gamma(\delta) - (\nabla_x F(x^*, \delta))^{-1} F(\gamma(\delta), \delta).$$

We will show that Φ is a contraction, so that its fixed point $x(\delta)$ satisfies $F(x(\delta), \delta) = 0$, as claimed. We define (with a small abuse of notation), for $x \in B_{\rho_0}(x^*)$,

$$\Phi(x, \delta) = x - (\nabla_x F(x^*, \delta))^{-1} F(x, \delta).$$

Note that

$$\nabla_x \Phi(x, \delta) = \text{Id} - (\nabla_x F(x^*, \delta))^{-1} \nabla_x F(x, \delta).$$

Moreover, expanding the sine and cosine, we get

$$(5.9) \quad \nabla_x F(x, \delta) = \pm \begin{pmatrix} nm + \mathcal{O}(N^{4-L}) & \mathcal{O}(N^{2-2L}) + \mathcal{O}(\delta) \\ \mathcal{O}(N^{2-2L}) + \mathcal{O}(\delta) & nm + \mathcal{O}(N^{4-L}) \end{pmatrix}$$

for all x such that $|x - x^*| < CN^{-L}$, namely taking

$$\rho_0 = CN^{-L},$$

where the constant C that depends only on $\|W\|_{C^1}$ will be chosen later. Note that this choice of ρ_0 automatically implies that the new critical points that we found do not collapse into each other, (taking, for instance, $L \geq 2$) for all N sufficiently large (depending on L and $\|W\|_{C^1}$.) Thus the perturbed vector field has more than $8nm$ (regular) critical points.

Now, fixing $\delta_0(N) = N^{-L}$ and taking L sufficiently large, we rewrite (5.9) as

$$(5.10) \quad \nabla_x F(x, \delta) = \pm \begin{pmatrix} nm + \mathcal{O}(N^{-L/2}) & \mathcal{O}(N^{-L/2}) \\ \mathcal{O}(N^{-L/2}) + & nm + \mathcal{O}(N^{-L/2}) \end{pmatrix},$$

and we can compute

$$(5.11) \quad \nabla_x \Phi(x, \delta) = \begin{pmatrix} 1 + \Omega^{-1}(n^2 m^2 + \mathcal{O}(N^{-L/3})) & \mathcal{O}(N^{-L/3}) \\ \mathcal{O}(N^{-L/3}) & 1 + \Omega^{-1}(n^2 m^2 + \mathcal{O}(N^{-L/3})) \end{pmatrix},$$

where

$$\Omega := -n^2 m^2 + \mathcal{O}(N^{-L/3}).$$

Then, noting that

$$\frac{n^2 m^2}{\Omega} = -1 + \mathcal{O}(N^{-L/3}),$$

we see that, for all N sufficiently large,

$$(5.12) \quad |\nabla_x \Phi(x, \delta)| \ll 1.$$

This implies that Φ contracts the distances. To show that Φ is a contraction, it now suffices to prove (5.14) below, that we will deduce from

$$(5.13) \quad |\Phi(x^*, \delta) - x^*| \ll \rho_0,$$

that is, the constant curve $\gamma(t): [-\delta_0, \delta_0] \rightarrow x^*$ does not change too much under Φ . Indeed, combining (5.12) and (5.13) and using the triangle inequality, we readily see that

$$(5.14) \quad \Phi(\Gamma_{\rho_0, \delta_0}(x^*, 0)) \subset \Gamma_{\rho_0, \delta_0}(x^*, 0).$$

On the other hand,

$$\Phi(x^*, \delta) - x^* = -(\nabla_x F(x^*, \delta))^{-1} F(x^*, \delta) \quad \text{and} \quad F(x^*, \delta) = \delta W(x^*),$$

recall (5.6) and $V_N(x^*) = 0$. Thus we can use (5.10) to estimate

$$\begin{aligned} |\Phi(x^*, \delta) - x^*| &= \left| \frac{1}{\Omega} \begin{pmatrix} nm + \mathcal{O}(N^{-L/2}) & \mathcal{O}(N^{-L/2}) \\ \mathcal{O}(N^{-L/2}) + & nm + \mathcal{O}(N^{-L/2}) \end{pmatrix} \begin{pmatrix} \delta W_1(x) \\ \delta W_2(x) \end{pmatrix} \right| \\ &\lesssim |\delta| \|W\|_{C^1} \leq |\delta| \|W\|_{C^1} \ll \rho_0, \end{aligned}$$

which leads to (5.13) for all N sufficiently large. In the last inequality, we have used that $\delta_0 = N^{-L}$ and $\rho_0 = CN^{-L}$, and we have taken the constant C sufficiently large (in fact, a large multiple of $\|W\|_{C^1}$). This concludes the argument for the hyperbolic critical points.

For the elliptic critical points, we proceed in the same way. Here we sketch the relevant calculations. First of all, we have that

$$\nabla F(\bar{x}, \delta) = \pm \begin{pmatrix} 0 & -m^2 \\ n^2 & 0 \end{pmatrix} + \delta W,$$

and, as above, we get thus

$$\det \nabla_x F(\bar{x}, \delta) = -n^2 m^2 + \mathcal{O}(\delta nm),$$

and so

$$|\det \nabla_x F(\bar{x}, \delta)| > \frac{N^2}{2}.$$

Defining Φ as

$$\Phi(x, \delta) = x - (\nabla_x F(\bar{x}, \delta))^{-1} F(x, \delta),$$

we can show that $\nabla_x \Phi(x, \delta)$ has the form (5.11) and that it contracts the distances. Then almost the same calculations as above allows us to prove the analogous of (5.13), with x^* replaced by \bar{x} , and to conclude with the same computations as in the previous case. ■

6. Stability of the MHD system in 2D

In this section, we prove some preliminary results on the system (MHD) in the two-dimensional setting.

6.1. Boundedness of the H^r norms

We start by proving an a priori estimate for the Sobolev norms of the solution. This result (Theorem 6.2) will provide us some useful estimates in order to prove Theorem 6.4, which quantifies the decay of the velocity as the viscosity becomes large. Moreover, a perturbative version of Theorem 6.2, namely Theorem 6.3, will be used directly in the proof of the 2D magnetic reconnection in Theorem 1.1.

Remark 6.1. While the propagation of the H^r regularity for 2D solutions is not surprising, the relevant information here is the fact that we control the growth exponentially in the L^2 norms, while only polynomially in the higher order Sobolev norms.

Define $\Gamma := 1 + \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2$ and take $\sigma < \min(\nu, \eta)$.

Theorem 6.2. *Let $u_0, b_0 \in H^r(\mathbb{T}^2)$ be two divergence-free vector fields with zero mean. Assume*

$$\|u_0\|_{H^k} + \|b_0\|_{H^k} \leq CN^k, \quad k = 0, \dots, r,$$

for some $N > 1$. Let (u, b, p) be the unique solution (MHD) with initial datum (u_0, b_0) . Then, for all $\kappa < 2$,

$$(6.1) \quad \|u(t, \cdot)\|_{H^r}^2 + \|b(t, \cdot)\|_{H^r}^2 + \kappa\nu \int_0^t \|\nabla^{r+1} u(s, \cdot)\|_{L^2}^2 ds \\ + \kappa\eta \int_0^t \|\nabla^{r+1} b(s, \cdot)\|_{L^2}^2 ds \leq N^{2r} e^{C\Gamma/\sigma^2},$$

and

$$(6.2) \quad \|u(t, \cdot)\|_{H^r}^2 + \|b(t, \cdot)\|_{H^r}^2 \leq N^{2r} e^{-2\sigma t} e^{C\Gamma/\sigma^2},$$

where the implicit constants C depend on σ, κ, r .

When $r = 0$, we have of course more precise estimates than (6.1), see, for instance, estimate (6.5) and the energy estimate (6.4). A similar comment applies at least to small values of r ; however, the general form (6.2), which is enough for our purposes, is well suited to be generalized to the corresponding stability estimate (see Theorem 6.3).

Proof. We divide the proof in several steps.

Step 1. Estimate for $r = 0$.

By standard arguments, we know that the following identity holds for smooth solutions:

$$(6.3) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} (|u(t, x)|^2 + |b(t, x)|^2) dx + \nu \int_{\mathbb{T}^2} |\nabla u(t, x)|^2 dx + \eta \int_{\mathbb{T}^2} |\nabla b(t, x)|^2 dx = 0.$$

By integrating the above identity in time, we obtain that

$$(6.4) \quad \|u(t, \cdot)\|_{L^2}^2 + \|b(t, \cdot)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(s, \cdot)\|_{L^2}^2 ds + 2\eta \int_0^t \|\nabla b(s, \cdot)\|_{L^2}^2 ds = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2,$$

which gives (6.1) for $r = 0$. Moreover, note that if we apply Poincaré's inequality to (6.3), we obtain that

$$\frac{d}{dt} (\|u(t, \cdot)\|_{L^2}^2 + \|b(t, \cdot)\|_{L^2}^2) \leq -2\nu \|u(t, \cdot)\|_{L^2}^2 - 2\eta \|b(t, \cdot)\|_{L^2}^2,$$

and by Gronwall's inequality (recall $\sigma < \min(\eta, \nu)$),

$$(6.5) \quad \|u(t, \cdot)\|_{L^2}^2 + \|b(t, \cdot)\|_{L^2}^2 \leq (\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2) e^{-2\sigma t}.$$

Step 2. Estimate for $r = 1$.

We differentiate the equations, and multiplying the first by ∇u and the second by ∇b , we obtain that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla u(t, \cdot)\|_{L^2}^2 + \|\nabla b(t, \cdot)\|_{L^2}^2) + \nu \|\nabla^2 u(t, \cdot)\|_{L^2}^2 + \eta \|\nabla^2 b(t, \cdot)\|_{L^2}^2 \\ & + \int_{\mathbb{T}^2} \nabla[(u \cdot \nabla)u] : \nabla u \, dx + \int_{\mathbb{T}^2} \nabla[(u \cdot \nabla)b] : \nabla b \, dx \\ & = \int_{\mathbb{T}^2} \nabla[(b \cdot \nabla)b] : \nabla u \, dx + \int_{\mathbb{T}^2} \nabla[(b \cdot \nabla)u] : \nabla b \, dx. \end{aligned}$$

Thanks to the properties of the transport operator, we have several cancellations which yield

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla u(t, \cdot)\|_{L^2}^2 + \|\nabla b(t, \cdot)\|_{L^2}^2) + \nu \|\nabla^2 u(t, \cdot)\|_{L^2}^2 + \eta \|\nabla^2 b(t, \cdot)\|_{L^2}^2 \\ & \leq \int_{\mathbb{T}^2} |\nabla u| |\nabla u|^2 \, dx + 3 \int_{\mathbb{T}^2} |\nabla u| |\nabla b|^2 \, dx. \end{aligned}$$

We now use Hölder's, Ladyzenskaya's, and Young's inequalities to get that

$$\begin{aligned} \int_{\mathbb{T}^2} |\nabla u| |\nabla u|^2 \, dx & \leq \|\nabla u(t, \cdot)\|_{L^2} \|\nabla u(t, \cdot)\|_{L^4}^2 \leq \|\nabla u(t, \cdot)\|_{L^2}^2 \|\nabla^2 u(t, \cdot)\|_{L^2} \\ & \leq \frac{C}{\nu \varepsilon} \|\nabla u(t, \cdot)\|_{L^2}^4 + \nu \varepsilon \|\nabla^2 u(t, \cdot)\|_{L^2}^2, \end{aligned}$$

and that

$$\begin{aligned} & 3 \int_{\mathbb{T}^2} |\nabla u| |\nabla b|^2 dx \\ & \leq \|\nabla u(t, \cdot)\|_{L^2} \|\nabla b(t, \cdot)\|_{L^4}^2 \leq 3 \|\nabla u(t, \cdot)\|_{L^2} \|\nabla b(t, \cdot)\|_{L^2} \|\nabla^2 b(t, \cdot)\|_{L^2} \\ & \leq \frac{C}{\eta \varepsilon} \|\nabla u(t, \cdot)\|_{L^2}^2 \|\nabla b(t, \cdot)\|_{L^2}^2 + \eta \varepsilon \|\nabla^2 b(t, \cdot)\|_{L^2}^2. \end{aligned}$$

Then, by “absorbing constants” and taking ε small we get, for all $\sigma < \min(\eta, \nu)$ and $\kappa < 2$,

$$\begin{aligned} (6.6) \quad & \frac{1}{2} \frac{d}{dt} (\|\nabla u(t, \cdot)\|_{L^2}^2 + \|\nabla b(t, \cdot)\|_{L^2}^2) + \kappa \nu \|\nabla^2 u(t, \cdot)\|_{L^2}^2 + \kappa \eta \|\nabla^2 b(t, \cdot)\|_{L^2}^2 \\ & \leq \frac{C}{\sigma} \|\nabla u(t, \cdot)\|_{L^2}^4 + \frac{C}{\sigma} \|\nabla u(t, \cdot)\|_{L^2}^2 \|\nabla b(t, \cdot)\|_{L^2}^2 \\ & \leq \frac{C}{\sigma} \|\nabla u(t, \cdot)\|_{L^2}^2 (\|\nabla u(t, \cdot)\|_{L^2}^2 + \|\nabla b(t, \cdot)\|_{L^2}^2), \end{aligned}$$

where C depends on κ and σ . Finally, by Poincaré’s inequality, we obtain

$$\begin{aligned} (6.7) \quad & \frac{d}{dt} (\|\nabla u(t, \cdot)\|_{L^2}^2 + \|\nabla b(t, \cdot)\|_{L^2}^2) \\ & \leq \left(-2\sigma + \frac{C}{\sigma} \|\nabla u(t, \cdot)\|_{L^2}^2 \right) (\|\nabla u(t, \cdot)\|_{L^2}^2 + \|\nabla b(t, \cdot)\|_{L^2}^2), \end{aligned}$$

and then, by Gronwall’s lemma and the energy inequality, we obtain that

$$\|\nabla u(t, \cdot)\|_{L^2}^2 + \|\nabla b(t, \cdot)\|_{L^2}^2 \leq (\|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2) e^{C\Gamma/\sigma^2} e^{-2\sigma t}.$$

Finally, we integrate in time (6.6) and by using the energy inequality and the estimate above, we obtain that

$$\begin{aligned} & \kappa \nu \int_0^t \|\nabla^2 u(s, \cdot)\|_{L^2}^2 ds + \kappa \eta \int_0^t \|\nabla^2 b(s, \cdot)\|_{L^2}^2 ds \\ & \leq (\|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2) \left(1 + \frac{CC_0}{\sigma^2} e^{C\Gamma/\sigma^2} \right), \end{aligned}$$

which implies (6.1).

Step 3. Inductive step.

Let $r \geq 2$ and assume that the estimate (6.2) holds for $r - 1$. Let $\alpha \in \mathbb{N}^2$ with $|\alpha| = r$. We differentiate the equation by ∂^α and we multiply by $\partial^\alpha u$ and $\partial^\alpha b$, respectively, the equations for u and b in (MHD), obtaining that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\partial^\alpha u(t, \cdot)\|_{L^2}^2 + \|\partial^\alpha b(t, \cdot)\|_{L^2}^2) + \nu \|\nabla \partial^\alpha u(t, \cdot)\|_{L^2}^2 + \eta \|\nabla \partial^\alpha b(t, \cdot)\|_{L^2}^2 \\ & \quad + \int_{\mathbb{T}^2} \partial^\alpha [(u \cdot \nabla) u] : \partial^\alpha u dx + \int_{\mathbb{T}^2} \partial^\alpha [(u \cdot \nabla) b] : \partial^\alpha b dx \\ & \quad = \int_{\mathbb{T}^2} \partial^\alpha [(b \cdot \nabla) u] : \partial^\alpha b dx + \int_{\mathbb{T}^2} \partial^\alpha [(b \cdot \nabla) b] : \partial^\alpha u dx. \end{aligned}$$

Then, by using again the properties of the transport operator, we have that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\|\partial^\alpha u(t, \cdot)\|_{L^2}^2 + \|\partial^\alpha b(t, \cdot)\|_{L^2}^2) + \nu \|\nabla \partial^\alpha u(t, \cdot)\|_{L^2}^2 + \eta \|\nabla \partial^\alpha b(t, \cdot)\|_{L^2}^2 \\
& + \sum_{\beta \leq \alpha, \beta \neq 0} \int_{\mathbb{T}^2} \partial^\beta u \cdot \nabla \partial^{\alpha-\beta} u : \partial^\alpha u \, dx + \sum_{\beta \leq \alpha, \beta \neq 0} \int_{\mathbb{T}^2} \partial^\beta u \cdot \nabla \partial^{\alpha-\beta} b : \partial^\alpha b \, dx \\
(6.8) \quad & = \sum_{\beta \leq \alpha, \beta \neq 0} \int_{\mathbb{T}^2} \partial^\beta b \cdot \nabla \partial^{\alpha-\beta} u : \partial^\alpha b \, dx + \sum_{\beta \leq \alpha, \beta \neq 0} \int_{\mathbb{T}^2} \partial^\beta b \cdot \nabla \partial^{\alpha-\beta} b : \partial^\alpha u \, dx.
\end{aligned}$$

By summing over $|\alpha| = r$ and $0 \neq \beta \leq \alpha$, we get the estimate

$$\begin{aligned}
(6.9) \quad & \frac{1}{2} \frac{d}{dt} (\|\nabla^r u(t, \cdot)\|_{L^2}^2 + \|\nabla^r b(t, \cdot)\|_{L^2}^2) + \nu \|\nabla^{r+1} u(t, \cdot)\|_{L^2}^2 + \eta \|\nabla^{r+1} b(t, \cdot)\|_{L^2}^2 \\
& \leq \sum_{k=1}^r \left(\int_{\mathbb{T}^2} |\nabla^k u| |\nabla^{r+1-k} u| |\nabla^r u| \, dx + \int_{\mathbb{T}^2} |\nabla^k u| |\nabla^{r+1-k} b| |\nabla^r b| \, dx \right) \\
& + \sum_{k=1}^r \left(\int_{\mathbb{T}^2} |\nabla^k b| |\nabla^{r+1-k} u| |\nabla^r b| \, dx + \int_{\mathbb{T}^2} |\nabla^k b| |\nabla^{r+1-k} b| |\nabla^r u| \, dx \right).
\end{aligned}$$

Now note that the second and the third terms on the right-hand side are the same. Then, consider the terms of the sum with $k = 1, r$; we have the following estimates:

$$\begin{aligned}
(6.10) \quad & \int_{\mathbb{T}^2} |\nabla u(t, x)| |\nabla^r u(t, x)|^2 \, dx \leq \|\nabla u(t, \cdot)\|_{L^2} \|\nabla^r u(t, \cdot)\|_{L^4}^2 \\
& \leq \|\nabla u(t, \cdot)\|_{L^2} \|\nabla^r u(t, \cdot)\|_{L^2} \|\nabla^{r+1} u(t, \cdot)\|_{L^2} \\
& \leq \frac{C}{\nu \varepsilon} \|\nabla u(t, \cdot)\|_{L^2}^2 \|\nabla^r u(t, \cdot)\|_{L^2}^2 + \nu \varepsilon \|\nabla^{r+1} u(t, \cdot)\|_{L^2}^2.
\end{aligned}$$

$$\begin{aligned}
(6.11) \quad & \int_{\mathbb{T}^2} |\nabla u(t, x)| |\nabla^r b(t, x)|^2 \, dx \leq \|\nabla u(t, \cdot)\|_{L^2} \|\nabla^r b(t, \cdot)\|_{L^4}^2 \\
& \leq \|\nabla u(t, \cdot)\|_{L^2} \|\nabla^r b(t, \cdot)\|_{L^2} \|\nabla^{r+1} b(t, \cdot)\|_{L^2} \\
& \leq \frac{C}{\eta \varepsilon} \|\nabla u(t, \cdot)\|_{L^2}^2 \|\nabla^r b(t, \cdot)\|_{L^2}^2 + \eta \varepsilon \|\nabla^{r+1} b(t, \cdot)\|_{L^2}^2.
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\mathbb{T}^2} |\nabla b(t, x)| |\nabla^r b(t, x)| |\nabla^r u(t, x)| \, dx \\
& \leq \int_{\mathbb{T}^2} |\nabla b(t, x)| |\nabla^r b(t, x)|^2 \, dx + \int_{\mathbb{T}^2} |\nabla b(t, x)| |\nabla^r u(t, x)|^2 \, dx \\
& \leq \frac{C}{\sigma \varepsilon} \|\nabla b(t, \cdot)\|_{L^2}^2 (\|\nabla^r u(t, \cdot)\|_{L^2}^2 + \|\nabla^r b(t, \cdot)\|_{L^2}^2) + \nu \varepsilon \|\nabla^{r+1} u(t, \cdot)\|_{L^2}^2 \\
& + \eta \varepsilon \|\nabla^{r+1} b(t, \cdot)\|_{L^2}^2.
\end{aligned}$$

If $r = 2$, this is enough to conclude since, taking ε small, we have obtained that

$$\begin{aligned} \frac{d}{dt} (\|\nabla^r u(t, \cdot)\|_{L^2}^2 + \|\nabla^r b(t, \cdot)\|_{L^2}^2) + \kappa\nu \|\nabla^{r+1} u(t, \cdot)\|_{L^2}^2 + \kappa\eta \|\nabla^{r+1} b(t, \cdot)\|_{L^2}^2 \\ \leq \frac{C}{\sigma} (\|\nabla u(t, \cdot)\|_{L^2}^2 + \|\nabla b(t, \cdot)\|_{L^2}^2) (\|\nabla^r u(t, \cdot)\|_{L^2}^2 + \|\nabla^r b(t, \cdot)\|_{L^2}^2), \end{aligned}$$

and as we did several times before, by Poincaré's and Gronwall's inequalities, we get that

$$\|\nabla^r u(t, \cdot)\|_{L^2}^2 + \|\nabla^r b(t, \cdot)\|_{L^2}^2 \leq (\|\nabla^r u_0\|_{L^2}^2 + \|\nabla^r b_0\|_{L^2}^2) e^{C\Gamma/\sigma^2} e^{-2\sigma t},$$

which also implies, after integration in time,

$$\begin{aligned} \kappa\nu \int_0^t \|\nabla^{r+1} u(s, \cdot)\|_{L^2}^2 ds + \kappa\eta \int_0^t \|\nabla^{r+1} b(s, \cdot)\|_{L^2}^2 ds \leq \|\nabla^r u_0\|_{L^2}^2 + \|\nabla^r b_0\|_{L^2}^2 \\ + \frac{C}{\sigma} \int_0^t (\|\nabla u(s, \cdot)\|_{L^2}^2 + \|\nabla b(s, \cdot)\|_{L^2}^2) (\|\nabla^r u(s, \cdot)\|_{L^2}^2 + \|\nabla^r b(s, \cdot)\|_{L^2}^2) ds \\ \leq (\|u_0\|_{H^r}^2 + \|b_0\|_{H^r}^2) e^{C\Gamma/\sigma^2}. \end{aligned}$$

Then, we assume that $r \geq 3$ and we estimate the remaining terms in (6.9) as follows: back to (6.8), we exploit the divergence-free condition, and by integrating by parts, we have to bound the following integrals:

$$\begin{aligned} \sum_{k=2}^{r-1} \int_{\mathbb{T}^2} |\nabla^k u(t, x)| |\nabla^{r-k} u(t, x)| |\nabla^{r+1} u(t, x)| dx \\ \leq \sum_{k=2}^{r-1} C \int_{\mathbb{T}^2} |\nabla^k u(t, x)|^2 |\nabla^{r-k} u(t, x)|^2 dx + \frac{\nu}{c} \int_{\mathbb{T}^2} |\nabla^{r+1} u(t, x)|^2 dx. \end{aligned}$$

By Ladyzenskaya's inequality,

$$\begin{aligned} \int_{\mathbb{T}^2} |\nabla^k u(t, x)|^2 |\nabla^{r-k} u(t, x)|^2 dx \leq \|\nabla^k u(t, \cdot)\|_{L^4}^2 \|\nabla^{r-k} u(t, \cdot)\|_{L^4}^2 \\ \leq \|\nabla^k u(t, \cdot)\|_{L^2} \|\nabla^{k+1} u(t, \cdot)\|_{L^2} \|\nabla^{r-k} u(t, \cdot)\|_{L^2} \|\nabla^{r-k+1} u(t, \cdot)\|_{L^2} \\ \leq \frac{1}{2} \|\nabla^k u(t, \cdot)\|_{L^2}^2 \|\nabla^{r-k+1} u(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \|\nabla^{k+1} u(t, \cdot)\|_{L^2}^2 \|\nabla^{r-k} u(t, \cdot)\|_{L^2}^2. \end{aligned}$$

Note that, choosing $k = r - 1$ in the second term, we have a contribution

$$\|\nabla^r u(t, \cdot)\|_{L^2}^2 \|\nabla u(t, \cdot)\|_{L^2}^2.$$

Thus, recalling the previous estimates for the contributions $k = 1, \dots, r$, after rearranging the indices in the sums, we arrive to

$$\begin{aligned} \frac{d}{dt} (\|\nabla^r u(t, \cdot)\|_{L^2}^2 + \|\nabla^r b(t, \cdot)\|_{L^2}^2) \leq -\kappa\nu \|\nabla^r u(t, \cdot)\|_{L^2}^2 - \kappa\eta \|\nabla^r b(t, \cdot)\|_{L^2}^2 \\ + \frac{C}{\sigma} (\|\nabla u(t, \cdot)\|_{L^2}^2 + \|\nabla b(t, \cdot)\|_{L^2}^2) (\|\nabla^r u(t, \cdot)\|_{L^2}^2 + \|\nabla^r b(t, \cdot)\|_{L^2}^2) \\ (6.12) \quad + \frac{C}{\sigma} \sum_{k=2}^{r-1} (\|\nabla^{r-k+1} u(t, \cdot)\|_{L^2}^2 \|\nabla^k u(t, \cdot)\|_{L^2}^2). \end{aligned}$$

We now estimate the last group of terms using the induction assumption (6.2) for $k = 2, \dots, r-1$. This yields

$$(6.13) \quad \|\nabla^{r-k+1}u(t, \cdot)\|_{L^2}^2 \|\nabla^k u(t, \cdot)\|_{L^2}^2 \leq N^{2(r-k+1)} e^{C\Gamma/\sigma^2} \|\nabla^k u(t, \cdot)\|_{L^2}^2.$$

Plugging this into (6.12), we arrive to

$$(6.14) \quad \begin{aligned} \frac{d}{dt} (\|\nabla^r u(t, \cdot)\|_{L^2}^2 + \|\nabla^r b(t, \cdot)\|_{L^2}^2) &\leq -\kappa v \|\nabla^r u(t, \cdot)\|_{L^2}^2 - \kappa \eta \|\nabla^r b(t, \cdot)\|_{L^2}^2 \\ &+ \frac{C}{\sigma} (\|\nabla u(t, \cdot)\|_{L^2}^2 + \|\nabla b(t, \cdot)\|_{L^2}^2) (\|\nabla^r u(t, \cdot)\|_{L^2}^2 + \|\nabla^r b(t, \cdot)\|_{L^2}^2) \\ &+ \frac{C}{\sigma} \sum_{k=2}^{r-1} N^{2(r-k+1)} e^{C\Gamma/\sigma^2} \|\nabla^k u(t, \cdot)\|_{L^2}^2. \end{aligned}$$

Using Poincaré's inequality and Gronwall's lemma, we obtain the desired estimate (6.2). Note that the time integral of the last contribution of the right-hand side of (6.14) is estimated using the induction assumption (6.1). Integrating (6.14), we also obtain (6.1), and the proof is concluded. \blacksquare

6.2. Boundedness of the H^r norms of the difference equation (6.15)

We also need a perturbative version of Theorem 6.2, in order to control the H^r norms of solutions of the difference system

$$(6.15) \quad \begin{cases} \partial_t v + \operatorname{div}(v \otimes v + 2v \otimes u) + \nabla P_{v,h} = v \Delta v + \operatorname{div}(h \otimes h + 2h \otimes b), \\ \partial_t h + (v \cdot \nabla)h + (u \cdot \nabla)h + (v \cdot \nabla)b = \eta \Delta h + (h \cdot \nabla)v + (h \cdot \nabla)u + (b \cdot \nabla)v, \\ \operatorname{div} v = \operatorname{div} h = 0, \end{cases}$$

where (u, b) is a solution of (MHD). Note that $w = u + v$, $m = b + h$ solves the (MHD) equation (with an appropriate choice of the pressure). To do so, we define

$$\tilde{\Gamma} := 1 + \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|h_0\|_{L^2}^2,$$

and we take $\sigma < \min(v, \eta)$.

Theorem 6.3. *Let $v_0, h_0 \in H^r(\mathbb{T}^2)$ be two divergence-free vector fields with zero mean. Let $(v, h, P_{v,h})$ be the unique solution (6.15) with initial datum (v_0, h_0) , where (u, b) is a solution of (MHD). Assume that*

$$\|u_0\|_{H^k} + \|b_0\|_{H^k} \leq CN^k \quad \text{and} \quad \|v_0\|_{H^k} + \|h_0\|_{H^k} \leq C\delta, \quad k = 0, \dots, r,$$

for some $N > 1$. Then

$$(6.16) \quad \|h(t, \cdot)\|_{H^r}^2 + \|v(t, \cdot)\|_{H^r}^2 \leq \delta^2 N^{2r} e^{-2\sigma t} e^{C\tilde{\Gamma}/\sigma^2},$$

where the implicit constants C depend on r and σ .

The proof is a straightforward generalization of that of Theorem 6.2, we left the details to the reader. Again, at least for small values of r , we may prove stronger estimates. For instance, for $r = 0$, we may set $\tilde{\Gamma} := \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2$. The bound (6.16) is however sufficient for our purposes.

6.3. Decay of the velocity

We conclude the section by proving that the H^r norm of the velocity decays in ν (if the viscosity is large). This result (Theorem 6.4) will be useful to prove 2D magnetic reconnection with arbitrary initial velocity and large viscosity, namely Theorem 1.3.

Theorem 6.4. *Let $u_0, b_0 \in H^r(\mathbb{T}^2)$ be two divergence-free vector fields with zero mean. Assume*

$$\|u_0\|_{H^k} + \|b_0\|_{H^k} \leq CN^k, \quad k = 0, \dots, r,$$

for some $N > 1$. Let (u, b, p) be the unique solution of (MHD) with initial datum (u_0, b_0) . Assume that $\nu > 3\eta$. Then

$$(6.17) \quad \|\nabla^r u(\cdot, t)\|_{L^2}^2 \leq \left(\|\nabla^r u_0\|_{L^2}^2 + \frac{C e^{\nu t} e^{-\eta t}}{\nu \eta} N^{2r} \right) e^{C\Gamma/\sigma^2} e^{-\nu t}.$$

Remark 6.5. This bound will be important for two reasons. The first is that the right-hand side goes to zero as $\nu \rightarrow \infty$. The second is that, as in all the 2D results of this paper, the estimate is exponential in the L^2 norm of the initial datum, while only polynomial in the higher order Sobolev norms.

Proof. We divide the proof in several steps.

Step 1. L^2 estimate.

Let us consider the equation for the velocity field, namely,

$$\partial_t u + (u \cdot \nabla) u + \nabla P = \nu \Delta u + (b \cdot \nabla) b.$$

Multiplying the equation above by u and integrating by parts, it is easy to show that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u(t, \cdot)\|_{L^2}^2 + \nu \|\nabla u(t, \cdot)\|_{L^2}^2 &= - \int_{\mathbb{T}^2} b \otimes b : \nabla u \, dx \leq \|b(t, \cdot)\|_{L^4}^2 \|\nabla u(t, \cdot)\|_{L^2} \\ &\leq \|b(t, \cdot)\|_{L^2} \|\nabla b(t, \cdot)\|_{L^2} \|\nabla u(t, \cdot)\|_{L^2} \\ &\leq \frac{C}{\nu} \|b(t, \cdot)\|_{L^2}^2 \|\nabla b(t, \cdot)\|_{L^2}^2 + \frac{\nu}{2} \|\nabla u(t, \cdot)\|_{L^2}^2, \end{aligned}$$

where in the third line we used Ladyzhenskaya's inequality, while in the fourth we used Young's inequality. Then, by Poincaré's inequality, we obtain that

$$\frac{d}{dt} \|u(t, \cdot)\|_{L^2}^2 \leq -\nu \|u(t, \cdot)\|_{L^2}^2 + \frac{C}{\nu} \|b(t, \cdot)\|_{L^2}^2 \|\nabla b(t, \cdot)\|_{L^2}^2.$$

Now, we use Gronwall's inequality and Theorem 6.2 with $r = 0, 1$ (and, say, $\kappa = 1$), to compute

$$\|u(t, \cdot)\|_{H^r}^2 + \|b(t, \cdot)\|_{H^r}^2 \leq N^{2r} e^{-2\sigma t} e^{C\Gamma/\sigma^2},$$

and

$$\begin{aligned}
\|u(t, \cdot)\|_{L^2}^2 &\leq \left(\|u_0\|_{L^2}^2 + \frac{C}{\nu} \int_0^t \|b(s, \cdot)\|_{L^2}^2 \|\nabla b(s, \cdot)\|_{L^2}^2 e^{\nu s} ds \right) e^{-\nu t} \\
&\leq \left(\|u_0\|_{L^2}^2 + \frac{C e^{C\Gamma/\sigma^2}}{\nu} \int_0^t \|\nabla b(s, \cdot)\|_{L^2}^2 e^{(\nu-2\sigma)s} ds \right) e^{-\nu t} \\
&\leq \left(\|u_0\|_{L^2}^2 + \frac{C e^{C\Gamma/\sigma^2}}{\nu} e^{(\nu-2\sigma)t} \int_0^t \|\nabla b(s, \cdot)\|_{L^2}^2 ds \right) e^{-\nu t} \leq \|u_0\|_{L^2}^2 e^{-\nu t} + \frac{C e^{C\Gamma/\sigma^2} e^{-\eta t}}{\nu \eta}
\end{aligned}$$

where we have taken $\sigma > \eta/2$; this is possible as we have assumed that $\eta = \min(\eta, \nu)$ (in fact, $\nu > 3\eta$).

Step 2. H^1 estimate.

We differentiate the equation and multiply by ∇u to get, after integration by parts

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla u(t, \cdot)\|_{L^2}^2 + \nu \|\nabla^2 u(t, \cdot)\|_{L^2}^2 &\leq \int_{\mathbb{T}^2} |\nabla u| |\nabla u|^2 dx + \int_{\mathbb{T}^2} |\nabla(b \otimes b)| |\nabla^2 u| dx \\
&\leq \frac{\nu}{2} \|\nabla^2 u(t, \cdot)\|_{L^2}^2 + \frac{C}{\nu} \|\nabla u(t, \cdot)\|_{L^2}^2 \|\nabla u(t, \cdot)\|_{L^2}^2 + \frac{C}{\nu} \int_{\mathbb{T}^2} |b|^2 |\nabla b|^2 dx \\
&\leq \frac{\nu}{2} \|\nabla^2 u(t, \cdot)\|_{L^2}^2 + \frac{C}{\nu} \|\nabla u(t, \cdot)\|_{L^2}^2 \|\nabla u(t, \cdot)\|_{L^2}^2 \\
&\quad + \frac{C}{\nu} \|b\|_{L^2}^2 \|\nabla b\|_{L^2}^2 + \frac{C}{\nu} \|\nabla b\|_{L^2}^2 \|\nabla^2 b\|_{L^2}^2.
\end{aligned}$$

We rewrite the inequality above by using Poincaré's inequality,

$$\begin{aligned}
\frac{d}{dt} \|\nabla u(t, \cdot)\|_{L^2}^2 &\leq \left(\frac{C}{\nu} \|\nabla u(t, \cdot)\|_{L^2}^2 - \nu \right) \|\nabla u(t, \cdot)\|_{L^2}^2 \\
&\quad + \frac{C}{\nu} \left(\|b(t, \cdot)\|_{L^2}^2 \|\nabla b(t, \cdot)\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \|\nabla^2 b\|_{L^2}^2 \right),
\end{aligned}$$

and as usual, Gronwall's lemma gives that

$$\begin{aligned}
&\|\nabla u(t, \cdot)\|_{L^2}^2 \\
&\leq \left(\|\nabla u_0\|_{L^2}^2 + \frac{C}{\nu} \int_0^t (\|b(s, \cdot)\|_{L^2}^2 \|\nabla b(s, \cdot)\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \|\nabla^2 b\|_{L^2}^2) e^{\nu s} ds \right) e^{C\Gamma/\sigma^2} e^{-\nu t} \\
&\leq \left(\|\nabla u_0\|_{L^2}^2 + \frac{C}{\nu} \int_0^t \|\nabla b(s, \cdot)\|_{L^2}^2 e^{(\nu-2\sigma)s} ds + \frac{CN^2}{\nu} \int_0^t \|\nabla^2 b(s, \cdot)\|_{L^2}^2 e^{(\nu-2\sigma)s} ds \right) \\
&\quad \times e^{C\Gamma/\sigma^2} e^{-\nu t} \\
&\leq \left(\|\nabla u_0\|_{L^2}^2 + \frac{C}{\nu} e^{(\nu-2\sigma)t} \int_0^t \|\nabla b(s, \cdot)\|_{L^2}^2 ds + \frac{CN^2}{\nu} e^{(\nu-2\sigma)t} \int_0^t \|\nabla^2 b(s, \cdot)\|_{L^2}^2 ds \right) \\
&\quad \times e^{C\Gamma/\sigma^2} e^{-\nu t} \\
&\leq \left(\|\nabla u_0\|_{L^2}^2 + \frac{C}{\nu \eta} e^{(\nu-2\sigma)t} + \frac{CN^4}{\nu \eta} e^{(\nu-2\sigma)t} \right) e^{C\Gamma/\sigma^2} e^{-\nu t}.
\end{aligned}$$

Then the conclusion follows as in the previous step.

Step 3. General case.

Let $\alpha \in \mathbb{N}^2$ with $|\alpha| = r$. We apply ∂^α to the equation and we multiply by $\partial^\alpha u$ to obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial^\alpha u(t, \cdot)\|_{L^2}^2 + \nu \|\nabla \partial^\alpha u(t, \cdot)\|_{L^2}^2 + \sum_{\beta \leq \alpha, \beta \neq 0} \int_{\mathbb{T}^2} \partial^\beta u \cdot \nabla \partial^{\alpha-\beta} u : \partial^\alpha u \, dx \\ = \sum_{\beta \leq \alpha} \int_{\mathbb{T}^2} \partial^\beta b \cdot \nabla \partial^{\alpha-\beta} b : \partial^\alpha u \, dx. \end{aligned}$$

We analyze the third and the fourth terms separately. First, we have that

$$\begin{aligned} \sum_{\beta \leq \alpha, \beta \neq 0} \int_{\mathbb{T}^2} \partial^\beta u \cdot \nabla \partial^{\alpha-\beta} u : \partial^\alpha u \, dx \\ \leq 2 \int_{\mathbb{T}^2} |\nabla u| |\nabla^r u|^2 \, dx + \sum_{1 < |\beta| < r} \int_{\mathbb{T}^2} \partial^\beta u \cdot \nabla \partial^{\alpha-\beta} u : \partial^\alpha u \, dx \\ \leq \frac{\nu}{4} \|\nabla^{r+1} u(t, \cdot)\|_{L^2}^2 + \frac{C}{\nu} \|\nabla u(t, \cdot)\|_{L^2}^2 \|\nabla^r u(t, \cdot)\|_{L^2}^2 \\ + \frac{C}{\nu} \sum_{k=2}^{r-1} \int_{\mathbb{T}^2} |\nabla^k u|^2 |\nabla^{r-k} u|^2 \, dx \\ \leq \frac{\nu}{4} \|\nabla^{r+1} u(t, \cdot)\|_{L^2}^2 + \frac{C}{\nu} \|\nabla u(t, \cdot)\|_{L^2}^2 \|\nabla^r u(t, \cdot)\|_{L^2}^2 + \frac{C}{\nu} N^{2r} e^{C\Gamma/\sigma^2} e^{-4\sigma t}, \end{aligned}$$

where we have integrated by parts in the sum $1 < |\beta| < r$ and we have applied Theorem 6.2. On the other hand, arguing in a similar way, for the last term we have that

$$\begin{aligned} \sum_{\beta \leq \alpha} \int_{\mathbb{T}^2} \partial^\beta b \cdot \nabla \partial^{\alpha-\beta} b : \partial^\alpha u \, dx = - \sum_{\beta \leq \alpha} \int_{\mathbb{T}^2} \partial^\beta b \cdot \partial^{\alpha-\beta} b : \nabla \partial^\alpha u \, dx \\ \leq \frac{\nu}{4} \int_{\mathbb{T}^2} |\nabla^{r+1} u|^2 \, dx + \frac{C}{\nu} \sum_{k=0}^r \int_{\mathbb{T}^2} |\nabla^k b(t, \cdot)|^2 |\nabla^{r-k} b(t, \cdot)|^2 \, dx \\ \leq \frac{\nu}{4} \|\nabla^{r+1} u(t, \cdot)\|_{L^2}^2 + \frac{C}{\nu} \sum_{k=0}^r \|\nabla^k b(t, \cdot)\|_{L^4}^2 \|\nabla^{r-k} b(t, \cdot)\|_{L^4}^2 \\ \leq \frac{\nu}{4} \|\nabla^{r+1} u(t, \cdot)\|_{L^2}^2 + \frac{C}{2\nu} \sum_{k=0}^r \|\nabla^k b(t, \cdot)\|_{L^2}^2 \|\nabla^{r-k+1} b(t, \cdot)\|_{L^2}^2 \\ + \frac{C}{2\nu} \sum_{k=0}^r \|\nabla^{r-k} b(t, \cdot)\|_{L^2}^2 \|\nabla^{k+1} b(t, \cdot)\|_{L^2}^2 \\ \leq \frac{\nu}{4} \|\nabla^{r+1} u(t, \cdot)\|_{L^2}^2 + \frac{C}{\nu} e^{C\Gamma/\sigma^2} \sum_{k=0}^r N^{2(r-k)} \|\nabla^{k+1} b(t, \cdot)\|_{L^2}^2 e^{-2\sigma t}. \end{aligned}$$

Then, by Poincaré's inequality, we obtain that

$$\begin{aligned} \frac{d}{dt} \|\nabla^r u(t, \cdot)\|_{L^2}^2 &\leq \frac{C}{\nu} (\|\nabla u(t, \cdot)\|_{L^2}^2 - \nu) \|\nabla^r u(t, \cdot)\|_{L^2}^2 + \frac{C}{\nu} N^{2r} e^{C\Gamma/\sigma^2} e^{-4\sigma t} \\ &\quad + \frac{C}{\nu} \sum_{k=0}^r N^{2(r-k)} \|\nabla^{k+1} b(t, \cdot)\|_{L^2}^2 e^{-2\sigma t}. \end{aligned}$$

Now, applying Gronwall's inequality and arguing as we did in the previous steps, we obtain that (again we choose $\sigma = \eta/2$)

$$\begin{aligned} \|\nabla^r u(t, \cdot)\|_{L^2}^2 &\leq \left(\|\nabla^r u_0\|_{L^2}^2 + \frac{C}{\nu} N^{2r} \int_0^t e^{(v-2\eta)s} ds \right) e^{C\Gamma/\sigma^2} e^{-\nu t} \\ &\quad + \left(\frac{C}{\nu} \sum_{k=0}^r N^{2(r-k)} \int_0^t \|\nabla^{k+1} b(s, \cdot)\|_{L^2}^2 e^{(v-\eta)s} ds \right) e^{C\Gamma/\sigma^2} e^{-\nu t} \\ &\leq \left(\|\nabla^r u_0\|_{L^2}^2 + \frac{C e^{\nu t} e^{-2\eta t}}{\nu(v-2\eta)} N^{2r} + \sum_{m=0}^r \frac{C N^{2r}}{\nu \eta} e^{(v-\eta)t} \right) e^{C\Gamma/\sigma^2} e^{-\nu t}, \end{aligned}$$

where we used (6.1) in the last inequality. Recalling $\nu > 3\eta$, this implies (6.17), and the proof is concluded. \blacksquare

7. Magnetic reconnection in 2D (small velocity)

We are ready to prove the 2D case of our main Theorem 1.1.

7.1. Construction of the initial datum

For our argument, we may choose as a reference solution any of the (large frequency) Taylor fields defined in Section 5. However, to fix the idea, we focus on

$$\mathcal{V}_{n\ell}^1 = (\ell \sin nx_1 \cos \ell x_2, -n \cos nx_1 \sin \ell x_2),$$

where $n^2 + \ell^2 = N^2$ and N will be taken large. We will set, with a small abuse of notation, $V_N = \mathcal{V}_{n\ell}^1$. Also, we take $\mathcal{V}_1 \in \mathbb{V}_1$ and we choose, among them, one of the Hamiltonian structurally stable vector field (we know that in \mathbb{V}_1 there are some). For example, we can use

$$V_1 = \left(\sin y, \frac{1}{2} \sin x \right).$$

We consider the initial data

$$m_0 = b_0 + h_0, \quad \text{with } b_0 := \frac{M}{N} V_N \quad \text{and } h_0 := \delta V_1,$$

where $M > 0$ (is possibly large) and $\delta > 0$ will be chosen to be very small. Indeed, the goal is to show that we can choose δ small and N big enough such that if $m(t, \cdot)$ is the solution of (MHD) with initial datum m_0 , then m_0 will have at least $8\ell n$ regular critical

points, while at some positive time $t = T > 0$, the topology of the integral lines of $m(T, \cdot)$ will be the same of V_1 ; in particular, $m(T, \cdot)$ will have 4 regular points. Thus a change of topology of the magnetic lines happened between $t = 0$ and $t = T$. Note that in the statement of the theorem, we denote with b the solution which reconnects, but in the proof it denotes the reference solution.

Proof of Theorem 1.1, case $d = 2$. We divide the proof in several steps.

Step 1. Auxiliary solutions of (MHD).

Recall that V_N is divergence free and that $\Delta V_N = -N^2 V_N$. Moreover, V_N solves

$$(V_N \cdot \nabla) V_N = \nabla P_{V_N},$$

with pressure (recall $N^2 = n^2 + \ell^2$)

$$P_{V_N} = \frac{1}{2} (\ell^2 (\sin(nx_1))^2 + n^2 (\sin(\ell x_2))^2).$$

Thus the couple

$$(u, b) = (0, e^{-\eta N^2 t} V_N)$$

is the unique smooth solution of (MHD) with data $(0, V_N)$; the pressure is $e^{-2\eta N^2 t} P_{V_N}$.

Step 2. Integral lines at time 0.

Consider the rescaled datum

$$\frac{N}{M} m_0 = V_N + \frac{N}{M} \delta V_1.$$

Taking

$$(7.1) \quad \delta < c \frac{M}{N^{L+1}},$$

where c is a suitable small constant, we have that $\frac{N}{M} \delta \ll \delta_0(N)$ from Lemma 5.4, thus the vector field $\frac{N}{M} m_0$, and so m_0 , has at least $8n\ell$ regular critical points.

Step 3. Evolution under the MHD flow.

We denote with (v, h) the solution of the difference equation (3.4), with initial datum

$$(0, h_0), \quad \text{where } h_0 := \delta V_1.$$

Thus, under our choices, the reference solution is $(u, b) = (0, \frac{M}{N} e^{-\eta N^2 t} V_N)$ and the solution with initial datum

$$(0, m_0), \quad \text{where } m_0 := \frac{M}{N} V_N + h_0, \quad \text{with } b_0 := \frac{M}{N} V_N \quad \text{and } h_0 := \delta V_1,$$

is denoted by (w, m) . Using Duhamel's formula, we can write

$$(7.2) \quad m(t, \cdot) = \frac{M}{N} e^{-\eta N^2 t} V_N + \delta e^{-\eta t} V_1 + D(t, \cdot),$$

where

$$D(t, \cdot) := L_h(t, \cdot) + L_b(t, \cdot)$$

and we recall that L_h, L_b introduced in (4.3)-(4.4) are defined as

$$\begin{aligned} L_h(t, \cdot) &:= \int_0^t e^{\eta(t-s)\Delta} \operatorname{div} (h(s) \otimes v(s) - v(s) \otimes h(s)) \, ds, \\ L_b(t, \cdot) &:= \int_0^t e^{\eta(t-s)\Delta} \operatorname{div} (b(s) \otimes v(s) - v(s) \otimes b(s)) \, ds. \end{aligned}$$

Since (the constant C here depends on M and r)

$$\|b_0\|_{H^k} \leq CN^{-1} \|V_N\|_{H^k} \leq CN^k \quad \text{and} \quad \|h_0\|_{H^k} = \delta \|V_1\|_{H^k} \leq C\delta,$$

by Theorem 6.3 we get that

$$\|v(t, \cdot)\|_{H^k} + \|h(t, \cdot)\|_{H^k} \leq \delta N^k e^{-\sigma t} e^{C\tilde{\Gamma}/\sigma^2},$$

for $k = 0, \dots, r$. Then, by using the above formula, we estimate the H^r norms of the tensorial products in L_h and L_b as follows:

$$\begin{aligned} \|h(s) \otimes v(s)\|_{H^{r+1}} &\leq \|h(s)\|_{L^\infty} \|v(s)\|_{H^{r+1}} + \|v(s)\|_{L^\infty} \|h(s)\|_{H^{r+1}} \\ &\leq \|h(s)\|_{H^2} \|v(s)\|_{H^{r+1}} + \|v(s)\|_{H^2} \|h(s)\|_{H^{r+1}} \\ &\leq C\delta^2 N^{r+3} e^{-\sigma s}, \\ \|b(s) \otimes v(s)\|_{L^2} &\leq \|b(s)\|_{L^\infty} \|v(s)\|_{L^2} \leq C\delta e^{-\eta N^2 s}, \\ \|b(s) \otimes v(s)\|_{H^{r+1}} &\leq C \|v(s)\|_{L^\infty} \|b(s)\|_{H^{r+1}} + C \|v(s)\|_{H^{r+1}} \|b(s)\|_{L^\infty} \\ &\leq C \|v(s)\|_{H^2} \|b(s)\|_{H^{r+1}} + C \|v(s)\|_{H^{r+1}} \|b(s)\|_{L^\infty} \\ &\leq C e^{-\eta N^2 s} (\delta N^2 + \delta N^{r+1}) \leq C e^{-\eta N^2 s} \delta N^{r+1}. \end{aligned}$$

Using (4.6), we estimate

$$\begin{aligned} (7.3) \quad \|L_h(t, \cdot)\|_{H^r} &\leq C \int_0^t \|e^{\eta(t-s)\Delta} (h(s) \otimes v(s))\|_{H^{r+1}} \, ds \\ &\leq C \int_0^t e^{-\eta(t-s)} \|h(s) \otimes v(s)\|_{H^{r+1}} \, ds \\ &\leq C\delta^2 N^{r+3} e^{-\sigma s} \int_0^t e^{-\eta(t-s)} e^{-\eta\sigma s} \, ds \leq C\delta^2 N^{r+3} e^{-\sigma s}. \end{aligned}$$

Using (4.7), we estimate

$$\begin{aligned} (7.4) \quad \|L_b(t, \cdot)\|_{H^r} &\leq C \int_0^{t/2} \|e^{\eta(t-s)\Delta} (b(s) \otimes v(s))\|_{H^{r+1}} \, ds + C \int_{t/2}^t \|e^{\eta(t-s)\Delta} (b(s) \otimes v(s))\|_{H^{r+1}} \, ds \\ &\leq C \int_0^{t/2} (t-s)^{-(r+1)/2} \|b(s) \otimes v(s)\|_{L^2} \, ds + C \int_{t/2}^t e^{-\eta(t-s)} \|b(s) \otimes v(s)\|_{H^{r+1}} \, ds \\ &\leq C\delta \int_0^{t/2} (t-s)^{-(r+1)/2} e^{-\eta N^2 s} \, ds + C\delta N^{r+1} \int_{t/2}^t e^{-\eta(t-s)} e^{-\eta N^2 s} \, ds \\ &\leq C\delta N^{-2} + C\delta N^{r+1} e^{-\eta N^2 t/2}. \end{aligned}$$

Step 4. Choice of the parameters.

In this step, we fix the parameters N and δ . Recall that δ must satisfy (7.1), so that $m(0, \cdot)$ has at least $8n\ell$ regular critical points. Then we consider the behavior of the fluid at time $t = T$. We rescale the magnetic field as

$$\delta^{-1} e^{\eta T} m(T, \cdot),$$

and then recalling (7.2), we get

$$\delta^{-1} e^{\eta T} m(T, \cdot) = V_1 + \delta^{-1} \frac{M}{N} e^{-\eta(N^2-1)T} V_N + \delta^{-1} e^{\eta T} D(T, \cdot).$$

Our goal is to choose N so large that

$$(7.5) \quad \|\delta^{-1} e^{\eta T} m(T, \cdot) - V_1\|_{H^r} \ll 1,$$

for a sufficiently large¹ r , so that using the structural stability of V_1 under C^1 perturbations (see Section 5) and Sobolev embedding, one can show that the set of the integral lines of $\delta^{-1} e^{\eta T} m(T, \cdot)$, and thus of $m(T, \cdot)$, is diffeomorphic to that of V_1 . In particular, $m(T, \cdot)$ has only 4 critical points, and we must have had magnetic reconnection between $t = 0$ and $t = T$. It remains to prove (7.5). We choose

$$\delta = M e^{-\eta T} N^{-(L+1)}, \quad L \geq r + 3,$$

for some sufficiently large L . This is compatible with (7.1), and combining (7.3)–(7.4), we get

$$\begin{aligned} \|\delta^{-1} e^{\eta T} D(T, \cdot)\|_{H^r} &\leq C \delta^{-1} e^{\eta T} (\delta^2 N^{r+3} + \delta N^{-2} + C \delta N^{r+1} e^{-\eta N^2 T/2}) \\ &\leq C e^{\eta T} \delta N^{r+3} + \frac{C e^{\eta T}}{N^2} + C N^{r+1} e^{-\eta N^2 T/2} \\ &\leq \frac{C}{N} + \frac{C e^{\eta T}}{N^2} + C N^{r+1} e^{-\eta N^2 T/2} \ll 1, \end{aligned}$$

where we have taken N sufficiently large, depending on T and η (in particular, N proportional to $\eta^{-1/2}$). Under this choice of δ and N , we also have

$$\left\| \delta^{-1} \frac{M}{N} e^{-\eta(N^2-1)T} V_N \right\|_{H^r} \leq C N^{L+r+1} e^{-\eta N^2 T} \ll 1,$$

and this concludes the proof. ■

We conclude the section with some remarks.

Remark 7.1. As for the three-dimensional case, the 2D solution provided by Theorem 1.1 does not reconnect instantaneously. To show this, let (w, m) be the solution constructed in Theorem 1.1 above. Then, by using Duhamel's formula, we can write

$$m(t, \cdot) = \frac{M}{N} e^{-\eta N^2 t} V_N + \delta e^{-\eta t} V_1 + D(t, \cdot),$$

where

$$D(t, \cdot) := L_h(t, \cdot) + L_b(t, \cdot),$$

¹It suffices $r = 3$ in order to control the norm $C^1(\mathbb{T}^2)$.

and L_h and L_b are defined as in (4.3)–(4.4). For the rescaled vector field $\tilde{m} = \frac{N}{M}m$, we have that

$$\|\tilde{m}(t, \cdot) - V_N\|_{H^r} \leq C(1 - e^{-\eta N^2 t} + \delta + \|D(t, \cdot)\|_{H^r}).$$

Then, since

$$\lim_{t \rightarrow 0} \frac{1 - e^{-\eta N^2 t}}{-\eta N^2 t} = 1,$$

given the estimate on the Duhamel's term, we have that for $t \leq \gamma/(\eta N^2)$, with γ small enough, it holds

$$(7.6) \quad \|\tilde{m}(t, \cdot) - V_N\|_r \leq C(\gamma + \delta + \delta^2 N^{r+3} + \delta N^{r+1}).$$

Thus, thanks to our choice of the parameter δ , i.e.,

$$\delta = M e^{-\eta T} N^{-(L+1)}, \quad L \geq r + 3,$$

we obtain that the quantity in (7.6) can be made as small as we want for N big enough. This implies that the solution $m(t, \cdot)$ has at least $8n\ell$ hyperbolic critical points if $t \leq \gamma/(\eta N^2)$, showing that the result is stable for small times and the reconnection is not instantaneous. On the other hand, in Theorem 1.1 we prove that $m(t, \cdot)$ is topologically equivalent to V_1 provided that we can make small the quantity

$$N^r e^{-\eta N^2 t} \ll 1,$$

and this holds for $\eta N^2 t \gg 1$, which is equivalent to require that

$$t \gg \frac{1}{\eta N^2}.$$

Therefore, the reconnection must take place in a resistive time scale, that is, at a time of order $1/(\eta N^2)$.

Remark 7.2. The choice $u_0 = 0$ simplifies the proof, but we may easily generalize the argument to small velocities, namely, taking $\|u_0\|_{H^r} = \varepsilon$ (for a sufficiently large r), where the size of the small parameter ε depends on all the relevant parameters we introduced in the proof. As in the 3D case (see Remark 4.2), we can consider large data u_0 (and ε -perturbations of them) introducing some extra structure. Moreover, in the 2D case we can also consider large velocities as long as we take the viscosity ν sufficiently large. This result is proved in the next section.

Remark 7.3. The result requires $\eta > 0$, and we cannot promote it to the zero resistivity limit $\eta \rightarrow 0$; see Remark 4.3. Remark 4.4 applies in the 2D case as well.

Remark 7.4. As explained in the introduction, we can obtain a 3D reconnection result by the 2D result, extending the 2D magnetic field to a 3D object in the natural way (see (1.1)). However, the 3D reconnection result obtained in this way would not be *structurally stable* (in the sense of Remark 1.4), since we would need some structural stability of (lines of) degenerate critical points (under 3D perturbations), which is in general false. Indeed, we can immediately recognize that the argument used in Lemma 5.4 fails for degenerate critical points.

8. Magnetic reconnection in 2D (large velocity)

We are now ready to show how to prove magnetic reconnection for any initial zero-average velocity field in $H^4(\mathbb{T}^2)$, but taking the viscosity ν very large.

Proof of Theorem 1.3. We consider initial data (u_0, b_0) , with $u_0 \in H^4(\mathbb{T}^2)$ with zero average, and we consider the initial data

$$b_0 = \frac{M}{N} V_N + \delta V_1,$$

where $M > 0$ (is possibly large) and $\delta > 0$ will be small (only depending on N and M), and N large (depending on η and T) so that we have, as in the proof above, that b_0 has at least $8n\ell$ (regular) critical points. More precisely, we fix (recall (7.1))

$$(8.1) \quad \delta = c \frac{M}{N^{L+1}},$$

for some small constant $c > 0$. Let then $T > 0$. Since (see again the previous section)

$$e^{\eta T \Delta} \left(\frac{M}{N} V_N + \delta V_1 \right) = \frac{M}{N} e^{-\eta N^2 t} V_N + \delta e^{-\eta t} V_1,$$

we can represent $b(\cdot, T)$, using Duhamel's formula, as

$$b(\cdot, T) = \frac{M}{N} e^{-\eta N^2 T} V_N + \delta e^{-\eta T} V_1 + D(t),$$

where now

$$D(t) := \int_0^T e^{\eta(T-s)\Delta} \operatorname{div} (b(s, \cdot) \otimes u(s, \cdot) - u(s, \cdot) \otimes b(s, \cdot)) ds,$$

Thus we rescale $e^{\eta T} \delta^{-1} b(T, \cdot)$, and we must prove

$$\|e^{\eta T} \delta^{-1} b(T, \cdot) - V_1\|_{H^r} \ll 1,$$

for a sufficiently large r , so that by the Sobolev embedding and the structural stability of V_1 under C^1 perturbations (see Section 5), we know that $b(T, \cdot)$ (as the rescaled field) has only 4 (regular) critical points. Since

$$\|e^{\eta T} \delta^{-1} b(T, \cdot) - V_1\|_{H^r} = \left\| e^{\eta T} \delta^{-1} \frac{M}{N} e^{-\eta N^2 T} V_N + e^{\eta T} \delta^{-1} D(t) \right\|_{H^r},$$

we only need to bound the Sobolev norms of these two terms in a suitable way. If

$$R := \|u_0\|_{H^4},$$

then by (6.17) we have, after taking ν sufficiently large compared to T ,

$$\|u(\cdot, t)\|_{H^4} \leq e^{\frac{c}{\eta^2}(R^2 + N^2 + 1)} R^2 \frac{e^{-\eta t}}{\nu \eta}.$$

Thus,

$$\begin{aligned}
\|e^{\eta T} \delta^{-1} D(t)\|_{H^3} &\leq C e^{\eta T} \delta^{-1} \int_0^T e^{-\eta(T-s)} \|b(s, \cdot)\|_{H^4} \|u(s, \cdot)\|_{H^4} ds \\
&\leq C \delta^{-1} \int_0^T e^{\eta s} \|b(s, \cdot)\|_{H^4} \|u(s, \cdot)\|_{H^4} ds \\
&\leq \frac{e^{C(R^2+N^2+1)} R^2}{\delta \nu \eta} \int_0^T \|b(s, \cdot)\|_{H^4} ds \leq \frac{e^{C(R^2+N^2+1)} R^2 N^{2r}}{\delta \nu \eta \sigma} \ll 1,
\end{aligned}$$

where we used (6.1) in the penultimate inequality and we have then taken again ν sufficiently large (compared to all the fixed parameters). Also, we have

$$\left\| e^{\eta T} \delta^{-1} \frac{M}{N} e^{-\eta N^2 T} V_N \right\|_{H^3} \leq C \delta^{-1} M N^2 e^{-\eta T(N^2-1)} \ll 1,$$

where, recalling the definition (8.1) of δ , the last inequality follows choosing N sufficiently large. This concludes the proof. \blacksquare

Remark 8.1. The assumption $u_0 \in H^4(\mathbb{T}^2)$ can be improved, but our goal was simply to provide examples in which we can consider large velocities at the initial time $t = 0$.

Remarks 7.3 and 7.4 apply here too.

9. Instantaneous reconnection

The aim of this section is to exhibit an example of instantaneous magnetic reconnection, proving Theorem 1.5. The 3D argument is exactly the same used in [10] to prove instantaneous vortex reconnection for Navier–Stokes, so we will only sketch it for the sake of completeness. The 2D case requires some new ideas, but it is in some extent simpler. Indeed, while the 3D argument relies upon the subharmonic Melnikov theory developed by Guckenheimer and Holmes in [18], in 2D one can use the more classical homoclinic/heteroclinic Melnikov method, in the heteroclinic formulation of Bertozzi, see p. 1276 of [3]. Moreover, a significant simplification of the argument is possible, based on the stream function formulation (we are grateful to Daniel Peralta-Salas for this observation), and we present it below.

Proof of Theorem 1.5, $d = 3$. We consider initial data (u_0, b_0) , with b_0 given by

$$\begin{aligned}
(9.1) \quad b_0 &:= M \left(\sin(x_3 + \varepsilon h), \cos(x_3 + \varepsilon h), -\varepsilon(\partial_{x_1} h) \sin(x_3 + \varepsilon h) \right. \\
&\quad \left. - (\varepsilon \partial_{x_2} h) \cos(x_3 + \varepsilon h) \right),
\end{aligned}$$

where $h = h(x_1, x_2)$ is a 2D periodic function and ε is a small parameter, both to be fixed later. Note that b_0 is zero-average and divergence-free. Since u_0 may be large, we only have local solutions. However, this is acceptable for our purposes, since we aim to prove instantaneous reconnection of some of the magnetic lines of b_0 . Noting that b_0 is the pullback of the Beltrami field

$$M(\sin x_3, \cos x_3, 0),$$

under the volume preserving diffeomorphism

$$(9.2) \quad \Phi(x) = (x_1, x_2, x_3 + \varepsilon h(x)),$$

we see that the integral lines of b_0 are periodic or quasi-periodic and they form families of invariant tori, like that one of (9.2). We expand in Taylor series finding, for small times $t > 0$,

$$(9.3) \quad B(t) = b_0 + t(\eta \Delta b_0 - (u_0 \cdot \nabla) b_0 + (b_0 \cdot \nabla) u_0) + \mathcal{O}(t^2).$$

The goal is then to show that the Melnikov function associated to one of the periodic orbits (living on one of the resonant tori) has a non-simple zero. The computation of the Melnikov function only depends on the linear part of the PDE above, since in the non-resistive case $\eta = 0$, the magnetic lines are transported, thus the Melnikov function must be identically zero. Once we have noted this fact, we can proceed exactly as in the proof of Theorem 1.4 of [10] to show that, choosing $h(x_1, x_2) = \cos(pX_1 - qx_2)$, where p/q is rational, for all $\varepsilon > 0$ sufficiently small and for all sufficiently small times $t > 0$, the resonant torus corresponding to $\cot X_3 = p/q$ is instantaneously broken. ■

Remark 9.1. By some standard PDEs considerations, and invoking a slightly more general version of the stability Theorem 3.1 (which takes into account small initial velocities rather than zero ones), one can show that if we restrict to small (say) H^1 velocities, the (strong) solutions constructed in the previous proof are indeed *global*.

We conclude the paper with the proof of the 2D instantaneous reconnection result.

Proof of Theorem 1.5, $d = 2$. Let us consider the following vector field:

$$(9.4) \quad b_0^\varepsilon = M(-\sin(x_1) \sin(x_2 - \varepsilon x_1), -\varepsilon \sin(x_1) \sin(x_2 - \varepsilon x_1) - \cos(x_1) \cos(x_2 - \varepsilon x_1)).$$

Note that $\operatorname{div} b_0^\varepsilon = 0$. Note also that for $\varepsilon = 0$, the vector field b_0^ε corresponds to $-\mathcal{V}_{11}^4$, and we denote it by b_0 . The integral lines of b_0 are the solutions of

$$(9.5) \quad \begin{cases} \dot{x}_1 = -M \sin(x_1) \sin(x_2), \\ \dot{x}_2 = -M \cos(x_1) \cos(x_2), \end{cases}$$

where the dot denotes the derivative with respect to the parametrization. The integral lines (of the system above) present heteroclinic orbits connecting four saddle points. This is a peculiarity shared by the phase diagram of the Taylor fields. Note that, if we considered $u_0 = 0$, the solution of (MHD) would be given by $(u, b) = (0, -M e^{-2t} b_0(x))$: obviously, the magnetic field b is topologically equivalent to b_0 (that is $-\mathcal{V}_{11}^4$ from Section 5.1), and there would be no reconnection at any time. However, if we perturb b_0 as in (9.4), we are able to provide an example of instantaneous change of the topology of the integral lines.

We start noting that b_0^ε is given by the pullback of b_0 ,

$$b_0^\varepsilon = \Phi^* b_0,$$

under the change of variables

$$\Phi(x_1, x_2) = (x_1, x_2 - \varepsilon x_1).$$

It is clear that Φ is a volume preserving diffeomorphism, thus, in particular, the integral lines of b_0^ε are topologically equivalent to those of b_0 . However, b_0^ε is not a Taylor field (in particular, it is not an eigenvector of the Laplacian), and its structure can change during the evolution of the fluid.

We consider a smooth vector field u_0 with zero divergence. The system (MHD) with initial datum (u_0, b_0^ε) admits a global smooth solution that we denote by $(u^\varepsilon, b^\varepsilon)$. By using the equation, we Taylor expand b^ε with respect to time, obtaining that

$$(9.6) \quad b^\varepsilon(t, x) = b_0^\varepsilon + t(\eta \Delta b_0^\varepsilon + (b_0^\varepsilon \cdot \nabla)u_0 - (u_0 \cdot \nabla)b_0^\varepsilon) + \mathcal{O}(t^2),$$

here t has to be seen as the perturbative parameter. Of course, this representation holds for short times, that is a harmless restriction since we want to prove an instantaneous reconnection result.

Since $\operatorname{div} b^\varepsilon(t, \cdot) = 0$ (and the average on \mathbb{T}^2 is zero as well), there exists a stream function (or Hamiltonian) $\psi^\varepsilon(t, \cdot)$, namely, a scalar function such that

$$b^\varepsilon(t, x) = \nabla^\perp \psi^\varepsilon(t, x).$$

We denote with ψ_0^ε the stream function at time $t = 0$, and recalling the definition (9.4) of b_0^ε , it is easy to verify that

$$\psi_0^\varepsilon(x_1, x_2) = -\sin x_1 \cos(x_2 - \varepsilon x_1).$$

To fix the ideas, we consider the saddle connection given by

$$x_2 = \varepsilon x_1 + \frac{\pi}{2},$$

with $x_1 \in [0, \pi]$. This is an heteroclinic orbit connecting the saddle points $A = (0, \pi/2)$ and $B = (\pi, \varepsilon\pi + \pi/2)$. Thus for the stream function, we have $\psi_0^\varepsilon(A) = \psi_0^\varepsilon(B)$.

Taking this into account, it is easy to show that, since the Hamiltonian is constant along heteroclinic orbits, the connection is (instantly) broken if we can prove that, at least for small values of t ,

$$(9.7) \quad \psi^\varepsilon(t, A) \neq \psi^\varepsilon(t, B).$$

In particular, the inequality (9.7) is implied by the following:

$$(9.8) \quad (\partial_t \psi^\varepsilon)(0, A) \neq (\partial_t \psi^\varepsilon)(0, B),$$

as we now explain. Applying Taylor's formula to ψ^ε , we get

$$(9.9) \quad \psi^\varepsilon(t, x) = \psi_0^\varepsilon(x) + (\partial_t \psi^\varepsilon)(0, x)t + \mathcal{O}(t^2).$$

Moreover, from the second equation in (MHD), we have that

$$(\partial_t \psi^\varepsilon)(0, A) = \eta \Delta \psi_0^\varepsilon(A) + (u_0 \cdot \nabla) \psi_0^\varepsilon(A),$$

and similarly for B . Since both A and B are critical points of ψ_0^ε ,

$$\nabla \psi_0^\varepsilon(A) = \nabla \psi_0^\varepsilon(B) = 0,$$

and then

$$\partial_t \psi^\varepsilon(0, A) = \eta \Delta \psi_0^\varepsilon(A),$$

and the same for B . Thus, by a direct computation we obtain that

$$\Delta \psi_0^\varepsilon(A) = -2\varepsilon \quad \text{and} \quad \Delta \psi_0^\varepsilon(B) = 2\varepsilon \cos(\varepsilon\pi),$$

and in particular, $\Delta \psi_0^\varepsilon(A) \neq \Delta \psi_0^\varepsilon(B)$ for $\varepsilon \in (0, 1)$, that concludes the proof. \blacksquare

Remark 9.2. The previous arguments cannot be promoted to the zero resistivity limit $\eta \rightarrow 0$, since they work in a regime where $\eta \gg t$. Indeed, otherwise we could not consider the $\mathcal{O}(t^2)$ terms in (9.3) and (9.9) as higher order perturbations of the leading terms

$$t(\eta \Delta b_0^\varepsilon + (b_0^\varepsilon \cdot \nabla) u_0 - (u_0 \cdot \nabla) b_0^\varepsilon) \quad \text{and} \quad t(\eta \Delta \psi_0^\varepsilon + (u_0 \cdot \nabla) \psi_0^\varepsilon).$$

Thus, as $\eta \rightarrow 0$, also the range of times $t > 0$ for which we have proved a change of topology (with respect to the initial time $t = 0$) shrinks to zero.

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