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Geometrie

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Abstract. The workshop Geometrie, organized by Aaron Naber (Evanston), André Neves (Chicago), Eleonora Di Nezza (Paris), and Burkhard Wilking (Münster), was well attended with over 45 participants whose interests covered a considerable part of the current research in Differential Geometry. We had talks in such diverse subjects as geometric evolution equations, Kähler geometry, minimal surfaces, RCD spaces, hyperbolic metrics, or scalar curvature metrics.

Mathematics Subject Classification (2020): 30Lxx, 32Q20, 53A10, 53C25, 53E10, 53E30.

Introduction by the Organizers

There were 19 one hour talks equally spread across five days but with one less talk on Wednesday because of the traditional Wednesday afternoon hike. The participants reflected a broad geographic/gender representation and contributed to a very active and convivial atmosphere.

There were three talks on geometric evolution equations. Vogiatzi spoke about high codimension mean curvature flow. In particular she showed how assuming some quadratic bounds one could guarantee that the first time blow ups are codimension one even if the initial condition is of higher codimension. Lafuente studied the blow-down of immortal solutions to Ricci flow on essential manifolds of any dimension, extending a result of Lott in the three dimensional case. Finally, Hein reported on his long time project with Tosatti where they aim to understand immortal solutions to Kähler-Ricci flow. The immortality assumption can be guaranteed if the first Chern class lies in the closure of the Kähler cone. He studied the case where the initial condition admits a holomorphic fibration, in which case the flow should contract the fiber but converge to a Kähler-Einstein metric on the base of the fibration.

On the subject of Kähler geometry, besides Hein's talk, we had two more talks. Sena-Dias started with an overview of the existence problem of Einstein metrics (not necessarily Kähler) on a Kähler manifold. She then talked about her new result (with Oliveira) regarding the existence of new Einstein metrics on blow-ups of $\mathbb{CP}^2$. Liu reported on his study of complete Kähler manifolds with nonnegative Ricci curvature. The first result he mentioned (under a stronger curvature condition) shows that the average of the scalar curvature (when suitably weighted) is well defined, answering a question of Lei Ni. His second result shows that (assuming Euclidean volume growth) if the one tangent cone at infinity is Ricci flat, then the manifold is Ricci flat. There are counterexamples if the manifold is not Kähler and hence its interest.

We had four talks addressing scalar curvature rigidity and one of them intersected with Kähler geometry. Klemmensen proved the stability of the positive mass theorem for Kähler manifolds. The three dimensional case was recently proven by Dong, a participant, and Song. The Hilbert-Einstein functional is a central quantity in the study of scalar curvature. Buttsworth showed in his talk that this functional tends to $-\infty$ along Ebin geodesics. Zeidler talked about rigidity for spin fill-ins with non-negative scalar curvature. Gromov had previously showed that a spin fill-in of a manifold satisfying the assumptions of Llarull theorem satisfies a geometric lower bound on the boundary mean curvature. Zeidler and co-authors show that equality is only achieved by the Euclidean ball. Finally, Lott used the Dirac operator following Cecchini-Zeidler to deduce some restrictions on non-compact complete manifolds which admit a metric with positive scalar curvature.

The studying of positive scalar curvature metrics is related with the study of minimal surfaces and on this topic we had three lectures. Rivière talked about the existence problem for Hamiltonian stationary surfaces. After surveying the subject he proposed a direct method for constructing Hamiltonian stationary discs with prescribed singularities. He ended his talk by developing a min-max method to produce Legendrian Surfaces in a closed 5 Sasakian manifold. The minmax method was also mentioned by Liokumovich in his talk. He presented recent developments by him and some of his students regarding the Weyl Law for the area growth of minimal surfaces of any codimension. The codimension one case had been previously settled him and co-authors. A different type of minimal surfaces were addressed by Jiang in her talk. She considered the area growth of area-minimizing essential surfaces in finite volume 3-manifolds having a metric whose sectional curvature is bounded from above by -1 or whose scalar curvature is bounded from below by -6. In each of these cases she showed that the area growth is always one-side bounded by the corresponding growth for the hyperbolic metric. The quantity that measures that growth is called the minimal surface entropy.

We had two talks regarding the study of hyperbolic metrics and their properties. For the case of closed surfaces, an important tool in the study of Teichmuller is the mapping class group. Fanoni presented her work on studying the mapping class group for hyperbolic surfaces which do not have finite topology. In higher dimensions it is hard to know apriori whether a given manifold admits a hyperbolic metric. A celebrated example of Gromov-Thurston gives a metric with sectional curvature everywhere close to -1 but for which the base manifold admits no hyperbolic metric. Hamenstad showed in her talk how to improve this construction and provide one example where such manifold admits a negatively curved Einstein metric.

The workshop finished with a talk of Petrunin on a very classical and beautiful subject which is to find isometric embedding of manifolds into Euclidean spaces with some geometric bonds. Such embeddings exist from the Nash Embedding Theorem.

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Workshop: Geometrie

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Scalar curvature along Ebin geodesics

Timothy Buttsworth
(joint work with Christoph Böhm, Brian Clarke)

1. Einstein-Hilbert action

Let M be a smooth, connected, closed and oriented manifold of dimension $n \geq 3$, and let $\mathcal{M}$ denote the set of smooth Riemannian metrics on M. An important function in Riemannian geometry is the *Einstein-Hilbert action* $\mathcal{S} : \mathcal{M} \to \mathbb{R}$ given by

$$\mathcal{S}(g) = \int_M \mathrm{S}(g) dV_g,$$

where $\mathrm{S}(g)$ is the scalar curvature of g, and dV_g is the volume form. The main source of interest in the Einstein-Hilbert action is the following formula for its first variation:

$$d\mathcal{S}_g(h) = \int_M \left\langle \frac{\mathrm{S}(g)g}{2} - \mathrm{Ric}(g), h \right\rangle_g dV_g.$$

We see from this formula that critical points of $\mathcal{S}$ on the restricted set

$$\mathcal{M}_1 = \left\{ g \in \mathcal{M} \mid \int_M dV_g = 1 \right\}$$

are precisely the *Einstein metrics*, i.e., solutions of

$$\mathrm{Ric}(g) = \lambda g$$

for some $\lambda \in \mathbb{R}$. Therefore, it is expected that understanding the *asymptotics* of the restricted Einstein-Hilbert action $\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}$ may be used to produce new Einstein metrics.

2. The restricted space $\mathcal{M}_\mu$

It turns out that for the purposes of producing Einstein metrics, we can actually restrict the domain of $\mathcal{S}$ even further. For a given volume form μ that has volume 1, we let $\mathcal{M}_\mu = \{g \in \mathcal{M} | dV_g = \mu\}$, i.e., the set of Riemannian metrics g that have volume form μ. A classical result of Moser [4] shows that for any $\tilde{g} \in \mathcal{M}_1$, there is a $g \in \mathcal{M}_\mu$ so that $\phi^* g = \tilde{g}$ for some diffeomorphism $\phi : M \to M$. Consequently, we do not lose any Riemannian structures by restricting $\mathcal{M}_1$ to $\mathcal{M}_\mu$. Furthermore, the Second Contracted Bianchi identity implies that for $g \in \mathcal{M}_\mu$, g is critical point of $\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}$ if and only if g is a critical point of $\mathcal{S} : \mathcal{M}_\mu \to \mathbb{R}$.

3. Geometry and topology of $\mathcal{M}_\mu$

It is clear that $\mathcal{M}_\mu$ is an infinite-dimensional Fréchet manifold. The tangent space of $\mathcal{M}_\mu$ at any given g is given by

$$T_g\mathcal{M}_\mu = \{h \in S^2(T^*M)|\ \mathrm{tr}_g(h) = 0\}.$$

We can then equip $\mathcal{M}_\mu$ with a geometric structure with the following L^2 or *Ebin* metric:

$$\mathcal{E}_g(h,k) = \int_M \langle h,k\rangle_g dV_g.$$

The critical points of the corresponding length functional are the so-called *Ebin geodesics*. For a given $g \in \mathcal{M}_\mu$ and $h \in T_g\mathcal{M}_\mu$, the Ebin geodesic $\gamma(t)$ with $\gamma(0) = g$ and $\gamma'(0) = h$ is given by

$$\gamma(t) = g(\exp(tH)\cdot,\cdot),$$

where $H = h^{\sharp_g}$. The study of the Einstein-Hilbert functional $\mathcal{S} : \mathcal{M}_\mu \to \mathbb{R}$ (especially its asymptotics), can be faciliated through understanding the asymptotic behaviour of $\mathcal{S}$ along the Ebin geodesics.

4. Compact homogeneous spaces

Understanding the asymptotics of the Einstein-Hilbert functional by examining its behaviour along Ebin geodesics has been enormously fruitful in the homogenoeus case. If we suppose that $M = G/H$ is a homogeneous space and we let $\mathcal{M}_1^G$ be the set of G-invariant Riemannian metrics on M that have volume 1, then critical points of the restricted functional $\mathcal{S} : \mathcal{M}_1^G \to \mathbb{R}$ are homogeneous Einstein metrics.

Let us construct a formula for the Einstein-Hilbert action. Since the scalar curvature is constant for homogeneous metrics, it suffices to find a formula for the scalar curvature. To this end, let Q be a unit-volume bi-invariant metric on G. Choose $\mathfrak{m}$ to be the Q-orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Then homogeneous Ebin geodesics starting at Q on M are given by

$$\gamma(t) = \sum_{i=1}^{l} e^{tv_i} Q|_{\mathfrak{m}_i},$$

where $\{v_i\}_{i=1}^l \subset \mathbb{R}$, and $\mathfrak{m} = \bigoplus_{i=1}^l \mathfrak{m}_i$ is a choice of $Ad(H)$-irreducible and Q-orthogonal decomposition. The volume 1 constraint is $\sum_{i=1}^l d_i v_i = 0$, where $d_i = \dim(\mathfrak{m}_i)$. By ignoring the stationary geodesics, we can also assume that $\sum_{i=1}^l d_i v_i^2 = 1$. The scalar curvature of the homogeneous metric $\gamma(t)$ is given by (see [1])

$$\mathrm{S}(\gamma(t)) = \frac{1}{2}\sum_{i=1}^{l} d_i b_i e^{-v_i t} - \frac{1}{4}\sum_{i,j,k=1}^{l} [ijk] e^{t(v_i - v_j - v_k)},$$

where b_i and $[ijk]$ are non-negative structure constants for the homogeneous space.

Having now competed the computation of the Einstein-Hilbert action, we turn to the question of existence of homogeneous Einstein metrics. Observe that the

$d_i, b_i, [ijk]$ terms are all non-negative, and the exponential terms with the largest growth are $e^{t(v_i - v_j - v_k)}$, where $v_j = \min\{v_1, \cdots, v_n\} < 0$, and $v_i > v_k$. Therefore, the scalar curvature will converge to $-\infty$, as long as at least one of the corresponding $[ijk]$ terms do not vanish. On the other hand, if these terms do vanish, and some of the $d_i b_i$ terms do not vanish, there are directions where scalar curvature converges to $+\infty$. This pursuit of these observations lead to the construction of new Einstein metrics in [2] as critical points of the Einstein-Hilbert action using the Mountain Pass Theorem.

5. Formula without symmetry

In this project, we demonstrate that the tendency of scalar curvature to converge to $-\infty$ along Ebin geodesics is not unique to the homogeneous case.

Theorem 1. *Let M be a compact, oriented smooth manifold of dimension at least 5, equipped with a Riemannian metric g_0 with volume form μ, and volume 1. There exists an open and dense set $\mathcal{N} \subset T_{g_0}\mathcal{M}_\mu$ (in the Whitney C^∞ topology) so that for each $h \in \mathcal{N}$, $\lim_{t\to\infty} S(\gamma(t)) = -\infty$ uniformly on M.*

Following the homogeneous case as closely as possible, we prove Theorem 1 in [3] by first finding a local g_0-orthonormal frame in which the presentation of H is as simple as possible. It is generally impossible to ask that the frame diagonalises H, but it is always possible to find a frame $e = \bigcup_{i=1}^{L} \{e_{i_a}\}_{a=1}^{m_i}$ in which H takes the block diagonal form

$$H = \begin{pmatrix} \lambda_1 I_{m_1} + S_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{m_2} + S_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_L I_{m_L} + S_L \end{pmatrix},$$

according to a clustering of eigenvalues of H. When we compute scalar curvature using this frame, we find, like in the homogeneous case, terms of the form $[ijk]e^{t(-\lambda_i - \lambda_j + \lambda_k)}$, where $[ijk]$ is a non-negative term, depending on the frame. We use tools from differential topology to show that at least one of these terms is generically non-zero, starting in dimension five, showing that scalar curvature tends to converge to $-\infty$ on the entire manifold as t tends to $+\infty$.

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Classification of mapping classes for surfaces of infinite type

Federica Fanoni
(joint work with Mladen Bestvina and Jing Tao)

To an (orientable, connected) surface S (without boundary), we can associate its *mapping class group* $\mathrm{MCG}(S)$, defined as the group of homotopy classes of orientation preserving homeomorphisms. Mapping class groups are fundamental objects in low-dimensional geometry and topology, as they for instance appear as orbifold fundamental groups of the moduli space of Riemann surface structures on the associated surface, and they also play an important role in the study of three-manifolds (via constructions such as Heegaard splittings and mapping tori).

Except for a few cases (e.g. the sphere), mapping class groups are interesting infinite groups which have been largely explored in the last decades. One fundamental result concerns the classification of mapping classes — elements of mapping class groups — of surfaces of finite type (i.e. whose fundamental group is finitely generated), proven by Nielsen [4–6] and Thurston [8]:

Theorem 1 (Nielsen, Thurston)**.** *Let S be a finite-type surface of negative Euler characteristic. Given $\varphi \in \mathrm{MCG}(S)$, there is a representative homeomorphism $f \in \varphi$ which is either periodic, reducible or pseudo-Anosov.*

Recall that a homeomorphism f is:

- *periodic* if there is some $n > 0$ so that f^n is the identity;
- *reducible* if there is a finite collection Γ of pairwise disjoint, pairwise non-homotopic curves, such that $f(\Gamma) = \Gamma$;
- *pseudo-Anosov* if there are two f-invariant transverse singular measured foliations $\mathcal{F}^{\pm}$ and $\lambda > 1$ so that f expands $\mathcal{F}^{+}$ by a factor λ and contracts $\mathcal{F}^{-}$ by a factor $\frac{1}{\lambda}$.

Here *curves* are assumed to be closed, simple and *essential* — that is, neither contractible, nor bounding a disk with a single puncture. In what follows, we will say that a mapping class is periodic, reducible or pseudo-Anosov if it has a representative which is periodic, reducible or pseudo-Anosov, respectively.

A consequence of this result is that there is a canonical decomposition of S into subsurfaces via a collection of pairwise disjoint and pairwise non-homotopic curves so that, up to passing to a power, each subsurface is preserved by f and the restriction of f to each subsurface is either periodic or pseudo-Anosov. Furthermore, we can essentially reconstruct the map on the whole surface by the maps on the subsurfaces. Said otherwise, we can think of periodic and pseudo-Anosov elements as Lego blocks which allow us to construct every mapping class.

Using Nielsen and Thurston's result, together with work of Hurwitz [3] and again Nielsen [7], one can show that:

- a mapping class φ is periodic if and only if there is a hyperbolic metric on S so that a representative of φ is an isometry for this metric; moreover, both conditions are equivalent to saying that for every pair of homotopy classes of curves α and β,

$$\{i(\varphi^n(\alpha), \beta) \mid n \in \mathbb{Z}\}$$

is bounded;

- a mapping class φ is pseudo-Anosov if and only if for every pair of homotopy classes of curves α and β,

$$i(\varphi^n(\alpha), \beta) \to \infty$$

as $n \to \infty$.

Recall that $i(\alpha, \beta)$ is the *geometric intersection number* of α and β, that is, the minimum number of intersections of representatives of the two classes.

If we drop the assumption of the surface being of finite type, mapping class groups are much less understood and there is no general classification of their elements. Together with Mladen Bestvina and Jing Tao, our goal is to generalize Nielsen and Thurston's result to this setup. Note that in the case of *endperiodic* maps (see [2] for definitions), a Nielsen–Thurston type classification has been given by Handel and Miller and extended by Cantwell, Conlon and Fenley [2].

In our work, the first step is to study maps which do not have any pseudo-Anosov behavior, i.e. mapping classes φ so that for every pair of homotopy classes of curves α and β,

$$\{i(\varphi^n(\alpha), \beta) \mid n \in \mathbb{Z}\}$$

is bounded. We call these maps *tame*. While for finite-type surfaces tameness is equivalent to periodicity and to having a representative which is an isometry, this is not true in the general setting. As a simple example, the map $(x, y) \to (x+1, y)$ of the surface $\mathbb{R}^2 \setminus \mathbb{Z}^2$ is tame and can be realized as an isometry, but it is not periodic.

The main result of our work [1] is the following classification theorem:

Theorem 2. *Let S be a surface and φ an extra tame mapping class. Then there is a canonical decomposition of S into three (possibly disconnected) subsurfaces S_0, S_{per} and S_∞, a representative f of φ and a hyperbolic metric on S so that for every connected component X of the decomposition, there is $n = n(X)$ such that $f^n(X) = X$ and $f^n|_X$ is homotopic to:*

- *an isometric translation, if $X \subset S_\infty$,*
- *a periodic isometry, otherwise.*

Moreover, each component of S_0 contains at most one curve, which is peripheral.

We can think of this theorem as saying that the building blocks of extra tame maps are translations and periodic elements. *Extra* tame maps are tame maps with an additional finiteness condition on the set of limits of iterates of curves — see [1] for the precise definitions. This assumption is needed: for tame maps, the theorem doesn't hold. The crucial issue is that the canonical decomposition we would like to construct is not a decomposition into subsurfaces, but into subsets with complicated accumulating behavior.

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Strong G_2-structures with torsion

Anna Fino

(joint work with Lucia Martín-Merchán and Alberto Raffero)

On a Riemannian n-dimensional manifold (M, g), a metric connection ∇ is said to have totally skew-symmetric torsion if its 3-covariant torsion tensor $g(T(\cdot,\cdot),\cdot)$ is skew-symmetric. In such a case, ∇ is related to the Levi-Civita connection ∇^{LC} of g as follows

$$g(\nabla_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} g(T(X,Y), Z), \quad X, Y, Z \in \Gamma(TM).$$

When M is oriented and the structure group of its frame bundle admits a reduction to a closed subgroup $H \subseteq \mathrm{SO}(n)$, the existence of an H-connection with totally skew-symmetric torsion can be characterized in terms of the intrinsic torsion of the H-structure ([3, Prop. 4.1]). In particular, by [3, Thm. 4.7], a 7-manifold with a G_2-structure φ admits a G_2-connection ∇ with totally skew-symmetric torsion if and only if $d * \varphi = \theta \wedge *\varphi$, where θ is the Lee form of the G_2-structure. The torsion 3-form T of ∇ can be written in terms of φ as follows

$$T = \frac{1}{6} \star(d\varphi \wedge \varphi)\varphi - \star d\varphi + \star(\theta \wedge \varphi).$$

In particular, T vanishes identically if and only if the G_2-structure is *parallel*, i.e. $\nabla\varphi = 0$ or, equivalently, $d\varphi = 0$ and $d \star \varphi = 0$ ([1]).

In this talk we will present recent results, obtained in the paper [2], on 7-diimensional manifolds admitting strong G_2-structures with torsion (shortly strong G_2T-structures), i.e. a G_2-connection with closed totally skew-symmetric torsion. In particular, we will discuss the twisted G_2 equation, which is given by the following system of equations

$$d\varphi \wedge \varphi = 0, \quad d \star \varphi = \theta \wedge \star\varphi, \quad dT = 0, \quad d\theta = 0$$

and represents the G_2-analogue of the twisted Calabi-Yau equation for $SU(n)$-structures introduced by Garcia-Fernández, Rubio, Shahbazi and Tipler in [4].

Moreover, we will show that if a 7-dimensional compact, connected homogeneous space M for the almost effective action of a connected Lie group G admits an invariant strong G_2T-structure, then M is diffeomorphic either to $S^3 \times T^4$ or to $S^3 \times S^3 \times S^1$, up to a covering. On the spaces $S^3 \times T^4 \cong SU(2) \times U(1)^4$ and $S^3 \times S^3 \times S^1 \cong SU(2) \times SU(2) \times U(1)$, we will also present homogeneous examples of strong G_2T-structures solving the twisted G_2 equation and whose associated G_2-connection with totally skew symmetric torsion is flat. It remains an open problem to see whether there exist compact 7-manifolds admitting a strong G_2T-structure inducing a non-flat G_2-connection with totally skew-symmetric torsion.

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Einstein metrics on Gromov Thurston manifolds

Ursula Hamenstädt

(joint work with Frieder Jäckel)

Gromov and Thurston constructed in 1987 for every dimension $n \geq 4$ and any $\epsilon > 0$ a closed manifold of dimension n which admits a Riemannian metric of curvature contained in the interval $[-1-\epsilon, -1+\epsilon]$ but which does not admit a hyperbolic metric. Such manifolds are called Gromov-Thurston manifolds.

Fine and Premoselli constructed for a family of Gromov Thurston manifolds in dimension 4 negatively curved Einstein metrics. In the talk, we explain the following extension of this result.

Theorem 1. *For every $n \geq 4$ and every $\epsilon > 0$ there exists a closed manifold M with the following properties.*

(1) M admits a metric of curvature in $[-1-\epsilon, -1+\epsilon]$ but no hyperbolic metric.
(2) M admits a negatively curved Einstein metric.

The basic strategy for the construction of the metric is taken from the work of Fine and Premoselli. It consists in starting with a Gromov Thurston manifold which is obtained from an arithmetic hyperbolic manifold by a covering branched along a codimension two submanifold which is homologous to zero. The idea is to glue a specific Einstein metric near the branch locus to the hyperbolic metric on the complement and deform the resulting metric to an Einstein metric using an implicit function theorem.

We embark from an arithmetic hyperbolic manifold and use subgroup separation for geometrically finite subgroups of the fundamental group as well as virtual retraction of the group onto its stabilizer subgroup of a hyperplane, as established by Bergeron and Bergeron, Haglund and Wise to construct in any dimension specific Thurston manifolds which are geometrically well controlled. We then show how to use this control to apply the implicit function theorem to the linearization of the Einstein operator to deform the glued metric to an Einstein metric. Suitably chosen examples do not admit hyperbolic metric.

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Immortal solutions of the Kähler-Ricci flow

Hans-Joachim Hein

(joint work with Man-Chun Lee, Valentino Tosatti)

Setup. On a given compact Kähler manifold X, consider the Kähler-Ricci flow

$$\frac{\partial\omega}{\partial t} = -\mathrm{Ric}(\omega) - \omega, \ \ \omega(t=0) = \omega_0. \tag{1}$$

Note that this is normalized so that Einstein metrics with Ricci curvature -1 are exactly the fixed points of the flow. Also note that $\hat{\omega}(s) := e^t\omega(t)$ with $s := e^t - 1$ then solves the ordinary unnormalized Ricci flow equation $\partial_s\hat{\omega} = -\mathrm{Ric}(\hat{\omega})$.

By Tian–Zhang [12], the equation (1) is solvable if and only if it is solvable at the level of cohomology classes. One easily checks that

$$[\omega(t)] = e^{-t}[\omega_0] + (1-e^{-t})c_1(K_X). \tag{2}$$

Since the Kähler cone of X is an open convex cone in $H^{1,1}(X,\mathbb{R})$, it follows that the maximal existence time of the Kähler-Ricci flow (1) is exactly the time when the parametrized straight line (2) from $[\omega_0]$ to $c_1(K_X)$ exits the Kähler cone. In particular, the flow (1) is immortal if and only if (the unnormalized flow $\hat{\omega}(s)$ is immortal, if and only if) $c_1(K_X)$ lies in the closure of the Kähler cone of X.

Two simple examples of this situation are (a) when $c_1(K_X)$ is a Kähler class and (b) when $c_1(K_X) = 0$. In fact, this is the setting of Cao's foundational paper [1] on the Kähler-Ricci flow, where he showed that as $t \to \infty$, $\omega(t)$ converges smoothly to the unique Kähler-Einstein metric on X in case (a) and $e^t\omega(t)$ converges smoothly to the unique Calabi-Yau metric in the class $[\omega_0]$ in case (b).

In general, a major difficulty in the subject is that the structure of classes on the boundary of the Kähler cone is poorly understood. However, for the particular class $c_1(K_X)$ the situation is expected to be quite transparent: the Abundance Conjecture predicts that if $c_1(K_X)$ lies in the closure of the Kähler cone, then

$c_1(K_X)$ admits a smooth nonnegative representative of the form $f^*(\omega_{\mathrm{FS}}|_B)$, where $f : X \to B \subset \mathbb{C}P^N$ is a holomorphic fibration of X with connected Calabi-Yau fibers. More precisely, the Abundance Conjecture predicts that $K_X^{\otimes p}$ is globally generated for p sufficiently large and divisible, and then for any such p the sections of $K_X^{\otimes p}$ define such a map f thanks to classical work of Iitaka.

Let $S \subset B$ denote the discriminant locus of f, a proper subvariety of B, so that $f|_{f^{-1}(B\setminus S)}$ is a C^∞ submersion onto $B \setminus S$. Let Y denote a general fiber of f, so that Y is a connected Calabi-Yau manifold, and write $m := \dim B$, $n := \dim Y$. Examples (a) and (b) are characterized by the properties that f is an isomorphism (which implies $n = 0$) and that $m = 0$, respectively. We will assume from now on that $n > 0$ and $m > 0$. Then based on (a) and (b) one would expect that for $t \to \infty$ the Kähler-Ricci flow $(X, \omega(t))$ collapses along the fibers of f to an Einstein metric on B. Strictly speaking the correct limit is $f^*\omega_B$ with ω_B a *twisted* Einstein metric on B. This satisfies $\mathrm{Ric}(\omega_B) + \omega_B = \omega_{\mathrm{WP}}$, the Weil-Petersson form describing the variation of complex structure of the fibers of f. Song–Tian [10] proved that $\omega(t)$ converges to $f^*\omega_B$ weakly as currents on X.

Main Theorem. *Given any compact set $K \subset B \setminus S$, we have that $|\mathrm{Ric}(\omega(t))|_{\omega(t)}$ remains uniformly bounded on $f^{-1}(K)$ as $t \to \infty$.*

Remarks. (1) Our bound is an a priori estimate: it only depends on lower-order quantities that were known to be bounded from previous work.

(2) This is a development of a series of papers [6–8] on an analogous problem for a 1-parameter family of elliptic PDEs rather than a parabolic PDE.

(3) The only two previously known cases are the following:

- The regular fibers of f are flat rather than Ricci-flat. By Fong–Zhang [4], this is equivalent to $|\mathrm{Sect}(\omega(t))|$ remaining uniformly bounded on $f^{-1}(K)$. Fong–Zhang use a local covering trick and Yau's estimates, developing an analogous elliptic PDE result due to Gross–Tosatti–Zhang [5].
- The regular fibers of f are pairwise isomorphic ($\Longleftrightarrow \omega_{\mathrm{WP}} = 0$). In this case the result was proved by Fong–Lee [3] by developing the elliptic PDE result in [7], which uses techniques that go beyond Yau's estimates.

(4) We conjecture that $|\mathrm{Ric}(\omega(t))|_{\omega(t)}$ stays uniformly bounded globally on X. The corresponding statement for scalar curvature rather than Ricci curvature was proved by Song–Tian [11] using Perelman's estimates for the Kähler-Ricci flow.

Heuristic idea of proof. If the Ricci curvature bound is false, then there exist $t_i \to \infty$ and $x_i \in f^{-1}(K)$ such that $\lambda_i := |\mathrm{Ric}(\omega(t_i))(x_i)|_{\omega(t_i)}$ maximizes the Ricci curvature on $[0, t_i] \times f^{-1}(K)$ and such that $\lambda_i \to \infty$ as $i \to \infty$. Then we rescale

$$\tilde{\omega}_i(\tilde{t}) := \lambda_i \omega(t_i + \lambda_i^{-1}\tilde{t}).$$

If it was possible to pass to a limit $\tilde{\omega}_\infty(\tilde{t})$, this would be an eternal unnormalized Kähler-Ricci flow. By the result of Song–Tian [11] mentioned in (4), it would also be scalar-flat, hence (by the evolution equation of the scalar curvature) Ricci-flat. But hopefully the property $|\mathrm{Ric}(\tilde{\omega}_i(0))(x_i)|_{\tilde{\omega}_i(0)} = 1$ can be passed to the limit, by using the regularity of the flows $\tilde{\omega}_i$ implied by a uniform Ricci curvature bound on $[-\lambda_i t_i, 0] \times f^{-1}(K)$, and this would be an obvious contradiction.

Input to make the heuristics more rigorous. By Fong–Zhang [4] there exists a constant $C = C(K) \geq 1$ such that for all $t \geq 0$ it holds on $f^{-1}(K)$ that

$$\frac{1}{C}\omega_{\mathrm{cyl}}(t) \leq \omega(t) \leq C\omega_{\mathrm{cyl}}(t), \ \ \omega_{\mathrm{cyl}}(t) := f^*\omega_B + e^{-t}\omega_0.$$

This provides a uniform parabolicity of the Kähler-Ricci flow equation relative to a cylindrical background geometry with fiber diameter $\sim e^{-t/2}$. There are now two easy cases: $\lambda_i/e^{t_i} \to \infty$, and $\lambda_i/e^{t_i} \to c \in (0,\infty)$. In these cases, the blown-up reference geometries $(f^{-1}(K), \lambda_i\omega_{\mathrm{cyl}}(t_i + \lambda_i^{-1}\tilde{t}), x_i)$ converge to flat Euclidean $\mathbb{C}^{m+n}$ and to a cylinder $\mathbb{C}^m \times (Y_\infty, c\omega_0|_{Y_\infty})$, $Y_\infty := f^{-1}(f(x_\infty))$, respectively. By the uniform parabolicity of the Kähler-Ricci flow and by parabolic regularity, our heuristic argument is then justified in these two cases.

The difficulty in the remaining case. In the collapsing case, $\delta_i := \lambda_i/e^{t_i} \to 0$, it is well-understood after Song–Tian [10, 11] how to pass to a weak limit of $\tilde{\omega}_i(\tilde{t})$ and that this weak limit is actually the pullback of a static Euclidean flow on $\mathbb{C}^m$. The issue is the absence of a suitable parabolic regularity theory in this case. To see more concretely what this means, observe that in a dream scenario,

$$\tilde{g}_i(\tilde{t}) \sim g_{\mathbb{C}^m} + \delta_i g_{Y_\infty} + \mathbf{err} \ \text{ with } g_{\mathbb{C}^m} \text{ flat and } g_{Y_\infty} \text{ Ricci-flat.}$$

But even then, $|\mathrm{Ric}(\tilde{g}_i)|_{\tilde{g}_i} \sim \delta_i^{-2} \cdot$(2nd derivatives of $\mathbf{err}$ in fixed local coordinates), so discussing the boundedness of Ricci amounts to understanding the asymptotics of $\tilde{g}_i(\tilde{t})$ relative to a Ricci-flat comparison cylinder to order δ_i^2.

At this point it is clear that one should probably abandon the ansatz of proving boundedness of Ricci by contradiction and instead aim for precise asymptotics of $\omega(t)$ relative to the obvious Ricci-flat comparison cylinders. This is what was done in [7, 8] in the elliptic case. The proof is again by contradiction, but by aiming for a stronger statement we gain more control in the contradiction argument.

Carrying this out still requires on the order of 100 pages. Here we only mention that the source of the contradiction changes slightly compared to the above. For example, already in the two easy cases we could have observed that a Calabi-Yau metric on $\mathbb{C}^{m+n}$ or $\mathbb{C}^m \times Y_\infty$ uniformly equivalent to the natural reference metric is actually isometric to it, which would contradict the failure of good asymptotics with respect to these reference metrics in each case. This is a Liouville theorem for the complex Monge-Ampère equation, which was proved in [6] in the cylinder case. Following Li–Li–Zhang [9] and Chen–Wang [2], this line of thinking already leads to a new proof of the usual Evans-Krylov estimate on a ball.

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Inverse Mean Curvature Flow and Geometric Inequalitiess on 3-Manifolds

Gerhard Huisken

The lecture uses inverse solutions ${\Sigma^2_t}_{t\geq 0}$ of inverse mean curvature flow $\frac{d}{dt}X = \frac{1}{H}\nu$ of 2-dimensional hypersurfaces in Riemannian 3-manifolds to sweep out exterior regions of some initial boundary $\Sigma^2_0 = \partial\Omega_0, \Omega_0 \in (M^3, g)$. One application (joint with Thomas Körber, Vienna) is an alternative proof of Hamilton's conjecture, stating that 3-manifolds with pinched Ricci-curvature $\mathrm{Ric}_g \geq \epsilon R g, \epsilon > 0$ are compact (first proven by Deruelle-Schulze-Simon using earlier work of Lott in the bounded curvature case; general case completed my Lee-Topping). A second application uses solutions from a point in (M^3, g) to define a quasi-local notion of radius for $\Omega \in (M^3, g)$, leading to a result that a lower bound on scalar curvature can hold only in regions with a sharp upper bound on their radius - reminiscent of work by Schoen-Yau concerning the formation of black holes due to concentration of matter.

Ricci lower bounds and nonmanifold structure

Erik Hupp

(joint work with Aaron Naber, Kai-Hsiang Wang)

This talk presented a negative result about the manifold structure of collapsed Ricci limits $(X, d, \mu, p) \overset{\text{mpGH}}{\longleftarrow} (M^n_i, g_i, d\mathrm{Vol}_{g_i}/\mathrm{Vol}_{g_i}(B_1(p_i)), p_i)$ (where (M^n_i, g_i) are smooth, complete Riemannian manifolds with $\mathrm{Ric}_{g_i} \geq \Lambda > -\infty$). In the noncollapsed setting, i.e. if $\mathrm{Vol}_{g_i}(B_1(p_i)) \geq \nu > 0$ for ν independent of i, the work of

Cheeger-Colding guarantees some topological regularity for limit spaces (with an anologue in the non-collapsed RCD setting–see [3, Thm. 4.11 & 5.1]):

Theorem 1 (Cheeger-Colding [1])**.** *If (X, d, μ, p) is a non-collapsed Ricci limit as above, then $X = \hat{\mathcal{R}} \cup \mathcal{S}$ where:*

- *$\hat{\mathcal{R}}$ is open, dense, and bi-Hölder to a smooth Riemannian manifold.*
- *$\mathcal{S}$ has Hausdorff dimension $\dim_{\mathcal{H}}(\mathcal{S}) \leq n-2$.*
- *$\mu = c \cdot \mathcal{H}^n$ for some constant $c > 0$.*

However, the first bullet point of Theorem 1 may not hold without the non-collapsing assumption:

Theorem 2 (Hupp-Naber-Wang [2])**.** *Consider an arbitrary smooth, closed Riemannian 4-manifold (X^4, h) with a lower bound on the Ricci curvature $\mathrm{Ric}_h > \Lambda > -\infty$. Then for any $\varepsilon > 0$, there exists a sequence of smooth Riemannian 6-manifolds (M_i^6, g_i) that GH-converge to a metric space (M_∞, d_∞) with the following properties:*

- *The (M_i^6, g_i) have the same lower Ricci bound as (X, h), i.e. $\mathrm{Ric}_{g_i} > \Lambda$.*
- *(M_∞, d_∞) is GH ε-close to the original Riemannian manifold (X^4, h).*
- *(M_∞, d_∞) is not a manifold near any point. In fact, every open set $U \subset M_\infty$ has nontrivial homology: $H_2(U)$ is infinitely generated.*
- *(M_∞, d_∞) is 4-rectifiable.*

The proof of Theorem 2 proceeds via a warped product construction; one produces smooth base spaces (X_i^4, h_i) and positive functions $f_i : X_i \to (0, \infty)$ to form warped product spaces $(M_i^6, g_i) := (X_i^4, h_i) \times_{f_i} S^2 = (X_i \times S^2, h_i + f_i^2 g_{S^2})$. Roughly speaking, one iteratively glues copies of $\mathbb{C}P^2 \setminus B^4$ into the reference manifold X so that the added 2-cycles are independent, decreasing in diameter, and increasing in density (compare with [5]). The concavity properties of the warping factors $f_i \searrow 0$ are used to reconcile the geometry of the original manifold with that of $\mathbb{C}P^2 \setminus B^4$ near its boundary without violating the Ricci lower bound (compare with [4]).

Remark. Without too much additional work, by gluing $\mathbb{C}P^n \setminus B^{2n}$'s Theorem 2 holds for $(2n+2)$-dimensional manifolds collapsing to dimension $2n$, with H_2 replaced by H_{2k}, $0 < 2k < 2n$ and $n \geq 2$. Moreover, recent work in [6] has obtained the same statement but for $(n+2)$-dimensional manifolds collapsing to dimension n, with H_2 replaced by H_{n-1}, $n \geq 3$ of any parity. Essentially, if one can produce a Ricci-positive $5 \leq (n+2)$-dimensional manifold-with-boundary $(\mathcal{B}^{n+2}, g)$ that has the warped-product structure $(Y^n, h_Y) \times_f S^2$ on an open dense set, with a certain standardized conical structure near the boundary, then it can be glued in densely and at arbitrarily small scales using the methods of Theorem 2.

This raises the question of whether the first bullet point of Theorem 1 can be recovered in the collapsed case if one adds additional assumptions:

Question. For Ricci limits $(X, d, \mu, p) \overset{\mathrm{mpGH}}{\leftarrow} (M_i^n, g_i, d\mathrm{Vol}_{g_i}/\mathrm{Vol}_{g_i}(B_1(p_i)), p_i)$ with $\mathrm{Ric}_{g_i} \geq \Lambda > -\infty$, is there a *soft* (topological) assumption on the smooth manifolds M_i that ensures that X has a manifold structure on an open dense set?

Soft here can be interpreted broadly. This question was originally posed in [2, Qu. 1.1] with "soft" specified to mean uniformly finitely generated homology (i.e. there exist surjective homomorphisms $H_*(M_i) \twoheadleftarrow \mathbb{Z}^N$ for some $N < \infty$ independent of i). The point of being more vague here is to allow for possibly stronger assumptions if they are sufficient to recover a dense manifold structure for collapsed limits–the goal being to see what is needed to obtain a *positive* result.

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Minimal Surface Entropy of Hyperbolic Manifolds of Finite Volume

Ruojing Jiang

(joint work with Franco Vargas Pallete)

On a closed hyperbolic manifold M, Besson-Courtois-Gallot [1] studied the topological entropy of the geodesic flow and proved that the hyperbolic metric attains its minimum among all negatively curved metrics on M with the same volume. Recently, Calegari-Marques-Neves [2] introduced the concept of the minimal surface entropy of closed hyperbolic 3-manifolds, building on the construction and calculation of surface subgroups by Kahn-Markovic [5] [6]. The minimal surface entropy measures the number of essential minimal surfaces in M with respect to different metrics, shifting the focus from one-dimensional entities (geodesics) to two-dimensional objects.

Let $M = \mathbb{H}^n/\pi_1(M)$ be an n-manifold ($n \geq 3$) that admits a hyperbolic metric h_0, a closed surface immersed in M with genus at least two is said to be *essential* if the immersion is π_1-injective, and the image of its fundamental group in $\pi_1(M)$ is called a *surface subgroup*. Let $S(M, g)$ denote the set of surface subgroups of genus at most g up to conjugacy, and let the subset $S(M, g, \epsilon) \subset S(M, g)$ consist of the conjugacy classes whose limit sets are $(1+\epsilon)$-quasicircles. Moreover, $S_\epsilon(M) = \bigcup_{g\geq 2} S(M, g, \epsilon)$. Suppose h is an arbitrary Riemannian metric on M. For any $\Pi \in S(M, g)$, we set $\text{area}_h(\Pi) = \inf\{\text{area}_h(\Sigma) : \Sigma \in \Pi\}$.

Definition 1. The *minimal surface entropy of M with respect to h* is defined as follows.

$$\underline{E}(h) = \lim_{\epsilon \to 0} \liminf_{L \to \infty} \frac{\ln \#\{area_h(\Pi) \le 4\pi(L-1) : \Pi \in S_\epsilon(M)\}}{L \ln L},$$
$$\overline{E}(h) = \lim_{\epsilon \to 0} \limsup_{L \to \infty} \frac{\ln \#\{area_h(\Pi) \le 4\pi(L-1) : \Pi \in S_\epsilon(M)\}}{L \ln L}.$$

Additionally, we write $E(h)$ if $\underline{E}(h) = \overline{E}(h)$.

According to [2], when M is a closed hyperbolic 3-manifold, among metrics with sectional curvature less than or equal to -1, $\underline{E}(h)$ attains its minimum value at h_0, and $E(h_0) = 2$.

For higher dimensional closed hyperbolic manifolds, if the dimension of M is odd, Hamenstädt [3] verified the existence of surface subgroups and constructed an essential surface Σ_ϵ which is sufficiently well-distributed. Based on this result, we broadened the definition of minimal surface entropy to include a wider range of scenarios.

Theorem 2 (Jiang, [4])**.** *Let (M, h_0) be a closed hyperbolic manifold whose dimension $n \ge 3$ is odd, and let h be another metric on M with sectional curvature less than or equal to -1, then*

$$\underline{E}(h) \ge E(h_0) = 2.$$

The equality holds if and only if h is isometric to h_0.

In a recent work joint with Franco Vargas Pallete, we focused on hyperbolic 3-manifold (M, h_0) of finite volume. By utilizing the construction of surface subgroups by Kahn-Wright [7], as well as the existence of *closed* essential minimal surfaces corresponding to each subgroup, we calculated the minimal surface entropy of the hyperbolic metric.

Theorem 3 (Jiang-Vargas Pallete, in preparation)**.** *Let M be a hyperbolic 3-manifolds of finite volume, we have*

$$E(h_0) = 2.$$

However, for an arbitrary metric h, (M, h) may not contain an area-minimizing surface. Therefore, we need the following conditions for h to ensure the existence of such a surface.

Definition 4. A Riemannian metric h on M is *weakly cusped* if there exists a mean-convex foliation on each cusp $T \times [0, \infty)$ satisfying:

(1) The coordinate vector field ∂_s induced by the $[0, \infty)$ factor satisfies $\frac{1}{2s} \le \|\partial_s\|_h \le \frac{2}{s}$.
(2) The inverse of the systole of $T \times \{s\}$ is $O(s)$ as $s \to \infty$.
(3) The area of $T \times \{s\}$ is $O(\frac{1}{s^2})$ as $s \to \infty$.
(4) In the universal cover $\mathbb{R}^2 \times [0, \infty)$, each plane $\mathbb{R}^2 \times \{s\}$ has *finite altitude barriers* for balls. This means that for any $s, r > 0$ there exists a height $a = a(s, r) > 0$ so that for any $s' < s$ and any ball B in $\mathbb{R}^2 \times \{s'\}$ of radius

less than r (with respect to the induced metric in $\mathbb{R}^2 \times \{s'\}$) there exists an open set U in $\mathbb{R}^2 \times [s', a)$ so that $U \cap (T \times \{s'\}) = B$ and the region $\big(\mathbb{R}^2 \times [s', \infty)\big) \setminus U$ admits a mean convex foliation.

Theorem 5 (Jiang-Vargas Pallete)**.** *Let (M, h_0) be a hyperbolic 3-manifold of finite volume, and let h be a weakly cusped metric on M. If the sectional curvature of h is less than or equal to -1, then*

$$\underline{E}(h) \geq 2.$$

Furthermore, the equality holds if and only if h is isometric to h_0.

On the other hand, regarding the impact of the scalar curvature, Lowe-Neves [8] proved the following result using the tool of Ricci flow on a closed hyperbolic 3-manifold. If the scalar curvature of a metric h satisfies $R_h \geq -6$, then $\overline{E}(h) \leq 2$, and equality holds if and only if h and h_0 are isometric. We also derive this result for the finite volume version.

Theorem 6 (Jiang-Vargas Pallete)**.** *Let h be a weakly cusped metric on M. If the scalar curvature of h is greater than or equal to -6, then*

$$\overline{E}(h) \leq 2.$$

The equality holds if and only if h is isometric to h_0.

Furthermore, we can pose a similar question when M is a closed hyperbolic manifold of odd dimension:

Question 7. Is there a neighborhood $\mathcal{U}$ of h_0 in the metric space of M, such that if $h \in \mathcal{U}$ and $R_h \geq -n(n-1)$, then $\overline{E}(h) \leq 2$?

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Mass Inequality and Stability of the Positive Mass Theorem For Kähler Manifolds

Johan Jacoby Klemmensen

Let (M^n, g) be a complete asymptotically Euclidean manifold of dimension $n \geq 3$. If the scalar curvature R_g is integrable, Bartnik and Chruściel showed that the *ADM mass* $\mathfrak{m}(g)$ for each end, defined as

$$\mathfrak{m}(g) = \lim_{\rho\to\infty} \frac{\Gamma(\frac{n}{2})}{4(n-1)\pi^{n/2}} \int_{S_\rho} (g_{kl,k} - g_{kk,l})\, \nu_l dA_{\mathrm{eucl}}, \tag{1}$$

is well-defined and independent of the chosen asymptotically Euclidean coordinate chart at infinity. The conjectured Positive Mass Theorem (PMT) was first proven by Schoen-Yau resp. Witten for $\dim M \leq 7$ resp. M spin and states the following: if the scalar curvature is nonnegative and integrable, then $\mathfrak{m}(g) \geq 0$ for every end and $\mathfrak{m}(g) = 0$ for some end if and only if (M, g) is isometric to $(\mathbb{R}^n, g)$. Hein-LeBrun later proved the theorem for AE *Kähler* manifolds [5], a paper important for this work as we will study the *stability* of the Positive Mass Theorem for AE Kähler manifolds.

The question of stability of the PMT is the following: if the ADM mass is small, is (M, g) close to $(\mathbb{R}^n, g)$ in some sense? The answer was complicated by the discovery of a sequence of AE rotationally symmetric three-dimensional manifolds (M_i^3, g_i) such that $\mathfrak{m}(M_i^3, g_i) \to 0$, but (M_i^3, g_i) develops infinitely deep gravitational wells where the GH-distance diverges [7]. It then became clear that other weaker norms need to be considered, of which Sormani-Wenger's *intrinsic flat distance* is probably the most utilized.

Another direction in the investigation of stability came after Bray-Kazaras-Khuri-Stern [1] proved their integral inequality, inspired by the inequality of Stern [8]. This inequality bounds the ADM mass from below by an integral involving the scalar curvature and the Hessian of certain harmonic functions. Using this, Dong [3], and later Dong-Song [4], proved that any sequence of AE Riemannian three-manifolds with ADM mass going to zero converge in the Gromov-Hausdorff topology to Euclidean three-space after cutting out sets with vanishing boundary in the limit. Similar results were also proven using this integral inequality, although under other assumptions, including a curvature bound, by Allen-Bryden-Kazaras [2] and Kazaras-Khuri-Lee [6] on three-manifolds.

In this talk, we prove a stability results for the Positive Mass Theorem and a new integral inequality for the ADM mass on AE Kähler manifolds (X^{2m}, g, J). This is the first stability result of the Positive Mass Theorem for Kähler manifolds.

The stability result follows from the following integral inequality

Theorem 1. *Let (X^{2m}, g, J) be an asymptotically Euclidean Kähler manifold with nonnegative and integrable scalar curvature R_g. Let $z = x_1 + ix_2$ be one of the* *holomorphic coordinate functions from the work of Hein-LeBrun. Then*

$$\mathfrak{m}(g) \geq \frac{(m-1)!}{4(2m-1)\pi^m} \int_X \frac{1}{2}|\nabla^2 x_1|^2 + \frac{1}{2}|\nabla^2 x_2|^2 + |\nabla x_1|^2 R_g \,\mathrm{dvol}_g \,. \tag{2}$$

The formula resembles the result [1, Theorem 1.2] on three-dimensional AE Riemannian manifolds.

Given the integral estimate, we prove a new stability result for the Positive Mass Theorem. The theorem applies to general sequences of AE Kähler manifolds and requires the excision of sets with vanishing boundaries in the limit, and potentially accounts for the presence of gravitational wells known to appear in the Riemannian case:

Theorem 2. *Let (X_i^{2m}, g_i, J_i) be a sequence of AE Kähler manifolds of fixed complex dimension with nonnegative and integrable scalar curvatures, and suppose that the ADM masses $\mathfrak{m}(g_i) \to 0$ as $i \to \infty$. Then for all i there exists a domain Z_i such that $M_i \setminus Z_i$ converge in the pointed Gromov-Hausdorff sense to $(\mathbb{R}^{2m}, d_{eucl})$:*

$$(X_i \setminus Z_i, d_{g_i}, p_i) \to (\mathbb{R}^{2m}, d_{eucl}, 0) \tag{3}$$

where $p_i \in M_i \setminus Z_i$ is any choice of base point and d_{g_i} is the induced length metric of g_i on $M_i \setminus Z_i$. Furthermore, for any continuous $\xi\colon (0,\infty) \to (0,\infty)$ such that $\lim_{x\to 0^+} \xi(x) = 0$, we get for i large enough:

$$\mathrm{area}(\partial Z_i) \leq \frac{\mathfrak{m}(g_i)^{\frac{m}{2m-2}+\frac{1}{2}}}{\xi(\mathfrak{m}(g_i))}. \tag{4}$$

For three-dimensional Riemannian manifolds, the theorem was proven by Dong-Song [4]. Some parts of their proof can be generalized from the three-dimensional case in a straightforward manner, while others require new work: specifically, one of the essential ingredients in Dong-Song was the integral inequality of Bray-Kazaras-Khuri-Stern [1, Theorem 1.2], and a major part in proving the theorem in the Kähler setting is to prove the inequality in Theorem 1.

Finally, we present a new sequence of AE Kähler metrics on $\mathbb{C}^2$ satisfying the requirements of Theorem 1. The family is generated by a smoothing of the Burns metric and is given by

$$\omega_\lambda^{\log} = i\partial\overline{\partial}(r^2 + \lambda \log(r^2+\lambda)) = \begin{pmatrix} \frac{2\lambda^2+\lambda(2r^2+|z_2|^2)+r^4}{(r^2+\lambda)^2} & -\frac{\lambda \overline{z_1} z_2}{(r^2+\lambda)^2} \\ -\frac{\lambda z_1 \overline{z_2}}{(r^2+\lambda)^2} & \frac{2\lambda^2+\lambda(2r^2+|z_1|^2)+r^4}{(r^2+\lambda)^2} \end{pmatrix}. \tag{5}$$

As the ADM masses vanish along the sequences, we obtain an explicit family of Kähler metrics with Ricci curvature unbounded from below and for which Theorem 2 applies. The family has the property of global GH convergence to $(\mathbb{C}^2, g_{\mathrm{eucl}})$ in the limit, and we could not find families of Kähler metrics with nonnegative scalar curvature developing a gravitational well. Finding a family developing a gravitational well and with vanishing mass would be very interesting for the application of Theorem 2.

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On the long-time behaviour of collapsing Ricci flows

Ramiro Lafuente

(joint work with Francesco Pediconi)

Let $(M^n, g(t))_{t\in[0,\infty)}$ be an *immortal* Ricci flow solution. Recent groundbreaking work of Bamler [1], extending to higher dimensions previous results of Hamilton [4] and Perelman [9] in dimension 3, implies that for $t \gg 1$ there is a thick-thin decomposition

$$M = M_{\mathsf{thick}}(t) \,\dot\cup\, M_{\mathsf{thin}}(t)$$

such that, after parabolic rescaling, the flow on $M_{\mathsf{thick}}(t)$ converges to a —possibly singular— Einstein metric with negative scalar curvature, and the flow on $M_{\mathsf{thin}}(t)$ is *collapsed*, in the sense that there is no uniform lower bound for the pointed Nash entropy. In particular, the thin region has no uniform lower bound for the injectivity radius.

We are interested in the asymptotic behaviour of the flow in the collapsed regions. In dimension 3, a very satisfactory answer was obtained by Lott in [7] in the case of bounded curvature and diameter, i.e.

$$(1) \qquad \|\mathrm{Rm}_{g(t)}\| \le C\, t^{-1}, \qquad \mathrm{diam}(g(t)) \le C\sqrt{t}, \qquad \forall t > 0.$$

Recall that these are precisely the bounds that are invariant under parabolic rescaling. Assuming (1), Lott proves that the pull-back to the universal cover of any sequence of parabolic blow-downs $g_s(t) := s^{-1}g(st)$ subconverges to an expanding homogeneous Ricci soliton.

Our main result is an extension of Lott's result to higher dimensions $n \ge 4$, under an additional topological assumption:

Theorem. [5] Let M^n be a closed essential manifold, and let $(M^n, g(t))_{t\in[0,\infty)}$ be an immortal Ricci flow satisfying (1). Then, any sequence of parabolic blow-downs subconverges to an expanding Ricci soliton with cocompact nilpotent symmetry.

Recall that a closed manifold M^n is essential if the image of the fundamental class $[M]$ under the classifying map for the universal cover $M \to K(\pi_1(M), 1)$ is non-trivial in $H_n(K(\pi_1(M), 1), \mathbb{Z})$. For example, aspherical manifolds are trivially essential, and so is any manifold which is the connected sum of a closed manifold and an essential one.

Regarding the limits, the underlying manifold X can be described as follows: let B be a closed orbifold with universal cover $\tilde{B} \to B$. Let N be a simply-connected Lie group admitting a left-invariant Ricci soliton metric (also called nilsoliton), and let $\rho : \pi_1(B) \to \mathrm{Aut}(\mathsf{N})$ be a representation. Then $X := \tilde{B} \times_\rho \mathsf{N}$ is the corresponding associated fiber bundle; it is known as a *twisted principal bundle*. There is a well-defined notion of a free local action by N on X, which is simply-transitive on the fibers, and with a compact orbit space B, but which cannot in general be extended to a global action.

The limit geometry is that of an expanding Ricci soliton, which is of gradient type if and only if N is abelian. The horizontal distribution is integrable, and the metric satisfies the harmonic-Einstein equations from [6]. In the non-abelian case, the fibers are pairwise isometric nilsolitons, and the image of ρ commutes with the nilsoliton derivation (after identifying $\mathrm{Aut}(\mathsf{N}) \simeq \mathrm{Aut}(\mathfrak{n})$). Several families of examples solving these equtions, where B is any closed orientable surface, have been recently found by Adam Thompson. There is an interesting connection with branched minimal immersions of B into symmetric spaces of non-compact type, which may be of independent interest.

The convergence in our main theorem is in the sense of étale Riemannian groupoids, due to the collapse, as developed by Lott. The proof of our main theorem indeed starts with Lott's extension of Hamilton compactness theorem for Ricci flows, which yields precompactness —in the Riemannian groupoid topology— of the set of parabolic blow-downs of any Ricci flow satisfying (1). Essentially, we obtain smooth limits Ricci flows on an n-dimensional manifold, with additional data in the form of a sheaf of Lie algebras consisting of germs of Killing fields, whose 'orbit space' is precisely the Gromov-Hausdorff limit of the original sequence. Thanks to the Cheeger-Fukaya-Gromov theory of collapsing with bounded curvature [3], these Lie algebras are nilpotent, and their orbits consist precisely of all collapsing directions. The assumption of M being essential is critical to ensure that the Killing field germs do not vanish at any point; thus, we may assume that the limit groupoid is *locally free*. The local structure of such an object is the same as that of a twisted principal bundle. The latter is in turn nothing but a principal bundle locally, with a global twisting induced by a representation of the fundamental group of B.

The arguments in the previous paragraph allow us to reduce the main theorem to the study of Ricci flows on twisted principal bundles with nilpotent symmetry and compact orbit space. To show that the limits are expanding solitons, we construct a new scale-invariant, monotone quantity for these flows, which generalises Lott's generalized $\mathcal{W}$-entropy [7] in the abelian case to arbitrary nilpotent symmetry. The monotonicity of this functional along (gauged) Ricci flow uses an

L^2-version of the GIT curvature estimates from [2]. Indeed, in the case where B is a point, this monotone quantity coincides with the Lyapunov function for immortal homogeneous Ricci flows from [2]. On the other hand, when N is trivial and $B = M$ is a closed smooth manifold, it reduces to the functional

$$(g, f) \mapsto \mathcal{F}(g, f) \exp\left(\tfrac{2}{n}\mathcal{N}(g, f)\right).$$

Here $\mathcal{F}$ and $\mathcal{N}$ are respectively Perelman's energy functional [8] and Nash's entropy, defined for any Riemannian metric g on M and any smooth function $f \in \mathcal{C}^\infty(M)$ by

$$\mathcal{F}(g, f) = \int_B \left(R_g + |\nabla f|_g^2\right) e^{-f} d\mathrm{vol}_g \ , \qquad \mathcal{N}(g, f) = \int_B f\, e^{-f} d\mathrm{vol}_g.$$

This quantity is monotone under the Ricci flow gauged by ∇f and constant precisely on expanding gradient solitons, provided f also evolves to keep the weighted measure $e^{-f} d\mathrm{vol}_g$ fixed.

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Asymptotics of the volume spectrum

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(joint work with Larry Guth, Fernando Marques, André Neves, Bruno Staffa)

Let M be a compact Riemannian manifold of dimension $n+1$. The eigenvalues of the Laplacian Δ on M have the following variational characterization. Let $V = W^{1,2}(M) \setminus \{0\}$ and consider the Rayleigh quotient $E : V \to [0, \infty]$, $E(f) = \int_M \frac{|\nabla f|^2}{f^2} dV$. Functional E is homogeneous, $E(af) = E(f)$, so it descends to the quotient $P = V/\{\text{lines in V}\} \simeq \mathbb{R}P^\infty$. Then

$$\lambda_p = \inf_{P_p \subset P} \sup_{f \in P_p} E(f)$$

where the infimum runs over all linear subspaces of P of dimension p.

Weyl's law states that the eigenvalues $\{\lambda_p\}$ have asymptotic behaviour which only depends on the volume of M:

$$lim_{p\to\infty}\lambda_p p^{-\frac{2}{n+1}} = \alpha(n)vol(M)^{-\frac{2}{n+1}},$$

where $\alpha(n) = 4\pi^2 Vol(B)^{-\frac{2}{n+1}}$ and B is the unit disc in $\mathbb{R}^{n+1}$.

Gromov ([3], [4, Section 8], [5, Section 5.2], [6]) proposed studying widths of Riemannian manifolds as a non-linear analog of the spectral problem on M. The definition of width is similar to the above min-max characterization of the eigenvalues, but with the space of cycles on M as the underlying space and the mass as the energy. We will work with the spaces $\mathcal{Z}_k(M;\mathbb{Z}_2)$ of mod 2 flat k-cycles in M and $\mathcal{Z}_{k,\mathbb{R}}(M,\partial M;\mathbb{Z}_2)$ of relative mod 2 flat cycles whenever M has boundary.

It follows from the work of Almgren [1] on the topology of $\mathcal{Z}_k(M;\mathbb{Z}_2)$ that there exists a cohomology class $\lambda \in H^{n-k}(\mathcal{Z}_k(M;\mathbb{Z}_2);\mathbb{Z}_2)$, such that all cup powers $\lambda^p \neq 0$. We say that a family of cycles $\Phi : X \to \mathcal{Z}_k(M;\mathbb{Z}_2)$ is a p-sweepout if $\Phi^*(\lambda^p) \neq 0$.

The p-width for k-cycles on a manifold M can be defined as follows

$$\omega_p^k(M) = \inf_{\Phi:X\to Z_k(M;\mathbb{Z}_2)} \sup_{x\in X} Vol_k(\Phi(x)),$$

where the infimum is over all p-sweepouts Φ.

Gromov's conjecture can then be stated stated as follows: there exists a constant $a(n,k)$, such that for any compact manifold M we have

$$lim_{p\to\infty}\omega_p(M)p^{-\frac{n-k}{n}} = a(n,k)vol(M)^{\frac{k}{n}}$$

The current status of the conjecture is the following:

- for domains in $\mathbb{R}^n$ and all k the conjecture was proved by Liokumovich-Marques-Neves [9];
- for Riemannian manifolds and $k = n-1$ it was proved in the same paper;
- for Riemannian manifolds, $k = 1$ and all $n3$ it was very recently proved by Staffa [11].

The key difficult in extending from $k = n-1$ to the higher codimension case is the necessity to prove two highly-nontrivial parametric versions of some fundamental geometric inequalities: the parametric isoperimetric inequality and parametric coarea inequality that were formulated in [7].

The conjecture is related to existence questions about minimal submanifolds of Riemannian manifolds. In the case when $k = 1$ the widths ω_p^1 correspond to the length of a stationary geodesic network (and for $n = 2$ closed geodesics by [2]). The asymptotic distribution of widhts has been used by Liokumovich-Staffa and Li-Staffa to prove density and equidistribution results for stationary geodesic nets and closed geodesics in [10] and [8].

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Complete Kähler manifolds with nonnegative Ricci curvature

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The talk is based on [5]. We consider complete Kähler manifolds with nonnegative Ricci curvature. The main results are:

1. When the manifold has nonnegative bisectional curvature, we show that $\lim\limits_{r\to\infty}\frac{r^2}{vol(B(p,r))}\int_{B(p,r)} S$ exists. In other words, it depends only on the manifold. This solves a question of Ni in [8]. Also, we establish estimates among volume growth ratio, integral of scalar curvature, and the degree of polynomial growth holomorphic functions. The new point is that the estimates are sharp for *any* prescribed volume growth rate. As a byproduct, we show that $\lim\limits_{r\to\infty}\frac{r^2}{vol(B(p,r))}\int_{B(p,r)} S$ $< \epsilon$ iff the asymptotic volume ratio of the universal cover is almost maximal.

2. We discover a strong rigidity for complete Ricci flat Kähler metrics. Let $M^n (n \geq 2)$ be a complete Kähler manifold with nonnegative Ricci curvature and Euclidean volume growth. Assume either the curvature has quadratic decay, or the Kähler metric is dd^c-exact with quadratic decay of scalar curvature. If one tangent cone at infinity is Ricci flat, then M is Ricci flat. In particular, the tangent cone is unique. In other words, *we can test Ricci flatness of the manifold by checking one single tangent cone*. This seems unexpected, since apriori, there is no equation on M and the Bishop-Gromov volume comparison is not sharp on Ricci flat (nonflat) manifolds. Such result is in sharp contrast to the Riemannian setting: Colding and Naber [3] showed that tangent cones are quite flexible when $Ric \geq 0$ and $|Rm|r^2 < C$. This reveals subtle differences between Riemannian case and Kähler case. The result contains a lot of examples, such as all noncompact Ricci flat Kähler

surfaces of Euclidean volume growth (hyper-Kähler ALE 4-manifolds classified by Kronheimer [6]), higher dimensional examples of Tian-Yau type [10]. It also covers Ricci flat Kähler metrics of Euclidean volume growth on Stein manifolds with $b_2 = 0$, such as Ricci flat Kähler metrics on $\mathbb{C}^n$ [7] [9] [4] [1]. Note in this case, the cross section is singular.

We also propose a conjecture: Given a complete noncompact Kähler manifold with nonnegative Ricci curvature and Euclidean volume growth, if one tangent cone at infinity is Ricci flat (this means the metric is smooth and Ricci flat away from a real codimension 4 set), then the manifold is Ricci flat. Such should be compared with a theorem of Colding [2] in Riemannian geometry: For a complete Riemannian manifold with nonnegative Ricci curvature, if a tangent cone at infinity is Euclidean, then the manifold is Euclidean.

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Positive scalar curvature on noncompact manifolds

John Lott

There are many results on whether a given compact manifold admits a Riemannian metric with positive scalar curvature (psc). We focus on the noncompact case.

An open conjecture for compact manifolds says that an aspherical compact smooth manifold cannot admit a psc metric. There are two main approaches to this conjecture. The first one uses minimal hypersurfaces, following the work of Schoen-Yau [10], and μ-bubbles as introduced by Gromov [5, Section $5\frac{5}{6}$]. Recent advances are by Chodosh-Li [3] and Gromov [6]. The other approach, which we follow, uses Dirac operators.

The above conjecture has an extension to compact manifolds that may not be aspherical. If M is a compact connected oriented n-dimensional smooth manifold, choose a basepoint m_0 and consider the fundamental group $\Gamma = \pi_1(M, m_0)$. There is a pointed connected CW-complex $B\Gamma$ with the property that $\pi_1(B\Gamma) = \Gamma$ and the universal cover of $B\Gamma$ is contractible. There is a classifying map $\nu : M \to B\Gamma$, unique up to homotopy, that induces an isomorphism on π_1. If $[M] \in \mathrm{H}_n(M;\mathbb{Q})$ is the fundamental class in rational homology then the extended conjecture says that nonvanishing of the pushforward $\nu_*[M] \in \mathrm{H}_n(B\Gamma;\mathbb{Q})$ is an obstruction for M to admit a psc metric. If M is aspherical then one recovers the previous conjecture. There are many results on this extended conjecture, using Dirac operators [9].

We are concerned with obstructions to complete psc metrics on noncompact manifolds. There is also a long history to this problem, going back to the Gromov-Lawson paper [7]. Our motivation comes from conjectures relating scalar curvature to simplicial volume, for compact manifolds.

Here is a test question. Suppose that Y is a compact connected oriented n-dimensional manifold-with-boundary, with connected boundary ∂Y. Choosing a basepoint $y_0 \in \partial Y$, put $\Gamma = \pi_1(Y, y_0)$ and $\Gamma' = \pi_1(\partial Y, y_0)$. There is a classifying map of pairs $\nu : (Y, \partial Y) \to (B\Gamma, B\Gamma')$, unique up to homotopy. Let $[Y, \partial Y] \in \mathrm{H}_n(Y, \partial Y;\mathbb{Q})$ be the fundamental class. Is nonvanishing of the pushforward $\nu_*[Y, \partial Y] \in \mathrm{H}_n(B\Gamma, B\Gamma';\mathbb{Q})$ an obstruction for the interior $\mathrm{int}(Y) = Y - \partial Y$ of Y to admit a complete psc metric, provided that
(a) The homomorphism $\Gamma' \to \Gamma$ is injective, or
(b) The metric has finite volume?

Here if $\Gamma' \to \Gamma$ is not injective then we define $\mathrm{H}_n(B\Gamma, B\Gamma';\mathbb{Q})$ using the algebraic mapping cone complex; it could more accurately be written as $\mathrm{H}_n(B\Gamma' \to B\Gamma;\mathbb{Q})$.

One needs some condition like (a) or (b), as can be seen if $Y = D^2$. Then $\pi_1(\partial Y) \to \pi_1(Y)$ is not injective, $\nu_*[Y, \partial Y] \neq 0$ and $\mathrm{int}(Y)$ does admit a complete psc metric, such as a paraboloid, but not one of finite volume.

A special case of the above question is when Y and ∂Y are aspherical, in which case $\nu_*[Y, \partial Y]$ is automatically nonzero.

We give results in the direction of (a) and (b). One main tool is almost flat bundles in the relative setting. Almost flat bundles were introduced by Connes-Gromov-Moscovici [4] and give obstructions for compact spin manifolds to have psc metrics. Almost flat bundles in the relative setting were introduced by Kubota [8]. There are actually two versions: almost flat relative bundles and almost flat stable relative bundles. They are relevant for (a) and (b), respectively.

Our other main technical tool is Callias-type Dirac operators, as were used for example by Cecchini-Zeidler in [1,2]. This allows us to give localized obstructions to positive scalar curvature, that apply to incomplete manifolds. Some of the statements involve the mean curvature of a boundary.

For notation, R denotes scalar curvature. Our convention for mean curvature is such that $S^{n-1} \subset D^n$ has mean curvature $H = n - 1$.

The geometric setup for our localized statements is that we have a region in a manifold that is assumed to have positive scalar curvature, then there is an annulus around it with quantitatively positive scalar curvature, followed by a larger annulus of a certain size that can have some negative scalar curvature. What happens outside of the second annulus doesn't matter. The precise assumption is the following.

Assumption 1. *Given $r_0, D > 0$, put $r_0' = \frac{1}{256} r_0^2 D^2$ and $D' = D + \frac{32}{r_0 D}$. Let M be a connected Riemannian spin manifold, possibly with boundary and possibly incomplete. Let K be a compact subset of M containing ∂M. Suppose that*

- *The distance neighborhood $N_{D'}(K)$ lies in a compact submanifold-with-boundary $\mathcal{C}$,*
- *$R > 0$ on K,*
- *$R \geq r_0$ on $N_D(K) - K$ and*
- *$R \geq -r_0'$ on $N_{D'}(K) - N_D(K)$.*

The first main result just uses almost flat K-theory classes, denoted by $\mathrm{K}^*_{af}(\cdot)$.

Theorem 1. *Suppose that Assumption 1 holds, where ∂M has nonnegative mean curvature. Given $\beta \in \mathrm{K}^{-1}_{af}(\mathcal{C})$ if M is even dimensional, or $\beta \in \mathrm{K}^0_{af}(\mathcal{C})$ if M is odd dimensional, we have*

$$\int_{\partial M} \widehat{A}(T\partial M) \wedge \mathrm{ch}\left(\beta\big|_{\partial M}\right) = 0. \tag{1}$$

Our main application of Theorem 1 is to finite volume complete manifolds M of dimension at most seven. We show that there is an exhaustion of M by compact submanifolds K_i whose boundaries ∂K_i have nonnegative mean curvature as seen from $M - K_i$. (It would be interesting if the dimension restriction could be removed.) Rather than applying Theorem 1 to K_i, we apply it to a suitable compact region of $\overline{M - K_i}$ containing ∂K_i. In this way we obtain end obstructions to the existence of finite volume psc metrics. As a simple example, there is no complete finite volume psc metric on $[0, \infty) \times T^{n-1}$ if $n \leq 7$.

The next main result uses almost flat relative K-theory classes, denoted by $\mathrm{K}^*_{af}(\cdot, \cdot)$. The geometric setup is similar to the previous one, except that now there is no boundary.

Theorem 2. *Suppose that Assumption 1 holds, where $\partial M = \emptyset$ and K is a compact submanifold-with-boundary in M. Then given $\beta \in \mathrm{K}^*_{af}(\mathcal{C}, \mathcal{C} - \mathrm{int}(K))$, we have*

$$\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0. \tag{2}$$

Theorem 2 is relevant to part (a) of the test question above.

The third main result combines Theorems 1 and 2. It uses almost flat stable relative K-theory classes, denoted by $\mathrm{K}^*_{af,st}(\cdot, \cdot)$ The generators of $\mathrm{K}^0_{af,st}(\cdot, \cdot)$ differ from generators of $\mathrm{K}^0_{af}(\cdot, \cdot)$ essentially by having additional K^{-1}-generators for the second factor. This meshes well with the compact exhaustions of finite volume manifolds. In the geometric assumptions, it is now assumed that the boundary of the inner compact region has nonnegative mean curvature as seen from the complement.

Theorem 3. *Suppose that Assumption 1 holds, where $\partial M = \emptyset$, K is a compact codimension-zero submanifold-with-boundary in M, and ∂K has nonnegative mean* *curvature as seen from $M - K$. Then given $\beta \in \mathrm{K}^*_{af,st}(\mathcal{C}, \mathcal{C} - \mathrm{int}(K))$, we have*

$$\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0. \tag{3}$$

When combined with the result about compact exhaustions of complete finite volume Riemannian manifolds, Theorem 3 is relevant to part (b) of the test question above.

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A sharp isoperimetric-type inequality for Lorentzian spaces with timelike Ricci curvature bounded below

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(joint work with Fabio Cavalletti)

The isoperimetric problem is one of the most classical problems in Mathematics, tracing its origins to the Greek legend of Dido, Queen of Carthage. In the context of Riemannian geometry, it seeks to answer the following question:

"What is the maximal volume that can be enclosed by a given surface area?"

Equivalently, it can be framed as the problem of finding the *maximal* function $I_{(M,g)}(\cdot) : [0,\infty) \to [0,\infty)$ such that for every subset $E \subset M$ with smooth boundary ∂E in the $(n+1)$-dimensional Riemannian manifold (M^{n+1}, g), the following inequality holds:

$$\mathrm{Vol}^n_g(\partial E) \geq I_{(M,g)}(\mathrm{Vol}^{n+1}_g(E)), \tag{1}$$

where $\mathrm{Vol}_g^{n+1}(E)$ and $\mathrm{Vol}_g^n(\partial E)$ denote he $(n+1)$-dimensional measure of E with respect to g and the n-dimensional measure of ∂E with respect to the restriction of g, respectively.

The literature on the isoperimetric problem in Riemannian geometry is vast. Even in Euclidean spaces, a complete solution is relatively recent and required several significant breakthroughs. In the framework of Riemannian manifolds with Ricci curvature bounded below, an isoperimetric inequality of the form (1) (where the function $I_{(M,g)}(\cdot)$ depends only on the dimension and the Ricci lower bound) was proved by Gromov in the case of a positive Ricci lower bound, following earlier work by Lévy.

If (M^{n+1}, g) is a *Lorentzian* manifold, the *maximal* function $I_{(M,g)}(\cdot) : [0, \infty) \to [0, \infty)$ satisfying (1) is identically zero, at least for small volumes and, in several examples (including Minkowski space-time), for all volumes. This occurs because causal diamonds have positive $(n+1)$-volume, but their boundary is a null hypersurface (with singularities of negligible measure) having zero n-volume, with respect to the restriction of the ambient Lorentzian metric.

Indeed, due to the different signature, a geometric *minimization* problem in the *Riemannian* context becomes a *maximization* problem in the *Lorentzian* context. A landmark example is given by geodesics, which locally *minimize* length in *Riemannian* geometry but locally *maximize* time separation (i.e., Lorentzian length) in *Lorentzian* geometry. The same phenomenon appears in the isoperimetric problem, which, in Lorentzian signature, reads as:

"What is the maximal area that can be used to enclose a given volume?"

Equivalently, it can be stated as the problem of finding the *minimal* function $J_{(M,g)}(\cdot) : [0, \infty) \to [0, \infty)$ such that for every subset $E \subset M$ with a smooth boundary ∂E in the $(n+1)$-dimensional Lorentzian manifold (M^{n+1}, g), the following inequality holds:

$$\mathrm{Vol}_g^n(\partial E) \leq J_{(M,g)}(\mathrm{Vol}_g^{n+1}(E)), \tag{2}$$

where $\mathrm{Vol}_g^{n+1}(E)$ and $\mathrm{Vol}_g^n(\partial E)$ denote the $(n+1)$-dimensional measure of E with respect to $|g|$ and the n-dimensional measure of ∂E with respect to the restriction of $|g|$, respectively.

In sharp contrast to the Riemannian setting, where the literature on the isoperimetric problem is extensive, the literature on the isoperimetric problem in Lorentzian signature is rather limited: Bahn-Ehlrich [3] for cones in Minkowski space-time, Bahn [2] for simply connected domains in 2-dimensional Lorentzian surfaces with Gaussian curvature bounded above, Abedin-Corvino-Kapita-Wu [1] in warped product space-times with non-negative timelike Ricci curvature, Lambert-Scheuer [7] in warped product space-times with non-negative Ricci curvature in null directions.

One of the main reasons for the relatively limited bibliography compared to the Riemannian case is the lack of a regularity theory for critical points of the area functional, which may fail to be elliptic due to the Lorentzian signature of the ambient metric. We overcome this issue by adopting an optimal transport approach, which bypasses the regularity problems. Below, some notation before stating a simplified version of the main result obtained in collaboration with Cavalletti [5].

Let (M^{n+1}, g) be a smooth globally hyperbolic Lorentzian manifold. Let V, S be two Cauchy hypersurfaces, with $S \subset I^+(V)$, where

$$I^+(V) = \{y \in M \colon \exists x \in V \text{ such that } x \ll y\}$$

denotes the chronological future of V. Let

$$\tau_V(y) := \sup_{x \in V} \tau(x, y), \ \forall y \in I^+(V), \qquad \mathrm{dist}(V, S) := \inf_{y \in S} \tau_V(y),$$
$$C(V, S) := \{\gamma(t) \mid t \in [0,1], \text{ s.t. } \gamma \text{ is a timelike } \tau_V\text{-maximizing geodesic with } \gamma(0) \in V, \, \gamma(1) \in S\},$$

denote the time-separation from V, the "distance" from V to S, and the region spanned by timelike τ_V-maximising geodesics from V to S, respectively.

Theorem 1 [Cavalletti-M., [5]]. Let (M^{n+1}, g) be a globally hyperbolic Lorentzian manifold satisfying Hawking-Penrose's strong energy condition (i.e., Ric ≥ 0 on timelike vectors). Let $V, S \subset M$ be Cauchy hypersurfaces with $S \subset I^+(V)$. Then

$$\mathrm{Vol}_g^n(S) \, \mathrm{dist}(V, S) \leq (n+1) \, \mathrm{Vol}_g^{n+1}(C(V, S)).$$

Some comments are in order (for more details, we refer to [5]):

- Previous results in the literature about isoperimetric-type inequalities in Lorentzian manifolds (or in Riemannian space-like slices) assume the metric g to be a warped product. There is no symmetry assumption on the space-time in Theorem 1, but merely a lower bound on the Ricci curvature in the timelike directions.
- Theorem 1 is stated for non-negative Ricci curvature just for the sake of simplicity. A completely analogous statement holds for Ricci curvature bounded below by $K \in \mathbb{R}$ in the timelike directions.
- The isoperimetric-type inequality in Theorem 1 is sharp and rigid: the equality is attained if and only if the space-time is conical.
- The isoperimetric-type inequality in Theorem 1 is proved in the higher generality of Loretzian pre-length spaces satisfying timelike Ricci curvature lower bounds in a synthetic sense via optimal transport, the so-called $\mathrm{TCD}(K, N)$ spaces [4]. Also the assumption on V can be relaxed considerably: it is enough to assume that V is a Borel, achronal, timelike complete subset.

As applications, in [5] we establish:

- An upper bound on the area of Cauchy hypersurfaces S inside the interior of a black hole, involving the time-distance from S to the center of the black-hole.
- An upper bound on the area of Cauchy hypersurfaces in cosmological space-times. The novelty with respect to previous results (see for instance [1]) is that no spatial symmetry is assumed; such a higher generality seems to have advantages also for from the physical point of view [6].

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Tubed embeddings

Anton Petrunin

A smooth isometric embedding of one Riemannian manifold into another will be called *tubed* if the image admits a uniformly thick tubular neighborhood. We consider the following question: *When does a Riemannian manifold admit a tubed embedding in a Euclidean space of large dimension?*

The necessary conditions include: (1) bounded sectional curvature, (2) positive injectivity radius, and (3) uniformly polynomial growth. The latter means that the volume of any ball of radius R in our manifold is bounded by a fixed polynomial of R.

We prove a partial converse. Namely, if in addition to the three properties the covariant derivatives of the curvature tensor are bounded, then the manifold admits an embedding into Euclidean spaces of sufficiently large dimension.

The proof combines the Nash embedding theorem and the Krauthgamer–Lee theorem about graph embeddings.

We also discuss related questions with other target spaces.

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Einstein metrics from the Calabi Ansatz via Derdzinski duality

Rosa Sena-Dias
(joint work with Gonçalo Oliveira)

A metric is Einstein if it is a constant multiple of its Ricci tensor. On a Kähler manifold, the Ricci tensor can be conveniently encapsulated in a $(1,1)$-form.

Definition 1. *(M, J, ω, g) Kähler is Kähler-Einstein if there is a constant λ such that* $\mathrm{Ric}(g) = \lambda\omega$.

Kähler-Einstein implies $c_1(M) = \lambda[\omega]$. When $\lambda > 0$, the Fano case, there are non trivial conditions on existence of solutions. Chen-Donaldson-Sun (see [CDS]) recently proved:

Theorem 1. *(M, J, ω) Kähler admits a Kähler-Einstein metric in the class $c_1(M, J)$ iff it is K-stable.*

We will not define K-stability. The proof of this theorem involves a continuity method though metrics with *a cone angle* along a divisor ([Do]): cone angle metrics have become popular in Kähler geometry. There are Kähler manifolds, such as $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$ or $\mathbb{CP}^2\sharp 2\overline{\mathbb{CP}}^2$, which do not carry Kähler-Einstein metrics.

Question 1. *Can Kähler geometers help find Einstein metrics on $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2\sharp 2\overline{\mathbb{CP}}^2$?*

1. Calabi's extremal metrics

To attack question (1) one can replace "Einstein" by "good".

Definition 2 (Calabi). *A compact Kähler manifold (M^{2n}, J, ω_0) is extremal in the sense of Calabi if it is a critical point for the Calabi functional defined by*

$$\mathcal{C}(\omega) = \int_M \mathrm{Scal}^2(g_\omega)\omega^n, \quad \forall \omega \in \mathcal{K}(M) \cap [\omega_0].$$

Here $\mathcal{K}(M)$ is the Kähler cone and g_ω is the metric associated to ω. The Euler Lagrange equations for $\mathcal{C}$ show that ω is extremal iff $\bar\partial\nabla^{(1,0)}\mathrm{Scal}(g_\omega) = 0$. This can be taken as a definition in the non-compact setting. In particular, constant scalar curvature Kähler metrics hence Kähler-Einstein metrics are extremal. Calabi was the first to write down explicit compact examples of non cscK extremal metrics. He devised an ansatz for constructing such metrics. We need a very special case here ([C]).

Theorem 2 (Calabi). *For all $m \in \mathbb{Z}_+$, there is a 2-parameter family of extremal metrics on the total space of $\mathcal{O}(-m)$, and an extremal metric on $\mathbb{H}_m = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-m))$.*

Note that $\mathbb{H}_1$ is biholomorphic to $\mathbb{CP}^2\sharp\overline{\mathbb{CP}^2}$.

2. Conformally Kähler-Einstein metrics

If in addressing Question (1) one insists on keeping the Einstein condition, then one can consider relaxing the Kähler condition.

Definition 3. *(M, g) is conformally Kähler/Einstein if there is a smooth function $\sigma : M \to \mathbb{R}$ and a Kähler/Einstein metric g_0 on M such that $g = e^{\sigma} g_0$.*

Page wrote down an explicit conformally Kähler, Einstein metric on $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$.

Question 2 (Derdzinski)**.** *What Kähler manifolds are conformally Einstein?*

Theorem 3 (Derdzinski)**.** *Let (M^4, J, ω, g) be Kähler. There is an Einstein metric conformal to g on the set where $\mathrm{Scal}(g) \neq 0$ iff g is Bach-flat, in which case $\frac{g}{\mathrm{Scal}^2(g}$ is Einstein where defined.*

3. Bach-flat metrics

Let (M, g) be Riemannian. The Weyl curvature W is a $(3, 1)$ tensor such that

$$W(g) = 0 \implies g \text{ is locally conformally flat.}$$

If M is compact, we further define the Weyl energy $\mathcal{W}$ as the total Weyl curvature.

Definition 4. *(M, g) compact is Bach-flat iff it is critical for the Weyl energy.*

The Euler Lagrange equations for $\mathcal{W}$ yield that g is Bach-flat iff its Bach-tensor B vanishes. Letting Ricci_0 denote the traceless Ricci tensor,

$$B_{ab} = W_{ab}^{cd} (\mathrm{Ricci}_0)_{cd} + \nabla^c \nabla_c (\mathrm{Ricci}_0)_{ab} - \nabla^c \nabla_a (\mathrm{Ricci}_0)_{bc}.$$

The above formula allows us to define Bach-flat metrics in the non-compact setting and shows that Einstein metrics are Bach-flat. It is not hard to see Bach-flatness is a conformally invariant property and this proves the easy part of Theorem (3). But how to find Bach-flat Kähler metrics?

Theorem 4 (LeBrun)**.** *Let (M^4, J, ω, g) be a Kähler surface. Then g is Bach-flat iff $[\omega]$ is critical for $\mathcal{A}$ on $\mathcal{K}(M)$ and g is extremal in its Kähler class.*

Next we define $\mathcal{A}$. Given (M, J, ω, g) Kähler:

$$\mathcal{C}(g) = \int_M (\mathrm{Scal}(g) - \underline{\mathrm{Scal}}(g))^2 \, d\,\mathrm{vol} + 32\pi^2 \frac{(c_1(M) \wedge [\omega])^2}{[\omega]^2}, \tag{1}$$

where $\underline{\mathrm{Scal}}(g)$ is the average of the scalar curvature which is topologically determined. When $\nabla^{(1,0)} \mathrm{Scal}(g)$ is holomorphic, for g an extremal metric in $[\omega]$,

$$\mathcal{C}(g) = \mathcal{F}(\nabla^{(1,0)} \mathrm{Scal}(g), [\omega]) + 32\pi^2 \frac{(c_1(M) \wedge [\omega])^2}{[\omega]^2},$$

where $\mathcal{F}$ denotes the Futaki invariant. Now $\xi = \nabla^{(1,0)} \mathrm{Scal}(g)$ is called the extremal vector field of M. It exists even when $[\omega]$ does not admit an extremal representative and only depends on $[\omega]$. We set $\mathcal{A}$ to be the right hand side of (1). Theorem 4 was used by Chen-Lebrun-Weber ([CLW]) to prove existence of a conformally Kähler-Einstein metric on $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}^2}$. Elsewhere, LeBrun has showed no other compact complex surfaces can carry conformally Einstein Kähler metrics so we look for them in the non-compact setting.

4. Non-compact conformally Kähler-Einstein metrics

Our idea was as follows.

- For $m > 1$, the Calabi metrics on $\mathbb{H}_m$ are never conformally Einstein.
- The Calabi ansatz can be carried out on $\mathbb{H}_m$ with a cone angle along the divisor at infinity. For the right choice of a cone angle, the metric will be Bach-flat.
- If the vanishing locus of Scal occurs "before" the cone angle divisor, the conformally Kähler, Einstein metric will not "see" the cone angle.

This idea essentially works for $m \geq 3$. We proved the following result ([OS]).

Theorem 5 (Oliveira-Sena-Dias). *For $m = 1$ and $\beta \in]0, 4\pi[$, there is a conformally Kähler, Einstein metric with cone angle β along the divisor at infinity in $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}}^2$. For $m \geq 3$, there are two conformally Kähler, Einstein metrics on the disk bundle $\mathbb{D}_m$ inside $\mathcal{O}(-m)$, $g_m^{\pm}$, with the same conformal infinity; g_m^+ is smooth, whereas g_m^- has a cone angle along the zero section.*

We shall explain the notion of conformal infinity in the next section.

5. Poincaré-Einstein metris

Definition 5 ([An2]). *Let M be a manifold with boundary ∂M and ρ a defining function for ∂M. A metric g is Poincaré-Einstein if it is Einstein on M and $\rho^2 g$ extends smoothly over ∂M. The conformal class of $(\rho^2 g)_{|\partial M}$ is independent of ρ. It is the conformal infinity of g, which is a Poincaré-Einstein filling of (M, g).*

The Poincaré model for the hyperbolic metric is Poincaré-Einstein. In this language, g_m^+ is a Poincaré-Einstein filling of $S^3/\mathbb{Z}_m$ and g_m^- is its "cap".

In Witten's AdS/CFT correspondence, Poincaré-Einstein fillings play a fundamental role ([W]). Page-Pope have constructed a Poincaré-Einstein fillings of $S^3/\mathbb{Z}_m$, ([PP]) which look different from ours but as it turns out, must be isometric to it. We hope the description we have for the fillings may be of use for instance, to determine renormalised volumes in the sense of [An].

6. New metrics out of rescaled limits?

Since there is a great deal of interest in degenerating families of Einstein metrics, we calculated limits in our construction. For $m \geq 3$, there is a 1-parameter subfamily of our family of Bach-flat metrics along which the scalar curvature tends to zero; the corresponding family of Einstein metrics degenerates. We find a convergent rescaling whose limit is Einstein and conformal to a scalar-flat Kähler metric so cannot come from Theorem (3). Is this metric new/interesting?

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Hamiltonian Stationary Surfaces in Sasakian geometry. Variational approaches.

Tristan Rivière

In the early 90's Yong-Geun Oh introduced the problem of studying critical points of the area among Lagrangian surfaces in a symplectic riemannian manifold. Such surfaces are called Hamiltonian stationary or sometimes H-minimal surfaces. This variational problem is motivated by natural questions such as the study of the Plateau problem in Lagrangian homology classes, the construction of calibrated minimal surfaces in Calabi Yau geometry (Thomas-Yau conjecture) or even the minmax constructions of minimal surfaces in spheres. We will first present the difficulties of dealing with the Hamiltonian stationary equation in general and present as a "warning" the construction of "pathological solutions" to this equation in 2 dimension which are nowhere continuous. Then we will turn to the special case of area minimizing H-minimal surfaces and the discovery in the early 2000 of a family of singularities of conical type by Schoen and Wolfson around which the Maslov class realized by the Lagrangian planes is non trivial. We will raise the question of the possible location of these singularities and whether they "interact" or not. In relation with this question, we will mention a direct method for constructing Hamiltonian stationary discs with prescribed Schoen Wolfson cones (joint work with Filippo Gaia and Gerard Orriols). In the second part of the talk we will prove that every non trivial minmax operation for the area of Legendrian Surfaces in a closed 5 Sasakian manifold is realised by a smooth branched Hamiltonian surface with possibly isolated conical singularities.

Singularity Models for High Codimension Mean Curvature Flow

Artemis Aikaterini Vogiatzi
(joint work with Huy The Nguyen)

Mean curvature flow is a geometric evolution equation that describes how a submanifold embedded in a higher-dimensional space changes its shape over time. We establish a codimension estimate that enables us to prove at a singular time of the flow, there exists a rescaling that converges to a smooth codimension one limiting flow in Euclidean space, regardless of the original flow's codimension. Under a cylindrical type pinching, we show that this limiting flow is weakly convex and either moves by translation or is a self-shrinker. Considering manifolds, such as the $\mathbb{C}P^n$, we go beyond the finite timeframe of the mean curvature flow, by proving that the rescaling converges smoothly to a totally geodesic limit in infinite time. Our approach relies on the preservation of the quadratic pinching condition along the flow and a gradient estimate that controls the mean curvature in regions of high curvature.

We suppose $n \geq 5$ with initial data $\mathcal{M}_0 = F_0(\mathcal{M})$, when $\mathcal{N}$ is an n-dimensional, closed, immersed submanifold satisfying a quadratic pinching condition of the form $f := -|A|^2 + c_n|H|^2 - d_n \geq 0$, where $c_n \leq \frac{4}{3n}$, if $n \geq 8$ and $c_n \leq \frac{3(n+1)}{2n(n+2)}$, if $n = 5, 6$ or 7 and d_n a positive constant that depends on the background curvature. Our focus was on the case of high codimension, $m \geq 2$. In this work, we successfully generalised results that concerned hypersurfaces assuming initial conditions such as convexity and two-convexity to higher codimension and extended results of submanifolds of higher codimension in the Euclidean and spherical spaces to Riemannian manifolds. We considered submanifolds of arbitrarily high codimension immersed into any Riemannian manifold, assuming a specific quadratic bound. We proved pointwise gradient estimates for the mean curvature flow, which we derived directly from the quadratic curvature bound of f and they played a key role in our analysis. The importance of the gradient estimates is that they allowed us to control the mean curvature and hence the full second fundamental form on a neighbourhood of fixed size. These gradient estimates rely solely on the mean curvature at a point and not on the maximum of curvature, as is the case with more general parabolic-type derivative estimates. This ensures that we have control over the curvature of the submanifold during a blow-up procedure. Using these gradient estimates, we were able to obtain the most important result of our work: we established a codimension estimate that shows that in regions of high curvature, with the assumption of the quadratic pinching, singularity models for this pinched flow must always be codimension one, regardless of the original flow's codimension. This crucial estimate enabled us to prove that at a singular time of the flow, using the gradient estimates, there exists a rescaling that converges to a smooth codimension one limiting flow in Euclidean space. Also, we proved cylindrical estimates that demonstrate an improvement in curvature pinching, as we approach a singularity. Under a cylindrical type pinching, this limiting flow is weakly convex and moves by translation or is a self-shrinker. These estimates allowed us to analyse the behaviour of the flow near singularities and establish the existence of the limiting flow. Specifically, we showed that these models can be classified up to homothety. From these theorems we derived that at the first singular time of the flow, the only possible blow-up limits are codimension one shrinking round spheres, shrinking round cylinders, and translating bowl solitons.

Assuming the $\mathbb{C}P^n$ as the background space in particular, we derived a new behaviour of the flow. By proving a decay estimate for the traceless part of the second fundamental form, we derive that the solution exists forever and converges to a totally geodesic submanifold as $t \to \infty$.

Mean curvature flow with surgery was initially introduced by Huisken and Sinestrari in 2009 for hypersurfaces in $\mathbb{R}^{n+1}$, assuming 2-convexity. In 2018, Nguyen extended this concept to submanifolds of higher codimension in Euclidean space $\mathbb{R}^{n+m}$, where $m \geq 2$ represents the codimension and n the dimension of the submanifold, under a quadratic pinching condition. Nguyen's work is pivotal for understanding mean curvature flow with surgery in high codimension. Using our results in our arsenal, we can now generalize mean curvature flow with surgery in high codimension from the Euclidean space, as first demonstrated by Nguyen in 2018, to any Riemannian manifold that satisfies the quadratic pinching condition that we use.

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Einstein metrics on Spheres

Matthias Wink
(joint work with Jan Nienhaus)

An important question in geometry is if a given manifold admits a Riemannian metric which is Einstein. Any Einstein metric on the spheres S^2 and S^3 is round with positive scalar curvature. In higher dimensions, much less is known. For example, it is a famous open question if there exists a Ricci flat metric on S^4.

In the case of odd dimensional spheres, a program initiated by Boyer-Galicki-Kollár [BGK05] has eventually led to many examples of Sasaki-Einstein metrics on every S^{2m+1}, for $m \geq 1$, by the efforts of many authors.

In contrast, the only known examples of even-dimensional spheres admitting a non-round Einstein metric are S^6, S^8 and S^{10}. In fact, both S^6 and S^8 admit infinitely many non-isometric Einstein metrics due to Böhm [Böh98]. A nearly Kähler metric on S^6 was constructed by Foscolo-Haskins [FH17] and Chi [Chi24] found an additional example on S^8.

In joint work with Jan Nienhaus, in [NW23] we provide a new construction of Einstein metrics on spheres and prove:

Theorem 1. *The ten-dimensional sphere S^{10} admits three non-round, non-isometric Einstein metrics with positive scalar curvature.*

As in the previous works of Böhm, Foscolo-Haskins and Chi, the new examples on S^{10} are also of cohomogeneity one. Specifically, the metric is invariant under the action of $SO(d_1+1) \times SO(d_2+1)$ on $S^{d_1+d_2+1}$ and for each pair $(d_1, d_2) = (2,7), (3,6), (4,5)$ we find one non-round Einstein metric. In fact, based on numerical investigations, we conjecture that these are the only $SO(d_1+1) \times SO(d_2+1)$-invariant metrics on S^{10}. The principal orbit of the action is an $S^{d_1} \times S^{d_2}$ and one obtains a metric on $S^{d_1+d_2+1}$ by smoothly collapsing the S^{d_1} on one end and the S^{d_2} on the other end (or vice versa).

One may also try to construct an Einstein metric on $S^{d_1+1} \times S^{d_2}$ by collapsing the S^{d_1}-factor and then reflecting the local solution along a principal orbit. Any such solution is called a symmetric solution. A common feature of the previous constructions of Einstein metrics on even-dimensional spheres due to Böhm, Foscolo-Haskins and Chi is the existence of symmetric solutions. In fact, Böhm constructs infinitely many Einstein metrics on $S^{d_1+1} \times S^{d_2}$ for $5 \leq d_1+d_2+1 \leq 9$ and $d_1, d_2 \geq 2$, Foscolo-Haskins find a nearly Kähler metric on $S^3 \times S^3$ and Chi's

Einstein metric on S^8 is itself symmetric (the principal orbit is an S^7 which collapses to a point at both ends). The existence of these symmetric solutions is particularly helpful as it provides important insights into the dynamics of the associated differential equation and, in combination with further arguments, yields the Einstein metrics on S^6 and S^8.

In contrast, numerical studies indicate that in the situation of Theorem 1 there is no corresponding symmetric (non-product) $SO(d_1+1)\times SO(d_2+1)$-invariant Einstein metric on $S^{d_1+1}\times S^{d_2}$ for $(d_1,d_2)=(3,6),\ldots,(7,2)$ and only one for $(d_1,d_2)=(2,7)$. In particular, for the proof of Theorem 1 we use a new technique based on a rotation index for curves. With our techniques we can also recover Böhm's Einstein metrics on $S^5,\ldots,S^9$ as well as the corresponding symmetric solutions, since Böhm's examples are also $SO(d_1+1)\times SO(d_2+1)$-invariant (for $5\le d_1+d_2+1\le 9$ and $d_1,d_2\ge 2$).

The construction roughly proceeds as follows. With a suitable coordinate change we transform the Einstein equation into an ODE system defined on a set homeomorphic to the cylinder $\overline{D}^2\times[-1,1]$. The interval $[-1,1]$ is parametrized by the rescaled mean curvature H of the principal orbit, which decreases monotonically from $H=+1$ to $H=-1$. Trajectories in the interior correspond to local solutions of the Einstein equations, and the boundary of the cylinder is invariant under the ODE.

The conditions for the smooth collapse of an S^d–factor correspond to stationary points of the ODE and at $H=1$ (resp. $H=-1$) there is a 1-parameter family of solutions emanating from (resp. converging to) these stationary points. Furthermore we note that $\operatorname{cone}(h)=(0,0,h)$ is a parametrization of the singular Einstein metric induced by the sine suspension over the principal orbit. Therefore one can try to analyze if trajectories wind around the cone solution.

In dimension ten, one proves that trajectories in the $\{H=1\}$ – slice approach the base point $(0,0,1)$ of the cone solution in a specific tangent direction. Furthermore, trajectories in the interior of the cylinder that are close to $(0,0,1)$ remain close to $\operatorname{cone}(h)=(0,0,h)$ (at least for $h\ge 0$) and indeed exhibit a winding behaviour around the cone solution which can be quantified.

For a given pair (d_1,d_2) one now considers the intersection of all trajectories that emanate from the stationary point corresponding to the smooth collapse of the S^{d_1} at $H=1$ and all trajectories that converge to the stationary point corresponding to the smooth collapse of the S^{d_2} at $H=-1$ within the $\{H=0\}$ – slice. Based on the winding behaviour of the trajectories, it follows that the resulting curve has a winding number greater than two and thus at least two intersection points, each of which corresponds to an Einstein metric on S^{10}. One intersection point corresponds to the round metric, the other corresponds to a non-round Einstein metric.

In order to recover Böhm's metrics on $S^5,\ldots,S^9$ one observes that in these dimensions the base point $(0,0,1)$ of the cone solution is a spiral, hence trajectories in the $\{H=1\}$ – slice wind around it infinitely often. Therefore one can find trajectories in the interior of the cylinder that wind arbitrarily often around the cone solution until they reach the $\{H = 0\}$ – slice. As a consequence, in the previous construction, one exhibits infinitely many intersection points and thus infinitely many Einstein metrics on $S^5, \ldots, S^9$.

We note that in the case of S^{11} we numerically exhibited Einstein metrics for $(d_1, d_2) = (2, 8), (3, 7)$ but not for $(d_1, d_2) = (4, 6), (5, 5)$.

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Rigidity of spin fill-ins with non-negative scalar curvature

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(joint work with Simone Cecchini, Sven Hirsch)

Let (Σ^{n-1}, g_Σ) be a closed, connected non-empty Riemannian manifold. A *non-negative scalar curvature* ("NNSC") *fill-in* of (Σ, g_Σ) is a connected Riemannian manifold (M, g) with boundary $\partial M = \Sigma$ such that $\mathrm{Scal}_g \geq 0$ and the metric g restricts to g_Σ on $\partial M = \Sigma$.

A recent result of Shi–Wang–Wei [9] states that on a connected manifold with non-empty boundary, every Riemannian metric on the boundary extends to a positive scalar curvature metric in the interior. Consequently, every (Σ^{n-1}, g_Σ) admits a NNSC fill-in provided that Σ is null-bordant as a smooth manifold. However, the mean curvature of the boundary with respect to a NNSC fill-in obtained from Shi–Wang–Wei's theorem is always negative. Moreover, Miao [8, Theorem 3] observed that there exists an a priori constant $C = C(\Sigma, g_\Sigma) > 0$, such that any NNSC fill-in (M, g) of (Σ, g_Σ) satisfies $\min_\Sigma \mathrm{H}_{\partial M, g} \leq C$, at least if $n \leq 7$ or M is spin.

Taking the above as a starting point, we present more precise geometric bounds on the minimum of the boundary mean curvature of spin NNSC fill-in (M, g) of a given Riemannian spin manifold (Σ, g_Σ). The first is defined in terms of the hyperspherical radius.

Definition 1. *The* hyperspherical radius *of (Σ, g_Σ) is defined in terms of distance non-increasing maps to round spheres as follows*

$$\mathrm{Rad}_{\mathrm{S}^{n-1}}(\Sigma, g_\Sigma) = \sup\{R > 0 \mid \exists f \colon (\Sigma, g_\Sigma) \to \mathrm{S}^{n-1}_R \colon \deg(f) \neq 0, \mathrm{Lip}(f) \leq 1\}.$$

Our main result states that the minimum of the boundary mean curvature of a spin fill-in is bounded in terms of the hyperspherical radius, where equality is achieved only for Euclidean discs.

Theorem 1 (Cecchini–Hirsch–Z. [2, Theorem 1.5]). *Let* (M, g) *be a* spin *fill-in of a Riemannian spin manifold* (Σ, g_Σ) *such that* $\mathrm{H}_{\partial M} \geq (n-1)/R$*, where* $R = \mathrm{Rad}_{\mathrm{S}^{n-1}}(\Sigma, g_\Sigma)$ *is the hyperspherical radius. Then* M *is isometric to the round disc of radius* R*.*

Note that the estimate without rigidity has previously been proved by Gromov [4, p. 3] and the rigidity result was an open conjecture. The proof of this theorem involves establishing an almost-rigidity result for maps to convex Euclidean domains in the spirit of the comparison results of Llarull [6], Goette–Semmelmann [3] and Lott [7]. The main difficulty lies in the fact that the hyperspherical radius is defined in terms of a supremum so that a priori there may not exist a *smooth* map realizing the hyperspherical radius.

Theorem 2 (Cecchini–Hirsch–Z. [2, Theorem 1.6]). *For* $n \geq 3$*, let* $\Omega \subseteq \mathbb{R}^n$ *be a compact domain with smooth strictly convex boundary. Let* M *be a connected compact Riemannian spin manifold with connected boundary such that* $\mathrm{Scal}_M \geq 0$*.*

Furthermore, let $(f_i \colon \partial M \to \partial\Omega)_{i \in \mathbb{N}}$ *be a sequence of smooth maps satisfying*

- $\mathrm{Lip}(f_i) \leq 1 + \frac{1}{i}$,
- $\mathrm{H}_{\partial M} \geq \mathrm{H}_{\partial\Omega} \circ f_i - \frac{1}{i}$,
- $\deg(f_i) \neq 0$.

Then there exists an isometry $\phi \colon \partial M \to \partial\Omega$ *with* $\mathrm{II}_{\partial M} = \phi^* \mathrm{II}_{\partial\Omega}$ *such that, after passing to a subsequence,* $f_i \to f$ *in* $\mathrm{W}^{1,p}(\partial M, \mathbb{R}^n)$ *for every* $p < \infty$*. Moreover,* M *is flat and isometric to* Ω*.*

While the above provides a precise geometric bound, the upper bound on the mean curvature obtained in this way is always positive. Miao [8, Question 2] asked if there exist null-bordant Riemannian manifolds which do not admit any NNSC fill-in with positive mean curvature. We answer this question in the spin setting for NNSC spin fill-ins via the following result.

Theorem 3 (Cecchini–Hirsch–Z. [2, Theorem 1.2]). *Let* (Σ, g_Σ) *be a closed spin manifold which admits a non-trivial harmonic spinor. Then any NNSC spin fill-in of* (Σ, g_Σ) *is Ricci-flat with minimal boundary.*

This theorem is based on a completely elementary extension result for spinors involving APS boundary conditions. Classical results of Hitchin [5] and Bär [1] show that there are many null-bordant Riemannian manifold which admit harmonic spinors. Thus there are many examples of Riemannian spin manifolds (Σ, g_Σ) which do admit some NNSC spin fill-in but none with positive mean curvature. For instance, this includes certain Berger spheres (S^3, g) with large fiber.

A more general extension statement is provided in our paper which also has a number of further applications, see [2, Theorems 1.3 and 3.5].

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