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# Mini-Workshop: New Horizons in Linear Dynamics, Universality, and the Invariant Subspace Problem

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Abstract. The mini-workshop *New Horizons in Linear Dynamics, Universality, and the Invariant Subspace Problem* discussed recent advances in the study of dynamical properties of continuous linear operators, and in the related study of universal properties of holomorphic functions. Ideas from topological dynamics, ergodic theory, and very recent advances on the invariant subspace problem were also considered.

*Mathematics Subject Classification (2020):* 30K05, 30K15, 37B02, 37B05, 47A15, 47A16.



## Introduction by the Organizers

The mini-workshop *New Horizons in Linear Dynamics, Universality, and the Invariant Subspace Problem*, organized by Sophie Grivaux (Lille), Karl Grosse-Erdmann (Mons), Etienne Matheron (Lens) and Alfred Peris (Valencia), was attended by 17 participants with broad geographic representation from 9 countries, including three participants from South America. It brought together researchers from various backgrounds (linear dynamics, universal functions, and topological dynamics). The main motivation for the organization of the workshop was the fact that the area of linear dynamics has recently seen, on the one hand, major advances and, on the other hand, the arrival of a very strong new generation of young researchers. This not only guided the choice of participants (more than half of them were junior researchers with at most 10 years after their PhD), but also the set-up of the workshop. In fact, it had the form of a *groupe de travail* or

*Arbeitsgruppe* with a proposed schedule but no chairpersons, which led to a very informal meeting and many discussions during and after each presentation. Apart from 14 talks, several of which presented work in progress and preliminary ideas, there was an open-ended problem-and-discussion session every day that had been prepared in advance by one or two junior researchers. The fact that the problem sessions were only ended by the necessity to go for lunch or dinner attests to the great success of this format and the excellent atmosphere between the participants.

Topics discussed in the talks and in the problem sessions included the following:

- properties of hypercyclic and recurrent operators
- examples of hypercyclic and chaotic operators
- notions and properties related to frequent hypercyclicity
- ergodic theoretic aspects of linear dynamics
- notions motivated by topological dynamics
- problems on universal holomorphic functions

By coincidence, Per Enflo posted a new version of his solution of the Invariant Subspace Problem on Hilbert spaces during the workshop. This was also briefly considered by the participants during the meeting, but it was decided that an in-depth discussion was beyond the scope of the workshop.

The classical notions of hypercyclicity, chaos and recurrence were discussed in several talks. *Dimitris Papathanasiou* presented a characterization of chaos for weighted bachward shifts on directed trees, while *Fernando Costa Jr.* discussed the existence of algebras of hypercyclic vectors for such operators. In a related talk, *Clifford Gilmore* studied in which sense operators possessing an algebra of hypercyclic vectors can be considered to be typical. *Jürgen Müller* discussed the hypercyclicity of the Taylor shift on various Bergman spaces and the importance of the Cauchy transform in this context. *Antoni López-Martínez* presented recent work on recurrence and the related notion of quasi-rigidity. *Quentin Menet* presented a new take on $C$-type operators that have been introduced and studied by him in recent years and that have helped to solve several major open problems in linear dynamics.

The notion of frequent hypercyclicity, which is by now central in linear dynamics, was discussed in four talks. *Frédéric Bayart* presented a new, strong form of this notion, called hereditary frequent hypercyclicity, with the aim of studying the $T \oplus T$-problem of frequent hypercyclicity. *Rodrigo Cardeccia* presented a characterization of frequently recurrent weighted backward shifts and a corresponding zero-one law, while *Romuald Ernst* discussed recent work on the delicate problem of finding common frequently hypercyclic vectors. This last problem was also the starting point of some ergodic theoretic discussions in the talk by *Étienne Matheron*, who studied, among others, the existence of equivalent ergodic measures for given operators.

The notion of universality of holomorphic functions is closely related to linear dynamics. In her talk, *Myrto Manolaki* studied and compared two notions of universality, that of universal Taylor shifts and the recent notion of Abel universality.

Another area that has recently had a strong influence on linear dynamics is that of topological dynamics. *Nilson C. Bernardes Jr.* discussed the notion of structural stability coming from topological dynamics, and he reported that an open problem on this notion was recently solved by methods from linear dynamics. *Martina Maiuriello* studied several notions coming from topological dynamics and from linear dynamics in the context of composition operators on spaces of measurable functions. This was followed by a presentation by *Emma D'Aniello*, who considered the concept of supercyclicity in that context.

The problem-and-discussion sessions had the following topics: Common hypercyclicity (organized by *Romuald Ernst* and *Fernando Costa Jr.*), *Composition operators from the perspective of topological dynamics* (Martina Maiuriello and Dimitris Papathanasiou), *Universality* (Myrto Manolaki and Clifford Gilmore), *Furstenberg families in linear dynamics* (Antoni López-Martínez and Rodrigo Cardeccia), and an *Open problem session* (Quentin Menet). In the latter session, which concluded the workshop, each participant presented one of his or her favourite problems.

**Mini-Workshop: New Horizons in Linear Dynamics, Universality, and the Invariant Subspace Problem**

**Table of Contents**

Frédéric Bayart (joint with Sophie Grivaux, Étienne Matheron)
*Hereditarily frequent hypercyclic operators* ........................ 1075

Romuald Ernst (joint with Stéphane Charpentier, Monia Mestiri, Augustin Mouze)
*Common frequent hypercyclicity* ........................ 1076

Dimitris Papathanasiou (joint with Karl Grosse-Erdmann)
*Chaotic shifts on directed trees.* ........................ 1077

Fernando Costa Jr. (joint with Arafat Abbar)
*Hypercyclic algebras for shifts on trees* ........................ 1077

Martina Maiuriello (joint with Emma D'Aniello, Udayan B. Darji)
*Preserved dynamics: visualizing the connection between composition operators and weighted shifts* ........................ 1078

Emma D'Aniello (joint with Martina Maiuriello)
*Preserved dynamics: a focus on hypercyclic and supercyclic behaviors* .. 1081

Nilson C. Bernardes Jr.
*Some recent aspects of linear dynamics* ........................ 1084

Clifford Gilmore (joint with William Alexandre, Sophie Grivaux)
*Typicality of operators admitting a hypercyclic algebra* ........................ 1085

Myrto Manolaki (joint with Stéphane Charpentier, Konstantinos Maronikolakis)
*Universal Taylor series VS Abel universal functions* ........................ 1085

Antoni López-Martínez (joint with Sophie Grivaux, Alfred Peris)
*Quasi-rigidity: the recurrence version of weak-mixing* ........................ 1086

Rodrigo Cardeccia (joint with Santiago Muro)
*Frequently recurrent backward shifts* ........................ 1087

Jürgen Müller (joint with Peter Beise)
*Cauchy integrals at the service of hypercyclicity* ........................ 1088

Quentin Menet
*Operators of $C$-type* ........................ 1090

Étienne Matheron (joint with Sophie Grivaux, Quentin Menet)
*Measure-theoretic (non-)orthogonality for weighted shifts* ........................ 1091

# Abstracts

## Hereditarily frequent hypercyclic operators

Frédéric Bayart
(joint work with Sophie Grivaux, Étienne Matheron)

An important open question in the theory of frequently hypercyclic operators is the following: given two frequently hypercyclic operators $T_1$ and $T_2$, is is true that $T_1 \oplus T_2$ is frequently hypercyclic? Motivated by this problem, we introduce the class of hereditarily frequently hypercyclic operators. By its very definition, if $T_1$ is frequently hypercyclic and $T_2$ is hereditarily frequently hypercyclic, then their direct sum $T_1 \oplus T_2$ is frequently hypercyclic. Having introduced this definition, we are immediately faced with some obvious questions:

**Do there exist hereditarily frequently hypercyclic operators?** We answer this in the affirmative, by two different methods. Indeed, there are two "standard" ways of proving that an operator is frequently hypercyclic: either by showing that it satisfies the so-called Frequent Hypercyclicity Criterion (see [1, Theorem 6.18]) or by exhibiting a large supply of eigenvectors associated to unimodular eigenvalues (see [2]). In both cases, one gets in fact hereditary frequent hypercyclicity.

**Is hereditary frequent hypercyclicity strictly stronger than frequent hypercyclicity?** The answer is "Yes", and we prove this with the following strategy: we construct two frequently hypercyclic weighted shifts $B_w$ and $B_{w'}$ on $c_0(\mathbb{Z}_+)$ such that $B_w \oplus B_{w'}$ is not frequently hypercyclic. Therefore, we also answer the aforementioned question about the direct sum of two frequently hypercyclic operators.

**What are hereditarily frequently hypercyclic operators good for?** We shall use them in the context of "disjoint hypercyclicity". The notion of disjointness in hypercyclicity was introduced independently in [3] and [4]. Let $N \geq 1$ and let $T_1, \dots, T_N \in \mathfrak{L}(X)$. Following [4], we say that $T_1, \dots, T_N$ are *disjoint*, or that the tuple $(T_1, \dots, T_N)$ is *diagonally hypercyclic*, if there exists $x \in X$ such that $\{(T_1^n x, \dots, T_N^n x) : n \geq 0\}$ is dense in $X^N$; in other words, $x \oplus \cdots \oplus x$ is hypercyclic for $T_1 \oplus \cdots \oplus T_N$. Similarly, $(T_1, \dots, T_N)$ is said to be *d-frequently hypercyclic* if there exists $x \in X$ such that $x \oplus \cdots \oplus x$ is frequently hypercyclic for $T_1 \oplus \cdots \oplus T_N$.

A natural problem regarding $d$-hypercyclicity is that of the *extension* of $d$-hypercyclic tuples. It was shown in [5] that given any $N \geq 1$, any Banach space $X$ and any $T_1, \dots, T_N \in \mathfrak{L}(X)$ such that $(T_1, \dots, T_N)$ is $d$-hypercyclic, there exists $T_{N+1} \in \mathfrak{L}(X)$ such that $(T_1, \dots, T_{N+1})$ is also $d$-hypercyclic. We obtain a similar result if we replace hypercyclicity by frequent hypercyclicity, on every Banach space supporting a hereditarily frequently hypercyclic operator.

## References

[1] F. Bayart and É. Matheron, *Dynamics of linear operators*, Cambridge Tracts in Math, vol. 179, Cambridge University Press, 2009.

[2] F. Bayart and É. Matheron, *Mixing operators and small subsets of the circle*, J. Reine Angew. Math. **715** (2016), 75–123.

[3] L. Bernal-González, *Disjoint hypercyclic operators*, Studia Math. **182** (2007), no. 2, 113–131.

[4] J. P. Bès and A. Peris, *Disjointness in hypercyclicity*, J. Math. Anal. Appl. **336** (2007), 297–315.

[5] Ö. Martin and R. Sanders, *Extending families of disjoint hypercyclic operators*, preprint (2023), arXiv:2312.07054.


# Common frequent hypercyclicity

Romuald Ernst

(joint work with Stéphane Charpentier, Monia Mestiri, Augustin Mouze)

This talk aims at giving intuitions on the recent results on common frequent hypercyclicity obtained in [3].

Only a few authors have studied the existence and non-existence of common frequently hypercyclic vectors. Indeed, even for just two frequently hypercyclic operators there was no condition to guarantee that they do share a frequently hypercyclic vector or not. It was not even known if there exists a countable family of frequently hypercyclic operators that do not share such a common vector. The only negative known result was from Bayart and Grivaux [1, 2] who proved that an uncountable family of multiples of a single operator cannot admit a common frequently hypercyclic vector.

We present in this talk some positive results applied to the special case of weighted shifts on $\ell^p(\mathbb{N})$ allowing to obtain common frequently hypercyclic vectors for countable families in a wide class of weighted shifts. We also present a method to obtain necessary conditions for the existence of common frequently hypercyclic vectors for countable or finite families. This method allowed us to explain how we have been able to give the first example of two frequently hypercyclic operators that do not share any common frequently hypercyclic vector.


## References

[1] F. Bayart and S. Grivaux, *Frequently hypercyclic operators*, Trans. Amer. Math. Soc. **358** (2006), 5083–5117.

[2] F. Bayart, *Common hypercyclic vectors for high-dimensional families of operators*, Int. Math. Res. Not. IMRN **21** (2016), 6512–6552.

[3] S. Charpentier, R. Ernst, M. Mestiri, and A. Mouze, *Common frequent hypercyclicity*, J. Funct. Anal. **283** (2022), Paper No. 109526, 51 pages.

## Chaotic shifts on directed trees.

Dimitris Papathanasiou
(joint work with Karl Grosse-Erdmann)

An operator on a Fréchet space is called chaotic provided it is hypercyclic (i.e. it admits a dense orbit) and has a dense set of periodic points. Grosse-Erdmann characterized in [1] when a classical unilateral and bilateral weighted backward shift acting on a Fréchet sequence space is chaotic.

One way to generalize weighted shifts is to change the indexing set of sequences on which the operator is acting, from $\mathbb{N}$ or $\mathbb{Z}$ to a directed tree. We are calling the resulting class of operators weighted shifts on directed trees. In this talk we present the results obtained in [2], where we extend Grosse-Erdmann's result and characterize when a weighted shift on a directed tree is chaotic (if it is acting on a Fréchet sequence space of the tree).

In particular: we discuss the similarities and differences of the characterizing conditions that occur while passing from $\mathbb{N}$ or $\mathbb{Z}$ to a directed tree; and when we restrict ourselves to spaces of the form $\ell^p$, $1 \leq p < \infty$ or $c_0$ of a directed tree, then we get quantitative characterizations for when a weighted shift is chaotic expressed by branched continued fractions that reflect the geometry of the tree and involve the weights of the shift. Those fractions can be expressed in terms of capacities of the Poisson boundary of the tree and yield a surprising connection between chaotic shifts on directed trees and harmonic analysis on directed trees.

### References

[1] K.-G. Grosse-Erdmann, *Hypercyclic and chaotic weighted shifts*, Studia Math. **139** (2000), 47–68.

[2] K.-G. Grosse-Erdmann and D. Papathanasiou, *Chaotic shifts on directed trees*, arXiv preprint (2023), arXiv:2303.03980.

## Hypercyclic algebras for shifts on trees

Fernando Costa Jr.
(joint work with Arafat Abbar)

The class of weighted backward shift operators on sequences spaces is one of the most studied in the field of Linear Dynamics. This is not a surprise, for these operators are simple to define, rather simple to work with and can present even the most wild chaotic behaviour. In some recent works of Martínez-Avendaño [5], Grosse-Erdmann and Papathanasiou [3] and [4], this class has been studied in the context of sequence spaces of a tree, where some simpler examples/counterexamples of classical problems can be constructed. This context also allows the consideration of classical operators with completely distinguished behaviour on trees. For instance, Rolewicz operators $\lambda B$ on the $\ell^2$-space of a rooted tree can be hypercyclic even when $|\lambda| < 1$: one only has to play with the geometry of the tree to obtain this behaviour which is impossible for the classical shift operators.

Every $\ell^p$-space of a tree is a Banach algebra for the coordinatewise product, thus one can also investigate whether the set of hypercyclic vectors of such spaces is algebrable. This question has been deeply studied for the trees $\mathbb{N}$ and $\mathbb{Z}$ in the works [1] and [2]. In this talk, we discuss this exact problem. One of the most surprising facts is that, unlike for hypercyclicity, the geometry of the tree does not help to get hypercyclic algebras.

We present a characterization for the algebrability of the set of hypercyclic vectors for weighted backward shift operators on rooted directed trees. As a direct application, we conclude that there are trees on which one can consider Rolewicz operators that are mixing (and even chaotic) but with no hypercyclic algebra. The unrooted case is also discussed and remains open for general trees.

## References


[1] F. Bayart, F. Costa Jr., and D. Papathanasiou, *Baire theorem and hypercyclic algebras*, Adv. Math. **376** (2021), Paper No. 107419, 58 pages.

[2] J. Falcó and K.-G. Grosse-Erdmann, *Algebrability of the set of hypercyclic vectors for backward shift operators*. Adv. Math. **366** (2020), 107082, 25 pages.

[3] K.-G. Grosse-Erdmann and D. Papathanasiou, *Dynamics of weighted shifts on directed trees*, Indiana Univ. Math. J. **72** (2023), 263–299.

[4] K.-G. Grosse-Erdmann and D. Papathanasiou, *Chaotic weighted shifts on directed trees*, arXiv preprint (2023), arXiv:2303.03980.

[5] R. A. Martínez-Avendaño, *Hypercyclicity of shifts on weighted $L^p$ spaces of directed trees*, J. Math. Anal. Appl. 446 (2017), 823–842.


## Preserved dynamics: visualizing the connection between composition operators and weighted shifts


Martina Maiuriello

(joint work with Emma D'Aniello, Udayan B. Darji)


We investigate the intimate relation between composition operators on $L^p$ and bilateral weighted shifts on $\ell^p$, $1 \leq p < \infty$, via a factor map satisfying the *strong bounded selector property.*

Let $X$ be a separable Banach space. By an *operator* $T$ on $X$ we mean a bounded linear operator $T : X \to X$; $\mathbb{D}$ and $\mathbb{T}$ denote, respectively, the open unit disk and the unit circle in the complex plane $\mathbb{C}$ and $S_X = \{x \in X : \|x\| = 1\}$. The standard notation $\sigma(T)$ is used for the spectrum of $T$. An operator $T$ is said to be

- *topologically transitive* if for any pair of non-empty open subsets $U$, $V$ of $X$, there is $k \in \mathbb{N}$ such that $T^k(U) \cap V \neq \emptyset$;
- *topologically mixing* if for any pair of non-empty open subsets $U$, $V$ of $X$, there is $k_0 \in \mathbb{N}$ such that $T^k(U) \cap V \neq \emptyset$, for all $k \geq k_0$;
- *chaotic* if it is topologically transitive and has a dense set of periodic points;
- *Li-Yorke chaotic* if there is an uncountable set $U \subset X$ such that if $x \neq y \in U$, then $\liminf_{n\to\infty} \|T^n(x) - T^n(y)\| = 0$ and $\limsup_{n\to\infty} \|T^n(x) - T^n(y)\| > 0$;
- *hypercyclic* if it admits a *hypercyclic vector,* i.e. if there exists a vector $x \in X$ whose orbit $Orb(x, T) = \{T^n x, n \in \mathbb{N}\}$ is dense in $X$;

- *frequently hypercyclic* if it admits a *frequently hypercyclic vector,* i.e. if there exists $x \in X$ such that for each non-empty open subset $U$ of $X$, the set of integers $N(x, U) = \{n \in \mathbb{N} : T^n(x) \in U\}$ has positive lower density, where
$$\underline{dens}(N(x,U)) = \liminf_{N\to\infty} \frac{\#\{1 \leq n \leq N : T^n(x) \in U\}}{N} > 0;$$
- *supercyclic* if it admits a *supercyclic vector,* i.e., if there exists a vector $x \in X$ whose projective orbit $\{\lambda T^n x; n \geq 0, \lambda \in \mathbb{C}\}$ is dense in $X$.

When invertible, the operator $T$ is said to be

- *expansive* if for each $x \in S_X$ there exists $n \in \mathbb{Z}$ such that $\|T^n(x)\| \geq 2$;
- *uniformly expansive* if there exists $n \in \mathbb{N}$ such that for each $x \in S_X$ it is $\|T^n(x)\| \geq 2$ or $\|T^{-n}(x)\| \geq 2$;
- with the *shadowing property* if, for each $\epsilon > 0$ there exists $\delta > 0$ such that every $\delta$-pseudotrajectory $\{x_n\}_{n\in\mathbb{Z}}$ of $T$ (i.e. $\|T(x_n) - x_{n+1}\| \leq \delta$ for all $n \in \mathbb{Z}$) is $\epsilon$-shadowed by a $T$-orbit (i.e. $\|T^n(x) - x_n\| < \epsilon,$ for all $n \in \mathbb{Z}$ for some $x \in X$);
- *hyperbolic* if $\sigma(T) \cap \mathbb{T} = \emptyset$;
- *generalized hyperbolic* if $X = M \oplus N$, where $M$ and $N$ are closed subspaces of $X$ with $T(M) \subseteq M$ and $T^{-1}(N) \subseteq N$, and $\sigma(T_{|_M}) \subset \mathbb{D}$ and $\sigma(T^{-1}_{|_N}) \subset \mathbb{D}$;
- *structurally stable* if there exists $\epsilon > 0$ such that $T + \phi$ is topologically conjugate to $T$ whenever $\phi \in C_b(X)$ is a Lipschitz map with $\|\phi\|_\infty \leq \epsilon$ and $Lip(\phi) \leq \epsilon$.

Recall that two operators $T : X \to X$ and $S : Y \to Y$ are said to be *topologically semi-conjugate* if there exists a linear, bounded, surjective map $\Pi : X \to Y$, called *factor map,* for which $S \circ \Pi = \Pi \circ T$. In such case, $(Y, S)$ is called a *factor* of $(X, T)$, while $(X, T)$ is an *extension* of $(Y, S)$. In particular, if $\Pi$ is a homeomorphism, the two dynamical systems are *topologically conjugate.* By the Open Mapping Theorem we have that
$$\exists L > 0 \text{ s.t.}, \forall y \in Y, \exists x \in \Pi^{-1}(y) \text{ with } \|x\| \leq L\|y\|.$$
Moreover, $\Pi$ is said to admit a *strong bounded selector* if
$$\exists L > 0 \text{ s.t.}, \forall y \in Y, \exists x \in \Pi^{-1}(y) \text{ with } \frac{1}{L}\|S^n(y)\| \leq \|T^n(x)\| \leq L\|S^n(y)\|, \forall n \in \mathbb{Z}.$$
**Lemma**([3, 4]): *Given two semi-conjugate dynamical systems $(X, T)$ and $(Y, S)$:*

- *if $T$ is hypercyclic, mixing, chaotic, freq. hypercyclic, supercyclic, then so is $S$;*
- *if $T$ has the shadowing property, then so does $S$.*

*Moreover, if $\Pi$ admits the strong bounded selector, then*

- *if $T$ is expansive, uniformly expansive, hyperbolic, then so is $S$;*
- *if $S$ is Li-Yorke chaotic, then so is $T$.*

We now present some results from [1, 2]. Let $(X, \mathcal{B}, \mu)$ be a $\sigma$-finite measure space and let $f : X \to X$ be a bijective bimeasurable transformation satisfying
$$(\star) \qquad \exists c > 0 \quad : \quad \mu(f^{-1}(B)) \leq c\mu(B) \ \text{ for every } B \in \mathcal{B}.$$
For $1 \leq p < \infty$ the *composition operator induced by $f$* is $T_f$ acting on $L^p(X, \mathcal{B}, \mu)$ as $\varphi \mapsto \varphi \circ f$. We assume that also $f^{-1}$ satisfies condition $(\star)$, so that $T_f$ is a

well-defined, bounded, invertible, linear operator. As introduced in [1], and using the well-known Hopf Decomposition Theorem, $T_f$ is said to be a

- *dissipative composition operator* if $X = \dot{\bigcup}_{k\in\mathbb{Z}} f^k(W)$ for some $W \in \mathcal{B}$, $0 < \mu(W) < \infty$. In this case, the set $W$ is called *wandering set*;
- *conservative composition operator* if for each measurable set $B \subseteq X$ with $\mu(B) > 0$, there is $n > 0$ such that $\mu(B \cap f^{-n}(B)) > 0$.

Moreover, a dissipative composition operator $T_f$ is said to be of *bounded distortion on $W$* if there exists $K > 0$ such that

$$(\Diamond) \qquad \frac{1}{K}\mu(f^k(W))\mu(B) \leq \mu(f^k(B))\mu(W) \leq K\mu(f^k(W))\mu(B),$$

for all $k \in \mathbb{Z}$ and $B \in \mathcal{B}(W)$, where $\mathcal{B}(W) = \{B \cap W, B \in \mathcal{B}\}$.

Most of the above mentioned properties are characterized for composition operators, while chaos, frequent hypercyclicity, shadowing and hyperbolicity are well understood and characterized only in the dissipative setting with bounded distortion. We recall that the class of composition operators contains the one of weighted shifts, for which the above mentioned chaotic and hyperbolic properties are completely characterized. Recall that, letting $A = \mathbb{Z}$ or $A = \mathbb{N}$ and $X = \ell^p(A)$, $1 \leq p < \infty$ or $X = c_0(A)$, and given a bounded sequence of scalars $w = \{w_k\}_{k\in A}$, then the *weighted backward shift $B_w$ on $X$* is defined by $B_w(\{x_k\}_{k\in A}) = \{w_{k+1}x_{k+1}\}_{k\in A}$. The shift is called *bilateral* when $A = \mathbb{Z}$, and *unilateral* when $A = \mathbb{N}$. Clearly, a unilateral $B_w$ is not invertible, while in the bilateral case $B_w$ is invertible if and only if $\inf_{n\in\mathbb{Z}} |w_n| > 0$. We have that:

**Lemma** $(*)$. *Let $T_f$ be a dissipative composition operator of bounded distortion, and $W$ being the wandering set. Consider the weighted backward shift $B_w$ on $\ell^p(\mathbb{Z})$ with weights $w_k = \big(\mu(f^{k-1}(W))/\mu(f^k(W))\big)^{\frac{1}{p}}$. Then, $B_w$ is a factor of $T_f$ by a factor map $\Pi$ admitting a strong bounded selector.*

We obtain the following:

**Theorem.** *Let $T_f$ be a dissipative composition operator of bounded distortion, and $W$ being the wandering set. Let $B_w$ be the weighted backward shift on $\ell^p(\mathbb{Z})$ given in Lemma $(*)$. Then, the operator $T_f$ has the Property P if and only if $B_w$ has the Property P, where Property P denotes one of: Li-Yorke chaos; hypercyclicity; topological mixing; chaos; frequent hypercyclicity; expansivity; uniform expansivity; the shadowing property; hyperbolicity.*

One of the main ingredients of the above theorem is the fact that every weighted backward shift $B_w$ is conjugate, by an isometry, to a particular well-selected composition operator (see [1, 2] for more details).

**Open Problems:**

(1) In a dissipative setting without bounded distortion, does the characterization provided in [1] for the shadowing property still hold? Is it still true that a composition operator is generalized hyperbolic if and only if it has the shadowing property?

(2) In the dissipative setting of bounded distortion, does the structural stability have a shift-like behavior?

## References

[1] E. D'Aniello, U. B. Darji, and M. Maiuriello, *Generalized hyperbolicity and shadowing in $L^p$ spaces*, J. Differential Equations **298** (2021), 68–94.
[2] E. D'Aniello, U. B. Darji, M. Maiuriello, *Shift-like operators on $L^p(X)$*, J. Math. Anal. Appl. **515** (2022), 13 pages.
[3] K.-G. Grosse-Erdmann and A. Peris Manguillot, *Linear Chaos*, Springer, 2011.
[4] M. Maiuriello, *Dynamics of Linear Operators*, Aracne Editore, 2022.

## Preserved dynamics: a focus on hypercyclic and supercyclic behaviors

Emma D'Aniello
(joint work with Martina Maiuriello)

We investigate conditions for a composition operator on $L^p$, $1 \leq p < \infty$, to be hypercyclic or $\mathbb{R}$-supercyclic. We introduce a characterization of $\mathbb{R}$-supercyclic composition operators, in parallelism with what is known for hypercyclicity. These results allow us to show the "shift-like" behaviour of composition operators in hypercyclicity and supercyclicity. Let $X$ be a separable Banach space. An *operator* will be a continuous linear map $T : X \to X$. An operator $T$ is said to be:

- *hypercyclic* if it admits a *hypercyclic vector,* i.e. if there exists a vector $x \in X$ whose orbit $Orb(x, T) = \{T^n(x) : n \geq 0\}$ is dense in $X$;
- *topologically transitive* if for any pair of non-empty open subsets $U$, $V$ of $X$, there is $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$.

By the well-known Birkhoff Transitivity Theorem, in the separable context, transitivity and hypercyclicity are the same thing. Moreover, $T$ is said to be:

- *cyclic* if it admits a *cyclic vector*, i.e. if there exists $x \in X$ such that the linear span of its orbit span$\{T^n(x) : n \geq 0\}$ is dense in $X$;
- $\mathbb{C}$-*supercyclic* (or *supercyclic*) if it admits a $\mathbb{C}$-*supercyclic* (or *supercyclic*) *vector*, i.e. if there exists $x \in X$ such that $\{\lambda T^n(x) : n \geq 0, \lambda \in \mathbb{C}\}$ is dense in $X$;
- $\mathbb{R}$-*supercyclic* if it admits an $\mathbb{R}$-*supercyclic vector*, i.e. if there exists $x \in X$ such that $\{\lambda T^n(x) : n \geq 0, \lambda \in \mathbb{R}\}$ is dense in $X$;
- $\mathbb{R}^+$-*supercyclic* if it admits an $\mathbb{R}^+$-*supercyclic vector*, i.e. if there exists $x \in X$ such that $\{\lambda T^n(x) : n \geq 0, \lambda \in \mathbb{R}^+\}$ is dense in $X$;
- $\Gamma$-*supercyclic* for a set $\Gamma \subseteq \mathbb{C}$ (as in [4]) if it admits a $\Gamma$-*supercyclic vector*, i.e. $x \in X$ such that the set $Orb(\Gamma x, T) = \{\lambda T^n(x) : n \geq 0, \lambda \in \Gamma\}$ is dense in $X$.

**Question.** What are the relations among these properties?

**Answer.** As recalled in details in [7], they are the following

Hypercyclicity $\Rightarrow$ $\mathbb{R}^+$-supercyclicity $\Leftrightarrow$ $\mathbb{R}$-supercyclicity $\Rightarrow$ Supercyclicity $\Rightarrow$ Cyclicity

We also have the following analogue to Birkhoff's Transitivity Theorem:

**Theorem** ([2, Theorem 1.12])**.** *Let $T : X \to X$. The following are equivalent:*

(i) *$T$ is $\mathbb{C}$-supercyclic ($\mathbb{R}$-supercyclic);*

(ii) *for each pair of non-empty open subsets $U, V$ of $X$ there exist $n \geq 0$ and $\lambda \in \mathbb{C}$ ($\lambda \in \mathbb{R}$) such that $\lambda T^n(U) \cap V \neq \emptyset$.*

**Theorem** ([7, Theorem 2.3])**.** *Let $T : X \to X$. The following are equivalent:*

(i) *$T$ is $\mathbb{R}^+$-supercyclic;*

(ii) *for each pair of non-empty open subsets $U, V$ of $X$ there exist $n \geq 0$ and $\lambda \in \mathbb{R}^+$ such that $\lambda T^n(U) \cap V \neq \emptyset$.*

Two operators $T : X \to X$ and $S : Y \to Y$ are said to be *topologically semi-conjugate* if there exists a linear, bounded, surjective map $\Pi : X \to Y$, called a *factor map,* for which $S \circ \Pi = \Pi \circ T$. Then $S$ is called a *factor* of $T$. It is well-known that if $T$ is cyclic, hypercyclic, $\mathbb{R}$-supercyclic, $\mathbb{C}$-supercyclic, then so is $S$ (see [2]).

**Question.** What about the other way around? That is, the implication:

$$S \text{ hypercyclic (resp. supercyclic)} \Rightarrow T \text{ hypercyclic (resp. supercyclic)}$$

**Answer.** In some cases, it holds.

Hypercyclicity was studied in [6], jointly with U. B. Darji and M. Maiuriello. Supercyclicity was studied more recently in [7], jointly with M. Maiuriello. As mentioned in Martina Maiuriello's talk (previous talk), if $(X, \mathcal{B}, \mu, f, T_f)$ is a dissipative system of bounded distortion on a set $W$, then we have that $T_f : L^p(X) \to L^p(X)$ is topologically semi-conjugate to $B_w : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$, i.e. $B_w \circ \Gamma = \Gamma \circ T_f$, where $B_w$ is the weighted backward shift with weights

$$w_k = \left( \frac{\mu(f^{k-1}(W))}{\mu(f^k(W))} \right)^{\frac{1}{p}} \quad \text{and} \quad \Gamma(\varphi) := \left\{ \frac{\mu(f^k(W))^{\frac{1}{p}}}{\mu(W)} \int_W \varphi \circ f^k d\mu \right\}_{k \in \mathbb{Z}}$$

and where $\Gamma : L^p(X) \to \ell^p(\mathbb{Z})$ is the factor map (see [5, Lemma 4.2.3]). Hence, by the previous comments, if $T_f$ is hypercyclic (resp. supercyclic) then $B_w$ is hypercyclic (resp. supercyclic). The converse also holds! Let us focus on hypercyclicity first. This is the result:

**Theorem** ([6, Theorem M])**.** *If $B_w$ is hypercyclic then so is $T_f$.*

The main "ingredients" of the proof are [1, Theorem 1.2] (characterization of hypercyclicity for composition operators) and [6, Proposition 5.1] (sufficient condition for hypercyclicity of composition operators with bounded distortion). Hence, $T_f$ is hypercyclic if and only if $B_w$ is hypercyclic, so that we can characterize hypercyclicity of $T_f$ by using the one already known for $B_w$.

**Question.** How about supercyclicity and $\mathbb{R}$-supercyclicity? Is the following true?

$$T_f \text{ is supercyclic (resp. } \mathbb{R}\text{-supercyclic)} \Leftrightarrow B_w \text{ is supercyclic (resp. } \mathbb{R}\text{-supercyclic)}$$

**Answer.** Yes!

In order to prove this, we need some preliminary results:

**Theorem** ([3])**.** *An operator on a complex separable infinite dimensional Banach space $X$ is $\mathbb{R}$-supercyclic if and only if it is $\mathbb{R}^+$-supercyclic.*

**Theorem** ([7, Theorem 3.1]) [Characterization of $\mathbb{R}$-supercyclicity - General case]**.** *Let $(X, \mathcal{B}, \mu, f, T_f)$ be a composition dynamical system. The composition operator*

*$T_f$ is $\mathbb{R}$-supercyclic if and only if for all $\epsilon > 0$, for all $B \in \mathcal{B}$ of finite measure, there exist $B' \subseteq B$, $k \geq 1$ and $\lambda > 0$, such that*

$$\mu(B \setminus B') < \epsilon, \quad \mu(f^k(B')) < \lambda^p \epsilon \text{ and } \mu(f^{-k}(B')) < \lambda^{-p}\epsilon.$$

**Proposition** ([7, Proposition 4.4]) [$\mathbb{R}$-supercyclicity Sufficient Condition]
*Let $(X, \mathcal{B}, \mu, f, T_f)$ be a dissipative composition dynamical system of bounded distortion generated by the set $W$. If, for each $\epsilon > 0$ and for each $N \in \mathbb{N}$, there exists $k \geq 1$, $\lambda \in \mathbb{R}^+$ such that*

$$\mu(f^k(\bigcup_{|j|\leq N} f^j(W))) < \epsilon\lambda^p \quad \text{and} \quad \mu(f^{-k}(\bigcup_{|j|\leq N} f^j(W))) < \epsilon\lambda^{-p}$$

*then $T_f$ is $\mathbb{R}$-supercyclic and, therefore, $\mathbb{C}$-supercyclic.*

The same as for hypercyclicity happens with $\mathbb{R}$- and $\mathbb{C}$-supercyclicity:

**Theorem** ([7, Theorem 4.5]) [$\mathbb{R}$-supercyclicity: shift-like behaviour]
*Let $(X, \mathcal{B}, \mu, f, T_f)$ be a dissipative composition dynamical system of bounded distortion generated by $W$. The composition operator $T_f$ is $\mathbb{R}$-supercyclic if and only if the weighted shift $B_w$ given in (WS) is so.*

**Corollary** ([7, Corollary 4.6]) [$\mathbb{C}$-supercyclicity: shift-like behaviour]
*Let $(X, \mathcal{B}, \mu, f, T_f)$ be a dissipative composition dynamical system of bounded distortion generated by $W$. The composition operator $T_f$ is $\mathbb{C}$-supercyclic if and only if the weighted shift $B_w$ given in (WS) is so.*

We finally get that the following holds:

**Theorem** ([7, Theorem 4.8]) [Characterization of $\mathbb{C}$-supercyclicity - Dissipative case] *Let $(X, \mathcal{B}, \mu, f, T_f)$ be a dissipative composition dynamical system of bounded distortion generated by $W$. The composition operator $T_f$ is $\mathbb{C}$-supercyclic if and only if, for any $q \in \mathbb{N}$,*

$$\liminf_{n\to+\infty} \mu(f^{q-n}(W)) \cdot \mu(f^{q+n}(W)) = 0.$$

As an example, we look at [7, Example 4.9]: Let $X = \mathbb{R}$, $\mathcal{B}$ the collection of the Borel subsets of $\mathbb{R}$, $f(x) = x + 1$ and $W = [0, 1[$. Observe that, for any measure $\mu$ on $\mathbb{R}$ such that $0 < \mu(W) < \infty$, the system $(X, \mathcal{B}, \mu, f)$ is a dissipative system generated by $W$. The measure $\mu$ is a probability measure, the density of which is given by

$$h(x) = \frac{1}{2}e^{-|x|}.$$

By applying the previous theorem, we have that $T_f$ is $\mathbb{C}$-supercyclic.

**Open Problem.** Let $\Gamma \subseteq \mathbb{C}$. As we have seen, when $\Gamma = \{1\}, \mathbb{C}, \mathbb{R}, \mathbb{R}^+$ (and, by the results in [4], for other types of $\Gamma$ as well), we have the equivalence

"$T_f$ is $\Gamma$-supercyclic $\Leftrightarrow$ $B_w$ is $\Gamma$-supercyclic"

What about any $\Gamma \subset \mathbb{C}$?

### References

[1] F. Bayart, U.B. Darji, and B. Pires, *Topological transitivity and mixing of composition operators*, J. Math. Anal. Appl. **465** (2018), 125–139.
[2] F. Bayart and E. Matheron, *Dynamics of linear operators*, Cambridge Tracts in Mathematics, vol. 179, Cambridge University Press, 2009.
[3] T. Bermúdez, A. Bonilla, A. Peris, *$\mathbb{C}$-supercyclic versus $\mathbb{R}$-supercyclic operators*, Archiv der Mathematik **79** (2002), 125–130.
[4] S. Charpentier, R. Ernst, Q. Menet, *$\Gamma$-supercyclicity*, J. Funct. Anal. **270** (2016), 4443–4465.
[5] E. D'Aniello, U.B. Darji, and M. Maiuriello, *Generalized hyperbolicity and shadowing in $L^p$ spaces*, J. Differential Equations **298** (2021), 68–94.
[6] E. D'Aniello, U.B. Darji, and M. Maiuriello, *Shift-like operators on $L^p(X)$*, J. Math. Anal. Appl. **515** (2022), 13 pages.
[7] E. D'Aniello, M. Maiuriello, *$\mathbb{R}$- and $\mathbb{C}$-supercyclicity for some classes of operators*, arXiv preprint (2024), arXiv:2404.04028.


## Some recent aspects of linear dynamics

Nilson C. Bernardes Jr.

A classic theorem in linear dynamics from the 1960s asserts that *every invertible hyperbolic operator on a Banach space is structurally stable.* This result is known as *Hartman's theorem* and is a major tool for proving the celebrated *Grobman-Hartman theorem* on the local linearization of diffeomorphisms around hyperbolic fixed points. It was originally established by Philip Hartman in 1960 for operators on finite-dimensional euclidean spaces. The extension to arbitrary Banach spaces was independently obtained by Jacob Palis and Charles Pugh in the late 1960s, motivated by an argument due to Jürgen Moser.

It was soon realized that the converse of Hartman's theorem is true in the finite-dimensional setting (Joel Robbin, 1972), but whether or not this converse is always true remained open for more than 50 years. This problem was finally settled in a 2021 joint paper of the speaker with A. Messaoudi (see [2]).

In our talk we will give an overview of the solution to this problem and its relationship with the notion of *generalized hyperbolicity* and the *generalized Grobman-Hartman theorem* (see [1]). We will also present a recent result due to the speaker asserting that *every invertible generalized hyperbolic operator on a Banach space is time-dependent stable* (see [3] based on the notions introduced in [4]).

The following basic open problem will be proposed:

The Stability Problem in Linear Dynamics: *Characterize the notion of structural stability for invertible operators on Banach (or Hilbert) spaces.*


### References

[1] N. C. Bernardes Jr. and A. Messaoudi, *A generalized Grobman-Hartman theorem*, Proc. Amer. Math. Soc. **148** (2020), 4351–4360.
[2] N. C. Bernardes Jr. and A. Messaoudi, *Shadowing and structural stability for operators*, Ergodic Theory Dynam. Systems **41** (2021), 961–980.
[3] N. C. Bernardes Jr., *Time-dependent stable operators*, arXiv preprint (2024), arXiv:2404.10410.
[4] J. M. Franks, *Time dependent stable diffeomorphisms*, Invent. Math. **24** (1974), 163–172.

## Typicality of operators admitting a hypercyclic algebra

Clifford Gilmore

(joint work with William Alexandre, Sophie Grivaux)

This talk is concerned with the study of typical properties (in the Baire category sense) of certain classes of continuous linear operators acting on Fréchet algebras, and it is based on [1]. We would like to recall that the investigation of the typicality of operators possessing particular linear dynamical properties was initiated in the monograph by Grivaux, Matheron and Menet [3].

I will begin by introducing the setting of closed balls $\mathcal{B}_M(X)$ of bounded linear operators $T\colon X \to X$ with $\|T\| \leq M$, for $M > 0$. Here $X$ denotes the complex Fréchet algebras $X = \ell_p(\mathbb{N})$, $1 \leq p < +\infty$, or $X = c_0(\mathbb{N})$. When endowed with the topology of pointwise convergence, i.e. the Strong Operator Topology (SOT), the space $(\mathcal{B}_M(X), \texttt{SOT})$ is Polish, which allows us to employ Baire catetory techniques. We say that a property of elements of $X$ is *typical* if the set of all $x \in X$ that possesses the property is comeagre in $X$.

During the talk, I will recall some pertinent results from the area of hypercyclic algebras that will be of use, in particular a well-known and powerful criterion from Bayart, Costa and Papathanasiou [2]. To conclude, I will give an idea of the proof of the following result: whenever $M > 1$, a typical operator in $(\mathcal{B}_M(X), \texttt{SOT})$ admits a hypercyclic algebra.

### References

[1] W. Alexandre, C. Gilmore, and S. Grivaux, *Typicality of Operators on Fréchet Algebras Admitting a Hypercyclic Algebra*, arXiv preprint (2023), arXiv:2312.08321.

[2] F. Bayart, F. Costa Jr., and D. Papathanasiou, *Baire theorem and hypercyclic algebras*, Adv. Math. **376** (2021), Paper No. 107419, 58 pages.

[3] S. Grivaux, E. Matheron, and Q. Menet, *Linear Dynamical Systems on Hilbert Spaces: Typical Properties and Explicit Examples*, Mem. Amer. Math. Soc. **269** (2021).

## Universal Taylor series VS Abel universal functions

Myrto Manolaki

(joint work with Stéphane Charpentier, Konstantinos Maronikolakis)

This talk is concerned with the comparison of the boundary behaviour of holomorphic functions on the unit disc $\mathbb{D}$ and the behaviour of their Taylor polynomials on the boundary. A classical result of this nature is Abel's Limit Theorem, which says that, if the sequence $(S_n(\zeta))$ of Taylor polynomials of a holomorphic function $f$ on $\mathbb{D}$ converges for some $\zeta \in \partial\mathbb{D}$, then the nontangential limit of $f$ at $\zeta$ (denoted by $\operatorname{nt}\lim_{z\to\zeta} f(z)$) exists, and the two limits agree. If we merely know that a subsequence $(S_{n_k}(\zeta))$ converges, no conclusion about the boundary behaviour of $f$ at $\zeta$ may be drawn. Nevertheless, as it is shown in [4], $\lim_{k\to\infty} S_{n_k}(\zeta)$ and $\operatorname{nt}\lim_{z\to\zeta} f(z)$ must agree almost everywhere on the set where they simultaneously exist. As we will see, this result answers a question that naturally arises from a theorem of Beise, Meyrath and Müller in [1] and finds applications in the theory of universality.

The second (and main) part of the talk will focus on the comparison of two classes of holomorphic functions on $\mathbb{D}$ which are universal with respect to dilations and partial sums respectively: the class of Abel universal functions (which possess a chaotic boundary behaviour) and the class of universal Taylor series (which possess a chaotic behaviour on the boundary). In particular, we will discuss some recent results from [2] and [3] about their boundary properties, their local and global growth, their Taylor polynomials outside $\mathbb{D}$, their gap structure, their invariance under composition (from the left and the right) and their algebrability.

## References


[1] H.-P. Beise, T. Meyrath, and J. Müller, *Mixing Taylor shifts and universal Taylor series*, Bull. Lond. Math. Soc. **47** (2015), 136–142.

[2] S. Charpentier, M. Manolaki, and K. Maronikolakis, *Abel universal functions: boundary behaviour and Taylor polynomials*, arXiv preprint (2023), arXiv:2310.05611.

[3] S. Charpentier, M. Manolaki, and K. Maronikolakis, *Invariance of Abel universality under composition and applications*, arXiv preprint (2024), arXiv:2401.02367.

[4] S. J. Gardiner and M. Manolaki, *A convergence theorem for harmonic measures with applications to Taylor series*, Proc. Amer. Math. Soc. **144** (2016), 1109–1117.


## Quasi-rigidity: the recurrence version of weak-mixing


Antoni López-Martínez

(joint work with Sophie Grivaux, Alfred Peris)


Linear Dynamics studies the orbits generated by the iterations of a continuous linear operator $T : X \longrightarrow X$ acting on a (usually infinite-dimensional) topological vector space $X$, and the two most studied properties in this branch of mathematics during the last 40 years have been hypercyclicity and chaos. However, the notion of *recurrence* has recently been considered in this linear context (see [2]), generating a new line of research which is full of natural questions and open problems.

In this talk we focus on the so-called $T \oplus T$*-recurrence problem* originally posed in [2, Question 9.6]: **If $T$ is a recurrent operator, does it follow that $T \oplus T$ is also a recurrent operator?** The respective hypercyclicity problem was a very long-standing question posed by Herrero in 1992 that finally got a solution with the paper of De La Rosa and Read [3].

For our case, following the preprint [4] (joint work with S. Grivaux and A. Peris) and in order to give a complete answer to the recurrence version of question, we cover the next contents:

**1.** We introduce the notion of *quasi-rigidity* and we compare it with that of *weak-mixing*, showing that *quasi-rigidity* behaves for recurrence as *weak-mixing* for hypercyclicity: these properties characterize the recurrence/hypercyclicity of the direct sum operator $\underbrace{T \oplus T \oplus \cdots \oplus T}_{N}$ for every $N \in \mathbb{N}$.

**2.** We construct a recurrent but not quasi-rigid operator by modifying the ideas shown in [1], and solving the already mentioned $T \oplus T$*-recurrence problem*. In particular, given any (separable, infinite-dimensional) Banach space and any $N \in \mathbb{N}$ we can construct a recurrent operator $T \in L(X)$ such that

$$\underbrace{T \oplus T \oplus \cdots \oplus T}_{N} \text{ is recurrent but } \underbrace{T \oplus T \oplus \cdots \oplus T}_{N+1} \text{ is not recurrent.}$$

**3.** We discuss how to use quasi-rigidity to study the linear structure of the set of recurrent vectors (dense lineability [6] and spaceability [5]), and with the presented results we again justify that "quasi-rigidity" is the recurrence version of "weak-mixing". Indeed, for operators acting on Banach spaces it was well-known that a weakly-mixing operator admits a hypercyclic subspace if and only if its essential spectrum intersects the unit disc, and we obtain that a quasi-rigid operator admits a recurrent subspace if and only if its essential spectrum intersects the unit disk. As a consequence we obtain the curious result: *a weakly-mixing operator admits a hypercyclic subspace if and only if it admits a recurrent subspace.*

## References


[1] J.-M. Augé, *Linear Operators with Wild Dynamics*, Proc. Amer. Math. Soc. **140** (2012), 2103–2116.

[2] G. Costakis, A. Manoussos, and I. Parissis, *Recurrent linear operators*, Complex Anal. Oper. Theory **8** (2014), 1601–1643.

[3] M. De La Rosa and C. Read, *A hypercyclic operator whose direct sum $T \oplus T$ is not hypercyclic*, J. Oper. Theory **61** (2009), 369–380.

[4] S. Grivaux, A. López-Martínez, and A. Peris, *Questions in linear recurrence: From the $T \oplus T$-problem to lineability*, arXiv preprint (2022), arXiv.2212.03652.

[5] A. López-Martínez, *Recurrent subspaces in Banach spaces*, Int. Math. Res. Not. IMRN 2024;, rnad321, https://doi.org/10.1093/imrn/rnad321.

[6] A. López-Martínez and Q. Menet, *Two remarks on the set of recurrent vectors*, arXiv preprint (2023), arXiv:2309.07573.


## Frequently recurrent backward shifts


Rodrigo Cardeccia

(joint work with Santiago Muro)


Given a linear operator $T : X \to X$, a vector $x \in X$ is said to be frequently recurrent for $T$ provided that for every non-empty open set $U$ that contains the vector $x$ we have that $\{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density. If the set $\mathrm{FRec}(T)$ of frequently recurrent vectors for $T$ is dense in the space, then we say that the operator $T$ is frequently recurrent.

In [1] it was asked whether every frequently recurrent backward shift is frequently hypercyclic. If the basis is completely bounded, then the problem is simple: every frequently recurrent backward shift is reiteratively hypercyclic and since the basis is completely bounded it is frequently hypercyclic [2]. However, for more general spaces the answer is still unclear.

In this talk we will study a zero-one law for frequently recurrent backward shifts. If a backward shift supports one frequently recurrent vector, then the operator is frequently recurrent.

**Theorem.** *Let $X$ be a Fréchet space with unconditional basis $\{e_n\}_{n\geq 0}$ and suppose that the backward shift map $B : X \to X$ is well defined. The following are equivalent:*

*(1)* $\mathrm{FRec}(B) \neq 0$*;*
*(2)* $\overline{\mathrm{FRec}(B)} = X$*.*

As a consequence of our analysis, we provide a characterization for frequently recurrent backward shifts.

**Theorem.** *Let $X$ be a Fréchet space with unconditional basis $\{e_n\}_{n\geq 0}$, $\{\|\cdot\|_p\}_{p\geq 1}$ an increasing sequence of seminorms defining the topology of $X$, and suppose that the backward shift map $B : X \to X$ is well defined. The following are equivalent:*

*(1) $B$ is frequently recurrent;*
*(2) There is a set $A = \{N_k\}$, with $N_0 = 0$, a sequence of subsets $A_p \subset A$ with positive lower density and $\varepsilon_p \to 0$ such that*
  *(a) for every $p$, $\sum_{n\in A_p} e_{n+p} \in X$ and*
  *(b) for every $q$ and every $m \in A_q$ we have that*

$$\left\| \sum_{p=1}^{\infty} \sum_{n\in A_p, n>m} \sum_{j=0}^{p} e_{n-m+N_j} \right\|_q < \varepsilon_q$$

### References


[1] A. Bonilla, K.-G. Grosse-Erdmann, A. López-Martínez, and A. Peris, *Frequently recurrent operators*, J. Funct. Anal. **283** (2022), 36 pages.

[2] S. Charpentier, K. Grosse-Erdmann, and Q. Menet, *Chaos and frequent hypercyclicity for weighted shifts*, Ergodic Theory Dynam. Systems **41** (2019), 3634–3670.


## Cauchy integrals at the service of hypercyclicity


Jürgen Müller

(joint work with Peter Beise)


### 1. Bergman spaces and Cauchy integrals

Let $\mathbb{C}_\infty$ denote the Riemann sphere and $m_2$ the normalized surface measure on $\mathbb{C}_\infty$. For $1 \leq p < \infty$ and $\Omega \subset \mathbb{C}_\infty$ an open set we write $A^p(\Omega) = A^p(\Omega, m_2)$ for the spherical Bergman space, that is the set of all functions $f \in H(\Omega)$, where $H(\Omega)$ is the space of functions holomorphic in $\Omega$ and vanishing at $\infty$ (if $\infty \in \Omega$), fulfilling

$$\|f\|_p := \left( \int_\Omega |f|^p \, dm_2 \right)^{1/p} < \infty .$$

Then $(A^p(\Omega), \|\cdot\|_p)$ is a Banach space and with the natural scalar product a Hilbert space in the case $p = 2$. Moreover, for compact sets $E \subset \mathbb{C} \setminus \{0\}$ we denote by $M(E)$ the set of complex measures on the Borel subsets of $\mathbb{C}$ with support in $E$. In the sequel, let $\Omega = \mathbb{C}_\infty \setminus E$.

Writing $\gamma(\zeta) = 1/(1-\zeta)$ for $\zeta \neq 1$, the Cauchy integral $C\mu \in H(\Omega)$ of $\mu \in M(E)$ is defined by

$$(C\mu)(z) := \int \gamma(w^{-1}z)\, d\mu(w) = \int \frac{d\mu(w)}{1 - w^{-1}z} = \int \frac{w d\mu(w)}{w - z} \quad (z \in \Omega).$$

In particular, for the Dirac measure $\delta_w$ at $w \in E$ the Cauchy integral is $\gamma(w^{-1}\,\cdot)$. Additionally, we define the absolute Cauchy integral by

$$(C^*\mu)(z) := \int \frac{d|\mu|(w)}{|1 - w^{-1}z|} \quad (z \in \mathbb{C}),$$

where $|\mu|$ denotes the total variation measure of $\mu$. Writing $M_p(E)$ for the set of complex Borel measures on $E$ such that $C^*\mu \in L^p(m_2)$, we have $C\mu \in A^p(\Omega)$ for $\mu \in M_p(E)$. It turns out that $A^p(\Omega)$ is non-trivial if and only if $M_p(E)$ is non-trivial and that $\{C\mu : \mu \in M_p(E)\}$ densely spans $A^p(\Omega)$ (cf. [1, Section 11]).

The Hilbert space $A^2(\Omega)$ is non-trivial if and only if $E$ has positive logarithmic capacity, and then it is infinite-dimensional. In contrast, for $p < 2$ the functions $C\delta_w$ belong to $M_p(E)$ and densely span $A^p(\Omega)$, in other words, the rational functions with simple poles are dense in $A^p(\Omega)$ (cf. [1, Section 11]). Also, in this case $M(E) = M_p(E)$. If $E$ is finite, the spaces $A^p(\Omega)$ are finite dimensional. It might be interesting to study the structure of $A^p(\Omega)$ in the case that $E$ is a countable set having only one limit point.

## 2. Taylor shift on $A^p(\Omega)$

If $0 \in \Omega$, the (backward) Taylor shift $T : A^p(\Omega) \to A^p(\Omega)$ is defined by

$$(Tf)(z) := \begin{cases} (f(z) - f(0))/z, & z \neq 0 \\ f'(0), & z = 0 \end{cases}.$$

In particular, for $|z| < \operatorname{dist}(0, \partial\Omega)$ we have $(Tf)(z) = \sum_{k=0}^{\infty} a_{k+1} z^k$, that is, near 0 the Taylor shift acts as backward shift on the Taylor coefficients. If $\infty \in \Omega$, then $T$ is an isomorphism and, in particular, the Taylor shift on $A^2(\mathbb{C}_\infty \setminus \mathbb{T})$ turns out to be conjugated to the bilateral backward shift on the weighted $\ell^2$-space with weight sequence $((|k| + 1)^{-1})_{k \in \mathbb{Z}}$.

The relevance of Cauchy integrals for the dynamics of the Taylor shift stems from the following basic observation: If $\mu \in M_p(E)$ then

$$TC\mu = CR\mu,$$

where $d(R\mu)(w) = w^{-1}\, d\mu(w)$. In particular, if $p < 2$ then $TC\delta_w = w^{-1} C\delta_w$ for $w \in E$, so that $w^{-1}$ is an eigenvalue. Actually, in this case the spectrum of $T$ and the point spectrum both equal $1/E$.

In this talk we focus on sets $E \subset \mathbb{T}$. In this case the spaces $A^p(\Omega)$ may be seen as $T$-invariant closed subspaces of $A^p(\mathbb{C}_\infty \setminus \mathbb{T})$. In contrast to the case $p < 2$, where due to the sufficient supply of eigenvectors the dynamics of the Taylor shifts are quite well-understood (see [2]), only first partial results are detected for $p \geq 2$. Here, the Taylor shift has no eigenvalues at all. Denoting for $\mu \in M_p(E)$ the closure of $\operatorname{span}\{CR^j\mu : j \in \mathbb{Z}\}$ in $A^p(\Omega)$ by $A^p_\mu$, the spaces $A^p_\mu$ are again $T$-invariant closed subspaces of $A^p(\mathbb{C}_\infty \setminus \mathbb{T})$. We have that:

**Lemma.** *Let $E \subset \mathbb{T}$ be a compact set, $\mu \in M_p(E)$ a Rajchman measure and pick $h \in \operatorname{span}\{CR^j\mu : j \in \mathbb{Z}\}$. Then $T^n h \to 0$ in $(A^p_\mu, \|\cdot\|_p)$ as $n \to \pm\infty$.*

According to Kitai's criterion, this leads to

**Proposition.** *Let $E \subset \mathbb{T}$ be a compact set and let $\mu \in M_p(E)$ be a non-zero Rajchman measure. Then the system $(T, A^p_\mu)$ is topologically mixing.*

If $E \subset \mathbb{T}$ has positive arc-length measure on $E$, then the trace measure $\mu$ on $E$ is a non-zero Rajchman measure in $M_p(E)$ for arbitrary $p$. Hence, $(T, A^p_\mu)$ is topologically mixing for all $p$. It would be interesting to know if for $p \geq 2$ the Taylor shift is always hypercyclic on $A^p_\mu$ and under which conditions on the measure $\mu \in M_p(E)$ the space $A^p_\mu$ equals $A^p(\Omega)$. The same questions may be asked for the closure of

$$\mathbb{C}[T]C\mu = \{P(T)C\mu : P \text{ polynomial}\}$$

in $A^p(\Omega)$, for which $C\mu$ is a cyclic vector by definition.

## References

[1] D. R. Adams and L. I. Hedberg, *Functions Spaces and Potential Theory*, Springer, 1999.

[2] J. Müller and M. Thelen, *Dynamics of the Taylor shift on Bergman spaces*, J. Operator Theory **87** (2022), 25–40.

## Operators of $C$-type

Quentin Menet

An important question in Linear dynamics concerned the link between linear chaos and frequent hypercyclicity. A frequently hypercyclic operator that is not chaotic was obtained by Bayart and Grivaux in 2007 by considering a weighted shift on $c_0$ [1]. However, in order to exhibit a chaotic operator that is not frequently hypercyclic, it was necessary to introduce a new family of chaotic operators. These operators are now called "Operators of $C$-type". The first example of an operator of $C$-type that is chaotic but not frequently hypercyclic was obtained in [4]. Actually it is also possible to prove that this operator is not distributionally chaotic and possesses only countably many unimodular eigenvalues (answering other open questions). In [2], the study of operators of $C$-type has also allowed to highlight the existence on Hilbert spaces of $\mathcal{U}$-frequently hypercyclic operators that are not frequently hypercyclic and of frequently hypercyclic operators that are not ergodic. More surprisingly, these operators can also be used to get some examples of $\mathcal{U}$-frequently hypercyclic invertible operators such that the inverse is not $\mathcal{U}$-frequently hypercyclic [5] and some examples of frequently hypercyclic invertible operators such that the inverse is not frequently hypercyclic [6].

In addition to these operators, other families of operators have emerged in Linear dynamics. We can for instance mention the study of dynamical properties of weighted shifts on trees [3] or of the "diagonal plus shift" operators in [2] (that can be seen as weighted shifts with loops). In this talk, we give another interpretation of operators of $C$-type by using the structure of trees. Indeed we can show that the operators of $C$-type can be seen as weighted shifts on trees with loops and where each weight is not a scalar but an operator. This gives us a large family of operators and we show that under some simple conditions on the loops, we can succeed to control the orbits of sequences with finite support. This control can then be used to get (or not) some dynamical properties as mentioned before.

## References


[1] F. Bayart and S. Grivaux, *Invariant Gaussian measures for operators on Banach spaces and linear dynamics*, Proc. Lond. Math. Soc. **94** (2007), 181–210.

[2] S. Grivaux, E. Matheron and Q. Menet, *Linear dynamical systems on Hilbert spaces: typical properties and explicit examples*, Mem. Amer. Math. Soc. **269** (2021), 147 pages.

[3] K. Grosse-Erdmann and D. Papathanasiou, *Dynamics of weighted shitfs on directed trees*, Indiana Univ. Math. J. **72** (2023), 263–299.

[4] Q. Menet, *Linear chaos and frequent hypercyclicity*, Trans. Amer. Math. Soc. **369** (2017), 4977–4994.

[5] Q. Menet, *Inverse of $\mathcal{U}$-frequently hypercyclic operators*, J. Funct. Anal. **279** (2020), 20 pages.

[6] Q. Menet, *Inverse of frequently hypercyclic operators*, J. Inst. Math. Jussieu **21** (2022), 1867–1886.


## Measure-theoretic (non-)orthogonality for weighted shifts


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(joint work with Sophie Grivaux, Quentin Menet)


The initial motivation for this work was to find a "soft" proof of the following result from [1]: *if $B$ is the canonical backward shift on $\ell_2 = \ell_2(\mathbb{Z}_+, \mathbb{C})$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ have moduli $> 1$, then the operators $\lambda_1 B$ and $\lambda_2 B$ share common frequently hypercyclic vectors.* Even though it is a very special case of a much more general theorem proved in [1], this is a non-trivial result with a rather technical proof.

One could cheat and give a very short proof by applying a result from [3]: assuming as we may that $1 < |\lambda_1| < |\lambda_2|$, the pair $(\lambda_1 B, \lambda_2^2 B^2)$ is *diagonally frequently hypercyclic* because $|\lambda_2^2| > |\lambda_1|$; in particular, $\lambda_1 B$ and $(\lambda_2 B)^2$ share a common frequently hypercyclic vector, and hence the same holds for $\lambda_1 B$ and $\lambda_2 B$. This is really cheating since the result of [3] we are using is itself quite technical.

The kind of "soft" proof we had in mind was rather one using measure theoretic arguments. For example, assume that we were able to find ergodic measures with full support $m_1$, $m_2$ for $\lambda_1 B$, $\lambda_2 B$ such that one of them is absolutely continuous with respect to the other, say $m_1 \ll m_2$. Then, using the pointwise ergodic theorem, we would conclude that $m_1$-almost every $x \in \ell_2$ is frequently hypercyclic for both $\lambda_1 B$ and $\lambda_2 B$. However, this approach is doomed to failure in view of the following lemma. Here and afterwards, if $X$ is a Polish topological vector space and $T \in \mathcal{L}(X)$ is a continuous linear operator on $X$, then we denote by $\mathcal{P}_T(X)$ the set of all $T$-invariant Borel probability measures on $X$.

**Lemma.** *Let $X$ be a Polish topological vector space, and let $T_1, T_2 \in \mathcal{L}(X)$. Assume that there exists no $x \in X$ such that $T_1^{n_k} x \to u \neq 0$ and $T_2^{n_k} x \to v \neq 0$, for some increasing sequence $(n_k) \subset \mathbb{N}$. Then, for any $m_1 \in \mathcal{P}_{T_1}(X)$ and $m_2 \in \mathcal{P}_{T_2}(X)$ such that $m_1(\{0\}) = 0 = m_2(\{0\})$, we have $m_1 \perp m_2$.*

We say that two operators $T_1$ and $T_2$ are *measure-theoretically orthogonal*, and we write $T_1 \perp T_2$, if they satisfy the conclusion of the lemma. With this notation, we immediately get from the lemma that for any operator $T \in \mathcal{L}(X)$, if $\lambda_1, \lambda_2$ are scalars such that $|\lambda_1| \neq |\lambda_2|$, then $\lambda_1 T \perp \lambda_2 T$. The main question we consider in [2] is the following. For any weight sequence $w = (w_n)_{n\geq 1} \in \mathbb{C}^{\mathbb{N}}$, let us denote by $B_w$ the associated weighted backward shift acting on $\ell_p = \ell_p(\mathbb{Z}_+)$, $1 \leq p < \infty$.

Question. *Characterize the pairs of weight sequences $(u, v)$ such that $B_u \perp B_v$.*

We are very far from being able to answer this question; but we do obtain some results related to it. Note that the question is meaningful only for weighted shifts on $\ell_p$ admitting non-trivial invariant measures, *i.e.* invariant measures different from the point mass $\delta_0$.

It can be shown that a weighted shift $B_w$ acting on $\ell_p$ admits such invariant measures if and only if

$$(*) \qquad \sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty;$$

and we will always assume that our weighted shifts satisfy this condition. Incidentally, we do not know if a weighted shift $B_w$ acting on $c_0$ admits non-trivial invariant measures if and only if $|w_1 \cdots w_n| \to \infty$.

Recall that two operators $T_1, T_2 \in \mathcal{L}(X)$ are said to be *similar* if there exists an invertible operator $V : X \to X$ such that $T_2 = V T_1 V^{-1}$.

**Proposition.** *Let $B_u$ and $B_v$ be two weighted shifts on $\ell_p$.*

(a) *If $B_u$ and $B_v$ are not similar and $m_u \in \mathcal{P}_{B_u}(\ell_p)$, $m_v \in \mathcal{P}_{B_v}(\ell_p)$ are such that $m_u(\{e_0^* = 0\}) = 0 = m_v(\{e_0^* = 0\})$, then $m_u \perp m_v$.*
(b) *If $B_u$ and $B_v$ are not similar and $u, v$ are bounded below, then $B_u \perp B_v$.*
(c) *It may happen that $B_u$ and $B_v$ are not similar and yet $B_u \not\perp B_v$.*

In view of this proposition, an obvious question arises: if $B_u$ and $B_v$ are similar, does it follow that they are not orthogonal? When $B_u$ and $B_v$ act on $\ell_2$ and are *unitarily equivalent*, it follows from the next theorem that indeed $B_u \not\perp B_v$ (since in that case $|u_n| = |v_n|$ for all $n \geq 1$).

In what follows, by a "product measure on $\ell_p$", we mean an infinite product probability measure $m = \otimes_{n\geq 0} \mu_n$ on $\mathbb{C}^{\mathbb{Z}_+}$ such that $m(\ell_p) = 1$.

**Theorem 1.** *The following are equivalent for two weighted shifts $B_u$ and $B_v$ acting on $\ell_p$.*

(1) *$B_u$ and $B_v$ admit non-orthogonal invariant product measures $m_u$ and $m_v$ with absolutely continuous factors* (*wrt Lebesgue measure on* $\mathbb{C}$).

(2) *There exists $\kappa > 0$ such that* $\sum_{n=0}^{\infty} \left(1 - \kappa \left|\frac{u_1 \cdots u_n}{v_1 \cdots v_n}\right|\right)^2 < \infty$.

(3) *$B_u$ and $B_v$ admit equivalent Gaussian ergodic product measures with full support.*

Note that if $w$ is a weight sequence satisfying $(*)$, then $B_w$ admits a "canonical" ergodic Gaussian measure with full support, namely

$$m \sim g_0 e_0 + \sum_{n=1}^{\infty} \frac{g_n}{w_1 \cdots w_n} e_n,$$

where $(g_n)$ is an i.i.d. sequence with $g_n \sim \mathcal{N}(0,1) = \frac{1}{\pi} e^{-|z|^2} dxdy$. So, in view of Kakutani's theorem on the equivalence of infinite product measures, it should be reasonably clear that (2) $\Longrightarrow$ (3). Moreover, it is obvious that (3) $\Longrightarrow$ (1); so the "non-trivial" part in Theorem 1 is the implication (1) $\Longrightarrow$ (2).

**Theorem 2.** *Let $B_u$ and $B_v$ acting on $\ell_p$ admit invariant product measures $m_u = \otimes_{n\geq 0} \mu_{n,u}$ and $m_v = \otimes_{n\geq 0} \mu_{n,v}$ such that $m_u \not\perp m_v$.*

(a) *Assume that $\ell_p = \ell_p(\mathbb{Z}_+, \mathbb{R})$, that the weights are positive, and that $\mu_{0,u} = p(t)dt$ and $\mu_{0,v} = q(t)dt$. If $f = \sqrt{p}$ is $\mathcal{C}^1$-smooth except at a finite number of points with $tf'(t) \in L_2(\mathbb{R})$ and if $f$ is* not *continuous, then "one can remove the square in Theorem 1": there exists $\kappa > 0$ such that* $\sum_{n=0}^{\infty} \left(1 - \kappa \frac{u_1 \cdots u_n}{v_1 \cdots v_n}\right) < \infty$.

(b) *If $\mu_{0,u}$ and $\mu_{0,v}$ have non-zero discrete parts and $\mu_{0,u}(\{0\}) = 0 = \mu_{0,v}(\{0\})$, then $\left|\frac{u_1 \cdots u_n}{v_1 \cdots v_n}\right|$ is eventually constant.*

In view of Theorems 1 and 2, it seems plausible that as soon as two weighted shifts $B_u$ and $B_v$ admit non-orthogonal invariant product measures $m_u = \otimes_{n\geq 0} \mu_{n,u}$ and $m_v = \otimes_{n\geq 0} \mu_{n,v}$ such that $\mu_{0,u}(\{0\}) = 0 = \mu_{0,v}(\{0\})$, then they admit equivalent Gaussian ergodic product measures with full support.

## References

[1] S. Charpentier, R. Ernst, M. Mestiri, and A. Mouze, *Common frequent hypercyclicity*, J. Funct. Anal. **283** (2022), paper 109526.

[2] S. Grivaux, Q. Menet, and É Matheron, *Orthogonality of invariant measures for weighted shifts*, arXiv preprint (2023), arXiv:2312.13920.

[3] Ö. Martin, Q. Menet, and Y. Puig, *Disjoint frequently hypercyclic pseudo-shifts*, J. Funct. Anal. **283** (2022), paper 109474.

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