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Arbeitsgemeinschaft: Geometry and Representation Theory around the P=W Conjecture

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Abstract. Given a smooth projective curve C, nonabelian Hodge theory gives a diffeomorphism between two different moduli spaces associated to C. The first is the moduli space of Higgs bundles on C of rank n, which is equipped with the structure of an algebraic completely integrable Hamiltonian system. The second is the character variety of representations of the fundamental group of C into $\mathrm{GL}(n)$. In 2012, de Cataldo, Hausel, and Migliorini [1] proposed the $P = W$ conjecture which identifies the perverse filtration on the cohomology of the Higgs moduli space with the weight filtration on the cohomology of the character variety. Recently, in 2022, two independent proofs of the $P = W$ Conjecture appeared, in work of Maulik & Shen [2] and Hausel, Mellit, Minets & Schiffmann [6]. The aim of the Arbeitsgemeinschaft was to understand the $P = W$ Conjecture and these two recent proofs.

Mathematics Subject Classification (2020): 14D20, 14H60.

Introduction by the Organizers

The workshop *Geometry and Representation theory around the* $P = W$ *Conjecture*, organized by Tamas Hausel (Klosterneuburg), Davesh Maulik (Cambridge), Anton Mellit (Vienna), Olivier Schiffmann (Orsay), and Junliang Shen (New Haven), was well attended with 48 participants with broad geographic representation and a blend of students and researchers with various backgrounds. Following the traditional format of an Arbeitsgemeinschaft, the organizers prepared a detailed program in advance for the participants, who prepared and delivered the lectures. We had a total of 16 talks of one hour each, with ample time for discussions and questions during the rest of the week. On Wednesday afternoon, despite some rainy weather, we took a short excursion to a nearby town for cake. On Thursday evening, Peter Scholze moderated the vote for the next Arbeitsgemeinschaft in the series.

The topic of the workshop was the $P = W$ conjecture and its recent proofs. This conjecture involves two moduli spaces associated to a smooth projective curve C. The first space is the moduli space $\mathcal{M}^n_{Dol}$ of certain Higgs bundles - pairs (E, Φ) of a rank n vector bundle E and a Higgs field $\Phi : E \to E \otimes K_C$. This carries the structure of an algebraic completely integrable system, induced by the Hitchin map

$$\mathcal{M}^n_{Dol} \to \mathcal{A},$$

a proper map to an affine space known as the Hitchin base. The second space is the character variety $\mathcal{M}^n_B$ of representations of the fundamental group of C into GL_n. After suitable twistings, both of these spaces are smooth varieties. While they are far from isomorphic, nonabelian Hodge theory [7, 13] provides a diffeomorphism

$$\mathcal{M}^n_{Dol} \cong \mathcal{M}^n_B, \tag{1}$$

which underlies the change of complex structures in the hyperkähler metric on $\mathcal{M}^n_{Dol}$. In particular, we have an identification of cohomology

$$H^*(\mathcal{M}^n_{Dol}; \mathbb{Q}) \cong H^*(\mathcal{M}^n_B; \mathbb{Q}). \tag{2}$$

Each side of this equality carries a natural filtration. For the Higgs moduli space, the Hitchin map induces a perverse Leray filtration [1]

$$P_0 \subset \cdots \subset P_i \subset \cdots \subset P_k = H^k(\mathcal{M}^n_{Dol}; \mathbb{Q}). \tag{3}$$

For the character variety, Deligne's [5] mixed Hodge structure gives a weight filtration on its cohomology

$$W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(\mathcal{M}^n_B; \mathbb{Q}). \tag{4}$$

In 2012, de Cataldo, Hausel, and Migliorini [3] formulated the $P = W$ conjecture, which proposed that these filtrations essentially coincide. More precisely, after reindexing, we have the

$$P_*(H^*(\mathcal{M}^n_{Dol}, \mathbb{Q})) = W_*(H^*(\mathcal{M}^n_B; \mathbb{Q}))$$

under the isomorphism induced by the non-abelian Hodge theorem.

Earlier results proved this conjecture for $n = 2$ [3] and $g = 2$ [4]. Most recently two complete proofs of the $P = W$ Conjecture appeared in 2022, by Maulik–Shen [9] and by Hausel–Mellit–Minets–Schiffmann [6]. The goal of the Arbeitsgemeinschaft was to understand the $P = W$ Conjecture and these proofs. The talks were structured to begin with an introduction to the $P = W$ conjecture and then to discuss, in parallel, concepts and techniques used in its resolution, concluding with presentations of the proofs in the final lectures.

On Monday, the focus was on introducing the main players in the conjecture as well as some of the necessary background. This included presentations on mixed Hodge structures and the geometry of the character variety as well as presentations on the moduli space of Higgs bundles and the definition and basic properties of the perverse filtration associated to any proper map. The day concluded with the statement of the $P = W$ conjecture.

The next topic featured lectures on the singular cohomology of the two moduli spaces. This included lectures on Markman's proof [8] that tautological classes generate the ring, Shende's proof [12] that tautological classes lie in a specific part of the weight filtration, and Mellit's proof [10] of the curious Hard Lefschetz property for character varieties. Using these results, the $P = W$ conjecture can be reduced to a statement on the Higgs moduli space, regarding the interaction between tautological classes and the perverse filtration. We then had lectures on variants of the Higgs moduli space which appear in the proofs, allowing poles in the Higgs field or parabolic structure at marked points.

After these, we covered techniques from geometric representation theory which appear in the arguments. This included lectures on Springer theory, both traditional finite/affine Springer theory and the more recent global Springer theory of Zhiwei Yun [14], which plays a key role in [9]. We then had two lectures on the cohomological Hall algebra of zero-dimensional sheaves on a surface, following [11] and used in [6]. We first had a lecture defining this object and studying a presentation of this algebra; this was followed by a lecture on applying this construction to give an action of the algebra $\mathcal{H}_2$ on cohomology of the Higgs moduli space.

With this background in place, we concluded the workshop with presentations on both proofs of the $P = W$ conjecture, illustrating how to combine the different techniques presented during the week to yield the final result.

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Arbeitsgemeinschaft: Geometry and Representation Theory around the P=W Conjecture

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Abstracts

Recollections on Hodge theory, examples

Miguel González

We recall the fundamental aspects of Hodge theory in algebraic geometry. The main reference used is [3]. Other important references include [1, 2].

1. Classical Hodge theory

For a smooth complex projective variety X, we can use the existence of a Kähler form $\omega \in H^2(X, \mathbb{C})$, coming from restricting the Fubini–Study form on $\mathbb{P}^n$, and the fact that X has a compact complex manifold structure. Then we have the results of classical Hodge theory [3, Chapters 5 and 6] which are obtained analytically by studying elliptic differential operators on the spaces $\Omega^k(X, \mathbb{C})$ of complex-valued forms:

- On a compact manifold there is a Laplacian $\Delta : \Omega^k(X, \mathbb{C}) \to \Omega^k(X, \mathbb{C})$ which is an order two elliptic differential operator.
- The cohomology classes $H^k(X, \mathbb{C})$ are in bijection with $\ker \Delta$.
- On a Kähler manifold Δ preserves the subspaces $\Omega^{p,q}(X)$ of complex-valued forms of type (p, q).

Combining these, the following result is deduced:

Theorem 1.1. *Let $p + q = k$ be integers and $H^{p,q} \subseteq H^k(X, \mathbb{C})$ be the subset of cohomology classes that can be represented by a complex valued differential form of type (p, q). Then we have the* ***Hodge decomposition:***

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

It satisfies the symmetry $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

This gives rise to a finer invariant of X called the **Hodge numbers** $h^{p,q} = \dim H^{p,q}(X)$. Because of conjugation, they satisfy the symmetry $h^{p,q} = h^{q,p}$.

Example 1.2. We have the following easy examples of hodge numbers.

- Projective spaces $\mathbb{P}^n$. In this case there is only even cohomology, and $H^{2k}(\mathbb{P}^n, \mathbb{C})$ is one-dimensional, generated by ω^k. From the symmetry in the Hodge numbers (or the fact that ω is real, hence of type $(1, 1)$) we deduce that the genereators are of type (k, k) and $h^{p,q} = 0$ for $p \neq q$.
- Projective curves of genus g. Here $h^{0,0} = h^{1,1} = 1$ and, because of $\dim H^1(X, \mathbb{C}) = 2g$, we must have $h^{1,0} = h^{0,1} = g$.

Set $V := H^k(X,\mathbb{C})$. We have an equivalent formulation in terms of filtrations: given the direct sum decomposition above, we get a descending filtration

$$F^pV := \bigoplus_{l\geq p} V^{l,k-l}$$

and the symmetry condition is rephrased as $F^pV \oplus \overline{F^{n+1-p}V} = V$.

From the filtration we recover the decomposition as

$$V^{p,q} = F^pV \cap \overline{F^qV} \simeq F^pV/F^{p+1}V.$$

This structure is called a **Hodge structure of weight** k.

2. Hard Lefschetz theorem

Consider the Lefschetz operator on forms

$$L : \Omega^k(X,\mathbb{C}) \to \Omega^{k+2}(X,\mathbb{C}).$$

given by exterior product with ω. Since ω is closed, it descends to cohomology

$$L : H^k(X,\mathbb{C}) \to H^{k+2}(X,\mathbb{C}).$$

Now consider the finite-dimensional vector space $H := \bigoplus H^k(X,\mathbb{C})$. Using the L_2-adjoint of L, denoted by Λ and defined via the Hodge star operator as $\Lambda := (-1)^{k(n-k)} \star L \star$, H becomes an $\mathfrak{sl}_2$-representation via $\{L, \Lambda, [L,\Lambda]\}$, and the eigenspaces of $[L,\Lambda]$ are the $H^k(X,\mathbb{C})$ with eigenvalue $(k-n)$.

From the structure of $\mathfrak{sl}_2$-representations, the following theorem is deduced.

Theorem 2.1 (Hard Lefschetz theorem)**.** *If the (complex) dimension of X is n, then for every $k \leq n$ the map*

$$L^{n-k} : H^k(X,\mathbb{C}) \to H^{2n-k}(X,\mathbb{C})$$

is an isomorphism.

Since ω is of type $(1,1)$, we have in fact an isomorphism between $H^{p,q}(X)$ and $H^{n-q,n-p}(X)$, which corresponds to another symmetry on the Hodge numbers ($h^{p,q} = h^{n-q,n-p}$). Notice that this theorem also implies that L is injective on $H^i(X,\mathbb{C})$ for $i \leq n/2$, so that the first half of the Betti numbers are non-decreasing.

3. Mixed Hodge theory

When we try to work with a non-projective or non-smooth complex variety the previous results do not hold, since the compactness and the Kähler form are required for classical Hodge theory. For example, we have:

$$H^1(\mathbb{C}^\times,\mathbb{C}) = \left\langle \left[\frac{dz}{z}\right] \right\rangle,$$

which cannot be decomposed in $H^{1,0}$ and $H^{0,1}$: it is one dimensional but $\dim H^{1,0}$ should equal $\dim H^{0,1}$. Moreover, $[\frac{dz}{z}] = [\frac{d\bar{z}}{\bar{z}}]$ so, unlike before, a class can be represented by forms of different (p,q) types.

In order to address this, Deligne [1] introduced an extension to Hodge theory, known as **mixed Hodge theory**, which solves the problem by allowing different Hodge-theoretical weights (other than k) on $H^k(X, \mathbb{C})$.

To give an idea of how this should work on $\mathbb{C}^\times$, write $P = 0$, $Q := \infty$ so that $\mathbb{C}^\times = \mathbb{P}^1 \setminus \{P, Q\}$. Then we have the long exact sequence on relative cohomology:

$$\cdots \to H^1(\mathbb{P}^1, \mathbb{C}^\times) \to H^1(\mathbb{P}^1) \to H^1(\mathbb{C}^\times) \to H^2(\mathbb{P}^1, \mathbb{C}^\times) \to H^2(\mathbb{P}^1) \to \dots$$

Thus, any reasonable definition of weight should imply that $H^1(\mathbb{C}^\times)$ has weight 2, as it is isomorphic to $\ker H^2(\mathbb{P}^1, \mathbb{C}^\times) \to H^2(\mathbb{P}^2)$. Moreover, since it is one-dimensional it should have type $(1, 1)$.

The following is the structure that arises in general.

Theorem 3.1 (Deligne)**.** *The space $V := H^k(U, \mathbb{Q})$ acquires the following natural structure:*

(1) *An ascending filtration, $W_k V$, called the* ***weight filtration****, and*
(2) *a descending filtration on $V_\mathbb{C} := V \otimes \mathbb{C} = H^k(U, \mathbb{C})$, denoted $F^p V_\mathbb{C}$, called the* ***Hodge filtration****,*

such that F^p induces a Hodge structure of weight k on each graded piece $\mathrm{Gr}_k^W V_\mathbb{C} = W_k V_\mathbb{C} / W_{k-1} V_\mathbb{C}$.

These structures are called **mixed Hodge structures**, and they form an abelian category. For example, kernels and cokernels of morphisms of mixed Hodge structures (i.e. preserving the filtrations) have an induced mixed Hodge structure.

We then have the following invariants:

- The **mixed Hodge numbers** $h^{p,q;k}(X) := \dim \mathrm{Gr}_p^F \mathrm{Gr}_{p+q}^W H^k(X, \mathbb{C})$.
- The **mixed Hodge polynomial** $H_X(x, y, t) := \sum h^{p,q;k}(X) x^p y^q t^k$.

We now list some useful properties for computations.

Proposition 3.2. *The following hold:*

- *Given a morphism $f : X \to Y$, the map $f^* : H^*(Y) \to H^*(X)$ preserves the mixed Hodge structures.*
- *The cup product $H^k(X) \times H^l(X) \to H^{k+l}(X)$ respects the mixed Hodge structures.*
- *The Künneth isomorphism $H^\bullet(X \times Y) \simeq H^\bullet(X) \otimes H^\bullet(Y)$ respects the mixed Hodge structures.*

Example 3.3. As mentioned before, for $\mathbb{C}^\times$ we have that $H^0(\mathbb{C}^\times)$ is one dimensional of type $(0, 0)$ and $H^1(\mathbb{C}^\times)$ is one dimensional of type $(1, 1)$. The mixed Hodge polynomial is

$$H_{\mathbb{C}^\times}(x, y, t) = 1 + xyt.$$

We can work out the example of $(\mathbb{C}^\times)^a \times \mathbb{C}^b$ using the Künneth formula property. Clearly the mixed Hodge polynomial of $\mathbb{C}$ is $H_\mathbb{C}(x, y, t) = 1$, so that the mixed Hodge polynomial of $U := (\mathbb{C}^\times)^a \times \mathbb{C}^b$ is

$$H_U(x, y, t) = (1 + xyt)^a.$$

This is a polynomial in xyt so the nontrivial elements in k-th cohomology have type (k,k). If the coordinates are $(z_1,\dots,z_a,w_1,\dots,w_b)$, they are the ones of the form $\frac{dz_{i_1}}{z_{i_1}}\wedge\cdots\wedge\frac{dz_{i_k}}{z_{i_k}}$, and there are $\binom{a}{k}$ of them.

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Recollections on perverse filtrations, examples

Matthew Huynh

1. Perverse sheaves

The reference for this section is [1]. Let X be a separated scheme over $\mathbb{C}$ of finite type (i.e. a variety), and let $D(X)=D^b_c(X^{\mathrm{an}},\mathbb{Q})$ be the bounded derived category of $\mathbb{Q}_X$-modules with constructible cohomology sheaves. Recall that $D(X)$ is a triangulated category equipped with a six-functor formalism, and that the category of perverse sheaves on X is the heart of the perverse t-structure $({}^pD^{\leq 0}(X), {}^pD^{\geq 0}(X))$, where

$$K\in {}^pD^{\leq 0}(X) \iff \dim\operatorname{supp}\mathcal{H}^i(K)\leq -i,\quad \forall i\in\mathbb{Z},$$
$$K\in {}^pD^{\geq 0}(X) \iff \dim\operatorname{supp}\mathcal{H}^i(K^\vee)\leq -i,\quad \forall i\in\mathbb{Z}.$$

(In the formula above, $K^\vee$ means the Verdier dual of the complex $K\in D(X)$). The theory of t-structures on triangulated categories produces perverse truncation functors ${}^p\tau_{\leq i}$, ${}^p\tau_{\geq i}$, and a perverse cohomological functor ${}^p\mathcal{H}^0$.

For ease of exposition, assume from now on that X is smooth and irreducible. Note that under these assumptions, $\mathbb{Q}_X[\dim X]$ is a perverse sheaf on X.

Theorem 1.1 (Decomposition Theorem, [1, Théorème 6.2.5])**.** *Let X be as above, and let Y be another variety. Let $f: X\to Y$ be a proper morphism of algebraic varieties. Then there exists a (non-canonical) isomorphism in $D(Y)$:*

$$f_*\mathbb{Q}_X[\dim X]\simeq\bigoplus_{i\in\mathbb{Z}}({}^p\mathcal{H}^i f_*\mathbb{Q}_X[\dim X])[-i].$$

Moreover, the perverse sheaves ${}^p\mathcal{H}^i f_*\mathbb{Q}_X$ are semisimple.

The perverse cohomology sheaves of $f_*\mathbb{Q}_X[\dim X]$ have a symmetry that we describe next. First, recall that any cohomology class $\gamma\in H^\ell_{\mathrm{sing}}(X,\mathbb{Q})$ induces a morphism $\gamma:\mathbb{Q}_X\to\mathbb{Q}_X[\ell]$ in $D(X)$.

Theorem 1.2 (Relative Hard Lefschetz Theorem, [1, Théorème 6.2.10])**.** *Let X, Y be varieties with X smooth and irreducible, let $f : X \to Y$ be a projective morphism, and let $\eta \in H^2(X, \mathbb{Q})$ be the first Chern class of a relatively-ample line bundle on X. Then for each $i \in \mathbb{Z}_{\geq 0}$, the morphism*

$$ {}^p\mathcal{H}^0 \circ f_* \circ \eta^i : {}^p\mathcal{H}^{-i} f_*\mathbb{Q}_X[\dim X] \to {}^p\mathcal{H}^i f_*\mathbb{Q}_X[\dim X], $$

is an isomorphism.

2. The perverse (Leray) filtration

Let X be as above, let Y be another variety, and let $f : X \to Y$ be a morphism between them. Then we have a diagram

$$ \cdots \to {}^p\tau_{\leq i} f_*\mathbb{Q}_X \to {}^p\tau_{\leq i+1} f_*\mathbb{Q}_X \to \cdots \to f_*\mathbb{Q}_X. $$

By taking cohomology, we obtain the following

Definition 2.1. With the above notation, the perverse (Leray) filtration on $H^*(X, \mathbb{Q})$ is defined by

$$ P_i H^*(X, \mathbb{Q}) = \operatorname{Im}(H^*(Y, {}^p\tau_{\leq i} f_*\mathbb{Q}_X) \to H^*(Y, f_*\mathbb{Q}_X) = H^*(X, \mathbb{Q})). $$

The following theorem gives us a way to compute the perverse filtration in certain situations.

Theorem 2.2 ([2, Theorem 4.1.3])**.** *Let X be a smooth, irreducible variety, let Y be an affine variety, and let $f : X \to Y$ be a morphism between them. Then any generic (full) flag of linear sections of Y, denoted $Y = Y_0 \supset Y_1 \supset \cdots \supset Y_{\dim Y} \supset Y_{\dim Y+1} = \varnothing$, satisfies*

$$ P_i H^*(X, \mathbb{Q}) = \ker(H^*(X, \mathbb{Q}) \to H^*(f^{-1}(Y_{i-*+1}), \mathbb{Q})). $$

3. Strong perversity of a cohomology class

The reference for this section is [4, §1]. For simplicity, let us assume further that $f : X \to Y$ has equidimensional fibers, with $\dim X = 2 \dim Y$. We normalize the perverse filtration so that it starts at step 0:

$$ \begin{aligned} &P_i H^*(X, \mathbb{Q}) = \\ &\qquad \operatorname{Im}(H^{*-\dim Y}(Y, {}^p\tau_{\leq i}(f_*\mathbb{Q}_X[\dim Y])) \to H^{*-\dim Y}(Y, f_*\mathbb{Q}_X[\dim Y])). \end{aligned} $$

The notion of strong perversity helps us understand how the perverse filtration behaves with respect to cupping with a cohomology class.

Definition 3.1. Let c be a non-negative integer, and let $\gamma \in H^\ell(X, \mathbb{Q})$. We say γ has strong perversity c if for all $i \in \mathbb{Z}$, the composition

$$ {}^p\tau_i(f_*\mathbb{Q}_X) \to {}^p\tau_{\leq i}(f_*\mathbb{Q}_X[\ell]) = ({}^p\tau_{i+\ell} f_*\mathbb{Q}_X)[\ell] \to f_*\mathbb{Q}_X[\ell], $$

factors through ${}^p\tau_{i+(c-\ell)}(f_*\mathbb{Q}_X[\ell]) = ({}^p\tau_{i+c} f_*\mathbb{Q}_X)[\ell] \to f_*\mathbb{Q}_X[\ell]$.

Note that any cohomology class of degree ℓ automatically has strong perversity ℓ. Furthermore, by taking cohomology, we see that if γ has strong perversity c, then cupping with γ sends $P_iH^*(X,\mathbb{Q})$ to $P_{i+c}H^{*+\ell}(X,\mathbb{Q})$. A simple computation shows that if $\gamma_1,\dots,\gamma_s$ have strong perversities $c_1,\dots,c_s$ respectively, then $\gamma_1 \cup \dots \cup \gamma_s$ has strong perversity $\sum_{j=1}^s c_j$.

The notion of strong perversity behaves well with respect to the vanishing cycles functor. Before stating the proposition, we introduce some more notation. Let $g : X \to \mathbb{A}^1$ be a morphism, and let $X_0 = g^{-1}(0)$, yielding the vanishing cycles functor $\varphi_g : D(X) \to D(X_0)$. Let $X' = \operatorname{supp}(\varphi_g(\mathbb{Q}_X[\dim X])) \subset X_0$, and assume that $\varphi_g(\mathbb{Q}_X[\dim X]) = \mathbb{Q}_{X'}[\dim X']$. Finally, assume that g fits into the following commutative diagram,

$$\begin{tikzcd} X' \arrow[r, "i"] \arrow[d, "f'"] & X \arrow[d, "f"] \arrow[dr, "g"] & \\ Y' \arrow[r] & Y \arrow[r, "\nu"] & \mathbb{A}^1, \end{tikzcd}$$

where f is proper, and $Y' = \operatorname{supp}(\varphi_\nu(\mathbb{Q}_Y[\dim Y]))$.

Proposition 3.2 ([4, Proposition 1.4]). *We use the notation from the paragraph above. Suppose that $\gamma \in H^\ell(X,\mathbb{Q})$ has strong perversity c with respect to the morphism $f : X \to Y$. Then the class $i^*\gamma \in H^\ell(X',\mathbb{Q})$ has strong perversity c with respect to the morphism $f' : X' \to Y'$.*

4. Lefschetz structures

The reference for this section is [3].

Definition 4.1. A Lefschetz structure is the data of

(1) a finite-dimensional rational vector space V,
(2) an increasing filtration $P_\bullet V$ on V,
(3) and a linear endomorphism $\omega : V \to V$,

such that for all $i \in \mathbb{Z}$,

(1) $\omega(P_iV) \subset P_{i+2}V$,
(2) and $\omega^i : P_{-i}V/P_{-i-1}V \to P_iV/P_{i-1}V$ is an isomorphism.

A morphism of Lefschetz structures is a vector space morphism that is compatible with the filtrations and intertwines the endomorphisms.

The data of a Lefschetz structure $(V, P_\bullet, \omega)$ gives rise to a finite-dimensional $\mathfrak{sl}_2$-representation on $\mathrm{Gr}V_{\mathbb{C}} = \oplus_i P_iV_{\mathbb{C}}/P_{i-1}V_{\mathbb{C}}$ by defining the action of H on Gr_iV as multiplication by i, defining the action of X on $\mathrm{Gr}V$ by ω, and defining the action of Y by the relations $[X,Y] = H, [H,X] = 2X, [H,Y] = -2Y$. This assignment is functorial, and in fact we have the following

Proposition 4.2 ([3, Proposition 8.3]). *The category of Lefschetz structures is abelian, and the functor from the category of Lefschetz structures to the category of finite-dimensional $\mathfrak{sl}_2$-representations is exact and faithful.*

The Decomposition Theorem 1.1 and Relative Hard Lefschetz Theorem 1.2 together imply that if X is a smooth, irreducible variety, Y is another variety, $f : X \to Y$ is projective, and η is the first Chern class of a relatively-ample line bundle on X, then the perverse filtration on $H^{*+\dim X}(X, \mathbb{Q})$ and the cup product with η form a Lefschetz structure.

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The Higgs moduli space, the Hitchin map

Huishi Yu

Let C be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. Let n and d be integers that coprime to each other. Let D be any divisor of C. In this talk, we first introduce the moduli space of stable D-Higgs bundles over C with rank n and degree d. Then we describe this moduli space as the moduli space of stable pure one-dimensional sheaves over the total space of $\mathcal{O}_C(D)$. The supports of these one-dimensional sheaves are spectral curves. We can use them to describe the fiber of the Hitchin map.

We recall that a *D-Higgs bundle* is a pair (E, θ), where E is a vector bundle over C and θ is a D-twisted Higgs field

$$\theta : E \to E \otimes \mathcal{O}_C(D).$$

The D-Higgs bundle (E, θ) is *stable* if

$$\mu(F) < \mu(E) = \frac{\deg E}{\mathrm{rk} E}$$

holds for any non-zero proper θ-stable subbundle F of E. Then the moduli problem of stable Higgs bundles with rank n and degree d has coarse moduli space.

Theorem 1 ([2, 4, 5])**.** *Let the scheme $M^D_{n,d}$ be the moduli space of stable Higgs bundles over C with rank n and degree d. Then it is a normal, quasi-projective variety. This moduli space is called the* Dolbeault moduli space of C. *When D is the canonical divisor K_C, we denote $M_{n,d} = M^D_{n,d}$. In this case $M_{n,d}$ is smooth and of dimension $n^2(2g-2)+2$.*

For a Higgs bundle (E, θ) the Higgs field θ can be regarded as a twisted endomorphism of E. Its characteristic polynomial is

$$\chi_\theta = x^n + s_1 x^{n-1} + \cdots + s_n.$$

Then the *Hitchin map* is defined to be the morphism

$$\begin{aligned} h^D : M^D_{n,d} &\longrightarrow A^D = \bigoplus_{i=1}^{n} H^0(C, \mathcal{O}_C(iD)) \\ (E,\theta) &\longmapsto (s_1, \ldots, s_n). \end{aligned}$$

Theorem 2 ([2, 4, 5])**.** *The Hitchin map is proper.*

When $D = K_C$ deformation theory shows that $T^*N \subset M_{n,d}$, where N is the moduli space of stable vector bundles of rank n degree d over C. The natural symplectic structure on T^*N can extend to $M_{n,d}$. Moreover, Hitchin [3] proved that the Hitchin map is a Lagrangian fibration in this case.

From now on for simplicity, we take $D = K_C$. Let S° be the total space of the cotangent bundle Ω_C of C. Let $S = \mathrm{Proj}(Sym^\bullet(\Omega_C^\vee \oplus \mathcal{O}_C))$ be the natural compactification of S° and $D_\infty = S - S^\circ$. Take N sufficiently large such that $H = \mathcal{O}_S(D_\infty) \otimes \pi^*\mathcal{O}_C(N)$ is an ample divisor of S. Simpson's identification says that stable Higgs bundles over C should be identified with H-Gieseker stable pure 1-dimensional sheaves $\mathcal{E}$ over S satisfying $Supp(\mathcal{E}) \cap D_\infty = \emptyset$.

To prove this we first show that coherent Higgs sheaves over C are identified with coherent sheaves over S with support that doesn't intersect with D_∞. Then we observe that this correspondence preserves support dimension and subsheaves. Thus it identifies pure 1-dimensional sheaves on both sides. A direct calculation also shows that $\mathcal{O}_C(1)$-Gieseker stability is identified with H-Gieseker stability. Finally, slope-stable Higgs bundles over a smooth curve are just $\mathcal{O}_C(1)$-Gieseker stable coherent Higgs bundles of pure dimension 1 and we are done with the proof.

We can conclude the above arguments into a theorem of Simpson.

Theorem 3. *Take $\xi =: n[C] \in H_2(S,\mathbb{Z})$ and $\chi =: d + n(1-g)$. Let $M^H_{\xi,\chi}$ be the moduli space of H-Gieseker stable pure 1-dimensional sheaves $\mathcal{E}$ over S with*

$$[Supp(\mathcal{E})] = \xi, \quad \chi(\mathcal{E}) = \chi,$$

where the Fitting support $[Supp(\mathcal{E})]$ *is the dual element of* $c_1(\mathcal{E})$*. Then there exists an open embedding*

$$M_{n,d} \to M^H_{\xi,\chi}$$

whose image consists of those with

$$Supp(\mathcal{E}) \cap D_\infty = \emptyset.$$

The supporting curves of these 1-dimensional sheaves lead us to the definition of spectral curves. Let $a \in A$ be an element and $x \in f^*\Omega^1_C$ be the tautological section of the bundle. The zero-locus of the section

$$x^n + f^*a_1 \cdot x^{n-1} + \cdots + f^*a_n$$

of $f^*\Omega^n_C$ is a curve we denote by C_a. It is called the *spectral curve* over a. If $h((E\theta)) = a$, then by Cayley–Hamilton theorem we see that $\mathcal{E}$ is set-theoretically supported on C_a.

Following the proof of Simpson's identification, we can prove the theorem below which is often called the BNR correspondence.

Theorem 4 ([1]). *Let $a \in A_{int} = \{ a \in A \mid C_a \text{ is integral} \}$. Then torsion-free rank 1 sheaves over C_a are identified with Higgs bundles (E, θ) satisfying $h(E, \theta) = a$.*

Since torsion-free rank 1 sheaves over an irreducible curve are stable, we get that

$$\bar{J}^{\chi}(C_a) = h^{-1}(a), \quad \forall a \in A_{int}.$$

When $a \in A_{sm} = \{ s \in A \mid C_s \text{ is smooth} \}$, we get

$$J^{\chi}(C_a) = h^{-1}(a), \quad \forall a \in A_{sm}$$

is an abelian variety. When $D = K_C$ or $\deg D > 2g - 2$, the smooth locus is open dense in A. Thus we conclude that general fibers of the Hitchin map are abelian varieties. Moreover the BNR correspondence shows that the smooth locus lies in the image of h, thus h is dominant. Combining this with h is proper yields that the Hitchin map h is surjective.

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The twisted character variety and introduction to the $P = W$ conjecture

Anne Larsen

We introduce the twisted character variety and proceed to state the $P = W$ conjecture, which we verify in the easy cases when rank or genus equals 1. The main reference for this talk is [2].

1. The twisted character variety

Let C be a smooth projective complex curve of genus $g \geq 1$, and fix positive integers n, d with $(n, d) = 1$.

Definition 1.1. The twisted (Betti) character variety $\mathcal{M}^B_{n,d,g}$ is defined to be the affine GIT quotient

$$\mathcal{M}^B_{n,d,g} := \{(A_1, B_1, \dots, A_g, B_g) \in GL_n(\mathbb{C})^{2g} : [A_1, B_1] \cdots [A_g, B_g] = e^{\frac{2\pi i d}{n}} \operatorname{Id}\} /\!\!/ GL_n(\mathbb{C}),$$

where $GL_n(\mathbb{C})$ acts on $GL_n(\mathbb{C})^{2g}$ by simultaneous conjugation in each factor.

We may think of $\mathcal{M}^B_{n,d,g}$ as parametrizing local systems of rank n and degree d on $C \setminus \{pt\}$ with monodromy $e^{2\pi i d/n}\,\mathrm{Id}$ on the loop around the puncture. More generally, one may replace $GL_n(\mathbb{C})$ by other reductive algebraic groups, such as $SL_n(\mathbb{C})$ and $PGL_n(\mathbb{C})$. One can also consider local systems on a multiply punctured curve with prescribed monodromy around each puncture; this leads to the definition of a parabolic character variety.

Theorem 1.2. *If $(n,d) = 1$, the twisted character variety $\mathcal{M}^B_{n,d,g}$ is smooth of dimension $n^2(2g-2)+2$.*

Proof. Let $\zeta := e^{2\pi i d/n}$, and consider the map

$$\mu : GL_n(\mathbb{C})^{2g} \to SL_n(\mathbb{C}), (A_1, B_1, \ldots, A_g, B_g) \mapsto \prod_i [A_i, B_i].$$

One proves that $\zeta\,\mathrm{Id}$ is a regular value of this map and that the action of $GL_n(\mathbb{C})$ on $\mu^{-1}(\zeta\,\mathrm{Id})$ factors through a free action of $PGL_n(\mathbb{C})$. Then $\mu^{-1}(\zeta\,\mathrm{Id})$ is a principal PGL_n-bundle over $\mathcal{M}^B_{n,d,g}$, which is thus smooth. For the regularity and freeness assertions, one relies on the observation that $A_1, \ldots, B_g \in GL_n(\mathbb{C})$ such that $\mu(A_1, \ldots, B_g) = \zeta\,\mathrm{Id}$ must act irreducibly on $\mathbb{C}^n$, as for any subspace V fixed by all the A_i, B_i we would have

$$1 = \prod_i 1 = \det\left(\prod_i [A_i|_V, B_i|_V]\right) = \det(\zeta\,\mathrm{Id}_V) = \zeta^{\dim V}$$

and thus $\dim V = 0$ or n. $\square$

A similar argument can be used to show the smoothness of SL_n twisted character varieties, as well as parabolic varieties with sufficiently general choice of monodromy (see [3] for details). One can then describe the PGL_n character variety as the quotient of the SL_n version by μ_n^{2g}, where μ_n is the group of nth roots of unity, and μ_n^{2g} acts on $SL_n(\mathbb{C})^{2g}$ by multiplication on each factor.

2. The $P = W$ conjecture

We start by recalling the necessary ingredients from the first three talks: first, the mixed Hodge structure on the rational cohomology of any complex variety, and in particular the increasing weight filtration $W_\bullet H^i(\mathcal{M}^B_{n,d,g}, \mathbb{Q})$, where $W_\bullet$ has weights in $[0, 2i]$; second, the moduli space $\mathcal{M}^{\mathrm{Dol}}_{n,d,g}$ of semistable Higgs bundles of rank n and degree d on C; and third, the perverse Leray filtration on $H^*(\mathcal{M}^{\mathrm{Dol}}_{n,d,g}, \mathbb{Q})$ associated to the Hitchin map $\mathcal{M}^{\mathrm{Dol}}_{n,d,g} \to \mathcal{A}$, where $\mathcal{A}$ is the affine Hitchin base. For the purposes of the $P = W$ conjecture, we will take the following definition of the perverse Leray filtration (shifted so as to be graded in degrees matching those of the weight filtration):

Definition 2.1. Let $f : X \to Y$ be a proper map of smooth quasi-projective varieties. We define the perverse filtration

$$P_i^f H^j(X,\mathbb{Q}) := \mathrm{Im}(H^{j-d_X+r_f}(Y, {}^{\mathfrak{p}}\tau_{\leq i} Rf_*\mathbb{Q}_X[d_X - r_f]) \to H^j(X,\mathbb{Q})),$$

where $d_X := \dim X$ and $r_f := \dim X \times_Y X - \dim X$ is the defect of semismallness.

By counting points over finite fields, Hausel and Rodriguez-Villegas [2] were able to describe the weight filtration $W_\bullet H^*(\mathcal{M}^B_{n,d,g},\mathbb{Q})$ in some cases. In particular, they noticed a certain symmetry in the dimensions of graded pieces of the weight filtration ("curious Poincaré duality"), and in the case $n = 2$, they were able to prove that this resulted from a "curious Hard Lefschetz" isomorphism induced by cup product with a class $\alpha \in H^2(\mathcal{M}^B_{n,d,g},\mathbb{Q})$. Although the statement in this form does not appear to follow from the Hard Lefschetz theorem, as $\mathcal{M}^B_{n,d,g}$ is affine and α is of weight 4, the connection was clarified in [1] by the following conjecture:

Conjecture (P=W). *Under the non-abelian Hodge diffeomorphism $\mathcal{M}^B_{n,d,g} \cong \mathcal{M}^{Dol}_{n,d,g}$, the weight filtration on $H^*(\mathcal{M}^B_{n,d,g};\mathbb{Q})$ is identified with the perverse filtration on $H^*(\mathcal{M}^{Dol}_{n,d,g};\mathbb{Q})$ by*

$$P_i H^*(\mathcal{M}^{Dol}_{n,d,g},\mathbb{Q}) = W_{2i} H^*(\mathcal{M}^B_{n,d,g},\mathbb{Q}) = W_{2i+1} H^*(\mathcal{M}^B_{n,d,g},\mathbb{Q}).$$

That is, under the $P = W$ conjecture, curious Hard Lefschetz on the Betti side is explained by relative Hard Lefschetz on the Dolbeault side.

Although both the weight and perverse filtrations are difficult to understand in general, here are two cases in which both are as simple as possible:

Example 2.2. If $n = 1$, then on the Betti side we have by definition

$$\mathcal{M}^B_{1,d,g} = GL_1(\mathbb{C})^{2g} = (\mathbb{C}^*)^{2g}$$

and so by compatibility of the mixed Hodge structure with the Künneth isomorphisms and the fact that $H^1(\mathbb{C}^*)$ is of weight 2, we conclude that

$$W_{2i} H^*(\mathcal{M}^B_{1,d,g},\mathbb{Q}) = W_{2i+1} H^*(\mathcal{M}^B_{1,d,g},\mathbb{Q}) = \oplus_{j\leq i} H^j(\mathcal{M}^B_{1,d,g},\mathbb{Q}).$$

On the Dolbeault side, we have the Hitchin fibration

$$\mathcal{M}^{\mathrm{Dol}}_{1,d,g} = \mathrm{Pic}^d(C) \times H^0(C,\omega_C) \to H^0(C,\omega_C)$$

given by projection onto the second factor. Again we get that

$$P_i H^*(\mathcal{M}^{\mathrm{Dol}}_{1,d,g}) = \oplus_{j\leq i} H^j(\mathcal{M}^{\mathrm{Dol}}_{1,d,g},\mathbb{Q}).$$

Example 2.3. If $g = 1$, then on the Betti side one can show that

$$\mathcal{M}^B_{n,d,1} \cong (\mathbb{C}^*)^2, (A_1, B_1) \mapsto (A_1^n = \alpha\,\mathrm{Id}, B_1^n = \beta\,\mathrm{Id})$$

and thus the weight filtration is as described in the previous example. On the Dolbeault side, we note that a Higgs bundle in this case consists of a pair $(\mathcal{E}, \theta \in \mathrm{End}(\mathcal{E}))$, which is semistable as a Higgs bundle exactly when $\mathcal{E}$ is semistable as a vector bundle. Moreover, since $(n, d) = 1$, semistability is equivalent to stability, and so for $\mathcal{E}$ semistable we have that $\mathrm{End}(\mathcal{E}) \cong \mathbb{C}$. In addition, the space of stable vector bundles of rank n and degree d on an elliptic curve C is isomorphic to C (via the determinant map), and thus we get a Hitchin fibration

$$\mathcal{M}^{\mathrm{Dol}}_{n,d,1} \cong C \times \mathbb{C} \to \mathbb{C} \hookrightarrow \oplus_{i=1}^{n} H^0(C, \omega_C^{\otimes i}) = \mathbb{C}^n,$$

where the first map is projection onto the factor $\mathbb{C}$ and the second is the embedding $\mathbb{C} \hookrightarrow \mathbb{C}^n$ given by taking a constant diagonal matrix to its characteristic polynomial. Again the perverse filtration is trivial.

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Tautological classes for the Moduli of Higgs Bundles

Soumik Ghosh

1. Notations and Definitions

Let C be a smooth projective curve of genus $g > 1$ over $\mathbb{C}$. We fix integers r, d such that $\gcd(r, d) = 1$. We denote by $\mathcal{M}^{Dol}_{r,d}$ the Dolbeault moduli space of stable Higgs bundles on C of rank r and degree d. Under the above assumptions, $\mathcal{M}^{Dol}_{r,d}$ is a smooth quasi-projective variety, which we shall denote henceforth by $\mathcal{M}^{Dol}$. Furthermore, we have a universal Higgs bundle $\mathcal{E}$ on $\mathcal{M}^{Dol} \times C$.

Definition 1.1 (Tautological classes)**.** Given an integer $k \geq 0$ and $\gamma \in H^\bullet(C; \mathbb{Q})$, we define the tautological classes $c_k(\gamma) := p_{\mathcal{M}^{Dol},*}[ch_k(\mathcal{E}).p_C^*\gamma] \in H^\bullet(\mathcal{M}^{Dol}; \mathbb{Q})$ where $p_{\mathcal{M}^{Dol}}$ and p_C are the two projections from $\mathcal{M}^{Dol} \times C$ to $\mathcal{M}^{Dol}$ and C respectively.

We shall sketch, following the arguments of Beauville and Markman, the proof of the following theorem due to Markman: (see **Theorem** 7 of [Mar01] or [HT00])

Theorem 1.2. *$H^\bullet(\mathcal{M}^{Dol}; \mathbb{Q})$ is generated by the tautological classes $\{c_k(\gamma) : k \in \mathbb{Z}_{\geq 0}, \gamma \in H^\bullet(C; \mathbb{Q})\}$.*

The key idea is to use appropriate modifications of Beauville's diagonal trick.
Remark: $c_k(\gamma)$'s give the Kunneth components of the Chern characters of the universal Higgs bundle.

2. Beauville's Diagonal Trick

2.1. **Moduli of Stable Vector Bundles on a Curve.** We illustrate Beauville's Diagonal trick, following [Bea92], to find generators of the cohomology ring of $\mathcal{M}(r,d) = \mathcal{M}$ the moduli space of s-6table vector bundles on C of rank r, degree d. In this case $\mathcal{M}$ is a smooth projective variety. Let $\mathcal{E}'$ be a universal bundle on $\mathcal{M} \times C$ and $\mathcal{E}''$ is a universal bundle on $C \times \mathcal{M}$. Consider the diagram

$$\begin{tikzcd} & \mathcal{M}\times C\times\mathcal{M} \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ \mathcal{M}\times C & \mathcal{M}\times\mathcal{M} & C\times\mathcal{M} \end{tikzcd}$$

. Define $\mathcal{E}xt^!_{\pi_{13}}(\pi_{12}^*\mathcal{E}', \pi_{23}^*\mathcal{E}'') = \sum_{i\geq 0}(-1)^i \left[\mathcal{E}xt^i_{\pi_{13}}(\pi_{12}^*\mathcal{E}', \pi_{23}^*\mathcal{E}'')\right] \in K(\mathcal{M}\times\mathcal{M})$, the K-class corresponding the element $R\pi_{13*}R\mathcal{H}om(\pi_{12}^*\mathcal{E}', \pi_{23}^*\mathcal{E}'')$ in the derived category. Then by Grothendieck-Riemann-Roch, we get

$$ch\left(\mathcal{E}xt^!_{\pi_{13}}(\pi_{12}^*\mathcal{E}', \pi_{23}^*\mathcal{E}'')\right) = \pi_{13,*}\left(ch(\pi_{12}^*\mathcal{E}')^\vee ch(\pi_{23}^*\mathcal{E}'')\pi_2^*\operatorname{Td}_C\right) \tag{1}$$

Since C is a curve, $Ext^i_C(E,F) = 0$ for $i \geq 2$ and for two stable vector bundles E, F we have the isomorphism $Ext^0(E,F) \cong \begin{cases}\mathbb{C} \text{ if } E \cong F \\ 0 \text{ if o.w.}\end{cases}$. We have that $R\pi_{13,*}R\mathcal{H}om\left(\pi_{12}^*\mathcal{E}', \pi_{23}^*\mathcal{E}''\right)$ is represented in the derived category by a 2-term complex of vector bundles on $\mathcal{M}\times\mathcal{M}$ given by $\cdots \to 0 \to K^0 \xrightarrow{\theta} K^1 \to 0 \to \cdots$. So at the point $x = (E,F) \in \mathcal{M}\times\mathcal{M}$, we get the exact sequence $0 \to Ext^0_C(E,F) \to K^0(x) \to K^1(x) \to Ext^1_C(E,F) \to 0$. This identifies the diagonal $\Delta_\mathcal{M}$ in $\mathcal{M}\times\mathcal{M}$ (at least set-theoretically but since we are working with $\mathbb{Q}$ coefficients, we can afford to be a little sloppy) as a degeneracy locus of the morphism between vector bundles $\theta : K^0 \to K^1$ of the set of points $x \in \mathcal{M}\times\mathcal{M}$ such that $K^0(x) \xrightarrow{\theta(x)} K^1(x)$ is not injective. We check that the degeneracy locus has expected co-dimension. To this end, we see that $\operatorname{rk} K^1 - \operatorname{rk} K^0 + 1 = ext^1_C(E,E) - ext^0_C(E,E) + 1 = \dim\mathcal{M} = \operatorname{codim}_{\mathcal{M}\times\mathcal{M}}\Delta_\mathcal{M} =: m$. So by Porteous formula, we get

$$[\Delta_M] = c_m(K^1 - K^0) = c_m(-R\pi_{13,*}R\mathcal{H}om\left(\pi_{12}^*\mathcal{E}', \pi_{23}^*\mathcal{E}''\right)) \tag{2}$$

It follows from equations (1) and (2) that $[\Delta] = \sum_i n_i p_1^* B_i \cup p_2^* C_i$ where $B_i, C_i \in$ the ring generated by the Kunneth components of the Chern characters of the universal bundle $\mathcal{E}$ on $\mathcal{M}\times C$ and p_i is the ith projection from $\mathcal{M}\times\mathcal{M}$. This proves the classical result of Atiyah-Bott, see [AB83] and [Bea92]

Theorem 2.1. *With notations as above, the Kunneth components of Chern classes of $\mathcal{E}$, a universal bundle on $\mathcal{M}\times C$ generate $H^\bullet(\mathcal{M};\mathbb{Q})$.*

2.2. **Moduli of Stable Sheaves on a Symplectic Surface.** We now consider the case where S is a projective symplectic surface (so $\omega_S \cong \mathcal{O}_S$), $\mathcal{L}$ is an ample line bundle on S and $\mathcal{M} = \mathcal{M}_\mathcal{L}(r, c_1, c_2)$ is the moduli space of $\mathcal{L}$-stable sheaves on S of rank r, 1st Chern class c_1 and 2nd Chern class c_2. Then by a theorem of Mukai (see [Muk84]), we know $\mathcal{M}$ is smooth quasi-projective and symplectic, so $\dim \mathcal{M} =: m$ is even. We assume $\mathcal{M}$ is complete and say $\mathcal{E}'$ is a universal sheaf on $\mathcal{M} \times S$, $\mathcal{E}''$ is a universal sheaf on $S \times \mathcal{M}$. We consider the analogous diagram

$$\begin{tikzcd} & \mathcal{M} \times S \times \mathcal{M} \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ \mathcal{M} \times S & \mathcal{M} \times \mathcal{M} & S \times \mathcal{M} \end{tikzcd}$$

. Define $\mathcal{E}xt^{!}_{\pi_{13}}(\pi_{12}^{*}\mathcal{E}', \pi_{23}^{*}\mathcal{E}'')$ as before. Then we have the following theorem due to Markman

Theorem 2.2. *With notations as above,* $[\Delta_{\mathcal{M}}] = c_m(-\mathcal{E}xt^{!}_{\pi_{13}}(\pi_{12}^{*}\mathcal{E}', \pi_{23}^{*}\mathcal{E}''))$

As a corollary, we get

Corollary. *With notations as above, the Kunneth components of Chern classes of $\mathcal{E}$, a universal sheaf on $\mathcal{M} \times S$ generate $H^{\bullet}(\mathcal{M}; \mathbb{Q})$.*

Some comments on the proof of Theorem 2.2. Firstly, we observe that for $(E, F) \in \mathcal{M} \times \mathcal{M}$, $Ext^0_S(E, F) \cong \begin{cases} \mathbb{C} \text{ if } E \cong F \\ 0 \text{ o.w.} \end{cases}$, $Ext^2_S(E, F) \cong Ext^0_S(F, E \otimes \omega_S)^{\star} = Ext^0_S(F, E)^{\star} \cong \begin{cases} \mathbb{C} \text{ if } E \cong F \\ 0 \text{ o.w.} \end{cases}$ and $Ext^1_S(E, F) \cong Ext^1_S(F, E \otimes \omega_S)^{\star} \cong Ext^1_S(F, E)^{*}$. Using these calculations, one can show that in the derived category, $R\pi_{13,*}R\mathcal{H}om\,(\pi_{12}^{*}\mathcal{E}', \pi_{23}^{*}\mathcal{E}'')$ is represented by a three term complex of vector bundles $\cdots \to V_{-1} \xrightarrow{g} V_0 \xrightarrow{f} V_1 \to \cdots$ with the i-th cohomology sheaf, $\mathcal{E}xt^{i+1}_{\pi_{13}}(\pi_{12}^{*}\mathcal{E}', \pi_{23}^{*}\mathcal{E}'')$. Furthermore, the dual complex has i-th cohomology sheaf $\mathcal{E}xt^{i+1}_{\pi_{13}}(\pi_{23}^{*}\mathcal{E}'', \pi_{12}^{*}\mathcal{E}')$, g is injective and both $\operatorname{Coker} f$, $\operatorname{Coker} g^{*}$ are supported as line bundles on the diagonal $\Delta_{\mathcal{M}}$. Further, if $r_i = \operatorname{rk} V_i$, we see that $-r_{-1} + r_0 - r_1 = -\chi(E, E) = 2 - m$. One then uses the following technical lemma due to Markman, namely **Lemma** 4 of [Mar01]

Lemma 2.3. *Let $V_{-1} \xrightarrow{g} V_0 \xrightarrow{f} V_1$ be a complex of vector bundles of ranks r_{-1}, r_0, r_1 respectively on a smooth variety M such that g is injective,* $\operatorname{Coker} f$, $\operatorname{Coker} g^{*}$ *are supported as line bundles on a smooth sub-variety Δ of pure co-dimension d and $-r_{-1} + r_0 - r_1 = d - 2$. Then we have* $c_d(V_{\bullet}) = \begin{cases} [\Delta], & m \text{ even} \\ 0, & m \text{ odd} \end{cases}$.

3. Back to Moduli Space of Higgs Bundles

Set $S := \mathbb{P}(T^{*}C \oplus 1)$, $\pi : S \to C$ is the projection. Then there exists an ample line bundle A on S such that $\mathcal{M}^{Dol}$ is an open sub-variety of $\mathcal{M}$ the moduli space of A-stable sheaves of rank 0, 1st Chern class $c_1 = r.c_1[\pi^{*}\omega_C \otimes \mathcal{O}_S(D_{\infty})]$ and Euler characteristic $\chi = d + r(1 - g)$ by the BNR correspondence. So $\mathcal{M}^{Dol}$ consists of those sheaves whose support is disjoint from D_{∞}. Moreover $\mathcal{M}$ can be chosen to be compact. Let $\mathcal{F}_{\mathcal{M}}$ be a universal sheaf on $\mathcal{M} \times S$ and $\mathcal{F}$ is its restriction to $\mathcal{M}^{Dol} \times S$. Note that $\mathcal{F}$ has support in $\mathcal{M}^{Dol} \times T^{*}C$ and $T^{*}C$ is symplectic.

Using this observation and Serre Duality, we see that if $F \in \mathcal{M}^{Dol}, G \in \mathcal{M}$,

$$Ext^i_S(F,G) \cong Ext^{2-i}_S(G,F)^* \tag{3}$$

So the proof of Theorem 2.2.1 goes through and we see that the diagonal in $\mathcal{M} \times \mathcal{M}^{Dol}$ is given by $c_m\left(-\mathcal{E}xt^!_{\pi_{13}}(\pi^*_{12}\mathcal{F}_{\mathcal{M}}, \pi^*_{23}\mathcal{F})\right)$. Note that even though $\mathcal{M}$ is not smooth, the diagonal is contained in the smooth locus so the technical lemma still applies. We refer to [Sim94b], [Sim94a] and [Mar01] for more details.

$\tilde{\mathcal{M}}$ is a smooth compactification of $\mathcal{M}$ together with a morphism $f_{\mathcal{M}} : \tilde{\mathcal{M}} \to \mathcal{M}$ which restricts to identity on $\mathcal{M}^{Dol}$. Using the fact that the cohomology of $\mathcal{M}^{Dol}$ is pure (see [Hei14]), one concludes that $H^\bullet(\tilde{\mathcal{M}};\mathbb{Q}) \to H^\bullet(\mathcal{M}^{Dol};\mathbb{Q})$ is surjective. The Kunneth factors of the diagonal in $\tilde{\mathcal{M}} \times \tilde{\mathcal{M}}$ generate $H^\bullet(\tilde{\mathcal{M}};\mathbb{Q})$ since $\tilde{\mathcal{M}}$ is smooth compact. Therefore the Kunneth factors of the diagonal in $\tilde{\mathcal{M}} \times \mathcal{M}^{Dol}$ generate $H^\bullet(\mathcal{M}^{Dol};\mathbb{Q})$. But the diagonal in $\tilde{\mathcal{M}} \times \mathcal{M}^{Dol}$ is given by $c_m\left[-\mathcal{E}xt^!_{\pi_{13}}(\pi^*_{12}f^*_{\mathcal{M}}\mathcal{F}_{\mathcal{M}}, \pi^*_{23}\mathcal{F})\right]$. It follows that Kunneth components of Chern classes of $\mathcal{F}$ generate $H^\bullet(\mathcal{M}^{Dol};\mathbb{Q})$.

The morphism $\begin{matrix} H^\bullet(S;\mathbb{Q}) \to H^\bullet(\mathcal{M}^{Dol};\mathbb{Q}) \\ \alpha \mapsto \pi_{\mathcal{M}^{Dol},*}[\pi^*_S(\alpha.\,\mathrm{Td}_S).ch(\mathcal{F})] \end{matrix}$ factors through $H^\bullet(T^*C;\mathbb{Q})$ and hence through $H^\bullet(C;\mathbb{Q})$. By the projection formula and Grothendieck-Riemann-Roch, we see that for $\alpha \in H^\bullet(C;\mathbb{Q})$

$$\pi_{\mathcal{M}^{Dol},*}[ch(\mathcal{F}).\pi^*_S(\mathrm{Td}_S).(id \times \pi)^* p^*_C \alpha] = p_{\mathcal{M}^{Dol},*}[ch(\mathcal{E}).p^*_C(\alpha.\,\mathrm{Td}_C)].$$

It follows that the Künneth components of the Chern classes of $\mathcal{E}$ generate the ring $H^\bullet(\mathcal{M}^{Dol};\mathbb{Q})$ which proves Theorem 1.2.

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Weights for the (Betti) tautological classes

Ko Aoki

In this talk, we discuss Shende's result explained in [2].

We fix a (based) curve X of genus $g \geq 1$. We also fix n and d which is coprime to n. We consider the Betti moduli space (aka twisted character variety) M^{B} and the Dolbeault moduli space M^{Dol} as in the previous lectures.

As we have seen in the previous lectures, the $P = W$ conjecture (for GL_n) is a statement about the two filtrations on the ring

$$H^*(M^{\mathrm{B}}; \mathbf{Q}) \simeq H^*(M^{\mathrm{Dol}}; \mathbf{Q}).$$

Note that the right-hand side (and thus the left-hand side) is generated by so-called *tautological classes* by Markman's theorem. Shende's result is about the Betti side of the story so that we omit the superscript B from now on. Shende's theorem identifies the weights of the tautological classes: Namely, it says that the tautological class $c_k(\gamma)$ for $\gamma \in H^i(X)^\vee$ sits inside

$$F^k H^{2k-i}(M_{\mathrm{PGL}}) \cap \bar{F}^k H^{2k-i}(M_{\mathrm{PGL}}) \cap W_{2k} H^{2k-i}(M_{\mathrm{PGL}}).$$

In particular, it says that this twisted character variety is Hodge–Tate. Combined with Mellit's curious hard Lefschetz theorem, the $P = W$ conjecture is reduced to checking that products of tautological classes have the correct perversities.

We discuss the proof of this theorem. Once the machinery of (higher) stacks is set up, the proof is quite straightforward. First, for us, a *stack* means an étale hypersheaf on Aff, the category of affine varieties. We write Stk for the $(\infty, 1)$-category of stacks. Note that in this proof, only 1-truncated stacks are (at least a posteriori) used, but the $(2, 1)$-categorical structure there is still important.

An important stack for us is the *stack of (Betti) local systems* $\mathrm{Loc}_G(Y)$. It is defined for a homotopy type Y and an algebraic group (or any group stack in general) G. It is simply $\mathrm{Hom}(Y, BG)$, where Y denotes the constant stack associated with Y. When Y is the classifying space of a group Γ, it is computed as the stack $\mathrm{Hom}_{\mathrm{grp}}(\Gamma, BG)/G$.

Then we talk about twisted character varieties. Recall that M_{PGL} is defined as a GIT quotient, but there is also a version $\mathcal{M}$ which is obtained by taking the stack quotient. It is a Deligne–Mumford stack and the map to the coarse moduli space M_{PGL} induces an isomorphism on rational cohomology. What is nice about this version is that $\mathcal{M}$ is a connected component of $\mathrm{Loc}_{\mathrm{PGL}_n}(Y)$ where Y is the homotopy type of X.

We then review Deligne's theory. Let MHS be the abelian category of mixed Hodge structures over $\mathbf{Q}$. Essentially by Deligne, we have a functor

$$\mathsf{Aff}^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}(\mathrm{MHS})$$

that sends an affine variety to a complex of mixed Hodge structures whose cohomology computes the rational cohomology of the variety with a mixed Hodge structure. Since this satisfies étale hyperdescent, we have a unique (up to contractible choices) extension

$$\mathsf{Stk}^{\mathrm{op}} \to \operatorname{Ind} \mathrm{D}^{\mathrm{b}}(\mathrm{MHS})$$

that carries colimits to limits.

With all these preparations, now Shende's result simply follows from the computation of the mixed Hodge structure of $B\mathrm{PGL}_n$, which is already done in [1].

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Curious Lefschetz property

Mikhail Gorsky

The goal of this talk is to present the definition of the curious Lefschetz property introduced in [3] and to give an overview of the proof of its validity for the cohomology of character varieties due to A. Mellit [5].

Let X be an algebraic variety of dimension d over $\mathbb{C}$. Recall that the cohomology $H^k(X)$ carries an increasing weight filtration $W_\bullet$ and two decreasing filtrations $F_\bullet, \overline{F}_\bullet$. Consider a class $\omega \in H^2(X)$. It is known that $H^2(X) = W_4 H^2(X)$, which implies that the projection $[\omega]$ of ω to W_4/W_3 belongs to $F_2 \cap \overline{F}_2$.

The cup product with ω acts on the cohomology with contact support as

$$\cup\omega : W_i H_c^j \to W_{i+4} H_c^{j+2}.$$

One says that $\omega \in H^2(X)$ satisfies the *curious Lefschetz property* with middle weight $2d$ if all weights of X are even and for all $k, i \geq 0$, the i-th power of the cup product with ω induces an isomorphism

$$(\cup\omega)^i : W_{2d-2i} H_c^k(X)/W_{2d-2i-i} H_c^k(X) \xrightarrow{\sim} W_{2d+2i} H_c^{k+2i}(X)/W_{2d+2i-1} H_c^{k+2i}(X).$$

For smooth X, this property can also be naturally translated to the setting of ordinary cohomology by means of Poincaré duality.

The notion of the curious Lefschetz property was first introduced and it was conjectured to hold for twisted character varieties of closed Riemann surfaces by T. Hausel and F. Rodriguez-Villegas in [3]. Such varieties fit into a more general class of generic parabolic character varieties, whose basic properties were proved in [2], where the same conjecture was formulated for this larger class. In [5], the curious Lefschetz property was proved for all generic parabolic character varieties.

Let $g \geq 0, n > 0, k > 0$ be integers. Let $C_1, \ldots, C_k \in \mathrm{GL}_n(\mathbb{C})$ be a tuple of diagonal matrices satisfying the *genericity* assumption of [2], i.e. such that $\prod_{j=1}^k \det(C_j) = 1$ and for every $1 \leq r < n$ and every choice of r eigenvalues $\alpha_{j,1}, \ldots, \alpha_{j,r}$ out of n eigenvalues of C_j for each $j = 1, \ldots, k$, we have

$$\prod_{j=1}^{k}\prod_{l=1}^{r}\alpha_{j,l} \neq 1.$$

Given such data, we first define the variety $\mathcal{X}_{\mathrm{par}}$ as follows:

$$\mathcal{X}_{\mathrm{par}} := \left\{A_1,\dots,A_g,B_1,\dots,B_g,\gamma_1,\dots,\gamma_k \in \mathrm{GL}_n(\mathbb{C}) \mid \prod_{i=1}^{g}[A_i,B_i]\prod_{j=1}^{k}(\gamma_j^{-1}C_j\gamma_j) = \mathrm{Id}\right\}.$$

The variety $\mathcal{X}_{\mathrm{par}}$ admits an action of the group $G_{\mathrm{par}} := \mathrm{GL}_n(\mathbb{C} \times Z(C_1) \times \cdots \times Z(C_k)$, and the *parabolic character variety* X_{par} is defined as the GIT quotient

$$X_{\mathrm{par}} := \mathcal{X}_{\mathrm{par}} /\!\!/ G_{\mathrm{par}}.$$

It is proved in [2] that, under the genericty assumption, both varieties $\mathcal{X}_{\mathrm{par}}$ and X_{par} are non-singular, and the map $\mathcal{X}_{\mathrm{par}} \to X_{\mathrm{par}}$ is a principal $(G_{\mathrm{par}}/\mathbb{C}^\times)$-bundle.

Twisted character varieties are special cases of the varieties X_{par}: they correspond to $k = 1, C_1 = e^{\frac{2\pi i}{n}}\mathrm{Id}_n$.

Theorem 1. [5] *For an arbitrary genus $g \geq 0$, any $k \geq 0$, and any collection $C_1,\dots,C_k \in GL_n(\mathbb{C})$ of diagonal matrices satisfying the genericity assumption, the corresponding character variety X_{par} carries a canonical holomorphic symplectic 2-form ω whose class in $H^2(X_{par})$ satisfies the curious Lefschetz property with middle weight equal to the dimension. In particular, the curious Lefschetz property holds for all twisted character varieties.*

The following are the main steps of the proof:

(1) An explicit construction of the 2-form ω is given, adapting techniques and results of [6]. This form agrees with the one in [1].

(2) Assume now that the matrix C_k has distinct eigenvalues. Instead of considering the principal bundle $\mathcal{X}_{\mathrm{par}} \to X_{\mathrm{par}}$, Mellit [5] introduces a fiber bundle $\widetilde{X}_{\mathrm{par}} \xrightarrow{\epsilon} X_{\mathrm{par}}$ with fiber the group U of upper unitriangular matrices (this bundle is thus in particular a homotopy equivalence), where $\widetilde{X}_{\mathrm{par}}$ can be defined in suitable changed coordinates

$$A'_1,\dots,A'_g,B'_1,\dots,B'_g,\gamma'_1,\dots,\gamma'_{k-1} \in \mathrm{GL}_n(\mathbb{C}), u \in U$$

by the matrix equation

$$\prod_{i=1}^{g}[A'_i,B'_i]\prod_{j=1}^{k-1}(\gamma_j^{-1}C_j\gamma'_j)u = C_k^{-1}.$$

(3) Keep the same assumption on C_k, and assume further that $g = 0$ and that the centralizers of all the matrices C_j are formed by block-diagonal matrices. Denote by $W = S_n$ the permutation group and by $W_j \subseteq W$ the stabilizer of C_j. Then one shows that X_{par} carries a stratification $X_{\mathrm{par}} = \sqcup_{\bar{\pi}} X_{\bar{\pi}}$ indexed by tuples $\bar{\pi} = (\pi_1,\dots,\pi_{k-1}) \in W/W_1 \times \cdots \times W/W_{k-1}$.

Further, for each $\bar{\pi}$, we have a diagram

$$\begin{tikzcd}[column sep=small] & \widetilde{X}_{\bar{\pi}} = \epsilon^{-1}(\bar{\pi}) \arrow[dl] \arrow[dr] & \\ X_{\bar{\pi}} & & Y_{\beta(\bar{\pi})}(C_{\bar{\pi}}), \end{tikzcd} \tag{1}$$

where $C_{\bar{\pi}} = C_k^{-1} \prod_{j=1}^{k-1} \pi_j(C_j^{-1})$, the element σ_{π_j} is the positive braid lift of the minimal permutation representing π_i, $\beta(\pi)$ is the product braid $\sigma_{\pi_1}\sigma_{\pi_1^{-1}} \cdots \sigma_{\pi_{k-1}}\sigma_{\pi_{k-1}^{-1}}$. The arrow on the right is a vector bundle with the base $Y_{\beta(\bar{\pi})}$ being a so-called *braid variety*. Pullback of the form ω to $\widetilde{X}_{\bar{\pi}}$ coincides with the pullback of a certain explicit 2-form on $Y_{\beta(\bar{\pi})}$, also denoted by ω.

(4) Each braid variety $Y_{\beta(\bar{\pi})}$ is shown to admit a decomposition (in a certain precise sense) into pieces of the form $(\mathbb{C}^\times)^{2a} \times \mathbb{C}^b$, equivalent to a certain decomposition in [7], such that the restriction of ω to each cell is the pullback of a form which can be written as $\sum_{u \le v} \omega_{uv} d\log x_u \wedge d\log x_u$ in some coordinates on $(\mathbb{C}^\times)^{2a}$. Such coordinates on a torus are called *log-canonical*. In fact, for each cell of this decomposition, there exists a closed surface such that the cocharacter lattice of the torus is isomorphic to its first homology and ω can be identified with the intersection form.

(5) It is verified that (that class of) any 2-form on a torus which admits log-canonical coordinates satisfies the curious Lefschetz property, and the same is true for its pullback onto $(\mathbb{C}^\times)^{2a} \times \mathbb{C}^b$.

(6) The middle weights for ω on different pieces of the decomposition $Y_{\beta(\bar{\pi})}$ all coincide. It is proved that in this case, the existence of a decomposition implies the curious Lefschetz property for the entire $Y_{\beta(\bar{\pi})}$.

(7) By using diagram (1), we transfer this result to obtain the curious Lefschetz property for $[\omega]$ on $X_{\bar{\pi}}$.

(8) The middle weight for ω for each $\bar{\pi}$ agrees with $\dim X_{\mathrm{par}}$. The existence of the stratification then implies the curious Lefschetz property for X_{par}.

(9) By carefully taking the genus contributions into account, the above can be adapted to prove the curious Lefschetz property for X_{par} for arbitrary g, still under the assumption that C_k has distinct eigenvalues.

(10) Assume now that C_k may have eigenvalues with nontrivial multiplicities. By adding one more matrix to the initial data if necessary, we may assume without loss of generality that $C_k = \mathrm{Id}_n$. We further perturb it and consider a family $X_{U_\varepsilon/W}$ over U_ε/W of character varieties given by $C_k = \lambda \in U_\varepsilon$, where

$$U_\varepsilon := \left\{ (\lambda_1, \ldots, \lambda_n) \in T \subset \mathrm{GL}_n \mid \prod_{i=1}^n \lambda_i = 1, \lambda_i \neq \lambda_j, |\lambda_i - 1| \le \varepsilon \right\}$$

for sufficiently small $\varepsilon > 0$. The cohomology spaces of fibers form local systems over U_ε, and so the cohomology of the fiber over a base point x_0

carries an action of $\pi_1(U_\varepsilon/W, x_0)$. It is shown that it comes from a W-action, and the class of the 2-form ω and the weight filtration are invariant under the W-action. Further, this W-action can be identified with the W-action coming from the Grothendieck-Springer sheaf. Together, these results are shown to verify related conjectures from [4] and imply the desired curious Lefschetz property for the variety corresponding to $C_k = \mathrm{Id}_n$. This completes the proof of Theorem 1.

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Moduli of twisted Higgs bundles

Junhui Qin

In this talk we discuss the moduli space of Higgs bundles twisted by an effective divisor with degree bigger than $2g-2$. Even far from being a Lagrangian fibration after twisting, the Hitchin fibration will have good geometry in another viewpoint. For example, the support theorem can be extended to the whole Hitchin base [1]. We will review this theorem and give some applications.

1. The case of GL_n

Let X be a curve with genus g. We fix $(n, d) = 1$. We fix an effective divisor D with $\deg(D) > 2g-2$, and consider several moduli spaces of D-twisted Higgs bundles.

Definition 1.1. A D-twisted Higgs bundle is a vector bundle $\mathcal{E}$ with a morphism $\mathcal{E} \to \mathcal{E}(D)$. Denote $\mathcal{M}^D_{n,d}$ the moduli stack of D-twisted (GL_n)-Higgs bundle with rank n and degree d.

Definition 1.2. The Hitchin base is

$$\mathcal{A}^D_n = \bigoplus_{i=1}^{n} H^0(X, \mathcal{O}(iD)).$$

The Hitchin fibration is by taking the characteristic polynomial

$$h^D\colon \mathcal{M}^D_{n,d} \longrightarrow \mathcal{A}^D_n.$$

Definition 1.3. For a point $a \in \mathcal{A}^D_n$, the spectral curve X_a is the zero locus of

$$u^n + p^*(a_1) + \cdots + p^*(a_n) = 0$$

in $\Sigma = \mathbb{V}(-\mathcal{O}(D))$, where $p\colon \Sigma \to X$. The elliptic (resp. smooth) locus $\mathcal{A}^{D,ell}_n$ (resp. $\mathcal{A}^{D,sm}_n$) is the open subvariety of $\mathcal{A}^D_n$ consisting those a such that X_a is integral (resp. smooth).

Recall from the previous talk, we have:

Theorem 1.4 (Spectral correspondence)**.** *There is a correspondence between the rank* 1 *coherent torsion-free sheaf on X_a (which means $j_{a*}j_a^*\mathcal{F} \cong \mathcal{F}$ for the generic embedding j_a) and D-twisted Higgs bundles of rank n lving over $a \in \mathcal{A}^D_n$. More precisely, there is an isomorphism of stacks between the former of degree e and $h_D^{-1}(a) \subset \mathcal{M}^D_{n,d}$ where $d = e - \frac{n(n-1)}{2}\deg(D)$.*

Over the stable part of $\mathcal{M}^D_{n,d}$, we can take the coarse moduli scheme $M^{D,st}_{n,d}$ which is quasi-projective, smooth and with proper Hitchin fibration $h^D\colon M^{D,st}_{n,d} \to \mathcal{A}^D_n$. We denote $M^{D,ell}_{n,d}$ (resp. $M^{D,ell}_{n,d}$) the restriction on elliptic (resp. smooth) locus.

Some dimension computations by Riemann–Roch theorem:

- $\dim \mathcal{A}^D_n = n(1-g) + \frac{n(n+1)}{2}\deg(D)$.
- $\dim M^D_{n,d} = n^2\deg(D) + 1$.

Now we consider the universal natural component of Picard stack $\mathcal{J}_n$ over $\mathcal{A}^D_n$ which is defined as $\mathcal{J}(X_a)$ over a pointwisely. Denote J_n the associated coarse moduli space. For example, over a smooth point $a \in \mathcal{A}^D_n$, it is the Jacobian of X_a. Every $\mathcal{L} \in \mathcal{J}(X_a)$ acts on rank 1 torsion-free coherent sheaf on X_a, and does not change the degree of the sheaf, hence we obtain an action of $\mathcal{J}_n$ on $\mathcal{M}_{n,d}$ by spectral correspondence. This action preserves the stable part. Moreover, over the smooth locus, this action makes $M^{sm}_{n,d}$ a J^{sm}_n-torsor, hence the cohomology of $h^{D,sm}$ is the same of $p^{D,sm}\colon J^{sm}_n \to \mathcal{A}^{D,sm}_n$. By general theory of abelian variety, we compute

$$R^ih^{D,sm}_*\mathbb{C} \cong \bigwedge^i R^1h^{D,sm}_*\mathbb{C},$$

with $R^1h^{D,sm}_*\mathbb{C}$ a local system of rank $n(g-1) + \frac{n(n-1)}{2}\deg(D) + 1$. We conclude that the cohomologies are local systems over the smooth locus.

Next we state the main theorem of this report, which says that when $\deg(D) > 2g-2$, we can extend Ngô's support theorem [2], to the whole $\mathcal{A}^D_n$ instead of only the elliptic locus.

Theorem 1.5 (Chaudouard–Laumon, [1]). *The complex $Rh_*^{D,st}\mathbb{C}$ is semisimple, and all simple factors of $Rh_*^{D,st}\mathbb{C}[n^2 \deg D + 1]$ have full support. In particular, we can write*
$$Rh_*^{D,st}\mathbb{C} \cong \bigoplus_i \mathrm{IC}_{\mathcal{A}_n^D}(Rf_*^i h^{D,sm}\mathbb{C})[-i].$$

Proof. The semisimplicity is due to Beilinson–Bernstein–Deligne–Gabber decomposition. For having full support, since we have Ngô's result on elliptic locus, we only need to show that there is no generic point of supports of simple factors outside the elliptic locus. We have a stratification of $\mathcal{A}_n^D$ as follows. Define Λ_n the set consisting
$$n_1 \geq n_2 \geq \cdots \geq n_s,\, n = \sum n_i m_i.$$
For every such partition, we have a finite morphism
$$\mathcal{A}_{n_1}^D \times \cdots \times \mathcal{A}_{n_s}^D \longrightarrow \mathcal{A}_n^D$$
by sending characteristic polynomial $(P_1, \cdots, P_s)$ to $P_1^{m_1} \cdots P_s^{m_s}$. Since every polynomial can be factorized uniquely, this is a stratification of $\mathcal{A}_n^D$. Notice $\mathcal{A}_n^{D,ell}$ is the locus defined by $((n), (1))$. Denote η a generic point of support of simple factors, and suppose it to be in the locus of $((n_s), (m_s))$. Take $(a_1, \cdots, a_s)$ lying over a, then a_i lies in the elliptic locus of $\mathcal{A}_{n_i}^D$. Define $n' = n_1 + \cdots + n_s$ and a' the corresponding point in $\mathcal{A}_{n'}^D$.

Some estimation of corresponding commutative group schemes:

- There exists a surjective homomorphism $J_{n,a} \to J_{n',a'}$ with affine kernel.
- There exists a surjective homomorphism $J_{n',a'} \longrightarrow J_{n_1,a_1} \times \cdots \times J_{n_s,a_s}$ with affine kernel.

Therefore, if we decompose $J_{n,\bar{a}}$ as its affine part and abelian variety part (denote their dimensions as d^{ab} and d^{aff}),
$$d^{\mathrm{ab}}(J_{n,a}) = \sum d^{\mathrm{ab}}(J_{n_i,a_i}).$$

Denote d_a the dimension of $\overline{\{a\}}$. Denote d_h the relative dimension of the map h and $d_{\mathcal{A}}$ the dimension of $\mathcal{A}$. We have two crucial inequalities:

- Ngô's result on weak abelian fibration gives
$$d_{h_n^{D,st}} - d_{\mathcal{A}_n^D} + d_a \geq d^{\mathrm{ab}}(J_{n,a}).$$
Here, a weak abelian fibration is in the sense of .
- Severi inequality over elliptic locus
$$d^{\mathrm{aff}}(J_{n_i,a_i}) = \delta_{X_{a_i}} := \mathrm{length}(\mathcal{O}_{\widetilde{X}_{a_i}}/\mathcal{O}_{X_{a_i}}) \leq d_{\mathcal{A}_{n_i}^D} - d_{a_i}$$
Hence $d^{\mathrm{ab}}(J_{n_i,a_i}) \geq d_{h_{n_i}^{D,st}} - d_{\mathcal{A}_{n_i}^D} + d_{a_i}$.

Combining these two, we obtain
$$d_{h_n^{D,st}} - d_{\mathcal{A}_n^D} + d_a \geq \sum_i \left(d_{h_{n_i}^{D,st}} - d_{\mathcal{A}_{n_i}^D} + d_{a_i} \right),$$
i.e.
$$1 - s \geq (n - n_1 - \cdots - n_s)(\deg(D) - 2g + 2).$$

It only happens when $s = 1$ and $n_1 = n$ since $\deg(D) > 2g - 2$. □

Remark 1.6. If $D = \Omega_X$, then the theorem is no longer true. In fact, from [4], we know at least every Levi subgroup of GL_n contributes a support in the above stratification.

2. The case of SL_n and endoscopic decomposition

We briefly discuss the case of SL_n. Fix a line bundle $\mathcal{L}$ of degree d. Let $\Gamma := \mathrm{Pic}^0(X)[n]$. We identify Γ with its dual by Weil pairing and identify $\gamma \in \Gamma$ with κ. We can form the moduli space $M^D_{n,\mathcal{L}}$ of stable D-twisted SL_n-Higgs bundle following [5]. Denote $h^D \colon M^D_{n,\mathcal{L}} \to \mathcal{A}^D$ the associated Hitchin fibration. Γ acts on $M^D_{n,\mathcal{L}}$ and we denote the γ-fix subvariety by M^γ which has Hitchin base $\mathcal{A}^D_\gamma$. Denote $i^D_\gamma \colon \mathcal{A}^D_\gamma \hookrightarrow \mathcal{A}^D$.

Instead of having full support, the support of h^D is more complicated due to the phenomenon of endoscopy. However, we can still extend Ngô's result from elliptic locus to whole Hitchin base when $\deg(D) > 2g - 2$. From this extension, we obtain the following theorem:

Theorem 2.1 (Endoscopic decomposition, [5]). *Denote d_γ the codimension of $\mathcal{A}^D_\gamma$ in $\mathcal{A}^D$. Then we have*

$$(Rh^D_* \mathbb{C})_\kappa \cong i^D_{\gamma *}((Rh^D_{\gamma *}\mathbb{C})_\kappa)[-2d_\gamma].$$

We can even do more. We have the picture

$$\begin{tikzcd}
\mathcal{M}^{D-p}_{n,\mathcal{L}} \arrow[r] \arrow[d] & \mathcal{M}^D_{n,\mathcal{L}} \arrow[r, "ev_p"] \arrow[d] & {[\mathfrak{sl}_n/\,\mathrm{SL}_n]} \arrow[d] \arrow[dr, "f"] & \\
\mathcal{A}^{D-p} \arrow[r] & \mathcal{A}^D \arrow[r, "ev_p"] & \mathfrak{sl}_n/\!/\,\mathrm{SL}_n \arrow[r, "a_2"] & \mathbb{A}^1
\end{tikzcd}$$

and $\mathcal{M}^{D-p}_{n,\mathcal{L}}$ is identified as the critical locus of $f \circ ev_p$ in $\mathcal{M}^D_{n,\mathcal{L}}$. Moreover, by a computation of vanishing cycle functor ϕ_f, we obtain

$$\phi_f(\mathrm{IC}(\mathcal{M}^D_{n,\mathcal{L}})) \cong \mathcal{M}^{D-p}_{n,\mathcal{L}}.$$

The same holds for M^γ. By deducing the case Ω_X from $\Omega_X + p + q$, we have:

Theorem 2.2 (Endoscopic decomposition for $D = \Omega_X$). *Back to the case $D = \Omega_X$, i.e. all Higgs bundle are non-twisted. Then*

$$(Rh_* \mathbb{C})_\kappa \cong i_{\gamma *}((Rh_{\gamma *}\mathbb{C})_\kappa)[-2d_\gamma].$$

Here by h we mean the Hitchin fibration for $D = \Omega_X$. So are i_γ and h_γ.

As explained in [5], this can be strengthened as an enhanced version of topological mirror symmetry.

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Moduli spaces of parabolic Higgs bundles

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Parabolic (Higgs) bundles are (Higgs) bundles with the extra data of a choice of flag in the fiber over some points. They were introduced by Simpson in [2], in order to generalize the nonabelian Hodge correspondence to noncompact curves. Their moduli space was then constructed by Yokogawa in [3]. In this talk, we explain their role in the nonabelian Hodge correspondence, and how they are used in the proof of $P = W$ by [4].

Let C be a smooth projective connected curve over $\mathbb{C}$. Let $p \in C$ be a point. One could more generally replace p by a reduced effective divisor; the present discussion would carry over without any change. We restrict ourselves to consider only *full* flags, but the theory is also available for any partial flag.

1. Parabolic Higgs bundles and the nonabelian Hodge correspondence

Definition 1.1. A parabolic vector bundle on (X, p) is the data of

(i) a vector bundle $\mathcal{E}$ on C of rank n
(ii) a full flag $\mathcal{E}_{|p} = E_1 \supset E_1 \supset \cdots \supset E_n \supset 0$ on the fiber of $\mathcal{E}$ at p
(iii) weights $d \leq \alpha_1 \leq \cdots \leq \alpha_n < d+1$ for some $d \in \mathbb{Z}$

The weights are also called a *stability condition* and sometimes considered to be additional data. As this suggest, they are mainly used to define stability:

Definition 1.2. The *parabolic degree* of a parabolic bundle $(\mathcal{E}, E_\bullet)$ is $\mathrm{pdeg}(\mathcal{E}, \alpha) := \deg \mathcal{E} + \sum_{i=1}^{n} \alpha_i \dim \mathrm{Gr}_i E_\bullet$ The *parabolic slope* is defined for $\mathcal{E} \neq 0$ by $\mathrm{p}\mu(\mathcal{E}, \alpha) = \mathrm{pdeg}(\mathcal{E}, \alpha)/\operatorname{rk} \mathcal{E}$. A parabolic bundle is *stable* (resp. *semistable*) if for any subbundle $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$, one has $\mathrm{p}\mu(\mathcal{F}, \alpha) < \mathrm{p}\mu(E, \alpha)$ (resp. $\mathrm{p}\mu(\mathcal{F}, \alpha) \leq \mathrm{p}\mu(\mathcal{E}, \alpha)$), where $\mathcal{F}$ inherits a filtration and weights from those of $\mathcal{E}$. When α is considered as an extra data, one can write $\mu_\alpha(\mathcal{E}) := \mathrm{p}\mu(\mathcal{E}, \alpha)$ and talk about α-(semi)stability.

Definition 1.3. A parabolic Higgs bundle on (X, p) is a pair $(\mathcal{E}, \theta)$ where $\mathcal{E}$ is a parabolic vector bundle, and $\theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_C(p)$ must respect the filtration in the sense that $\operatorname{Res}_p \theta(E_i) \subset E_i$ for all i. (Semi)stability is defined as for parabolic bundles, by only considering subbundles which are preserved by θ.

Remark 1.4. Often in the literature, the condition $\operatorname{Res}_p \theta(E_i) \subset E_i$ is strengthened into $\operatorname{Res}_p \theta(E_i) \subset E_{i+1}$. We will adopt the convention that this defines a *strongly* parabolic Higgs bundle.

Simpson proves in [2] the following theorem:

Theorem 1.5. *Let α be a stability condition. There is a natural bijection between*

(i) *α-stable strongly parabolic Higgs bundles on C with parabolic degree* 0
(ii) *irreducible local systems on $C \setminus \{p\}$ with monodromy around p semisimple with eigenvalues* $\mathrm{e}^{2i\pi\alpha_j}$.

Actually, Simpson proves a more general statement with arbitray flags and non-strongly parabolic bundles, but it is more difficult to state and involves filtered (=parabolic) local systems.

2. Moduli spaces and the proof of $P = W$

Fix coprime integers $n \geq 1$, $d \in \mathbb{Z}$ and a stability condition α which we assume to be generic (in the sense of [5]; this implies in particular that stability = semistability). Yogokawa construct a moduli space of parabolic Higgs bundles in [3]:

Theorem 2.1. *α-stable parabolic Higgs bundles of rank n and degree d on (C, p) form a smooth quasiprojective variety which we call $\overline{M}_{n,d,p}$. It comes with a Hitchin map $h : \overline{M}_{n,d,p} \to \overline{A} := \bigoplus_i H^0(C, (\Omega^1_C(p))^{\otimes i})$ which is proper.*

We can now define a whole zoo of moduli spaces, which all enter the proof of $P = W$ from [4] :

Definition 2.2. Let:

(i) $\overline{M}^{ell}_{n,d,p}$ be the open subset of $\overline{M}_{n,d,p}$ consisting of points whose underlying Higgs bundle has an integral spectral curve
(ii) $M^{ell}_{n,d,p}$ be the moduli space of rank n, degree d (non-parabolic) p-twisted Higgs bundles whose spectral curve is integral
(iii) $\overline{M}^{nil}_{n,d,p}$ be the closed subset of $\overline{M}_{n,d,p}$ of parabolic Higgs bundles whose Higgs field has nilpotent residue at p
(iv) $\overline{M}_{n,d}$ be the closed subset of $\overline{M}_{n,d,p}$ of parabolic Higgs bundles whose Higgs field has zero residue at p
(v) $M_{n,d}$ be the usual moduli space of rank n, degree d stable Higgs bundles

They fit together in the following diagram:

$$\begin{tikzcd}
\overline{M}^{ell}_{n,d,p} \arrow[r, hook, "\text{open}"'] \arrow[d, "\text{cover}"] & \overline{M}_{n,d,p} & \overline{M}^{nil}_{n,d,p} \arrow[l, hook', "\text{closed}"'] & \overline{M}_{n,d} \arrow[l, hook', "\text{closed}"'] \arrow[d, "\text{fibration}"] \\
M^{ell}_{n,d,p} & & & M_{n,d}
\end{tikzcd}$$

The reason we consider this diagram is that the $\mathcal{H}_2$ action, which is key to the proof of $P = W$ by [4], can only be constructed on the cohomology of $M^{ell}_{n,d,p}$. Then, the statement about perverse degrees of tautological classes must be transferred all the way around this diagram to obtain the $P = W$ statement for $M_{n,d}$. We now explain the following isomorphism :

Theorem 2.3.

$$H^\bullet_{\text{pure}}(\overline{M}^{ell}_{n,d,p}) \simeq H^\bullet(\overline{M}^{nil}_{n,d,p})$$

where by $H^\bullet_{\text{pure}}(X)$ for X a smooth variety we mean the image of the cohomology of a smooth compactification by the restriction map , which coincides with the sum $\bigoplus_n W_n H^n(X)$ where $W_\bullet$ is the weight filtration from mixed Hodge theory.

Observe that $\overline{M}_{n,d,p}$ carries a $\mathbb{C}^*$ action, by scaling θ. The Hitchin map is $\mathbb{C}^*$-equivariant with respect to a contracting (=positive weight) action on $\overline{A}$. In particular the fixed point locus, which is closed in the proper $h^{-1}(0)$, is also proper, and the action can be used to retract the whole $\overline{M}_{n,d,p}$ onto the fixed point locus. They then have isomorphic cohomologies. Hence, $\overline{M}_{n,d,p}$ has the cohomology of a compact variety, and because it is smooth, it has pure cohomology (more precisely, use that for compact M, the weights of $H^k(M)$ are $\leq k$, and for smooth M, the weights of $H^k(M)$ are $\geq k$). More formally, the $\mathbb{C}^*$-action on $\overline{M}_{n,d,p}$ turns it into a semi-projective variety (=smooth, quasiprojective, with a contracting $\mathbb{C}^*$ action with proper fixed point locus) in the sense of [1], where it is proved that a semi-projective variety has pure cohomology.

The key for continuing is the map

$$\chi : \overline{M}_{n,d,p} \to \Big\{ \sum_i \lambda_i = 0 \Big\} \subset \mathbb{C}^n$$

whose value at $(\mathcal{E}, \theta)$ is the list of eigenvalues of $\operatorname{Res}_p \theta$, ordered thanks to the flag.

In [5], a regular Poisson structure is constructed on $\overline{M}_{n,d,p}$, whose symplectic leaves are the fibers of χ, and they prove:

Theorem 2.4. *The map χ is smooth and surjective.*

Moreover, this map is $\mathbb{C}^*$-equivariant for the scaling action of $\mathbb{C}^*$ on $\mathbb{C}^n$. In this situation, it is proved in [1] that every inclusion map $\chi^{-1}(\mu) \hookrightarrow \overline{M}_{n,d,p}$ induces an isomorphism in cohomology - the idea is that the action allows one to construct a nice compactification and apply Erehsmann theorem. Applying this first to $\mu = 0$, we see that $H^\bullet(\overline{M}^{nil}_{n,d,p}) = H^\bullet(\overline{M}_{n,d,p})$. Then, choose μ to be generic, in the sense that $\sum_{i \in I} \mu_i \neq 0$ for $I \neq \{1, \dots, n\}$. This implies that any $(E, \theta) \in \chi^{-1}(\mu)$ has no nontrivial subbundle or quotient, because of the equality $\sum_x \operatorname{tr} \operatorname{Res}_x \theta = 0$. In turn, this implies that the corresponding spectral curve must be integral : any subcurve could be used to produce a quotient bundle. We proved that $\chi^{-1}(\mu) \subset \overline{M}^{ell}_{n,d,p}$. Hence the map $H^\bullet(\overline{M}_{n,d,p}) \to H^\bullet(\overline{M}^{ell}_{n,d,p})$, which becomes an isomorphism after restricting to $\chi^{-1}(\mu)$, must be injective. By using a smooth compactification, we also show that this map surjects on the pure part. The theorem is proven.

References

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Finite and affine Springer fibers

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1. Finite Springer fibers

Let G be a connected reductive group over an algebraically closed field k and $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{B}$ be the flag variety of G which is the G-homogeneous projective variety parametrizing the Borel subgroups of G. If B is a Borel subgroup of G, then $\mathfrak{B}$ can be identified with G/B via the map $gB \to gBg^{-1}$ if $g \in G$. Let $\mathfrak{N} \subset \mathfrak{g}$ be the subvariety of nilpotent elements. Let us define $\widehat{\mathfrak{N}}$ as the set consisting of pairs (e, B) such that $e \in \mathfrak{N}$ and e is contained in the nilpotent radical of Lie$B$. The springer resolution is the projection map

$$\pi : \widehat{\mathfrak{N}} \to \mathfrak{N}$$

where π takes (e, B) to e. For a nilpotent element $e \in \mathfrak{N}$, the springer fiber $\mathfrak{B}_e$ of e is the inverse image of e under the projection map π i.e. the springer fiber of e is the set of Borel subgroups B of G such that e is in the nilpotent radical of Lie$B$.

Example: If $G = SL(V)$ for some vector space of dimension n then $\mathfrak{B}$ is the moduli space of full flags $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = V$ because $SL(V)$ acts transitively on set of full flags and the stablizer of the standard flag is the Borel subgroup consisting of the set of upper triangular matrices. Then for a nilpotent element e, the springer fiber of e consists of those flags such that $eV_i \subset V_{i-1}$.

1.1. **Grothendieck alteration.** Let $\widetilde{\mathfrak{g}}$ be the variety consisting of pairs (x, B) where $x \in \mathfrak{g}$ and $B \in \mathfrak{B}$ such that $x \in$ Lie$B$. The projection map $\pi_{\mathfrak{g}} : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ sending the pair (x, B) to x is called the Grothendieck alternation. For $e \in \mathfrak{N}$, if we denote $\widehat{\mathfrak{B}}_e$ as the inverse image of e under $\pi_{\mathfrak{g}}$, then $\mathfrak{B}_e$ is a subscheme of $\widehat{\mathfrak{B}}_e$ and the two schemes have same reduced structure.

2. Affine Grassmannian and affine flag variety

Let k be an algebraically closed field. If R is a k-algebra, then we denote $R[[\epsilon]]$ the ring of formal power series over R and by $R((\epsilon))$ the ring of Laurent series over R. The affine Grassmannian for G(denoted as Gr_G) is the quotient k-space LG/L^+G where the quotient in the category of k-spaces is the sheafification of the presheaf quotient $R \to LG(R)/L^+G(R)$ where for R a k-algebra, $LG(R) = G(R((\epsilon)))$ and $L^+G = G(R[[\epsilon]])$. For a Borel subgroup B of G, an Iwahori subgroup $I \subset L^+G$ is by definition the inverse image of B under the projection map $L^+G \to G$ taking ϵ to zero. Then the affine flag variety for G is the quotient k-space LG/I.

The $R[[\epsilon]]$-submodule $R[[\epsilon]]^n \subset R((\epsilon))^n$ is called the standard lattice. A lattice $\mathfrak{L} \subset R((\epsilon))^n$ is a projective $R[[\epsilon]]$-submodule such that there exists $N \in \mathbb{Z}_{\geq 0}$ with

$$(\epsilon^N R[[\epsilon]])^n \subset \mathfrak{L} \subset (\epsilon^{-N} R[[\epsilon]])^n$$

The affine Grassmannian can be identified with the set of lattices in $R((\epsilon))^n$ as a k-space.

A lattice chain inside $R((\epsilon))^n$ is a chain

$$\epsilon\mathfrak{L}_0 \subset \mathfrak{L}_{n-1} \subset \mathfrak{L}_{n-2} \subset \cdots \subset \mathfrak{L}_1 \subset \mathfrak{L}_0$$

such that each $\mathfrak{L}_i$ is a lattice in $R((\epsilon))^n$ and such that each quotient $\mathfrak{L}_{i+1}/\mathfrak{L}_i$ is a locally free R-module of rank 1. If R is a k-algebra then the standard lattice chain is defined to be

$$\Lambda_i = \bigoplus_{j=0}^{n-i-1} R[[\epsilon]]e_{j+1} \bigoplus \bigoplus_{j=n-i}^{n-1} \epsilon R[[\epsilon]]e_{j+1}$$

where $e_1, \cdots, e_n$ is the standard basis of $R((\epsilon))^n$. Action of each element of $LG(R)$ on the standard lattice chain gives rise to another lattice chain inside $R((\epsilon))^n$ and the stabilizer of the standard lattice chain under this action is the Iwahori subgroup $I(R) \subset GL_n(R[[\epsilon]])$ from which we can conclude that the affine flag variety for GL_n is the space of lattice chains in $R((\epsilon))^n$.

3. Affine springer fiber

If $F = k((\epsilon))$, then $\mathfrak{g}(F) = \mathfrak{g} \otimes_k F$ is the Lie algebra of the loop group $LG = G(k((\epsilon)))$. For $\gamma \in \mathfrak{g} \otimes_k F$ a regular semisimple element, the affine springer fiber of γ in the affine Grassmannian Gr_G is the underlying reduced ind-scheme $\mathfrak{X}_\gamma$ of $\widehat{\mathfrak{X}}_\gamma$ where $\widehat{\mathfrak{X}}_\gamma$ is a subfunctor of Gr_G whose value on the k-algebra R is

$$\widehat{\mathfrak{X}}_\gamma(R) = \{[g] \in Gr_G(R) | Ad(g^{-1})\gamma \in \mathfrak{g}(R[[\epsilon]])\}$$

where $Ad(g^{-1})$ is its adjoint action on $\mathfrak{g} \otimes_k F$. Alternatively, in terms of lattices if $G = GL_n$ then Gr_G is identified with the moduli space of lattices in $R((\epsilon))^n$ and $\widehat{\mathfrak{X}}_\gamma$ is identified with those lattices $\Lambda \subset R((\epsilon))^n$ such that $\gamma\Lambda \subset \Lambda$ i.e. those lattices that are stable under the endomorphism of $R((\epsilon))^n$ given by γ.

Example: If we consider the case $G = SL_2$ and $\gamma = \begin{bmatrix} 0 & \epsilon^2 \\ \epsilon & 0 \end{bmatrix}$, then $\mathfrak{X}_\gamma$ consists of exactly those lattices $\Lambda \in \mathrm{Gr}_G$ such that $\epsilon k[[\epsilon]] \oplus k[[\epsilon]] \subset \Lambda \subset k[[\epsilon]] \oplus \epsilon^{-1} k[[\epsilon]]$.

Parahoric subgroups of the loop group LG are the connected group subschemes of LG containing an Iwahori subgroup with finite codimension. Conjugacy classes of parabolic subgroups of G are in bijection with the subsets of the Dynkin diagram of G and LG-conjugacy classes of parahoric subgroups of LG are in bijection with proper subsets of the vertices of the extended Dynkin diagram of G. For each parahoric subgroup P there exists a canonical exact sequence of group schemes

$$1 \to \mathbf{P}^+ \to \mathbf{P} \to L_{\mathbf{P}} \to 1$$

where $\mathbf{P}^+$ is the pro-unipotent radical of $\mathbf{P}$ and $L_{\mathbf{P}}$ is a reductive group over k which is the Levi quotient of $\mathbf{P}$. For each parahoric subgroup $\mathbf{P} \subset LG$, the affine partial flag variety $Fl_{\mathbf{P}}$ is defined to be the sheafification of the functor $R \to LG(R)/\mathbf{P}(R)$ in the category of k-algebras. Then the affine Grassmannian and the affine flag variety correspond to the special cases for $\mathbf{P} = L^+G$ and $\mathbf{P} = \mathbf{I}$(an Iwahori subgroup) respectively. If $\mathbf{P} \subset \mathbf{Q}$ are two parahoric subgroups of G, then there exists a natural projection $Fl_{\mathbf{P}} \to Fl_{\mathbf{Q}}$.

For a parahoric subgroup $\mathbf{P} \subset LG$, the closed sub-scheme $\widehat{\mathfrak{X}}_{\mathbf{P},\gamma}$ is defined by

$$\widehat{\mathfrak{X}}_{\gamma}(R) = \{[g] \in Fl_{\mathbf{P}} | Ad(g^{-1})\gamma \in (Lie\mathbf{P})\widehat{\otimes}_k R\}$$

The reduced ind-scheme $\mathfrak{X}_{\mathbf{P},\gamma}$ of $\widehat{\mathfrak{X}}_{\mathbf{P},\gamma}$ is said to be the affine springer fiber of γ of type $\mathbf{P}$. If $\mathbf{P} \subset \mathbf{Q}$ are two parahoric subgroups of LG, then the natural projection map $Fl_{\mathbf{P}} \to Fl_{\mathbf{Q}}$ induces a map $\mathfrak{X}_{\mathbf{P},\gamma} \to \mathfrak{X}_{\mathbf{Q},\gamma}$. In particular, for all parahoric subgroups $\mathbf{P}$, we always have a map $\mathfrak{X}_{\mathbf{I},\gamma} \to \mathfrak{X}_{\mathbf{P},\gamma}$.

4. Action of Weyl group on finite springer fibers

The map $\pi_{\mathfrak{g}} : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is small implies that $\mathbf{R}\pi_{\mathfrak{g},*}\mathbb{Q}_\ell[\dim\mathfrak{g}]$ is a perverse sheaf which is the middle extension of its restriction to any open dense subset of $\mathfrak{g}$. If $\mathfrak{g}^{\mathrm{rs}}$ is the regular semisimple locus of $\mathfrak{g}$, then $\pi_{\mathfrak{g}}$ restricted to $\mathfrak{g}^{\mathrm{rs}}$ is a W-torsor. Hence $\mathbf{R}\pi_{\mathfrak{g},*}\mathbb{Q}_\ell[\dim\mathfrak{g}]|_{\mathfrak{g}^{\mathrm{rs}}}$ is a local system shifted in degree -dim$\mathfrak{g}$ and admits an action of the Weyl group W. By functoriality of middle extension, W also acts on $\mathbf{R}\pi_{\mathfrak{g},*}\mathbb{Q}_\ell[\dim\mathfrak{g}]$. Therefore W acts on stalks of $\mathbf{R}\pi_{\mathfrak{g},*}\mathbb{Q}_\ell[\dim\mathfrak{g}]$ hence acts on $H^*(\widehat{\mathfrak{B}}_x)$ for all $x \in \mathfrak{g}$. For a nilpotent element e, both $\widehat{\mathfrak{B}}_e$ and $\mathfrak{B}_e$ have same reduced structure implies that $H^*(\widehat{\mathfrak{B}}_e) = H^*(\mathfrak{B}_e)$ for which we get an action of W on $H^*(\mathfrak{B}_e)$.

5. Action of affine Weyl group on affine springer fibers

Let us assume that the affine Weyl group is generated by the affine simple reflections $s_0, s_1, \cdots, s_r$. For $\mathbf{P}$ a parahoric subgroup of LG, let $L_{\mathbf{P}}$ be the Levi quotient of $\mathbf{P}$ and $\mathfrak{l}_{\mathbf{P}} = \mathrm{Lie}L_{\mathbf{P}}$ and we define the evaluation map $\mathrm{ev}_{\mathbf{P},\gamma}$ as follows. For $[g] \in Fl_{\mathbf{P}}$ such that $\mathrm{Ad}(g^{-1})\gamma \in \mathrm{Lie}\mathbf{P}$, the coset $[g] = g\mathbf{P}$ is sent to the image of $\mathrm{Ad}(g^{-1})\gamma$ under the projection $\mathrm{Lie}\mathbf{P} \to \mathfrak{l}_{\mathbf{P}}$ and this map is well defined upto the adjoint action of $L_{\mathbf{P}}$. If $\pi_{\mathfrak{l}_{\mathbf{P}}}$ is the Grothendieck alteration for the reductive group $\mathfrak{l}_{\mathbf{P}}$ then we have the following cartesian diagram:

$$\begin{tikzcd} \widehat{\mathfrak{X}}_{\mathbf{I},\gamma} \arrow[r, "\mathrm{ev}_{\mathbf{I},\gamma}"] \arrow[d, "\pi_{\mathbf{P},\gamma}"'] & \tilde{\mathfrak{l}}_{\mathbf{P}}/L_{\mathbf{P}} \arrow[d, "\pi_{\mathfrak{l}_{\mathbf{P}}}"] \\ \widehat{\mathfrak{X}}_{\mathbf{P},\gamma} \arrow[r, "\mathrm{ev}_{\mathbf{P},\gamma}"] & \mathfrak{l}_{\mathbf{P}}/L_{\mathbf{P}} \end{tikzcd}$$

By the action of finite Weyl group on finite springer fibers, we get an action of the Weyl group $W(L_{\mathbf{P}})$ of $L_{\mathbf{P}}$ on the direct image complex $\mathbf{R}\pi_{\mathfrak{l}_{\mathbf{P}},*}\mathbf{D}$ where $\mathbf{D}$ is the dualizing complex for $\tilde{\mathfrak{l}}_{\mathbf{P}}/L_{\mathbf{P}}$. Therefore by proper base change we get an action of $W(L_{\mathbf{P}})$ on $\mathbf{R}\pi_{\mathbf{P},\gamma,*}\mathbf{D}_{\widehat{\mathfrak{X}}_{\mathbf{I},\gamma}}$ and hence on $H_*(\mathfrak{X}_{\mathbf{I},\gamma})$. If we take the standard parahoric subgroup $\mathbf{P}$ corresponding to the i'th node of the extended Dynkin diagram, then $W(L_{\mathbf{P}}) =< s_i >$ and we have s_i acting on $H_*(\mathfrak{X}_{\mathbf{I},\gamma})$. To check the braid relations between s_i and s_{i+1}, we can choose a standard parahoric $\mathbf{P}$ such that $W(L_{\mathbf{P}}) = \langle s_i, s_{i+1}\rangle$ and braid relation holds due to the action of $W(L_{\mathbf{P}})$ on $H_*(\mathfrak{X}_{\mathbf{I},\gamma})$.

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Global Springer Theory

Cheng Shu

The global Springer theory of Yun [3, 4] provides an action of the graded double affine Hecke algebra on the direct image complex for a parabolic version of the Hitchin map. Some ingredients in this theory, notably the Weyl group action and the strong perversity of certain Chern classes, are used in Maulik-Shen's proof [2] of the P=W conjecture [1]. The purpose of this talk is to explain this in some more details.

Let C be a smooth projective algebraic curve over $\mathbb{C}$ with genus $g > 1$. Let L be a line bundle on C with $\deg L > 2g$. We fix coprime integers $n > 0$ and $d > 0$, as well as a line bundle D on C with $\deg D = d$. The smooth fine moduli space of L-twisted Higgs bundles with determinants isomorphic to D is denoted by $\check{M}$. Let Γ denote the subgroup of n-torsion points of the Jacobian variety of C. Then Γ acts on $\check{M}$ by tensor product. Let $\hat{M} := [\check{M}/\Gamma]$ be the quotient stack. It is a connected component of the moduli stack of L-twisted PGL_n-Higgs bundles. The Hitchin base for $\hat{M}$ is the affine space $\hat{A} := \oplus_{i=2}^n H^0(C, L^{\otimes i})$. The Hitchin map $\hat{h} : \hat{M} \to \hat{A}$ is a proper, flat and surjective.

We fix a Borel subgroup $B \subset G = \mathrm{PGL}_n$. Denote by $\hat{\mathfrak{M}}^{par}$ the moduli stack of parabolic PGL_n-Higgs bundles. Its groupoid of $\mathbb{C}$-points consists of the quadruples (E, θ, x, E_x^B), where (E, θ) is an L-twisted PGL_n-Higgs bundle on C, x is a closed point of C and E_x^B is a B-reduction of E at x. If we represent E by a vector bundle on C, then E_x^B is equivalent to a full flag on its fibre over x. We will denote by

$\hat{M}^{par}$ the open substack of $\hat{\mathfrak{M}}^{par}$ where the underlying PGL_n-Higgs bundles lie in $\hat{M}$.

1. Weyl group action (see [2, §3.4 (C)])

Let $\pi : \hat{M}^{par} \to \hat{M} \times C$ be the morphism that sends (E, θ, x, E_x^B) to (E, θ, x). Let $T \subset B$ be a maximal torus and let W be the Weyl group of G defined by T. We claim that

- there is an action of W on the complex $\pi_* \mathbb{Q}$,

where $\mathbb{Q}$ is the constant sheaf on $\hat{M}^{par}$, and the functor π_* is understood to be derived. Moreover,

- $(\pi_* \mathbb{Q})^W \cong \mathbb{Q}$.

It suffices to notice that π fits into the following Cartesian diagram:

$$\begin{tikzcd} \hat{M}^{par} \arrow[r] \arrow[d, "\pi"'] & {[\mathfrak{b}/B]_L} \arrow[d, "r_L"] \\ \hat{M} \times C \arrow[r, "ev"'] & {[\mathfrak{g}/G]_L}. \end{tikzcd}$$

Some explanations are in order. There is a $\mathbb{G}_m$-action on the Lie algebra $\mathfrak{g}$ by homothety, which commutes with the adjoint action. It induces an action of $\mathbb{G}_m$ on the quotient stack $[\mathfrak{g}/G]$. Regarding L as a $\mathbb{G}_m$-torsor on X, we can form $[\mathfrak{g}/G]_L$, the twist of $[\mathfrak{g}/G]$ by L, which is a fibration over X with fibres isomorphic to $[\mathfrak{g}/G]$. Similarly, we have $[\mathfrak{b}/B]_L$. The *evaluation map* sends $(E, \theta, x) \in \hat{M} \times C$ to the endomorphism $\theta_x : E_x \to E_x$, regarded as a section of $[\mathfrak{g}/G]_L$ over x. The vertical arrow r_L on the right hand side is the L-twisted version of the Grothendieck simultaneous resolution $r : [\mathfrak{b}/B] \to [\mathfrak{g}/G]$. If we take a smooth atlas $\mathfrak{g} \to [\mathfrak{g}/G]$, then r pullbacks to the morphism

$$r_0 : \tilde{\mathfrak{g}} := \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\} \longrightarrow \mathfrak{g}$$

that sends $(x, \mathfrak{b})$ to x, where $\mathcal{B}$ is the flag variety parametrising Borel subalgebras. It follows from the classical Springer theory that $r_{0*}\mathbb{Q}$ admits an action of W such that $(r_{0*}\mathbb{Q})^W \cong \mathbb{Q}$. By base change, we get the desired properties for $\pi_* \mathbb{Q}$.

2. Splitting of the universal Higgs bundle (see [2, §3.4 (A)])

Let $(\mathcal{U}, \Theta)$ be the universal Higgs bundle on $\hat{M} \times C$. Consider the pullback $\pi^* \mathcal{U}$. The fibre of this bundle over (E, θ, x, E_x^B) is E_x. But it is automatically equipped with a full flag E_x^B. This means that $\pi^* \mathcal{U}$ admits a filtration whose associated graded is a direct sum of line bundles $L_1, \ldots, L_n$. The fibres of these line bundles over (E, θ, x, E_x^B) are the successive quotients of the flag E_x^B. Consequently, we have a relation between Chern classes:

$$c_1(\pi^* \mathcal{U}) = \sum_{i=1}^{n} c_1(L_i).$$

We can give a more precise description of the line bundles L_i. For this purpose, we regard E_x^B as a B-torsor, instead of a flag. Then the surjective homomorphism $B \to T$ allows us to obtain a T-torsor E_x^T from E_x^B by extension of structure groups. This defines a tautological T-torsor L^T on $\hat{M}^{par}$, whose fibre over (E, θ, x, E_x^B) is E_x^T. Let $T_0 = (\mathbb{G}_m)^n$ be the rank n torus and regard T as the quotient of T_0 by the centre Z of GL_n, which is a $\mathbb{G}_m$ diagonally embedded in T_0. For $1 \leq i \leq n$, let $x_i \in Hom(T_0, \mathbb{G}_m)$ be the character that sends $(t_1, \dots, t_n)$ to t_i. Let x_0 be the character of Z that sends t to t. Since any subtorus is a direct factor, we may write $T_0 = Z \times S$ for some subtorus S, and let x be the character of T_0 that restricts to x_0 on Z and the trivial character on S. Now, $\lambda_i := x_i - x$ descends to a character of T, and each L_i is the $\mathbb{G}_m$-torsor obtained from L^T by extension of structure groups via λ_i.

3. Strong perversity of $c_1(L_i)$ (see [2, §3.4 (B)]).

Any cohomology class $c \in H^2(\hat{M}^{par})$ is canonically identified with a morphism of complexes $\mathbb{Q} \to \mathbb{Q}[2]$ in the constructible derived category $\mathcal{D}_c^b(\hat{M}^{par})$. Let $\bar{h} : \hat{M}^{par} \to \hat{A} \times C$ be the composition $(\hat{h} \times \mathrm{Id}) \circ \pi$. Then, c induces a morphism between the direct image complexes $\bar{h}_*\mathbb{Q} \to \bar{h}_*\mathbb{Q}[2]$. We will denote this morphism by the same letter. Applying the perverse truncation functor ${}^{\mathfrak{p}}\tau_{\leq k}$, we get a morphism between the truncated complexes ${}^{\mathfrak{p}}\tau_{\leq k} c : {}^{\mathfrak{p}}\tau_{\leq k}\bar{h}_*\mathbb{Q} \to {}^{\mathfrak{p}}\tau_{\leq k}\bar{h}_*\mathbb{Q}[2]$. If ${}^{\mathfrak{p}}\tau_{\leq k} c$ factors through the complexes ${}^{\mathfrak{p}}\tau_{\leq k-1}\bar{h}_*\mathbb{Q}[2]$ that is one-step smaller, then we say that c has stronger perversity 1. It tuns out that the line bundles $c_1(L_i)$ in (2) have strong perversity 1.

We explain this strong perversity result in the particular case of GL_n and PGL_n. A first reduction is to restrict the strong perversity statement to an open dense subset $U \subset \hat{A} \times C$ where $\bar{h}$ is smooth. Now, the decomposition theorem says that the complex $\bar{h}_*\mathbb{Q}$ is isomorphic to the direct sum of its shifted perverse cohomology sheaves, which are in fact shifted local systems on U. Then, the strong perversity over U amounts to saying that the morphism $\mathcal{H}^i(\bar{h}_*\mathbb{Q}) \to \mathcal{H}^i(\bar{h}_*\mathbb{Q}[2])$ between local systems is zero. It suffices to show that for some $u \in U$, we have a zero map between the fibres $H^i(\bar{h}^{-1}(u)) \to H^{i+2}(\bar{h}^{-1}(u))$. This is the cup product by $c_1(L_i|_{\bar{h}^{-1}(u)})$. Indeed, the Chern class $c_1(L_i|_{\bar{h}^{-1}(u)})$ vanishes. To see this, we use the expression for L_i given in (2); i.e., L_i is associated to the character $\lambda_i = x_i - x$. It remains to notice that the line bundles (on the moduli space of GL_n-Higgs bundles) associated to x_i and x have the same Chern class, whence the vanishing of $c_1(L_i|_{\bar{h}^{-1}(u)})$. In the case of GL_n, the vanishing of $c_1(L_i|_{\bar{h}^{-1}(u)})$ can be achieved by an appropriate choice of the normalisation of the universal bundle. To extend this vanishing result to larger open subsets, we need a support theorem saying that $\bar{h}_*\mathbb{Q}$ is the intermediate extension of its restriction to any open. In [4] this is proved for the anisotropic locus, and in [2] over the entire base.

Finally, we briefly recall how the vanishing of Chern classes is achieved in [4]. This is the content of [4, Lemma 3.2.3], which eventually relies on [4, Lemma A.2.1]. The key arguments go as follows. The latter lemma says that the map of multiplication by N on a smooth commutative group scheme induces on its (co)homology a linear operator whose eigenvalues are completely determined by the (co)homology degree. As a degree 2 cohomology class, $c_1(L_i|_{\bar{h}^{-1}(u)})$ should be an eigenvector of this linear operator with eigenvalue N^2, whereas the universal preperty that L_i satisfies implies that it should has eigenvalue N. The only possibility is that $c_1(L_i|_{\bar{h}^{-1}(u)}) = 0$, and the strong perversity follows.

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Cohomological Hall algebras and Hecke operators on surfaces

Miguel Moreira

This talk will be based on the papers [1, 3]. In the proof of $P = W$ in [1], one of the fundamental steps is to construct the action of a large algebra $W(S)$ on homology/cohomology of (the elliptic loci of) moduli spaces of Higgs bundles. This action interacts in a controlled way with tautological classes, and hence with the Chern filtration – which, by a result of Schende presented in a previous talk, matches the W filtration. Using this action one, constructs a $\mathfrak{sl}_2$ triple, which is later used to prove that the perverse filtration matches the Chern filtration (on the elliptic loci; further work is required to reduce the statement to the elliptic loci).

This talk focuses on the construction of this algebra action. I will explain how the algebra $W(S)$ comes from the theory of Cohomological Hall algebras (CoHA), following [3].

1. Cohomological Hall algebra

Cohomological Hall algebras are algebra structures that one can define on the homology of a moduli stack parametrizing objects in abelian conditions satisfying appropriate conditions. The one that we are interested in is the cohomological Hall algebra of 0-dimensional sheaves on a surface S, which for simplicity we will assume is projective. The main example in applications to Higgs bundles is $S = T^*C$, but we can also compactify it to $\mathbb{P}(T^*C \oplus \mathcal{O}_C)$.

Let

$$\mathfrak{C}oh^0(S) \subseteq \mathfrak{C}oh(S)$$

be the derived stacks parametrizing sheaves on S and 0-dimensional sheaves on S. The CoHA of (0-dimensional) sheaves on S is the Borel–More homology of the stack:

$$\mathbb{H}(S) \coloneqq H_*(\mathfrak{C}oh(S)) \supseteq H_*(\mathfrak{C}oh^0(S)) =: \mathbb{H}^0(S)\,.$$

We define a product on $\mathbb{H}(S)$ and $\mathbb{H}^0(S)$ as follows: let $\mathcal{E}\mathrm{xt}$ be a stack parametrizing short exact sequences of the form

$$0 \to F_1 \to F_2 \to F_3 \to 0\,;$$

this stack admits a map q to $\mathfrak{C}oh \times \mathfrak{C}oh$ remembering F_1, F_3 and another map p to $\mathfrak{C}oh$ remembering F_2. The product on the CoHA is defined by

$$H_*(\mathfrak{C}oh) \otimes H_*(\mathfrak{C}oh) \to H_*(\mathfrak{C}oh \times \mathfrak{C}oh) \xrightarrow{q^!} H_*(\mathcal{E}\mathrm{xt}) \xrightarrow{p_*} H_*(\mathfrak{C}oh)\,.$$

where $q^!$ is a "virtual pullback".

The goal of the talk is to understand $\mathbb{H}^0(S)$ and its action on the (co)homology of moduli spaces.

2. Hecke patterns

Let $\mathfrak{M}$ be a substack of $\mathfrak{C}oh(S)$. Under some conditions on $\mathfrak{M}$, it is possible to define a left (and also a right) action of $\mathbb{H}^0(S)$ on $H_*(\mathfrak{M})$. When these appropriate conditions are met, $\mathfrak{M}$ is called a Hecke pattern. The most crucial property of a Hecke pattern is that it should be closed under point modifications. Note that this forces $\mathfrak{M}$ to have many connected components, since point modifications change the topological type of sheaves. The definition of the action resembles the definition of the CoHA product.

Example 2.1. For $S = \mathbb{P}(T^*C \oplus \mathcal{O}_C)$, pure 1-dimensional sheaves on S with integral support away from the ∞ divisor (=the elliptic loci of the moduli of Higgs bundles) form a Hecke pattern,

$$\mathfrak{M}_r^{\mathrm{ell}} = \bigsqcup_{d \in \mathbb{Z}} \mathfrak{M}_{r,d}^{\mathrm{ell}}\,.$$

On the other hand, the stack of all semistable 1-dimensional sheaves (with support away from the ∞-divisor) is not a Hecke pattern. This is why the proof of $P = W$ in [1] requires a reduction to the elliptic loci.

3. Length 1 Hecke patterns and Negut's lemma

The first step to understand $\mathbb{H}^0(S)$ and its action on $H_*(\mathfrak{M})$ more concretely is to start with the action of elements in

$$H_*(\mathfrak{C}oh_\delta) \subseteq \mathbb{H}^0(S)\,,$$

where $\mathfrak{C}oh_\delta$ is the stack of 0-dimensional sheaves of length 1. Note that $\mathfrak{C}oh_\delta \simeq S \times B\mathbb{G}_m$, so

$$H_*(\mathfrak{C}oh_\delta) \simeq H^*(S)[u]$$

by Poincaré duality. The (left) action of $u^n\lambda \in \mathbb{H}^0(S)$, for $n \geq 0$ and $\lambda \in H^*(S)$, produces an operator

$$T_n^+(\lambda)\colon H_*(\mathfrak{M}) \to H_*(\mathfrak{M})\,.$$

Similarly, there is another operator $T_n^-(\lambda)$ coming from the right action.

Negut's lemma [2, Proposition 2.19] gives a very concrete understanding of these operators by identifying some of the maps that come up in the definition of the action with (virtual) projective bundles. In particular, it gives

(1) A formula for the image of the fundamental class $[\mathfrak{M}]$ under $T_n^{\pm}(\lambda)$;
(2) A formula for the commutator between $T_n^{\pm}(\lambda)$ and the operators of capping with tautological classes.

4. (Deformed) W-algebras

The main result of [3], and of this talk, is an isomorphism

$$\mathbb{H}^0(S) \simeq W^+(S)$$

with an explicitly defined algebra.

Let $W(S)$ be the algebra generated by

$$\psi_n(\lambda), T_n^+(\lambda), T_n^-(\lambda) \quad n \geq 0, \lambda \in H^*(S)$$

and a central element c, modulo certain explicit relations which have the following shape:

(1) ψ commute.
(2) $[T^{\pm}, \psi] = T^{\pm}$.
(3) Quadratic relation on $T^{\pm}$.
(4) Cubic relation on $T^{\pm}$.
(5) $[T^+, T^-] = \psi$.

Let $W^0, W^{\pm}$ be the algebras generated by $\psi_n(\lambda)$ and $T_n^{\pm}(\lambda)$, respectively. Let $W^{\geq}$ be the algebra generated by W_0 and W^+, and $W^{\leq}$ similarly defined.

The algebra $W(S)$ acts on the (tautological part of the) homology $H_*(\mathfrak{M})$ of a Hecke pattern; the actions of $W^+(S) \simeq \mathbb{H}^0(S)$ and $W^-(S) \simeq \mathbb{H}^0(S)^{\mathrm{op}}$ are identified with the left and right CoHA actions previosuly mentioned, while the action of $W^0(S)$ is essentially capping with tautological classes.

There are a few important consequences that we can extract from this description of the algebra $W(S)$: the CoHA is generated as an algebra by $H_*(\mathfrak{C}oh_\delta)$; $W(S)$ contains copies of the Heisenberg and Virasoro Lie algebras; when $c_1 = 0$, $W^{\geq}(S)$ is the universal envelopping algebra of a certain Lie algebra of differentials;

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Construction of the action of $\mathfrak{sl}_2$ and $\mathcal{H}_2$

Wille Liu

A main ingredient of the proof of Hausel–Mellit–Minets–Schiffmann [1] of the $P = W$ conjecture for GL_n is the construction of an action of an $\mathfrak{sl}_2$-triple on the cohomology $\mathrm{H}^*(M_{r,d})$ of the elliptic locus of the moduli space of Higgs bundles of rank $r > 0$ and degree $d \in \mathbf{Z}$ over a smooth complex projective curve C. The construction makes use of the algebra W for the cotangent bundle T^*C of a C, which is formed by the *cohomological Hall algebra* of 0-dimensional Higgs sheaves and the *tautological classes* (namely, Künneth coefficients of the Chern character of a universal sheaf F_α on M_α).

According to the calculation of Mellit–Minets–Schiffmann–Vasserot [2], the algebra W is the universal enveloping algebra of a Lie-super-algebra with generators $\{D_{m,n}(\xi)\}_{m,n\geq 0,\ \xi\in \mathrm{H}^*(C)}$ and relations

$$[D_{m,n}(\xi), D_{m',n'}(\xi')] = (nm' - mn')D_{m+m',n+n'-1}(\xi\xi').$$

By the formalism of cohomological Hall algebras, there is a natural action of W on $\bigoplus_{d\in\mathbf{Z}} \mathrm{H}^*(\mathcal{M}_{r,d})$, where $\mathcal{M}_{r,n}$ is the moduli *stack* of Higgs bundles.

In order to obtain an action of operators similar to $D_{m,n}(\xi)$ on $\mathrm{H}^*(M_{r,d})$, Hausel–Mellit–Minets–Schiffmann perform firstly degeneration on the operators $\{D_{m,n}(\xi)\}_{m,n,\xi}$ to obtain a new set of operators $\left\{\tilde{D}_{m,n}(\xi)\right\}_{m,n,\xi}$ which act on $\mathrm{H}^*(\mathcal{M}_{r,d})$ for fixed $r > 0$ and $d \in \mathbf{Z}$. If one thinks of $D_{m,n}(\xi)$ as linear differential operators $\sum_i z_i^{m+1}(\partial/\partial z_i)$ on an algebraic torus $(\mathbf{C}^\times)^k$, then the degeneration procedure is similar to the change of coordinates via the exponential map $\exp : \mathbf{C}^k \to (\mathbf{C}^\times)^k$. The new set of operators satisfy

$$[\tilde{D}_{m,n}(\xi), \tilde{D}_{m',n'}(\xi')] = (nm' - mn')\tilde{D}_{m+m'-1,n+n'-1}(\xi\xi') + \cdots ,$$

where secondary terms are omitted. These relations resemble those of the Lie algebra $\mathcal{H}_2$ of Hamiltonian currents on $\mathbf{C}^2$.

As it is known that there exists a non-canonical isomorphism $\mathcal{M}_{m,n} \cong M_{r,d} \times \mathrm{B}\mathbf{G}_{\mathrm{m}}$, one has to get rid of the extra variable from the cohomology ring $\mathrm{H}^*(\mathrm{B}\mathbf{G}_{\mathrm{m}}) \cong \mathbf{C}[c_1(\mathcal{O}(1))]$. This is done by finding a Weyl algebra $\mathbf{C}[y, \partial_y]$-action on $\mathrm{H}^*(\mathcal{M}_{r,d})$ from the operators $\{\tilde{D}_{0,1}(\eta), \tilde{D}_{1,0}(1)\}$ for some $\eta \in \mathrm{H}^2(C)$. The $\mathfrak{sl}_2$-triple (e, h, f) are then defined as modified version of the operators $(\tilde{D}_{0,2}(\eta), \tilde{D}_{1,1}(1), \tilde{D}_{1,0}(1))$ on $\mathrm{H}^*(\mathcal{M}_{r,d})$, restricted to the kernel $\ker(\partial_y) \subset \mathrm{H}^*(\mathcal{M}_{r,d})$.

The advantage of this $\mathfrak{sl}_2$-triple is that one can calculate explicitly their action on tautological classes and thereby determine the filtration defined by the nilpotent element e. An argument based on perturbation of ample line bundles shows that this filtration coincides with the perverse filtration. This concludes the $P = C$ part of the $P = W$ conjecture.

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P=W conjecture after Maulik and Shen

Mirko Mauri

1. Statement of the P=W conjecture

Let C be a smooth projective complex curve, $n \in \mathbb{N}$ and $d \in \mathbb{Z}$ with $\gcd(n, d) = 1$. The *Dolbeault moduli space* $M_{\mathrm{Dol}} = M_{\mathrm{Dol}}(n, d)$ is the coarse moduli space which parametrises S-equivalence classes of semistable Higgs bundles on C of rank n and degree d, i.e., polystable pairs (E, ϕ) consisting of a vector bundle E of rank n and degree d and a section $\phi \in H^0(C, End(E) \otimes \omega_C)$, called Higgs field.

Via the non-abelian Hodge correspondence Ψ, M_{Dol} is diffeomorphic to the *Betti moduli space* M_{B}, i.e., the affine GIT quotient

$$M_B := \left\{(A_1, B_1, \ldots, A_g, B_g) \in \mathrm{GL}_n^{2g} \mid \prod_{j=1}^{g} [A_j, B_j] = e^{2\pi i \frac{d}{n}} 1_{\mathrm{GL}_n} \right\} /\!\!/ \mathrm{GL}_n \,.$$

It parametrises isomorphism classes of semi-simple representations of the fundamental group of C with prescribed monodromy $e^{2\pi i \frac{d}{n}} 1_{\mathrm{GL}_n}$ around a base point.

The P=W conjecture predicts that two filtrations of very different origins on the rational singular cohomology $H^*(M_{\mathrm{Dol}}) = \Psi^* H^*(M_{\mathrm{B}})$ coincide. On the Betti side, the cohomology of M_{B} is endowed with Deligne's weight filtration

$$W_\bullet H^*(M_{\mathrm{B}}, \mathbb{Q}),$$

a measure of the topology of algebraic compactifications of M_{B}. The Dolbeault moduli space is equipped with a projective fibration called *Hitchin fibration*

$$h \colon M_{\mathrm{Dol}} \to A := \bigoplus_{i=1}^{n} H^0(C, \omega_C^{\otimes i}),$$

which assigns to (E, ϕ) the characteristic polynomial $\mathrm{char}(\phi)$ of the Higgs field ϕ. As a result, the cohomology of $M_{\mathrm{Dol}}(n, d)$ is endowed with the perverse Leray filtration

$$P_\bullet H^*(M_{\mathrm{Dol}}(n, d)),$$

a measure of the singularities of the fibers of the Hitchin fibration.

Theorem 1.1 (P=W conjecture)**.** *For any $k \geq 0$, we have*

$$P_k H^*(M_{\mathrm{Dol}}) = \Psi^* W_{2k} H^*(M_{\mathrm{B}}).$$

The P=W conjecture was formulated by de Cataldo, Hausel and Migliorini in [3] in 2010, and proved for the first time by Maulik and Shen [15] in September 2022. The week after the submission of [15] on arXiv, a second proof was made available by Hausel, Mellit, Minets and Schiffmann [10]. A third proof by Maulik, Shen and Yin appeared in [16] in the summer 2023.

2. Proof of the P=W conjecture after Maulik and Shen

The proof of Maulik and Shen is a clever reduction to a result by Yun [22, Lemma 3.2.3] concerning the perversity of the first Chern class of tautological line bundles on a moduli space of parabolic Higgs bundles. Delving deeper into the proof of the statement, this boils down to the classical fact that the first Chern class of the normalized Poincaré line bundle over $\mathrm{Jac}(C) \times C$ is the class of the identity in

$$\mathrm{End}(H^1(C)) = H^1(C)^* \otimes H^1(C) = H^1(\mathrm{Jac}(C)) \otimes H^1(C) \subset H^2(\mathrm{Jac}(C) \times C). \tag{1}$$

In particular, it does not have any component in the Künneth summand

$$H^2(\mathrm{Jac}(C)) \otimes H^0(C).$$

Curiously, this was also a crucial ingredient already in the earlier paper [3]; see in particular [3, Prop. 5.1.2].

The proof of the P=W conjecture of Maulik and Shen consists of four steps.

2.1. **Step 1. Strong perversity of Chern classes.** It has already been observed in [6, Thm 0.6] that the P=W conjecture is equivalent to the multiplicativity of the perverse filtration.[1] Indeed, in [12], Markman showed that the cohomology ring of $H^*(M_{\mathrm{Dol}})$ is generated as an algebra by tautological classes, i.e., Künneth components of the universal PGL_n-bundle $U \to C \times M_{\mathrm{Dol}}$. It was known by the work of Shende [21] and de Cataldo–Maulik–Shen [6, Thm 0.4 and 0.5] that tautological classes have matching weights and perversities, but it was unclear that the same holds for monomials in tautological classes. The weights of these monomials pose no issue, since the weight filtration is multiplicative (use the definition via logarithmic differential forms). The multiplicativity of the perverse filtration was the missing ingredient. To this end, Maulik and Shen introduce the notion of strong perversity, i.e., a sheaf-theoretic version of multiplicativity which in particular ensures the desired multiplicativity in cohomology. In conclusion, to show the P=W conjecture, it is enough to show that, for all $k \geq 0$, the Chern classes $\mathrm{ch}_k(U)$ have strong perversity k, with respect to the extended Hitchin fibration $(h, \mathrm{id})\colon M_{\mathrm{Dol}} \times C \to A \times C$.

[1] When the authors of [3] first presented the conjecture in early 2010s, this was one of the very first questions asked to them, according to Hausel. We now understand that this is a very delicate question; see [23, 24, 16, 1].

2.2. **Step 2. Splitting principle via parabolic Higgs bundles.** The product $M_{\mathrm{Dol}} \times C$ is dominated by the moduli space of parabolic Higgs bundles. For each point $((E,\phi),c) \in M_{\mathrm{Dol}} \times C$, the parabolic structure consists of the additional choice of a complete filtration on E_c, preserved by ϕ_c. The advantage is that the grading of the filtration gives a splitting of the universal vector bundle into line bundles. In this way, one can describe the Chern classes of the tautological bundle as homogeneous polynomials in the first Chern classes of parabolic tautological line bundles. Now, over a dense open set of the Hitchin base, Yun has already computed that these line bundles have strong perversity 1. Via the homogeneity, this implies that $\mathrm{ch}_k(U)$ have strong perversity k, at least generically over the Hitchin base. Recall that Yun's result follows essentially from (1), since the general fiber of the Hitchin fibration h is a Jacobian.

2.3. **Step 3. Support theorem for meromorphic Higgs bundles.** In the third step, we extend Yun's result to the whole base of the Hitchin fibration. To this purpose, it is convenient to enlarge M_{Dol} to the moduli space $M_{\mathrm{Dol}}(D)$ of meromorphic Higgs bundles, i.e., pairs (E,ϕ) with $\phi\colon E \to E \otimes \omega_C \otimes D$ for some effective divisor $D > 0$. Indeed, the cohomology of $H^*(M_{\mathrm{Dol}}(D))$ is determined by the geometry of any nonempty open set of $M_{\mathrm{Dol}}(D)$, saturated with respect to the Hitchin fibration, in particular the open set where Yun controls the strong perversity of tautological line bundles. More precisely, one proves that the direct image $Rh(D)_*\mathbb{Q}_{M_{\mathrm{Dol}}(D)}$ is the intermediate extension of its restriction to any open set, equivalently that it does not contain any direct summand supported on a proper subvariety of $A(D)$. Classically, this property is called *full support*. Note that the full support property fails for the ordinary Dolbeault moduli space M_{Dol}; see [4, 19, 18]. The fourth and last step of the proof must take care of this issue.

Full support phenomena have been detected and exploited by Ngô for the proof of the Fundamental Lemma [20]. Using the full support property, Chaudouard–Laumon [2] and Maulik–Shen [14] showed that the cohomology $H^*(M_{\mathrm{Dol}}(n,d))$ is independent of the degree d. The same circle of ideas have been used by de Cataldo, Rapagnetta and Saccà [7] to determine the Hodge numbers of the O'Grady 10 manifold.

2.4. **Step 4. Vanishing cycle techniques.** The strong perversity of the Chern classes of the universal bundle on $M_{\mathrm{Dol}}(D)$ implies the analogous result on M_{Dol}, via the formalism of vanishing cycles. The key idea is to enlarge the original space M_{Dol} to $M_{\mathrm{Dol}}(D)$, whose geometry is more amenable (see Step 3), and then pullback the strong perversity property to M_{Dol}, via the vanishing cycle functor.

This idea comes from Donaldson–Thomas theory, but it has been successfully used also for the proof of the topological mirror symmetry conjecture (or Hausel–Thaddeus conjecture) by Maulik and Shen [13], the description of the BPS sheaf in enumerative geometry by Kinjio and Koseki [11], or the study of the K-theory of M_{Dol} by Groechenig and Shen [9].

2.5. **Singular Dolbeault moduli spaces.** It would be desirable to drop the coprimality assumption on n and d, in particular for the original and fundamental case $d = 0$. In all these cases, however, the moduli space $M_{\mathrm{Dol}}(n, d)$ is singular, and Poincaré duality and relative hard Lefschetz theorems fail in general for singular cohomology. These symmetries can be restored taking instead intersection cohomology. In [5, Question 4.1.7], de Cataldo and Maulik proposed the PI=WI conjecture for the intersection cohomology of singular Dolbeault moduli spaces, proved only for $(d, g) = (0, 1)$ and $(d, g, n) = (0, 2, 2)$ in [8, Main Thm] (leaving aside the smooth cases). See [17, §5] for partial results in rank two, in particular a numerical evidence for curious hard Lefschetz in [17, Cor. 1.5].

At the moment, curious hard Lefschetz is still unknown in arbitrary degree d. The current proofs of the P=W conjecture rely on the multiplicative structure of singular cohomology. However, intersection cohomology does not have a canonical ring structure. This means that the proofs in the smooth context do not naturally extend to the singular case, thus challenging the community to understand better P=W phenomena.

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Hausel-Mellit-Minets-Schiffmann's proof, (I)

Raphaël Picovschi (extended abstract by Olivier Schiffmann)

This talk and the following one provided an overview of the actual proof of the $P = W$ conjecture following the approach of Hausel, Mellit, Minets and Schiffmann [1]. As in the proof by Maulik and Shen, one is reduced to computing the perverse degree of tautological classes. The starting point is to use the relative Hard Lefschetz theorem, which describes the perverse filtration on the cohomology $H^*(M_{r,d})$ of the moduli space of stable Higgs bundles of rank r and degree d in terms of an action of an $\mathfrak{sl}_2$-triple (e, h, f) for e a class in $H^2(M_{r,d})$, relatively ample with respect to the Hitchin morphism. The desired property of tautological classes is a consequence of the commutation relation

$$[h, \psi_m(\xi)] = m\psi_m(\xi)$$

for $\psi_m(\xi)$ a Kunneth component of the mth Chern character of the tuatological bundle.

The main idea of [1] is to construct the above $\mathfrak{sl}_2$-triple using Hecke operators of punctual modifications. The algebra generated by these Hecke operators, which is identified with a certain $W_{1+\infty}$-algebra $W(C)$ modeled on the cohomology of the curve C, is not quite the right object. The aim of this talk was to explain how to modify the Hecke operators in order to obtain an action

$$\mathcal{H}_2 \otimes H^*(M^{ell}_{r,d})[x, y] \to H^*(M_{r,d})[x, y]$$

of the Lie algebra $\mathcal{H}_2$ of Hamiltonian vector fields on the plane on the cohomology of the moduli space of stable Higgs bundles –over the elliptic locus and extended by two formal variables. The Lie algebra $\mathcal{H}_2$ admit the following presentation : it is linearly generated by elements $D_{n,m}(\xi)$, for $n, m \geq 0$ and $\xi \in H^*(C)$ subjected to the relations

$$[D_{m,n}(\xi), D_{m',n'}(\mu)] = (mn' - m'n)D_{m+m'-1,n+n'-1}(\xi\mu).$$

The algebra $\mathcal{H}_2$ contains a Weyl algebra (generated by $D_{1,0}(\eta), D_{0,1}(1), D_{0,0}(\eta)$) and a copy of $\mathfrak{sl}_2$ (generated by $D_{0,2}(1), D_{1,1}(1), D_{2,0}(1)$).

The construction of the action of $\mathcal{H}_2$ from that of $W(C)$ is intricate and was not presented in details. It relies on certain identifications between $H^*(M^{ell}_{r,d})$ and $H^*(M^{ell}_{r,d+1})$ –which necessitates the restriction to the elliptic locus– as well as some rather involved computations of commutation relations between Hecke operators. However, the commutation relation between Hecke operators and multiplication by tautological classes, as well as the compatibility between Hecke operators and the perverse filtration follow from standard geometric considerations, and were presented in some detail. A proof of the $P = W$ conjecture over the elliptc locus, for moduli spaces of stable (possibly twisted) Higgs bundles ensues.

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Hausel-Mellit-Minets-Schiffmann's proof, (II)

Jiangfan Yuan

We begin with some remarks on the surface W-algebras. Since the Hecke operators are destabilizing, we need to work over elliptic locus of moduli stacks.

The strategy to prove $P = W$ is to study perverse filtration of $\mathfrak{M}^{\mathrm{parell}}_{r,d,D}$ and make reduction steps to the classical case.

Let $y_{p,i}$ be the first chern class of the line bundle over $\mathfrak{M}^{\mathrm{parell}}_{r,d,D}$, with fiber the i-th subquotient of the flag $\mathcal{E}^{\bullet}_p$. Let $X_{p,i}$ be the pullback of (adjoint) Hecke modification: $\mathrm{Mod}_{p,i} : \mathfrak{M}^{\mathrm{parell}}_{r,d,D} \to \mathfrak{M}^{\mathrm{parell}}_{r,d-1,D}, \left(\mathcal{E}, \theta, (\mathcal{E}^{\bullet}_p)_{p\in D}\right) \mapsto (\mathrm{Ker}(\mathcal{E} \to \delta_{p,i}), \theta', (\mathcal{E}'^{\bullet}_p)_{p\in D})$, where $\mathcal{E} \to \delta_{p,i}$ sends to the i-th eigenspace of residue at p. Let $H_{r,D}$ be the orbifold cohomology $H^*_{\mathrm{orb}}\left(C_{(r,D)}\right)$ of the stacky curve $C_{(r,D)}$ with r-foldings at D, it is isomorphic to $H^*(C)[p_i]_{p\in D, i=1,\dots,r}$ $(\deg p_i = 2)$ modulo relations:

$$\sum_i p_i = \omega, \quad p_i H^{>0}(C) = 0, \quad p_i q_j = 0$$

Proposition 1. *We set $\psi_n(p_i) := y^n_{p,i}$ and $T_n(p_i) := y^n_{p,i} X_{p,i}$, then the surface W-algebra generated by $\tilde{D}_{m,n}(\pi), \pi \in H_{r,D}$ satisfies the relations as the usual one. The enlarged surface W-algebra acting on $H^*(\mathfrak{M}^{\mathrm{parell}}_{r,d,D})$.*

We conclude the proof of $P = W$ conjecture in [1] through the following steps:

Step 1. Recall from the previous lecture we have seen the $P = C$ holds for elliptic parabolic moduli space. Namely, we have:

Theorem 2. *The subspace $P_m H^*_{\mathrm{pure}}(M^{\mathrm{parell}}_{r,d,D})$ is spanned by products $\prod_i \psi_{m_i}(\xi_i)$ with $\sum_i m_i \leq m + N$, where $-N$ is the $\mathfrak{h}$-weight of $1 \in H^0_{\mathrm{pure}}(M^{\mathrm{parell}}_{r,d,D})$.*

By definition, for any stability condition K, the elliptic locus of parabolic Higgs moduli stack (resp. space) is open in $\mathfrak{M}^{\mathrm{par},K}_{r,d,D}$ (resp. $M^{\mathrm{par},K}_{r,d,D}$). We fix one stability condition and consider the following diagram

$$\begin{tikzcd}
M^{\mathrm{parell}}_{r,d,D} \arrow[r, hook, "\iota^{\mathrm{parell}}"] \arrow[d] & M^{\mathrm{par}}_{r,d,D} \arrow[d, "\chi^{\mathrm{par}}"'] \arrow[dr, "\chi_D"] & & \chi_D^{-1}(\mu) \arrow[ll, hook', "\iota_\mu"'] \arrow[d] \\
A^{\mathrm{parell}} \arrow[r, hook] & A^{\mathrm{par}} \cong A_D \times A' \arrow[r] & A_D \cong \mathbb{A}^{r|D|-1} \ni & \mu
\end{tikzcd}$$

where $\chi_D : \left(\mathcal{E}, \theta, (\mathcal{E}^\bullet_p)_{p\in D}\right) \mapsto \left(\mathrm{res}_p(\theta)\right)_{p\in D}$ sends an element in parabolic Higgs moduli space to an ordered collection with coordinates being eigenvalues of residue of its Higgs field at each $p \in D$. Moreover $\sum_{p\in D} \mathrm{tr}(\mathrm{res}_p(\theta)) = 0$.

We can take good $\mu = (\mu_{p,i}) \in A_D$ so that $\chi_D^{-1}(\mu) \subseteq M^{\mathrm{parell}}_{r,d,D}$. Thus ι_μ^* factors as:

$$H^*(M^{\mathrm{par}}_{r,d,D}) \xrightarrow{(\iota^{\mathrm{parell}})^*} H^*(M^{\mathrm{parell}}_{r,d,D}) \longrightarrow H^*(\chi_D^{-1}(\mu))$$

Recall Markman's argument [2], $H^*_{\mathrm{taut}}(M^{\mathrm{par}}_{r,d,D}) \cong H^*_{\mathrm{pure}}(M^{\mathrm{par}}_{r,d,D})$. By semiprojectivity of $M^{\mathrm{par}}_{r,d,D}$, $H^*_{\mathrm{pure}}(M^{\mathrm{par}}_{r,d,D}) \cong H^*(M^{\mathrm{par}}_{r,d,D}) \cong H^*(\chi_D^{-1}(0))$.

As a result, we see that the restriction map $\left(\iota^{\mathrm{parell}}\right)^*$ is injective and induces an isomorphism $H^*(M^{\mathrm{par}}_{r,d,D}) \cong H^*_{\mathrm{pure}}(M^{\mathrm{parell}}_{r,d,D})$. Moreover, since the elliptic locus is open, $\left(\iota^{\mathrm{parell}}\right)^*$ is also a map of Lefschetz structures. We get

Corollary 3. *$P = C$ holds for $M^{\mathrm{par}}_{r,d,D}$.*

Step 2. We know that the $\mathbb{C}^\times$-action on $M^{\mathrm{par}}_{r,d,D}$ (by scaling the Higgs field) endows it a structure of semi-projective variety and all the fibers of χ_D has isomorphic pure cohomology $H^*(M^{\mathrm{par}}_{r,d,D})$. We consider one particular fiber $\chi_D^{-1}(0)$.

Define the projectivization $\overline{M}^{\mathrm{par}}_{r,d,D} := \left(M^{\mathrm{par}}_{r,d,D} \times \mathbb{C} \backslash (\chi^{\mathrm{par}})^{-1}(0) \times \{0\}\right)/\mathbb{C}^\times$ of M^{par}, and we denote by $\bar{\chi}$ the extended Hitchin map $\overline{M}^{\mathrm{par}}_{r,d,D} \to \overline{A}^{\mathrm{par}}$. We also have $\chi_D^{-1}(0) \cong (\chi^{\mathrm{par}})^{-1}(A_0)$ for some linear subspace $A_0 \subseteq A^{\mathrm{par}}$. Then $\chi_D^{-1}(0)$ also admits a projectivization $\overline{M}_0$ as above, fits into the following diagram:

$$\begin{tikzcd}
\overline{M}_0 \arrow[r, hook, "\bar{\iota}_0"] \arrow[d] & \overline{M}^{\mathrm{par}} \arrow[d, "\bar{\chi}^{\mathrm{par}}"] \\
\overline{A}_0 \arrow[r, hook] & \overline{A}^{\mathrm{par}}
\end{tikzcd}$$

which extends ι_0 and χ^{par}. Let $\partial M := \overline{M}^{\mathrm{par}} \backslash M^{\mathrm{par}}$ be the boundary.

Proposition 4. *Let $L \in H^2(\overline{M}^{\mathrm{par}})$ be the cycle corresponding to $[\partial M]$, we have*

$$H^*(M^{\mathrm{par}}) \cong H^*(\overline{M}^{\mathrm{par}})/LH^*(\overline{M}^{\mathrm{par}}); \quad H^*(\chi_D^{-1}(0)) \cong H^*(\overline{M}_0)/LH^*(\overline{M}_0)$$

here the later L is the restriction to boundary class on $\overline{M}_0$ and $(\bar{\iota}_0)^*$ induces an isomorphism on the quotient

$$\iota_0^* : H^*(M^{\mathrm{par}}) \xrightarrow{\cong} H^*(\chi_D^{-1}(0))$$

Proposition 5. *(i)Let* $n := \dim M^{\mathrm{par}} - \dim \chi_D^{-1}(0) = \mathrm{codim}_{\overline{A}^{\mathrm{par}}}(\overline{A}_0)$*. Then we have* $L^n = c\left[\overline{M}_0\right] = c \cdot \overline{\iota}_0^* \overline{\iota}_{0,*}$ *for some multiplicity* c *and* $\mathrm{Im}(\overline{\iota}_0^*) \subseteq \mathrm{Im}(L^n)$*.*
(ii) For any $i \geq 0$*,* $(\iota_0^*)^{-1}\left(\mathrm{Ker}\, L^i + \mathrm{Im}\, L\right) \subseteq \mathrm{Ker}\, L^{n+i} + \mathrm{Im}\, L$*.*

Suppose $\alpha \in P_i H^j(\chi_D^{-1}(0))$, by definition, it can be represented by an element in $W_{i+n-j} H^*(\overline{M}_0) \subseteq \mathrm{Ker}\, L^{i+\dim \chi_D^{-1}(0)-j+1} + \mathrm{Im}\, L$, recall $W_\bullet$ is the canonical filtration with respect to the nilpotent operator L. By Proposition 5, via $(\iota_0^*)^{-1}$, it can be represented in $H^*(\overline{M}^{\mathrm{par}})$ by an element in $\mathrm{Ker}\, L^{i+\dim M^{\mathrm{par}}-j+1}$. This shows that α in fact lies in $P_i H^j(M^{\mathrm{par}})$. We can conclude that

Corollary 6. *The inverse of* ι_0^* *is an isomorphism of Lefschetz structure, and thus* $P = C$ *holds for* $\chi_D^{-1}(0)$*.*

Step 3. Finally, we take in particular $D = (p)$ to be one point. Let $\widetilde{M}_{r,d}$ be the closed subvariety of $\chi_p^{-1}(0)$, determined by the condition $\mathrm{res}_p(\theta) = 0$. We have:

$$\begin{tikzcd}
\chi_p^{-1}(0) \arrow[rr, bend left, "A"] \arrow[dr] & \widetilde{M}_{r,d} \arrow[l, hook', "\iota"'] \arrow[r, "\pi"] \arrow[d] & M_{r,d} \arrow[ll, bend left, "B"] \arrow[dl] \\
& A^{\mathrm{par}} &
\end{tikzcd}$$

where π is the projection map forgetting the flag structure on $\widetilde{M}_{r,d}$. Note that $\mathrm{codim}\, \iota = \mathrm{reldim}\, \pi = \binom{r}{2}$, we set

$$A := \pi_* \iota^* : H^*(\chi_p^{-1}(0)) \to H^{*-\binom{r}{2}}(M_{r,d}),$$

$$B := \iota_* \pi^* : H^*(M_{r,d}) \to H^{*+\binom{r}{2}}(\chi_p^{-1}(0))$$

Define $\Delta := \prod_{1\leq i<j\leq r}(y_{p,i} - y_{p,j}) \in H^{2\binom{r}{2}}\left(\chi_p^{-1}(0)\right)$, it is the Euler class of relative tangent bundle of π, as well as the normal bundle of $\widetilde{M}$ in $\chi_p^{-1}(0)$. Thus we have:

$$\pi_*(\Delta) = \pm r!, \quad \iota^* \iota_* = \pm\Delta, \quad AB = \pi_* \iota^* \iota_* \pi^* = \pm r!$$

Both operators A, B preserve perverse filtration. By the $\mathfrak{h}$-homogeneity of Δ, we can deduce

Proposition 7. *Let* $-N$ *be the perversity of* $1 \in H^*(M_{r,d})$ *and* $-N'$ *be the perversity of* $1 \in H^*(\chi_p^{-1}(0))$*, then* $N' = N + \binom{r}{2}$*.*

Apply AB to $f = \prod_i \psi_{m_i}(\xi_i) \in H^*(M_{r,d}), \xi_i \in H^*(C)$, combining the previous Proposition and $P = C$ for $\chi_p^{-1}(0)$, we can deduce that $f \in P_{\sum_i m_i - N} H^*(M_{r,d})$. Conversely, for any $f \in P_m H^*(M_{r,d})$, $Bf = \Delta f \in H^*(\chi_p^{-1}(0))$ is of the form

$$\sum_k \lambda_k g_k, \text{ where } g_k \text{ monomial of the form } \prod_i \psi_{m_i}(\xi_i), \xi_i \in H_{r,p} \supseteq H^*(C)$$

Define an operator $\mathrm{Asym} := \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} (-1)^{\ell(\sigma)} \sigma$ on $H^*(\chi_p^{-1}(0))$, then

$$Bf = \Delta f = \sum_k \lambda_k \,\mathrm{Asym}(g_k)$$

where $\mathrm{Asym}(g_k)$ is of the form $\Delta \cdot \prod_l \psi_{m_l}(\xi_l)$, with $\xi_l \in H^*(C)$. We conclude that

Theorem 8. *The subspace $P_m H^*(M_{r,d})$ is spanned by products $\prod_i \psi_{m_i}(\xi_i)$ with $\sum_i m_i \leq m + N$, where $-N$ is the perversity of $1 \in H^0(M_{r,d})$.*

Theorem 8 is in fact the $P = C$ for the stable Higgs moduli space. Combine the result of Shende [3], we finally reach the following statement for $P = W$:

Corollary 9. *The subspace $W_{2m} H^*(M_{r,d})$ is the span of products $\prod_i \psi_{m_i}(\xi_i)$ with $\sum_i m_i \leq m$. Hence we have $P_k = W_{2(k+N)}$.*

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