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# Hochschild (Co)Homology and Applications

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Abstract. In 1945 Gerhard Hochschild published *On the cohomology groups of an associative algebra* in the *Annals of Mathematics* and thereby created what is now called Hochschild theory. The subject not only provides interesting homological invariants; it also serves as a link connecting algebra, topology, and geometry. The focus of the meeting was on recent developments, for instance in the study of singularities, deformations, and representations.

*Mathematics Subject Classification (2020):* Primary: 16Exx. Secondary: 13Dxx, 19Dxx, 20Jxx, 53Dxx, 55Nxx.



## Introduction by the Organizers

At the end of the nineteenth century, Poincaré created invariants to distinguish different topological spaces and their features, foreshadowing the homology and cohomology groups that would appear later. Towards the middle of the twentieth century, these notions were imported from topology to algebra. The subject of group homology and cohomology was founded by Eilenberg and MacLane, and the subject of Hochschild homology and cohomology by Hochschild.

The meeting was devoted to Hochschild (co)homology, which now appears in the settings of representation theory, algebraic geometry, category theory, functional analysis, topology, and beyond. There are strong connections to cyclic homology and K-theory. Many mathematicians use Hochschild (co)homology in their research, and many continue to develop theoretical and computational techniques for better understanding. Hochschild (co)homology is a broad and growing field, with connections to diverse parts of mathematics.

A similar meeting with title *Hochschild (co)homology and derived categories* was planned as a Satellite for the ICM 2022 in St. Petersburg (`https://indico.eimi.ru/event/315/`). The meeting was cancelled, but the organizers are grateful that the Oberwolfach Institute provided a second chance for this event.

There were 22 talks and the extended abstracts provide more details. Almost all talks were given in person, except the ones of Claude Cibils and Travis Schedler who joined via Zoom (the latter giving his talk at SLMath in Berkeley). The following additional short contributions were presented during a *Gong Show*:

- Tekin Karadag: Nonabelian Hochschild cohomology and abelian Hopf cohomology
- Naageswaran Manikann: Chromatic graph homology and its relation to Hochschild homology
- Matt Booth: Nonsmooth Calabi–Yau algebras
- Isambard Goodbody: Reflexivity and Hochschild cohomology
- Julie Symons: An equivalence of derived deformations
- Miantao Liu: Categorification of Goncharov–Shen's basic triangle
- Jules Besson: Categorification of cluster algebras through hereditary extriangulated categories
- Arne Mertens: Quasi-categories in modules as weak dg-categories
- Violeta Borges Marques: Hochschild cohomology for quasi-categories in modules
- Øyvind Solberg (report on work by Mads Hustad Sandøy): Radical cube zero selfinjective algebras with finitely generated Hochschild cohomology

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## Workshop: Hochschild (Co)Homology and Applications

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# Abstracts

## Keller's conjecture for singular Hochschild cohomology

Xiao-Wu Chen
(joint work with Huanhuan Li, Zhengfang Wang)

Let $A$ be a finite dimensional algebra over a field $k$. Denote by $A^e = A \otimes_k A^{\mathrm{op}}$ its enveloping algebra. Recall that the *Hochschild cohomology algebra* of $A$ is defined to be the graded algebra

$$\mathrm{HH}^*(A) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_{A^e}(A, A),$$

whose multiplication is known as the cup product which makes $\mathrm{HH}^*(A)$ a graded-commutative algebra.

Let $\mathcal{T}$ be a small $k$-linear triangulated category with $\Sigma$ its suspension functor. For any integer $n$, we denote by $\mathrm{Hom}(\mathrm{Id}_\mathcal{T}, \Sigma^n)$ the $k$-space formed by all natural transformations $\eta\colon \mathrm{Id}_\mathcal{T} \to \Sigma^n$ between triangle functors. The *graded center* of $\mathcal{T}$ is a graded algebra

$$Z^*(\mathcal{T}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(\mathrm{Id}_\mathcal{T}, \Sigma^n),$$

whose multiplication is defined such that $\eta'\eta = \Sigma^n(\eta') \circ \eta$ with $\eta' \in Z^m(\mathcal{T})$. We observe that $\eta\eta' = (-1)^{mn}\eta'\eta$, that is, $Z^*(\mathcal{T})$ is also graded-commutative.

Denote by $A$-mod the abelian category of finite dimensional left $A$-modules, and by $\mathbf{D}^b(A\text{-mod})$ its bounded derived category. The *characteristic morphism* of $A$ is the following homomorphism between graded algebras

$$\chi^A\colon \mathrm{HH}^*(A) \longrightarrow Z^*(\mathbf{D}^b(A\text{-mod})),\ \zeta \mapsto \zeta \otimes^{\mathbb{L}}_A -.$$

The homomorphism $\chi^A$ plays a role in support varieties and deformation theory.

Let $\mathcal{C}$ be a small dg category. Its Hochschild cohomolgy algebra is defined to be

$$\mathrm{HH}^*(\mathcal{C}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathbf{D}(\mathcal{C}^e)}(\mathcal{C}, \Sigma^n(\mathcal{C})),$$

where $\mathbf{D}(\mathcal{C}^e)$ is the derived category of right dg modules over the enveloping dg category $\mathcal{C}^e = \mathcal{C} \otimes_k \mathcal{C}^{\mathrm{op}}$.

Denote by $\mathbf{D}(\mathcal{C})$ the derived category of right dg $\mathcal{C}$-modules. By the Yoneda embedding, the homotopy category $H^0(\mathcal{C})$ is viewed as a full subcategory of $\mathbf{D}(\mathcal{C})$. The dg category $\mathcal{C}$ is called pretriangulated if $H^0(\mathcal{C})$ is a triangulated subcategory of $\mathbf{D}(\mathcal{C})$, in which case, $\mathcal{C}$ is called a *dg enhancement* of $H^0(\mathcal{C})$. We have a canonical morphism

$$\mathrm{can}\colon \mathrm{HH}^*(\mathcal{C}) \longrightarrow Z^*(H^0(\mathcal{C})),\ \zeta \mapsto \zeta \otimes^{\mathbb{L}}_\mathcal{C} -.$$

Here, each morphism $\zeta\colon \mathcal{C} \to \Sigma^n(\mathcal{C})$ in $\mathbf{D}(\mathcal{C}^e)$ gives rise to a natural transformation

$$\zeta \otimes^{\mathbb{L}}_\mathcal{C} -\colon \mathrm{Id}_{\mathbf{D}(\mathcal{C})} \longrightarrow \Sigma^n,$$

which restricts to the required element $\zeta \otimes^{\mathbb{L}}_\mathcal{C} -\colon \mathrm{Id}_{H^0(\mathcal{C})} \to \Sigma^n$ in the graded center.

The bounded dg derived category $\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})$ is a canonical dg enhancement of $\mathbf{D}^b(A\text{-mod})$. Furthermore, we have the following well known result.

**Proposition 1.** *There is an isomorphism $\phi^A$ of graded algebras making the following triangle commutative.*

$$\begin{tikzcd} \mathrm{HH}^*(A) \arrow[rr, "\phi^A"] \arrow[dr, "\chi^A"'] & & \mathrm{HH}^*(\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})) \arrow[dl, "\mathrm{can}"] \\ & Z^*(\mathbf{D}^b(A\text{-mod})) & \end{tikzcd}$$

Set $\mathcal{D} = \mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})$. By [2], the isomorphism $\phi^A$ is induced by a fully-faithful triangle functor

$$\mathbf{D}^b(A^e\text{-mod}) \longrightarrow \mathbf{D}(\mathcal{D}^e),\ X \mapsto \mathcal{D}(-, X \otimes^{\mathbb{L}}_A -).$$

Denote by $C^*(A,A)$ the Hochschild cochain complex of $A$, and by $C^*(\mathcal{D},\mathcal{D})$ the Hochschild cochain complex of $\mathcal{D}$. They are both *brace $B_\infty$-algebras* [1], with their cup products and brace operations. We recall that a $B_\infty$-algebra structure on a graded space $V$ is equivalent to a dg bialgebra structure $(T^c(sV), \Delta, D, \mu)$ on the tensor coalgebra $(T^c(sV), \Delta)$. The inverse of $\phi^A$ is induced by the restriction $C^*(\mathcal{D},\mathcal{D}) \to C^*(A,A)$, where we identify $A$ with the full dg subcategory of $\mathcal{D}$ given by the single object $A$.

We have the following fundamental result.

**Theorem 2.** (Keller, Lowen-Van den Bergh) *The isomorphism $\phi^A$ above lifts to an isomorphism*

$$C^*(A^{\mathrm{op}}, A^{\mathrm{op}}) \simeq C^*(\mathcal{D},\mathcal{D})$$

*in the homotopy category of $B_\infty$-algebras.*

To better understand the appearance of the opposite algebra $A^{\mathrm{op}}$ above, we define the *transpose $B_\infty$-algebra* $V^{\mathrm{tr}}$ of a given $B_\infty$-algebra $V$: they have the same underlying graded space, and the dg bialgebra corresponding to $V^{\mathrm{tr}}$ is isomorphic to $(T^c(sV), \Delta^{\mathrm{op}}, D, \mu)$. This definition is motivated by the following fact: there is a strict $B_\infty$-isomorphism

$$C^*(A,A)^{\mathrm{tr}} \simeq C^*(A^{\mathrm{op}}, A^{\mathrm{op}}).$$

For a $B_\infty$-algebra $V$, its *opposite $B_\infty$-algebra* $V^{\mathrm{opp}}$ corresponds to the dg bialgebra $(T^s(sV), \Delta, D, \mu^{\mathrm{opp}})$. In particular, $V^{\mathrm{opp}}$ and $V$ have the same underlying $A_\infty$-algebra structure. We have the following duality theorem [1].

**Theorem 3.** *Let $V$ be a $B_\infty$-algebra. Then there is a $B_\infty$-quasi-isomorphism $V^{\mathrm{tr}} \to V^{\mathrm{opp}}$.*

Combining the two theorems above, we obtain an isomorphism

$$C^*(A,A)^{\mathrm{opp}} \simeq C^*(\mathcal{D},\mathcal{D})$$

in the homotopy category of $B_\infty$-algebras.

Recall that the *singularity category* of $A$ is defined by the Verdier quotient triangulated category

$$\mathbf{D}_{\mathrm{sg}}(A) = \mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-proj}).$$

Denote by $\mathcal{P}$ the full dg subcategory of $\mathcal{D}$ formed by perfect complexes. Then the dg singularity category $\mathbf{S}_{\mathrm{dg}}(A) = \mathcal{D}/\mathcal{P}$ canonically enhances $\mathbf{D}_{\mathrm{sg}}(A)$.

The *singular Hochschild cohomology algebra* of $A$ is defined to the graded algebra

$$\mathrm{HH}^*_{\mathrm{sg}}(A) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A^e)}(A, \Sigma^n(A)).$$

It is graded-commutative. The following result is analogous to Theorem 2.

**Theorem 4.** (Keller [2]) *Assume that $\mathcal{D}$ is smooth. Then there is an isomorphism*

$$\psi^A\colon \mathrm{HH}^*_{\mathrm{sg}}(A) \simeq \mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(A))$$

*of graded algebras.*

In view of Theorems 2 and 4, Keller conjectures that the isomorphism $\psi^A$ lifts to the $B_\infty$-level. To make it more precise, we recall that both the *left singular Hochschild cochain complex* $\overline{C}^*_{\mathrm{sg},L}(A,A)$ and *right singular Hochschild cochain complex* $\overline{C}^*_{\mathrm{sg},R}(A,A)$ compute $\mathrm{HH}^*_{\mathrm{sg}}(A)$, and are brace $B_\infty$-algebras.

**Conjecture.** (Keller [2]) *Assume that $\mathcal{D}$ is smooth and set $\mathcal{S} = \mathbf{S}_{\mathrm{dg}}(A)$. Then there is an isomorphism*

$$\overline{C}^*_{\mathrm{sg},L}(A^{\mathrm{op}}, A^{\mathrm{op}}) \simeq C^*(\mathcal{S},\mathcal{S})$$

*in the homotopy category of $B_\infty$-algebras.*

There is a stronger version of Keller's conjecture, which claims that the isomorphism above lifts $\psi^A$. We only treat the weak version. The following invariance theorem [1] justifies Keller's conjecture to some extent.

**Theorem 5.** *Keller's conjecture is invariant under one-point (co-)extensions and singular equivalences with level.*

Since any derived equivalence induces a singular equivalence with level, then Keller's conjecture is invariant under derived equivalences.

Let $Q$ be a finite quiver without sinks. Denote by $A_Q = kQ/J^2$ the corresponding algebra with radical square zero. The *Leavitt path algebra* $L(Q)$ is naturally $\mathbb{Z}$-graded, and is viewed as a dg algebra with trivial differential. We verify Keller's conjecture for $A_Q$; see [1].

**Theorem 6.** *Let $Q$ be a finite quiver without sinks. Write $\mathcal{S}_Q = \mathbf{S}_{\mathrm{dg}}(A_Q)$. Then there are isomorphisms*

$$\overline{C}^*_{\mathrm{sg},L}(A_Q^{\mathrm{op}}, A_Q^{\mathrm{op}}) \simeq C^*(L(Q), L(Q)) \simeq C^*(\mathcal{S}_Q,\mathcal{S}_Q)$$

*in the homotopy category of $B_\infty$-algebras.*

Theorems 5 and 6 imply that finite dimensional gentle algebras satisfy Keller's conjecture.

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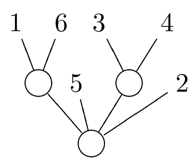
## A generalization of cyclic homology for operads

Vladimir Dotsenko

Similar to how associative algebras give an abstraction of the notion of an endomorphism of a vector space $V$, (symmetric) operads [6] give an abstraction of the notion of a multilinear map. Matrices of the given size can be multiplied, and the product is bilinear and associative, which is precisely how one defines an associative algebra. A multilinear map has a certain number of arguments, say $n$, and one has the following structural features:

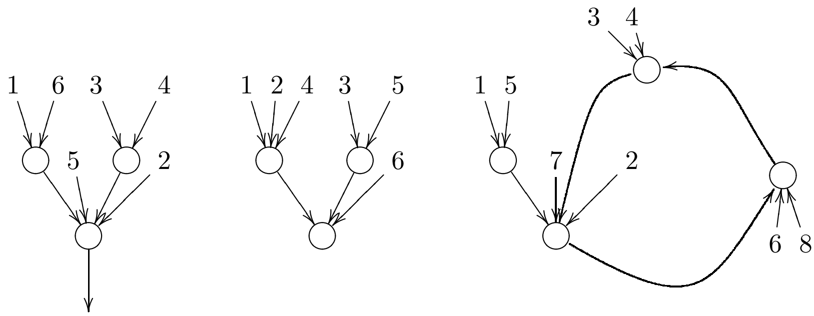
- an action of the symmetric group $S_n$ on multilinear operations with $n$ arguments,
- if we consider all multilinear operations together, one can substitute operations into one another, forming operations with more arguments,
- moreover, substitutions of multilinear operations in one another are linear in each of the operations, are, in a sense, associative, and are reasonably equivariant with respect to the symmetric group actions.

Graphically, it is convenient to visualize iterated substitutions of multilinear operations using rooted trees.




Here, one can decorate each vertex with $k$ incoming edges by a multilinear map with $k$ arguments, and then compose them "along the tree", and the properties above (associativity and equivariance) simply mean that the result of such a composition does not depend on the order of "partial" calculations that contract edges of a tree one by one. Now if we replace multilinear maps by a collection $\mathcal{O} = \{\mathcal{O}(n)\}$ of representations of symmetric groups that can be composed along trees, we obtain an *operad*. Moreover, if we assume that $\mathcal{O}$ is augmented, one can define the *bar construction* $\mathsf{B}(\mathcal{O})$, which is the chain complex made of rooted trees whose vertices with $k$ inputs are decorated by elements of $s\overline{\mathcal{O}}(k)$, the homological shift of the $k$-th component of the augmentation ideal of $\mathcal{O}$, with the differential that computes the alternating sum of edge contractions. This chain complex carries all crucial information on the homotopy theory of $\mathcal{O}$. If $\mathcal{O}(k) = 0$ for $k \neq 1$, this recovers the usual bar construction of an augmented associative algebra $A = \mathcal{O}(1)$; that chain complex computes $\mathrm{Tor}^A_\bullet(\Bbbk, \Bbbk)$.

Suppose now that the vector space $V$ is finite-dimensional. In this case, there is one more bit of structure one can consider: computing traces with respect to one of the arguments, thus obtaining multilinear scalar functions from multilinear maps. The combinatorial objects that one need to add to rooted trees to encode the properties of compositions and traces are now as follows: one needs rooted trees without an output (then the root vertex can be decorated by a scalar multilinear function) and the "wheel graphs", which are directed graphs of genus one where each vertex has only one outgoing edge. Here are examples of these:




Now if we replace multilinear maps by a collection $\mathcal{O} = \{\mathcal{O}(n)\}$ of representations of symmetric groups and multilinear functions by another collection $\mathcal{W} = \{\mathcal{W}(n)\}$ of the same kind, and require that those can be evaluated along the graphs above, we obtain a *wheeled operad* [7]. Moreover, if we assume that $(\mathcal{O}, \mathcal{W})$ is augmented, one can define the *wheeled bar construction* $\mathsf{B}^{\circlearrowleft}(\mathcal{O}, \mathcal{W})$, which is the chain complex made of our graphs whose vertices with $k$ inputs and one output are decorated by elements of $s\overline{\mathcal{O}}(k)$ and whose vertex with $k$ inputs and no outputs (if exists) is decorated by an element of $s\overline{\mathcal{W}}(k)$, with the differential that computes the alternating sum of edge contractions.

In particular, for an operad $\mathcal{O}$ one can construct two canonical wheeled operads: one can consider $\mathcal{O}$ as a wheeled operad where all contractions along graphs without an output vertex are zero, and one can consider the *wheeled completion* $\mathcal{O}^{\circlearrowleft}$ which is the left adjoint functor (we add all formal evaluations along our graphs, and nothing else). If $\mathcal{O}(k) = 0$ for $k \neq 1$, this recovers the usual bar construction of an augmented associative algebra $A = \mathcal{O}(1)$ (computing the Tor groups as mentioned above) *and* the cyclic bar complex (computing the cyclic homology of $A$). It is thus not unreasonable to say that the wheeled bar construction generalizes cyclic homology.

Let $\mathcal{O}$ be an augmented operad over a field $\Bbbk$ of zero characteristic. Consider the Lie algebra $\mathrm{Der}(\mathcal{O}(x_1, \ldots, x_n))$ of all derivations of the free $\mathcal{O}$-algebra with $n$ generators, and its subalgebra $\mathrm{Der}^+(\mathcal{O}(x_1, \ldots, x_n))$ that is the kernel of the Lie algebra homomorphism $\mathrm{Der}(\mathcal{O}(x_1, \ldots, x_n)) \to \mathfrak{gl}_n$ arising from the augmentation of $\mathcal{O}$. Using the above Lie algebra homomorphism, one can view every $\mathfrak{gl}_n$-module as a $\mathrm{Der}(\mathcal{O}(x_1, \ldots, x_n))$-module. In my recent preprint [1], the following result was proved.

**Theorem 1.** *As $n \to \infty$, the homology $H_\bullet(\mathrm{Der}(\mathcal{O}(x_1, \ldots, x_n)), (\Bbbk^n)^{\otimes p} \otimes (\Bbbk^{n*})^{\otimes q})$ stabilizes, and can be explicitly out of the homology of the wheeled bar construction $\mathsf{B}^{\circlearrowleft}(\mathcal{O})$: the collection of these homology groups for all $p, q$ is the coPPROP completion $S^c(H_\bullet(\mathsf{B}^{\circlearrowleft}(\mathcal{O})))$. Moreover, the multiplicity of each finite-dimensional irreducible $\mathfrak{gl}_n$-module in the homology $H_\bullet(\mathrm{Der}_+(\mathcal{O}(x_1, \ldots, x_n)))$ stabilizes and can be explicitly computed out of the same coPPROP completion $S^c(H_\bullet(\mathsf{B}^{\circlearrowleft}(\mathcal{O})))$.*

In particular, this theorem implies the theorem of Loday–Quillen [5] and Tsygan [8] on the homology of the Lie algebra of infinite matrices, its non-unital version of Feigin–Tsygan [2] and Hanlon [4], and the theorem of Fuchs [3] on stability of the homology of the Lie algebra of vector fields on $\Bbbk^n$. The proof relies on classical invariant theory for $\mathfrak{gl}_n$, and brings the wealth of operadic methods into matters of Lie algebra homology, allowing to prove various new results of similar flavour.

It turns out [1] that if one considers the bar construction of the wheeled completion $\mathsf{B}^{\circlearrowleft}(\mathcal{O}^{\circlearrowleft})$, it has a similar relationship to another, arguably even more important Lie algebra, the algebra $\mathrm{SDer}(\mathcal{O}(x_1, \ldots, x_n))$ of divergence zero derivations of the free algebra, for appropriately defined divergence. In many interesting cases, tangent derivations of automorphisms of free algebras have zero divergence, so these results can be viewed as some infinitesimal K-theory computations.

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## On the Hochschild homology of the category of finitely generated projective modules

Antoine Touzé
(joint work with A. Djament)

If $\mathcal{K}$ be a $k$-linear category over a field $k$, we denote by $\mathrm{HH}_*(\mathcal{K}, B)$ the Hochschild homology of $\mathcal{K}$ with coefficient in the $\mathcal{K}$-$\mathcal{K}$-bimodule $B$, as defined by Mitchell [7]. In this talk, we focus on the Hochschild homology of the $k$-linear category $k[\mathrm{P}_R]$. The objects of $k[\mathrm{P}_R]$ are the finitely generated projective right modules over a ring $R$, its vector spaces of morphisms are given by:

$$\mathrm{Hom}_{k[\mathrm{P}_R]}(P,Q) := k[\mathrm{Hom}_R(P,Q)]$$

(where $k[X]$ is a notation for the free $k$-vector space on a set $X$), and its composition law is the unique $k$-bilinear map which extends the composition of $R$-linear maps. The data of a $k[\mathrm{P}_R]$-$k[\mathrm{P}_R]$-bimodule is equivalent to the data of a (non-necessarily additive) bifunctor

$$B : \mathrm{P}_R \times \mathrm{P}_R^{\mathrm{op}} \to \mathrm{Vect}_k \ ,$$

where $\mathrm{P}_R \subset \mathrm{Mod}_R$ is the full subcategory of finitely generated projective modules.

One reason to consider the Hochschild homology $\mathrm{HH}_*(k[\mathrm{P}_R], B)$ is its close relation with the homology of general linear groups. Namely, for all positive integers $n$, the $k$-vector space

$$B_n := B(R^n, R^n)$$

is naturally endowed with an action of $GL_n(R)$, where every $g \in GL_n(R)$ acts as $B(g, g^{-1})$ on $B_n$. These $k$-linear representations $B_n$ assemble into a a $k$-linear representation $B_\infty = \bigcup_{n>0} B_n$ of $GL_\infty(R) = \bigcup_{n>0} GL_n(R)$. Scorichenko has shown [8] that if the bifunctor $B$ is *polynomial* in the sense of Eilenberg and Mac Lane [3], there is an isomorphism of graded vector spaces:

$$\mathrm{H}_*(GL_\infty(R), B_\infty) \simeq \mathrm{H}_*(GL_\infty(R), k) \otimes_k \mathrm{HH}_*(k[\mathrm{P}_R], B). \qquad (*)$$

(Scorichenko's result is actually formulated in terms of stable $K$-theory, we refer the reader to [1] for a published proof formulated in terms of group homology.)

The remarkable isomorphism $(*)$ is a strong motivation to try to understand and compute $\mathrm{HH}_*(k[\mathrm{P}_R], B)$ for polynomial coefficients $B$. The next example gives a typical polynomial bifunctor, for which it would be interesting to have a computation.

**Example 1.** Given a $(k, R)$-bimodule $M$ and a $(R, k)$-bimodule $N$, one can consider the bifunctor

$$B(P,Q) = S^d(\mathrm{Hom}_R(P,M)) \otimes_k S^e(Q \otimes_R N) \ ,$$

where $S^i$ denotes the $i$-th symmetric power of a $k$-vector space. This bifunctor is polynomial, of degree $d$ with respect to $P$ and of degree $e$ with respect to $Q$.

Until now, the Hochschild homology was understood only when $k$ is a field of characteristic zero (folklore) or when $k = R = \mathbb{F}_q$ [5, 4, 6]. In a recent work [2] with A. Djament, we obtain explicit formulas computing $\mathrm{HH}_*(k[\mathrm{P}_R], B)$ , when $k$ is a perfect field of positive characteristic $p$, $R$ is a ring of characteristic $p$, and for a wide family of polynomial coefficients $B$ (including the ones given in example 1). Our results show that for these coefficients, the Hocschild homology of $k[\mathrm{P}_R]$ is controlled by two classical quantities, namely:

(1) $\mathrm{Tor}_*$ of modules over $R \otimes_{\mathbb{Z}} k$,
(2) $\mathrm{Tor}_*$ of modules over classical Schur algebras.

In this talk, we explain the computations of [2] in the "easy case", that is, when the characteristic of $k$ is big enough. In the case of the typical polynomial coefficients $B$ given in example 1, we obtain the following computation.

**Example 2.** Assume that $k$ is a perfect field of characteristic $p$, that $R$ is a ring of characteristic $p$ and that $B$ is the polynomial bifunctor of example 1. Assume furthermore that $d$ and $e$ are less than $p$. Then

$$\mathrm{HH}_*(k[\mathrm{P}_R], B) = 0 \quad \text{if } d \neq e.$$

If $d = e$, let $T_*$ denote the graded $k$-vector space which is equal to $k$ in nonnegative even degrees and zero elsewhere, and let $\mathbb{T}(M,N)_{\mathrm{even}}$ and $\mathbb{T}(M,N)_{\mathrm{odd}}$ denote the even degree summand and the odd degree summand of the graded $k$-vector space

$$\mathbb{T}(M,N)_* = T_* \otimes_k \mathrm{Tor}_*^{R\otimes_{\mathbb{Z}} k}(M,N)\ .$$

Then there is a graded isomorphism:

$$\mathrm{HH}_*(k[\mathrm{P}_R], B) = \bigoplus_{0\le i\le d} \Lambda^i(\mathbb{T}(M,N)_{\mathrm{odd}}) \otimes_k S^{d-i}(\mathbb{T}(M,N)_{\mathrm{even}})$$

where $\Lambda^i$ refers to the $i$-th exterior power of a $k$-vector space.

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## How to enhance categories, and why

Dmitry Kaledin

It has become an accepted wisdom by now that when one localizes a category $\mathcal{C}$ with respect to a class of maps $W$, the resulting object should be treated as more than just a category $h^W(\mathcal{C})$ — it has to be equipped with an additional structure colloquially known as "enhancement". A well-known example of this is the derived category $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$, obtained by localizing the category of chain complexes in $\mathcal{A}$ with respect to the class of quasiisomorphisms, where one should at least equip $\mathcal{D}(\mathcal{A})$ with a triangulated structure, and even this is not enough for many practical applications. However, the phenomenon is actually much more widespread and deep.

One example of localization that is so ubiquitous that it passes almost unnoticed is that of the category Cat of small categories, with $W$ being the class of equivalences of categories. Namely, recall that a commutative square of categories and functors is commonly understood as a square

$$\begin{array}{ccc} \mathcal{C}_{01} & \xrightarrow{\gamma_{01}^1} & \mathcal{C}_1 \\ {\scriptstyle\gamma_{01}^0}\downarrow & & \downarrow{\scriptstyle\gamma_1} \\ \mathcal{C}_0 & \xrightarrow{\gamma_0} & \mathcal{C} \end{array} \tag{*}$$

of categories and functors equipped with an isomorphism $\alpha : \gamma_0 \circ \gamma_{01}^0 \to \gamma_1 \circ \gamma_{01}^1$ that is rarely written explicitly but always implied. A square can be cartesian or cocartesian if it satisfies the usual universal properties (in particular, localization is an example of a cocartesian square). A square (*) is an honest commutative square in Cat if all the categories are small and $\alpha = \mathrm{id}$, but this is not what we want in practice, and gives the wrong version of the universal property. If one localizes Cat with respect to $W$, the resulting category $\mathrm{Cat}^0$ is easy to describe explicitly – its objects are small categories, and morphisms are isomoprhism classes of functors – an a square (*) of small categories always defines a commutative square in $\mathrm{Cat}^0$, but the universal property of a cartesian square is lost: for uniqueness, one needs to remember $\alpha$, while passing to $\mathrm{Cat}^0$ forgets it, and only remembers the fact that some $\alpha$ exists. This behaviour is typical; it also occurs for cones in triangulated categories, and for homotopy cartesian and cocartesian products.

In practice, for Cat, we know what to do: we actually work in $\mathrm{Cat}^0$, but we remember that Cat is somewhere in the background, and go up to that level whenever needed. For a more general pair $\langle \mathcal{C}, W \rangle$, the problem is more complicated. At the very least, we expect to have a "homotopy type" of morphisms $\mathcal{H}^W(\mathcal{C})(c, c')$ for any objects $c, c' \in h^W(\mathcal{C})$ in the localized category such that the set of map $h^W(\mathcal{C})(c, c')$ is $\pi_0$ of this homotopy type (for Cat, the corresponding homotopy types are 1-truncated and correspond to the groupoids of functors and isomorphisms between functors). This is difficult to formalize since already a "homotopy type" is something only defined up to an equivalence of some sort, thus sits in a category obtained by localization, so the argument becomes unpleasantly circular.

A common perception is that this curcularity is unavoidable. Namely, there are categories of "models for homotopy theories", equipped with a model structure in the sense of Quillen [4], and in particular, with a class of "weak equivalences". The simplest to state is the category Top-Cat of small categories enriched in topological spaces, considered modulo some rather complicated notion of a weak equivalence. We have the truncation functor $\pi_0$ : Top-Cat $\to$ Cat, and a topological category is taken to provide an enhancement for its truncation. There are many other models that are more convenient to work with, such as "quasicategories" of A. Joyal and J. Lurie [3], or "complete Segal spaces" of Ch. Rezk [R]. All these categories of models are "Quillen-equivalent", and it is accepted as an axiom that this means that they define the "same" homotopy theory. In particular, it is a theorem that the naive localized categories are then canonically equivalent, and a localized category $\mathcal{H}^W(\mathcal{C})$ is defined as an object in this "localized category of homotopy theories". The curcularity of such a definition is accepted as a necessary evil: if one cannot do better, it is better to do something rather than not do anything at all.

The point of my talk is that this perception is not quite correct: one *can* do better. The idea actually goes back to Grothendieck [2], and it is quite simple. For any category $\mathcal{C}$ and small category $I$, we have the category $I^o\mathcal{C}$ of functors from the opposite category $I^o$ to $\mathcal{C}$, and for any class $W$ of maps in $\mathcal{C}$, we have the class $W(I)$ of maps in $I^o\mathcal{C}$ that are pointwise in $W$. Thus we not only have the localized category $h^W(\mathcal{C})$, but a whole bunch of categories $h^{W(I)}(I^o\mathcal{C})$ indexed by $I \in$ Cat; the question is, is this family of categories enough to recover the enhanced for $h^W(\mathcal{C})$?

We have found out that the answer to this question is positive. Moreover, it is not necessary, nor in fact desirable, to use all small categories $I \in$ Cat, and it suffices to restrict out attention to the full subcategory Pos $\subset$ Cat of partially ordered sets. A family of categories indexed by $J \in$ Pos is conveniently axiomatized by a Grothendieck fibration $\mathcal{C} \to$ Pos in the sense of [1], with morphisms between families given by functors cartesian over Pos. For any $\mathcal{C} \in$ Top-Cat and $J \in$ Pos, we have the functor category $J^o\mathcal{C} \in$ Top-Cat and its truncation $\pi_0(J^o\mathcal{C}) \in$ Cat, and taken together, these define a Grothendieck fibration $K(\mathcal{C}) \to$ Pos with fibers $K(\mathcal{C})_J \cong \pi_0(J^o\mathcal{C})$. Our main result is the following.

- For any $\mathcal{C}, \mathcal{C}' \in$ Top-Cat fibrant with respect to the model structure, and any functor $\gamma : K(\mathcal{C}) \to K(\mathcal{C}')$ cartesian over Pos, there exists a map $f : \mathcal{C} \to \mathcal{C}'$ such that $\gamma \cong K(f)$, and two maps $f, f' : \mathcal{C} \to \mathcal{C}'$ are homotopic if and only if $K(f)$ and $K(f')$ are isomorphic over Pos.

Informally, we can consider the category $(\mathrm{Cat} / \mathrm{Pos})^0$ of categories fibered over Pos, with morphisms given by isomorphism classes of cartesian functors, and then our main result claims that $K$ provides a fully faithful embedding $K : h^W(\text{Top-Cat}) \to (\mathrm{Cat} / \mathrm{Pos})^0$. This is not quite correct since Pos is large, so isomorphism classes of cartesian functors can form a proper class and not a set; part of our result is that this does not happen for fibrations of the form $K(\mathcal{C}) \to$ Pos.

For the category Cat, our main results gives nothing: at the end of the day, our enhancements are again categories considered up to an equivalence. What we do, however, is reduce the localization problem for a general category to the case $\langle \mathrm{Cat}, W\rangle$ – where, as mentioned above, we in any case know what to do.

Of course, our main result is just the beginning of the story. Here are some other things one can do: characterize the essential image of the full embedding $K$ (this requires about six axioms, similar to the Steenrod-Eilenberg axiomatization of generalized cohomology theories), construct an enhancement for the category of enhanced categories (that is, of fibrations $\mathcal{C} \to \mathrm{Pos}$ lying in the image of $K$), describe homotopy cartesian and cocartesian squares by a universal property. Some of the important categorical notions are generalized *verbatim* – this includes fully faithful functors and adjoint pairs. In general, developing the whole enhanced category theory in this setting becomes a rather pleasant exercise, and the arguments are the usual categorical arguments; only very rarely one has to remember the representing objects in the category Top-Cat.

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## On the ring theory of differential graded rings

Alexander Zimmermann

Let $R$ be a commutative ring. Cartan defined in [3] a differential graded $R$-algebra as a $\mathbb{Z}$-graded algebra $A$ together with an $R$-linear endomorphism $d$ of degree 1 with $d^2 = 0$ satisfying $d(ab) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$ for all homogeneous elements $a, b \in A$ of degree $|a|$, resp. $|b|$. A differential graded $(A, d)$-module $(M, \delta)$ (or dg-module over a dg-algebra for short) is a $\mathbb{Z}$-graded $A$-module with an $R$-linear endomorphism $\delta$ of degree 1 satisfying $\delta^2 = 0$ and $\delta(a \cdot m) = d(a) \cdot m + (-1)^{|a|} a \cdot \delta(m)$ for all homogeneous $a \in A$ and $m \in M$. Similarly we define dg-right modules and dg-bimodules. A dg-submodule of a dg-module $(M, \delta)$ is a graded submodule stable under the action of $\delta$. A dg-module is dg-simple $(S, \delta)$ if there is no dg-submodule of $S$ other than $S$ or 0. A dg-algebra is dg-simple if it is simple as dg-bimodule over itself.

The ring theory of dg-algebras laid unexplored until very recently. Aldrich and Garcia Rozas characterised in [1] dg-algebras with semisimple dg-module categories. Orlov [7, 8] studied finite dimensional dg-algebras over a field mainly under a geometric perspective. Goodbody [4] used Orlov's work to define dg-Jacobson radicals and shows a version of Nakayama's lemma for finite dimensional dg-$R$-algebras over a field $R$, such that the quotient modulo the ordinary Jacobson radical is separable.

We propose a more systematic concept. Consider the set of dg-left ideals of $(A, d)$. The intersection of the maximal elements in this poset will produce a dg-left ideal. If $d = 0$ and the grading is trivial, then this is the classical Jacobson ideal, whence two-sided. However, consider the following

**Example 1.** [9] The endomorphism complex of a dg-module $(M, \delta)$ is a dg-algebra, as is well-known. In case of a field $K$ with trivial grading and 0 differential we consider the complex $K \stackrel{\mathrm{id}}{\to} K$, being a dg-K-module concentrated in degree $-1$ and 0. Its endomorphism complex $(A, d)$ is then $\begin{pmatrix} K & K \\ K & K \end{pmatrix}$ with differential $d(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $d(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}) = \begin{pmatrix} 0 & y-x \\ 0 & 0 \end{pmatrix}$, and of course the differential of the upper right corner elements being 0. Then there is only one dg-left module over $A$, namely the right column.

**Definition 2.** [9] The dg-Jacobson radical $\mathrm{dgrad}_2(A, d)$ of a dg-algebra $(A, d)$ is the intersection of the left annihilators of the dg-simple dg-left modules.

It is not hard to see that $\mathrm{dgrad}_2(A, d)$ is a twosied dg-ideal. A dg-module is said to be dg-Noetherian (resp. dg-Artinian) if it satisfies the ascending (resp. descending) chain condition on dg-submodules.

**Theorem 3.** (dg-version of Nakayama's lemma) [9] Let $(A, d)$ be a dg-algebra over a commutative ring $R$, and let $(M, \delta)$ be a dg-Noetherian and dg-Artinian dg-module over $(A, d)$. Then for any dg-submodule $(N, \delta)$ of $(M, \delta)$ we get

$$N + \mathrm{dgrad}_2(A, d) \cdot M = M \Rightarrow N = M.$$

In a similar philosophy dgnil(A,d) is defined to be the sum of nilpotent twosided dg-ideals of $(A, d)$, and $\mathrm{dgPrad}(A, d)$ is the intersection of twosided dg-prime ideals. Here a dg-ideal is said $(P, d)$ to dg-prime if whenever $(T, d)$ and $(S, d)$ are twosided dg-ideals with $ST \subseteq P$, then $S \subseteq P$ or $T \subseteq P$.

**Proposition 4.** [10] Let $(A, d)$ be a left dg-Noetherian and left dg-Artinian dg-algebra. Then $\mathrm{dgnil}(A, d) = \mathrm{dgrad}_2(A, d) = \mathrm{dgPrad}(A, d)$.

I do not know yet if any dg-Artinian algebra is dg-Noetherian, whence if Hopkins' theorem holds in the dg-version.

**Proposition 5.** [9] If $(A, d)$ is dg-Noetherian and dg-Artinian then $\mathrm{dgrad}_2(A, d)$ is the smallest twosided dg-ideal such that the quotient is a finite direct product of dg-simple dg-algebras. Hence the left and the right version of $\mathrm{dgrad}_2(A, d)$ coincide.

So, how to produce dg-simple dg-algebras? Of course, simple algebras which happen to be dg-algebras, such as Example 1, are dg-simple. Orlov [7] calls these algebras formally dg-simple. However, there are more, such as the algebra of dual numbers $K[X]/X^2$ with $|X| = -1$ and $d(X) = 1$. Note that Example 1 is dg-simple, but by [1] the dg-module category is not semisimple.

In order to do produce more dg-simple algebras, we first study Ore localisation for dg-algebras.

**Theorem 6.** [10] Let $(R, d)$ be a dg-ring, and let $S$ be a multiplicative set of homogeneous elements. Let $\mathrm{ass}_\ell(S) := \{r \in R \mid \exists s \in S : \ sr = 0\}$. Assume that either $S$ consists of regular elements, or else $S \subseteq \ker(d)$ is a left Ore set and the image of $S$ in $R/\mathrm{ass}_\ell(S)$ consists of regular elements of $R/\mathrm{ass}_\ell(S)$. Then

$$d_S(b, s) := (-1)^{|s|+1}(d(s), s) \cdot (b, s) + (-1)^{|s|}(d(b), s)$$

defines a differential graded structure on $R_S$, and the natural homomorphism is a dg ring homomorphism $\lambda : (R, d) \to (R_S, d_S)$ such that $\lambda(S) \in R_S^\times$, the group of invertible elements of $R_S$, and such that for any $q \in R_S$ there exists $s \in S$ with $\lambda(s) \cdot q \in \mathrm{im}(\lambda)$. Similar statements hold for the right version.

Note that Braun-Chuang-Lazarev [2] gave a very abstract construction for lifting an Ore localisation of the homology algebra of a dg-algebra at a multiplicative Ore set $\overline{S}$ to an Ore localisation of the dg-algebra. We give here an explicit version, for an Ore set $S$ formed by elements of $\ker(d)$, giving then by reduction the set $\overline{S}$.

We now use this result to construct dg-simple dg-algebras. Recall that a graded algebra is gr-prime if the zero ideal is a gr-prime ideal (i.e. dg-prime for the zero differential). A graded ring is called graded left Goldie if it does not allow an infinite direct sum of graded left ideals, and in addition it satisfies the ascending chain condition on left annihilators.

**Theorem 7.** [10] Let $R$ be a commutative ring and let $(A, d)$ be a differential graded $R$-algebra. Suppose that $\ker(d)$ is a gr-prime ring and suppose that $\ker(d)$ is left gr-Goldie.

- If $(A, d)$ is dg-Noetherian as bimodule, then the localisation of $A$ at the homogeneous regular elements $S_A$ of $A$ exists and is dg-simple.
- If the homogeneous regular elements $S_{\ker(d)}$ of $\ker(d)$ form a left Ore set in $A$,
  - then the left Ore localisation of $(A, d)$ at $S_{\ker(d)}$ is a dg-simple differential graded $R$-algebra.
  - Further, $S_{\ker(d)} \subseteq S_A$ and hence in case $S_A$ is left Ore as well, $A_{S_A}$ and $A_{S_{\ker(d)}}$ both exist, are dg-simple rings, and the natural homomorphism $A_{S_{\ker(d)}} \to A_{S_A}$ is injective.

Recall that the classical Goldie theorem only asks for a semiprime ring obtaining then a semisimple Artinian algebra. However, Goodearl and Stafford [5] show that for $A = K[X, Y]/XY$ where $K$ is a field, $X$ is of degree 1 and $Y$ is of degree 0 is graded Goldie, graded semiprime, but not graded semisimple. Though, the only non invertible homogeneous regular elements are the elements of $K$. Hence, the assumption of being prime is necessary. Goodearl and Stafford show in [5] a Goldie's theorem of group graded graded prime rings, graded by an abelian group. We use their result in an essential way.

We finally mention that we are able to prove in [11] a differential graded version of Posner's theorem using Karasik's result [6]. This then gives again dg-simple algebras using the theory of polynomial identity algebras.

We note that in [11] again, as in Theorem 7, and as in [1], the hypothesis of the classical theorem we want to generalise is assumed in the graded version for $\ker(d)$, and under a few additional technical assumptions we show the dg-version of the classical results. We suspect that this is a general pattern.

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## Cohomology of monoidal categories

Sarah Witherspoon

The Hochschild cohomology of an associative algebra and the cohomology of a finite tensor category may each be viewed as a special case of the cohomology of an exact monoidal category. In this talk, we briefly introduced the types of monoidal categories and cohomology in which we are interested, and made some general definitions and statements while mostly focusing on these two special cases.

A *monoidal category* is a category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that is associative up to a natural isomorphism (satisfying a pentagon axiom) and a unit object $\mathbf{1}$ that is a multiplicative identity for $\otimes$ up to natural isomorphisms. It is an *exact monoidal category* if it is abelian (or more generally additive) and $\otimes$ preserves exact sequences (for a suitable notion of exact sequences). See e.g. [1, 3, 11] for details. We briefly explain next the two main classes of examples mentioned above. For our purposes here, in case the algebra $A$ is finite dimensional as a vector space over the field, it suffices to restrict to finitely generated (bi)modules if desired.

(1) Let $A$ be an associative algebra over a field $k$. Let $A^{op}$ denote its opposite algebra and $A^e = A \otimes_k A^{op}$, the *enveloping algebra* of $A$. The category of $A$-bimodules is equivalent to the category of left $A^e$-modules, where the tensor factor $A^{op}$ of $A^e$ acts on the right as $A$. To obtain an exact monoidal category, we take the full subcategory $\mathrm{lrp}(A^e)$ of *left-right projective* $A^e$-modules, those that are projective as left $A$-modules and as right $A$-modules. The category $\mathrm{lrp}(A^e)$ is an exact monoidal category with tensor product $\otimes_A$ and unit object $A$.

(2) Let $A$ be a Hopf algebra over a field $k$. That is, there are linear maps $\Delta : A \to A \otimes_k A$ (coproduct), $\varepsilon : A \to k$ (counit), and $S : A \to A$ (antipode), satisfying some properties. See e.g. [7] for details. The category $A$-Mod of left $A$-modules has tensor product $\otimes_k$ of underlying vector spaces with $A$-module structure given by $\Delta$. The field $k$ is an $A$-module via $\varepsilon$. The category $A$-Mod is an exact monoidal category with tensor product $\otimes_k$ and unit object $k$. As a particular case, let $G$ be a finite group. The group algebra $kG$ is a Hopf algebra with $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, $S(g) = g^{-1}$ for all $g \in G$, and $kG$-Mod is an exact monoidal category.

**Cohomology.** Assume an exact monoidal cagtegory $\mathcal{C}$ has enough projectives, that is each object is a homomorphic image of some projective object of $\mathcal{C}$. The *cohomology* of $\mathcal{C}$ is

$$\mathrm{H}^*(\mathcal{C}) := \mathrm{Ext}^*_{\mathcal{C}}(\mathbf{1}, \mathbf{1}) := \bigoplus_{n \geq 0} \mathrm{Ext}^n_{\mathcal{C}}(\mathbf{1}, \mathbf{1}),$$

graded by the natural numbers. When $\mathcal{C} = \mathrm{lrp}(A^e)$ for an associative algebra $A$ over the field $k$, the cohomology $\mathrm{H}^*(\mathcal{C})$ is also known as $\mathrm{HH}^*(A, A)$, the *Hochschild cohomology* of $A$. When $\mathcal{C} = A$-Mod for a Hopf algebra $A$ over $k$, the cohomology $\mathrm{H}^*(\mathcal{C})$ is also known as $\mathrm{H}^*(A, k)$, the *Hopf algebra cohomology* of $A$, and in particular when $A = kG$, this is the *group cohomology* $\mathrm{H}^*(G, k)$.

**Cup product.** The cohomology $\mathrm{H}^*(\mathcal{C})$ has a graded associative multiplication that may be defined in more than one way. We will give a definition that uses the monoidal structure. Let $P$ be a projective resolution of the unit object $\mathbf{1}$ in $\mathcal{C}$: $\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbf{1} \longrightarrow 0$. Assume that (the total complex of) $P \otimes P$ is also a projective resolution of $\mathbf{1} \cong \mathbf{1} \otimes \mathbf{1}$, and more generally that $P^{\otimes r}$ is as well for all $r \geq 2$. By the comparison theorem, there is a chain map $\Delta_P : P \to P \otimes P$ lifting $\mathbf{1} \xrightarrow{\sim} \mathbf{1} \otimes \mathbf{1}$. Let $f \in \mathrm{Hom}_{\mathcal{C}}(P_m, \mathbf{1})$ and $g \in \mathrm{Hom}_{\mathcal{C}}(P_n, \mathbf{1})$ be cocycles. They can be extended to chain maps (abusing notation) $f \in \mathrm{Hom}_{\mathcal{C}}(P, P[-m])$ and $g \in \mathrm{Hom}_{\mathcal{C}}(P, P[-n])$ where a number in brackets indicates a degree shift. Define the *convolution product* $fg$ by the composition of functions

$$P_{m+n} \xrightarrow{\Delta_P} (P \otimes P)_{m+n} \xrightarrow{\pi} P_m \otimes P_n \xrightarrow{f \otimes g} \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1},$$

where $\pi$ is projection onto the indicated summand, or equivalently as a map in the Hom complex, $fg \in \mathrm{Hom}_{\mathcal{C}}(P, P[-m-n])$. This induces a well defined multiplication on cohomology $\mathrm{H}^*(\mathcal{C})$ that turns out to be graded commutative. This fact enables support variety theory—a geometric tool for studying the objects and the structure of the category—that is particularly rich in settings where the cohomology $\mathrm{H}^*(\mathcal{C})$ is also finitely generated as an algebra.

**Lie bracket.** The cohomology $\mathrm{H}^*(\mathcal{C})$ has a second useful binary operation, a graded Lie bracket, that may also be defined in more than one way. We will give here a definition that is a direct generalization of the Gerstenhaber bracket on Hochschild cohomology, under the above assumptions on a projective resolution $P$ of the unit object $\mathbf{1}$. The following theorem, a compilation of results from the indicated papers, defines a Lie bracket.

**Theorem.** [6, 10, 11] *Let $f \in \mathrm{Hom}_{\mathcal{C}}(P, P[-m])$ and $g \in \mathrm{Hom}_{\mathcal{C}}(P, P[-n])$ be cocycles. There exist morphisms $\widetilde{f} \in \mathrm{Hom}_{\mathcal{C}}(P, P[1-m])$ and $\widetilde{g} \in \mathrm{Hom}_{\mathcal{C}}(P, P[1-n])$ such that $[f, g] := f\,\widetilde{g} - (-1)^{(m-1)(n-1)} g\,\widetilde{f}$ induces a well defined graded Lie bracket on cohomology $\mathrm{H}^*(\mathcal{C})$. Moreover: (i) up to chain homotopy, the morphism $\widetilde{f}$ (and similarly $\widetilde{g}$) is determined by the equation*

$$d\widetilde{f} + (-1)^m \widetilde{f} d = (f \otimes \mathrm{id_P} - \mathrm{id_P} \otimes \mathrm{f})\Delta_{\mathrm{P}}$$

*where $d$ is the differential on $P$, together with another technical equation involving the augmentation, and (ii) in the cohomology $\mathrm{H}^*(\mathcal{C})$, the image of $[f, g]$ is independent of choices of $P$, $\Delta_P$, $\widetilde{f}$, and $\widetilde{g}$.*

The morphism $\widetilde{f}$ described in the theorem is termed a *homotopy lifting* of $f$. This graded Lie bracket $[f, g]$ generalizes the historical Gerstenhaber bracket on Hochschild cohomology, as may be shown by taking $P$ to be the bar resolution with some standard choices for $\Delta_P$, $\widetilde{f}$, and $\widetilde{g}$. This operation arises, for example, in algebraic deformation theory. One advantage of the definition of the bracket $[f, g]$ in the theorem above is its flexibility in allowing a choice of projective resolution.

**Gerstenhaber algebra.** The following theorem combines the two binary operations on cohomology $\mathrm{H}^*(\mathcal{C})$ described above. For details, see the indicated papers.

**Theorem.** [6, 11] *Under the above assumptions,* $\mathrm{H}^*(\mathcal{C})$ *is a Gerstenhaber algebra.*

In particular, the Lie bracket is a graded derivation with respect to the cup product. The proof of the theorem realizes the Lie bracket as a graded commutator of infinity coderivations on $P$; each homotopy lifting $\widetilde{f}$ is the first map in a sequence defining an infinity coderivation [6, 11]. In [11] the Lie bracket is shown to be the same as that defined topologically in [3], a generalization of results for Hochschild cohomology [8].

We close with a brief summary of some recent results on Lie brackets and some open questions. If $\mathcal{C}$ is braided, that is the tensor product is commutative up to natural isomorphism, then the graded Lie structure on the cohomology $\mathrm{H}^*(\mathcal{C})$ is abelian in positive degrees (i.e. all brackets are 0) [3]. This generalizes [2, 9] for quasitriangular Hopf algebras. For Hopf algebras that are not quasitriangular (i.e. their module categories are not braided), it is unknown whether this Lie bracket is always 0. Some nonquasitriangular examples for which it is known to be 0 are the quantum elementary abelian groups [4]. For Hopf algebras $A$ having bijective antipode $S$, the cohomologies $\mathrm{H}^*(A, k)$ and $\mathrm{HH}^*(A, A)$ are related via the functor $F : A\text{-Mod} \to \mathrm{lrp}(A^e)$ given by tensor induction, $F(M) = A^e \otimes_A M$, where $A$ is embedded as a subalgebra of $A^e$ by the map $(\mathrm{id}_A \otimes_k S)\Delta$. This functor induces an embedding of Hopf algebra cohomology $\mathrm{H}^*(A,k)$ into Hochschild cohomology $\mathrm{HH}^*(A,A)$ under which the Gerstenhaber bracket on $\mathrm{HH}^*(A,A)$ restricts to the bracket on $\mathrm{H}^*(A,k)$ defined as above or as in [3]. See [5] for details.

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# Hochschild cohomology and deformation theory via reduction systems


Severin Barmeier

(joint work with Zhengfang Wang, Philipp Schmitt)


Reduction systems go back to the theory of abstract rewriting systems which are used in a variety of disciplines including computer science, logic and linguistics. In algebra reduction systems were popularized by Bergman's 1978 paper titled "The Diamond Lemma in ring theory" and they belong to the circle of ideas including (noncommutative) Gröbner bases.

## 1. The geometry of reduction systems

*Throughout* $\Bbbk$ *denotes a field of characteristic* 0*.* In our work [BW$^{\mathrm{a}}$] on deformations of reduction systems, a conceptual viewpoint on the geometry of reduction systems emerged, paralleling the geometry of associative algebras. To draw this analogy let us consider associative structures on a finite-dimensional $\Bbbk$-vector space $V$ and finite reduction systems for a finite quiver $Q$ with a set $S \subset Q_{\geq 2}$ of "leading terms" of path length $\geq 2$. We have the following parallel description:

| **associative algebras** | **reduction systems** |
|---|---|
| $\mu \in \mathrm{Hom}_{\Bbbk}(V^{\otimes_{\Bbbk} 2}, V)$ | $\varphi \in \mathrm{Hom}_{\Bbbk Q_0^e}(\Bbbk S, \Bbbk \mathrm{Irr}_S)$ |
| associativity of $A = (V, \mu)$ | confluence of $R = \{(s, \varphi(s))\}_{s \in S}$ |
| *affine variety of associative algebras* $\mathrm{Alg}_V \subset \mathrm{Hom}_{\Bbbk}(V^{\otimes_{\Bbbk} 2}, V) \simeq \Bbbk^{(\dim V)^3}$ | *set of confluent reduction systems* $\mathrm{Red}_S \subset \mathrm{Hom}_{\Bbbk Q_0^e}(\Bbbk S, \Bbbk \mathrm{Irr}_S)$ |
| *algebra isomorphism* $\mathrm{GL}(V) \curvearrowright \mathrm{Alg}_V$ group action | *equivalence of reduction systems* $G_S \curvearrowright \mathrm{Red}_S$ groupoid action |

(If $\Bbbk$ is not algebraically closed, then "variety" should be replaced by "scheme of finite type over $\Bbbk$".)

Deformations of associative algebras can be understood as moving a point inside $\mathrm{Alg}_V$ and similarly for deformations of reduction systems. Although the set $\mathrm{Red}_S$ carries no immediately obvious geometric structure, it contains finite-dimensional varieties as subsets

$$\mathrm{Red}_S^{<} \subset \mathrm{Red}_S^{\prec} \subset \mathrm{Red}_S.$$

Here $^{<}$ and $^{\prec}$ correspond to compatibility of the points in $\mathrm{Red}_S$ with respect to path length or with respect to a chosen noncommutative Gröbner basis.

Although the $\Bbbk$-linear world of associative multiplications is well-behaved, it can be difficult to work with in practice, whereas the description via reduction systems is often much more tractable.

## 2. Deformation theory

In [BW$^{\mathrm{a}}$] we proved the following general result giving a formal counterpart to the non-formal geometric picture outlined in §1

**Theorem 1** ([BW$^{\mathrm{a}}$]). Let $A = \Bbbk Q/I$ be the path algebra of a finite quiver $Q$ with ideal of relations $I$. Let $R = \{(s, \varphi(s))\}_{s \in S}$ be any reduction system satisfying the diamond condition for $I$. Then the following are equivalent

(1) formal deformations of $A$ as associative algebra
(2) formal deformations of $R$ as confluent reduction system.

When looking at first-order deformations — from the geometric viewpoint of §1 at the *tangent spaces* of $\mathrm{Alg}_V$ and $\mathrm{Red}_S$ — we have the following corollary.

**Corollary 2** ([BW$^{\mathrm{a}}$]). In the setting of Theorem 1 there is an isomorphism

$$\mathrm{HH}^2(A, A) \simeq \{\text{first-order deformations of } R\}/\text{equivalence}.$$

The right-hand side is often easily computed from any reduction system for $A$.

## 3. Applications

3.1. **Strict deformation quantization of Poisson structures.** Let $\Bbbk = \mathbb{C}$ (for quantum-mechanical reasons) and consider the algebra $A = \mathbb{C}[x_1, \ldots, x_d]$ of $\mathbb{C}$-valued polynomial functions on $\mathbb{R}^d$. Then $A$ admits a natural reduction system

$R = \{(x_j x_i, x_i x_j)\}_{1 \le i < j \le d}$ whose associated bimodule resolution is the Koszul resolution.

In the context of deformation quantization, one considers formal deformations of $A$ over $\mathbb{C}[[t]]$, where the formal parameter $t$ stands in for the Planck constant $\hbar$ and the first-order term is given by a Poisson structure on $\mathbb{R}^d$. It is a widely open problem to find "strict" convergent quantizations for (classes of) Poisson structures. From Theorem 1 we obtain the following result.

**Theorem 3** ([BW$^{\mathrm{a}}$])**.** Let $\eta$ be any polynomial Poisson structure on $\mathbb{R}^d$. Any formal deformation of $R$ gives a formal deformation quantization of $\eta$ and the resulting star product can be given by the graphical formula

$$f \star g = \sum_{n \ge 0} \sum_{\Lambda \in \mathfrak{G}^*_{2,n}} C_\Lambda(f, g) \qquad \text{for all } f, g \in \mathbb{C}[x_1, \dots, x_d]$$

where $C_\Lambda$ is a formal power series of differential operators associated to a certain graph $\Lambda$.

In [BS23] we used the results of [BW$^{\mathrm{a}}$, Ch. 10] in combination with tools from functional analysis to extend convergent star products obtained from algebraizations of Theorem 3 from polynomial functions to analytic functions with infinite radius of convergence. The resulting algebras of quantum observables can in examples even be represented as adjointable operators on pre-Hilbert spaces [BS23, §3.4], as dictated by the standard formulation of quantum mechanics.

3.2. **Deformations of Abelian categories of quasi-coherent sheaves.** Let $X$ be any separated scheme of finite type over $\Bbbk$. Then $X$ admits a finite affine open cover $\mathfrak{U}$ closed under $\cap$ and the restriction $\mathcal{O}_X|_{\mathfrak{U}}$ of the structure sheaf to the open sets in $\mathfrak{U}$ can be viewed as a presheaf of commutative algebras on $\mathfrak{U}$. This presheaf can be encoded into a single associative algebra $A = \mathcal{O}_X|_{\mathfrak{U}}!$. Building on work of Lowen–Van den Bergh [LVdB05], we have the following result.

**Theorem 4** ([BW24])**.** Let $X$ be any separated scheme of finite type over a field $\Bbbk$ of characteristic 0. There exists a finite reduction system $R$ for $A = \mathcal{O}_X|_{\mathfrak{U}}!$. In particular, there is an equivalence between

(1) formal deformations of $R$ as confluent reduction system
(2) formal deformations of $\mathrm{Qcoh}(X)$ and $\mathrm{coh}(X)$ as Abelian categories.

The non-formal deformations using the setup of §1 can be used to find non-formal deformation of Abelian categories of (quasi)coherent sheaves which play a central role in noncommutative algebraic geometry. This setup can also produce noncommutative deformations of singularities, for example when $X$ is the geometric resolution of a cyclic surface singularity [BW24, §5].

3.3. **$\mathbf{A}_\infty$ deformations of extended Khovanov arc algebras.** Let $m, n \ge 1$ and let $\mathrm{K}^n_m$ be the extended Khovanov arc algebra introduced by Stroppel. These algebras are finite-dimensional Koszul algebras with a diagrammatic basis and a multiplication rule derived from a 2D TQFT. The module category of $\mathrm{K}^n_m$ describes perverse sheaves on Grassmannians (Stroppel), parabolic category $\mathcal{O}$ for $\mathfrak{gl}_{m+n}(\mathbb{C})$

(Brundan–Stroppel) and their perfect derived category describes the Fukaya–Seidel category of Hilbert schemes of points on type A Milnor fibres (Mak–Smith).

**Theorem 5** ([BW[b]])**.** For all $m, n \geq 2$ we have the following:

(1) $\mathrm{HH}^2_{i-2}(\mathrm{K}^n_m, \mathrm{K}^n_m)$ is 1-dimensional for $i = 2mn - 4$.
(2) $\mathrm{K}^n_m$ admits an explicit nontrivial $\mathrm{A}_\infty$ deformation.

Theorem 5 settles Stroppel's Conjecture given in her ICM 2010 address [Str10] negatively for all $m, n \geq 2$ but at the same time proves the existence of nontrivial algebraic deformations of Fukaya–Seidel categories of Hilbert schemes of type A Milnor fibres. It is not (yet) known, whether the $\mathrm{A}_\infty$ deformations of Theorem 5 may give rise to new or already-known knot invariants.

Further applications of [BW[a]] to symplectic geometry [BSW[a], BSW[b]] are reported by Zhengfang Wang in this volume.

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## Filtered derived categories of curved deformations

Alessandro Lehmann
(joint work with Wendy Lowen)

Let $A$ be a dg-algebra over a field $k$. It is a well-known issue [1, 2, 3, 4] that the Hochschild complex of $A$ does not govern the deformations of $A$ as a dg-algebra, but as a curved dg-algebra. Since modules over a curved algebra have differentials which do not square to zero, it is impossible to define a derived category of a curved deformation using classical methods. In particular, this creates a significant obstacle in developing a satisfactory deformation theory for differential graded algebras and, by extension, triangulated categories.

In my talk I reported on the recent preprint [5], where we associate to a curved deformation an invariant which takes the place of the nonexistent derived category.

Let $R_n$ be the commutative ring $k[t]/(t^{n+1})$, and let $A_n$ be a curved deformation of $A$ over $R_n$. Then if $M$ is a cdg $A_n$-module, one has the $t$-adic filtration

$$0 = t^{n+1}M \subseteq t^n M \subseteq \ldots \subseteq tM \subseteq M;$$

since $A$ has no curvature, the curvature of $A_n$ lies in $tA_n$ and each associated piece $\mathrm{Gr}_t^i(M) = \frac{t^i M}{t^{i+1}M}$ is an $A$-module. We define the *$n$-derived category* $D^n(A_n)$ of the deformation $A_n$ as the quotient of the category of $A_n$-modules by the subcategory of those modules for which the whole associated graded is acyclic; we call these modules $n$-acyclic. This quotient turns out to be surprisingly well behaved, exhibiting the following properties:

- There exist $n+1$ explicit compact modules $\Gamma_1, ..., \Gamma_n$ that generate $D^n(A_n)$ as a triangulated category;
- The quotient functor
$$H^0(A_n\text{-Mod}) \to D^n(A_n)$$
from the homotopy category of cdg $A_n$-modules admits both a left and a right adjoint, which are then automatically fully faithful;
- Denoting with $A_m$ the induced deformation of order $m \leq n$, there is a system of fully faithful embeddings
$$D(A) \hookrightarrow D^1(A_1) \hookrightarrow \ldots \hookrightarrow D^{n-1}(A_{n-1}) \hookrightarrow D^n(A_n)$$
where each functor is induced by the restriction of scalars along the projection $A_n \to A_m$.
- Denoting with $D^{\mathrm{si}}(A_n)$ the semiderived category from [10], there exists an admissible embedding
$$D^{\mathrm{si}}(A_n) \hookrightarrow D^n(A_n)$$
whose essential image is given by the $A_n$-modules which are free as graded $R_n$-modules;
- If $A_n$ has no curvature, all $n$-acyclic modules are acyclic and there exists a localization $D^n(A_n) \to D(A_n)$ which has both a left and a right adjoint. In particular, there exist two fully faithful embeddings $D(A_n) \hookrightarrow D^n(A_n)$ - one left admissible, one right admissible.

Define the full subcategories $\mathcal{T}_i \subseteq D^n(A_n)$ as

$$\mathcal{T}_i = \{M \in D^n(A_n) |\ \mathrm{Gr}_t^j(M) \cong 0 \text{ in } D(A) \text{ for all } j \neq i\}.$$

The main feature of the $n$-derived category is the existence of a semiorthogonal decomposition

$$D^n(A_n) = \langle \mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n \rangle;$$

moreover, the functors $\mathrm{Gr}_t^i \colon D^n(A_n) \to D(A)$ induce equivalences $\mathcal{T}_i \xrightarrow{\sim} D(A)$. One sees that the embeddings $\mathcal{T}_i \hookrightarrow D^n(A_n)$ are both left and right admissible, so the semiorthogonal decomposition also has a recollement-type formulation.

This has two interesting consequences. First of all, an analysis of the gluing bimodules induced by the decomposition shows that as soon as $D(A)$ is smooth, the same holds for $D^n(A_n)$. In the case where $A_n$ has no curvature, this implies that $D^n(A_n)$ is a categorical resolution of $D(A_n)$ in the sense of [7] (see also [8] and [9] for similar constructions).

Moreover, the decomposition suggests a way in which $D^n(A_n)$ can be interpreted as a deformation of $D(A)$. Assume $n = 1$ for simplicity: in this case, the semiorthogonal decomposition is expressed via the following recollement

$$\begin{tikzcd} D(A) \arrow[r, hook] & D^1(A_1) \arrow[l, bend right, "\mathrm{Coker}\, t"'] \arrow[l, bend left, "\mathrm{Ker}\, t"] \arrow[r, "\mathrm{Im}\, t"] & D(A). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

This should be thought as a categorification of the short exact sequence

$$0 \to A \to A_1 \to A \to 0$$

witnessing $A_1$ as a first order deformation of $A$. Our current goal is to fully understand what kind of categorical extensions can emerge as $n$-derived categories of deformations, and to classify all such extensions in terms of Hochschild cohomology. The final picture we expect to obtain is a threefold correspondence between $HH^2(A)$, cdg Morita deformations of $A$ (in the spirit of [1]) and deformations of $D(A)$ as an (enhanced) triangulated category in the sense above.

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## Finite generation of cohomology: the things we know (a little) and the ones we don't (a lot)

Julia Pevtsova

Disclaimer: the title for this talk was suggested as a joke; I did not expect the organizers to take it seriously. But when it was already on the program, pinned to the wall in the Oberwolfach dining hall, it was official and too late to change! I would still replace the word "we" with "I" so that all omissions can only be blamed on my own ignorance.

Let $\mathfrak{C}$ be a finite tensor category over a field $k$. That is, we put the following conditions on $\mathfrak{C}$ (see [4]):

(1) $\mathfrak{C}$ is an abelian rigid monoidal category
(2) Hom-sets are finite
(3) $\mathfrak{C}$ has finitely many simple modules; each simple has a projectve cover
(4) length of any object in $\mathfrak{C}$ is finite
(5) The tensor unit $\mathbb{1}$ is simple

These conditions in particular imply that the tensor product on $\mathfrak{C}$ is exact in each variable.

**Conjecture** (Etingof-Ostrik, '04)**.** For any finite tensor category $\mathfrak{C}$,

(CFG1) $\mathrm{Ext}^*_{\mathfrak{C}}(\mathbb{1}, \mathbb{1})$ is Noetherian,
(CFG2) For any $M, N \in \mathfrak{C}$, $\mathrm{Ext}^*_{\mathfrak{C}}(M, N)$ is a finite $\mathrm{Ext}^*_{\mathfrak{C}}(\mathbb{1}, \mathbb{1})$-module.

Whenever the conditions of the conjecture hold, we say that $\mathfrak{C}$ satisfies the *Cohomological Finite Generation* (CFG) property. One might compare this to the terminology introduced in [3].

One motivation for the finite generation question is that one expects to express the Balmer spectrum of the corresponding stable category, which is small tensor triangulated, in terms of the cohomology ring:

**Conjecture** (Nakano-Vashaw-Yakimov, '21)**.** Let $\mathfrak{C}$ be a finite tensor category. Then

$$\mathrm{Spec}_{\mathrm{Bal}}(\mathrm{Stab}\,\mathfrak{C}) \cong \mathrm{Proj}\, Z_c\, \mathrm{Ext} *_{\mathfrak{C}}(\mathbb{1}, \mathbb{1}),$$

where $Z_c$ stands for the "categorical center of the cohomology ring of $\mathfrak{C}$" (see [9]).

The main example of a finite tensor category we'll consider is $\mathrm{Rep}_k A$, the category of finite–dimensional representations of a finite–dimensional Hopf $k$-algebra $A$. For the majority of the examples, $k$ is a field, but not always. Here is what is known about the CFG property in this case (this list is not claimed to be exhaustive).

(1) Suppose $A$ is cocommutative, and $k$ is a field. Then $\mathrm{Rep}_k A$ satisfies CFG ([6])
(2) This holds more generally, due to a recent result of W. van der Kallen; the notes with the outline of the complete proof are currently being written up by two participants of the workshop, Chris Parker and Juan Omar Gomez. Let $R$ be a Noetherian commutative ring, let $A$ be a finite projective cocommutative Hopf algebra over $R$. Then $\mathrm{Latt}_R(A)$, the category of *A-lattices* over $R$, satisfies CFG ([11]).
(3) Let $A$ be a finite dimensional pointed Hopf algebra over $\mathbb{C}$ with abelian group of group-like elements. Then (modulo several isolated cases) $\mathrm{Rep}_k A$ satisfies CFG ([1]). This last example uses the full force of the classification of such Hopf algerbas, due in particular to Andruskiewitsch, Angiono, Schneider, Heckenberger, and includes small quantum groups for which more precise information about cohomology is known ([7]).

**Remark.** In the first two cases, the Nakano-Vashaw-Yakimov conjecture is known to hold, see [5], [8], [2]. Note that in the cocommutative case it simplifies to the statement

$$\mathrm{Spec}_{\mathrm{Bal}}(\mathrm{Stab}(\mathrm{Rep}_k\, A)) \cong \mathrm{Proj}\,\mathrm{Ext}^*_A(\mathbb{1},\mathbb{1}),$$

Moving to a big open field of what we don't know, I discussed the Etingof Ostrik conjecture for other classes of Hopf algebras, mentioning Fomin-Kirillov algebras $FK_n$ associated to a symmetric group $S_n$. In the case of $FK_3$ the cohomology ring $H^*(FK_3,\mathbb{C})$ was computed in the case of $S_3$ in [10] - and is finitely generated - but the question is completely open for large $n$.

One can observe that *all* the examples already listed live "over vector spaces", that is, there is a forgetful (fiber = exact, tensor) functor $\mathrm{Rep}_k\, A \to \mathrm{Vect}_k$. One discovers a vast open field of unsolved problems in cohomological finite generation if one moves to – even cocommutative – finite dimensional algebras (or finite group schemes) living over other *incompressible* categories starting with Verlinde categories $\mathrm{Ver}_p$ for a prime $p$.

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## Entropies of Serre functors for higher hereditary algebras

Yang Han

In this talk, we report the main results of my paper [5].

A topological dynamical system $(X, f)$ consists of a topological space $X$ and a continuous function $f : X \to X$. The topological entropy $h_{\mathrm{top}}(f)$ measures the complexity of $(X, f)$. As a categorical analog of topological dynamical system, in [2], Dimitrov, Haiden, Katzarkov and Kontsevich introduced categorical dynamical system $(\mathcal{T}, F)$ which consists of a triangulated category $\mathcal{T}$ and a (triangle) endofunctor $F : \mathcal{T} \to \mathcal{T}$ of $\mathcal{T}$, and (categorical) entropy which measures the complexity of a categorical dynamical system. Roughly speaking, the entropy is the asymptotically exponential growth rate of the complexity of a categorical dynamical system. In [3], Fan, Fu and Ouchi introduced (categorical) polynomial entropy which is the asymptotically polynomial growth rate of the complexity of a categorical dynamical system. Moreover, in [9], Kikuta and Ouchi introduced Hochschild (co)homology entropy. Hochschild (co)homology entropy is defined not for a categorical dynamical system, but for a "dg categorical dynamical system". Meanwhile, Kikuta and Ouchi posed a question ([9, Question 2.13]): When does the Hochschild (co)homology entropy for a "dg categorical dynamical system" coincide with the entropy for the corresponding categorical dynamical system?

As a generalization of representation-finite hereditary algebras, Iyama and Oppermann introduced higher representation-finite algebras in [8]. As a generalization of representation-infinite hereditary algebras, Herschend, Iyama and Oppermann introduced higher representation-infinite algebras in [7]. Meanwhile, they also introduced higher hereditary algebras which are shown to be either higher representation-finite algebras or higher representation-infinite algebras ([7, Theorem 3.4]). Many classical results in the representation theory of hereditary algebras have higher dimensional analogs for higher hereditary algebras. Moreover, as a generalization of fractionally Calabi-Yau algebras, Herschend and Iyama introduced twisted fractionally Calabi-Yau algebras in [6], which contain higher representation-finite algebras as typical examples ([6, Theorem 1.1]).

For a higher hereditary algebra, we will calculate the entropy and polynomial entropy of Serre functor, and the Hochschild (co)homology entropy of Serre quasi-functor. Given Serre functor and higher hereditary algebra have congenital relationship, the calculations become feasible.

Our main results are the following.

**Theorem 1.** *Let $A$ be a twisted $\frac{q}{p}$-Calabi-Yau algebra. Then*

(1) *the entropy of Serre functor $h_t(S) = \frac{q}{p}t$.*

(2) *the polynomial entropy of Serre functor $h_t^{\mathrm{pol}}(S) = 0$.*

(3) *the Hochschild (co)homology entropy of Serre quasi-functor $h^{HH^\bullet}(\tilde{S}) = h^{HH_\bullet}(\tilde{S}) = 0$ if we assume further that $A$ is elementary.*

Theorem 1 (1) and (2) generalize the corresponding results [2, 2.6.1] and [3, Remark 6.3] for fractionally Calabi-Yau algebras. To date, it is not known whether every higher representation-finite algebra, or more general, twisted fractionally Calabi-Yau algebra, is fractionally Calabi-Yau or not ([1, Question 1.6]). So Theorem 1 should have its own place.

From Theorem 1 and [6, Theorem 1.1], we get immediately the following corollary.

**Corollary.** *Let $A$ be an indecomposable $d$-representation-finite algebra, $r$ the number of isomorphism classes of simple $A$-modules, and $p$ the number of indecomposable direct summands of the basic $d$-cluster tilting $A$-module. Then*

(1) *the entropy of Serre functor $h_t(S) = \frac{d(p-r)}{p}t$.*

(2) *the polynomial entropy of Serre functor $h_t^{\mathrm{pol}}(S) = 0$.*

(3) *the Hochschild (co)homology entropy of Serre quasi-functor $h^{HH^\bullet}(\tilde{S}) = h^{HH_\bullet}(\tilde{S}) = 0$ if we assume further that $A$ is elementary.*

Applying the Hirzebruch-Riemann-Roch type theorem ([4, Theorem 1]) and Wimmer's formula ([10, Theorem]), we can obtain the following Theorem 2 which gives the Yomdin type inequality on Hochschild homology entropy.

**Theorem 2.** *Let $A$ be a finite dimensional elementary algebra of finite global dimension, $M$ a perfect $A$-bimodule complex, and $\Psi_M := -C_M C_A^{-1}$ the dual Coxeter matrix of $M$. Then $h^{HH_\bullet}(M) \geq \log \rho(\Psi_M)$. Here, $\rho(\Psi_M)$ is the spectral radius of the square matrix $\Psi_M$.*

I do not know whether the Gromov type inequality on Hochschild homology entropy, that is, $h^{HH_\bullet}(M) \leq \log \rho(\Psi_M)$, and the Gromov and Yomdin type inequalities on Hochschild cohomology entropy, that is, $h^{HH^\bullet}(M) \leq \log \rho(\Psi_M)$ and $h^{HH^\bullet}(M) \geq \log \rho(\Psi_M)$, hold or not.

The Theorem 2 above will be applied to show the following Theorem 3.

**Theorem 3.** *Let $A$ be an elementary $d$-representation-infinite algebra, and $\Phi$ the Coxeter matrix of $A$. Then*

(1) *the entropy of (inverse) Serre functor: $h_t(S) = dt + \log \rho(\Phi)$ and $h_t(S^{-1}) = -dt + \log \rho(\Phi^{-1})$. Furthermore, $\rho(\Phi) = \rho(\Phi^{-1})$.*

(2) *the polynomial entropy of (inverse) Serre functor: $h_t^{\mathrm{pol}}(S) = s(\Phi)$ and $h_t^{\mathrm{pol}}(S^{-1}) = s(\Phi^{-1})$. Furthermore, $s(\Phi) = s(\Phi^{-1})$. Here, $s(\Phi)$ is the polynomial growth rate of the square matrix $\Phi$.*

(3) *the Hochschild (co)homology entropy of (inverse) Serre quasi-functor: $h^{HH^\bullet}(\tilde{S}) = h^{HH_\bullet}(\tilde{S}) = h(S) = \log \rho(\Phi) = \log \rho(\Phi^{-1}) = h(S^{-1}) = h^{HH_\bullet}(\tilde{S}^{-1}) = h^{HH^\bullet}(\tilde{S}^{-1})$. Here, $h(S)$ is the value $h_0(S)$ of the entropy $h_t(S)$ of the Serre functor $S$ at $t = 0$.*

Partial results of Theorem 3 (1) and (2) for representation-infinite hereditary algebras had been obtained in [2, Theorem 2.17] and [3, Proposition 4.4]. The equalities $\rho(\Phi) = \rho(\Phi^{-1})$ and $s(\Phi) = s(\Phi^{-1})$ seem to be new. Serre functor is a categorification of Coxeter matrix. Theorem 3 (1), (2) and (3) suggest that, for an elementary higher representation-infinite algebra, the entropy of Serre functor and the Hochschild (co)homology entropy of Serre quasi-functor are the categorifications of spectral radius of Coxeter matrix, and polynomial entropy is the categorification of polynomial growth rate of Coxeter matrix, in some sense.

Furthermore, our main results imply that the Kikuta and Ouchi's question on the relations between entropy and Hochschild (co)homology entropy has positive answer, that is, $h^{HH^\bullet}(\tilde{S}) = h^{HH_\bullet}(\tilde{S}) = h(S)$, and the Gromov-Yomdin type equalities on entropy and Hochschild (co)homology entropy hold, that is, $h(S) = \log\rho([S])$ and $h^{HH^\bullet}(\tilde{S}) = h^{HH_\bullet}(\tilde{S}) = \log\rho([H^0(\tilde{S})])$, for the Serre functor $S$ on the perfect derived category and the Serre quasi-functor $\tilde{S}$ on the perfect dg module category of an elementary twisted fractionally Calabi-Yau algebra or an indecomposable elementary higher hereditary algebra.

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## Tamarkin-Tsygan calculus for gentle algebras

Sibylle Schroll

(joint work with Christian Chaparro, Andrea Solotar, Mariano Suárez-Álvarez)

The Tamarkin-Tsygan calculus of an associative algebra is the comprehensive data of its Hochschild cohomology and homology and their algebraic structures. In general, it is next to impossible to obtain the entire information of this calculus, since already the explicit calculation of the Hochschild cohomology and homology are of exceeding computational complexity. However, in certain cases, when there is much structural information about the algebras available, one might attempt such a calculation. One such case is given by the class of gentle algebras.

Gentle algebras date from the 1980s where they first appeared in the context of iterated tilted algebras of type A. Since then, their representation theory has been intensely studied, both in terms of their module categories and their derived categories. Remarkably, gentle algebras appear in many different settings such as cluster theory where Jacobian algebras of quivers with potentials from ideal triangulations of surfaces with marked points in the boundary are shown to be gentle [1, 7], in $\mathcal{N} = 2$ gauge theories [4] and in the context of homological mirror symmetry of surfaces with stops where it is shown in [6, 8] that the perfect derived category of a graded gentle algebra is triangle equivalent to the partially wrapped Fukaya category of such a surface.

Conjecturally, the Hochschild cohomology of a gentle algebra should hence correspond to the symplectic cohomology of the associated marked surface, see for example, upcoming work of Abouzaid and Ganatra. One of our motivations to calculate the Tamarkin-Tsygan calculus of gentle algebras was to understand this statement from a representation theoretic point of view.

Another motivation was to see if it is possible to completely and explicitly calculate the whole calculus for the class of gentle algebras and what new information could be extracted from the knowledge of the Tamarkin-Tsygan calculus.

Let $k$ be a field, which for the purposes of this abstract we assume to be of characteristic not 2, a restriction which we do not impose in [5], on which this abstract is based.

**Definition 1.** A gentle algebra $kQ/I$ is given by a quiver $Q = (Q_0, Q_1)$ and an ideal $I$ of $kQ$ such that

(1) for every vertex $v \in Q_0$, there are at most two arrows ending and at most two arrows starting at $v$,
(2) for any arrow $a \in Q_1$ there is at most one arrow $b$ (resp. $b'$) such that $ab$ is non-zero in $kQ$ and $ab \in I$ (resp. $ab' \notin I$) and there is at most one arrow $c$ (resp. $c'$) such that $ca$ is non-zero in $kQ$ and $ca \in I$ (resp. $c'a \notin I$).
(3) The ideal $I$ is generated by the quadratic monomial relations in (2).

The Tamarkin-Tsygan calculus of an associative $k$-algebra $A$ is the data of

$$(\mathrm{HH}^*(A), \smile, [-,-], \mathrm{HH}_*(A), \frown, B)$$

where $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A)$, for $A^e = A \otimes_k A^{op}$, is the Hochschild cohomology of $A$, which, with the multiplication given by the cup product $\smile$, is a graded commutative algebra. Under the Gerstenhaber bracket $[-,-]$, the shifted cohomology $\mathrm{HH}^*(A)[1]$ is a graded Lie algebra. The Hochschild homology $\mathrm{HH}_*(A) = \mathrm{Tor}^{A^e}_*(A, A)$ is a (right) $\mathrm{HH}^*(A)$-module with the action given by the cap product $\frown$. Finally, the Connes differential $B$ is a morphism on the Hochschild homology of degree one and which squares to zero.

For monomial algebras, Bardzell has given a presentation of the minimal projective bimodule resolution of $A$ which we will denote by $\mathcal{R}$ [2]. Furthermore, Strametz [11] showed that there is a morphism of complexes of vector spaces $\mathrm{Hom}_{A^e}(\mathcal{R}, A) \simeq (k(\Gamma_m||B), d^m)$ where $\Gamma_0 = Q_0, \Gamma_1 = Q_1$,

$$\Gamma_m = \{a_1 a_2 \ldots a_m, a_i \in Q_1, a_i a_{i+1} \in I\}$$

and $B$ is a basis of paths of $kQ/I$. Then $k(\Gamma_m||B)$ is the $k$-vector space generated by all $(\gamma, \alpha) \in \Gamma_m \times B$ such that $\gamma$ and $\alpha$ begin and end at the same vertex in $Q$.

With this notation at hand we can present $(\mathrm{HH}^*(A), \smile)$ as a graded commutative algebra by giving a set of generators and their relations.

**Theorem 2.** [5] Let $A = kQ/I$ be a gentle algebra. Then $(\mathrm{HH}^*(A), \smile)$, as a graded commutative algebra is generated by the following elements, given here in terms of cohomology classes of cocyles in $k(\Gamma_*||B)$

(I) $\sum_{v\in Q_0}(v, v) \in \mathrm{HH}^0(A)$,

(II) $(s(\alpha), \alpha) \in \mathrm{HH}^0(A)$, for $\alpha$ a cycle in $Q$ with a unique relation at $s(\alpha)$,

(III) $\langle\!\langle\alpha\rangle\!\rangle = \sum_{i=0}^{r-1}(s(rot^i(\alpha)), rot^i(\alpha)) \in \mathrm{HH}^0(A)$, for $\alpha$ a primitive cycle in $Q$ of length $r$ with no relations and where $rot^i(\alpha)$ for $0 \le i \le r-1$ runs through the rotations of $\alpha$,

(IV) $(c, c) \in \mathrm{HH}^1(A)$, where $c$ is an arrow in $Q_1 \setminus T$ for a fixed spanning tree $T$ of $Q$,

(V) $(\gamma, \alpha) \in \mathrm{HH}^m(A)$, for $m \ge 1$ such that $\gamma$ and $\alpha$ do not start or end with the same arrow and where there is no arrow $a$ in $Q_1$ pre- or postcomposing with $\gamma$ such that $a\gamma \in I$ or $\gamma a \in I$,

(VI) $\langle\!\langle C\rangle\!\rangle = \sum_{i=0}^{r-1}((rot^i(C)), s(rot^i(C))) \in \mathrm{HH}^{\varepsilon m}(A)$ where $C = D^\varepsilon$ and $D = \gamma_m \dots \gamma_1$ is a primitive cycle in $Q$ with $\gamma_i\gamma_{i+1} \in I$ and $\gamma_m\gamma_1 \in I$ and where $\varepsilon = 1$ if $m$ is even and $\varepsilon = 2$ otherwise.

and with relations in terms of generators $g$ and $h$ of $\mathrm{HH}^*(A)$ given by

- all products $g \smile h$ except when
  - $g = h$ and both are of type $(III)$ or both of type $(VI)$
  - $g = (c, c)$ is of type $(IV)$ and $h$ is of type $(III)$ or $(VI)$ with the underlying cycle passing through the arrow $c$
- $(c, c) \smile h - (d, d) \smile h$, for any generator $h$ of type $(III)$ or $(VI)$ such that the cycle underlying $h$ contains two *distinct* arrows $c$ and $d$ in $Q_1 \setminus T$.

We note that the generators and, more generally, basis elements of $\mathrm{HH}^*(A)$ can be interpreted in terms of a surface associated to $A$. More precisely, by [10, 9, 3, 6, 8] a ribbon graph is associated to every gentle algebra, which, when embedded in the corresponding marked ribbon surface, cuts the surface into polygons that have exactly one boundary segment as an edge, or have no boundary segment as an edge but have a fully marked boundary component in the interior.

Building on this, we establish in [5] an explicit bijection of simple closed curves around the boundary components with zero or one marked point and fully marked boundary components and generators of $\mathrm{HH}^*(A)$ of types $(II), (III), (V)$ and $(VI)$ in such a way that the winding number of the closed curve gives the degree in $\mathrm{HH}^*(A)$ of the associated generator. The generator of type $(I)$ corresponds to the edges of the ribbon graph which have winding number zero. The generators of type $(IV)$ do not fit into this description in terms of boundary components but can also be given a surface interpretation. Namely, a generator of $\mathrm{HH}^1(A)$ of type $(IV)$ corresponds to a curve of winding number 1 connecting two unmarked boundary segments.

This geometric interpretation of generators as special curves in the surface turns out to be compatible with the algebraic structure of cohomology: the cup product of generators corresponds to the concatenation of the associated curves, and there is a similar way to interpret the Gerstenhaber bracket. More precisely, it is shown in [5], that the Gerstenhaber bracket of the generators of the Hochschild cohomology is almost always zero. The only non-zero brackets arise from brackets of generators of type $(IV)$ given by elements $(c, c)$ for $c$ an arrow in the complement of a (fixed) spanning tree of $Q$ and other generators passing through the arrow $c$. In this case, the Gerstenhaber bracket counts how often the curve defining $c$ and the curve corresponding to the other generator run in parallel. For more details, as well as the complete Tamarkin-Tsygan calculus of a gentle algebra, see [5].

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## Complexes with small homology

Petter Andreas Bergh

Our starting point is a conjecture from algebraic topology by Gunnar Carlsson. Let $p$ be a prime number and $G$ an elementary abelian $p$-group of rank $d$; in other words, $G$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^d$. It was conjectured in [3] that if $G$ acts freely on a non-trivial finite CW-complex $X$, then

$$\sum_{n \in \mathbb{Z}} \dim_k \mathrm{H}_n(X; k) \geq 2^d$$

where $k$ denotes the field $\mathbb{Z}/p\mathbb{Z}$ (note that the sum is actually finite). Carlsson settled the case when $p = 2$ and $d$ is at most 3 in [4], but the conjecture remains open at the time of writing.

In [3], Carlsson also stated an algebraic version of the conjecture. Namely, suppose that $p$, $d$ and $G$ are as before, and take now a non-trivial (meaning non-acyclic) finite complex $D$ of finitely generated free $kG$-modules. The conjecture then states that

$$\sum_{n\in\mathbb{Z}} \dim_k \mathrm{H}_n(D) \geq 2^d$$

Again, in [4] Carlsson settled the case when $p = 2$ and $d$ is at most 3.

As explained in [3], the algebraic version actually implies the topological one, in the sense that if the algebraic version were true, then so would the topological version be. However, the algebraic version turned out not to be true; in [6], Srikanth Iyengar and Mark Walker showed that a counterexample always exists when $p$ is odd and $d$ is at least 8. To construct these counterexamples, they used certain properties of exterior algebras, in particular the existence of so-called Lefschetz elements.

In the recent paper [2], Jon Carlson extended Iyengar and Walker's counterexamples to arbitrary groups. Namely, suppose that $k$ is a field of odd characteristic $p$, and that $G$ is an arbitrary finite group whose $p$-rank $d$ is at least 8. Carlson showed that there exist infinitely many non-isomorphic and non-trivial perfect complexes $D$ over $kG$, with

$$\sum_{n\in\mathbb{Z}} \dim_k \mathrm{H}_n(D) < 2^d$$

Here, a perfect complex means a finite complex of finitely generated projective left $kG$-modules. In the special case when $G$ is a $p$-group, the projective modules are free, and so Carlson's result provides an abundance of counterexamples to the original conjecture.

In recent work, we have proved a version of Carlson's result for complexes over a finite tensor category $(\mathscr{C}, \otimes, \mathbf{1})$. Thus $\mathscr{C}$ is a locally finite $k$-linear abelian category (for some field $k$), with a finite set of isomorphism classes of simple objects. There are enough projective objects, and every object admits a projective cover, and therefore also a minimal projective resolution. The tensor product is an associative bifunctor, and comes with a unit object $\mathbf{1}$ which is simple. Finally, the category is rigid, meaning that all objects have left and right duals. A typical example is the category of finitely generated left $kG$-modules, for $G$ a finite group, or, more generally, the category of finitely generated left modules over a finite-dimensional Hopf-algebra.

It was conjectured in [5] that the cohomology ring $\mathrm{H}^*(\mathscr{C})$ of $\mathscr{C}$, that is, the Ext-algebra of the unit object $\mathbf{1}$, is finitely generated as a $k$-algebra, and that the cohomology of $\mathscr{C}$ is finitely generated over this ring. When this holds, we say that $(\mathscr{C}, \otimes, \mathbf{1})$ has *finitely generated cohomology.* In [1], we proved the following result.

**Theorem.** Let $k$ be a field of characteristic not 2, and $(\mathscr{C}, \otimes, \mathbf{1})$ a finite tensor $k$-category with finitely generated cohomology. If the Krull dimension $d$ of $\mathrm{H}^*(\mathscr{C})$ is at least 8, then there exist infinitely many non-isomorphic and nontrivial perfect complexes $D$ over $\mathscr{C}$, with

$$\sum_{n\in\mathbb{Z}} \ell\left(\mathrm{H}_n(D)\right) \leq \sum_{n\in\mathbb{Z}} \mathrm{FPdim}_{\mathscr{C}}\left(\mathrm{H}_n(D)\right) < 2^d$$

Here $\ell(-)$ denotes length, and $\mathrm{FPdim}_{\mathscr{C}}(-)$ the so-called Frobenius-Perron dimension (which is always at least the length). For modules over a Hopf-algebra, the latter equals the vector space dimension, and so the theorem recovers Carlson's result for group algebras.

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## Koszul, Calabi–Yau deformations of q-symmetric algebras


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(joint work with Mykola Matviichuk, Brent Pym)


Let $\mathsf{A}_q := \mathbb{C}\langle x_0, \dots, x_{n-1}\rangle/(x_i x_j - q_{ij} x_j x_i)$ be the $q$-symmetric algebra, defined in terms of an $n \times n$ matrix $q = (q_{ij})$ with $q_{ij} \in \mathbb{C}^\times$, $q_{ij} q_{ji} = 1$ for all $i \neq j$, and $q_{ii} = 1$ for all $i$. This algebra is Koszul and has Koszul dual the $q$-exterior algebra, $A_q^! = \mathbb{C}\langle \partial_0, \dots, \partial_{n-1}\rangle/(\partial_i \partial_j + q_{ji} \partial_j \partial_i)$, with $(\partial_i)$ interpreted as the dual basis to $(x_i)$. In this talk we explained how to construct some filtered and non-filtered deformations of $A_q$. In some cases, notably where $A_q$ is related via deformation quantisation to toric log symplectic structures on $\mathbb{P}^{n-1}$, i.e., Poisson brackets of the form $\sum_{i<j} \pi_{ij} x_i \partial_i \wedge x_j \partial_j$ for some $\pi = (\pi_{ij})$ forming a skew-symmetric $n \times n$ matrix of corank one, we construct deformations which are formally universal. They give many new families of quadratic, Koszul, and twisted Calabi–Yau (and hence Artin–Schelter regular) algebras. These results are motivated by the work [2] of the authors on Poisson deformations of toric log symplectic structures (and more generally, log symplectic structures whose polar divisor is normal crossings).

The basic technique we use is the Hochschild cohomology of $A_q$ together with its $\mathbb{Z}^n$ grading coming from dilations on each of the variables $x_0, \dots, x_{n-1}$. We describe this structure (which is not new, and is a special case of results appearing in the literature, such as [1, Theorem 3.3]): $HH^*(A_q)$ is concentrated in the weights (indexing $\mathbb{Z}^n$ from 0 to $n-1$):

$$\Phi := \{w \in \mathbb{Z}^n \mid w_j \geq -1, \forall j, \prod_k q_{jk}^{w_k} = 1 \text{ whenever } w_j \geq 0\} \subseteq \mathbb{Z}^n.$$

Setting $\Phi_i := \{w \in \Phi \mid \#\{j \mid w_j = -1\} = i\}$ and $\Phi_{\leq i} := \bigcup_{j \leq i} \Phi_j$, we see that $HH^i(A_q)$ is concentrated in weights $\Phi_{\leq i}$. Let $\overline{\mathbb{Z}^n} := \{w \in \mathbb{Z}^n \mid \sum_i w_i = 0\}$ be the weights summing to zero, i.e., the ones appearing in the dilation invariant subspace $HH^*(A_q)^{\mathbb{C}^\times}$. Let $\overline{\Phi_{\leq i}} := \Phi_{\leq i} \cap \overline{\mathbb{Z}^n}$. It follows that infinitesimal graded deformations are in weights $\overline{\Phi_{\leq 2}}$ and the possible obstructions are in weights $\overline{\Phi_{\leq 3}}$.

Therefore, given a subset $\Psi \subseteq \overline{\Phi_{\leq 2}}$ of weights such that no sum of $\geq 2$ elements of $\Psi$ lies in $\overline{\Phi_{\leq 3}}$, the deformations in the direction $\Psi$ are unobstructed. We explained that these infinitesimal deformations extend to filtered quadratic deformations $\widetilde{A}_q$ over $\mathbb{C}$, with the property that $\operatorname{gr} \widetilde{A}_q \cong A_q$, and if no elements of $\Psi$ are sums of other elements of $\Psi$, then the filtered degree $-1$ part matches the original first-order deformation. The filtration is defined on $A_q$, with $F^m A_q$ the sum of all weight spaces with weights $w \in \mathbb{N}^n$ such that $w \geq w'$ for $w'$ a sum of some $m$ elements of $\Psi$ (with $w \geq w'$ meaning $w_i \geq w'_i$ for all $i$). As filtered deformations preserve the property of being both Koszul and twisted $n$-Calabi–Yau, these deformations enjoy these properties (in particular, are Artin–Schelter regular).

We give several new examples of such filtered deformations. These examples include filtered deformations of $A_q$ for $n = 3$ and $n = 4$ which give irreducible components of the moduli space of quadratic algebras with the same graded dimension as $A_1$ but which do not pass through $A_1$. One example had, for $\zeta$ a primitive seventh root of unity, $q = \begin{pmatrix} 1 & 1 & \zeta & \zeta^{-2} & \zeta \\ 1 & 1 & \zeta^{-1} & \zeta^{-2} & \zeta^3 \\ \zeta^{-1} & \zeta & 1 & \zeta & \zeta^{-1} \\ \zeta^2 & \zeta^2 & \zeta^{-1} & 1 & \zeta^{-3} \\ \zeta^{-1} & \zeta^{-3} & \zeta & \zeta^3 & 1 \end{pmatrix}$. This $A_q$ is untwisted Calabi–Yau and we found a family of filtered Calabi–Yau deformations given by replacing three of the $q$-commutation relations by the following:

$$x_0x_1 - x_1x_0 = ax_3x_4, \quad x_0x_2 - \zeta x_2x_0 = bx_4^2, \quad x_0x_3 - \zeta^5 x_3x_0 = cx_2^2.$$

Note that, for $a, b, c$ nonzero, these algebras are all isomorphic to the one for $a = b = c = 1$ by dilating the variables. We show moreover that the centres of these algebras form a flat family, generated by $x_i^7$ for all variables $x_i$, together with a deformation of the element $x_0x_1x_2x_3x_4$. For $a = b = c = 1$ this deformation is

$$x_0x_1x_2x_3x_4 + \frac{1}{7}(-5\zeta^5 - 3\zeta^4 - \zeta^3 + \zeta^2 - 4\zeta - 2)x_4x_2^3x_1 +$$
$$\frac{1}{7}(-\zeta^5 + 5\zeta^4 + 4\zeta^3 + 3\zeta^2 + 2\zeta + 1)x_4^2 - x_3^2x_2 + \frac{1}{7}(\zeta^5 - 5\zeta^4 - 4\zeta^3 - 3\zeta^2 - 2\zeta - 1)x_4^3x_3x_1.$$

For another example, with $n = 3$, we set $q = \begin{pmatrix} 1 & -1 & i & i \\ -1 & 1 & i & i \\ -i & -i & 1 & -1 \\ -i & -i & -1 & 1 \end{pmatrix}$, with weights of $HH(A_q)^{\mathbb{C}^\times}$ given by $0$, $(-1,-1,1,1)$, $(1,1,-1,-1)$, $(3,-1,-1,-1)$, $(-1,3,-1,-1)$, $(-1,-1,3,-1)$, and $(-1,-1,-1,3)$. This $A_q$ is untwisted Calabi–Yau, and we produce from this the filtered Calabi–Yau deformation of $A_q$ given by replacing the relation $x_0x_1 + x_1x_0$ by $x_0x_1 + x_1x_0 + tx_2x_3$ for a parameter $t \in \mathbb{C}$.

This example is particularly interesting because $q$ is conjugate by a diagonal matrix to the one $q' = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$. Here, $A_{q'}$ is a graded Clifford algebra (now twisted Calabi–Yau), and its deformations include a twelve-parameter family of graded Clifford algebras (thanks to Colin Ingalls for pointing this out). The weights for these deformations are given by $(-1, -1, 2, 0)$ and permutations of this, so they are different from the previous weights; generic deformations can be given by replacing relations $x_i x_j + x_j x_i$ by $x_i x_j + x_j x_i + a x_k^2 + b x_\ell^2$ for $i, j, k, \ell$ all distinct, and $a, b \in \mathbb{C}$. Since $q, q'$ are conjugate, $A_q$ and $A_{q'}$ define equivalent categories $\mathrm{qgr} A_q \simeq \mathrm{qgr} A_{q'}$ of $\mathbb{Z}$-graded modules modulo modules of elements annihilated by powers of the augmentation ideal $(x_0, x_1, x_2, x_3)$. Thus one can construct from the preceding an abelian category deformation $\mathrm{qgr} A_q$ which is not obtainable by deforming $A_q$ itself (as an algebra).

We then proceed to consider the case where $A_q$ is related to toric log symplectic structures on $\mathbb{P}^{n-1}$ by deformation quantisation. According to our paper [2], the weights occurring in the second Poisson cohomology $HP^*(\mathbb{P}^{n-1}, \pi)$ for the latter type of Poisson structures have a rigid structure: other than the zero weight, they are given by weights $w \in \mathbb{Z}^n$ with $w_i = -1$ for exactly two values of $i$. For each such weight $w$ with $w_i = w_j = -1$, colour the corresponding edge $i$—$j$ of the complete graph on $n$ vertices. Then in [2] we proved that the resulting couloured set has connected components which are cycles and segments. We additionally colour in the angles $\angle ikj$ opposite to coloured edges $i$—$j$ when the weight $w$ with $w_i = w_j = -1$ has $w_k > 0$; we colour it darkly if $w_k = 2$, and lightly if $w_k = 1$, so that there are either two lightly coloured angles or one darkly coloured angle opposite to each coloured edge. We call the resulting graph the *smoothing diagram* due to its geometric interpretation: it specifies which codimension-one singularities of the degeneracy locus of $\pi$ (i.e., the union of coordinate hyperplanes of $\mathbb{P}^{n-1}$) can be removed under deformation.

If $\pi = \sum_{i,j} \pi_{ij} x_i \partial_i \wedge x_j \partial_j$ is a toric Poisson structure on $\mathbb{P}^{n-1}$, then for generic $\hbar$, setting $q_{ij} := \exp(\hbar \pi_{ij})$, the weights of $HH^*(A_q)$ are the same as the weights of the Poisson cohomology of $(\mathbb{P}^{n-1}, \pi)$. In fact, by work in progress by Lindberg and Pym, for $\hbar$ a deformation parameter, $A_q$ is the Kontsevich canonical deformation quantisation of $(\mathbb{P}^{n-1}, \pi)$.

Our main theorem shows that, when $HH^*(A_q)$ has the same weights as those occurring in the Poisson cohomology of a toric log symplectic structure (for example, $q_{ij} = \exp(\hbar \pi_{ij})$ for $\pi$ log symplectic and $\hbar$ generic), then we can construct a formally universal family of actual quadratic deformations of $A_q$ by a combination of modifying $q$, applying filtered deformations over the coulored components of the smoothing diagram which are segments, and replacing couloured cycles by Feigin–Odesskii elliptic algebras [3], and finally tensoring the elliptic algebras together with the aforementioned filtered deformation on the complement of the couloured cycles. Note that, unlike the examples considered at the beginning of this report, all of these families are obtained by analytic continuation of deformation quantisations of $A_1$, and in particular have $A_1$ on the closure.

Unlike our paper [2], which considers more general varieties than $\mathbb{P}^{n-1}$ and which need not be toric, we do not need to consider the algebraic structures on the Hochschild cohomology ($L_\infty$ structures) and it is enough to only consider the weight decomposition. We also recover the corresponding statement to our main theorem for Poisson deformations of $(\mathbb{P}^{n-1}, \pi)$, a special case of the main result of [2], without requiring the algebraic structures on $\mathbb{P}^{n-1}$. The reason why we do not need the algebraic structure is because the unobstructedness follows by the explicit construction of the deformations (the filtered ones above and the Feigin–Odesskii ones).

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## Hochschild cohomology of Hilbert schemes of points

Pieter Belmans
(joint work with Lie Fu, Andreas Krug)

We will start from an algebro-geometric question which, a priori, has nothing to do with Hochschild cohomology. Yet, with the right approach, it turns out that (a generalization of) Hochschild cohomology is precisely the tool to answer this question, and at the same time the methods and tools also suggest interesting invariants to study outside this specific geometric setup.

**Geometric motivation.** Let $S$ be a smooth projective surface. Its *Hilbert scheme of points* $\mathrm{Hilb}^n S$ (where $n \geq 2$) is a smooth projective variety arising as an important example of a moduli space: the moduli space of length-$n$ subschemes, whilst at the same time it is a crepant resolution of singularities of $\mathrm{Sym}^n S = S^n \times / \mathrm{Sym}_n$, through the Hilbert–Chow morphism

$$\mathrm{Hilb}^n S \to \mathrm{Sym}^n S. \tag{1}$$

Its geometry has been the topic of significant interest.

We are interested in its *deformation theory*, which we will approximate by trying to understand the vector space $\mathrm{H}^1(\mathrm{Hilb}^n S, \mathrm{T}_{\mathrm{Hilb}^n S})$ classifying first-order deformations of the Hilbert scheme. It always contains the first-order deformations $\mathrm{H}^1(S, \mathrm{T}_S)$ of the surface, but what else might be in there?

The following intermediate results exist:

(a) Fantechi [5] has shown that if $\mathrm{H}^1(S, \mathcal{O}_S) = 0$ or $\mathrm{H}^0(S, \mathrm{T}_S) = 0$, and at the same time $\mathrm{H}^0(S, \omega_S^\vee) = 0$, then

$$\mathrm{H}^1(\mathrm{Hilb}^n S, \mathrm{T}_{\mathrm{Hilb}^n S}) = \mathrm{H}^1(S, \mathrm{T}_S),$$

i.e., they have the *same* deformation theory. These conditions hold, e.g., whenever $S$ is a surface of general type.

(b) Hitchin [6] has shown that if $\mathrm{H}^1(S, \mathcal{O}_S) = 0$, then

$$\mathrm{H}^1(\mathrm{Hilb}^n S, \mathrm{T}_{\mathrm{Hilb}^n S}) = \mathrm{H}^1(S, \mathrm{T}_S) \oplus \mathrm{H}^0(S, \omega_S^\vee),$$

thus linking the Poisson structures on $S$ to the deformations of $\mathrm{Hilb}^n S$.

Both proofs are very geometric and heavily rely on the geometry of (1). A more categorical proof of Hitchin's result, moreover assuming that $\mathrm{H}^2(S, \mathcal{O}_S) = 0$ is given in [3], which uses Hochschild cohomology and its limited functoriality.

But in complete generality, by [2, Corollary B] the answer for an arbitrary surface is given by

$$\mathrm{H}^1(\mathrm{Hilb}^n S, \mathrm{T}_{\mathrm{Hilb}^n S}) = \mathrm{H}^1(S, \mathrm{T}_S) \oplus \mathrm{H}^0(S, \omega_S^\vee) \oplus \left(\mathrm{H}^1(S, \mathcal{O}_S \otimes \mathrm{H}^0(S, \mathrm{T}_S)\right). \tag{2}$$

**Hochschild–Serre cohomology.** In order to prove (2) we (re)introduce a bigraded algebra that contains Hochschild cohomology and Hochschild homology as graded subspaces. This definition has an obvious analogue for an arbitrary smooth and proper dg category $\mathcal{A}$ (and we will come back to this later), with $\mathbf{D}^{\mathrm{b}}(X)$ for $X$ a smooth projective variety (or Deligne–Mumford stack) recovering the geometric definition we make now.

The *Hochschild–Serre cohomology* of $X$ is

$$\mathrm{HS}^*_\bullet(X) := \bigoplus_{j,k \in \mathbb{Z}} \mathrm{HS}^j_k(X)$$

where

$$\mathrm{HS}^j_k(X) := \mathrm{Ext}^{j+k\dim X}_{X \times X}(\Delta_* \mathcal{O}_X, \Delta_* \omega_X^{\otimes k}).$$

One recognizes the powers of the Serre functor of $\mathbf{D}^{\mathrm{b}}(X)$, which explains how to define this for every dg category which admits a Serre functor.

We have that

(1) $k = 0$ recovers the Hochschild cohomology of $X$,
(2) $k = 1$ recovers the Hochschild homology of $X$.

There are some obvious questions one should ask about this object. But first we explain the relation to the deformation theory of $\mathrm{Hilb}^n S$.

**Main result.** The Hochschild–Serre cohomology can be shown to be a categorical invariant. And we can extend (1) to include the stacky symmetric quotient

$$\begin{tikzcd} \mathrm{Hilb}^n S \arrow[dr, "\text{Hilbert–Chow}"'] & & {[\mathrm{Sym}^n S] = [S^n / \mathrm{Sym}_n]} \arrow[dl, "\text{coarse moduli space}"] \\ & \mathrm{Sym}^n S & \end{tikzcd} \tag{3}$$

which also acts as a crepant resolution, it just happens to be a Deligne–Mumford stack. The Bridgeland–King–Reid–Haiman equivalence gives the equivalence

$$\mathbf{D}^{\mathrm{b}}(\operatorname{Hilb}^n S) \cong \mathbf{D}^{\mathrm{b}}([\operatorname{Sym}^n S]) \tag{4}$$

making it possible to compute the Hochschild (and Hochschild–Serre) cohomology of $\operatorname{Hilb}^n S$ by computing it for $[\operatorname{Sym}^n S]$.

There is an orbifold Hochschild–Kostant–Rosenberg decomposition [4], which makes it possible to compute the Hochschild–Serre cohomology of any symmetric quotient stack $[\operatorname{Sym}^n X]$, which is where all the algebro-geometric work takes place. By suitably decomposing the computation using orbifold Hochschild–Kostant–Rosenberg, and then combining the components *for all n simultaneously* we can get a short answer. It depends on the parity of $\dim X$, so let us just give the conclusion for $\operatorname{Hilb}^n S$:

$$\bigoplus_{n\geq 0} \mathrm{HS}^*_k(\operatorname{Hilb}^n S)t^n \cong \operatorname{Sym}^* \left( \bigoplus_{i\geq 1} \mathrm{HS}^*_{1+(k-1)i}(S)t^i \right). \tag{5}$$

To prove (2) we take $k = 0$, so that *all* of the negative Hochschild–Serre cohomology of $S$ is used, and subsequently we take $* = 2$ to compute $\mathrm{HH}^2(\operatorname{Hilb}^n S)$. To obtain the geometric deformations in the Hochschild–Kostant–Rosenberg decomposition of $\mathrm{HH}^2(\operatorname{Hilb}^n S)$, one bootstraps from earlier results which describe the components $\mathrm{H}^2(\operatorname{Hilb}^n S, \mathcal{O}_{\operatorname{Hilb}^n S})$ and $\mathrm{H}^2(\operatorname{Hilb}^{\mathrm{n}} \mathrm{S}, \bigwedge^2 \mathrm{T}_{\operatorname{Hilb}^n S})$, and cancels these contributions in the Hochschild–Serre calculation.

**Questions.** We have the following obvious questions, which are of interest even if you do not care at all about (2):

1. Equip the Hochschild–Serre cohomology of a smooth and proper dg category with the structure of a Gerstenhaber algebra (and also a Connes differential), recovering the usual Gerstenhaber calculus structure on the *pair* $(\mathrm{HH}^\bullet(X), \mathrm{HH}_\bullet(X))$.
2. Relate this Gerstenhaber algebra structure to the geometric Gerstenhaber algebra structure on the Hochschild–Kostant–Rosenberg decomposition of the Hochschild–Serre cohomology, generalizing the work of Kontsevich, Căldăraru, Calaque–Van den Bergh, … These two questions are work-in-progress by Lie Fu and collaborators.
3. Extend the picture beyond smooth and proper dg categories.
4. There is a Heisenberg algebra controlling the properties of symmetric quotient stacks (and symmetric powers of dg categories). This originates in the computation of Betti and Hodge numbers of Hilbert schemes of points. Is there a Heisenberg algebra controlling the Hochschild–Serre cohomology of symmetric quotient stacks?

   There are some obvious problems that arise: Hochschild–Serre cohomology is not very functorial (yet), and the description of the Hochschild–Serre cohomology does not obviously fit in the usual description of a Fock space.

(5) Compute the Hochschild–Serre cohomology of the symmetric power of a dg category $\mathcal{A}$ in terms of the Hochschild–Serre cohomology of $\mathcal{A}$ in geometrically meaningful examples, e.g., for the noncommutative projective planes and quadrics from [1].

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## Some applications of Hochschild cohomology in physics

Evgeny Skvortsov
(joint work with Alexey Sharapov)

The main goal of the presentation is to explain what some of the applications of Hochschild cohomology in physics are. One of the biggest problems of the theoretical high energy physics is the quantum gravity problem, to which there exists a number of approaches, e.g. string theory, asymptotic freedom, higher spin gravity (HiSGRA), see e.g. [1], etc. HiSGRA is perhaps the unique topic in physics where the main structures are determined by Hochschild cohomology. Via AdS/CFT correspondence some HiSGRA are related to the second order phase transitions, which carries over the same mathematical structures.

Let us explain first why many problems in (quantum) field theory are controlled by Chevalley-Eilenberg cohomology and not by Hochschild one. A classical field theory is often a useful starting point to get a quantum one. The former is usually given as an action functional, which leads to classical equations of motion. The equations of motion can always be written in the form of a sigma-model $dw = Q(w)$, where fields $w$ are maps of zero degree $w : \mathcal{N}_1 \to \mathcal{N}_2$ between two $Q$-manifolds: $\mathcal{N}_1$ is the $Q$-manifold $(\Pi TM, d)$, the parity shifted tangent bundle of a spacetime manifold $M$, the algebra of functions being the exterior algebra of differential forms $\Omega^\bullet(M)$ and $d$ is the de Rham differential; $\mathcal{N}_2$ is a nonnegatively graded $Q$-manifold with a homological vector field $Q$. The maps $w$ (fields) are required to relate the $Q$-structures, $w_\star(d) = Q$, which can be written as a set of PDEs $dw = Q(w)$. Examples include numerous AKSZ-models [2]. Additional properties are $Q(0) = 0$. Then, as is well-known, the Taylor series of $Q$ can be interpreted as the structure maps of an $L_\infty$-algebra. One can also achieve $Q'(0) = 0$ and, hence, the $L_\infty$-algebra is minimal and its first (bilinear) map defines some (graded) Lie algebra $\mathfrak{g}$. The trilinear map is a Chevalley-Eilenberg cocycle of $\mathfrak{g}$. More generally, many questions about a given field theory (counterterms, charges, anomalies, etc.) can be reduced to the Chevalley-Eilenberg of $\mathfrak{g}$.

HiSGRAs are very special types of field theories that extend gravity with massless fields of all spins and are controlled by infinite-dimensional symmetries, called higher-spin symmetries. Higher-spin algebras are associative algebras that can be defined in many different ways, e.g. as the deformation quantization of coadjoint orbits of Lie groups that represent the spacetime symmetry. A toy model is the Moyal-Weyl star-product algebra on $\mathbb{R}^{2n}$, which realizes the Weyl algebra $A_n$. It is also important that any higher-spin algebra, say $A$, can be tensored with $\mathrm{Mat}_n(k)$ to get $\mathrm{Mat}_n(A)$ and the corresponding HiSGRA theory should exist for any $n$.

As a result, the dynamics of HiSGRA is determined by the Chevalley-Eilenberg cohomology of $gl_n(A) = L(\mathrm{Mat}_n(A))$, where $L$ is the canonical map that sends any associative algebra to a Lie algebra. Thanks to the Tsygan–Loday–Quillen theorem the Chevalley-Eilenberg cohomology of $gl_n(A)$ can be related to the cyclic cohomology of $A$ and the latter can be related to the Hochschild cohomology. As a matter of fact, interactions in HiSGRAs and various observables are directly related to/determined by the Hochschild cohomology of $A$ and there are no examples where any additional information enters. Not surprisingly, thanks to the effect of 'big matrices' the $L_\infty$-algebra ($Q$ structure) that determines the equations of motion originates from an $A_\infty$-algebra, say $\mathbb{A}$, via the symmetrization map.

Let us explain some of the technical details with one example — Chiral HiSGRA, see e.g. [3]. The higher-spin algebra is just $A_1$ ($A_1$ is the smallest Weyl algebra). The free theory is encoded by an $A_\infty$-algebra $\mathbb{A}_0$ that is built on $A_1$ and its bimodule $A_1^*$. It is convenient to replace $A_1$ with $A_0 = A_1 \rtimes \mathbb{Z}_2$ (skew group algebra, where the nontrivial element $r$ of $\mathbb{Z}_2$ realizes the reflection map on $\mathbb{R}^2$) since there is a direct relation between the twisted bimodule $A_1 r$ and $A_1^*$. As a graded vector space $\mathbb{A}_0$ is concentrated in degrees 0 and 1. The maps of $\mathbb{A}_0$ are

$$m_2(a,b) = a \star b \qquad m_2(a,u) = a \star u \qquad m_2(v,a) = -v \star a\,, \tag{1}$$

where $a, b, ... \in A_0$ (have degree 1) and $u, v, ... \in A_0$ its bimodule (have degree 0). The $A_\infty$-maps $m$ all have degree $(-1)$ and satisfy $m \circ m = 0$.

The interactions correspond to the deformations of $\mathbb{A}_0$. The first order deformation corresponds to trilinear structure maps $m_3(\bullet, \bullet, \bullet)$ and it does exist thanks to the AFLS (Alev-Farinati-Lambre-Solotar) theorem [4], which implies that $HH^2(A_0, A_0) \simeq k$. To give an example, we find for one of the $m_3$-components

$$m_3(a,b,u) = \phi_1(a,b) \star u\,, \tag{2}$$

where $\phi_1 \in HH^2(A_0, A_0)$. There are no obstructions $HH^3(A_0, A_0) \simeq 0$ and the deformation can be continued. In fact, it can be shown that any one-parameter family of associative algebras $A_\nu$ (the product in $A_0$ is denoted by $\star$)

$$a \diamond b = a \star b + \sum_{k \geq 1} \nu^k \phi_k(a,b) \tag{3}$$

can be used to construct an $A_\infty$-algebra whose associated field theory equations $dw = Q(w)$ are integrable [5, 6]. Here $\phi_1 \in HH^2(A_0, A_0)$ that determines the leading deformation. All $A_\infty$ structure maps can be constructed out of $\phi_i$.

There is an interesting relation to the deformation quantization. $A_1$ can be understood as the deformation quantization of the algebra $C(\mathbb{R}^2)$ of functions on $\mathbb{R}^2$. However, as we see, the field theory requires $B = C(\mathbb{R}^2) \rtimes \mathbb{Z}_2$. Another closely related algebra is the orbifold algebra $C(\mathbb{R}^2)/\mathbb{Z}_2 \sim C(\mathbb{R}^2/\mathbb{Z}_2)$. In general, the deformation quantization of orbifolds is an open problem, see e.g. [7, 8]. We see that $B$ admits two deformations, one deforming $C(\mathbb{R}^2)$ into $A_1$ and another one that relies on the reflection map from $\mathbb{Z}_2$. The Hochschild cocycle $\phi_1$ above is directly related to the Feigin-Felder-Shoikhet cocycle $\psi \in HH^2(A_1, A_1^*)$ [9]. The latter can be obtained as a consequence of the Kontsevich-Shoikhet-Tsygan formality [10].

Therefore, we observed that the first two 'floors', $m_{2,3}$, of the $A_\infty$-algebra are related to the Kontsevich and the Kontsevich-Shoikhet-Tsygan formalities. The higher structure maps can also be explicitly constructed as certain configuration space integrals of the Kontsevich-type. The configuration space is that of compact concave polygons and is related to the totally nonnegative Grassmannian [3]. The $A_\infty$-relations can be proven via Stokes theorem. This points towards an existence of a structure encompassing the known formality theorems.

Holographically Chiral HiSGRA is related to a rich class of 3d conformal field theories — Chern-Simons vector models. The latter were recently conjectured to exhibit a new duality — 3d bosonization duality. Via AdS/CFT correspondence the same $A_\infty$-algebras can be seen to realize a certain new type of a symmetry in the vector models. The $3d$ bosonization duality can be reduced to proving that the corresponding $L_\infty$-algebra admits a unique set of invariants that play the role of correlation functions. Again, the proof of the duality can be reduced to the Hochschild cohomology [5, 11, 12].

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## Deformations of graded gentle algebras and orbifold surfaces


Zhengfang Wang

(joint work with Severin Barmeier, Sibylle Schroll)


**Definition 1.** Let $\Bbbk$ be a field. An associative algebra is *graded gentle* if it is isomorphic to a graded algebra $A \simeq \Bbbk Q/I$, where $Q$ is a finite graded quiver and $I \subset \Bbbk Q$ is a two-sided ideal generated by quadratic monomial relations, where

- at each vertex of $Q$ there are at most two incoming and two outgoing arrows
- for each arrow $y$ of $Q$ there is at most one arrow $x$ such that $xy \notin I$, at most one arrow $x'$ such that the path $x'y \in I$, at most one arrow $z$ such that $yz \notin I$ and at most one arrow $z'$ such that the path $yz' \in I$.

A maximal configuration of arrows at a vertex is illustrated in Fig. 1, where we follow the widely adopted notation of marking the relations directly in the quiver by a dotted line, i.e.

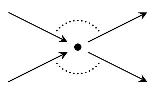

Figure 1. A maximal configuration of edges at a vertex of a gentle quiver

Given a gentle algebra $A = \Bbbk Q/I$, we may define a ribbon graph [10]. The combinatorial geometric model of $A$ consists of the surface associated to this ribbon graph which was shown in [9, 6, 8] to describe the derived category of $A$, together with a line field and a dissection into topological disks by curves on the surface connecting certain marked points, all of which can be determined from $A$. Moreover, in [6] it is shown that the derived category of $A$ is triangle equivalent to the partially wrapped Fukaya category of the associated graded surface with stops.

In this talk we give a complete and explicit description of the $\mathrm{A}_\infty$ deformation theory of graded gentle algebras which in the homologically smooth case gives a complete description of the algebraic deformation theory of the partially wrapped Fukaya category of graded surfaces with stops in the sense of [6, 8].

The deformation theory of associative algebras is naturally controlled by the Hochschild cohomology in degree 2. Inspired by [5] we have the following description of $\mathrm{HH}^2(A, A)$ for a graded gentle algebra $A$ in terms of its surface model.

**Theorem 2.** Let $A$ be a graded gentle algebra with surface model $(S, \Sigma, \eta)$. Then $\dim \mathrm{HH}^2(A, A)$ coincides with the number of boundary components $\partial_i S$ of $\Sigma$ satisfying one of the following conditions

- $\partial_i S$ has a single stop with winding number $-1$,
- $\partial_i S$ is fully stopped with winding number $-1$ or $-2$,
- $\partial_i S$ has no stops with winding number $-1$ or $-2$,

Note that the cocycles corresponding to boundary components without stops of winding numbers $-1$ or $-2$ give rises to curved $\mathrm{A}_\infty$ deformations, which will be excluded in this talk.

A natural question is that what the geometric meanings of the (un-curved) $\mathrm{A}_\infty$ deformations of graded gentle algebras are. To answer this question, we need to study the partially wrapped Fukaya categories of orbifold surfaces.

**Definition 3.** Let $\Gamma$ be an arc system in an orbifold surface $(S, \Sigma)$ and let $x \in \mathrm{Sing}(S)$ be an orbifold point. Denote by $\Gamma_x$ the set of half-edges of arcs in $\Gamma$ connecting to $x$. By a *linear order* $\prec_x$ at $x$ we shall mean a linear order on the set $\Gamma_x$ which is compatible with the natural cyclic order obtained from the orientation of $S$.

A *dissection* $\Delta = (\Gamma, \{\prec_x\}_{x \in \mathrm{Sing}(S)})$ of an orbifold surface with stops $(S, \Sigma)$ is given by an arc system $\Gamma$, where $\Gamma_x \neq \emptyset$ for each $x \in \mathrm{Sing}(S)$, together with a collection of linear orders at all orbifold points such that the complement of the arcs in $\Delta$ is a disjoint union of

- (topological) disks containing no orbifold points in the interior and at most one stop or one missing relation in their boundary
- (topological) annuli with one boundary component being a fully stopped boundary component and the other boundary component containing no stops or missing relations

See Fig. 2 for an illustration.

**Definition-Proposition 4.** Let $\mathbf{S} = (S, \Sigma, \eta)$ be a graded orbifold surface with stops. To any dissection $\Delta$ on $\mathbf{S}$ we associate an $\mathrm{A}_\infty$ category $\mathbf{A}_\Delta$ whose objects are the arcs in $\Gamma$.

Given two arcs $\gamma_i$ and $\gamma_j$ in $\Gamma$, a $\Bbbk$-linear basis of morphism from $\gamma_i$ to $\gamma_j$ is given by the boundary paths and orbifold paths from $\gamma_i$ to $\gamma_j$. There are three types of $\mathrm{A}_\infty$ products on $\mathbf{A}_\Delta$

- $\bar{\mu}_2$ for the (associative) concatenation of paths
- $\mathring{\mu}_n$ for $n \geq 3$, which is given by smooth disk sequences
- $\overset{\times}{\mu}_n$ for $n \geq 1$, which is given by an orbifold disk sequences

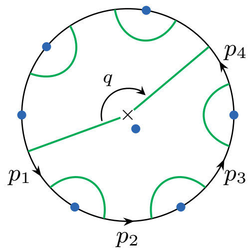


FIGURE 2. Arcs on a disk with a single orbifold point and 6 stops

For instance, for the dissection $\Delta$ in Figure 2 the only nonzero product in $\mathbf{A}_\Delta$ is given by

$$\overset{\times}{\mu}_4(p_1 \otimes p_2 \otimes p_3 \otimes p_4) = q.$$

**Theorem 5.** Let $\mathbf{S} = (S, \Sigma, \eta)$ be a graded orbifold surface with stops. We have the following

(1) The idempotent completion $\mathrm{tw}(\mathbf{A}_\Delta)^\natural$ of the DG category of twisted complexes of $\mathbf{A}_\Delta$ is independent of the choice of the dissection $\Delta$ on $\mathbf{S}$. We call $\mathrm{tw}(\mathbf{A}_\Delta)^\natural$ the partially wrapped Fukaya category of $\mathbf{S}$ and denote it by $\mathcal{W}(\mathbf{S})$.

(2) Let $\widetilde{\mathbf{S}} = (\widetilde{S}, \widetilde{\Sigma}, \widetilde{\eta})$ be the double cover of the orbifold surface $\mathbf{S}$. Then $\mathcal{W}(\mathbf{S})$ of $\mathbf{S}$ is quasi-equivalent to the partially wrapped Fukaya category $\mathcal{W}(\widetilde{\mathbf{S}})$ of the smooth graded surface $\widetilde{\mathbf{S}}$ studied in [6].

**Remark 6.** A good indication that this is the right construction is the fact we just learnt that there is in [4] a similar construction independently.

As an application of the above theorem we may describe the geometric meaning of $\mathrm{A}_\infty$ deformations of graded gentle algebras using orbifold surfaces.

**Theorem 7.** Let $A$ be a graded gentle algebra whose surface model is given by $\mathbf{S} = (S, \Sigma, \eta)$.

(1) Let $\theta$ be a linear combination of cocycles which correspond to boundary components of winding number $-1$ in Theorem 2. Denote by $A_\theta$ the corresponding (un-curved) $\mathrm{A}_\infty$ deformation. Then the idempotent completion $\mathrm{tw}(A_\theta)^\natural$ is quasi-equivalent to the partially wrapped Fukaya category $\mathcal{W}(\overline{\mathbf{S}})$ of the orbifold surface $\overline{\mathbf{S}} = (\overline{S}, \overline{\Sigma}, \overline{\eta})$. Here $\overline{\mathbf{S}}$ is obtained from $\mathbf{S}$ by compactifying the boundary components of $\theta$ into orbifold points.

(2) Any (uncurved) $\mathrm{A}_\infty$ deformation of $A$ is derived equivalent to a graded skew-gentle algebra, except the case where $\mathbf{S}$ is a genus zero surface with four boundary components each of which has exactly one stop and the winding number $-1$.

**Remark 8.** Conversely, we expect that any algebra $B$ which is derived equivalent to a skew gentle algebra comes from a *formal* dissection on the orbifold surface.

Note that some special dissections which give skew-gentle algebras were studied by [7, 1].

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## Higher structure on the Gerstenhaber-Schack complex: rich and (box) operadic


Lander Hermans

(joint work with Hoang Dinh Van, Wendy Lowen, Ricardo Campos)


The Gerstenhaber-Schack (GS) complex for prestacks takes up the pivotal role of the Hochschild complex for associative algebras: GS cohomology computes Ext-cohomology, the complex captures the algebraic structure of the prestack and it is endowed with a $\mathrm{L}_\infty$-structure governing its deformations. Prestacks generalize presheaves of associative algebras and are motivated by (noncommutative) algebraic geometry where they appear as structure sheaves and (noncommutative) deformations thereof. Indeed, Lowen and Van den Bergh observed in [1][2] that Hochschild cohomology of presheaves parametrizes their first order deformations, not as presheaves, but as prestacks.

We present our work on rich higher structure on the GS complex, from which the $\mathrm{L}_\infty$-structure is obtained as a shadow. We have, joint with Hoang Dinh Van and Wendy Lowen, the following main results:

(1) In [3], we establish an action of the operad $\mathsf{Quilt}$ from [7] on the GS complex. $\mathsf{Quilt}$ induces a $\mathrm{L}_\infty$-structure which we twist by a part of the algebraic structure of the prestack.
(2) In [4], we describe an action of a new operad $\Box\mathsf{p}$, whose algebras are called 'box operads', on an extension of the GS complex. In a next step, we show

that every box operad is endowed with a $L_\infty$-structure and prove that the GS complex is a $L_\infty$-subalgebra of its box operadic extension.

In this talk, I touch upon both (1) and (2), and I present (1) in detail via a calculus of rectangles. This allows us to describe the higher operations coming from $\mathsf{Quilt}$ explicitly, such as the resulting $L_\infty$-structure.

The above two approaches each have their own advantage as follows. Approach (2) establishes the right analogy with the seminal work of Gerstenhaber-Voronov on operads [5] which I recall in the talk. On the other hand, I explain how, in joint work with Ricardo Campos, we build on (1) to obtain our solution to the Deligne conjecture in [6] showing that the GS complex is an algebra over (an operad homotopy equivalent to) the chain little disks operad. Concretely, we solve Hawkins' conjecture by computing the homology of $\mathsf{Quilt}$ and show that it is isomorphic to the operad $\mathsf{Brace}$ encoding brace algebras. As a corollary, the twisted operad $\mathsf{TwQuilt}$ is homotopy equivalent to the chain little disks operad. This is work in preparation.

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# Hochschild-Pirashvili homology and outer functors


Christine Vespa

(joint work with Geoffrey Powell)


In 2000, Pirashvili [1] associated to any topological space $X$ a homology theory. For $X$ the circle, this homology identifies with the classical Hochschild homology and Pirashvili studied the case where $X$ is a sphere. For $X$ a wedge of n circles, Hochschild-Pirashvili homology has been first studied by Turchin and Willwacher [2]. In particular, they obtained that this homology gives rise to interesting representations of the groups $\mathrm{Out}(F_n)$ named *bead representations*. Recently Gadish and Hainaut proved that these representations appear naturally in the study of configuration spaces on a wedge of circles [3].

In this talk I will give an overview of the previous results and, noting that Hochschild-Pirashvili homology for a wedge of circles gives rise to a functor on finitely generated free groups, I will explain how functorial tools such as polynomial functors, exponential functors and, what we called, outer functors can be used to study Hochschild-Pirashvili homology for a wedge of circles. [4] (This is a joint work with Geoffrey Powell).

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## On the first $\tau$-tilting Hochschild cohomology of an algebra


Claude Cibils

(joint work with M. Lanzilotta, E.N. Marcos, A. Solotar)


Let $k$ be a field, $\Lambda$ be a finite dimensional $k$-algebra and $\tau$ be the Auslander-Reiten translation of $\Lambda$-bimodules, see for instance [6]. We introduce, according to one of the main ideas of $\tau$-tilting theory mentioned in [4], the $\tau$-tilting Hochschild cohomology

$$^{\tau}HH^1(\Lambda) = \mathsf{DHom}_{\Lambda-\Lambda}(\Lambda, \tau\Lambda).$$

This relies on the Auslander-Reiten duality formula, see for instance [1]. In a sense $^{\tau}HH^1(\Lambda)$ amounts to recover the missing morphisms from $\Lambda$ to $\tau\Lambda$ which factors through injective bimodules, see [2].

We define the excess $e(\Lambda) = \mathsf{dim}_k {}^{\tau}HH^1(\Lambda) - \mathsf{dim}_k HH^1(\Lambda)$.

One of the main results is that for a zero excess bound quiver algebra $\Lambda = kQ/I$, the Hochschild cohomology in degree two $HH^2(\Lambda)$ is isomorphic to the space of morphisms $\mathsf{Hom}_{kQ-kQ}(I/I^2, \Lambda)$. This may be useful to determine when $HH^2(\Lambda) = 0$ for these algebras.

We recall that the algebras the following about algebras $\Lambda$ with $HH^2(\Lambda) = 0$. Let $V$ be a -vector space of dimension $n$ over an algebraically closed field $k$. Let $\mathcal{A}lg_n$ be the affine open subscheme of algebra structures with 1 of the affine algebraic scheme defined by $\mathcal{S}_n(R) = \{$associative $R$-algebra structures on $R \otimes_k V\}$, where $R$ is a commutative $k$-algebra. Corollary 2.5 of [3] states that $HH^2(\Lambda) = 0$ if and only if the orbit of $\Lambda \in \mathcal{A}lg_n$ under the general linear group $\mathcal{G}L(V)$ is an open subscheme of $\mathcal{A}lg_n$. Moreover, P. Gabriel in [3, p. 140] mentions that it should be one of the main tasks of associative algebra to determine for every $n$ the number of irreducible components of $\mathcal{A}lg_n$. The determination of algebras with zero Hochschild cohomology in degree 2 makes it possible to obtain lower bounds for the number of irreducible components of $\mathcal{A}lg_n$, as G. Mazzola did in [5, p. 100].

In [2] we compute the excess for hereditary, radical square zero and monomial triangular algebras. For a bound quiver algebra $\Lambda$, a formula for the excess of $\Lambda$ is obtained. We also give a criterion for $\Lambda$ to be $\tau$-rigid.

Let $\Lambda = kQ/I$ a bound quiver algebra, and let $Z\Lambda$ be its center. We have

$$\dim_k {}^{\tau}HH^1(\Lambda) = \mathsf{dim}_k Z\Lambda - \sum_{x \in Q_0} \mathsf{dim}_k x\Lambda x + \sum_{a \in Q_1} \mathsf{dim}_k t(a)\Lambda s(a).$$

Questions arise about Morita invariance, Morita derived invariance or derived invariance of ${}^{\tau}HH^1(\Lambda)$. Also about a possible Lie structure, and an eventual prolongation towards a cohomological theory.

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## Universal Massey products in representation theory of algebras

Gustavo Jasso

(joint work with Fernando Muro)

We work over an arbitrary field. Recall Kadeishvili's Intrinsic Formality Criterion [Kad88]:

**Theorem.** Let $A$ be a graded algebra whose Hochschild cohomology vanishes in the following bidegrees:

$$\mathrm{HH}^{p+2,-p}(A, A) = 0, \qquad p \geq 1.$$

Then, every minimal $A_\infty$-algebra structure on $A$ is gauge $A_\infty$-isomorphic to the trivial $A_\infty$-structure, whose higher operations $m_{p+2} = 0$, $p \geq 1$, vanish.

In our joint work we generalise Kadeishvili's Criterion as follows.

**Definition.** Fix an integer $d \geq 1$. A graded algebra is *$d$-sparse* if it is concentrated in degrees that are multiples of $d$ (hence this condition is empty if $d = 1$). A *$d$-sparse Massey algebra* is a pair $(A, m)$ consisting of a $d$-sparse graded algebra $A$ and a Hochschild cohomology class

$$m \in \mathrm{HH}^{d+2,-d}(A, A), \qquad \mathrm{Sq}(m) = 0,$$

of bidegree $(d + 2, -d)$ whose Gerstenhaber square vanishes.

For example, if

$$(A, m_{d+2}, m_{2d+2}, m_{3d+2}, \dots)$$

is a minimal $A_\infty$-algebra structure on a $d$-sparse graded algebra $A$ (in which case $m_{i+2} = 0$, $i \notin d\mathbb{Z}$, for degree reasons), then $m_{d+2} \in \mathrm{C}^{d+2,-d}(A, A)$ is a Hochschild cocycle whose associated Hochschild cohomology class

$$\{m_{d+2}\} \in \mathrm{HH}^{d+2,-d},$$

its *universal Massey product (of length $d + 2$)*, has vanishing Gerstenhaber square

$$\mathrm{Sq}(\{m_{d+2}\}) = 0.$$

Consequently, the pair $(A, \{m_{d+2}\})$ is a $d$-sparse Massey algebra.

*Remark.* It is an easy consequence of the $d$-sparsity assumption that the universal Massey product of a minimal $A_\infty$-algebra is invariant under $A_\infty$-isomorphisms.

*Remark.* Universal Massey products of length 3 have been investigated previously in representation theory, see for example [BKS04].

**Definition.** The *Hochschild–Massey cohomology* of a $d$-sparse Massey algebra $(A, m)$ is the cohomology

$$\mathrm{HH}^{\bullet,*}(A, m)$$

of the *Hochschild–Massey (cochain) complex*, that is the bigraded cochain complex with components

$$\mathrm{HH}^{p+2,*}(A, A), \qquad p \geq 0,$$

and differential

$$\mathrm{HH}^{\bullet,*}(A, A) \longrightarrow \mathrm{HH}^{\bullet+d+1,*-d}(A, A), \qquad x \longmapsto [m, x],$$

in source bidegrees different from $(d+1, -d)$, where the differential is instead given by the formula by

$$\mathrm{HH}^{d+1,-d}(A, A) \longrightarrow \mathrm{HH}^{2(d+1),-2d}(A, A), \qquad x \longmapsto [m, x] + x^2.$$

*Remark.* That the differential of the Hochschild–Massey complex squares to zero is a consequence of the Gerstenhaber relations and the assumption $\mathrm{Sq}(m) = 0$.

**Theorem** ([JKM22, Theorem B])**.** Let $(A, m)$ be a $d$-sparse Massey algebra whose Hochschild–Massey cohomology vanishes in the following bidegrees:

$$\mathrm{HH}^{p+2,-p}(A, m) = 0, \qquad p > d.$$

Then, any two minimal $A_\infty$-algebras

$$(A, m_{d+2}, m_{2d+2}, m_{3d+2}, \dots) \qquad \text{and} \qquad (A, m'_{d+2}, m'_{2d+2}, m'_{3d+2}, \dots)$$

such that $\{m_{d+2}\} = m = \{m'_{d+2}\}$ are gauge $A_\infty$-isomorphic.

*Remark.* Kadeishvili's Intrinsic Formality Criterion is indeed a corollary of the above theorem: Take $d = 1$ and notice that the hypothesis in the criterion implies that every minimal $A_\infty$-algebra structure on $A$ has vanishing universal Massey product $\{m_3\} = 0$.

The proof of the theorem relies in an essential way on an enhanced $A_\infty$-obstruction theory developed by F. Muro in [Mur20a]. We also mention that the theorem is one of the key ingredients in the proof of the main theorem in [JKM22] which, as explained by B. Keller in the Appendix to *loc. cit.*, in a special case yields the final step in the proof of the Donovan–Wemyss Conjecture in the context of the Homological Minimal Model Program for threefolds [DW16, Wem23].

The aforementioned applications of the theorem rely on the following observation: The Hochschild–Massey cochain is equipped with a canonical bidegree $(d + 2, -d)$ endomorphism given by

$$\mathrm{HH}^{\bullet,*}(A, A) \longrightarrow \mathrm{HH}^{\bullet+d+2,*-d}(A, A), \qquad x \longmapsto m \smile x,$$

in source bidegrees different from $(d + 1, -d)$, where it is given by

$$\mathrm{HH}^{d+1,-d}(A, A) \longrightarrow \mathrm{HH}^{2(d+1)+1,-2d}(A, A), \qquad m \smile x + \{\delta_{/d}\} \smile x^2.$$

Here,

$$\delta_{/d} \in \mathrm{C}^{1,0}(A, A), \qquad x \longmapsto \tfrac{|x|}{d} x,$$

is the fractional Euler derivation (notice that $\frac{|x|}{d}$ is an integer due to the assumption that the graded algebra $A$ is $d$-sparse). The above endomorphisms is in fact null-homotopic. An explicit bidegree $(1, 0)$ null-homotopy is given by

$$\mathrm{HH}^{\bullet,*}(A, A) \longrightarrow \mathrm{HH}^{\bullet+1,*}(A, A), \qquad x \longmapsto \{\delta_{/d}\} \smile x.$$

Thus, a sufficient condition for the assumptions in the theorem to be satisfied is that the components of above endomorphism of the Hochschild–Massey complex of $(A, m)$ are bijective in all non-trivial source bidegrees. The latter condition is satisfied by the $d$-sparse Massey algebras investigated in [JKM22].

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## The homotopy theory of operated algebras

Guodong Zhou

A general philosophy of deformation theory of mathematical structures, as evolved from ideas of Gerstenhaber, Nijenhuis, Richardson, Deligne, Schlessinger, Stasheff, Goldman, Millson etc, is that the deformation theory of any given mathematical object can be described by a certain differential graded (=dg) Lie algebra or more generally an $L_\infty$-algebra associated to the mathematical object (whose underlying complex is called the deformation complex). This philosophy has been made into a theorem in characteristic zero by Lurie [17] and Pridham [18], expressed in terms of infinity categories. It is an important problem to construct explicitly the dg Lie algebra or $L_\infty$-algebra governing deformation theory of the mathematical object under consideration.

Another important problem about algebraic structures is to study their homotopy versions, just like $A_\infty$-algebras for usual associative algebras. From the perspective of operad theory, specifically, the task is to formulate a cofibrant resolution for the operad of an algebraic structure. The most desirable outcome would be providing a minimal model of the operad governing the algebraic structure. When this operad is Koszul, there exists a general theory, the so-called Koszul duality for operads [9, 8], which defines a homotopy version of this algebraic structure via the cobar construction of the Koszul dual cooperad, which, in this case, is a minimal model. However, when a operad is NOT Koszul, essential difficulties arise and few examples of minimal models have been worked out.

These two problems, say, describing controlling $L_\infty$-algebras and constructing homotopy versions, are closed related. In fact, given a cofibrant resolution, in particular a minimal model, of the operad in question, one can form the deformation complex of the algebraic structure and construct its $L_\infty$-structure as explained by Kontsevich and Soibelman [14] and van der Laan [24, 25]. However, in practice, a minimal model or a small cofibrant resolution is not known a priori.

Recently, we succeeded in resolving completely the above two problems for a large class of non-Koszul operads, say operads of operated algebras (that is, associative or Lie algebras endowed with certain kinds of linear operators), such as Rota-Baxter algebras with arbitrary weight and differential algebras with nonzero weight. Surprisingly, our method returns to the original method of Gerstenhaber [6, 7]. The method consists of several steps.

- Study formal deformations of an operated algebra and found its deformation equations, and inspired by the deformation equations, try to found the deformation complex;
- try to found the $L_\infty$-structure on the deformation complex which controls deformations and deduce from it the homotopy version of the operated algebra in question;
- try to show that the dg operad of the homotopy version is the minimal model of the operad of the operated algebra in question;
- use the method given in Kontsevich and Soibelman [14] and van der Laan [24, 25] to show that the deformation complex as well as its $L_\infty$-structure found above are exactly those deduced from the minimal model.

Note that there is another method for finding the $L_\infty$-structure, say, by derived bracket technique in the sense of Voronov [26, 27]; see [23, 22, 15].

Generalising classical derivation operation of smooth functions in Analysis (weight 0) and difference operators in Numerical Analysis (weight $\pm 1$), Guo and Keigher introduced differential operators of arbitrary weight. Let $\mathbf{k}$ be a base field of characteristic zero. A differential (associative) algebra of weight $\lambda \in \mathbf{k}$ is an associative algebra $(A, \mu_A)$ together with a linear operator $d_A : A \to A$ such that

$$d_A(ab) = d_A(a)b + ad_A(b) + \lambda\, d_A(a)d_A(b), \quad \forall a, b \in A.$$

Note that the defining relation of differential operators is not quadratic in case that $\lambda \neq 0$. In fact, Loday showed that the operad of differential algebras of weight zero is Koszul and he asked to extend Koszul duality from weight zero case to nonzero weight case. In a joint work with Guo, Li, Sheng, we develop a cohomology theory for differential algebras of arbitrary weight in [13] and finally with Chen, Guo and Wang, we found the minimal model [3] which in turn justifies the cohomology theory found in [13].

Another class of operated algebras are Rota-Baxter algebras. Closed related to Yang-Baxter equations [20], Rota-Baxter algebras emerged from Baxter's research in probability theory [1] and in Rota [19]. From 2000, renewed interest in the subject arose with the research of Connes and Kreimer [5], Guo and Keigher [11, 12] etc. Let $(A, \mu = \cdot)$ be an associative algebra over field $\mathbf{k}$ and $\lambda \in \mathbf{k}$. A linear operator $T : A \to A$ is said to be a Rota-Baxter operator of weight $\lambda$ if it satisfies

$$\mu \circ (T \otimes T) = T \circ (\mathrm{Id} \otimes T + T \otimes \mathrm{Id}) + \lambda\, T \circ \mu.$$

In this case, $(A, \mu, T)$ is called a Rota-Baxter (associative) algebra of weight $\lambda$. It is obvious that the operad for Rota-Baxter algebras of weight $\lambda$ is not a Koszul operad, not even a quadratic operad. Together with Wang [28, 29], we succeeded in finding the minimal model of the operad for Rota-Baxter (associative) algebras of weight $\lambda$, which enables us to develop cohomology theory as its $L_\infty$-structure controlling deformations and introduce homotopy Rota-Baxter algebras.

We also deal with other operated algebras such as Rota-Baxter Lie algebras [4], Nijenhuis algebras [21] with applications to Nijenhuis geometry [2] in mind.

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## Towards (Fg) for Brauer graph algebras

Karin Erdmann

Let $\Lambda$ be a finite-dimensional algebra over some field $K$. It satisfies the finite generation condition known as (Fg) if its Hochschild cohomology $HH^*(\Lambda) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$ is Noetherian, and its Ext algebra $E(\Lambda) = \mathrm{Ext}^*_{\Lambda}(\Lambda/J, \Lambda/J)$ is finitely generated as a module for $HH^*(\Lambda)$. If so then every $\Lambda$-module has a support variety with properties similar to support varieties from group cohomology. One of the consequences is that modules with bounded projective resolution must be $\Omega$-periodic (see [5]). One can show that this property holds for any tame symmetric algebra, and one might expect that such algebras satisfy (Fg). As one sees from [2], there are weakly symmetric tame algebras which do not satisfy (Fg).

We focus on Brauer graph algebras, that is, special biserial symmetric algebras. Inspired by past work on a class of such algebras which are local (see [3], [4]), we compute a minimal bimodule resolution for any Brauer graph algebra which has a 2-regular Gabriel quiver.

The shape of this bimodule resolution suggested that it may be possible to understand homomorphisms $\Omega^n_{\Lambda^e}(\Lambda) \to \Lambda$ well enough, and show that the (commutative) subalgebra $HH^{4*}(\Lambda)$ of the Hochschild cohomology is Noetherian, and that the ext algebra is finitely generated over this subalgebra. As explained in [8], this is sufficient to prove (Fg). We give a brief outline of results so far.

### 1. Brauer graph algebras

Assume $\Lambda = KQ/I$ where $Q$ is 2-regular, that is two arrows start and two arrows end at each vertex. Let $\overline{(\ )}$ be the involution on the arrows so that $\alpha \neq \overline{\alpha}$ start at the same vertex. We fix a permutation $f$ of the arrows such that an arrow $\alpha$ can be composed with $f(\alpha)$, so that $\alpha f(\alpha)$ is a path of length two. Then $g$ is the permutation on the arrows such that $g(\alpha) = \overline{f(\alpha)}$. For an arrow $\alpha$ let $n_\alpha$ be the length of the $g$-cycle containing $\alpha$. As well, for each $g$-cycle we fix a multiplicity $m_\alpha$. At each vertex, we fix two cyclic paths along $g$-cycles of $\alpha$ and $\overline{\alpha}$, denoted by these by $B_\alpha$ and $B_{\overline{\alpha}}$. The path $B_\alpha$ is assumed to have length $n_\alpha m_\alpha \geq 2$, and similarly for $B_{\overline{\alpha}}$.

**Definition 1.1.** The Brauer graph algebra $\Lambda$ is the algebra $\Lambda = KQ/I$ where $I$ is given by $\alpha f(\alpha) = 0$ and $B_\alpha = B_{\overline{\alpha}}$, for all arrows $\alpha$.

## 2. The minimal bimodule resolution

We construct an explicit minimal projective bimodule resolution

$$\dots \to \mathbb{P}_n \xrightarrow{d_n} \mathbb{P}_{n-1} \to \dots \xrightarrow{d_1} \mathbb{P}_0 \to \Lambda \to 0$$

By [7], the multiplicity of the indecomposable projective $\Lambda(e_i \otimes e_j)\Lambda$ as a direct summand in the $n$-th term $\mathbb{P}_n$ is equal to the dimension of $\mathrm{Ext}^n_\Lambda(S_i, S_j)$ where $S_i$ is the simple module corresponding to vertex $i$.

With this information, we find a set of minimal generators for $\mathbb{P}_n$ as follows.

**Lemma 2.1.** $\mathbb{P}_n$ has a set of generators labelled by the paths along $f$-cycles of lengths $t$ where $0 < t \leq n$ and $n - t$ even, together with one generator for each $i \in Q_0$.

Fix an $f$-cycle $(\alpha_1, \dots, \alpha_r)$ of length $r$, where $\alpha_s : s \to s+1$. We label the generators of $\mathbb{P}_n$ as $[h^n_{\alpha_s,t}]$ where $t = n, n-2, \dots, 2$ or $1$, they generate direct summand isomorphic to $\Lambda e_s \otimes e_{s+t}\Lambda$. In addition when $n$ is even we have generators $[h^n_i]$ generating a direct summand isomorphic to $\Lambda e_i \otimes e_i \Lambda$ of $\mathbb{P}_n$, for each vertex $i$ of $Q$.

## 3. Homomorphisms

To study homomorphisms, we assume that $f$-cycles have no self-crossing, and moreover that the socle elements $B_\alpha$ of $\Lambda$ have lengths $> 2$, so that paths of length 2 other than $\alpha f(\alpha)$ do not occur in minimal relations. We do not expect that this is essential but the conditions avoid additional technicalities.

With these, we define *elementary homomorphisms* from $\mathbb{P}_n \to \Lambda$ which induce homomorphisms $\Omega^n_{\Lambda^e}(\Lambda) \to \Lambda$, which are supported either on the set of elements $\{[h^n_{\alpha_s,t}] \mid 1 \leq s \leq r\}$ for a fixed $f$-cycle (with $t, n$ fixed), or are supported on the set $\{[h^n_i] \mid i \in Q_0\}$. The maps $\mathbb{P}_n \to \Lambda$ must take generators of $\Omega_{\Lambda^e}(\Lambda)$ to zero, that is they must satisfy explitly given identities. The images belong to a subspace $C_{\alpha,t}$ which depends only on the residue $\bar{t}$ of $t$ modulo $r$. We fix a basis $\mathcal{B}_{\alpha,t}$ for $C_{\alpha,t} \cap J^2$ such that $\mathcal{B}_{\alpha,t} = \mathcal{B}_{\alpha,\bar{t}}$. The crucial result is

**Proposition 3.1.** Every homomorphism $\varphi : \Omega^n_{\Lambda^e}(\Lambda) \to \Lambda$ is a $K$-linear combination of elementary homomorphisms.

We describe the elementary maps, and in each case we specify ones which we call minimal.

(i) Maps with support $[h^n_i]$ for $i \in Q_0$, and image in $KQ_0$. For $n = 4$ this is the special map $\pi$, which is minimal. In general, if $n = 4k$, this map is $\pi^k$.

(ii) Maps with support $[h^n_i]$ for $i \in Q_0$ labelled as $\varphi^n_z$ where $z$ varies through a basis of $Z(\Lambda) \cap J$. The minimal such maps occur when $n = 4$.

For each fixed cycle of $f$ as above we have the following elementary maps.

(iii) Maps with image in $J^2$: For each even $t$ with $0 \leq t \leq n$, we have maps are $\varphi^n_{\alpha,t:\underline{u}}$ where $\underline{u}$ varies through the basis $\mathcal{B}_{\alpha,t}$. For the minimal maps, we take $t = 2, 4, 6, \dots,\ 2r$ if $r$ is odd, and $t = 2, 4, \dots,\ r$ when $r$ is even. In each case, the minimal $n$ is either $n = 2t$ if $t \equiv 2$, or $n = t$ otherwise.

(iv) Maps with image in $KQ_0$: Let $r_\alpha$ be the least common multiple of $r$ and 4. Then we have for any $t \equiv 0$ which is a multiple of $r$ the map $\widetilde{\varphi}^n_{\alpha,t}$ which takes $[h^n_{\alpha_s,t}]$ to $e_s$. This is defined for any $n \geq r_\alpha$ and $n$ divisible by 4. The minimal such map is when $t = n = r_\alpha$.

(v) Maps with image in $KQ_0$: Assume $r$ is even but not divisible by 4. For any $t \equiv 2$ such that $r$ divides $t$ we have $\hat{\varphi}^n_{\alpha,t}$ which takes $[h^n_{\alpha_s,t}]$ to $(-1)^s e_s$. The minimal such map is $t = r$ and $n = 2r$.

(vi) Maps with image in $KQ_1$: Suppose $r$ divides $t-1$ (hence $r$ is odd), and $t \equiv 0$ modulo 4. Then we have the map $\varphi^n_{\alpha,t;\alpha}$ which takes $[h^n_{\alpha_s,t}]$ to $\alpha_s$. Here $r$ is odd and $t$ is of the form $t = r \cdot k + 1$. The minimal $t$ is $t = r + 1$ if $r \equiv 3$ modulo 4 and $t = 3r + 1$ otherwise. In each case $n = t$.

## 4. Finite generation

Using the explicit formulae we construct chain maps lifting the element $\pi$ in degree 4, and the polynomial generators $\widetilde{\varphi}^{r_\alpha}_{\alpha,r_\alpha}$.

**Proposition 4.1.** All elementary maps which are not minimal, are products of minimal maps and the polynomial generators. We have product formulae, such as

$$\varphi^n_{\alpha,t;\underline{u}} \circ \pi = \varphi^{n+4}_{\alpha,t:\underline{u}} \text{ and } \varphi^m_{\alpha,t:\alpha} \circ \pi = \varphi^{n+4}_{\alpha,t;\alpha}$$

for $0 \leq t \leq n$ and $t$ even.

**Corollary 4.2.** (a) The elements $\pi$ together with $\widetilde{\varphi}^{r_\alpha}_{\alpha,r_\alpha}$ and (when it exists) $\hat{\varphi}^{2r}_{\alpha,r}$ generate a polynomial subalgebra of $HH^{4*}(\Lambda)$.
(b) These together with the images of the minimal elementary maps in the above list generate $HH^{4*}(\Lambda)$.

The proof that the Ext algebra is finitely generated over $HH^{4*}(\Lambda)$ is in progress. The ext algebra is finitely generated, see [1], [6] which might help.

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