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Statistical Physics and Random Surfaces

Organized by
Christophe Garban, Villeurbanne
Jason Miller, Cambridge UK
Scott Sheffield, Cambridge MA

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Abstract. This conference featured a diverse group of participants, from various career stages, to study problems in several hot topics that have grown increasingly prominent and interrelated in recent years. These included the following: (1) random surface models (Liouville quantum gravity, random planar maps, etc.) (2) random curve models (Schramm–Loewner evolution, conformal loop ensembles, etc.) (3) gauge theory models (various forms of lattice Yang–Mills, other approximations) (4) spin models and height functions (including delocalization problems) (5) dimer models (two and higher dimensions versions) (6) conformal field theory (Liouville theory, other theories related to SLE) The conference enabled participants to communicate about the most recent break-throughs and lay the groundwork for further collaborative progress.

Mathematics Subject Classification (2020): 13-XX, 60-XX.

Introduction by the Organizers

The conference was organized by Christophe Garban, Jason Miller and Scott Sheffield. We proposed to study a range of hot and increasingly related topics including the following: (1) random surface models (Liouville quantum gravity, random planar maps, etc.) (2) random curve models (Schramm-Loewner evolution, conformal loop ensembles, etc.) (3) gauge theory models (various forms of lattice Yang-Mills, other approximations) (4) spin models and height functions (including delocalization problems) (5) dimer models (two and higher dimensions versions) (6) conformal field theory (Liouville theory, other theories related to SLE).

The international researchers came from a wide range of career stages and geographical locations and the talks and discussions on these topics were extremely fruitful. The organizers and participants thank the Mathematisches Forschungsinstitut Oberwolfach for making the event possible, and for their extremely helpful assistance in all aspects of the organization and logistics. We include here the abstracts in alphabetical order.

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Workshop: Statistical Physics and Random Surfaces

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Abstracts

Strong Characterization for the Airy Line Ensemble

Amol Aggarwal

1. Line Ensembles

In this talk we study infinite collections of random curves that "look like" non-intersecting Brownian motions. Specifically, a *line ensemble* is an infinite family $\mathsf{x} = (\mathsf{x}_1, \mathsf{x}_2, \ldots)$ of random functions $\mathsf{x}_j : \mathbb{R} \to \mathbb{R}$, which are ordered so that $\mathsf{x}_1 > \mathsf{x}_2 > \cdots$. A line ensemble satisfies the *Brownian Gibbs property* if the following holds, for any positive integers $i < j$ and real numbers $a < b$. Conditional on $\big(\mathsf{x}_k(s)\big)$ for $(k, s) \notin [i, j] \times [a, b]$, the law of $\big(\mathsf{x}_k(s)\big)$ for $(k, s) \in [i, j] \times [a, b]$ is given by $j - i + 1$ Brownian motions on $[a, b]$, conditioned to satisfy the below.

(1) *Boundary data*: For each $k \in [i, j]$, the k-th Brownian motion starts at $\mathsf{x}_k(a)$ and ends at $\mathsf{x}_k(b)$.
(2) *Ordering*: For each $s \in [a, b]$, we have $\mathsf{x}_{i-1}(s) > \mathsf{x}_i(s) > \cdots > \mathsf{x}_{j+1}(s)$ (where $\mathsf{x}_0 = \infty$).

A *Brownian line ensemble* is one satisfying the Brownian Gibbs property; one may informally view it as an "infinite family of non-intersecting Brownian motions."

A basic example of a Brownian line ensemble is the *parabolic Airy line ensemble* $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \ldots)$. It was originally introduced in the context of polynuclear growth by Prähofer–Spohn [7] through its finite-dimensional distributions, which form a determinantal point process with the extended Airy kernel; it was later realized as an infinite family of random continuous curves by Corwin–Hammond [5]. It arises as the edge scaling limit of the *Brownian watermelon*, which is a family of N standard Brownian bridges $\boldsymbol{B} = (B_1, B_2, \ldots, B_N)$ on $[-N, N]$, starting and ending at $B_i(-N) = B_i(N) = 0$, conditioned to not intersect (that is, to satisfy $B_1 \geq B_2 \geq \cdots \geq B_N$). At the middle of their domain $[-N, N]$, the top curves in this family lie near $2^{1/2}N$. After scaling around this point, these curves exhibit fluctuations of order $N^{1/3}$ and nontrivial spatial correlations on scale $N^{2/3}$; these are sometimes called the KPZ scaling exponents. In particular, the curves $\mathsf{x}_j^N(t) = 2^{1/2}N^{1/3} \cdot \big(B_j(N^{2/3}t) - 2^{1/2}N\big)$, admit an $N \to \infty$ limit $\mathsf{x}^N = (\mathsf{x}_1^N, \mathsf{x}_2^N, \ldots, \mathsf{x}_N^N) \longrightarrow \boldsymbol{\mathcal{R}}$, which is the parabolic Airy line ensemble.

The Airy line ensemble is broadly believed, and in various cases proven, to be a universal scaling limit in the context of random surfaces and stochastic growth models in the Kardar–Parisi–Zhang (KPZ) universality class (see the survey [4] of Corwin for further information and references). Therefore, a natural question is if there is an axiomatic characterization of the Airy line ensemble that could be useful for proving convergence to it.

2. Characterization

The following result from [3] indicates that, if $\boldsymbol{\mathcal{L}}$ is a Brownian line ensemble whose top curve is within a multiplicative error of $1 + o(1)$ from a parabola, then $\boldsymbol{\mathcal{L}}$ is an Airy line ensemble, up to rescaling and a (possibly random) affine shift.

Theorem 1 ([3, Theorem 2.9])**.** *Fix $\sigma > 0$ and a Brownian line ensemble $\boldsymbol{\mathcal{L}} = (\mathcal{L}_1, \mathcal{L}_2, \ldots)$ such that, for any $\varepsilon, \delta > 0$, there is $\mathfrak{K} = \mathfrak{K}(\varepsilon, \delta) > 0$ with*

$$\sup_{t \in \mathbb{R}} \mathbb{P}\Big[\big| \mathcal{L}_1(t) + \sigma t^2 \big| > \varepsilon t^2 + \mathfrak{K} \Big] < \delta. \tag{1}$$

Then there exists a parabolic Airy line ensemble $\boldsymbol{\mathcal{R}}$ and an independent pair of random variables $(\mathfrak{l}, \mathfrak{c}) \in \mathbb{R}^2$ such that $\mathcal{L}(t) = \sigma \cdot \boldsymbol{\mathcal{R}}(t/2\sigma^2) + \mathfrak{l}t + \mathfrak{c}$.

Let us make several comments on this theorem. First, the assumption (1) of approximate parabolicity for the top curve $\mathcal{L}_1$ of $\boldsymbol{\mathcal{L}}$ cannot be entirely omitted; as indicated by work of Adler–Ferrari–van Moerbeke [1], there do exist different Brownian line ensembles whose top curves decay linearly. However, it is plausible that any Brownian line ensemble whose top curve decays at least parabolically must be the Airy line ensemble (up to a random affine shift and scaling). Second, a quick consequence of Theorem 1 (which was predicted by Sheffield and Okounkov in 2006) is the following. If $\boldsymbol{\mathcal{L}}$ is extremal (that is, it cannot be expressed as a nontrivial mixture of Brownian line ensembles) and $\boldsymbol{\mathcal{L}}(x) - x^2$ is translation-invariant, then $\boldsymbol{\mathcal{L}}$ is the parabolic Airy line ensemble up to scaling and a shift.

Third, Theorem 1 requires approximate parabolicity of $\boldsymbol{\mathcal{L}}$, as opposed to its full translation-invariance; following the terminology of Sheffield [8], it may therefore be called a *strong characterization* of a Gibbs measure. Fourth, this theorem only makes an assumption on the top curve $\mathcal{L}_1$ of $\boldsymbol{\mathcal{L}}$. Along a similar vein, work of Dimitrov–Matetski [6] showed that the full law of $\mathcal{L}_1$ (namely, all of its finite-dimensional marginals) determines $\boldsymbol{\mathcal{L}}$, implying that $\boldsymbol{\mathcal{L}}$ is a parabolic Airy line ensemble, if one could manage to show in advance that $\mathcal{L}_1$ were an Airy$_2$ process.

The third and fourth remarks above are particularly helpful in proving random convergence of interfaces to the Airy line ensemble. Indeed, in various situations one might have a family of non-intersecting random walks satisfying a Gibbs property (either arising as level lines of a random surface, or from a stochastic growth model) that are discrete (thus not translation-invariant), whose top curve is the only one that is *a priori* controlled. If one could show that this family is tight under the KPZ scaling, then these non-intersecting random walks would plausibly converge to non-intersecting Brownian bridges, so that any limit point of this family is a Brownian line ensemble. Then, Theorem 1 could apply to uniquely identify its scaling limit as the Airy line ensemble. See the works of Aggarwal–Huang [3, Section 25] and Aggarwal–Corwin–Hegde [2] for situations in which this perspective has been useful in identifying the KPZ scaling limit of various discrete stochastic growth models, such as random polymers, exclusion processes, and stochastic vertex models.

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Cutting Liouville quantum gravity by SLE with mismatched central charge

Morris Ang
(joint work with Ewain Gwynne)

Schramm–Loewner evolution (SLE_κ) for $\kappa > 0$ is a one-parameter family of random fractal curves in the plane. It describes or is conjectured to describe the scaling limits of various discrete random curves which arise in statistical mechanics.

Liouville quantum gravity (γ-LQG) for $\gamma \in (0, 2]$ is a one-parameter family of random fractal surfaces (2d Riemannian manifolds) which arise in string theory and conformal field theory, and as the scaling limits of random planar maps. LQG surfaces are too rough to be Riemannian manifolds in the literal sense. Instead, a γ-LQG surface can be understood as a random metric measure space with conformal structure.

The first relationship between SLE_κ and γ-LQG, called the quantum zipper, was established by Sheffield in [1]. Roughly speaking, this result and its many extensions (including the mating of trees theorem [2]) say the following. Suppose we have a certain SLE_κ-type curve and a certain γ-LQG surface, sampled independently from each other, and that their parameters are matched in the sense that their *central charges* sum to 26:

$$\mathbf{c}_{\mathrm{SLE}}(\kappa) + \mathbf{c}_L(\gamma) = 26, \tag{1}$$

$$\mathbf{c}_{\mathrm{SLE}}(\kappa) := 1 - 6\left(\frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}\right)^2 \quad \text{and} \quad \mathbf{c}_L(\gamma) := 1 + 6\left(\frac{2}{\gamma} + \frac{\gamma}{2}\right)^2.$$

Then the sub-LQG surfaces parametrized by the complementary connected components of the SLE_κ curve are conditionally independent given the LQG lengths of their boundaries, and their laws can be described explicitly. See Figure 1. This independence property is the continuum analog of certain Markovian properties for random planar maps decorated by statistical physics models.

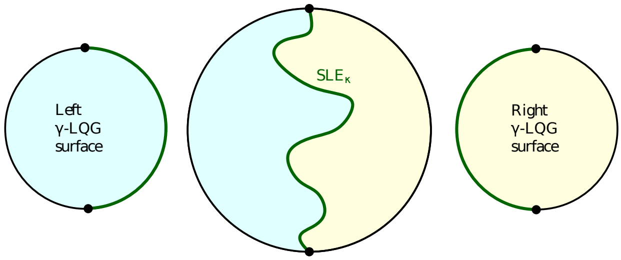

Figure 1. A Liouville quantum gravity surface parametrized by the disk together with an independent SLE_κ-type curve between two marked boundary points. Previous works (see in particular [1, 2]) have shown that if $\kappa = \gamma^2$ and the precise variant of SLE is chosen appropriately, then the sub-LQG surfaces parametrized by the regions to the left and right of the curve are conditionally independent given their LQG boundary lengths. We showed that if $\kappa \neq \gamma^2$, the sub-LQG surfaces parametrized by the left and right sides of the curve are conditionally independent given information about the values along the curve of certain auxiliary random fields, sampled independently from the SLE and the LQG.

Results of the above type have a huge number of applications, to such topics as SLE and LQG individually, conformal field theory (see, e.g., [3]), the geometry of random planar maps (see, e.g., [4]), random permutations (see, e.g., [5]), and the moduli of random surfaces (see, e.g., [6]). Particularly noteworthy consequences include the equivalence of $\gamma = \sqrt{8/3}$ LQG and the Brownian map [7] and convergences of conformally embedded random planar maps to LQG [8, 9].

In this talk, we presented the first relationships between SLE and LQG in the case when the parameters are *mismatched*, meaning that γ and κ are not related as in (1). Roughly speaking, we prove the following. Suppose we have an appropriate SLE_κ-type curve and a γ-LQG surface, sampled independently from each other as above, but whose parameters do not satisfy (1). Then the sub-LQG surfaces parametrized by the complementary connected components of the SLE_κ curve are not independent, but they are *conditionally* independent if we condition on certain extra information along the SLE_κ curve. The necessary extra information is described by one or more random generalized functions, sampled independently from the SLE and the LQG, with the property that the sum of $\mathbf{c}_{\mathrm{SLE}}$, $\mathbf{c}_L$, and the central charges associated with the extra generalized functions is equal to 26. These extra random generalized functions are described in terms of the theory of *imaginary geometry* [10]. Conditional independence statements of the above type were conjectured by Sheffield in private communication.

We also discussed similar conditional independence statements when the SLE curve is replaced by other interesting random sets, such as a conformal loop ensemble gasket, a Brownian motion path, or an LQG metric ball.

Analogously to the matched case, our results are continuum analogs of certain Markovian properties for random planar maps decorated by *multiple* statistical physics models. Roughly speaking, these properties say the following. Suppose we have a random planar map decorated by a two or more statistical physics models (e.g., a uniform spanning tree and two discrete Gaussian free fields) and we construct an interface from one of the models (e.g., a branch of the spanning tree). Then the planar maps in the complementary connected components of the interface are conditionally independent given the information about other statistical mechanics models along the interface (in our example, this corresponds to the restrictions of the discrete Gaussian free fields to the spanning tree branch). Similar Markovian properties also hold for related objects, such as uniform meanders.

We concluded the talk by discussing possible future directions related to our mismatched SLE-LQG theory. The first direction is to establish scaling limit results where the limiting object is expected to be LQG with more than one independent SLE or imaginary geometry field. In particular, the scaling limit of the *uniform meander* is expected to be $\gamma = \sqrt{\frac{1}{3}(17 - \sqrt{145})}$-LQG decorated by a pair of SLE_8 loops, which are all mutually independent [11, 12]. The second direction is the rigorous construction of a *Markovian string theory* in $d \in \{1, \dots, 25\}$ dimensional space corresponding to bosonic string theory, via a Liouville quantum gravity surface together with d independent Gaussian free fields. This is a problem posed by Sheffield, see [13] for more details.

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Near-critical dimers and massive SLE.

Nathanael Berestycki
(joint work with Levi Haunschmid-Sibitz)

We consider near-critical perturbations of the planar dimer model with Temperleyan boundary conditions [2]. On the square lattice, these models are obtained by changing the edge weights from 1 (corresponding to the critical case) to $1 + c_i\delta$, for some choice of constants $(c_i)_{1\le i\le 4}$, in a biperiodic manner, and where $\delta > 0$ is the mesh size. An analogous construction is also possible on the hexagonal lattice.

We analyse this model by making use of Temperley's bijection (see Figure 1). This leads us to the study of a pair of dual spanning trees; the primal tree with wired boundary conditions can be sampled from Wilson's algorithm, by erasing the loops of a random walk on a lattice with weights of the above form. In particular, the random walk does not scale to Brownian motion, but to Brownian motion with drift α determined by the constants $(c_i)_{1\le i\le 4}$. What can be said about the associated loop-erased random walk? We introduce and prove an exact discrete Girsanov identity on the hexagonal lattice (and an approximate one on the square lattice) which shows that, conditionally on the endpoint, the loop-erased random walk with drift coincides *exactly* with the loop-erasure of a massive (but isotropic) random walk with constant probability $\|\alpha\|^2\delta^2$ to be killed at each step, but conditioned to survive until leaving the domain. Making use of results of Chelkak and Wan [6] this implies that the scaling limit of LERW with drift is Makarov and Smirnov's [8] massive SLE_2.

This implies that the Temperleyan tree associated to the near-critical dimer model has a scaling limit in the Schramm topology. In turn, using the technology based on "imaginary geometry" developed by the author with Laslier and Ray [3, 4, 5] this implies that the height function itself converges. The law of the limit field depends only on the drift vector α and so would be the same on the square and hexagonal lattices.

By allowing the drift vector $\alpha = \alpha(x)$ to depend smoothly on the point $x \in D$, we obtain a larger and more interesting class of near-critical perturbations of the dimer model. Suppose $\alpha = \nabla\phi$ derives from a potential, and suppose that

$$\Delta\phi + \frac{1}{2}|\nabla\phi|^2 \ge 0. \tag{1}$$

Then the random walk converges to the solution of the Langevin stochastic differential equation

$$dX_t = dB_t + \nabla\phi(X_t)dt.$$

Under (1) we are able to prove that the loop-erasure of this random walk has a scaling limit, and thus so does the associated height function (still using results of

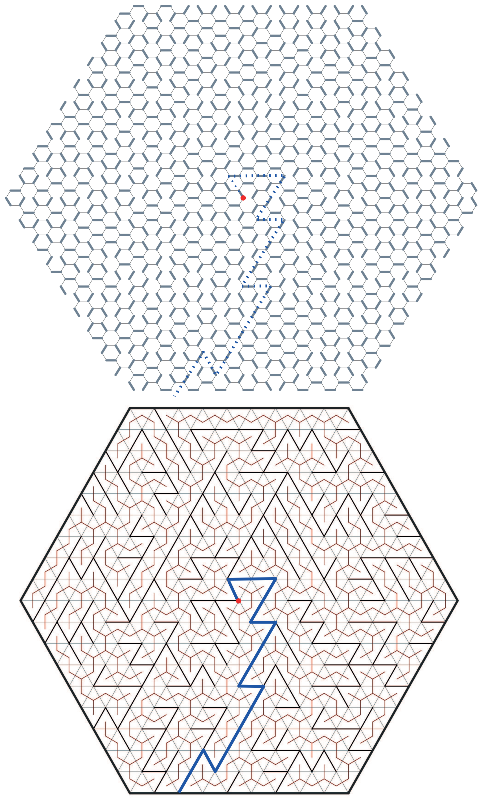

FIGURE 1. Temperley's bijection transforms a dimer configuration in a Temperleyan domain (left) into a pair of self-dual trees (right). The path in the primal tree from a designated vertex (the center of the hexagon) is highlighted in both pictures. In the scaling limit this path converges to a massive SLE_2.

[3, 4, 5]). This obeys a certain conformal covariance rule (note that the assumption (1) is in fact conformally invariant).

We obtain strong evidence supporting the following novel conjecture. Fix a smooth drift vector field α satisfying (1). Let $h^{\alpha;D}$ denote the above scaling limit of the dimer height function. Then $h^{\alpha;D}$ has a law which coincides with the following generalised Sine-Gordon field at the free fermion point, whose law $\mathbb{P}^{\mathrm{SG}}$ is informally given by the following description:

$$\mathbb{P}^{\mathrm{SG}}(dh) = \frac{1}{Z} \exp\left(\int_D \langle e^{ih(x)/\chi}; \alpha(x) \rangle dx \right) \mathbb{P}^{\mathrm{GFF}}(dh),$$

where $\chi = 1/\sqrt{2}$ is the imaginary geometry constant associated with the parameter $\kappa = 2$ of SLE, and $\mathbb{P}^{\mathrm{GFF}}(dh)$ is the law of a Gaussian free field with winding boundary conditions. (This is informal only as the integral appearing in the above

Radon–Nikodym derivative does not literally exist, see [7]. A rigorous construction in the full plane and for constant drift vector is provided in the recent work of Bauerschmidt and Webb; see in particular [9] for work in this direction in this particular case). In a sense, this conjecture says that massive SLE is a flow line of the Sine-Gordon field.

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Random triangulations in high genus

Thomas Budzinski
(joint work with Guillaume Chapuy, Baptiste Louf)

1. Random triangulations

The goal of the talk was to review several aspects of uniform random triangulations whose genus goes to infinity at the same time as their sizes. For $g \geq 0$ and $n \geq 2g-1$, let $T_{n,g}$ be a random triangulation picked uniformly at random among all triangulations with $2n$ triangles and genus g. The planar case $g = 0$ has been the object of lots of study in the last 20 years. In particular, graph distances in $T_{n,0}$ are typically of order $n^{1/4}$ [4], and a pioneer result of the theory was the proof of local convergence of $T_{n,0}$ to the UIPT [1].

2. Local limits in high genus

The Euler formula shows that the average vertex degree in $T_{n,g}$ is

$$\frac{6n}{n+2-2g} \approx \frac{6}{1-2g/n}.$$

In particular, the average degree in the UIPT is equal to 6, so it may be seen as a uniform randomized version of the 6-regular triangular lattice. On the other hand, if $\frac{g}{n}$ converges to $\theta > 0$, the average degree becomes larger than 6 like in the 7-regular hyperbolic triangulation, so it seems natural to observe random hyperbolic triangulations in the limit. Indeed, we proved [2] that if $\frac{g_n}{n} \to \theta \in \left[0, \frac{1}{2}\right)$, then T_{n,g_n} converges locally to a random infinite triangulation of the plane $\mathbb{T}_\theta$ whose distribution only depends on θ. For $\theta = 0$ we recover the UIPT. For $\theta > 0$, we obtained a one-parameter family of models that were introduced in [5] and shown to exhibit hyperbolic behaviour (exponential volume growth, positive speed of the random walk...).

A nice byproduct of this local convergence result is the asymptotic enumeration of triangulation of genus g and size n in any regime, up to a multiplicative error $e^{o(n)}$. This contrasts sharply with the planar case, where enumerative estimates are the main tool used to prove local convergence results.

3. Global properties in high genus

Given the hyperbolic nature of the local limit, it is natural to expect that in the high genus regime, the global properties of $T_{n,g}$ share a lot of similarities with those of random graph models such as uniform 3-regular graphs or the giant component of supercritical Erdös–Renyi random graphs. The end of the talk was devoted to some results inthis direction obtained in [3]. In the regime $\frac{g}{n} \to \theta > 0$, we showed that graph distances are typically logarithmic, that almost all pairs of points in $T_{n,g}$ have almost the same distance, and that $T_{n,g}$ satisfy a strong isoperimetric inequality: it has a positive Cheeger constant up to small defects whose size is $O(\log n)$. This isoperimetric inequality is the main tool to show the other two properties, and its proof relies heavily on the partial asymptotic results obtained in [2].

We concluded with a few open questions on the optimal constants. In particular, we conjecture that the distance between two typical vertices of $T_{n,g}$ is close to $D_\theta \log n$, whereas the diameter (i.e. the distance between the two furthest points) is close to $D'_\theta \log n$ with $D'_\theta = 3D_\theta$.

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Random surfaces and lattice Yang–Mills

Sky Cao
(joint work with Minjae Park, Scott Sheffield)

Euclidean Yang–Mills theory is given by the formal probability measure

$$Z^{-1}\exp(-S_{\mathrm{YM}}(A))dA,$$

where A ranges over a space of Lie algebra-valued 1-forms and $S_{\mathrm{YM}}(A)$ is the Yang–Mills action of A. As a first step towards rigorously defining this probability measure, we can replace continuous space $\mathbb{R}^n$ by a lattice $\mathbb{Z}^n$. Further replacing the infinite lattice $\mathbb{Z}^n$ by a finite subset $\Lambda \subseteq \mathbb{Z}^n$, we eventually obtain a well-defined probability measure $\mu_{\Lambda,\beta}$ on the space $E(\Lambda)^G$, where $\beta \geq 0$, G is a compact Lie group, and $E(\Lambda)$ is the set of edges of Λ. Then density of $\mu_{\Lambda,\beta}$ with respect to the product Haar measure $\prod_{e\in E(\Lambda)} dU_e$ on $E(\Lambda)^G$ is given by

$$d\mu_{\Lambda,\beta}(U) = Z_{\Lambda,\beta}^{-1}\prod_{p\in\mathcal{P}} e^{\beta\mathrm{Tr}(U_p)}\prod_{e\in E(\Lambda)} dU_e.$$

Here, $\mathcal{P}$ is the set of oriented plaquettes of Λ. Given a plaquette p with oriented boundary edges e_1, e_2, e_3, e_4, the plaquette variable $U_p := U_{e_1}U_{e_2}U_{e_3}U_{e_4}$. This model is known as lattice gauge theory, which was first introduced by Wilson [1].

The basic observables of lattice gauge theory are the Wilson loop observables. Given a loop $\gamma = e_1\cdots e_n$ in Λ, we define for $U \in E(\Lambda)^G$

$$U_\gamma := U_{e_1}\cdots U_{e_n}, \quad W_\gamma(U) := \mathrm{Tr}(U_\gamma).$$

In my talk, I discussed a recent result from [2], which gives a representation of $\mathbb{E}W_\gamma(U)$ in terms of a sum over embedded surfaces. We next discuss how these surfaces are constructed, and then give the main theorem.

Let γ be a loop. We will construct surfaces which can be interpreted as having boundary γ. First, fix a function $K : \mathcal{P} \to \mathbb{N}$, which should be interpreted as a plaquette count. I.e., K specifies how many copies of each plaquette I have to work with to build my surface. Eventually, we will sum over all K in the surface sum. To form the surfaces, we will never glue two plaquettes together, nor will we ever glue a plaquette to an edge of the loop γ. Instead, we introduce additional "abstract" faces, and we always glue these abstract faces to the existing plaquettes or loop edges. A surface formed in this way can be formally described by a pair $(\mathcal{M}, \phi)$, where $\mathcal{M}$ is a map and $\phi\colon \mathcal{M} \to \Lambda$ is a graph homomorphism such that the following hold.

(1) The dual graph of the map $\mathcal{M}$ is bipartite. This condition comes from the fact that plaquettes are never glued to each other. The faces in one partite class are called "edge-faces", and the faces in the other partite class are called "plaquette-faces".
(2) ϕ maps each plaquette face to a plaquette of Λ.
(3) ϕ maps each edge-face onto a single edge in Λ.

We refer to the pair $(\mathcal{M},\phi)$ as an edge-plaquette embedding. We say that $(\mathcal{M},\phi)$ has boundary γ if $\mathcal{M}$ is a map with boundary, which is mapped under ϕ to γ. Given a plaquette count $K\colon \mathcal{P}\to\mathbb{N}$, let $\operatorname{epe}(K,\gamma)$ be the set of all edge-plaquette embeddings such that the pre-image under ϕ of each plaquette $p\in\mathcal{P}$ has size $K(p)$.

Next, we describe the weights associated to a given edge-plaquette embedding. For $N,k\geq 1$, the Weingarten function Wg_N is a function on the symmetric S_k given by

$$\mathrm{Wg}_N(\sigma) := \frac{1}{k!}\sum_{\lambda\vdash k}\chi_\lambda(\mathrm{id})\chi_\lambda(\sigma)\prod_{(i,j)\in\lambda}\frac{1}{N+j-i}.$$

Here, the sum is over Young diagrams with k boxes, χ_λ is the character of S_k corresponding to λ, and $\mathrm{id}\in\mathrm{S}_k$ is the identity permutation. Since Wg_N is a linear combination of class functions, it itself is a class function. Thus, Wg_N is a function of the conjugacy class of σ, which is encoded by its cycle structure, which itself is a partition of $[k]$. In this way, Wg_N can be viewed as a function on partitions of size k.

Now, fix an edge-plaquette embedding $(\mathcal{M},\phi)$. For a lattice edge $e\in E(\Lambda)$, we can consider the set of edge-faces which are mapped under ϕ to e. The degrees of these edge faces form a partition $\mu_e(\mathcal{M},\phi)$ of some number $k_e(\mathcal{M},\phi)$ into $\ell_e(\mathcal{M},\phi)$ parts. Define

$$\mathcal{W}_e(\mathcal{M},\phi) := N^{2k_e(\mathcal{M},\phi)-\ell_e(\mathcal{M},\phi)}\mathrm{Wg}_N(\mu_e(\mathcal{M},\phi)).$$

We may finally state the main result.

1. Main theorem

For any loop γ, we have that

$$Z_{\Lambda,\beta}\mathbb{E}W_\gamma(U) = \sum_{K\colon \mathcal{P}\to\mathbb{N}}\;\sum_{(\mathcal{M},\phi)\in\operatorname{epe}(K,\gamma)}\beta^{\operatorname{area}(\mathcal{M},\phi)}N^{\chi(\mathcal{M})-1}\prod_{e\in E(\Lambda)}\mathcal{W}_e(\mathcal{M},\phi).$$

Here, $\operatorname{area}(\mathcal{M},\phi)$ is the area of $(\mathcal{M},\phi)$, which is simply $\sum_{p\in\mathcal{P}}K(p)$ if $(\mathcal{M},\phi)\in\operatorname{epe}(K,\gamma)$, and $\chi(\mathcal{M})$ is the Euler characteristic of $\mathcal{M}$. This is the desired surface sum representation of Wilson loop expectations. We remark that a more general version of this result holds for products of multiple loops.

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S-embeddings and planar Ising model

Dmitry Chelkak

Critical and near-critical Ising model on regular and isoradial grids. Motivated by the seminal work of Smirnov from mid-2000s, plenty of convergence results for the planar Ising model have been obtained during the last two decades. E.g., at the critical temperature T_{crit} on $\mathbb{Z}^2$, this includes convergence of interfaces to SLEs and CLEs (with $\kappa = 3$ for spin configurations and with $\kappa = 16/3$ for the Fortuin–Kasteleyn representation) as well as convergence of correlations of local fields (such as energy densities, spins, etc) towards CFT predictions. More recently, latter results were also generalized to the near-critical regime $T = T_{\mathrm{crit}} + m\delta$, where δ stands for the mesh size of the grid and $m \in \mathbb{R}$ is a fixed parameter called a *mass*. (It is worth noting that we still lack convergence results for interfaces if $m \neq 0$.) The name 'mass' is justified by the fact that (subsequential) scaling limits f of fermionic observables appearing in the model satisfy the massive holomorphicity equation $\overline{\partial} f = \frac{1}{2} m \overline{f}$; in particular, they are holomorphic if $m = 0$. Careful analysis of such observables is the cornerstone for all convergence results mentioned above; e.g., see [1] and references therein. Moreover, virtually all these results are proved to be *universal* for the class of Baxter's Z-invariant weights on isoradial graphs. However, further generalizations in this direction – e.g., even for general doubly periodic critical weights on $\mathbb{Z}^2$ – were not available until recently. The main goal of this talk is to discuss a tool – special drawings or embeddings of weighted planar graphs in the complex plane – that paves the way to such a progress.

Construction of s-embeddings. A relevant construction – *s-embeddings* of planar graphs carrying the Ising model – has been first suggested in [1, Section 6] and later developed in [2]. It plays the same role for the Ising model as Tutte's harmonic embeddings play for random walks on planar graphs. These two classes of embeddings are *not* immediately related to each other; however, both are particular cases of a more general construction, so-called *t-embeddings;* see [3, 4].

To construct an s-embedding of a given weighted planar graph $(G^\bullet, x)$, one starts with a complex-valued (equivalently, a pair of real-valued) solution $\mathcal{X}$ of the so-called propagation equation (see [2] for the rigorous definition)

$$\mathcal{X}(c_{00}) = \pm \mathcal{X}(c_{01}) \cdot \cos\theta_e \pm \mathcal{X}(c_{10}) \cdot \sin\theta_e \,. \tag{1}$$

Above, $c_{pq} = (v_p^\bullet v_q^\circ)$, $p = 0, 1$, $q = 0, 1$, are four *corners* of the graph $G^\bullet$ incident to its edge $e = (v_0^\bullet v_1^\bullet) = (v_0^\circ v_1^\circ)^*$, the $\pm$ signs come from a Kasteleyn orientation in the corresponding dimer model, and $\theta_e := 2 \arctan x_e$ is the parametrization of the Ising interaction constant assigned to e. Given (1), one defines

$$\mathcal{S}_{\mathcal{X}}(v_p^\bullet) - \mathcal{S}_{\mathcal{X}}(v_q^\circ) := (\mathcal{X}(c_{pq}))^2, \qquad \mathcal{Q}_{\mathcal{X}}(v_p^\bullet) - \mathcal{Q}_{\mathcal{X}}(v_q^\circ) := |\mathcal{X}(c_{pq})|^2 \tag{2}$$

and views $(\mathcal{S}_{\mathcal{X}}, \mathcal{Q}_{\mathcal{X}})$ as a discrete surface in the Minkowski space $\mathbb{R}^{2,1}$. The consistency of four increments along the sides of $(v_0^\bullet v_0^\circ v_1^\bullet v_1^\circ)$ follows from (1). In particular, four points $\mathcal{S}_{\mathcal{X}}(v_0^\bullet), \mathcal{S}_{\mathcal{X}}(v_0^\circ), \mathcal{S}_{\mathcal{X}}(v_1^\bullet), \mathcal{S}_{\mathcal{X}}(v_1^\circ)$ are vertices of a tangential quad; if there are no overlaps, then $\mathcal{S}_{\mathcal{X}}$ is a tiling formed by such quads. Real-linear transforms $\mathcal{X} \mapsto \alpha\mathcal{X} + \beta\overline{\mathcal{X}}$, $|\alpha|^2 - |\beta|^2 = 1$, correspond to isometries of $\mathbb{R}^{2,1}$.

Combinatorially, there are at least two natural setups to consider:
(i) one starts with an *infinite* graph $(G^\bullet, x)$ and aims to obtain a tiling of $\mathbb{C}$;
(ii) one starts with a *finite* planar graph, does not require (1) along its boundary, and aims to obtain a tiling of a finite polygon in $\mathbb{C}$.

A particular example of the 'disc' setup (ii) is a quadrangulation $G^\bullet \cup G^\circ$ of a *sphere* with a distinguished quad (equivalently, a *root* edge of $G^\bullet$), considered as the outer face. In this situation, there is essentially a unique (up to real-linear transforms) choice of $\mathcal{X}$ and it is known that $\mathcal{S}_\mathcal{X}$ does not have overlaps; in other words, in this case the discrete surface $(\mathcal{S}_\mathcal{X}, \mathcal{Q}_\mathcal{X})$ in $\mathbb{R}^{2,1}$ is 'canonical'. The simplest example of the 'plane' setup (i) is provided by doubly periodic graphs $G^\bullet$ equipped with *critical* Ising weights. Again, in this case there exists a unique (up to real-linear transforms) doubly periodic solution $\mathcal{X}$ of the equation (1), which gives rise to a 'canonical' s-embedding $\mathcal{S}_\mathcal{X}$ of such a model into the complex plane.

Scaling limits of fermionic observables. Given a sequence of s-embeddings $\mathcal{S}^\delta$ with 'mesh sizes' $\delta \to 0$ one can look for a differential equation – a generalization of the (massive) holomorphicity condition – that is satisfied by the limits of fermionic observables coming from the corresponding Ising models. The proper language to write such an equation comes from the discrete surfaces $(\mathcal{S}^\delta, \mathcal{Q}^\delta)$: if they approximate, as $\delta \to 0$, a space-like surface $\Sigma = \{(z, \theta(z))\} \subset \mathbb{C}\times\mathbb{R}$ and f is a (subsequential) limit of Ising fermionic observables on $\mathcal{S}^\delta$ as $\delta \to 0$, then $fdz + \overline{f}d\theta$ is a closed differential form. Moreover, if the limiting surface Σ is smooth and ζ is its *conformal parametrization,* then the function $\phi := f \cdot z_\zeta^{1/2} + \overline{f} \cdot \overline{z}_\zeta^{1/2}$ satisfies the massive holomorphicity equation $\phi_{\overline{\zeta}} = \frac{1}{2}m\overline{\phi}$, where $m = z_{\zeta\overline{\zeta}}/(z_\zeta z_{\overline{\zeta}})^{1/2}$ is the *mean curvature* of Σ multiplied by its metric element $|z_\zeta| - |z_{\overline{\zeta}}|$ at the point $(z(\zeta), \theta(\zeta))$.

Similarly to the progress obtained for the critical and near-critical Ising models on regular grids, one can now hope to prove convergence results for energy densities, spins, etc in the general s-embeddings setup. This is a work in progress with some results in this direction already obtained by Rémy Mahfouf and S. C. Park.

Convergence to SLE in the doubly periodic setup We conclude this talk by an informal version of the universality result in the doubly periodic setup (see [2]):

Theorem. *Let $(G^\bullet, x)$ be a doubly periodic graph equipped with critical Ising weigths x. Consider the corresponding 'canonical' doubly periodic s-embedding $\mathcal{S}$ with asymptotically horizontal profile of the third coordinate $\mathcal{Q}$. Then the FK-Ising interfaces on the scaled graphs $\delta\mathcal{S}$ converge, as $\delta \to 0$, to SLE(16/3) curves.*

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(Near-)critical percolation with long-range correlations on transient graphs

Alexander Drewitz
(joint work with A. Prévost, P.-F. Rodriguez)

Percolation models have been playing a fundamental role in statistical physics for several decades by now. They had initially been investigated in the gelation of polymers during the 1940s by chemistry Nobel laureate Flory and Stockmayer. From a mathematical point of view, the birth of percolation theory was the introduction of Bernoulli percolation by Broadbent and Hammersley in 1957 [BH57], motivated by research on gas masks for coal miners. One of the key features of this model is the inherent stochastic independence which simplifies its investigation, and which has led to deep mathematical results. On the one hand, this independence greatly simplifies the mathematical computations and as a consequence, the results obtained are impressively profound. On the other hand, this independence also poses a restriction prohibiting the investigation of more realistic models. Thus, one is naturally led to consider percolation models with correlations. While in the case of finite range or fast decaying correlations similar phenomena as in the case of Bernoulli percolation are to be observed, the situation changes drastically when considering models with stronger correlations, so-called *long-range correlations*. Models with long-range dependence exhibit interesting properties which sometimes are in stark contrast to what is observed in Bernoulli percolation. The lack of independence entails further obstacles such as the absence of the finite energy property. Even more dramatically, the BK inequality fails for these models. As a result, many of the techniques which were most essential in the investigation of Bernoulli percolation break down for percolation problems with long-range correlations.

It should be mentioned here that not only are such models oftentimes more realistic but they also lead to beautiful mathematics as well as interesting physics interpretations. Moreover, they sometimes exhibit certain integrability properties, which, somewhat surprisingly, makes them easier to study for some problems than their independent counterpart, and leads to deep results which are unknown for Bernoulli percolation.

As we will elaborate, even though from a probabilistic perspective the strength of correlations seems to make matters a priori only harder, they can also provide certain integrability properties which facilitate their rigorous mathematical study. This opens the door to the study of critical phenomena, notably in non-planar setups, and gives access to questions which so far have remained largely elusive.

Arguably one of the most important stochastic processes giving rise to percolation models with long-range correlations is the *Gaussian Free Field* (GFF). The GFF, which also goes by the name massless harmonic crystal, has been a fundamental model in statistical mechanics for over half a century, ever since the early days of constructive field theory, for which it serves as a fundamental building block. More recently, its intriguing geometric features have begun to be studied rigorously. One way to think of the GFF is as a generalization of a random walk with Gaussian increments to a process with d-dimensional time. For a long time already, the GFF has had many important applications to other branches of mathematics, with links in two dimensions to objects such as the Schramm-Loewner evolution [SS09], or to cover times [DLP12] and to theoretical physics (cf. [Car90, FFS92]). Literature is abundant, however, and we content ourselves with referring to [She07] and the references therein for further details.

The cable system. It has turned out that when endowing the discrete graph G with a certain continuum structure, the investigation of percolation problems for the GFF becomes in a loose sense more integrable. This continuous structure is the so-called *cable system*, also sometimes referred to as *metric graph*, and denoted by $\widetilde{G}$. Heuristically, $\widetilde{G}$ is obtained from G by adding line segments between neighboring vertices of G so that one obtains a metric graph which is a continuum object. While such a construction goes back to at least [Var85], it has been re-invigorated in this setting by [Lup16].

The reasons for this model being particularly amenable for a detailed investigation of its percolation are multiple, including among others its Gaussian and continuous character as well as the understanding of the law of the capacity of its level sets and its amenability to advanced isomorphism theorems connecting it to the model of random interlacements (see [DPR22] for the latter two items). The model of Random Interlacements (RI) has been introduced in 2007 by Sznitman, see the article [Szn10]. It has been motivated by investigations in mathematics and theoretical physics on the disconnection [BS08] and covering [BH91] of tori and boxes by simple random walk trajectories. In addition, RI serves as a mathematical model for corrosion, and it has found its way into the theoretical physics community also, see [SHS+16, GHS17]. In the context of level set percolation for the GFF it turns out particularly useful as it is supercritical in its entire range of parameters, thereby providing suitable connectivity properties for certain level sets of the GFF also by means of the isomorphism theorems.

We will report on the progress in understanding of (near-)critical percolation for the level sets of the GFF on the cable system for rather general transient graphs, in low dimensions. In particular, we provide explicit values for various critical exponents for rather general underlying graphs. Such detailed understanding has been restricted to either the case of Bernoulli percolation on very specific two-dimensional lattices (where techniques such as conformal invariance wield their full power) or else the mean-field regime. Surprisingly, the strong correlations of the GFF seem to be an advantage in the investigation of its nearly critical percolative properties.

What is more, these findings exhibit a certain universality of the critical exponents which we have been able to determine. More precisely, it turned out that – as conjectured in theoretical physics [Wei84] – they do not depend on the local structure of the underlying graph, but only on its large scale properties, in particular its volume growth and the Green function decay; these, however, are well-understood for a wide range of graphs of interest.

This talk is based on joint works with A. Prévost (U Cambridge) and P.-F. Rodriguez (Imperial College).

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Conformal removability of Schramm-Loewner Evolution

Konstantinos Kavvadias

(joint work with Jason Miller, Lukas Schoug)

The Schramm-Loewner evolution (SLE_κ) is a one parameter family ($\kappa > 0$) of curves which connect two boundary points of a simply connected domain. It was introduced by Schramm in [9] as a candidate to describe the scaling limit of the interfaces which arise in discrete models from statistical mechanics on planar lattices at criticality, such as loop-erased random walk and the percolation model. The value of $\kappa > 0$ determines the roughness of an SLE_κ curve. In particular, the

SLE_κ curves are simple if $\kappa \leq 4$, self-intersecting but not space-filling if $\kappa \in (4,8)$, and space-filling if $\kappa \geq 8$ [8]. The focus of this talk is on the critical value $\kappa = 4$, which corresponds to a curve which is simple but just barely so.

In recent years, there has been a substantial amount of work centered around the relationship between SLE_κ and Liouville quantum gravity (LQG) surfaces. Roughly speaking, LQG is the canonical model of a random two-dimensional Riemannian manifold and can formally be described by its metric tensor

$$e^{\gamma h(z)}(dx^2 + dy^2) \tag{1}$$

where h is an instance of (some form of) the Gaussian free field (GFF) on a planar domain D and $\gamma \in (0,2]$ is a parameter. Since the GFF is a distribution in the sense of Schwartz, it is non-trivial to make sense of (1) as the exponential is a non-linear operation and this has been the focus of a tremendous amount of mathematical work in the last two decades.

The manner in which SLE_κ arises in the context of LQG is that it describes the interfaces which are formed when one takes two independently sampled LQG surfaces and glues them together along their boundaries. The way that the gluing is constructed is using conformal welding, which we recall is defined as follows. Suppose that $\mathbf{D}_1$, $\mathbf{D}_2$ are two copies of the unit disk and $\phi\colon \partial\mathbf{D}_1 \to \partial\mathbf{D}_2$ is a homeomorphism. We say that a simple loop η on the two-dimensional sphere $\mathbf{S}^2$ together with conformal maps ψ_i for $i = 1,2$ from $\mathbf{D}_i$ to the left and right sides of $\mathbf{S}^2 \setminus \eta$ is a conformal welding with welding homeomorphism ϕ if $\phi = \psi_2^{-1} \circ \psi_1|_{\partial\mathbf{D}_1}$. For a given homeomorphism ϕ, it is not obvious if such a conformal welding exists but Sheffield [10] showed that for each $\gamma \in (0,2)$ it does exist if one takes the welding homeomorphism to be the one which associates points along the boundaries of two independent γ-LQG surfaces according to boundary length using the boundary measure from (1). In this case, the welding interface is an SLE_κ with $\kappa = \gamma^2 \in (0,4)$. This was extended to the case $\kappa = 4$ and $\gamma = 2$ in [3] and we also remark that a number of welding-type results which include the case $\kappa > 4$ were established in [1].

For a homeomorphism ϕ as above for which there exists a conformal welding, it is also a non-trivial question to determine whether the conformal welding is unique. Recall that a set $X \subseteq \mathbb{C}$ is said to be *conformally removable* if every homeomorphism $\varphi\colon \mathbb{C} \to \mathbb{C}$ which is conformal on $\mathbb{C} \setminus X$ is conformal on $\mathbb{C}$. It is not difficult to see that the welding interface being conformally removable implies that the conformal welding interface is unique up to Mobius transformations.

In order to prove that a set $X \subseteq \mathbb{C}$ which is equal to the boundary of a simply connected domain D (e.g., a simple curve) is conformally removable, one often makes use of a condition due to Jones and Smirnov [4]. We will not describe the Jones-Smirnov condition here, but remark that one often checks that it holds by using one of the sufficient conditions established in [4]. For example, it is shown in [4] that if the Riemann map $\mathbf{D} \to D$ is Hölder continuous up to $\partial\mathbf{D}$ (so that D is a so-called Hölder domain) then $X = \partial D$ is conformally removable; in fact it is shown in [4] that a much weaker modulus of continuity suffices (see also [6]). Rohde and Schramm proved that the complementary components of an SLE_κ curve for $\kappa \neq 4$ are Hölder domains hence conformally removable. The optimal Hölder exponent was later computed in [2] in which it was also shown that SLE_4 does not form the boundary of a Hölder domain. The precise modulus of continuity of the uniformizing map of the complementary component of an SLE_4 was determined in [5] and is given by $(\log \delta^{-1})^{-1/3+o(1)}$ as $\delta \to 0$. The reason for the difference in behavior for $\kappa \neq 4$ is that SLE_4 curves are barely non-self-intersecting and contain tight bottlenecks. Another way of formulating this is that if η is an SLE_κ for $\kappa \in (0,4)$ and $z \in \eta$ then the harmonic measure of $B(z,\epsilon)$ in the complement of η decays as fast as a power of ϵ as $\epsilon \to 0$ [2] but for $\kappa = 4$ can decay as fast as $\exp(-\epsilon^{-3+o(1)})$ as $\epsilon \to 0$ [5]. It was further shown in [5] that the Jones-Smirnov condition itself does not hold for SLE_4.

The main result of this paper is the conformal removability of SLE_4, which implies the uniqueness of the welding problem for critical ($\gamma = 2$) LQG. (We remark that a weaker version of the uniqueness of the welding problem for critical LQG was proved in [7].)

Theorem 1. *Suppose that η is an SLE_4 in $\mathbf{H}$ from 0 to ∞. Suppose that $f\colon \mathbf{H} \to \mathbf{H}$ is a homeomorphism which is conformal on $\mathbf{H} \setminus \eta$. Then f is a.s. conformal on η. In particular, the range of η is a.s. conformally removable.*

We note that the first assertion of Theorem 1 implies that the range of η is a.s. conformally removable because if $f\colon \mathbb{C} \to \mathbb{C}$ is a homeomorphism which is conformal on $\mathbb{C} \setminus \eta$ then by the Riemann mapping theorem we can post-compose its restriction to $\mathbf{H}$ with a conformal map so that we obtain a homeomorphism $\mathbf{H} \to \mathbf{H}$ which is conformal on $\mathbf{H} \setminus \eta$. We also remark that Theorem 1 applies if η is an SLE_4 in an arbitrary simply connected domain D since one can always conformally map D to $\mathbf{H}$.

One of the main steps in proving Theorem 1 is a new sufficient condition for a set $X \subseteq \mathbb{C}$ to be conformally removable. One of the novelties of the new condition is that it does not require $X = \partial D$ for $D \subseteq \mathbb{C}$ simply connected as in [4]. We also remark that in general the conformal removability for boundaries of domains which are not simply connected is less well-understood.

Next we give a rough description of the sufficient conformal removability condition that we introduce. Suppose that $X \subseteq \mathbb{C}$ has zero Lebesgue measure and $f\colon \mathbb{C} \to \mathbb{C}$ is a homeomorphism which is conformal on $\mathbb{C} \setminus X$. To show that f is conformal on $\mathbb{C}$ it suffices to show that f is absolutely continuous on lines (ACL), which we recall means that it is absolutely continuous on Lebesgue a.e. horizontal and vertical line. To prove that X is conformally removable, it therefore suffices to control the variation of f on the places where a Lebesgue typical horizontal or vertical line intersects X. We will fix $a > 0$ small, $M > 1$ large, and assume that X has upper Minkowski dimension at most $2 - 5a$. We will further assume that we have a family of sets $\mathcal{A} = \cup_{n=1}^{\infty} \mathcal{A}_n$ where each $A \in \mathcal{A}_n$ has the topology of an annulus with $\text{diam}(A) \le M 2^{-n}$ such that for each $z \in X$ and $n \in \mathbf{N}$ there exists $(1-a^2)n \le k \le n$ and $A \in \mathcal{A}_k$ so that $B(z, 2^{-n})$ is contained in the bounded component of $\mathbb{C} \setminus A$ and the following additional condition holds. There are finitely many open, simply connected, and pairwise disjoint sets $U_1, \dots, U_m$ for $1 \le m \le M$ in $A \setminus X$ with $\partial U_i \cap \partial U_{i+1} \neq \emptyset$ for $1 \le i \le m$ (with $U_{m+1} = U_1$) which satisfy some additional assumptions on the geometry of their pairwise intersections. This will allow us to construct a path γ which disconnects the inner and outer boundaries of A and whose image under f has diameter at most $2^{-(1-3a)k} + 2^{(1-a)k} \int_A |f'(w)|^2 d\mathcal{L}_2(w)$ (where $\mathcal{L}_2$ denotes two-dimensional Lebesgue measure). This gives an upper bound on $\mathrm{diam}(f([z-2^{-n}, z+2^{-n}]))$ which suffices because for (a compact part of) a typical line L the number of intervals of length 2^{-n} hit by X is $O(2^{(1-5a)n})$ (as X has upper Minkowski dimension at most $2-5a$) and the integral of $|f'|^2$ on the $2^{-(1-a^2)n}$-neighborhood of (a compact part of) L is $O(2^{-(1-a^2)n})$. In particular, the variation of f on (a compact part of) L in $X \cap L$ if $O(2^{-an/2})$ which tends to 0 as $n \to \infty$, hence f is absolutely continuous on L.

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Compactified Imaginary Liouville Theory

Antti Kupiainen

(joint work with Colin Guillarmou, Rémi Rhodes)

Statistical mechanics systems are expected to exhibit conformal symmetry at the critical temperature of second order phase transition. This phenomenon is expected to be *universal*: observable quantities such as the numerical values of critical exponents describing large scale properties of the system depend only on the underlying symmetries of the microscopic description. Universality classes of systems with same critical behaviour are believed to be described by Conformal Field

Theories (CFT). A CFT is characterised by a minimal set of so-called *primary* random fields V_i defined on $\mathbb{R}^d$ and their correlation functions

$$\langle\prod_{i=1}^{n} V_i(x_i)\rangle$$

Correlation functions of all the other fields in a CFT are determined by these and the conformal symmetry.

How does one find examples of CFTs? In physics there have been two complementary approaches: the axiomatic *Bootstraap approach* and the constructive *Path integral approach.* Bootstrap approach is based on the postulate that all the correlation functions of primary fields are determined by the 3-point functions $\langle\prod_{i=1}^{3} V_i(x_i)\rangle$. Then the bootstrap hypothesis implies constraints on these 3-point functions that one can use to guess solutions [1]. In the Path integral approach correlation functions are given as statistical averages over a Gibbs ensemble on fields $\phi(x)$, given by a formal path integral

$$\langle\prod_{i=1}^{n} V_i(x_i)\rangle = \int\prod_{i=1}^{n} W_i(\phi(x_i))e^{-S(\phi)}D\phi$$

where S a local action functional. The challenge then is to find examples of S which give rise to a CFT, give a rigorous meaning and construction of the path integral and then to justify the bootstrap hypothesis.

1. Liouville Theory

On a rigorous mathematical level these goals were achieved in the case of the *Liouville Conformal Field Theory* (LCFT) [4, 2, 3] which plays a prominent role in the theory of two dimensional random surfaces. A natural setup for two dimensional CFT is a smooth compact two dimensional surface Σ equipped with a Riemannian metric g. The Liouville action functional is defined for a smooth $\phi : \Sigma \to \mathbb{R}$ by

$$S_\Sigma(\phi, g) = \frac{1}{4\pi}\int_\Sigma (|d\phi|_g^2 + QK_g\phi + \mu e^{\gamma\phi})\mathrm{dv}_g$$

where the parameters are $\gamma \in (0, 2)$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ and $\mu \in \mathbb{R}_+$. K_g is the scalar curvature of the metric g, v_g the Riemannian volume measure and $|\cdot|_g$ the metric on 1-forms.

The rigorous definition of the path integral is given in terms of the *Gaussian Free Field* (GFF) X_g and the Gaussian Multiplicative Chaos (GMC) measure M_g. X_g is a random distribution $X_g \in H^{-s}(\Sigma)$ with $s > 0$ defined by

$$X_g(x) = \frac{1}{\sqrt{2\pi}}\sum_{n\geq 1}\frac{\alpha_n}{\sqrt{\lambda_n}}e_n(x)$$

where $(\alpha_n)_n$ are i.i.d unit Gaussians and e_n eigenfunctions of Laplacian Δ_g with eigenvalues $\lambda_n > 0$. M_g is defined as

$$M_g = \lim_{\epsilon\to 0}\epsilon^{\frac{\gamma^2}{2}}e^{\gamma X_{g,\epsilon}}v_g$$

where the convergence is the weak convergence of measures and the limit exists in probability. $X_{g,\epsilon}$ is a mollification of X_g. We then define LCFT path integral by setting $\phi = X_h + c$ with $c \in \mathbb{R}$ and defining

$$\int F(\phi) e^{-S_\Sigma(\phi,g)} D\phi := Z(g) \int_{\mathbb{R}} \mathbb{E}\left(F(\phi) e^{-\frac{1}{4\pi}\int_\Sigma QR_g \phi dv_g + \frac{\mu}{4\pi} e^{\gamma c} M_g(\Sigma)} \right) dc$$

for suitable functions F. $Z(g)$ is a normalisation constant "partition function of the GFF").

The primary fields of LCFT are so called vertex opeartors and given by $V_\alpha = e^{\alpha\phi}$ with $\alpha \in \mathbb{C}$. Their correlation functions are defined by a regularisation and renormalisation procedure akin to the GMC measure. In the Riemannian setup conformal invariance is formulated as a covariance of the correlation functions under rescaling of the metric $g \to e^\sigma g$ where $\sigma \in C^\infty(\Sigma)$:

$$\langle \prod_j V_{\alpha_j}(z_j) \rangle_{e^\sigma g} = e^{\frac{1+6Q^2}{96\pi}\int_\Sigma (|d\varphi|_g^2 + 2K_g \varphi) dv_g} e^{-\sum_j \Delta_{\alpha_j} \sigma(z_j)} \langle \prod_j V_{\alpha_j}(z_j)) \rangle_g$$

where $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$. This relation defines correlation functions of primary fields of a CFT with central charge $1 + 6Q^2$ and scaling exponents Δ_α.

2. Compact Imaginary Liouville Theory

Compact Imaginary Liouville Theory (CILT) is formally defined by the action functional of the LCFT by replacing γ by $i\beta$:

$$S_\Sigma(\Phi, g) = \frac{1}{4\pi}\int_\Sigma (|d\Phi|_g^2 + iQK_g\Phi + \mu e^{i\beta\Phi}) \mathrm{dv}_g$$

Here $\beta \in \mathbb{R}$ and $Q = \frac{\beta}{2} - \frac{2}{\beta}$. Furthermore the field Φ takes values in the *circle* $\mathbb{R}/(2\pi R\mathbb{Z})$ of radius R. In physics this model has been introduced as a putative scaling limit of so-called loop models (Potts and $O(N)$ models) and models of conformally invariant curves. It is believed to define a non-unitary Logarithmic CFT with primary fields having spin (non-trivial monodromy of correlation functions). It has also been argued to provide a path integral formulation of the celebrated Belavin-Polyakov-Zamolodchicov Minimal models. Many of its properties are not understood even on the physical level of rigour. Thus it provides a natural mathematical playground for these questions.

The probabilistic formulation of CILT uses again the GMC but in addition we need to explain how the fact that Φ takes values in the circle is taken into account. Let first $\Phi : \Sigma \to \mathbb{R}/(2\pi R\mathbb{Z})$ be smooth. Then $d\Phi$ is a closed 1-form on Σ which has a Hodge decompostion

$$d\Phi = d\Phi_0 + \omega$$

with $\Phi_0 : \Sigma \to \mathbb{R}$ and ω a harmonic 1 form satisfying

$$\int_\gamma \omega \in 2\pi R\mathbb{Z} \tag{1}$$

for all closed curves γ. Furthermore

$$\int_\Sigma |d\Phi|_g^2 \mathrm{dv}_g = \int_\Sigma |d\Phi_0|_g^2 \mathrm{dv}_g + \|\omega\|_2^2$$

Given a basis for the homology of Σ the set $\mathfrak{R}$ of harmonic 1 forms on Σ satisfying (1) can be identified with $\mathbb{Z}^{2\mathfrak{g}}$ so we can define the path integral of CILT by setting

$$\Phi = c + X_g + \Phi_\omega$$

where $\Phi_\omega = \int_{x_0}^x \omega$ is the primitive of ω and then

$$\langle F \rangle_{CILT} := Z(g) \sum_{\omega \in \mathfrak{R}} \int_{\mathbb{R}/2\pi R\mathbb{Z}} e^{-\frac{1}{4\pi}\|\omega\|^2} \mathbb{E}\Big[F(\Phi) e^{-\frac{1}{4\pi}\int_\Sigma (iQR_g\Phi + e^{i\beta\Phi}) dv_g}\Big] dc. \tag{2}$$

The formal expression $e^{i\beta\Phi}$ is defined as a random distribution in terms of the *Imaginary GMC* and the upshot is [5] that (2) is well defined for a large class of F provided $\beta < \sqrt{2}$. Furthermore the result is independent of the choice of the point x_0 used to define the primitive of ω.

However, in general the definition (2) depends on the choice of the homology basis due to the curvature term and requiring independence one is led to modifying the curvature term by adding to it term involving a choice of branch cuts such that the field is single valued in their complement. Upshot of this is that the independence forces a *quantisation* of β and Q:

$$\beta, Q \in \frac{1}{R}\mathbb{Z}.$$

Then the relation $Q = \frac{\beta}{2} - \frac{2}{\beta}$ implies $\beta^2 \in \mathbb{Q}$. Thus the central charge $1 - 6Q^2$ of this CFT is *rational.*

With analogy to to LCFT some primary fields of CILT are given by the vertex operators $V_e = e^{i\frac{e}{R}\Phi}$ where $2\pi R$ periodicity requires $e \in \mathbb{Z}$. These are called *electric operators* with electric charge e. CILT has however also *magnetic operators* $O_m(z)$ which produce a winding of the field Φ by $2\pi mR$ around the point z. To construct them let $\mathbf{z} = (z_1, \ldots, z_p)$, $\mathbf{m} = (m_1, \ldots, m_p)$ with $\sum_{j=1}^p m_j = 0$. There exists a harmonic 1-form $\omega_{\mathbf{z},\mathbf{m}}$ on $\Sigma \setminus \cup_i \{z_i\}$ s.t.

$$\int_{\gamma(z_j)} \omega_{\mathbf{z},\mathbf{m}} = 2\pi R m_j$$

with $\gamma(z_j)$ a small contour surrounding z_j. We then replace the field Φ in (2) by adding to it the primitive $\Phi_{\mathbf{m},\mathbf{z}}(x) = \int_{z_0}^x \omega_{\mathbf{z},\mathbf{m}}$. This defines the correlation functions of the magnetic operators $O_{m_i}(z_i)$. Finally the *electromagnetic* operators $V_{e,m}(z)$ are defined by having an electric and a magnetic charge at z. Due to the fact that X_g is a distribution and $\Phi_{\mathbf{m},\mathbf{z}}(x)$ is not continuous at z_i it turns out these fields depend on the direction the point x is taken to approach z_i so the electromagnetic field is defined at the tangent space of Σ at z_i. We denote it by $V_{e,m}(z, v)$ with $v \in T_z\Sigma$ a unit vector. The following theorem summarises our results, for more explicit formulation see [5]:

Theorem. *Let $\beta^2 \in \mathbb{Q}$ with $\beta^2 < 2$. The correlation functions $\langle \prod_j V_{e_j,m_j}(z_j, v_j))\rangle$ are well defined if*

$$\sum_i m_i = 0 \quad \text{and} \quad e_i > QR.$$

They are diffeomorphism invariant and Weyl covariant with central charge $1-6Q^2$ and conformal weights

$$\Delta_{\alpha,m} = \frac{\alpha}{2}(\frac{\alpha}{2} - Q) + \frac{m^2R^2}{4}$$

and spin

$$\langle \prod_j V_{\alpha_j,m_j}(z_j, r_{\theta_j} v_j))\rangle = \prod_j e^{iR(\alpha_j - Q)m_j\theta_j} \langle \prod_j V_{\alpha_j,m_j}(z_j, v_j))\rangle$$

where $r_\theta \in SO(2)$. The structure constants (three-point functions) of CILT have an explicit expression, a generalisation of the imaginary DOZZ formula. Finally, the correlation functions satisfy Segal's gluing axioms under cutting and pasting the surface Σ.

Open Questions

1. For $\beta^2 = 4\frac{p}{q}$ with p, q co-prime, $p > q$, $1 - 6Q^2 = 1 - 6\frac{(p-q)^2}{pq}$ are the central charges of BPZ minimal models. Minimal models have Coulomb gas representation for conformal blocks with two screening charges $e^{i\beta\phi}$ and $e^{\frac{1}{i\beta}\phi}$. For $q = 2$ only $e^{i\beta\phi}$ needed. Are these models related to CILT?

2. Spectrum of CILT consists all degenerate weights $\alpha_{r,s} = \frac{(r-1)p+(s-1)q}{\sqrt{pq}}$ whereas minimal models have $r < q - 1$, $s < p - 1$. Is there a path integral formulation of G. Felder's BRST reduction?

3. CILT Hamiltonian is non-self-adjoint with Jordan cells. Is it a logarithmic CFT?

4. For $\beta \in [\sqrt{2}, 2)$ the GMC renormalisation is not sufficient.How to construct CILT in this regime?

5. Is there a probabilistc theory for $\beta^2 \notin \mathbb{Q}$?

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The vertex-reinforced jump process with long range interactions

Franz Merkl and Silke Rolles
(joint work with Margherita Disertori)

The vertex-reinforced jump process (vrjp). Let $G = (\Lambda, E)$ be an undirected finite or infinite graph endowed with edge weights $W_e > 0$, $e \in E$. The vertex-reinforced jump process $Y = (Y_t)_{t\geq 0}$ is a process in continuous time, where given $(Y_s)_{s\leq t}$, the particle jumps from site i to site $j \sim i$ with rate $W_{ij} L_j(t)$, where

$$L_j(t) = 1 + \int_0^t 1_{\{Y_s = j\}}\, ds = \text{local time at } j \text{ with offset 1.}$$

The process was conceived by Wendelin Werner around 2000.

Transience on $\mathbb{Z}^d$ with long range interactions [3]. Fix a dimension $d \in \mathbb{N}$. Consider the vertex-reinforced jump process on the complete graph with vertex set $\mathbb{Z}^d$ and edge weights $W_{ij} = w(\|i-j\|_\infty)$, $i, j \in \mathbb{Z}^d$, with a decreasing weight function $w : [1,\infty) \to (0,\infty)$. If

$$\sum_{i\in\mathbb{Z}^d\setminus\{0\}} w(\|i\|_\infty) < \infty \quad \text{and} \quad w(x) \geq \overline{W}\frac{(\log_2 x)^\alpha}{x^{2d}} \text{ for all } x \geq 1$$

for some $\alpha > 3$ and $\overline{W} \geq \max\left\{80\log 2, \frac{15}{2}e^{\sqrt{2}\frac{\alpha-1}{\alpha-3}}\right\}$, then the expected number of visits to any given vertex is finite. In particular, the vrjp is a.s. transient.

Random walk in random conductances [6]. On any finite graph $G = (\Lambda, E)$, the discrete time process $(X_n)_{n\in\mathbb{N}_0}$ induced by the vertex-reinforced jump process starting at i_0 has the same distribution as a random walk in random conductances given by

$$W_{ij}e^{u_i+u_j}, \{i,j\} \in E,$$

where $(u_i)_{i\in\Lambda}$ are distributed according to the (u-marginal) of the non-linear hyperbolic supersymmetric sigma model μ^Λ_{W,i_0}, also called $H^{2|2}$ model, introduced by Zirnbauer in [8]. After integrating Grassmann variables out, it is given by

$$\mu^\Lambda_{W,i_0}(du\, ds) = e^{-\sum_{e\in E} W_e(B_e-1)} \sum_{S\in\mathcal{S}} \prod_{\{i,j\}\in S} W_{ij}e^{u_i+u_j} \prod_{i\in\Lambda\setminus\{i_0\}} \frac{e^{-u_i}}{2\pi}\, du_i\, ds_i,$$

where $u_{i_0} = s_{i_0} = 0$, $B_e := \cosh(u_i - u_j) + \frac{1}{2}(s_i - s_j)^2 e^{u_i+u_j}$ for $e = \{i,j\}$, $\mathcal{S}$ is the set of spanning trees of G, and $du_i\, ds_i$ denotes the Lebesgue measure on $\mathbb{R}^2$.

The $H^{2|2}$ model on $\mathbb{Z}^d$ with long range interactions [3]. Consider the $H^{2|2}$ model $\mu^{\Lambda_N}_{W,\rho}$ on the d-dimensional box $\Lambda_N := \{0, 1, \ldots, 2^N - 1\}^d$, $N \in \mathbb{N}$. It describes the random environment for the discrete time process of vrjp on $\Lambda_N \cup \{\rho\}$ starting at ρ. There exists $c > 0$ such that for all $m \in [0, c\overline{W}]$ one has

$$\mathbb{E}^{\Lambda_N}_{W,\rho}\left[(\cosh u_i)^m\right] \leq 2, \quad \mathbb{E}^{\Lambda_N}_{W,\rho}[(\cosh(u_i - u_j))^{m/2}] \leq 2 \cdot 2^{m/2}.$$

uniformly in $i, j \in \Lambda_N \cup \{\rho\}$ and N. In particular, for $\overline{W}$ large enough e^{u_i} are uniformly in $N \in \mathbb{N}$ and $i \in \Lambda_N$ integrable with respect to $\mu^{\Lambda_N}_{W,\rho}$.

Overview of the proof. The main steps to obtain bounds for $\mathbb{E}_{W,\rho}^{\Lambda_N}[(\cosh u_i)^m]$ are the following.

- We compare the $H^{2|2}$ model on Λ_N with long-range interactions with an $H^{2|2}$ model with hierarchical interactions. The distribution of u_i in the hierarchical $H^{2|2}$ model agrees with the distribution in an *effective* $H^{2|2}$ model. Studying the effective $H^{2|2}$ model can be reduced to studying a one-dimensional model with inhomogeneous weights. In this model, the bounds can be deduced in the spirit of [4]. In the effective $H^{2|2}$ model, all vertices interact with each other. However, using monotonicity from Poudevigne [5] or in a determinant in our argument, we only keep the relevant interactions. This yields an effective inhomogeneous one-dimensional model.
- The measure $\mu_{W,\rho}^{\Lambda_N}$ is the marginal of a supersymmetric model dealing with spin variables taking values in a supersymmetric extension of the hyperbolic plane. It involves both, bosonic and fermionic variables (Grassmann variables). Supersymmetry mixes bosonic and fermionic components. Using supersymmetry, one can evaluate expectations of certain observables explicity (Ward identities), which can then be used to derive bounds for expectations of other (non-supersymmetric) observables.

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The phase diagram of the Ising lattice Higgs model

Malin Palö Forsström

Ising lattice gauge theory is a spin models on the directed edges of hypercubic lattices, which takes spins in $\mathbb{Z}_2$. Lattice gauge theories were introduced independently by Wilson [11], as lattice approximations of the quantum field theories that appear in the standard model (known as Yang-Mills theories), and by Wegner in [10], as an example of a spin system with a phase transition without a local order parameter. In this talk we will focus on the Ising lattice Higgs model, which is the simplest example of a lattice gauge theory coupled to an external field. The action of this model is given by

$$-\beta \sum_{p\in C_2(B_N)} \rho(d\sigma(p)) - \kappa \sum_{e\in C_1(B_N)} \rho(\sigma(e)).$$

Since their introduction, lattice gauge theories and the lattice Higgs model have attracted great interest in the physics community, and have been successfully used both for simulations and as toy models for the Yang-Mills model [7, 9].

The natural observables in lattice Higgs models are Wilson loop observables, Wilson line observables, and ratios of such observables, such as the Marcu–Fredenhagen ratio ρ (see, e.g., [1, 2, 3, 8, 9]). These are all natural observables from a physics perspective (see, e.g. [2]), but are also interesting from a mathematical viewpoint since they are believed to undergo several phase transitions [9].

We draw the conjectured phase diagram of the Ising lattice Higgs model, in Figure 1.

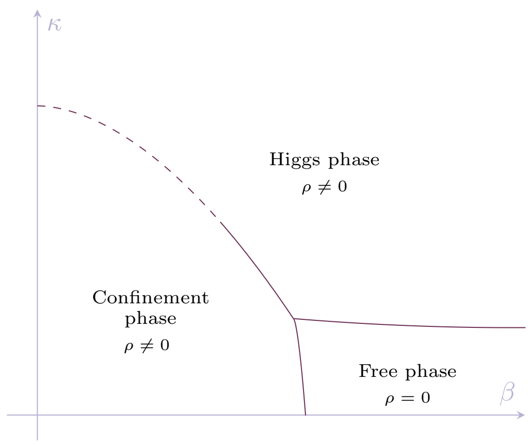

Figure 1. The conjectured phase diagram of the Ising lattice Higgs model. In the Higgs phase, the Marcu-Fredenhagen ratio is believed to be non-zero, and one expects exponential decay of correlations. In the free phase, one believes the Marcu-Fredenhagen ratio is identically zero, and expects exponential decay of correlations with polynomial correction.

In this phase diagram, there are two phases; a Higgs/confinement phase, where the expectaion of straight Wilson lines are conjectured ot have pure exponential decay in the length of the line, and the Marcu-Fredenhagen ratio is expected to be non-zero, and a free phase, where the Marcu-Fredenhagen is expected to be identically zero and the expectation of straight Wilson lines are conjectured to have exponential decay in the length of the line with polynomial corrections.

In this talk, we present recent results which confirms parts of this phase diagram in subsets of the three conjectured phases.

In particular, our main results show that the Marcu-Fredenhagen ratio is non-zero in non-trivial subsets of the Higgs and confinement phases, while identically zero in a non-trivial subset of the free phase. As a consequence, it follows that the model undergo at least one phase transition. In addition, we are able to show that the expected value of straight Wilson lines have a pure exponential decay. The main tools needed to prove these results are various high temperature and cluster expansion.

The talk is based on recent work in [4, 5, 6].

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Around the conformal anomaly

Eveliina Peltola

eveliina.peltola@aalto.fi
eveliina.peltola@hcm.uni-bonn.de

Department of Mathematics and Systems Analysis,
Aalto University, Finland,
&
Institute for Applied Mathematics,
University of Bonn, Germany.

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The *conformal restriction property* of Schramm-Loewner evolution (SLE) type measures has been a key feature — while still somewhat mysterious — of many developments involving SLE/CLE problems and their relationship to discrete lattice paths. For example, part of the conjecture that the self-avoiding walk (SAW) should converge in the scaling limit to an SLE(8/3) path is based on the exact conformal restriction invariance of SAW measures [1]. Generalizations of SLE type curves on Riemann surfaces, i.e., natural measures on paths and loops enjoying the symmetries underlying critical models and conformal field theory (CFT), were proposed by Kontsevich [2] and Werner [3] building on their known conformal restriction property [4] and on somewhat analogous conformal Markov properties of the discrete models (especially of their phase boundaries). Geometrically, an infinitesimal version of the conformal restriction property relates the loop measures to Malliavin's proposal [5] for a canonical diffusion on the diffeomorphism group $\mathrm{Diff}(S^1)$ of the circle. These measures, also termed "*MKS measures*" were soon thereafter investigated by a few people, involving Kontsevich & Suhov [6] Friedrich [7], and Dubédat [8, 9], among others. It is believed that the conformal restriction covariance property uniquely determines the MKS loop measures [6].

Pertaining to the general picture, the conformal restriction property can be formulated precisely in terms of an "*anomaly*" that describes the response of the system to deformations of the underlying space (domain, complex structure, etc.) For example, if μ_D is an SLE(κ) measure on a domain D and μ_U is an SLE(κ) measure on its subdomain $U \subset D$, then the two measures compare as

$$\frac{\mathrm{d}\mu_U}{\mathrm{d}\mu_D}(\gamma) = 1\{\gamma \subset U\} \; \exp\left(\tfrac{1}{2}c(\kappa)\, m_D(\gamma, D \setminus U)\right)$$

where $m_D(\gamma, D \setminus U)$ is a conformal invariant and $c(\kappa) = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$ is a function of the diffusivity parameter $\kappa \in (0,4]$, encoding the strength of the anomaly (being zero for $\kappa = 8/3$). It is well known [4, 10, 11] that for chordal or loop SLE(κ), the factor $m_D(\gamma, D \setminus U)$ equals the mass of Brownian loops in D intersecting both γ and $D \setminus U$. (It also has other alternative expressions, e.g., in terms of Schwarzian derivatives of uniformizing maps [4], relating it closer to infinitesimal deformations.) Brownian loop measure being infinite in some situations (though renormalizable), it may be preferable to seek other presentations of the anomaly.

Sid Maibach and I have investigated the universality of this anomaly [12], in the spirit of Kontsevich's geometric framework [2, 6], also motivated by the point of view of CFT à la Segal [13], and similar to the vertex operator algebra framework for CFT initiated by Friedan & Shenker [14]. Indeed, in order to fully understand the universal nature of the conformal restriction property and the associated anomaly (e.g., as a characterizing property of canonical MKS loop measures, or as an inherent feature of any conformal field theory), it seems necessary to investigate the relationship of MKS measures to the geometric content of CFT.

Consider the conformal anomaly concretely with both $U \subset D$ simply connected. If the Brownian loop measure were finite, the anomaly could also be written as

$$\exp\big(\tfrac{1}{2}c(\kappa)\, m_D(\gamma, D \setminus U)\big) = \text{“}\, \frac{\exp\big(\tfrac{1}{2}c(\kappa)\, m_{\mathbb{D}}(\varphi_D(\gamma))\big)}{\exp\big(\tfrac{1}{2}c(\kappa)\, m_{\mathbb{D}}(\varphi_U(\gamma))\big)}\,\text{”},$$

comparing the mass $m_{\mathbb{D}}$ of Brownian loops in the unit disk $\mathbb{D}$ intersecting $\varphi_D(\gamma)$ and those intersecting $\varphi_U(\gamma)$, where $\varphi_D \colon D \to \mathbb{D}$ (resp. φ_U) is a uniformizing map from D (resp. U) onto the disk. One would be thus led to considering the SLE(κ) curve γ on two conformal structures (Riemannian metrics) on the disk induced by the maps φ_D and φ_U. To make this more precise, one can consider the metric dependence of *CFT partition functions* $Z_g(\Sigma)$ on surfaces (Σ, g), also related to total masses of SLE (or MKS) measures (cf. [10, 15, 16]). Namely, two metrics g and $e^{2\sigma}g$ in the same conformal class are related by the anomaly functional

$$S_L^0(\sigma, g) := \frac{1}{12\pi}\iint_\Sigma \Big(\frac{1}{2}|\nabla_g \sigma|_g^2 + R_g\sigma\Big)\mathrm{vol}_g + \frac{1}{12\pi}\int_{\partial\Sigma} k_g\sigma\, \widetilde{\mathrm{vol}}_g, \quad \sigma \in C^\infty(\Sigma, \mathbb{R}),$$

where ∇_g, R_g, vol_g, k_g, $\widetilde{\mathrm{vol}}_g$ are respectively the divergence, Gaussian curvature, and volume form on Σ, and the boundary curvature and volume form on $\partial\Sigma$, induced by g. Changes of metrics are thus encoded into an exponential factor $\exp(c\, S_L^0(\sigma, g))$, where $c \in \mathbb{R}$ is the *central charge*. Any CFT partition function transforms as $Z_{e^{2\sigma}g}(\Sigma) = e^{c\, S_L^0(\sigma,g)} Z_g(\Sigma)$. Dubédat formalized [15] how a comparison of partition functions (e.g., determinants of Laplacians) gives the conformal restriction anomaly (note though that partition functions might not be finite). E.g., taking $\Sigma = \mathbb{D}$ with flat metric g_0 and $U \subset \mathbb{D}$ simply connected, we expect

$$\frac{Z_{g_0}(\mathbb{D})}{Z_{|(\varphi_U^{-1})'|^2 g_0}(\mathbb{D})}\, \frac{Z_{|(\varphi_U^{-1})'|^2 g_0}(\mathbb{D} \setminus \varphi_U^{-1}(\gamma))}{Z_{g_0}(\mathbb{D} \setminus \gamma)} = \exp\big(\tfrac{c}{2}\, m_{\mathbb{D}}(\gamma, \mathbb{D} \setminus U)\big)$$

for $\gamma \subset U$ a chord (or Jordan loop; in which case the identity should hold up to a multiplicative factor that only depends on the conformal moduli of $\mathbb{D} \setminus \gamma$, $U \setminus \gamma$).

The aim in [12] is to explicitly derive the *Virasoro algebra* — the Lie algebra of infinitesimal conformal symmetries — from complex deformations $\mathrm{Def}_{\mathbb{C}}(S^1)$ associated to Jordan loops on surfaces, parameterized by the circle S^1. Morally, $\mathrm{Def}_{\mathbb{C}}(S^1)$ should be thought of as a complexification of the infinite-dimensional Fréchet-Lie group $\mathrm{Diff}_+^{\mathrm{an}}(S^1)$ (comprising real-analytic, orientation-preserving diffeomorphisms of S^1), whose Lie algebra $\mathfrak{X}_{\mathbb{R}}^{\mathrm{an}}(S^1)$ consists of real-analytic vector fields on S^1. Its complexification $\mathfrak{X}_{\mathbb{C}}^{\mathrm{an}}(S^1) := \mathfrak{X}_{\mathbb{R}}^{\mathrm{an}}(S^1) \otimes \mathbb{C}$ is known as the *Witt algebra*. It can be thought of as the Lie algebra of the complex deformations

$\mathrm{Def}_{\mathbb{C}}(S^1)$, in the sense that flows of complex vector fields yield complex deformations. One is thus led to considering a moduli space of Riemann surfaces Σ with (analytically parametrized) boundary components (i.e., loops parameterized by the circle), endowed with a *sewing operation* across boundary components (viz. conformal welding), so boundary components become Jordan loops on the sewn surface.

In particular, the group $\mathrm{Diff}^{\mathrm{an}}_+(S^1)$ acts very naturally by reparameterization of the boundary components of Σ. However, in order to see the Virasoro structure which unifies the MKS loop measures, one defines a *real determinant line bundle* $\mathrm{Det}^c_{\mathbb{R}}$ on the moduli space, which is just a collection of real one-dimensional vector spaces associated to each equivalence class of surfaces Σ (encoding the conformal anomaly between metrics). The determinant line over Σ is defined as the collection of (formal) multiples of metrics under the equivalence $[e^{2\sigma}g] = e^{cS^0_L(\sigma,g)}[g]$. (By choosing a section for the determinant line bundle, the conformal anomaly $S^0_L(\sigma,g)$ can be written in a number of related but inequivalent ways [12], involving Loewner energy [17], Brownian loop measure [4, 11], or determinants of Laplacians [15].)

The moduli space carries an action of the complex deformations on the boundary components that truly change the complex structure of Σ, unlike mere reparameterizations. The (nontrivial) sewing operation on the moduli space and on the determinant line bundle then gives rise to an associative product for the associated central extension of $\mathrm{Def}_{\mathbb{C}}(S^1)$ by the multiplicative group of positive reals,

$$(\phi,\lambda)\cdot(\psi,\lambda') = (\phi\psi, \lambda\lambda'\,\Gamma_c(\phi,\psi)), \quad \phi,\psi\in \mathrm{Def}_{\mathbb{C}}(S^1),\ \lambda,\lambda'>0,$$

where Γ_c is a *cocycle* describing algebraically the relevant central extension, and geometrically the nontrivial twist in the sewing operation. (Note, however, that complex deformations cannot form a well-defined Lie group, so one cannot naively speak of their Lie algebra, nor of their central extensions.) From this structure, one can explicitly compute the conformal anomaly in terms of the (honest) Lie algebra cocycle by taking two flows $(\phi_t)_{t\in\mathbb{R}}$ and $(\psi_t)_{t\in\mathbb{R}}$ of complex deformations generated by two vector fields $v,w\in\mathfrak{X}^{\mathrm{an}}_{\mathbb{C}}(S^1)$ in the Witt algebra [12]:

$$\frac{1}{2}\frac{\partial^2}{\partial t\partial s}\Big(\log\Gamma_c(\phi_t,\psi_s)-\log\Gamma_c(\psi_s,\phi_t)\Big)\Big|_{t=s=0} = \frac{c}{24\pi}\mathrm{Im}\int_0^{2\pi} v'(\theta)w''(\theta)\,\mathrm{d}\theta.$$

In particular, we see that the cocycle is nontrivial whenever the central charge $c\neq 0$ is nonzero, and that it coincides with the imaginary part of the celebrated *Gel'fand-Fuks* (or Virasoro) *cocycle*. Note that while the determinant line bundle is topologically trivializable, the sewing operation is not. This is the key fact that gives rise to a *nontrivial central extension* of $\mathrm{Def}_{\mathbb{C}}(S^1)$ and its Lie algebra, yielding the Virasoro structure and the nontrivial conformal anomaly. However, perhaps surprisingly the cocycle vanishes for real vector fields $v,w\in\mathfrak{X}^{\mathrm{an}}_{\mathbb{R}}(S^1)$, which shows that the complex deformations are necessary in order to see the conformal anomaly.

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Minimal Surfaces in Random Environment

Ron Peled

(joint work with Michal Bassan, Barbara Dembin, Dor Elboim, Shoni Gilboa, and Daniel Hadas)

A minimal surface in a random environment (MSRE) is a surface which minimizes the sum of its elastic energy and its environment potential energy, subject to prescribed boundary conditions. Apart from their intrinsic interest, such surfaces are further motivated by connections with disordered spin systems, first-passage percolation models and minimal cuts in the $\mathbb{Z}^D$ lattice with random capacities.

We wish to study the geometry of d-dimensional minimal surfaces in a $(d+n)$-dimensional random environment. Specializing to a model that we term harmonic MSRE, in an "independent" random environment, we rigorously establish bounds on the geometric and energetic fluctuations of the minimal surface, as well as a scaling relation that ties together these two types of fluctuations. In particular, we prove, for all values of n, that the surfaces are delocalized in dimensions $d \leq 4$

and localized in dimensions $d \geq 5$. Moreover, the surface delocalizes with power-law fluctuations when $d \leq 3$ and sub-power-law fluctuations when $d = 4$. Our localization results apply also to harmonic minimal surfaces in a periodic random environment. Many of our results are new even for $d = 1$ (indeed, even for $d = n = 1$), corresponding to the well-studied case of (non-integrable) first-passage percolation.

Intrinsic uniqueness of gauge-covariant Yang–Mills dynamic

Hao Shen

(joint work with Ilya Chevyrev)

This talk is based on (part of) a joint work [5] with Ilya Chevyrev.

Fix a compact Lie group G with Lie algebra $\mathfrak{g}$. Recall that the Yang–Mills model (on $\mathbf{T}^2$) is defined by the functional $\int_{\mathbf{T}^2} |F_A|^2 dx$. Here A is a $\mathfrak{g}$ valued 1-form and $F_A = dA + [A, A]$ is the curvature of A. The functional is invariant under gauge transformation $A \mapsto g \circ A = gAg^{-1} - dgg^{-1}$.

The stochastic quantization equation is

$$\partial_t A_i = \Delta A_i + [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] + CA_i + \chi^\varepsilon * \xi_i$$

where the noise ξ is convolved with a mollifier function χ^ε. The equation is locally well-posed in 2D and 3D [1, 2]. Moreover, in [1, 2] it is also proved that there is a choice of C so that when two initial conditions a, b are gauge equivalent, in the limit $\varepsilon \to 0$ the solutions A^a and A^b from initial conditions a, b respectively are gauge equivalent in law. This is called the gauge covariant property, which allows us to project the dynamic to the quotient space. In [1, 2] we couple A^a and A^b by a particular time-dependent family of gauge transformations which solve a PDE.

In [5] it is proved that the gauge covariant property holds for only one choice of C. One of the important application is that the Langevin dynamics of a large class of lattice Yang–Mills models have the same (universal) limit. The proof of this result uses geometric arguments.

In the proof, we show that for a different choice of C, the solution is not gauge covariant. To this end, we argue that the Wilson loop gauge invariant observables are different for the two solutions starting from two delicately chosen gauge equivalent initial conditions. The loop is chosen to be the non-contractible loop around $\mathbf{T}^2$. To create the desired gauge equivalent initial conditions one uses ideas from sub-Riemannian geometry. A strictly positive lower bound for the difference between the two Wilson loop observables is established by perturbative arguments for the Yang–Mills SPDE and holonomy SDE.

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Scaling exponents for 2D percolation via Liouville quantum gravity

Xin Sun
(joint work with P. Nolin, W. Qian, Z. Zhuang)

This report is based on a joint work with P. Nolin, W. Qian, and Z. Zhuang [NQSZ23].

Bernoulli percolation is a simple and very natural process of statistical mechanics, defined on a lattice. It was introduced by Broadbent and Hammersley [BH57] to model the large-scale properties of a random material. Two-dimensional (2D) percolation is especially well understood, thanks to its connection to conformal invariance and Schramm-Loewner evolution (SLE). Introduced by Schramm in the groundbreaking work [Sch00], SLE is a one-parameter family of random non-self-crossing curves characterized by conformal invariance and domain Markov property, denoted by SLE_κ. It is conjectured to describe the scaling limits of a large class of two-dimensional random systems at criticality. In another breakthrough [Smi01] published shortly after, Smirnov proved that critical site percolation on the triangular lattice converges to a conformally invariant scaling limit, which can thus be described by SLE_6.

The arm exponents are a set of scaling exponents encoding important geometric information of percolation at and near its criticality. They describe the probability of observing connections across annuli of large modulus by disjoint connected paths of specified colors. When there is at least one arm of each of the two colors, such exponents are called polychromatic arm exponents. Otherwise, they are called monochromatic arm exponents. Based on the link with SLE, the exact values of the one-arm exponent and all the polychromatic arm exponents were derived rigorously for site percolation on the triangulation lattice in [LSW02] and [SW01], respectively. The value of these exponents were also predicted in the physics literature; see [ADA99] and references therein.

Despite the SLE connection, the evaluation of monochromatic arm exponents beyond the one-arm case has been a longstanding mystery. In this paper, we derive the exact value for the monochromatic two-arm exponent, namely with two disjoint connections of the same color. This exponent is also known as the backbone exponent, with a rich history. Prior to our work, there is no theoretical prediction for the backbone exponent in the literature that is consistent with numerical approximations. We show that it is the root of a simple elementary function. Moreover, it is transcendental.

Theorem 1. *The backbone exponent ξ is the unique solution in the interval $(\frac{1}{4}, \frac{2}{3})$ to the equation*

$$\frac{\sqrt{36x+3}}{4} + \sin\left(\frac{2\pi\sqrt{12x+1}}{3}\right) = 0. \tag{1}$$

The number ξ is transcendental. Namely, it is not a root of an integer-coefficient polynomial.

Using (1), we obtain the numerical value of ξ:

$$\xi = 0.35666683671288\ldots. \tag{2}$$

The best numerical result obtained so far is $\xi = 0.35661 \pm 0.00005$ in [FKZD22], which is based on Monte Carlo simulations. See also [Gra99, JZJ02, DBN04] for earlier numerical approximation results on ξ.

Our derivation consists of two steps. First, we express the backbone exponent in terms of a variant of SLE_6 call the SLE_6 bubble measure. Then we exactly solve the SLE_6 problem using the coupling between SLE and Liouville quantum gravity (LQG), and the exact solvability of Liouville conformal field theory (CFT). This approach to the exact solvability of SLE was developed by the third-named author with Ang, Holden, Remy in [AHS21, AS21, ARS21]. Our derivation can be viewed as an application of the KPZ relation to the derivation of scaling exponents for lattice models. The main novelty in our application is the usage of the exact solvability Liouville CFT.

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Coulomb gas on a Jordan domain

Fredrik Viklund
(joint work with Kurt Johansson)

Let D be a Jordan domain in the complex plane. Let $\beta > 0$ and consider the partition function of a planar Coulomb gas in D with a hard wall along $\eta := \partial D$,

$$Z_{n,\beta}(D) = \frac{1}{n!} \int_{D^n} e^{-\frac{\beta}{2} \sum_{1 \le k \neq \ell \le n} \log |z_k - z_\ell|^{-1}} \prod_{k=1}^{n} d^2 z_k.$$

We can view this as coming from a statistical mechanics model of n charged particles in the plane, interacting via the electrostatic energy at inverse temperature β, with all particles constrained to D. We are interested in how the geometry of D is reflected in the large n behavior of $Z_{n,\beta}(D)$. When $\beta = 2$ the model is determinantal and we consider this case from now on and write $Z_n(D) = Z_{n,2}(D)$. The starting point is the following result.

Proposition 1 (See [3]). *Let D be a bounded Jordan domain containing* 0. *We have the following identity for all $n \ge 1$,*

$$\log \frac{Z_n(D)}{Z_n(\mathbb{D})} = n(n+1) \log r_\infty(D) + \log \det(I - P_n B B^* P_n)_{\ell^2(\mathbb{Z}_+)}. \tag{1}$$

Here $\mathbb{D}$ is the unit disk, P_n is projection on the first n coordinates in $\ell^2(\mathbb{Z}_+)$, and $B = (\sqrt{k\ell} a_{k\ell})_{k,\ell}$ is the semi-infinite Grunsky matrix associated with D. Let $\mathbb{D}^*$ be the exterior unit disc and D^* the exterior of D and write $g : \mathbb{D}^* \to D^*$ for the conformal map normalized so that $g(z) = r_\infty z + O(1)$ as $z \to \infty$; $r_\infty = r_\infty(D)$ is the capacity of $\overline{D}$. Then the Grunsky coefficients $a_{k\ell}$ are defined via the expansion

$$\log \frac{g(\zeta) - g(z)}{\zeta - z} = - \sum_{k,\ell=1}^{\infty} a_{k\ell} \zeta^{-k} z^{-\ell}, \quad |z|, |\zeta| > 1.$$

Let $f : \mathbb{D} \to D$ be the interior conformal map normalized so that $f(0) = 0$, $f'(0) > 0$. The following result follows immediately from Proposition 1.

Theorem 1 (See [3]). *Suppose without loss in generality that $r_\infty(D) = 1$. Then $\eta = \partial D$ is a Weil-Petersson quasicircle if and only if*

$$-12 \limsup_{n \to \infty} \log \frac{Z_n(D)}{Z_n(\mathbb{D})} < \infty,$$

in which case the limit exists and equals the Loewner energy of η:

$$I^L(\eta) = \frac{1}{\pi}\int_{\mathbb{D}} |f''/f'|^2 d^2z + \frac{1}{\pi}\int_{\mathbb{D}} |g''/g'|^2 d^2z + 4\log|f'(0)|/|g'(\infty)| \tag{2}$$

is the Loewner energy of η.

Weil-Petersson quasicircles arise in a number of places in analysis and, recently, in probability, see, e.g., [4, 1, 6]. We refer to [7] for more on Wang's Loewner energy and various links to SLEs and random conformal geometry.

What happens if η is not a Weil-Petersson quasicircle? In this case η must have infinite Loewner energy and it is easy to see from (2) that one way this can happen is if η has corners. To state our main result, consider D with piecewise analytic boundary η with m corners of interior opening angles $\pi\alpha_p, p = 1, \dots, m$.

Theorem 2 (See [3])**.** *We have the following asymptotic formula.*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \frac{Z_n(D)}{Z_n(\mathbb{D})} = -\frac{1}{6}\sum_{p=1}^{m} \left(\alpha_p + \frac{1}{\alpha_p} - 2\right).$$

The expression on the right could be considered universal and appears for instance in small-t heat-trace formulas, see, e.g., [2, 5].

The proof of Theorem 2 starts from Proposition 1 and is based on careful asymptotic analysis of the Grunsky coefficients.

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Two optimization problems of the Loewner energy

Yilin Wang

Conformal welding encodes a Jordan curve into a circle homeomorphism. It is a classical subject in geometric function theory to study the correspondence between the analytic properties of the curve and homeomorphism. More precisely, let $\gamma \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be an oriented Jordan curve. We denote the two connected components of $\hat{\mathbb{C}} \setminus \gamma$ on the left and right of γ by Ω and Ω^*, respectively. We write $\mathbb{H}^2 := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and $\mathbb{H}^{2,*} := \{z \in \hat{\mathbb{C}} \mid \mathrm{Im}(z) < 0\}$. From the Carathéodory theorem, any conformal map f from $\mathbb{H}^2$ onto Ω extends continuously to a homeomorphism of their closures and defines a homeomorphism from $\mathbb{RP}^1 = \partial\mathbb{H}^2$ to γ. Similarly, any conformal map g from $\mathbb{H}^{2,*}$ onto Ω^* defines a homeomorphism from $\mathbb{RP}^1$ to γ. The *welding homeomorphism* (or simply the *welding*) of γ is defined as

$$h_\gamma := g^{-1} \circ f|_{\mathbb{RP}^1} \in \mathrm{Homeo}_+(\mathbb{RP}^1) \tag{1}$$

where we denote by $\mathrm{Homeo}_+(\mathbb{RP}^1)$ the space of orientation preserving homeomorphisms of $\mathbb{RP}^1$. There is ambiguity in defining the welding homeomorphism as one may possibly precompose f, g by $\mathrm{PSL}(2,\mathbb{R})$, and for any solution to the welding problem, one may post compose γ by $\mathrm{PSL}(2,\mathbb{C})$. Hence, the conformal welding is well-posed between the equivalent classes

$$\mathrm{PSL}(2,\mathbb{C})\backslash\{\text{Jordan curves}\} \Longleftrightarrow \mathrm{PSL}(2,\mathbb{R})\backslash \mathrm{Homeo}_+(\mathbb{RP}^1)/\,\mathrm{PSL}(2,\mathbb{R}).$$

We first explain that the graph of any $h \in \mathrm{Homeo}_+(\mathbb{RP}^1)$ is a positive curve in $\partial\mathrm{AdS}^3 \simeq \mathbb{RP}^1 \times \mathbb{RP}^1$, the boundary of the Anti-de Sitter space. Under this identification, the $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ action on $\mathrm{Homeo}_+(\mathbb{RP}^1)$ translates to the isometries of AdS^3 space. Similarly, viewing the Jordan curve as being on $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$, the $\mathrm{PSL}(2,\mathbb{C})$ action translates to the isometries of $\mathbb{H}^3$.

We then consider two optimizing problems for the Loewner energy, one under the constraint for the curve γ to pass through n given points on $\partial\mathbb{H}^3$; the other under the constraint for the graph of the welding homeomorphism h to pass through n given points on $\partial\mathrm{AdS}^3$. We observe that the answers to the two problems exhibit interesting symmetries: optimizing the Jordan curve in $\partial\mathbb{H}^3$ gives rise to a welding homeomorphism that is the boundary of a pleated plane in AdS^3, whereas optimizing the positive curve in $\partial\mathrm{AdS}^3$ gives rise to a Jordan curve that is the boundary of a pleated plane in $\mathbb{H}^3$.

This talk is based on the work [1].

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Loop-soup rewiring dynamics, double points, and Φ^4 models

Wendelin Werner

A critical Brownian loop-soup [3] in $\mathbf{R}^3$ or a bounded domain D in $\mathbf{R}^2$ is a Poissonian collection of unrooted unoriented Brownian loops with intensity given by the "natural" scale-invariant and translation-invariant measure on such Brownian loops. The properly renormalized occupation time measure of a critical loop-soup is known to be distributed like the (properly defined) square of a Gaussian free field [6, 7]. In two dimensions, it has been shown [1] that one can in fact construct a GFF in D out of a Brownian loop-soup by sampling additional independent ±1 signs for each cluster of loops (recall that the outermost boundaries of the clusters of loops are distributed like a Conformal Loop Ensemble CLE_4, see [11]). For more relations between the planar loop-soups, the GFF, its square and the CLE_4 structure, see for instance [10, 5] and the references therein. A similar relation between square of discrete GFF and random walk loop-soups exists also in the discrete setting, and to make the link with the actual discrete GFF and control the scaling limit, the cable-graph setup turns out to be very useful [8, 9].

The GFF is sometimes refered to as the bosonic free field. In the discrete setting, the indistinguishability of bosons can be viewed as mirroring the "rewiring" property [14] of the random walk loop-soups. Loosely speaking, if one sees that a given site is actually visited "twice" by the loop-soup (i.e., twice by one single loop or by two different loops once), then one can resample the connection probability at that site (thereby possibly merging the two different loops into one longer one, or splitting the one loop into two shorter ones) without changing the law of the loop-soup.

When trying to express the analog of this rewiring property in the continuum, one is facing the problem that the "quantity" of Brownian self-intersections corresponding to small loops is infinite. This feature has motivated the introduction (see [13, 2, 4]) of renormalized Brownian self-intersection local times some 40+ years ago, already in the context of Euclidean field theory [12].

In this presentation, motivated by extensions to higher dimensions, we outline why in two dimensions:

- For each fixed positive ϵ, when one performs the natural continuous-time rewiring dynamics on Brownian loop-soups where any two Brownian loops of time-length greater than ϵ can merge at a rate and point chosen according to the corresponding intersection local time between these loops, and on the other hand, any single Brownian loop can be split into two loops of time-length greater than ϵ at a rate given by its self-intersection local time, then the law of the loop-soup is invariant. This Markov process (when one considers a loop-soup in a bounded domain) makes almost surely only finitely many jumps during any finite time-interval.
- When ϵ tends to 0, the intensity of jumps does blow up, due to the exploding number of self-intersections. However, in two dimensions, it turns out to be possible to show that there exists a limiting non-trivial càdlàg Markovian dynamics with a dense set of rewiring-times.

This second point is actually closely related to following further results:

- It is possible to define for each given Brownian loop β and each loop-soup $\mathcal{L}$ (not containing β), the renormalized intersection local time of β with $\mathcal{L}$. In some sense, the existence of the dynamics mirrors the fact that the number of self-intersections of a Brownian loop is comparable to the number of intersections with the union of all other loops in a loop-soup.

- These double points and self-intersections in a loop-soup are very directly connected to the square of the occupation time, i.e., to the (properly defined) fourth power of the GFF (see e.g., [7] and the references therein). Indeed, the previous considerations make it actually possible to construct directly the reweighting between Brownian loop-soups (leading to the GFF) and reweighted Brownian loop-soups (leading to a construction of the so-called Φ_2^4 fields) – and thereby provide a construction of these Φ_2^4 fields via soups of interacting reweighted Brownian loops.

These results will be written up in an upcoming paper. How to adapt such ideas in $d = 3$ is work in progress.

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The 3D dimer model

Catherine Wolfram

(joint work with Nishant Chandgotia and Scott Sheffield)

The dimer model is one of the basic lattice models of statistical physics. A dimer tiling (a.k.a. perfect matching) τ of a graph G is a collection of edges such that every vertex is covered by exactly one edge in the collection. If G is a subgraph of $\mathbb{Z}^d$, then τ can be drawn as a tiling by dominoes. See Figure 1 for examples in two and three dimensions.

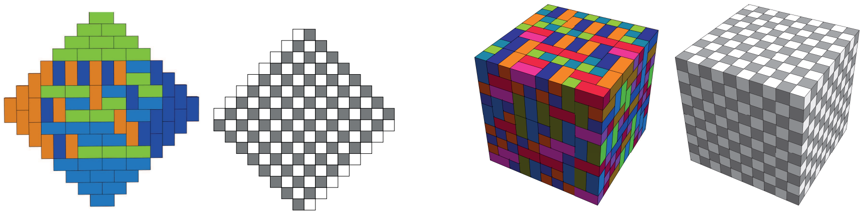

Figure 1. A dimer tiling of a region in $\mathbb{Z}^2$ called an *Aztec diamond* and the bipartite coloring of $\mathbb{Z}^2$ (left) and a dimer tiling of a cube and the bipartite coloring of $\mathbb{Z}^3$ (right).

The dimer model has been studied deeply in two dimensions, where it is a model of random surfaces through the correspondence between dimer tilings and Lipschitz "height functions" due to Thurston [1]. This talk is about the dimer model in three dimensions [4]. The main result is a large deviation principle and limit shape result for scaling limits of random dimer tilings of three-dimensional regions, analogous to the two-dimensional result of Cohn, Kenyon, and Propp [2].

1. Correspondence with discrete vector fields

The height function correspondence is unique to two dimensions. However for any dimension d, $\mathbb{Z}^d$ is a bipartite lattice, with underlying bipartition into white and black cubes as depicted in Figure 1. The colors of the tiles in Figure 1 indicate the cardinal direction of the tile, viewed as a vector from its white cube to its black cube. In this way, there is a correspondence between a dimer tiling τ and a discrete vector field v_τ, i.e., a function on oriented edges of the graph such that for any white-to-black oriented edge e,

$$v_\tau(e) = \begin{cases} 1 & e \in \tau \\ 0 & e \notin \tau. \end{cases} \tag{1}$$

If $-e$ denotes the same edge e with reversed orientation, then $v_\tau(-e) = -v_\tau(e)$. Subtracting a constant vector field from v_τ makes it divergence-free. When $d = 3$, this is

$$f_\tau(e) = \begin{cases} 5/6 & e \in \tau \\ -1/6 & e \notin \tau. \end{cases} \tag{2}$$

The height function in two dimensions can be constructed as the scalar potential of the curl-free dual of f_τ (i.e., the dual of f_τ is ∇h). The height function construction does not work in any higher dimension.

For the three-dimensional setting we consider in this talk, we think about dimer tilings (e.g. to compare them, talk about scaling limits, etc.) through the corresponding divergence-free vector field f_τ.

2. Set up for a large deviation principle

Fix a dimension $d = 2$ or 3. Suppose that $R \subset \mathbb{R}^3$ is a "reasonable" compact region and that b is a boundary condition on ∂R (specified e.g. using a vector field or height function). Also choose any sequence of discrete regions $R_n \subset \frac{1}{n}\mathbb{Z}^d$ such that R_n approximates R in Hausdorff distance and the boundary conditions of R_n converge to b as $n \to \infty$.

Question 1. What does a random dimer tiling of R_n look like as $n \to \infty$?

This vague question is made precise using the correspondence with divergence-free discrete vector fields f_τ, whose scaling limits are measurable divergence-free vector fields (in a suitable topology). A large deviation principle means quantifying: given a deterministic flow g, what is the probability that a tiling of R_n is close to g as $n \to \infty$? There is a *limit shape* if is there is one limiting flow that random tilings concentrate on as $n \to \infty$.

To set up the large deviation principle more formally requires the following ingredients.

(1) A sequence of probability measures $(\rho_n)_{n\geq 1}$ (e.g., uniform measure on dimer tilings of R_n)
(2) A topology (to say what the scaling limits are, and to compare things)
(3) A rate function $I(\cdot)$, where I measures, for any fixed $\delta > 0$,
"ρ_n(tiling flow f_τ is within δ of deterministic flow g) $\approx \exp(-n^d \cdot I(g))$"
(4) When $(\rho_n)_{n\geq 1}$ satisfy an LDP and the rate function $I(\cdot)$ has a unique minimizer, then the ρ_n-probability that a random tiling is close to minimizer goes to 1 as $n \to \infty$. When this holds, there is a unique limit shape which is given by the minimizer.

3. Differences between 2D and 3D

The large deviation principle for dimer tilings of two-dimensional regions proved by Cohn, Kenyon, and Propp is stated in terms of the corresponding height functions [2]. The limiting boundary conditions on ∂R are also specified using height functions. In the 2D result, the items mentioned above are:

(1) The measures ρ_n are uniform measure on tilings of R_n;
(2) The topology is the sup norm on the corresponding height functions h;
(3) The rate function $I(\nabla h)$ is a function of only the gradient ∇h and can be written as the integral of a "local entropy" function ent_2 which has an explicit formula;
(4) Through explicit analysis of the rate function formula, a unique limit shape exists given a region and boundary conditions.

Exact solvability (e.g., the fact that ent_2 has a formula) is an extremely powerful tool that plays a central role in many works about dimers in two dimensions. We expect that the 3D model is not solvable, which makes it very different to study. To give some intuition for the differences between two and three dimensions, we give three examples.

Example 2. (Kasteleyn determinants.) The partition function (=number of dimer tilings) of any bipartite planar graph can be computed as the determinant of a weighted adjacency matrix of the graph. This is called the Kasteleyn determinant formula. The necessary condition on a bipartite graph so that the number of perfect matchings (a.k.a. dimer tilings) is given by a Kasteleyn determinant is that the graph does not contain $K_{3,3}$ as a minor [3]. It is not hard to find an example showing that $\mathbb{Z}^3$ contains $K_{3,3}$ given only four lattice cubes.

Example 3. (Non-intersecting paths bijection.) Another way to compute the partition function for dimers in 2D is via the bijection with *non-intersecting paths* in $\mathbb{Z}^2$ by overlaying a tiling with a "brickwork tiling" where all tiles point in the same cardinal direction white-to-black. There is a formula to count 2D non-intersecting path patterns. There is an analogous bijection between dimer tilings of $\mathbb{Z}^3$ and non-intersecting paths in $\mathbb{Z}^3$, but no known formula to count 3D non-intersecting paths. The topology in three dimensions is much more complicated, and non-intersecting paths are not ordered, can be braided around each other, and so on.

Example 4. (Local move connectedness.) Any two dimer tilings of a simply connected region $D \subset \mathbb{Z}^2$ are connected by a finite sequence of flips, where two adjacent and parallel vertical tiles are swapped for horizontal ones or vice versa. Flip connectedness manifestly fails in 3D even regions as small as the $3 \times 3 \times 2$ box. This is also intimately related to the non-existence of Kasteleyn weights from the first example.

4. Main result

The main theorems stated below are a large deviation principle and the uniqueness of the limit shape in three dimensions. Examples of random tiling limit shapes can be seen in the simulations in Figure 2. We describe the set up and then state the theorems, i.e. specify the measures, topology (different from 2D without height functions), and describe the rate function (abstractly, as we do not have an explicit formula).

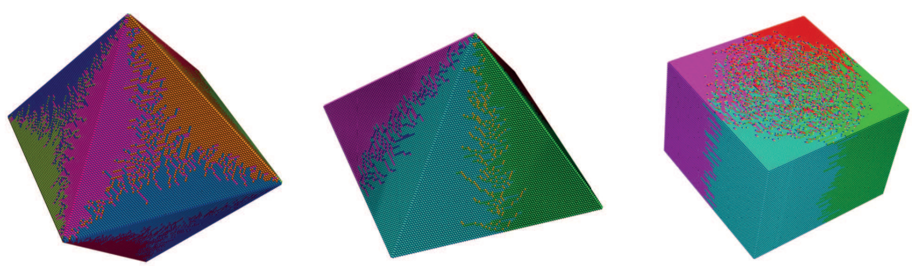

Figure 2. Simulations of limit shapes for 3D regions built by stacking 2D Aztec diamonds. See `https://github.com/catwolfram/3d-dimers` for videos of two-dimensional slices of these simulations as shown in the talk.

The topology we consider is the weak topology on the corresponding vector fields (equivalently, this is the induced topology of the Wasserstein metric defined component-wise). The scaling limits of f_τ are called asymptotic flows, which are measurable divergence-free vector fields valued in $\mathcal{O} = \{(s_1, s_2, s_3) : |s_1| + |s_2| + |s_3| \leq 1\}$.

Fix a compact region $R \subset \mathbb{R}^3$ and a boundary condition b of an asymptotic flow restricted to ∂R. Because of subtleties new to three dimensions, we consider two sequences of probability measures, one where a more general theorem holds and one more analogous to the set up in two dimensions.

- Hard boundary (HB): fix a sequence of regions $R_n \subset \frac{1}{n}\mathbb{Z}^3$ with boundary values b_n approximating b and let $\overline{\rho}_n$ be uniform measure on dimer tilings of R_n.
- Soft boundary (SB): choose a sequence of "thresholds" $(\theta_n)_{n\geq 0}$ with $\theta_n \to 0$ slowly enough and let ρ_n be uniform measure on free-boundary tilings of $R \cap \frac{1}{n}\mathbb{Z}^3$ with boundary values within θ_n of b.

Theorem 5 (Large deviation principles (Chandgotia, Sheffield, Wolfram [4])). *Fix a nice region $R \subset \mathbb{R}^3$ (compact, closure of a domain, ∂R piecewise smooth) and b a boundary value on ∂R.*

The measures ρ_n (resp. $\overline{\rho}_n$ with the mild additional condition that (R, b) is "flexible") satisfy a large deviation principle where the rate function I_b is, for g an asymptotic flow with boundary value b,

$$I_b(g) = C_b - Ent(g) := C_b - \frac{1}{Vol(R)} \int_R ent(g(x))\, \mathrm{d}x.$$

Here C_b is a constant and $ent(s) : \mathcal{O} \to [0, \infty)$ is defined to be

$$ent(s) = \max_{\mu \in \mathcal{P}^s} h(\mu), \tag{3}$$

where $h(\mu)$ is specific entropy and $\mathcal{P}^s$ is the set of probability measures on dimer tilings of $\mathbb{Z}^3$ which are invariant under even translations (i.e., translations that preserve the direction of flow) such that the μ-expected flow through the origin is $s \in \mathcal{O}$.

Theorem 6 (Limit shape (Chandgotia, Sheffield, Wolfram [4])). *For (R, b) "semi-flexible," the rate function I_b has a unique minimizer f with boundary value b. Combined with the large deviation principle, this implies that random tilings sampled from ρ_n (resp. $\overline{\rho}_n$ is (R, b) is flexible) concentrate on f exponentially fast as $n \to \infty$.*

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Winding probability for CLE and boundary CLEs

Pu Yu

The conformal loop ensemble (CLE_κ), as discovered by Sheffield [She09] and also with Werner [SW12], is a natural random collection of non-crossing planar loop. which is simple for $\kappa \in (8/3, 4]$ and non-simple for $\kappa \in (4, 8)$. For $\kappa \in (8/3, 4]$, the loops are simple and do not touch each other or the boundary; for $\kappa \in (4, 8)$, the loops are non-simple and may touch each other and the boundary. For $\kappa \in (2, 8)$, the boundary CLE (bCLE) is a collection of non-crossing boundary touching loops constructed via target invariant $\mathrm{SLE}_\kappa(\rho_-; \rho_+)$ by Miller-Sheffield-Werner [MSW17].

The first main result of this talk is to derive the exact probability that in a simply connected domain D and a given point z, the probability that the loop surrounding z in the non-simple CLE touches the boundary. In particular, using the conformal invariance property of CLE, we may assume D is the unit disk $\mathbb{D}$ and $z = 0$.

Theorem 1. *For $\kappa' \in (4, 8)$, let $\mathcal{L}$ be the loop in the (non-nested) $\mathrm{CLE}_{\kappa'}$ surrounding 0. Then we have*

$$\mathbb{P}[\mathcal{L} \cap \partial\mathbb{D} \neq \emptyset] = 1 - \frac{\sin(\pi(\frac{\kappa'}{4} + \frac{8}{\kappa'}))}{\sin(\pi\frac{\kappa'-4}{4})}. \tag{1}$$

We also derive the exact probability that z fall in to clockwise/counterclockwise (true/false) loops of the boundary CLE. Let $\mathrm{BCLE}_\kappa^{\circlearrowright}(\rho)$ be the boundary CLE with clockwise true loops defined in [MSW17].

Theorem 2. *For $\kappa \in (2, 4)$ and $\rho \in (-2, \kappa - 4)$, let $\{0 \in \mathrm{BCLE}_\kappa^{\circlearrowright}(\rho)\}$ (resp. $\{0 \notin \mathrm{BCLE}_\kappa^{\circlearrowright}(\rho)\}$) denote the event that the origin is surrounded by a clockwise true (resp. counterclockwise false) loop in $\mathrm{BCLE}_\kappa^{\circlearrowright}(\rho)$. Then we have*

$$\mathbb{P}[0 \in \mathrm{BCLE}_\kappa^{\circlearrowright}(\rho)] = \frac{\sin(\frac{2\pi}{\kappa}(\kappa - \rho - 4))\sin(\frac{\pi(4-\kappa)}{4\kappa}(\kappa - 2\rho - 4))}{\sin(\frac{\pi(4-\kappa)}{\kappa})\sin(\frac{\pi}{4}(\kappa - 2\rho - 4))}; \tag{2}$$

$$\mathbb{P}[0 \notin \mathrm{BCLE}_\kappa^{\circlearrowright}(\rho)] = \frac{\sin(\frac{2\pi}{\kappa}(\rho + 2))\sin(\frac{\pi(4-\kappa)}{4\kappa}(2\rho + 8 - \kappa))}{\sin(\frac{\pi(4-\kappa)}{\kappa})\sin(\frac{\pi}{4}(\kappa - 2\rho - 4))}. \tag{3}$$

Theorem 3. *For $\kappa' \in (4, 8)$ and $\rho' \in (\frac{\kappa'}{2} - 4, \frac{\kappa'}{2} - 2)$, let $\{0 \in \mathrm{BCLE}_{\kappa'}^{\circlearrowright}(\rho')\}$ (resp. $\{0 \notin \mathrm{BCLE}_{\kappa'}^{\circlearrowright}(\rho')\}$) denote the event that the origin is surrounded by a clockwise* *true (resp. counterclockwise false) loop in* $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\rho')$. *Then, we have*

(4)
$$\mathbb{P}[0 \in \mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\rho')] = \frac{\sin(\frac{2\pi}{\kappa'}(\kappa'-\rho'-4))\sin(\frac{\pi(\kappa'-4)}{4\kappa'}(\kappa'-2\rho'-4))}{\sin(\frac{\pi(\kappa'-4)}{\kappa'})\sin(\frac{\pi}{4}(\kappa'-2\rho'-4))};$$
$$\mathbb{P}[0 \notin \mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\rho')] = \frac{\sin(\frac{2\pi}{\kappa'}(\rho'+2))\sin(\frac{\pi(\kappa'-4)}{4\kappa'}(2\rho'+8-\kappa'))}{\sin(\frac{\pi(\kappa'-4)}{\kappa'})\sin(\frac{\pi}{4}(\kappa'-2\rho'-4))}.$$

In fact, for Theorems 1-3, we are going to prove the stronger result, which provides an exact formula for moments of the conformal radii seen from zero of the clockwise/counterclockwise loops.

Theorem 4. *For* $\kappa' \in (4,8)$, *let* $\mathcal{L}$ *be the loop in the (non-nested)* $\mathrm{CLE}_{\kappa'}$ *surrounding 0. Let* $\mathcal{D}_{\mathcal{L}}$ *be the connected component of* $\mathbb{D}\backslash\mathcal{L}$ *containing 0, and* $T = \{\mathcal{L}\cap\partial\mathbb{D}\neq\emptyset\}$. *We have:*

(1). *For* $\lambda \le \frac{\kappa'}{8}-1$, $\mathbb{E}[\mathrm{CR}(0,D_{\mathcal{L}})^{\lambda}\mathbb{1}_T]=\infty$, *and for* $\lambda > \frac{\kappa'}{8}-1$,

(5)
$$\mathbb{E}[\mathrm{CR}(0,D_{\mathcal{L}})^{\lambda}\mathbb{1}_T] = \frac{2\cos(\pi\frac{\kappa'-4}{\kappa'})\sin(\pi\frac{\kappa'-4}{4\kappa'}\sqrt{(\kappa-4)^2-8\kappa'\lambda})}{\sin(\frac{\pi}{4}\sqrt{(\kappa'-4)^2-8\kappa'\lambda})}.$$

(2). *For* $\lambda \le \frac{3\kappa'}{32}+\frac{2}{\kappa'}-1$, $\mathbb{E}[\mathrm{CR}(0,D_{\mathcal{L}})^{\lambda}\mathbb{1}_{T^c}]=\infty$, *and for* $\lambda > \frac{3\kappa'}{32}+\frac{2}{\kappa'}-1$,

(6)
$$\mathbb{E}[\mathrm{CR}(0,D_{\mathcal{L}})^{\lambda}\mathbb{1}_{T^c}] = \frac{\cos(\pi\frac{\kappa'-4}{\kappa'})\sin(\pi\frac{8-\kappa'}{4\kappa'}\sqrt{(\kappa'-4)^2-8\kappa'\lambda})}{\cos(\frac{\pi}{\kappa'}\sqrt{(\kappa'-4)^2-8\kappa'\lambda})\sin(\frac{\pi}{4}\sqrt{(\kappa-4)^2-8\kappa\lambda})}.$$

Theorem 5. *Fix* $\kappa \in (2,4)$ *and* $\rho \in (-2,\kappa-4)$. *Consider the* $\mathrm{BCLE}^{\circlearrowright}_{\kappa}(\rho)$ *boundary CLE in* $\mathbb{D}$. *Let* $\mathcal{L}$ *be the loop surrounding 0, and* $\mathcal{D}_{\mathcal{L}}$ *be the connected component of* $\mathbb{D}\backslash\mathcal{L}$ *containing 0. Let* $\lambda > \frac{\kappa}{8}-1$ *and* $\theta = \frac{\pi}{4}\sqrt{(4-\kappa)^2-8\kappa\lambda}$. *Then*

(7)
$$\mathbb{E}[\mathrm{CR}(0,D_{\mathcal{L}})^{\lambda}\mathbb{1}_{0\in\mathrm{BCLE}^{\circlearrowright}_{\kappa}(\rho)}] = \frac{\sin(\frac{\pi(4-\kappa)}{4})\sin(\frac{2\pi}{\kappa}(\kappa-\rho-4))}{\sin(\frac{\pi(4-\kappa)}{\kappa})\sin(\frac{\pi}{4}(\kappa-2\rho-4))}\cdot\frac{\sin(\frac{\kappa-2\rho-4}{\kappa}\theta)}{\sin(\theta)}.$$

(8)
$$\mathbb{E}[\mathrm{CR}(0,D_{\mathcal{L}})^{\lambda}\mathbb{1}_{0\notin\mathrm{BCLE}^{\circlearrowright}_{\kappa}(\rho)}] = \frac{\sin(\frac{\pi(4-\kappa)}{4})\sin(\frac{2\pi}{\kappa}(\rho+2))}{\sin(\frac{\pi(4-\kappa)}{\kappa})\sin(\frac{\pi}{4}(\kappa-2\rho-4))}\cdot\frac{\sin(\frac{2\rho+8-\kappa}{\kappa}\theta)}{\sin(\theta)}$$

Moreover, if $\lambda \le \frac{\kappa}{8}-1$, *then the left hand side of* (7) *and* (8) *are infinite.*

Theorem 6. *Fix* $\kappa' \in (4,8)$ *and* $\rho' \in (\frac{\kappa'}{2}-4,\frac{\kappa'}{2}-2)$. *Consider the* $\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\rho')$ *boundary CLE in* $\mathbb{D}$. *Let* $\mathcal{L}$ *be the loop surrounding 0, and* $\mathcal{D}_{\mathcal{L}}$ *be the connected component of* $\mathbb{D}\backslash\mathcal{L}$ *containing 0. Let* $\lambda' > \frac{\kappa'}{8}-1$ *and* $\theta' = \frac{\pi}{4}\sqrt{(4-\kappa')^2-8\kappa'\lambda'}$. *Then*

(9)
$$\mathbb{E}[\mathrm{CR}(0,D_{\mathcal{L}})^{\lambda'}\mathbb{1}_{0\in\mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\rho')}] = \frac{\sin(\frac{\pi(\kappa'-4)}{4})\sin(\frac{2\pi}{\kappa'}(\kappa'-\rho'-4))}{\sin(\frac{\pi(\kappa'-4)}{\kappa'})\sin(\frac{\pi}{4}(\kappa'-2\rho'-4))}\cdot\frac{\sin(\frac{\kappa'-2\rho'-4}{\kappa'}\theta')}{\sin(\theta')};$$

(10)

$$\mathbb{E}[\mathrm{CR}(0,D_{\mathcal{L}})^{\lambda'}\mathbb{1}_{0\notin \mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\rho')}] = \frac{\sin(\frac{\pi(\kappa'-4)}{4})\sin(\frac{2\pi}{\kappa'}(\rho'+2))}{\sin(\frac{\pi(\kappa'-4)}{\kappa'})\sin(\frac{\pi}{4}(\kappa'-2\rho'-4))}\cdot\frac{\sin(\frac{2\rho'+8-\kappa'}{\kappa'}\theta')}{\sin(\theta')}.$$

Moreover, if $\lambda' \leq \frac{\kappa'}{8}-1$*, then the left hand side of* (9) *and* (10) *are infinite.*

In [SSW09], the authors derived the moments of the conformal radii when $\mathcal{L}$ is the loop in CLE surrounding 0, and Theorems 4-6 are generalization of their result. Furthermore, it has been shown in [KSL22] that the fuzzy Potts model has close relation with the CLE percolations. Combining Theorem 5 and Theorem 6 with results from [MSW17, KSL22], we have the following result for the fuzzy Potts model one arm exponent.

Theorem 7. *Let* $q \in [1,4)$*, and suppose that the conformal invariance conjecture holds for the critical FK percolation with cluster weight* q*. Write* $\kappa = 4\arccos(-\sqrt{q}/2)/\pi \in [8/3,4)$*. For the fuzzy Potts model with red probability* r*, its blue (resp. red) bulk one-arm exponent* $\alpha_B(r)$ *(resp.* $\alpha_R(r)$*) is equal to* $\alpha_1(r)$ *(resp.* $\alpha_1(1-r)$*), where* $\alpha_1(r)$ *is given by the smallest positive solution in* $(0,1-2/\kappa)$ *to the equation*

$$\frac{\sin(\frac{\pi(\kappa+2\rho+8)}{4\kappa}\sqrt{(4-\kappa)^2+8\kappa x})}{\sin(\frac{\pi(\kappa-2\rho-8)}{4\kappa}\sqrt{(4-\kappa)^2+8\kappa x})} = \frac{\sin(\frac{\pi}{4}(\kappa+2\rho))}{\sin(\frac{\pi}{4}(\kappa-2\rho))}. \tag{11}$$

with $\rho = \frac{2}{\pi}\arctan\left(\frac{\sin(\pi\kappa/2)}{1+\cos(\pi\kappa/2)-1/(1-r)}\right) - 2 \in (-2,\kappa-4)$.

Finally, in [MSW17], the authors proved that the labeled $\mathrm{CLE}^{\beta}_{\kappa'}$ can be constructed using iterations of $\mathrm{BCLE}_\kappa(\rho)$, and the relation between β and ρ are derived in [MSW22, MSW21]. From the construction in [MSW17], one can show

$$\frac{1+\beta}{1-\beta} = \frac{\mathbb{P}[0\in \mathrm{BCLE}^{\circlearrowright}_{\kappa}(\rho)]\cdot\mathbb{P}[0\in \mathrm{BCLE}^{\circlearrowleft}_{\kappa'}(\rho'_R)]}{\mathbb{P}[0\notin \mathrm{BCLE}^{\circlearrowright}_{\kappa}(\rho)]\cdot\mathbb{P}[0\in \mathrm{BCLE}^{\circlearrowright}_{\kappa'}(\rho'_L)]}.$$

where $\rho'_R = -\frac{\kappa'}{2}-\frac{\kappa'}{4}\rho$, $\rho'_L = \kappa'-4+\frac{\kappa'}{4}\rho$. Therefore, by Theorems 2 and 3, $\frac{1+\beta}{1-\beta} = -\sin(\pi(\kappa-\rho)/2)/\sin(\pi\rho/2)$. This gives an alternative proof of [MSW21, Theorem 1.3]. [MSW22, Theorem 1.6] can be proved similarly.

Our proof is mainly based on couplings between SLE/CLE and Liouville quantum gravity (LQG), together with exact formulas from Liouville conformal field theory (LCFT). LQG is introduced by Polyakov in his seminal work [Pol81]. LCFT is a 2D quantum field theory rigorously developed in [DKRV16] and subsequent works. LCFT is closely related to LQG, as it has been demonstrated that many natural LQG surfaces can be described by LCFT [Cer21, AHS21, ASY22]. As observed by Sheffield [She16], one key aspect of random planar geometry is the *conformal welding* of random surfaces, where the interface under the conformal welding of two LQG surfaces is an SLE curve. Similar type of results were also proved in [DMS21, AHS23, ASY22].

The proof of Theorems 4-6 is another example of exact formula of SLE/CLE based on conformal welding of LQG surfaces and LCFT. In earlier works of

[MSW22, MSW21], the coupling between CLE and LQG was crucially used to derive properties of CLE. There the authors relied on the advanced exploration mechanisms for CLE percolations from [MSW17]. In contrast, we work directly with the classical construction of CLE in [She09] and boundary CLE in terms of the continuum exploration tree. Based on this construction, the boundary touching event along with the quantities in these theorems can be expressed in terms of radial $\mathrm{SLE}_\kappa(\rho;\kappa-6-\rho)$. In particular, we derive novel results on conformal welding of γ-LQG surfaces with radial $\mathrm{SLE}_\kappa(\rho;\kappa-6-\rho)$ being the interface. This allows us to express the quantities in Theorems 4-6 in terms of boundary lengths of LQG surfaces. The key LQG surfaces are the quantum triangles defined via LCFT in [ASY22], where we prove that by gluing two of its edges together, one gets a single disk decorated by independent radial SLE curve. Combining the structrual constants from LCFT will finish the proof.

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The chemical distance metric for non-simple CLE

Yizheng Yuan
(joint work with Valeria Ambrosio and Jason Miller)

In this talk, I explained what is the continuum analogue of the chemical distance metric in two-dimensional lattice models such as percolation. The chemical distance metric is the graph distance induced by the percolation clusters. Many authors have studied chemical distance in various models (mainly supercritical models where the metric is comparable to the Euclidean metric). The case of critical percolation, in contrast, is much more difficult. The critical model exhibits non-trivial scaling behaviour, and the exponent for the shortest length has been numerically estimated to have value approximately 1.13 [4]. However, the strongest rigorous results proved so far [1, 3] are that the exponent is strictly between its trivial bounds 1 and 4/3 (the latter is the dimension of $\mathrm{SLE}_{8/3}$ which is the inner boundary of a percolation cluster). Proving anything stronger for the chemical distance in critical percolation remains an open question. (This problem was asked by O. Schramm in his famous ICM 2006 article [7], and is one of the few questions that remain unsolved.)

In a joint work with Valeria Ambrosio and Jason Miller, we construct a chemical distance metric on the CLE gasket for each $\kappa \in]4, 8[$. Our metric should be the scaling limit of the chemical distance metric in lattice models that converge to CLE. In particular, we expect that the chemical distance metric in critical percolation converges to our CLE_6 metric. Moreover, our metric is the Euclidean analogue of the α-stable gasket constructed in [5, 2]. Our CLE metrics are determined by several natural properties that any notion of chemical distance on the gasket is expected to satisfy. Concretely, we show that our metric is uniquely characterised by being geodesic, Markovian, and conformally covariant.

In order to construct a CLE metric, we start with an approximation and show that the resulting spaces are tight in a suitable topology. Then we identify several natural properties that any subsequential limit satisfies, and show that they uniquely determine a metric on the CLE gasket. The characterisation and the proof outline are reminiscent to the LQG metric, however we emphasise that our objects behave very differently, and hence our proof techniques also differ significantly from those used in LQG. Two major challenges here are

1. The topology of the gasket forces paths to pass through intersections of CLE loops, and

2. Independence arguments are much more involved because the metric is sensitive to the shape of domain boundaries, and conditioning on regions strongly biases the shape of the remaining region.

In the regime $\kappa \in]8/3, 4[$, tightness of an approximation has been shown previously in [6]. We expect that our proof can be adapted to show uniqueness in that regime, too.

To state our theorems, we choose to work in a scale-invariant setup. We let $\mathcal{H}$ be the domain to one side of a two-sided whole-plane $\mathrm{SLE}_{16/\kappa}$, and let Γ a CLE_κ

in $\mathcal{H}$ where $\kappa \in]4,8[$. (The choice of a domain with $\mathrm{SLE}_{16/\kappa}$ boundary is natural considering that we want to have a metric that is defined on any region bounded between two CLE_κ loops.) We call a path in the gasket Υ_Γ of Γ admissible if it does not cross any loops of Γ.

We choose the following approximation. For an admissible path γ, let $\mathfrak{N}_\varepsilon(\gamma)$ denote the Lebesgue measure of the ε-neighbourhood of γ. We then let

$$\mathfrak{d}_\varepsilon(x,y;\Gamma) := \inf_{\substack{\gamma \text{ admissible} \\ x \to y}} \mathfrak{N}_\varepsilon(\gamma).$$

Theorem 1. *There exists a sequence* $\mathfrak{m}_\varepsilon$ *such that*

- *the laws of* $(\Upsilon_\Gamma, \mathfrak{m}_\varepsilon^{-1}\mathfrak{d}_\varepsilon(\cdot,\cdot;\Gamma))$ *are tight,*
- *any limit is non-trivial, geodesic, Markovian, and conformally covariant.*

We call such a metric a *$CLE_{\kappa'}$ metric*. Here, the metric being Markovian means that for any open set U, if we let $U^* \subseteq U$ be the points not inside any loop intersecting $\mathcal{H} \setminus U$, then the conditional law of the *internal metric* in $\overline{U^*}$ given U^* and the internal metric outside U is a translation-invariant function of U^*. Conformal covariance means that for any conformal map $\varphi\colon U \to \tilde{U}$, the lengths of $\varphi(\gamma)$ under $\mathfrak{d}^{\varphi(U^*)}(\cdot,\cdot;\varphi(\Gamma_{U^*}))$ are given by

$$\int |\varphi'|^\alpha \, dL_{\mathfrak{d}}(\gamma)$$

where $\alpha > 0$ is an exponent depending only on κ.

The (approximate) metrics $\mathfrak{m}_\varepsilon^{-1}\mathfrak{d}_\varepsilon$, $\mathfrak{d}$ are seen as Hölder-continuous functions on $\Upsilon_\Gamma \times \Upsilon_\Gamma$ with respect to the reference metric

$$d(x,y;\Gamma) = \inf_{\substack{\gamma \text{ admissible} \\ x \to y}} \operatorname{diam}(\gamma).$$

(To keep the report simple, I have not elaborated on the exact topology in which we consider our objects to lie in.)

Theorem 2. *Suppose* $\mathfrak{d}(\cdot,\cdot;\Gamma)$ *and* $\tilde{\mathfrak{d}}(\cdot,\cdot;\Gamma)$ *are $CLE_{\kappa'}$ metrics. Then (when both are coupled with* Γ*)*

$$\mathfrak{d} = c\tilde{\mathfrak{d}}$$

for some deterministic constant $c > 0$.

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Reporter: Ron Nissim

Participants

Dr. Amol Aggarwal
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Dr. Marie Albenque
IRIF, CNRS UMR 8243
Université Paris Cité
Case 7014
75205 Paris Cedex 13
FRANCE

Valeria Ambrosio
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Morris Ang
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Juhan Aru
Institute of Mathematics
EPFL
1015 Lausanne
SWITZERLAND

Prof. Dr. Nathanael Berestycki
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
AUSTRIA

Dr. Thomas Budzinski
ENS Lyon
UMPA
46, Allee d'Italie
69007 Lyon Cedex
FRANCE

Sky Cao
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Baptiste Cerclé
Institute of Mathematics
EPFL SB MATH
MA C2 647 (Bâtiment MA)
Station 8
1015 Lausanne
SWITZERLAND

Prof. Dr. Dmitry Chelkak
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Prof. Dr. Nicolas Curien
Institut de Mathématiques
Université Paris Saclay
Batiment 307
91405 Orsay
FRANCE

Cillian Doherty
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Prof. Dr. Alexander Drewitz
Department of Mathematics and
Computer Science
Universität zu Köln
Weyertal 86 - 90
50931 Köln
GERMANY

Prof. Dr. Christophe Garban
Institut Camille Jordan
Université de Lyon I
43 Blvd. du 11 Novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Colin Guillarmou
Département de Mathématiques
Faculté des Sciences d'Orsay
Université Paris-Saclay
91405 Orsay Cedex
FRANCE

Prof. Dr. Markus Heydenreich
Institut für Mathematik
Universität Augsburg
86135 Augsburg
GERMANY

Emmanuel Kammerer
Centre de Mathématiques
École Polytechnique
91128 Palaiseau Cedex
FRANCE

Dr. Konstantinos Kavvadias
Department of Mathematics
Massachusetts Institute of
Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Prof. Dr. Antti Kupiainen
Department of Mathematics and
Statistics
University of Helsinki
P.O. Box 68
00014 Helsinki
FINLAND

Dr. Piet Lammers
Laboratoire de Probabilites-Tour 56
Université P. et M. Curie
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Wenxuan Li
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge, MA 02138-2901
UNITED STATES

Sid Maibach
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Franz Merkl
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstraße 39
80333 München
GERMANY

Prof. Dr. Grégory Miermont
Unité de mathématiques pures et
appliquées
École normale supérieure de Lyon
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Jason P. Miller
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Ron Nissim
Department of Mathematics
Massachusetts Institute of Technology
143 Albany St. Apt. 412
Cambridge, MA 02139
UNITED STATES

Dr. Malin Palö Forsström
Department of Mathematics
Chalmers University of Technology
412 96 Göteborg
SWEDEN

Dr. Minjae Park
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Prof. Dr. Ron Peled
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv, Tel Aviv 69978
ISRAEL

Prof. Dr. Eveliina Peltola
Department of Mathematics
UCLA
405 Hilgard Ave.
Los Angeles, CA 90095-1555
UNITED STATES

Prof. Dr. Rémi Rhodes
Centre de Mathématiques et
d'Informatique
Université de Provence
39, Rue Joliot-Curie
13453 Marseille Cedex 13
FRANCE

Prof. Dr. Pierre-Francois Rodriguez
Imperial College London
Department of Mathematics
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Silke Rolles
Technische Universität München
School of Computation, Information and
Technology
Department of Mathematics
Boltzmannstraße 3
85748 Garching bei München
GERMANY

Dr. Avelio Sepulveda
Departamento de Ingeniería
Matemáticas
Universidad de Chile
Beaucheff 851
Estación Central Santiago 5555
CHILE

Prof. Dr. Scott Sheffield
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139
UNITED STATES

Prof. Dr. Hao Shen
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison, WI 53706-1388
UNITED STATES

Prof. Dr. Theo Sturm
Institut für angewandte Mathematik (IaM)
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Xin Sun
Beijing International Center for Mathematical Research (BICMR)
Beijing University
Beijing 100871
CHINA

Yi Tian
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Diederik van Engelenburg
Institut Camille Jordan
Université de Lyon I
43 Blvd. du 11 Novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Fredrik Viklund
Department of Mathematics
KTH
10044 Stockholm
SWEDEN

Prof. Dr. Yilin Wang
I.H.E.S.
Institut des Hautes Etudes Scientifiques
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Prof. Dr. Wendelin Werner
Center of Mathematical Sciences
University of Cambridge
Wilberforce Road
CB3 0WA Cambridge
UNITED KINGDOM

Catherine Wolfram
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Pu Yu
Department of Mathematics
Massachusetts Institute of Technology
235 Albany St, Apt 2021
Cambridge, MA 02139
UNITED STATES

Dr. Yizheng Yuan
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM