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Arrangements, Matroids and Logarithmic Vector Fields

Organized by
Takuro Abe, Tokyo
Graham Denham, London CA
Eva Maria Feichtner, Bremen
Gerhard Röhrle, Bochum

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Abstract. The focus of this workshop was on the ongoing interaction between geometric aspects of matroid theory with various directions in the study of hyperplane arrangements. A hyperplane arrangement is exactly a linear realization of a (loop-free, simple) matroid. While a matroid is a purely combinatorial object, though, an arrangement is associated with a range of algebraic and geometric constructions that connect closely with the combinatorics of matroids.

The meeting brought together researchers involved with complementary angles on the subject, many of whom had not met before, so an important underlying objective was to make introductions between groups with overlapping interests in order to facilitate new collaborations and advances in the subject.

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Introduction by the Organizers

The workshop "Arrangements, matroids and logarithmic vector fields" brought together a wide range of participants. Geographically, they came from Germany, Japan, Switzerland, France, Belgium, Spain, Poland, Sweden, Canada, the USA, the UK, Ireland, Israel, and Italy. They ranged over all career stages, from graduate students to senior faculty.

The central topics involved algebraic and geometric methods in the study of hyperplane arrangements and matroids. The former are (usually) finite sets of linear hyperplanes in affine space. Their linear dependencies carry the structure of a (realizable) matroid. Algebraic varieties constructed from hyperplane arrangements appear naturally in various contexts and have a considerable history in which singularity theory, reflection groups and configuration spaces figure prominently.

Perhaps the first such variety, historically, is the complement of a hyperplane arrangement in complex affine or projective space. The complement has nontrivial topology: its fundamental group generalizes the pure braid group and for important classes of examples, the higher homotopy groups vanish. Different arrangements that realize the same matroid may have homotopically inequivalent complements. However, the cohomology algebra of the complement was computed by Brieskorn, Orlik and Solomon [3, 6] and, in particular, depends only on the combinatorics of the matroid. From a contemporary point of view, the algebra they describe is a functorial invariant of a matroid that is defined regardless of whether or not there exists a hyperplane arrangement that realizes it.

With their wonderful compactifications, De Concini–Procesi [4] gave families of smooth varieties for which the hyperplane arrangement complement sits as the complement of a simple normal crossings divisor. The combinatorial nature of their construction led Feichtner and Yuzvinsky [5] to obtain a presentation of the Chow ring of the wonderful compactification. They showed that it depended only on combinatorics associated with the matroid. Adiprasito–Huh–Katz [1] later used this "Chow ring of a matroid" to great effect to prove the Rota–Heron–Welsh conjecture.

More recently and along the same lines, another arrangement construction, the matroid Schubert variety, was shown by Braden et al. [2] to give another matroid invariant, in this case its intersection cohomology. This was a key part in their celebrated proof of the top-heaviness conjecture, and gave an interpretation of the mysterious Kazhdan–Lusztig polynomial of a matroid.

To some extent, the workshop had its precursor in January 2021 with the workshop "Logarithmic Vector Fields and Freeness of Divisors and Arrangements: New perspectives and applications," which was held in online format due to Covid-19. It was wonderful and much rewarding to be able to bring people together now for an in-person meeting, complementing the group of participants with numerous early career scientists whose work has unfolded in the past 2 or 3 years. Notably, the study of geometry of matroids has expanded considerably since 2021, which was beautifully outlined in a number of talks.

One aim of the workshop, still maintained, was to bring together participants from slightly different mathematical communities within the subject involved with the themes above, have them share current developments and provide a platform for interaction. With this in mind, the program of talks was organized into loose themes:

The first day was devoted to topological aspects, with talks related to the $K(\pi,1)$ problem for arrangement complements and their generalizations by Delucchi, Yoshinaga, Jiang, Mücksch and Kyoji Saito. Suciu's subsequent talk on decomposable arrangments addressed the extent to which combinatorics predicts graded approximations to the fundamental group.

The second day was more combinatorial, with morning talks related to Chow rings of matroids by Ferroni, Nathanson and Larson. Kühne and Dinu spoke in the afternoon, the former highlighting some advances in computation that considerably extend our range to test conjectures about families of arrangements that realize the same matroid.

Talks by Mühlherr, Feigin and Tran presented new developments related to logarithmic derivations associated with arrangements derived from reflection arrangements. In a related spirit, Cuntz reported on recent work showing that restrictions of Weyl arrangements are precisely the generalized root systems of Dimitrov and Fioresi.

The combinatorial theme continued with talks related to matroids and Kazhdan–Lusztig–Stanley theory by Matherne and Coron, as well as an introduction to Berget–Eur–Spink–Tseng's notion of tautological bundles for matroids by Eur. Shiyue Li talked about combinatorics of Chow rings of moduli spaces of curves admitting an action by certain complex reflection groups.

The clustering approach helped form a coherent picture of latest developments, allowing for expository styles and thus making the material accessible also for people in the audience with different backgrounds and viewpoints. The discussions that evolved during the week, and away from the lecture hall, approved this rationale. The mix of participants led to some active discussions, informal presentations, and evening working groups. We are pleased to note that some new collaborations were forged at the meeting. At the time of writing, a new preprint by a participant has already appeared that credits this event for its origin: see [7].

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Arrangements, Matroids and Logarithmic Vector Fields

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Abstracts

Bernstein–Sato polynomials of hyperplane arrangements in $\mathbb{C}^3$

Daniel Bath

Given a reduced polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ (or analytic germ) the *Bernstein–Sato polynomial* $b_f(s) \in \mathbb{C}[s]$ is an invariant that miraculously captures almost all other singularity invariants of the hypersurface $\mathrm{V}(f) \subseteq \mathbb{C}^n$. For example, the roots $Z(b_f(s)) \subseteq \mathbb{C}$ of the Bernstein–Sato polynomial: contain the jumping numbers of f; determine the log canonical threshold of f; determine the minimal exponent of f (which is a refinement of the log canonical threshold); contain the Hodge spectrum; give, after exponentiation, the eigenvalues of the algebraic monodromy of the Milnor fiber; their multiplicities inform the number of terms in the weight filtration of $D_X f^{\alpha - k}$ (for $k \gg 0$), etc. Moreover, $Z(b_f(s)) \subseteq \mathbb{Q}_{<0} \cap (-n, 0)$, the Bernstein–Sato polynomial is always divisible by $s+1$, and f is smooth if and only if $b_f(s) = s + 1$. We refer to [5] for a survey of basic properties of Bernstein–Sato polynomials.

The major drawback of Bernstein–Sato polynomials is that they are basically impossible to compute. The relative impossibility is true computationally and theoretically: computer algebra systems exist for the task, but mostly fail to terminate; very few theorems exist computing Bernstein–Sato polynomials for families of hypersurfaces.

There is one classical family where $Z(b_f(s))$ is known. We say f is *positively weighted homogeneous* if there exists positive weights $\mathbf{w} = \{w_1, \ldots, w_n\}$ such that the derivation $E = \sum_{1 \leqslant i \leqslant n} w_i x_i \partial_{x_i}$ acts as $E \bullet f = (\text{non-zero scalar}) f$. In this case, the weights induce a non-standard, but positive, grading on $\mathbb{C}[x_1, \ldots, x_n]$ given by $\deg_{\mathbf{w}}(x_i) = w_i$, and $E \bullet f = \deg_{\mathbf{w}}(f) f$. Then we have the following result, attributed to too many independent actors to attribute easily: if f has an isolated singularity (at 0) and is positively weighted homogeneous, then

$$Z(b_f(s)) = \{-1\} \cup \{\bigcup_t \frac{-(t + \sum w_i)}{\deg_{\mathbf{w}}(f)} \mid [R/\partial f]_t \neq 0\}.$$

Here: $R = \mathbb{C}[x_1, \ldots, x_n]$, the *Jacobian ideal* $\partial f \subseteq R$ is the ideal generated by the partial derivatives $(\partial_{x_1} \bullet f, \ldots, \partial_{x_n} \bullet f)$; and $[R/\partial f]_t$ denotes the homogeneous degree t elements of $R/\partial f$ with respect to the aforementioned $\mathbf{w}$-grading.

Outside of case of positively weighted homogeneous isolated singularities, not much is known. For example, for hypersurfaces in $\mathbb{C}^2$ that are not positively weighted homogeneous, the current belief is that no similarly conclusive formula exists. Given this hopelessness, if one wants to find the *next* class of polynomials where there is hope for a "nice" formula for $Z(b_f(s))$, one arrives at the following restrictions: $f \in R = \mathbb{C}[x_1, x_2, x_3]$ is *positively weighted homogeneous locally everywhere*. Positively weighted homogeneous locally everywhere means that for all $\mathfrak{x} \in V(f)$, there exists an analytic local coordinate system so that the hypersurfaces can be, locally at $\mathfrak{x}$, defined by a positively weighted homogeneous polynomial. This class includes affine cones $C(Z) \subseteq \mathbb{C}^3$ of hypersurfaces $Z \subseteq \mathbb{P}^2$ whose only singularities are isolated positively weighted homogeneous.

This class also includes hyperplane arrangements (unions of hyperplanes) in $\mathbb{C}^3$. Since every hyperplane arrangement carries the numbers $Z(b_f(s))$ one is compelled to ask the canonical hyperplane arrangement question: are the zeroes of the Bernstein–Sato polynomial of a hyperplane arrangement combinatorially determined, i.e. are the a function of the intersection lattice? Walther [4] gave the answer "no" and showed Ziegler's pair of arrangements demonstrate the phenomenon. These are two hyperplane arrangements of degree 9 with the same intersection lattice which differ by the property of whether or not their six triple points lie on a quadric. In the special case where the triple points lie on a quadric, $-2 + \frac{2}{9}$ is a root of the Bernstein–Sato polynomial; in the generic case where the triple points do not lie on a quadric, $-2 + \frac{2}{9}$ is not a root. In fact, this is the *only* difference between the Bernstein–Sato polynomials of the two arrangements.

For the sake of this talk, our main result is the following:

Theorem 1 [1]: Let $f \in R = \mathbb{C}[x_1, x_2, x_3]$ define a central, essential, and indecomposable hyperplane arrangement. Let $Z \subseteq \mathbb{P}^2$ be its projectivization and consider the combinatorially determined numbers

$$\mathrm{CombR} = \left[\bigcup_{3 \leqslant k \leqslant 2\deg(f)-3} \frac{-k}{\deg(f)} \right] \cup \left[\bigcup_{z \in \mathrm{Sing}(Z)} \bigcup_{2 \leqslant j \leqslant 2m_z - 2} \frac{-j}{m_z} \right],$$

where m_z is the number of hyperplanes of Z containing z. Then

$$Z(b_f(s)) = \mathrm{CombR} \quad \text{OR} \quad Z(b_f(s)) = \mathrm{CombR} \cup \{-2 + \frac{2}{\deg(f)}\}.$$

Moreover, the following are equivalent:

(1) $-2 + \frac{2}{\deg(f)} \in Z(b_f(s))$;
(2) $[H^0_{\mathfrak{m}} R/\partial f]_{\deg(f)-1} \neq 0$;
(3) $\mathrm{reg} R/\partial f = 2\deg(f) - 5$;
(4) the hyperplane arrangement $V(f)$ is not formal.

In particular, the non-combinatorial phenomenon of the roots of the Bernstein–Sato polynomial exhibited by Ziegler's pair is the only pathology possible. To explain notation, for a R-module N, the *zeroeth local cohomology with respect to* $\mathfrak{m} = (x_1, \ldots, x_n)$ is $H^0_{\mathfrak{m}} N = \{n \in N \mid \mathfrak{m}^k n = 0 \text{ for } k \gg 0\}$; the i^{th} local cohomology $H^i_{\mathfrak{m}}$ is the i^{th}-comology of the right derived functor $H^0_{\mathfrak{m}}$. Regularity of a graded R-module N refers to Castelnuovo–Mumford regularity and equals $\max_t\{(\mathrm{maxdeg} H^t_{\mathfrak{m}} N) + t\}$. And formality is a well-studied hyperplane arrangement property we will not define for brevity's sake.

Let us briefly describe the ideas involved in **Theorem 1**'s proof. The Bernstein–Sato polynomial is defined as the monic minimal nonzero polynomial $b_f(s) \in \mathbb{C}[s]$ satisfying the functional equation

$$b_f(s) f^s \in D_X[s] f^{s+1}$$

where $X = \mathbb{C}^n$, D_X is the sheaf of $\mathbb{C}$-linear differential operators on X, $D_X[s]$ is a polynomial ring extension in a new indeterminant s, and D_X acts on f^{s+k} by formal application of the chain rule. In other words, $\mathbb{C}[s] \cdot b_f(s)$ equals the $\mathbb{C}[s]$-annihilator of $D_X[s]f^s/D_X[s]f^{s+1}$.

It can be proved that

$$Z(\operatorname{ann}_{\mathbb{C}[s]}(D_X[s]f^s/D_X[s]f^{s+1}) \\ = Z(\operatorname{ann}_{\mathbb{C}[s]}\operatorname{Ext}^{n+1}_{D_X[s]}(D_X[s]f^s/D_X[s]f^{s+1}, D_X[s]).$$

In [2] we used this fact to compute $Z(b_f(s))$ when f is a *free* hyperplane arrangement. Freeness helps because due to a result of Narváez Macarro [3], the above Ext-module is isomorphic $D_X[s]f^{-s-2}/D_X[s]f^{-s-1}$.

We have invented a complex of $D_X[s]$-modules that resolves (in the sense it is quasi-isomorphic to its terminal cohomology module) $D_X[s]f^s$ in many cases. This includes the case of positively weighted homogeneous divisors in $\mathbb{C}^3$. We can use this complex to compute the above Ext-module, affording a great deal of control over $Z(b_f(s))$. In particular, we find that non-vanishing degrees of $H^0_{\mathfrak{m}}(R/\partial f)$ give roots of the Bernstein–Sato polynomial and, after removing these roots, the leftovers are symmetric about -1.

None of this requires f being a hyperplane arrangement. But when f defines a hyperplane arrangement in $\mathbb{C}^3$, by earlier results of ours [2], $Z(b_f(s)) \cap [-1, 0)$ *is* combinatorially determined. Combining this with the data arising from the Ext-computation yields (with much labor) the proof of **Theorem 1**.

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Symmetries of the face monoid of the braid arrangement and extensions to left regular bands

Patricia Commins

(joint work with Benjamin Steinberg)

1. Introduction

The presentation corresponding to this abstract had two themes:

(1) The *face monoid* of the braid arrangement and its interactions with *Solomon's descent algebra* and the symmetric group, and

(2) Extensions to *left regular band* semigroups carrying symmetries.

Part (1) was based on a recent preprint [6] of the speaker and will be summarized in section 2. Perhaps surprisingly, the key tool [6] used to understand the interplay between the various algebraic structures was the poset topology of the *intersection lattice of the braid arrangement*. Hence, like many of the topics discussed at the workshop, the desired algebraic information was a *combinatorial property* of the braid arrangement, meaning it could be determined solely from its intersection semilattice.

Part (2) was based on the ongoing thesis work of the speaker, part of which is joint work with Benjamin Steinberg, and will be briefly summarized in section 3. The face monoid of the brain arrangement is a canonical example of an important type of semigroup called a left regular band (LRB) and its intersection lattice has an important semigroup theoretic meaning in this setting. Many popular combinatorial and geometric objects have associated LRB structures and thus intersection semilattice analogues. We investigate what the "intersection semilattices" can say about an LRB semigroup algebra under symmetry.

2. The face monoid of the braid arrangement

The faces of the braid arrangement $\mathscr{A}_{n-1}$ (or more generally, any real central hyperplane arrangement) form a monoid $\mathscr{F}_n$ via a product structure defined by Tits in [13]. The associated monoid algebra – the face algebra $\mathbb{C}\mathscr{F}_n$– is well-studied. It was first popularized for its connections to Markov chains (see [2]), and has since been studied extensively as an algebra whose representation theory is governed by the combinatorics of the arrangement (see for example [1, 3, 11]. The symmetric group S_n acts by algebra automorphisms on the face algebra.

In [12], Solomon proved the $\mathbb{C}$-span of the sums of permutations with the same descent set is closed under multiplication. Hence, this space is actually a *subalgebra* of the symmetric group algebra $\mathbb{C}S_n$ called Solomon's descent algebra and written Σ_n. The representation theory of Σ_n has been studied extensively, but as a rich, nonsemisimple algebra, many mysteries remain. In [3], Bidigare explains that Σ_n and $\mathbb{C}\mathscr{F}_n$ are initimately linked via the symmetric group action on $\mathbb{C}\mathscr{F}_n$:

Theorem 1 (Bidigare). *The S_n-invariant subalgebra of the face algebra $\mathbb{C}\mathscr{F}_n$ is antisomorphic to the descent algebra Σ_n.*

Inspired by classical invariant theory, Theorem 1 provides an opportunity to study new representations of Σ_n. Let G be a finite group acting on a finite dimensional $\mathbb{C}$-algebra A. Invariant theory studies the G-invariant subalgebra A^G as well as certain generalizations. For instance, A decomposes into a direct sum of its *G-isotypic subspaces* A^χ, one for each irreducible character χ of G. One standard question is to study each isotypic subspace A^χ as a module over A^G.

Bidigare's Theorem implies each S_n-isotypic subspace of $\mathbb{C}\mathscr{F}_n$ is a right module over Σ_n. In addition to studying the *trivial* isotypic subspace $(\mathbb{C}\mathscr{F}_n)^{S_n}$, Bidigare also studied the *sign* isotypic subspace. In [6], the speaker studies the following:

Question 2. What is the structure of each S_n-isotypic subspace of $\mathbb{C}\mathscr{F}_n$ as a right module over Σ_n?

We conclude this section with a very informal summary of the answer [6] gives to Question 2. The irreducible representations of both S_n and Σ_n are indexed by integer partitions of n; for partitions ν and λ of n, let χ^ν and M_λ denote the irreducible representations of S_n and Σ_n respectively.

Theorem 3. *The χ^ν-isotypic subspace of the face algebra $\mathbb{C}\mathscr{F}_n$ decomposes into Σ_n-submodules N^ν_μ labelled by integer partitions μ of n :*

$$(\mathbb{C}\mathscr{F}_n)^{\chi^\nu} = \bigoplus_{\mu \vdash n} N^\nu_\mu.$$

The $\mathbb{C}$-dimensions of the N^ν_μ have simple combinatorial formulas (see [6, Proposition 3.3]*) and the composition multiplicity of the Σ_n-simple M_λ within N^ν_μ is*

$$\dim(\chi^\nu) \cdot \langle \chi^\nu, F_{\lambda,\mu} \rangle,$$

where the $F_{\lambda,\mu}$'s are certain symmetric group representations whose Frobenius images can be described with an elegant generating function (see [6, Theorem 5.39]*).*

The composition multiplicities in the special cases $\nu = n$ and $\mu = 1^n$ recover work of Garsia–Reutenauer [7] on the Cartan invariants of the descent algebra and Uyemura-Reyes [14] on certain shuffling representations, respectively. The proof relies on interpreting $F_{\lambda,\mu}$ as (twisted) representations on the homology of intervals in the intersection lattice of $\mathscr{A}_{n-1}$.

3. Extensions to left regular bands

An LRB is a finite semigroup B for which (i) $x^2 = x$ and (ii) $xyx = xy$ for all $x, y \in B$. Seminal work of Brown in [5] generalized the connections between hyperplane face semigroups and Markov chains in [2] to all LRBs. There are LRBs associated to many beloved combinatorial and geometric objects including matroids, oriented matroids, complex hyperplane arrangements, CAT(0)-cube complexes, CAT(0)-zonotopal complexes, flags of $\mathbb{F}_q^n$, and more.

Each LRB has two associated posets coming from semigroup theory. These posets encode important algebraic information about the LRB semigroup algebra $\mathbb{C}B$ including whether $\mathbb{C}B$ has an identity element, the structure of the quiver of $\mathbb{C}B$, and projective resolutions of the simple $\mathbb{C}B$-modules (see [8, 9, 10]). One of the associated posets, which we write as $\Lambda(B)$, plays the role of the intersection semilattice of hyperplane arrangements. As an example, when B is the LRB associated to a matroid, $\Lambda(B)$ is the lattice of flats of the matroid.

Many LRBs in the literature carry natural group actions. For such an LRB B and group G, it is natural to consider the structure of the invariant subalgebra $(\mathbb{C}B)^G$, and more generally as in section 2, the simultaneous action of G and $(\mathbb{C}B)^G$ on $\mathbb{C}B$. See [4] which studies these questions for the *free left regular band* and a q-analogue. Studying these questions for general LRBs is ongoing thesis work of the speaker, joint with Benjamin Steinberg.

We quickly state two directions of this ongoing work as a sample. First, whether the invariant subalgebra $(\mathbb{C}B)^G$ is semisimple is based on the G-orbits of $\Lambda(B)$. Additionally, there is a large class of LRBs studied in [9] called *CW LRBs* which have analogues of the zonotopes associated to real, central hyperplane arrangements. For B a CW LRB, we explain the actions of G and $(\mathbb{C}B)^G$ on $\mathbb{C}B$ in terms of *twisted* G-representations on the cohomology of intervals in $\Lambda(B)$.

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Operads and Kazhdan–Lusztig–Stanley theory

Basile Coron

The theory of Kazhdan–Lusztig–Stanley polynomials was introduced by Stanley [5] in an attempt to unify the story of Kazhdan–Lusztig polynomials associated to Coxeter groups (Kazhdan-Lusztig [4]) and the story of g-polynomials associated to polytopes (Stanley [3]) from a purely combinatorial standpoint. This framework would later be seen to encompass similar "Kazhdan–Lusztig–like" polynomials associated to other combinatorial objects such as geometric lattices (Elias-Proudfoot-Wakefield [2]). In this latter case the definition of the Kazhdan-Lusztig-Stanley polynomial of a geometric lattice $\mathscr{L}$ revolves around the so-called characteristic polynomial associated to $\mathscr{L}$, which is the Poincaré series of the Orlik–Solomon algebra of $\mathscr{L}$ up to an alternating sign. If $\mathscr{L}$ is realised by a complex hyperplane arrangement then the Orlik–Solomon algebra of $\mathscr{L}$ is isomorphic to the cohomology of the complement of the hyperplane arrangement. In the particular case of the braid arrangement the arrangement complements are known to carry an interesting algebraic structure called an operad, which passes to the cohomology. A first objective of this work is to extend this structure to all geometric lattices. We call the extended structure a $\mathscr{GL}$-operad, which is axiomatised as follow.

Definition 1. A *$\mathscr{GL}$-operad O* is a collection $\{O(\mathscr{L}), \mathscr{L}\,\text{geometric lattice}\}$ of vector spaces over a given field, together with morphisms

$$\mu_{G,\mathscr{L}} : O([\hat{0}, G]) \otimes O([G, \hat{1}]) \to O(\mathscr{L}) \tag{1}$$

for any element G in a geometric lattice $\mathscr{L}$, such that for any pair $G_1 < G_2$ of elements in some geometric lattice $\mathscr{L}$ we have the equality

$$\mu_{G_2,\mathscr{L}} \circ (\mu_{G_1,[\hat{0},G_2]} \otimes \mathrm{Id}) = \mu_{G_1,\mathscr{L}} \circ (\mathrm{Id} \otimes \mu_{G_2,[G_1,\hat{1}]}). \tag{2}$$

One can develop the ground theory of $\mathscr{GL}$-operads, drawing inspiration from the theory of associative algebras. In particular, one can define the notions of a presentation of a $\mathscr{GL}$-operad, Grobner bases for $\mathscr{GL}$-operads and finally Koszulness of $\mathscr{GL}$-operad via Koszul complexes or bar constructions. We then show that the $\mathscr{GL}$-operad of Orlik–Solomon algebras is Koszul. Finally we show how one can find a subcomplex of the bar construction of the latter operad which categorifies the Kazhdan–Lusztig–Stanley polynomials of geometric lattices. The latter Koszulness result can then be used to show that this subcomplex has homology concentrated in one degree. As an immediate consequence we retrieve the celebrated result that the coefficients of the KLS polynomials of geometric lattices are positive (Braden-Huh-Matherne-Proudfoot-Wang [1]).

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Root systems for restrictions of Weyl arrangements

Michael Cuntz
(joint work with Bernhard Mühlherr)

Dimitrov and Fioresi introduced an object that they call a generalized root system. This is a finite set of vectors in a Euclidean space satisfying certain compatibilities between angles and sums and differences of elements. They conjecture that every generalized root system is equivalent to one associated to a restriction of a Weyl arrangement. In this talk we prove the conjecture and provide a complete classification of generalized root systems up to equivalence.

1. Generalized root systems

Definition 1 (Dimitrov, Fioresi [4]). Let $(V, (\cdot,\cdot))$ be a finite dimensional euclidean vector space, $\varnothing \neq R \subseteq V$ a finite subset. The pair (R, V) is called a *generalized root system* (GRS) if $V = \langle R \rangle$ and for all $\alpha, \beta \in R$:

$$\begin{aligned} (\alpha,\beta) < 0 &\implies \alpha+\beta \in R, \\ (\alpha,\beta) > 0 &\implies \alpha-\beta \in R, \\ (\alpha,\beta) = 0 &\implies (\alpha+\beta \in R \iff \alpha-\beta \in R). \end{aligned}$$

$\alpha \in R$ is called a *root*, the *rank* of (R, V) is the dimension of V.

Lemma 2. *Let (R, V) be a GRS. Then the following hold.*

(1) $R = -R$.
(2) $\forall 0 \neq \alpha \in R\ \exists \beta \in R,\ k \in \mathbb{N}$: $\quad \mathbb{R}\alpha \cap R = \{j\beta \mid j \in \mathbb{Z}, -k \leqslant j \leqslant k\}$.
β is called primitive*, k is the* multiplier *of β.*

Example 3. (i) If $\mathscr{A}$ is a Weyl arrangement with root system R, then $R \cup \{0\}$ is a GRS.
(ii) Let $B \subseteq \Delta$ be a subset of a simple system, $X := \langle B \rangle \leqslant V$, and $\pi : V \to V/X$ the projection. Then $\pi(R \cup \{0\})$ is a GRS. The corresponding arrangement is the restriction $\mathscr{A}^{X^\perp}$. Dimitrov and Fioresi call $\pi(R \cup \{0\})$ a *quotient of a root system.*

Problem: Classify all GRS.

2. Crystallographic arrangements

Definition 4 (C. [3]). Let $\mathscr{A}$ be a simplicial arrangement in V, $\mathscr{R} \subseteq V^*$ a finite set such that $\mathscr{A} = \{\ker \alpha \mid \alpha \in \mathscr{R}\}$, and $\mathbb{R}\alpha \cap \mathscr{R} = \{\pm\alpha\}$ for all $\alpha \in \mathscr{R}$. We call $(\mathscr{A}, V, \mathscr{R})$ a *crystallographic arrangement* if for all chambers $K \in \mathscr{K}(\mathscr{A})$:

$$\mathscr{R} \subseteq \sum_{\alpha \in B^K} \mathbb{Z}\alpha, \tag{2.1}$$

where

$$B^K = \{\alpha \in \mathscr{R} \mid \forall x \in K\ :\ \alpha(x) \geqslant 0,\ \langle \ker \alpha \cap \overline{K} \rangle = \ker \alpha\}$$

corresponds to the set of walls of K.

Proposition 5. *Let* (R, V) *be a GRS and* $\mathscr{A} := \{\alpha^\perp \mid \alpha \in R\}$*. We identify* V *with* V^* *via* $(\cdot,\cdot)$*. Then* $(\mathscr{A}, V, \mathscr{W}(R))$ *is a crystallographic arrangement where*

$$\mathscr{W}(R) := \{\alpha \in R \mid 0 \neq \alpha \text{ is primitive}\}$$

is the reduced root set *of* (R, V)*.*

Theorem 6 (C.-Heckenberger [1])**.** *There are (up to equivalence) exactly three families of irreducible crystallographic arrangements of rank at least* 2*:*

(1) The family of rank two parametrized by triangulations of convex n*-gons by non-intersecting diagonals.*
(2) For each rank $r > 2$*, arrangements of type* A_r*,* B_r*,* C_r *and* D_r*, and a further series of* $r-1$ *arrangements.*
(3) Another 74 *"sporadic" arrangements of rank* r*,* $3 \leqslant r \leqslant 8$*.*

3. Classification

Strategy: Consider each of the three families.

3.1. **Rank two.** This was already performed in [4].

3.2. **Series.** Let $n \geqslant 3$, $V = \mathbb{R}^n$, $B = (b_1, \ldots, b_n)$ a basis of V.

$$\begin{aligned} A_{n-1} &:= \{b_i - b_j \mid 1 \leqslant i \neq j \leqslant n\}, \\ D_n &:= \{\varepsilon b_i + \varepsilon' b_j \mid 1 \leqslant i \neq j \leqslant n,\ \varepsilon, \varepsilon' \in \{1, -1\}\}, \\ B_n &:= D_n \cup \{\varepsilon b_i \mid 1 \leqslant i \leqslant n,\ \varepsilon \in \{1, -1\}\} \end{aligned}$$

and for $J \subseteq \{1, \ldots, n\}$ we put

$$\begin{aligned} X_J &:= \{2\varepsilon b_j \mid j \in J,\ \varepsilon \in \{1, -1\}\}, \\ DC_n^J &:= D_n \cup X_J \quad \text{and} \quad BC_n^J := B_n \cup X_J. \end{aligned}$$

We put $C_n := DC_n^{\{1,\ldots,n\}}$. Note that $B_n = BC_n^{\varnothing}$ and $D_n = DC_n^{\varnothing}$.

Proposition 7. *Let* (R, V) *be a GRS of rank greater than two. Then the following hold.*

(1) If $\mathscr{W}(R) = A_{n-1}$*, then* $R = A_{n-1} \cup \{0\}$*.*
(2) If $\mathscr{W}(R) = B_n$*, then* $\exists J \subseteq \{1, \ldots, n\}: \ R = BC_n^J \cup \{0\}$*.*
(3) If $\mathscr{W}(R) = DC_n^J$*, then* $R = DC_n^J \cup \{0\}$*.*

Moreover, in each case, the GRS is equivalent to a quotient of a classic root system.

3.3. **Sporadic arrangements.** Write (r, i) for the crystallographic arrangement of rank r with label i. The following situations occur:

(1) The GRS is uniquely determined by the crystallographic arrangemets.
(2) The axioms of a GRS would imply the existence of $\alpha \neq 0$ with $(\alpha, \alpha) = 0$ on the elements of this reduced root set; in this case there is no corresponding GRS. Use $\forall \alpha, \beta \in R: \quad (\alpha + \beta \notin R \text{ and } \alpha - \beta \notin R) \Rightarrow (\alpha, \beta) = 0$.
(3) Particular cases: (3,6), (3,8), (3,9), (3,13), (3,20).

Note that the only sporadic case which does not uniquely determine a GRS is $(3, 6)$; this is a restriction of the root systems of types E_7 and E_8.

Theorem 8 (C.-Mühlherr [2]). *Each irreducible GRS of rank at least 2 is equivalent to a quotient of a classic root system of a finite Weyl group.*

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$K(\pi, 1)$ abelian arrangements and supersolvable posets

Emanuele Delucchi
(joint work with Christin Bibby)

Let $\mathscr{A}$ be a family of subspaces of a topological space X. Its *complement*

$$M(\mathscr{A}) := X \setminus \cup \mathscr{A}$$

is a topological space with deep connections to the incidence combinatorics of the elements of $\mathscr{A}$ that, at since at least [7], is represented by the poset

$$\mathscr{P}(\mathscr{A}) := \bigcup_{A \subseteq \mathscr{A}} \pi_0(\cap A); \quad L_1 \leqslant L_2 \text{ in } \mathscr{P}(\mathscr{A}) \text{ if } L_1 \supseteq L_2$$

of connected components of intersections of subfamilies of $\mathscr{A}$.
An arrangement is $K(\pi, 1)$ if the homotopy groups $\pi_i(M(\mathscr{A}))$ are trivial for $i > 1$.

Consider a connected abelian Lie group $\mathbb{G} := \mathbb{R}^p \times (S^1)^q$, where $p, q \in \mathbb{N}$, and let $\Gamma \simeq \mathbb{Z}^d$ be a finitely generated free abelian group. Every choice of a full-rank subset $\{a_1, \ldots, a_n\} \subseteq \Gamma \setminus \{0\}$ defines an **abelian arrangement** $\mathscr{A} = \{H_1, \ldots, H_n\}$ in $\mathbb{G}^d$, where

$$H_i := \ker\left(\operatorname{Hom}(\Gamma, \mathbb{G}) \to \mathbb{G},\ \phi \mapsto \phi(a_i)\right) \subseteq \operatorname{Hom}(\Gamma, \mathbb{G}) \simeq \mathbb{G}^d$$

Let $\mathscr{A}^\upharpoonright$ denote the (infinite, periodic) set of affine subspaces obtained by lifting the elements of $\mathscr{A}$ to the universal cover $\mathbb{R}^{d(p+q)}$ of $\mathbb{G}^d$. The covering $\mathbb{R}^{d(p+q)} \to \mathbb{G}^d$ restricts to a covering map $M(\mathscr{A}^\upharpoonright) \to M(\mathscr{A})$. The arrangement $\mathscr{A}$ is called **linear** if $\mathbb{G} \simeq \mathbb{C}$ ($p = 2$, $q = 0$), **toric** if $\mathbb{G} \simeq \mathbb{C}^*$ ($p = 1$, $q = 1$), **elliptic** if $\mathbb{G} \simeq S^1 \times S^1$ ($p = 0$, $q = 2$). We call $\mathscr{A}$ a **Coxeter** arrangement if $a_1, \ldots, a_n \in \mathbb{Z}^d$ is a set of positive roots of a Coxeter system of type ABCD.

The $K(\pi, 1)$**-problem** is to decide whether the property of being $K(\pi, 1)$ is determined by $\mathscr{P}(\mathscr{A})$. If $\mathscr{A}$ is a linear arrangement, this is an important classical problem. Outside the linear case, we know the following.

- **Toric Coxeter arrangements** are $K(\pi, 1)$. In this case $\mathscr{A}^{\upharpoonright}$ is the complexification of the arrangement of reflecting hyperplanes of the associated affine Coxeter system and $M(\mathscr{A}^{\upharpoonright})$ is $K(\pi, 1)$ by a recent result of Salvetti and Paolini [5]. Since $M(\mathscr{A}^{\upharpoonright})$ covers $M(\mathscr{A})$, $\mathscr{A}$ is $K(\pi, 1)$ as well.
- **Toric arrangements of "large type"** are $K(\pi, 1)$. We call an abelian arrangement $\mathscr{A}$ of "large type" if for every rank-two element $x \in \mathscr{P}(\mathscr{A})$ satisfies $|\mathscr{P}(\mathscr{A})_{\leqslant x}| > 4$. If $\mathscr{A}$ is of large type, then so is $\mathscr{A}^{\upharpoonright}$, and a result by Hendricks [3] shows that $\mathscr{A}^{\upharpoonright}$ is $K(\pi, 1)$. Thus, as in the previous item, $\mathscr{A}$ is $K(\pi, 1)$ as well.
- **Fiber-type toric and elliptic arrangements** are $K(\pi, 1)$. Below we define this class of arrangements and explain how it can be characterized via the poset $\mathscr{P}(\mathscr{A})$.

The notion of "fiber-type" linear arrangements was introduced by Falk and Randell [2]. It is a much-studied class of hyperplane arrangements. In particular, Falk and Randell proved that fiber-type linear arrangements are $K(\pi, 1)$.

Definition 1 (Fiber-type abelian arrangement, generalizing [2])**.** An abelian arrangement $\mathscr{A}$ in $\mathbb{G}^d = \mathrm{Hom}(\Gamma, \mathbb{G})$ is fiber-type if either $d = 1$ or there exists a rank-one split direct summand $\Gamma' \subseteq \Gamma$ and an abelian arrangement $\mathscr{B}$ in $\mathrm{Hom}(\Gamma/\Gamma', \mathbb{G})$ such that $\mathscr{B}$ is fiber-type and the projection $\mathrm{Hom}(\Gamma, \mathbb{G}) \to \mathrm{Hom}(\Gamma/\Gamma', \mathbb{G})$ restricts to a fibration $M(\mathscr{A}) \to M(\mathscr{B})$ whose fibers are homeomorphic to $\mathbb{G}$ with finitely many points removed.

Remark 2. By definition, if $\mathscr{A}$ is fiber-type there is a sequence of arrangements $\mathscr{A} = \mathscr{A}_d, \mathscr{A}_{d-1}, \ldots, \mathscr{A}_1$ with fibrations $\pi_i : M(\mathscr{A}_{i+1}) \to M(\mathscr{A}_i)$ for $i = 1, \ldots, d-1$.

Let $\mathscr{A}$ be either linear, toric or elliptic. In this case the homotopy type of $\mathbb{G}$ with finitely many points removed (i.e., of the fiber of each π_i) is that of a wedge of circles. An iterated application of the homotopy long exact sequence of a fibration then yields the following.

Corollary 3. *Fiber-type linear, toric and elliptic arrangements are $K(\pi, 1)$.*

We have an analogue of Terao's fibration theorem [6], showing that the class of fiber-type arrangements is indeed combinatorially determined. In order to state the theorem, we need to define supersolvable posets. We refer, e.g., to [1, §2.1, §2.2] for the relevant poset terminology, and we recall here the notions that are not standard.

We call a poset P **locally geometric** if it is bounded-below, pure (i.e., every maximal chain has the same, finite length) and every interval of P is a geometric lattice. The *length* of such a poset is defined to be one less than the cardinality of any maximal chain if P. Call $\hat{0}$ the unique minimal element of P and write $A(P)$ for the set of *atoms* of P, i.e., $A(P) = \{a \in P \mid \hat{0} \leqslant z < a \Rightarrow z = \hat{0}\}$.

An **M-ideal** of a locally geometric poset P is a pure, join-closed, order ideal $Q \subseteq P$ such that:

(1) $a \vee y \neq \varnothing$ whenever $y \in Q$ and $a \in A(P) \backslash A(Q)$.

(2) for every $x \in \max(P)$, there is some $y \in \max(Q)$ such that y is a modular element in the geometric lattice $P_{\leqslant x}$.

An M-ideal is called a **TM-ideal** if item (1) is strengthened by requiring that $|a \vee y| = 1$ whenever $y \in Q$ and $a \in A(P) \backslash A(Q)$.

Definition 4. A locally geometric poset P is **supersolvable** if there is a chain $\hat{0} = Q_0 \subset Q_1 \subset \cdots \subset Q_n = P$ where each Q_i is an M-ideal of Q_{i+1} of length i. If each Q_i is a TM-ideal of Q_i we call P **strictly supersolvable**.

Remark 5 ([1, Theorem 5.2.1])**.** If P is strictly supersolvable, then the characteristic polynomial of P factors linearly with positive integer roots.

Theorem 6 (Fibration theorem for abelian arrangements)**.** *Let $\mathscr{A}$ be an abelian arrangement. Then $\mathscr{A}$ is fiber-type if and only if $\mathscr{P}(\mathscr{A})$ is supersolvable.*

Remark 7. As a consequence of Remark 5 and [4, Theorem 7.8], the Poincaré polynomial of linear and toric arrangements (in fact, any abelian arrangement with $p > 0$) factors completely with positive integer roots.

Remark 8. The fibrations π_i associated with a fiber-type arrangement $\mathscr{A}$ as in Remark 2 have trivial monodtromy if and only if $\mathscr{P}(\mathscr{A})$ is strictly supersolvable. If this is the case, and $\mathscr{A}$ is toric, then a Lower Central Series formula holds for $\mathscr{A}$ [1, Theorem 5.3.10], generalizing Falk and Randell's result in the linear case [2].

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Algebraic degrees of phylogenetic varieties

Rodica Andreea Dinu

(joint work with Martin Vodička)

Phylogenetics is a science that models evolution. One central object in phylogenetics is the phylogenetic trees and they have become of interest to mathematicians, as relationships have been found between these objects and algebraic varieties. Such a variety is represented by a phylogenetic tree and a model of evolution by which we mean certain constraints on the probability of mutation. We consider the probabilities of the different mutations as entries of a matrix, which we call a *transition matrix*. By fixing a particular type of transition matrix, we obtain a probability distribution on the species' states. We get an algebraic map by fixing a model and then varying the entries of the transition matrices to obtain different probability distributions. The Zariski closure of the image of this map is a variety, that we call a *phylogenetic variety*, [6, 1]. We will assume that an abelian group G acts transitively and freely on the set of states. A general group-based model is a maximal subspace of transition matrices invariant under this group action. A subspace of this space is called a *group-based model*, see [10, 11, 16, 2]. We point out some known facts from the literature:

- (*Toric varieties*) The varieties coming from group-based models are toric, so they contain an algebraic torus as a dense open subset [7, 15].
- (*Normality*) Not much is known about the normality of these varieties. When G is one of $\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3$, the corresponding phylogenetic varieties for any tree are normal, see [16, 2]. On the other hand, when $|G| = 2k$, $k \geqslant 3$ the corresponding phylogenetic variety for any tree is not normal, see [2, Proposition 2.1]. However, when the tree is a tripod, there is a complete classification for normal phylogenetic varieties, see [4].
- (*Reduction to claw trees*) The defining ideals of these varieties associated to any phylogenetic tree can be seen as toric fiber products of the defining ideals of the phylogenetic varieties associated to claw trees (i.e. trees that have only one node and n leaves), see [14]. This fact shows that, in some cases, one can reduce checking a property for the phylogenetic variety associated to any phylogenetic tree to checking that property only in the case when the tree is a claw tree, see [5] and [10, Lemma 5.1]. As the Gorenstein property behaves well with respect to special toric fiber products, a classification of Gorenstein Fano phylogenetic varieties coming from any $G \in \{\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3\}$ and any (trivalent) tree was obtained in [2, Theorem 5.1]. When $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, the corresponding group-based model is also called *the 3-Kimura parameter model*, and it is of the greatest biological importance, as it corresponds to the action given by the Watson-Crick complementarity.
- (*Facet description*) The vertex description of the polytopes defining the group-based models is known, [1, 6, 10]. However, in general it is hard to deduce the facet descriptopn from the vertex description, and, in fact, the facet description is known only for small groups, [6, 9, 2].

We are interested in investigating the algebraic degrees of the phylogenetic varieties coming from group-based models. We call them *phylogenetic degrees*. In the literature, the phylogenetic degrees are known only for some computational examples, see [12]. As already mentioned, these varieties are toric; hence, computing the phylogenetic degrees relied on computing the volume of the associated polytopes in the lattice spanned by the vertices of the polytope, see [8, Section 5.3]. We present here our main results obtained in [3].

Theorem. The phylogenetic degree of the projective algebraic variety $X_{\mathbb{Z}_2,n}$ is equal to

$$\frac{n!}{2} - 2^{n-2}.$$

Theorem. The phylogenetic degree of the projective algebraic variety $X_{\mathbb{Z}_2\times\mathbb{Z}_2,n}$ is equal to

$$\frac{(3n)!}{4\cdot 6^n} - 3\cdot 2^{n-3}\cdot \sum_{i=0}^{n}(-2)^i\binom{n}{i}\frac{(3n)!}{(2n+i)!} + 3\cdot 4^{n-2}\binom{2n}{n} - n\cdot 4^{n-1}.$$

Theorem. The phylogenetic degree of the projective algebraic variety $X_{\mathbb{Z}_3,n}$ is equal to

$$\frac{(2n)!}{3\cdot 2^n} - 2^{n+1}\cdot 3^{n-2} + 3^{n-1}\cdot n.$$

For proving these results, we apply a combinatorial strategy: we regard the corresponding polytopes as embedded in higher dimensional cubes and we cut off parts that do not lie in our polytopes. As these parts often intersect each other, we apply the principle of inclusion and exclusion to determine their volume and the desired phylogenetic degrees.

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Toric vector bundles from hyperplane arrangements

Christopher Eur

According to [8, Introduction], the "most important algebraic geometric invariant of [a hyperplane arrangement] $\mathscr{A}$ is the module $D(\mathscr{A})$ [of the sheaf logarithmic derivations]." Questions about the extent to which the combinatorics of the arrangement determines properties of $D(\mathscr{A})$ has led to rich developments. Here, we describe another sheaf, in fact a toric vector bundle, associated with a hyperplane arrangement. Like the sheaf of logarithmic derivations, questions about the extent to which the combinatorics determine the properties of the vector bundle leads to rich interaction between combinatorics and geometry.

Let $E = \{1, \dots, n\}$ be a finite set, and let H_i be the i-th coordinate hyperplane $\{x_i = 0\}$ of $\mathbb{C}^E$. Let $T = (\mathbb{C}^*)^E$ be the algebraic torus, (i.e. the complement of the union of coordinate hyperplanes), which acts on $\mathbb{C}^E$ in the standard way. An r-dimensional linear subspace $L \subseteq \mathbb{C}^E$ that is not contained in a coordinate hyperplane defines a central and essential hyperplane arrangement $\mathscr{A}$ in L consisting of the intersections $L \cap H_i$. The arrangement complement, denoted $\mathring{L}$, is then the intersection $L \cap (\mathbb{C}^*)^E$. Let M denote the associated matroid, whose set of bases is $\{B \subseteq [n] :$ the composition $L \hookrightarrow \mathbb{C}^E \to \mathbb{C}^B$ is an isomorphism$\}$, where $\mathbb{C}^E \to \mathbb{C}^B$ is the coordinate projection.

We construct a T-equivariant vector bundle from the data $L \subseteq \mathbb{C}^E$, as follows. First, let us describe the base space of the vector bundle. Let $\mathbb{P}T$ be the projectivization of T, i.e. the quotient of T by its diagonal torus. Write $\overline{t}$ for the image of $t \in T$ in $\mathbb{P}T$. The base space will be the *permutohedral variety* X_n, which is the closure of the image of the rational map $\mathbb{P}(\mathbb{C}^E) \dashrightarrow \prod_{\varnothing \neq S \subseteq E} \mathbb{P}(\mathbb{C}^S)$. We note two properties of this variety:

- It is a smooth projective toric variety, whose open dense torus is $\mathbb{P}T$. We view X_n as a T-variety.
- It resolves the indeterminacy of the Cremona rational map $\mathbb{P}(\mathbb{C}^E) \dashrightarrow \mathbb{P}(\mathbb{C}^E)$ defined by $[x_1 : \dots : x_n] \mapsto [x_1^{-1} : \dots : x_n^{-1}]$. Let π_1 and π_2 be the two maps from X_n to $\mathbb{P}(\mathbb{C}^E)$, and denote α and β in $H^2(X_n; \mathbb{Z})$ (respectively) to be the pullbacks of the hyperplane class.

Let $\underline{\mathbb{C}}^E = X_n \times \mathbb{C}^E$ be a trivial rank n vector bundle on X_n.

Definition 1. For a linear subspace $L \subseteq \mathbb{C}^E$, define $\mathscr{S}_L$ to be the subbundle of $\underline{\mathbb{C}}^E$ whose fiber over a point $\overline{t} \in \mathbb{P}T \subset X_n$ is the linear subspace $t^{-1}L$. Define $\mathscr{Q}_L$ to be the quotient bundle $\underline{\mathbb{C}}^E/\mathscr{S}_L$.

These vector bundles were introduced in a joint work of the author with Berget, Spink, and Tseng [1], and were named *tautological bundles* of the hyperplane arrangement of $L \subseteq \mathbb{C}^E$. One may ask: which properties of the bundles $\mathscr{S}_L$ and $\mathscr{Q}_L$ depend only on the matroid M of $L \subseteq \mathbb{C}^E$?

Theorem 2. We highlight a few results from [1].

- The (T-equivariant) K-classes (and thus the Chern classes) of $\mathscr{S}_L$ and $\mathscr{Q}_L$ depend only on the matroid M. Moreover, for an arbitrary, not necessarily realizable, matroid M, one can construct K-classes $[\mathscr{S}_{\mathrm{M}}]$ and $[\mathscr{Q}_{\mathrm{M}}]$.
- The bundle $\mathscr{Q}_L$ admits a global section whose vanishing locus is the *wonderful compactfication* of the arrangement complement $\mathbb{P}\mathring{L}$ introduced in [3].
- Let $\int_{X_n} : H^*(X_n;\mathbb{Z}) \to \mathbb{Z}$ be the degree map provided by Poincaré duality on X_n. Then, the Chern classes of the vector bundles satisfy

$$\sum_{i+j+k+\ell=n-1} \left(\int_{\Sigma_E} \left(\alpha^i \beta^j c_k(\mathscr{S}_L) c_\ell(\mathscr{Q}_L) \right) \right) x^i y^j (-z)^k w^\ell$$
$$= (x+y)^{-1}(y+z)^r (x+w)^{|E|-r} \mathrm{T}_{\mathrm{M}}\left(\frac{x+y}{y+z}, \frac{x+y}{x+w} \right)$$

  where T_{M} is the Tutte polynomial of the matroid M.

These results explain in a unified way several previous results concerning the interaction between combinatorics and geometry in matroid theory. See [1, Section 1] for details. Moreover, the framework of studying matroids through these vector bundles as opened new doors: For instance, it has led to the development of the K-theory of matroids [7], the stellahedral geometry of matroids [6], the Gromov-Witten theory of matroids [9], and the tropical geometry of "type B" generalizations of matroids known as delta-matroids [5].

Let us conclude by considering the sheaf cohomologies of $\mathscr{S}_L$ and $\mathscr{Q}_L$, or of the bundles constructed from them (for example their tensor powers or duals). In general, it is expected that the sheaf cohomologies may depend on more than the matroid of L. Matt Larson has communicated to the author that due to deformation theory considerations, one should expect the cohomologies $H^i(X_n, \mathscr{Q}_L^\vee \otimes \mathscr{Q}_L)$ to depend on more than the matroid of L. On the other hand, we have the following.

Theorem 3. [4] Let c denote the number of coloops of the matroid M. Then, we have the following:

- $H^i(\bigwedge^p \mathscr{S}_L) = 0$ for all $i > 0$ and $p \geqslant 0$, and $\sum_{p\geqslant 0} \dim H^0(\bigwedge^p \mathscr{S}_L) u^p = (u+1)^c$.
- $H^i(\bigwedge^p \mathscr{Q}_L) = 0$ for all $i > 0$ and $p \geqslant 0$, and $\sum_{p\geqslant 0} \dim H^0(\bigwedge^p \mathscr{Q}_L) u^p = u^{n-r}\mathrm{T}_{\mathrm{M}}(1, 1+u^{-1})$.
- $H^i(\mathrm{Sym}^p \mathscr{Q}_L) = 0$ for all $i > 0$ and $p \geqslant 0$, and $\sum_{p\geqslant 0} \dim H^0(\mathrm{Sym}^p \mathscr{Q}_L) u^p = (\frac{1}{1-u})^{n-c}$.

Our discussion have been over the field $\mathbb{C}$, but all the results so far are valid over any algebraically closed field of arbitrary characteristic. With the restriction to only characteristic zero fields, Berget and Fink showed that $H^i(S^\lambda \mathscr{Q}_L) = 0$ for all $i > 0$ and S^λ the Schur functor of a partition λ [2]. Whether this remains true over positive characteristic, and other questions about sheaf cohomologies of the tautological bundles remain open; see [4, Section 5] for a partial list.

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Quasi-invariants and free arrangements

Misha Feigin

Given a divisor X in an algebraic variety M one can define a sheaf of so-called logarithmic vector fields which is a sheaf of modules over the sheaf of functions $\mathscr{O}_M$. Logarithmic fields is a sheaf of vector fields on M which are tangential to X at non-singular points of X. The divisor X is a Saito divisor if the the sheaf of modules of logarithmic vector fields is locally free (see e.g. [1] and references therein).

In the case of affine space $M = \mathbb{C}^n$ one may be interested in a stronger property of (global) freeness of logarithmic vector fields for a hypersurface X as a module over polynomial functions on M. Saito established freeness of the discriminant hypersurface of a finite Coxeter group W [2]. This case may also be treated at the level of the original vector space $V = \mathbb{C}^n$ where the group W acts by reflections rather than in the orbit space $M = V//W$.

More generally, for an arrangement of hyperplanes given by the equation

$$F := \prod_{\alpha \in \mathscr{A}} \alpha = 0,$$

where $\mathscr{A} \subset V^*$ is a finite set of non-collinear covectors, define the space of logarithmic vector fields

$$D = \{L \in \Gamma(TV) \colon F | L(F)\}. \tag{1}$$

Here divisibility is understood in the space of polynomials $R = S(V^*) \cong \mathbb{C}[x_1, \ldots, x_n]$, and D is an R–module. This module is free in the case when $\mathscr{A}$ is a positive half of a root system of a finite Coxeter group [2]. If an R-module D is free then the corresponding arrangement (1) is called free.

There is also a generalisation of free arrangements to the case of arrangements with multiplicities [3]. In the case of reflection arrangements the corresponding W-invariant vector fields are closely related to m-quasi-invariants of Coxeter groups introduced and studied in [4, 5, 6], see [7] for such a relation.

An arrangement with defining polynomial

$$F = F_r = \prod_{i=1}^{n} x_i \prod_{i<j}^{n} \prod_{k=-m}^{m} (x_i^r - q^k x_j^r)$$

was recently considered in [8] for $r = 1$, where its interesting combinatorial properties were established. Its freeness was established in [9] for $r = 1$ and generic $q \in \mathbb{C}^\times$. Moreover, the following vector fields form a free basis of the arrangement $F_1 = 0$ [10]:

$$E = \sum_{i=1}^{n} x_i \partial_i, \quad L_k = \sum_{i=1}^{n} x_i p_i^{(k)} \partial_i,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, $0 \leqslant k \leqslant n-2$, and

$$p_i^{(k)} = p_i^{(k)}(x_1, \ldots, x_n) = \sum_{s=1}^{n} \int_{x_i}^{x_s} t^k g(t)^m_{(q)} d_q t.$$

Here

$$g(t)^m_{(q)} = g(t) g(q^{-1}t) \ldots g(q^{-m+1}t),$$

$$g(t) = \prod_{j=1}^{n} (t - x_j),$$

and

$$\int_{x_i}^{x_s} h(t) d_q t := H(x_s) - H(x_i),$$

where H is such a function that

$$\frac{H(qx) - H(x)}{(q-1)x} = h(x).$$

These q-multplicative integral formulas for a free basis may be compared with additive discrete integral formulas for coefficients of a basis of the extended Catalan arrangement of type A_{n-1} [11], and to the ordinary integral formulas in case of type A_{n-1} multiarrangement (see [11] and references therein).

Let us now recall the definition of q-deformed quasi-invariants following [12]. Let $\zeta = e^{\frac{2\pi i}{r}}$, $r \in \mathbb{N}$. A polynomial $p \in S(V^*)$ is called a q-deformed cyclotomic m-quasi-invariant, $m \in \mathbb{N}$, if

$$p(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) - p(x_1, \ldots, \zeta^s x_j, \ldots, \zeta^{-s} x_i, \ldots, x_n)$$

is divisible by

$$\prod_{k=-m}^{m} (x_i^r - q^k x_j^r)$$

for all $i < j$ and $s \in \mathbb{N}$. Let $Q_m^{(q)}$ be the space of such polynomials p. It was established in [12] that $Q_m^{(q)}$ is a free $S(V^*)^{S_n}$-module for generic values of q.

Previous considerations may be unified into the following statement.

Theorem 1. *The arrangement $F_r = 0$, where $r \in \mathbb{N}$, is free except for possibly finitely many values of $q \in \mathbb{C}^\times$. A free basis is given by vector fields*

$$E = \sum_{i=1}^{n} x_i \partial_i, \quad L_k = \sum_{i=1}^{n} x_i p_i^{(k)}(x_1^r, \ldots, x_n^r) \partial_i,$$

where $0 \leqslant k \leqslant n-2$. Polynomials $p_i^{(k)}(x_1^r, \ldots, x_n^r) \in Q_m^{(q)}$.

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Chow functions for partially ordered sets

Luis Ferroni
(joint work with Jacob P. Matherne, Lorenzo Vecchi)

In 1992, Stanley initiated the study of Kazhdan–Lusztig–Stanley (KLS) functions associated to a P-kernel in a partially ordered set. In this talk I will introduce a new class of functions, called “Chow functions”, that interact in a remarkable way with the KLS functions. The name stems from the case in which the poset P is the lattice of flats of a matroid and the P-kernel is given by the characteristic polynomial—in this scenario, the Chow function encodes the Hilbert–Poincaré series of the Chow ring of the matroid. Although in the full generality of posets these Chow rings need not exist, a notable number of inequalities between the coefficients for Chow functions can be extracted from the KLS functions (and viceversa). Among other features, our theory sets a common ground for approaching some (a priori unrelated) conjectures in combinatorics, related to the real-rootedness of polynomials associated to Gorenstein* posets, face lattices of polytopes, and geometric lattices.

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Real matroid Schubert varieties, zonotopes, and virtual Weyl groups

Leo Jiang
(joint work with Yu Li)

Let V be a finite-dimensional vector space over a field k, and let $\mathscr{A} = (\alpha_e)_{e\in E} \subset V^*$ be a vector configuration indexed by a finite set E and spanning V^*. The *Schubert variety of the arrangement* $\mathscr{A}$ (or *matroid Schubert variety*) $Y_{\mathscr{A}}$ is the closure of V under the embedding $V \to \mathbb{R}^E \subset (\mathbb{R} \cup \{\infty\})^E = (\mathbb{P}^1)^E$ defined by $v \mapsto (\alpha_e(v))_{e\in E}$. Since its introduction by Ardila and Boocher [1], several recent advances in matroid theory (in particular, the proof of the Dowling–Wilson top-heavy conjecture and the development of Kazhdan–Lusztig theory for matroids) have been motivated by studying the topology of this singular variety. However, many of the results use tools from algebraic geometry with the requirement that k is algebraically closed. In our work, we study matroid Schubert varieties over the field $k = \mathbb{R}$. We are able to construct a cell complex which is a topological model for $Y_{\mathscr{A}}(\mathbb{R})$.

Theorem 1. *The real locus $Y_{\mathscr{A}}(\mathbb{R})$ of the matroid Schubert variety associated to a real arrangement $\mathscr{A}$ is homeomorphic to the zonotope $Z_{\mathscr{A}} = \sum_{e\in E}[-1,1]\alpha_e$ with parallel faces identified.*

To understand the definition of this equivalence relation on the zonotope, recall that the faces of $Z_{\mathscr{A}}$ are in bijection with the covectors of $\mathscr{A}$ (see for example [4, Proposition 2.2.2]). Explicitly, if $C \in \{+,-,0\}^E$ is a covector, then the corresponding face is

$$\sum_{C(e)=+} \alpha_e - \sum_{C(e)=-} \alpha_e + \sum_{C(e)=0} [-1,1]\alpha_e.$$

The faces associated to covectors with the same zero set are translates of each other, and the relation is simply the identification of faces by these translations.

This quotient of the zonotope was considered by Bartholdi–Enriquez–Etingof–Rains [2] in the special case that $Z_{\mathscr{A}}$ is the permutohedron, and Ilin–Kamnitzer–Li–Przytycki–Rybnikov [5] proved Theorem 1 for Weyl arrangements (when $\mathscr{A}$ is the set of positive roots of a root system). Our proof of Theorem 1 is independent of [5] and constructs an explicit homeomorphism (in fact, one for each homeomorphism $\mathbb{R} \to (-1,1)$) from the matroid Schubert variety to the quotient of the zonotope. These maps send the strata in the affine paving of $Y_{\mathscr{A}}$ (first observed by Proudfoot–Xu–Young [6]) to the cells of the combinatorial model.

In ongoing work, we use this combinatorial model to understand the topological invariants of real matroid Schubert varieties. For example, it is straightforward to obtain a presentation for the fundamental group of $Y_{\mathscr{A}}(\mathbb{R})$ using Theorem 1. When $\mathscr{A}$ is the braid arrangement, the fundamental group $\pi_1(Y_{\mathscr{A}}(\mathbb{R}))$ is a quotient of the pure virtual braid group called (among other names) the pure flat braid group [2]. We show that when $\mathscr{A}$ is more generally a Coxeter arrangement, the fundamental group $\pi_1(Y_{\mathscr{A}}(\mathbb{R}))$ is the analogous quotient of the pure virtual Artin group recently defined by Bellingeri–Paris–Thiel [3].

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From matroids to moduli spaces

Lukas Kühne

(joint work with Daniel Corey, Dante Luber, Piotr Pokora, Benjamin Schröter)

1. Matroids

A matroid is a combinatorial abstraction of independence, e.g., linear independence of vectors or spanning sets of edges in a graph. Matroids were introduced by Whitney [6] and independently by Nakasawa, see [5]. They are a central object in mathematics connecting multiple disciplines such as combinatorics, algebra, and geometry. Of particular importance is the work of Adiprasito, Huh and Katz [1], who demonstrate that matroids admit a Hodge theory originally stemming from algebraic geometry.

Let V be a d-dimensional vector space, E a finite set, and $(v_i)_{i \in E}$ a sequence of spanning vectors of V. There are various ways to record the linear dependencies among these vectors.

- *Independent sets*: $\mathscr{I} = \{A \subseteq E : (v_i)_{i \in A} \text{ are linearly independent}\}$.
- *Bases*: $\mathscr{B} = \{A \subseteq E : (v_i)_{i \in A} \text{ is a basis of } V\}$.
- *Rank*: $\mathrm{rk} : 2^E \to \mathbb{Z}\mathbb{Z}_{\geqslant 0}$; $\mathrm{rk}(A) = \dim \mathsf{span}(v_i : i \in A)$.
- *Flats*: $\mathscr{F} = \{A \subseteq E : v_j \notin \mathsf{span}(v_i : i \in A) \text{ for all } j \notin A\}$.
- *Circuits*: $\mathscr{C} = \{A \subseteq E : (v_i)_{i \in A} \text{ are minimally linearly dependent}\}$.

The notions of independent sets, bases, rank, flats, and circuits may each be axiomitized, each of which leads to a definition of a matroid. We favor the description in terms of bases:

Definition 1. A *matroid* M consists of a finite set E and a non-empty collection $\mathscr{B} \subset 2^E$ that satisfies the *basis exchange axiom*: for each pair A, B of distinct elements of $\mathscr{B}$ and $x \in A \backslash B$, there is a $y \in B \backslash A$ such that $A \backslash \{x\} \cup \{y\}$ is in $\mathscr{B}$.

Consider a vector configuration whose elements are the columns of a $r \times n$ full-rank matrix X. The matroid of this configuration, denoted $\mathsf{M}[X]$, is the matroid whose groundset is $\{1, 2, \ldots, n\}$ and its bases are the collections of r columns that are of full rank. Equivalently, these columns can be regarded as normal vectors of a hyperplane arrangement. In this interpretation, the bases are the collections of columns of size r which intersect in the origin only. Let $\mathbb{F}$ be a field. A matroid M is $\mathbb{F}$-*realizable* if there is a matrix X with entries in the field $\mathbb{F}$ such that $\mathsf{M} \cong \mathsf{M}[X]$.

2. The moduli space of a matroid

The *realization space* or *moduli space* $\mathscr{R}(\mathsf{M}; \mathbb{F})$ is the space of all realizations over a field $\mathbb{F}$ of a matroid M. It is an affine scheme that can be explicitly computed. An implemented to compute the realization space is contained in a new `OSCAR` module for matroids [3]. We showcase the computation in the following two examples.

Example 2. Let's consider the Fano matroid F:

```
julia> realization_space(fano_matroid())
The realization space is
[0   1   1   1   1   0   0]
[1   0   1   1   0   1   0]
[1   0   1   0   1   0   1]
in the integer ring
within the vanishing set of the ideal
2ZZ
```

The output shows a matrix which parametrizes the realizations. Because the vanishing ideal is $\langle 2 \rangle$, the matroid is realizable only over fields of characteristic 2. Over any such field a realization of the Fano matroid is given by this matrix. And every realization is projectively unique to this given one. Thus over every field of characteristic 2, the realization space of this matroid is just one point

Example 3. Another prominent matroid is the *Pappus* matroid as its configurations of rank two flats is a Pappus configuration. Let's begin by computing its realization space over $\mathbb{C}$:

```
julia> RS = realization_space(pappus_matroid(), char=0)
The realization space is
[1   0   1   0   x2   x2                   x2^2    1    0]
[0   1   1   0    1    1   -x1*x2 + x1 + x2^2    1    1]
[0   0   0   1   x2   x1                 x1*x2   x1   x2]
in the Multivariate polynomial ring in 2 variables over QQ
avoiding the zero loci of the polynomials
RingElem[x1 - x2, x2, x1, x2 - 1, x1 + x2^2 - x2,
  x1 - 1, x1*x2 - x1 - x2^2]
```

Thus the realization space is the affine space $\mathbb{A}^2$ over $\mathbb{C}$ with the seven specified curves removed. One can obtain a specific realization of this matroid by picking a point in that space that avoids these seven exceptional curves:

```
julia> realization(RS)
One realization is given by
[1   0   1   0   2   2   4   1   0]
[0   1   1   0   1   1   1   1   1]
[0   0   0   1   2   3   6   3   2]
in the Rational field
```

3. Application to freeness of line arrangements

Recently, Corey and Luber discovered explicit rank-3 matroids on 12 elements exhibiting singularities in their realization spaces [2]. This phenomenon can be used to construct novel Ziegler pairs, i.e., pairs of line arrangements having the same underlying matroid but their modules of logarithmic derivations have different shapes of free resolutions[4].

Example 4. Let M be the rank-3 matroid with groundset $\{1,\dots,12\}$ whose non-bases are given by

$$\{1,2,3\},\{1,4,5\},\{1,6,10\},\{2,4,6\},\{2,5,8\},\{2,9,10\},\{3,4,7\},$$
$$\{3,5,9\},\{3,6,11\},\{4,10,12\},\{5,6,7\},\{5,11,12\},\{7,8,11\},\{8,9,12\}.$$

This matroid can be realized over the complex numbers and the coordinate ring of $\mathscr{R}(\mathsf{M};\mathbb{C})$ is isomorphic to the ring

$$R = \frac{P^{-1}\mathbb{C}[x^{\pm}, y^{\pm}, z^{\pm}]}{I},$$

where P is a multiplicative semigroup with 32 generators, and I_1 is the principal ideal

$$I = \langle (xy + xz - x - y - z^2 + 1)(x - y - z) \rangle.$$

Hence, $\mathscr{R}(\mathsf{M};\mathbb{C})$ has two maximal components, C_1 and C_2, corresponding to $xy + xz - x - y - z^2 + 1 = 0$ and $x - y - z = 0$, respectively. The singular locus of $\mathscr{R}(\mathsf{M};\mathbb{C})$ is given by the one dimensional subvariety at the intersection. This subvariety is a smooth conic that is not excluded in the multiplicative semigroup P.

We sample points on the maximal components and in the singular locus, computing the corresponding module of logarithmic derivations and their free resolutions. We found that the minimal degree of a derivation of line arrangements on these two components is 8. In the singular locus however, we found that the minimal degree of a derivation of line arrangements is 7. Thus, the algebraic structure of these modules changes depending on the location in the realization space.

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Augmented geometry of matroids

Matt Larson
(joint work with Christopher Eur, June Huh)

In recent years, "augmented" versions of several geometric constructions related to hyperplane arrangements and matroids have been introduced in the literature. These constructions typically involve replacing the projectivization of a vector space by its projective completion. For example, suppose that $\{L_1, \ldots, L_m\}$ is a subspace arrangement in a vector space V of dimension r. For simplicity, assume that $\cap L_i = 0$ and that each L_i is proper. We allow L_i to be contained in L_j. Let $\underline{W}$ be the closure of the image of the rational map

$$\mathbb{P}L \dashrightarrow \prod_i \mathbb{P}(L/L_i).$$

Note that the dimension of $\underline{W}$ is $r-1$. Then the homology class of $\underline{W}$ is difficult to describe, see [6].

Let W be the closure of the image of the rational map

$$\mathbb{P}(L \oplus \mathbb{C}) \dashrightarrow \prod_i \mathbb{P}((L/L_i) \oplus \mathbb{C}).$$

Note that the dimension of W is r. Then the homology class of W admits a simple description, as follows. The *polymatroid* $\mathrm{rk}\colon 2^{[m]} \to \mathbb{Z}_{\geqslant 0}$ associated to the subspace arrangement is given by

$$\mathrm{rk}(S) = \mathrm{codim}\left(\bigcap_{i \in S} L_i\right).$$

A vector $(a_1, \ldots, a_m) \in \mathbb{Z}^m_{\geqslant 0}$ with $\sum a_i = r$ is a *basis* of this polymatroid if, for all $S \subseteq [m]$, we have $\sum_{i \in S} a_i \leqslant \mathrm{rk}(S)$. By Poincaré duality, we can determine the homology class of W by intersecting it with any monomials in hyperplane classes $H_1, \ldots, H_m$ of the factors. We have

$$\int [W] \cap H_1^{a_1} \cdots H_m^{a_m} = \begin{cases} 1 & (a_1, \ldots a_m) \text{ is a basis} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

I.e., the homology class is given by the bases of the polymatroid associated to the subspace arrangement. Note that, because the dimension of $\underline{W}$ is $r-1$, there isn't even an obvious guess for the homology class of $\underline{W}$.

The study of these problems can be reduced to the case of hyperplane arrangement by choosing $\mathrm{codim}(L_i)$-many generic hyperplanes containing each L_i; we then obtain a hyperplane arrangement which contains our subspace arrangement as some of the flats (see [2, Section 2]). Likewise, we can obtain many interesting examples of subspace arrangements by taking the flats of a hyperplane arrangement. In the case of a subspace arrangement arising in this fashion, the variety $\underline{W}$ is called the *wonderful variety* of the hyperplane arrangement and the variety W is called the *augmented wonderful variety.* Then (1) is a computation in the cohomology ring of the augmented wonderful variety. This ring only depends on the matroid M associated to the subspace arrangement, and is called the *augmented Chow ring* $A(\mathrm{M})$. It was introduced in [1] for unrelated reasons. It has the presentation

$$A(\mathrm{M}) = \frac{\mathbb{Z}[h_F]_{F \text{ nonempty flat of M}}}{((h_F - h_{F\vee G})(h_G - h_{F\vee G})) + (h_a^2, h_a h_F - h_a h_{F\vee a} : a \text{ atom})},$$

where F and G vary over the nonempty flats of M, and $\vee$ is the join in the lattice of flats of M. Then $A(\mathrm{M})$ is equipped with a degree map $\int$, and (1) is generalized by the following computation:

$$\int h_{F_1} \cdots h_{F_r} = \begin{cases} 1 & \text{for all } S \subseteq [r],\ |S| \leqslant \mathrm{rk}(\cup_{i\in S} F_i) \\ 0 & \text{otherwise.} \end{cases}$$

This formula was proved in [4]. While the proof in [4] is conceptual, it uses many tools from [3]. A simple combinatorial argument, using only definition of $A(\mathrm{M})$, was given in [5].

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Multimatroids and rational curves with cyclic action

Shiyue Li

(joint work with Emily Clader, Chiara Damiolini, Christopher Eur, Daoji Huang, Rohini Ramadas)

I will share with you a connection between multimatroids and moduli spaces of rational curves with cyclic action. Multimatroids are generalizations of matroids and delta-matroids that naturally arise from topological graph theory. The main result is a combinatorial formula for certain intersection numbers on the moduli space by relating to the volumes of independence polytopal complexes of multimatroids. Based on joint works with Emily Clader, Chiara Damiolini, Chris Eur, Daoji Huang, and Rohini Ramadas.

Poincaré polynomials associated to geometric lattices

Jacob P. Matherne

(joint work with Tom Braden, Luis Ferroni, June Huh, Nicholas Proudfoot, Matthew Stevens, Lorenzo Vecchi, Botong Wang)

A number of graded objects associated to a matroid have played a recent role in the resolution of long-standing conjectures in the field of matroid theory. The goal of this report is to survey what is known about their Poincaré polynomials.

To a (loopless) matroid M, one may associate the following graded objects:

(1) the intersection cohomology module $\mathrm{IH}(\mathsf{M})$ [3],
(2) its "stalk at the empty flat" $\mathrm{IH}(\mathsf{M})_\varnothing$ [3],
(3) the augmented Chow ring $\mathrm{CH}(\mathsf{M})$ [4], and
(4) the Chow ring $\underline{\mathrm{CH}}(\mathsf{M})$ [13].

Each of these objects has a topological interpretation when the matroid M is realizable by a collection of vectors in a complex vector space V. These interpretations hinge on a certain singular projective variety $Y_{\mathscr{A}}$, introduced in [1] and now called the matroid Schubert variety, that is constructed from the vector space V. It gets its name from the analogous role it plays in the Kazhdan–Lusztig theory of matroids [8] that the classical Schubert varieties play in the Kazhdan–Lusztig theory of Coxeter groups [18, 19].

The matroid Schubert variety $Y_{\mathscr{A}}$ has a canonical resolution of singularities $\pi_{\mathscr{A}}\colon \widetilde{Y}_{\mathscr{A}} \to Y_{\mathscr{A}}$, where $\widetilde{Y}_{\mathscr{A}}$ is the so-called augmented wonderful variety. In the realizable case, the respective graded objects in the numbered list above are isomorphic (with a degree-doubling isomorphism) to the following topological objects:

(1) the intersection cohomology $\mathrm{IH}(Y_{\mathscr{A}})$ of $Y_{\mathscr{A}}$,
(2) the local intersection cohomology $\mathrm{IH}_{(\infty,\ldots,\infty)}(Y_{\mathscr{A}})$ of $Y_{\mathscr{A}}$ at the point $(\infty,\ldots,\infty) \in Y_{\mathscr{A}}$,
(3) the cohomology $\mathrm{H}(\widetilde{Y}_{\mathscr{A}})$ of $\widetilde{Y}_{\mathscr{A}}$, and
(4) the cohomology $\mathrm{H}(\pi_{\mathscr{A}}^{-1}(\infty,\ldots,\infty))$ of the fiber $\pi_{\mathscr{A}}^{-1}(\infty,\ldots,\infty)$.

Although most matroids are not realizable [22], the miracle is that arbitrary matroids *behave as if they were geometric objects.* Indeed, $\mathrm{IH}(\mathsf{M})$, $\mathrm{CH}(\mathsf{M})$, and $\underline{\mathrm{CH}}(\mathsf{M})$ satisfy the Kähler package [3, 2, 4], a trio of important results consisting of Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations. The Heron–Rota–Welsh conjecture on the log-concavity of the characteristic polynomial of M [17, 25, 27] follows from the Hodge–Riemann relations for $\underline{\mathrm{CH}}(\mathsf{M})$, and both the Dowling–Wilson top-heavy conjecture on the shape of the lattice of flats of M [6, 7] and the nonnegativity of the Kazhdan–Lusztig and Z-polynomials of a matroid [8, 15] (see the bulleted list below) follow from the hard Lefschetz theorem for $\mathrm{IH}(\mathsf{M})$.

There has been an industry of recent interest in studying the respective Poincaré polynomials of the graded objects in the first bulleted list:

(1) the Z-polynomial $Z_{\mathsf{M}}(t)$ of a matroid M [23],
(2) the Kazhdan–Lusztig polynomial $P_{\mathsf{M}}(t)$ of a matroid M [8],

(3) the augmented Chow polynomial $\mathrm{H}_{\mathsf{M}}(t)$ of a matroid M [9], and
(4) the Chow polynomial $\underline{\mathrm{H}}_{\mathsf{M}}(t)$ of a matroid M [9].

Conjecture 1 ([23, 15, 26, 11])**.** *The polynomials $Z_{\mathsf{M}}(t)$, $P_{\mathsf{M}}(t)$, $\mathrm{H}_{\mathsf{M}}(t)$, and $\underline{\mathrm{H}}_{\mathsf{M}}(t)$ are real-rooted for every matroid M.*

Real-rootedness is the strictest condition in a sequence of implications involving interesting combinatorial patterns for single-variable polynomials whose coefficient sequence consists of nonnegative integers and has no internal zeros:

$$\begin{tikzcd}
 & \gamma\text{-positivity}^{1} \arrow[dr, Rightarrow] & \\
\text{real-rootedness} \arrow[ur, Rightarrow] \arrow[r, Rightarrow] & \text{log-concavity} \arrow[r, Rightarrow] & \text{unimodality}
\end{tikzcd}$$

The polynomials $P_{\mathsf{M}}(t)$ and $Z_{\mathsf{M}}(t)$ are real-rooted in the following cases: when $\mathsf{M} = \mathsf{U}_{d,n}$ is uniform of rank d on n elements for all $d \geqslant 1$ and all $2 \leqslant n - d \leqslant 15$ [14]; when M is a fan, wheel, or whirl matroid [20]; and when M is a sparse paving matroid with at most 30 elements [12]. Log-concavity of $P_{\mathsf{M}}(t)$ holds for all uniform matroids [28].

Poincaré duality and the hard Lefschetz theorem for $\underline{\mathrm{CH}}(\mathsf{M})$ and $\mathrm{CH}(\mathsf{M})$ imply unimodality for $\underline{\mathrm{H}}_{\mathsf{M}}(t)$ and $\mathrm{H}_{\mathsf{M}}(t)$, and the same theorems for $\mathrm{IH}(\mathsf{M})$ imply unimodality for $Z_{\mathsf{M}}(t)$. In [9], the semi-small decomposition of $\underline{\mathrm{CH}}(\mathsf{M})$ and $\mathrm{CH}(\mathsf{M})$ from [4] are used to prove the γ-positivity of $\underline{\mathrm{H}}_{\mathsf{M}}(t)$ and $\mathrm{H}_{\mathsf{M}}(t)$; and, a result of Braden–Vysogorets [5] is used to prove γ-positivity for $Z_{\mathsf{M}}(t)$.[2]

Whereas $P_{\mathsf{M}}(t)$ and $Z_{\mathsf{M}}(t)$ were defined recursively in [8, 23], and their interpretation as Poincaré polynomials was established later [3], the story for $\mathrm{H}_{\mathsf{M}}(t)$ and $\underline{\mathrm{H}}_{\mathsf{M}}(t)$ is the reverse. In [9] a recursive formula was given for the Poincaré polynomials $\mathrm{H}_{\mathsf{M}}(t)$ and $\underline{\mathrm{H}}_{\mathsf{M}}(t)$, paralleling the definition of $P_{\mathsf{M}}(t)$ and $Z_{\mathsf{M}}(t)$. This formula leads to several consequences [9]: $\mathrm{H}_{\mathsf{M}}(t)$ and $\underline{\mathrm{H}}_{\mathsf{M}}(t)$ are real-rooted for M sparse paving with at most 40 elements; $\mathrm{H}_{\mathsf{M}}(t)$ is real-rooted for M uniform since it is an example of a generalized binomial Eulerian polynomial of Haglund–Zhang [16]; and $\mathrm{H}_{\mathsf{M}}(t)$ (respectively $\underline{\mathrm{H}}_{\mathsf{M}}(t)$) is real-rooted for all M with rank less than five (respectively six) by using the fact that $\underline{\mathrm{CH}}(\mathsf{M})$ and $\mathrm{CH}(\mathsf{M})$ are Koszul algebras [21] together with results from [24].

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Combinatorial models of fibrations for hyperplane arrangements and oriented matroids

Paul Mücksch
(joint work with Masahiko Yoshinaga)

Let $\mathscr{A}$ be a hyperplane arrangement in the real vector space $V = \mathbb{R}^\ell$ with defining linear forms $\alpha_i \in V^*$ $(i = 1, \ldots, n)$, i.e. $\mathscr{A} = \{\ker(\alpha_i) \mid i = 1, \ldots, n\}$, $Q = \prod_{i=1}^n \alpha_i$ its defining polynomial and $L(\mathscr{A})$ the *intersection lattice* of $\mathscr{A}$. A fundamental problem is to decide when the complexified complement $\mathfrak{X}(\mathscr{A}) = \mathbb{C}^\ell \backslash Q^{-1}(0)$ is a $K(\pi, 1)$-space.

For special classes, such as the *braid arrangements* or more generally *supersolvable* arrangements, this can be achieved by utilizing fibrations which connect complements of arrangements of different ranks. The fundamental result due to Falk and Randell [5] and Terao [18] states that the natural projection of the ambient complex space of $\mathscr{A}$ to the quotient space V/X by a *modular* intersection $X \in L(\mathscr{A})$ (i.e. $X + Y \in L(\mathscr{A})$ for all $Y \in L(\mathscr{A})$) of corank one, restricted to the complement space is a fiber bundle map, its base space the complement of the localization $\mathscr{A}_X/X = \{H/X \mid H \in \mathscr{A} \text{ and } X \subseteq H\}$ and fiber a punctured complex plane $\mathbb{C}\backslash\{z_1, \ldots, z_k\}$. An arrangement is supersolvable if there is a maximal flag $X_0 < X_1 < \ldots < X_\ell \subseteq L(\mathscr{A})$ of modular elements. A successive application of Falk, Randell and Terao's theorem and the long exact sequences of homotopy groups associated to the fibrations then yields asphericity.

Another seminal positive results regarding the $K(\pi, 1)$-problem is Deligne's theorem [4], stating that complexified real simplicial arrangements are aspherical.

A finer combinatorial invariant of a real arrangement $\mathscr{A}$ is its *oriented matroid* which is encoded by the subposet $\mathscr{L}(\mathscr{A}) := \{(\operatorname{sgn}(\alpha_i(v) \mid i = 1, \ldots, n) \mid v \in V)\} \subseteq \{0, +, -\}^n$ where the partial order is given component-wise by declaring $0 < +$, $0 < -$ and $+$ and $-$ to be incomparable. An abstract oriented matroid $\mathscr{M} = (E, \mathscr{L})$ consists of a finite ground set E and a subposet $\mathscr{L} \subseteq \{0, +, -\}^E$ subject to certain axioms mimicking the geometry of the realizable case, see [2].

Subsequently, Salvetti [15] introduced a regular cell complex modeling the homotopy type of $\mathfrak{X}(\mathscr{A})$. Gel'fand and Rybnikov [7] realized that the construction of the *Salvetti complex* $\mathscr{S}(\mathscr{A})$ only depends on the oriented matroid associated to the real arrangement. This in turn led to an extension of Deligne's result to oriented matroids in general, i.e. the Salvetti complex of an oriented matroid whose covector complex yields a simplicial cell decomposition of the sphere is indeed aspherical, independently established by Cordovil [3] and Salvetti [16].

The natural question appears whether the "real" version of Falk, Randell and Terao's theorem extends to all oriented matroids as does Deligne's theorem.

In recent work [9], a complete answer is given in terms of combinatorial models of fibrations for Salvetti complexes of oriented matroids. The following notion is motivated by Quillen's seminal Theorem B from the homotopy theory of categories [14].

Definition 1 ([9, Def. 5.2])**.** Let $f : P \to Q$ be a poset map such that for all $y \leqslant y'$ $(y, y' \in Q)$ the inclusion $(f \downarrow y) := f^{-1}(Q_{\leqslant y}) \hookrightarrow (f \downarrow y')$ is a homotopy equivalence. Then f is called a *poset quasi-fibration.*

The important property of a poset quasi-fibration $f : P \to Q$ implied by Quillen's Theorem B is that the homotopy fiber of the realization of f (i.e. order complex and geometric realization) is homotopy equivalent to the realization of the poset $(f \downarrow y)$ for any $y \in Q$. In particular, a poset quasi-fibration yields a long exact sequence of homotopy groups involving the realizations of P, Q and $(f \downarrow y)$ analogous to a topological fibration.

Regarding a general oriented matroid $\mathscr{M} = (E, \mathscr{L})$ on a ground set E given by the poset $\mathscr{L} \subseteq \{0, +, -\}^E$, we denote its *geometric lattice* by $L(\mathscr{M}) = \{z(\sigma) = \{e \in E \mid \sigma_e = 0\} \mid \sigma \in \mathscr{L}\} \subseteq 2^E$. We have a similar notion of modularity for *flats* $X \in L$. Further, for $X \in L(\mathscr{A})$ we have a natural projection map $\rho_X : \mathscr{L} \to \mathscr{L}_X = \{\sigma|_X \mid \sigma \in \mathscr{L}\}, \sigma \mapsto \sigma|_X$ which extends to a map $\widetilde{\rho}_X : \mathscr{S} \to \mathscr{S}_X$, where $\mathscr{S} = \mathscr{S}(\mathscr{M})$ is the Salvetti-complex of the oriented matroid $\mathscr{M}$ and $\mathscr{S}_X = \mathscr{S}(\mathscr{M}_X)$ the complex of the localization $\mathscr{M}_X = (X, \mathscr{L}_X)$. The main result of [9] is as follows.

Theorem 2 ([9, Thm. 6.4])**.** *Let $X \in L(\mathscr{M})$ be a modular flat of corank 1. Then the map $\widetilde{\rho}_X : \mathscr{S} \to \mathscr{S}_X$ is a poset quasi-fibration. For $a \in \mathscr{S}_X$ the poset fiber $(\widetilde{\rho}_X \downarrow a)$ is homotopy equivalent to a wedge of circles $S^1 \vee \ldots \vee S^1$ ($|E \backslash X|$ circles).*

As a direct consequence, applying the long exact homotopy sequence, we obtain the following.

Theorem 3 ([9, Thm. 1.1])**.** *The Salvetti complex of a supersolvable oriented matroid is aspherical.*

A further important topological object associated to an arrangement is its *Milnor fibration* $f : \mathfrak{X}(\mathscr{A}) \to \mathbb{C}^\times, z \mapsto Q(z)$ which is a smooth fibration due to Milnor [8], and the fiber $F(\mathscr{A}) := f^{-1}(1)$ is called the *Milnor fiber* of $\mathscr{A}$. In contrast to the complement $\mathfrak{X}(\mathscr{A})$, whose cohomology algebra $H^\bullet(\mathfrak{X}(\mathscr{A}), \mathbb{Z})$ is completely described in terms of the poset $L(\mathscr{A})$ thanks to Orlik and Solomon's fundamental result [12], it is still open if the first Betti number $b_1(F(\mathscr{A}))$ of the Milnor fiber only depends on $L(\mathscr{A})$, cf. [17]. But there are a lot of partial results due to many authors; the strongest up to date on this problem are due to Papadima and Suciu [13], giving a formula for $b_1(F(\mathscr{A}))$ if either $|\mathscr{A}_X| \leqslant 3$ for all codimension 2 intersections $X \in L(\mathscr{A})$ or $3 \nmid |\mathscr{A}_X|$ for all X of codimension 2.

Regarding the homotopy type of $F(\mathscr{A})$, a complete description was given for generic arrangements by Orlik and Randell [11] and very recently by Brady, Falk and Watt [1] for real *reflection arrangements.*

In joint work in progress with Masahiko Yoshinaga [10], we describe a concrete combinatorial model for the Milnor fibration of any complexified real arrangement. Our construction works more generally for oriented matroids and yields a "combinatorial Milnor fibration" for any oriented matroid in terms of a poset quasi-fibration. The key is to consider a certain natural subdivision of the Salvetti-complex which we call the *(tope-)rank subdivision* $\operatorname{rksd}\mathscr{S}$. It supports a combinatorial map $\widetilde{Q} : \operatorname{rksd}\mathscr{S} \to \mathscr{C} \simeq S^1$ to the Salvetti-complex $\mathscr{C}$ of the rank 1 arrangement modeling the circle.

Our main results in [10] are as follows.

Theorem 4. *The complex* $\operatorname{rksd}\mathscr{S}$ *is PL-homeomorphic to* $\mathscr{S}$*. In particular, if* $\mathscr{S} = \mathscr{S}(\mathscr{A})$ *is the Salvetti-complex of a real arrangement* $\mathscr{A}$*, then* $\operatorname{rksd}\mathscr{S}$ *is homotopy equivalent to the complexified complement* $\mathfrak{X}(\mathscr{A})$*.*

Theorem 5. *The map* $\widetilde{Q} : \operatorname{rksd}\mathscr{S} \to \mathscr{C}$ *is a poset quasi-fibration.*

Theorem 6. *There is a (homotopy) commutative square*

$$\begin{tikzcd} \operatorname{rksd}\mathscr{S}(\mathscr{A}) \arrow[r, "\tilde{Q}"] \arrow[d, "\simeq"] & \mathscr{C} \arrow[d, "\simeq"] \\ \mathfrak{X}(\mathscr{A}) \arrow[r, "f"] & \mathbb{C}^{\times}, \end{tikzcd}$$

where the vertical maps are homotopy equivalences.

We can now define the *combinatorial Milnor fiber* of an oriented matroid $\mathscr{M}$ as $\widetilde{F}(\mathscr{M}) := \widetilde{Q}^{-1}(y)$ where y is a vertex of the circle complex $\mathscr{C}$. Then, Theorem 6 and some standard results from homotopy theory yield the following.

Theorem 7. *The combinatorial Milnor fiber* $\widetilde{F}(\mathscr{M})$ *is homotopy equivalent to the geometric Milnor fiber* $F(\mathscr{A})$*.*

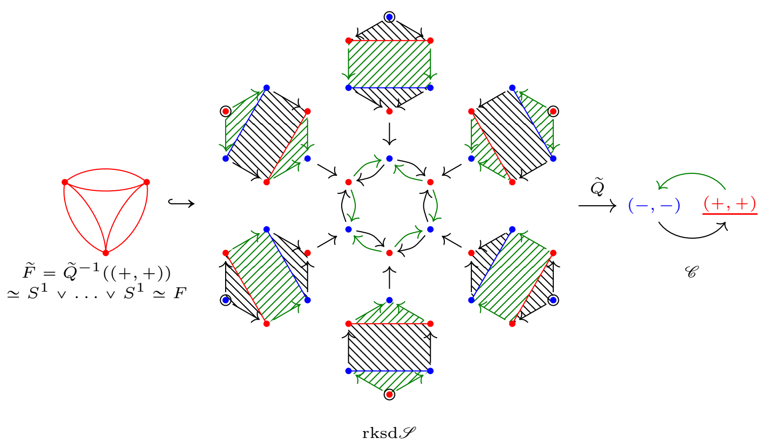

FIGURE 1. The combinatorial Milnor fibration on the tope-rank subdivision of the Salvetti complex, and its fiber, homotopy equivalent to a wedge of 4 circles, consistent with the geometric Milnor fiber.

We illustrate this central result with the following example.

Example 8. Let $\mathscr{A}$ be the arrangement in $\mathbb{R}^2$ with defining polynomial $Q = xy(x-y)$. Then one can see that the Milnor fiber $F = Q^{-1}(1)$ is a 3-punctured torus which is homotopy equivalent to a wedge of 4 circles. Figure 1 displays the tope-rank subdivision $\operatorname{rksd}\mathscr{S}$ of $\mathscr{S}(\mathscr{A})$ and the combinatorial Milnor fibration map $\widetilde{Q}$ where the preimages of cells in $\mathscr{C}$ are colored accordingly. We see that the combinatorial Milnor fiber $\widetilde{F}$ is homotopy equivalent to a wedge of 4 circles as well, in accordance with Theorem 7.

In conclusion, our regular CW-complex $\widetilde{F}$ respectively its face poset can be explicitly described in terms of the combinatorial data given by an oriented matroids. Moreover, it can easily be implemented e.g. in the computer algebra system GAP [6] to compute (co)homology or other invariants of the Milnor fiber for examples. We thus obtain a new tool to study the behavior of Milnor fiber invariants, in particular in view of the several long standing open questions regarding their combinatorial nature.

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Generation of the derivation module of graphic hyperplane arrangements

Leonie Mühlherr

A hyperplane arrangement $\mathscr{A}$ can be studied through various lenses, e.g., its combinatorial structure given by the intersection lattice $L(\mathscr{A})$ or its algebraic side by analysing the module of logarithmic derivations $D(\mathscr{A})$.

We are interested in freeness of graphic hyperplane arrangements:

Definition 1. Let $\mathbb{K}$ be a field and let $V = \mathbb{K}^l$. Let $x_1, .., x_l$ be a basis for the dual space V^*. Given a simple, undirected graph $G = (V, E)$, define an arrangement $\mathscr{A}(G)$ by

$$\mathscr{A} = \{\ker(x_i - x_j) | \{i, j\} \in E\}$$

In this case we are able to apply results from graph theory for our characterizations. Moreover, they are specific examples of Weyl sub-arrangements.

Definition 2 (Saito '79)**.** A $\mathbb{K}$-linear map $\theta : S \to S$ is a derivation if for $f, g \in S$:

$$\theta(f \cdot g) = f \cdot \theta(g) + g \cdot \theta(f).$$

Let $Der_{\mathbb{K}}(S)$ be the S-module of derivations of S.
Define an S-submodule of $Der_{\mathbb{K}}(S)$, called the module of $\mathscr{A}$-derivations, by

$$D(\mathscr{A}) = \{\theta \in Der_{\mathbb{K}}(S) \,|\, \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathscr{A}\}.$$

The arrangement $\mathscr{A}$ is called free if $D(\mathscr{A})$ is a free S-module.

A well-established result by Stanley, Edelman and Reiner ([3]) equates the class of chordal graphs with the class of free graphic hyperplane arrangements. More recently, Tran and Tsujie ([5]) proved that the subgraph class of strongly chordal graphs corresponds to the class of MAT-free arrangements. A result of a joint work with Abe, Kühne and Mücksch is that the class of weakly chordal graphs correspond to graphic arrangements with projective dimension 1, the latter being defined again as those with derivation module of projective dimension 1 (see [1]).

This last proof partially relied on finding explicit generators for $D(\mathscr{A}(G))$ for G so-called k-antiholes. The structure of the generators is closely related to the connectivity of the antiholes and their separators and with a focus on these properties, we can generalize the concept to all graphs.

1. Graph theory background

Based on [2], we first define a few graph theoretical objects that will be important going forward, starting with the basic notion of a graph:

Definition 3. A simple graph G on a set V is a tuple (V, E), such that $E \subseteq \binom{V}{2}$. E is the set of edges connecting the vertices in V.

The graph $G^C = \left(V, \binom{V}{2} \backslash E\right)$ is called the *complement graph* to G.

The complement of a cycle $C_k, k \geqslant 5$ is called an *antihole*.

For sets A, B in $V(G)$ we say that $X \subseteq V(G)$ separates A and B if every $(A-B)$-path in G contains a vertex in X. A set of vertices X separates two vertices a, b if it separates the sets $\{a\}, \{b\}$, but $a, b \notin X$ and we say that X separates G if it separates some two vertices $a, b \in V(G)$. X is called a *(a,b)-separator* or *separator*.

Definition 4. A set $T \subset V(G)$ is called a minimal (a, b)-separator if it is an (a, b)-separator and no proper subset of X separates a and b.

T is called a minimal separator of G if it is a minimal (a, b)-separator for some vertices a, b.

Closely connected to the notion of separators is the connectivity of a graph:

Definition 5. A graph G is called k-connected, $k \in \mathbb{N}$ if $|V(G)| > k$ and $G \backslash X$ is connected for every $X \subset V(G)$ with $|X| < k$.

Stated differently, no two vertices in a k-connected graph are separated by fewer than k other vertices.

The greatest integer, such that G is k-connected is called the *connectivity* of G, denoted $\kappa(G)$. For instance, the connectivity of the complete graph is $n-1$ for $|V(G)| = n$.

2. Main characters

The motivation for the definition of these new types of generators is the following result of a previous work (see [1]):

Theorem 6 (Abe, Kühne, Mücksch, M. '23)**.**

$$D(\mathscr{A}(C_\ell^C)) = \langle \theta_0, \ldots, \theta_{\ell-3}, \varphi_1, \ldots, \varphi_\ell \rangle_S,$$

where

$$\theta_i := \sum_{j=1}^{\ell} x_j^i \partial_{x_j} \ (i \geqslant 0) \text{ and } \varphi_i := \prod_{j \in [\ell] \backslash \{i-1, i, i+1\}} (x_i - x_j) \partial_{x_i}.$$

The connectivity of the l-antihole is $l-3$. The sets $[\ell] \backslash \{i-1, i, i+1\}$ are its minimal separators.

The following definition generalizes this:

Definition 7. Let T be a separator of G and C a connected component of $G\backslash T$. We can define a derivation of $D(\mathscr{A}(G))$ for each component by

$$\theta_C^T = \sum_{j\in C}\prod_{t\in T}(x_j - x_t)\cdot\partial_{x_j}.$$

These derivations are elements of the derivation module for all separators T and their components.

The following of our results establish the relations between the graph theoretical properties mentioned in the previous section and the structure of the module of logarithmic $\mathscr{A}$-derivations.

Proposition 8 (M., '24$^+$)**.** *Let G be a k-connected graph on l vertices, then*

$$\bigoplus_{i<k} D(\mathscr{A}(G))_i = \bigoplus_{i<k} D(\mathscr{A}(K_l))_i$$

That is, the derivations of polynomial degree smaller than k of $D(\mathscr{A}(G))$ coincide with the ones from the braid arrangement.

Theorem 9 (M., '24$^+$)**.** *Let G be a graph and $\mathscr{A}(G)$ the associated graphic hyperplane arrangement. Then $D(\mathscr{A}(G))$ can be generated using only $\theta_0,\dots,\theta_{\kappa(G)}$ and separator-based $\mathscr{A}$-derivations for a specific set of separators and components.*

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Permutation action on Chow rings of matroids

Anastasia Nathanson

(joint work with Robert Angarone and Victor Reiner)

In groundbreaking work, Adiprasito, Huh and Katz [1] affirmed long-standing conjectures of Rota–Heron–Welsh and Mason about vectors and matroids via a new methodology. Their work employed a certain graded $\mathbb{Z}$-algebra $A = \bigoplus_{k=0}^{r} A^k$ called the *Chow ring* for a matroid M of rank $r+1$, introduced by Feichtner and Yuzvinsky [2] as a generalization of the Chow ring of DeConcini and Procesi's *wonderful compactifications* for hyperplane arrangement complements. A remarkable integral Gröbner basis result proven by Feichtner and Yuzvinsky [2, Thm. 2] shows that each homogeneous component of A is free abelian: $A^k \cong \mathbb{Z}^{a_k}$ for a positive integer sequence $(a_0, a_1, \ldots, a_r)$.

A key step in [1] shows the sequence $(a_0, a_1, \ldots, a_r)$ is not only *symmetric* and *unimodal*, that is,

$$a_k = a_{r-k} \text{ for } r \leqslant k/2 \tag{1}$$

$$a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{\lfloor \frac{r}{2} \rfloor} = a_{\lceil \frac{r}{2} \rceil} \geqslant \cdots \geqslant a_{r-1} \geqslant a_r, \tag{2}$$

but in fact proves that A enjoys a trio of properties referred to as the *Kähler package* The first of these properties is *Poincaré duality*, proving (1) via a natural $\mathbb{Z}$-module isomorphism $A^{r-k} \cong \mathrm{Hom}_{\mathbb{Z}}(A^k, \mathbb{Z})$. The second property, called the *Hard Lefschetz Theorem*, shows that after tensoring A over $\mathbb{Z}$ with $\mathbb{R}$ to obtain $A_{\mathbb{R}} = \bigoplus_{k=0} A^k_{\mathbb{R}}$, one can find *Lefschetz elements* ω in $A^1_{\mathbb{R}}$ such that multiplication by ω^{r-2k} gives $\mathbb{R}$-linear isomorphisms $A^k_{\mathbb{R}} \to A^{r-k}_{\mathbb{R}}$ for $k \leqslant \frac{r}{2}$. In particular, multiplication by ω maps $A^k_{\mathbb{R}} \to A^{k+1}_{\mathbb{R}}$ *injectively* for $k < \frac{r}{2}$, strengthening the unimodality assertion (2).

Feichtner and Yuzvinsky defined the Chow ring $A(\mathscr{L}_M, \mathscr{G})$ for any choice of a *building set* $\mathscr{G}$ inside the *lattice of flats* $\mathscr{L}_M$ for the matroid M and gave their integral Gröbner basis presentation in that context. While the results of [1] were proven for the *maximal building set* $\mathscr{G} = \mathscr{L}_M \backslash \{\hat{0}\}$, the Chow ring satisfies the Kähler package for any building set (and even for Chow rings of *polymatroids*), as shown by Pagaria and Pezzoli [3, Thm. 4.21].

We are interested in how the *Poincaré duality* and *Hard Lefschetz* properties interact with symmetry. We consider any subgroup G of the group $\mathrm{Aut}(M)$ of symmetries of the matroid M, assuming that the building set $\mathscr{G}$ is also setwise G-stable. We observe that in this situation, G acts via graded $\mathbb{Z}$-algebra automorphisms on $A(\mathscr{L}_M, \mathscr{G})$, giving $\mathbb{Z}G$-module structures on each A^k, and $\mathbb{R}G$-module structures on each $A^k_{\mathbb{R}}$. One can also check that $A^r \cong \mathbb{Z}$ with trivial G-action, under one additional technical assumption, that $\mathscr{G}$ contains the ground set of the matroid. From this, the Poincaré duality pairing immediately gives rise to a $\mathbb{Z}G$-module isomorphism

$$A^{r-k} \cong \mathrm{Hom}_{\mathbb{Z}}(A^k, \mathbb{Z}) \tag{3}$$

where g in G acts on φ in $\mathrm{Hom}_{\mathbb{Z}}(A^k, \mathbb{Z})$ via $\varphi \mapsto \varphi \circ g^{-1}$; and $A^{r-k} \cong \mathrm{Hom}_{\mathbb{R}}(A^k, \mathbb{R})$ as $\mathbb{R}G$-modules. Furthermore, we observe that one can pick a Lefschetz element ω which is G-fixed, giving $\mathbb{R}G$-module isomorphisms and injections

$$\begin{aligned} A^k_{\mathbb{R}} &\xrightarrow{\sim} A^{r-k}_{\mathbb{R}} \quad \text{for } r \leqslant \frac{k}{2} \\ a &\longmapsto a \cdot \omega^{r-2k} \end{aligned} \tag{4}$$

$$\begin{aligned} A^k_{\mathbb{R}} &\hookrightarrow A^{k+1}_{\mathbb{R}} \quad \text{for } r < \frac{k}{2} \\ a &\longmapsto a \cdot \omega. \end{aligned} \tag{5}$$

Our goal is to use Feichtner and Yuzvinsky's Gröbner basis result, along with some combinatorics of *nested sets* , to prove a combinatorial strengthening of the isomorphisms and injections (3), (4), (5). The building set $\mathscr{G}$ distinguishes certain subsets $N = \{F_1, \ldots, F_\ell\} \subset \mathscr{G}$ called $\mathscr{G}$*-nested sets.* To each flat F in the $\mathscr{G}$-nested set N, we need a crucial quantity

$$m_N(F) := \mathrm{rk}(F) - \mathrm{rk}(\vee N_{<F}) \tag{6}$$

where $\vee N_{<F}$ denotes the lattice join in $\mathscr{L}_M$ of all elements of N strictly below F. Then the *Chow ring* $A(\mathscr{L}_M, \mathscr{G})$ of M with respect to the building set $\mathscr{G}$ is presented as a quotient of the polynomial ring $S := \mathbb{Z}[x_F]$ having one variable x_F for each flat F in $\mathscr{G}$. The presentation takes the form $A(\mathscr{L}_M, \mathscr{G}) := S/(I+J)$ where I, J are certain ideals of S. Feichtner and Yuzvinsky exhibited a Gröbner basis for $I + J$ that leads to the following standard monomial $\mathbb{Z}$-basis for $A(\mathscr{L}_M, \mathscr{G})$, which we will call the *FY-monomials* of M:

$$\begin{aligned}\mathrm{FY} := \{x_{F_1}^{m_1} \cdots x_{F_\ell}^{m_\ell} : N := \{F_1, \cdots, F_\ell\} \text{ is } \mathscr{G}\text{-nested},\\ \text{and } 0 \leqslant m_i < m_N(F_i) \text{ for all } i.\}\end{aligned}$$

The subset FY^k of FY-monomials $x_{F_1}^{m_1} \cdots x_{F_\ell}^{m_\ell}$ of total degree $m_1 + \cdots + m_\ell = k$ then gives a $\mathbb{Z}$-basis for A^k. One can readily check that the group G permutes the $\mathbb{Z}$-basis FY^k for A^k, endowing A^k with the structure of a *permutation representation*, or *G-set.*

Theorem 1. *Let M be a simple matroid rank $r + 1$ on ground set E. Let G be a group automorphisms of M, and $\mathscr{G}$ a building set in $\mathscr{L}_M$ that contains E, is setwise G-stable, and satisfies this stabilizer condition*[1]*:*

$$\begin{aligned}&\textit{any } \mathscr{G}\textit{-nested set } N = \{F_i\}_{i=1,2,\ldots,\ell} \textit{ and } g \in G \textit{ with } g(N) = N\\ &\qquad\qquad \textit{will have } g(F_i) = F_i \textit{ for } i = 1, 2, \ldots, \ell.\end{aligned} \tag{7}$$

Then there exist

(i) *G-equivariant bijections $\pi : \mathrm{FY}^k \xrightarrow{\sim} \mathrm{FY}^{r-k}$ for $k \leqslant \frac{r}{2}$, and*
(ii) *G-equivariant injections $\lambda : \mathrm{FY}^k \hookrightarrow \mathrm{FY}^{k+1}$ for $k < \frac{r}{2}$.*

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Second homotopy classes for elliptic Weyl group orbit spaces?

Kyoji Saito

We start with the back grounds and the motivation of the present work in progress.

The fundamental group of the regular orbit space for a complexified finite reflection group action is determined by E. Brieskorn [1], where the relations are homogeneous [1], called Artin braid relations, and the groups are called the Artin groups (of finite type). The higher homotopy groups of the orbit spaces are shown to vanish, i.e. they are $K(\pi,1)$-spaces, by P. Deligne [3], whose proof is based on the fact that the monoid associated with the Artin relation, so called Artin monoid, is embeddable into the Artin group and carries the lattice property [2].

The regular orbit space for a complexified affine Weyl group action was shown to be $K(\pi,1)$-space by G. Paolini and M. Salvetti [6]. In this case, the affine Weyl group is still a Coxeter group and the fundamental groups are presented by the homogeneous Artin braid relations, called an affine Artin group. However due to the lack of the least common mulitiple of all simple generators, one need to consider the dual Artin monoid instead of the Artin monoid [5].

As the next stage, we are interested in the topology of the regular orbit spaces of elliptic Weyl groups, described as follows [9]. An elliptic root system (R,I) is defined as a generalized root system whose associated form I is positive semi-definite of radical rank 2. The elliptic Weyl group $\widetilde{W}(R)$, generated by the reflections for the roots in R, acts properly discontinuously on a complex half space $\widetilde{\mathbb{E}}$, called the elliptic period domain, so that the quotient variety $\widetilde{W}(R)\backslash\!\backslash\widetilde{\mathbb{E}}$ is isomorphic to a smooth complex half space. The set of irregular orbits (=the quotient image in $\widetilde{W}(R)\backslash\!\backslash\widetilde{\mathbb{E}}$ of reflection hyperplanes of $\widetilde{W}(R)$) form a discriminant divisor $\widetilde{D}$. Here, we note that the elliptic Weyl group is no-longer a Coxeter group, but is presented by relations, called elliptic Coxeter relations, defined on the elliptic diagram [8].

Recently, a homogeneous presentation of the fundamental group of the elliptic regular orbit space $(\widetilde{W}(R)\backslash\!\backslash\widetilde{\mathbb{E}})\backslash\widetilde{D}$ is given by a joint work of Yoshihisa Saito and the author [7], where the relations are some generalizations, called elliptic braid relations, of Artin braid relations (which are some homogenizations of elliptic Coxeter relations). However, the elliptic Artin monoid defined by the elliptic braid relations is no-longer canncellative [10] [2] Further more, it was observed that such non-cancellative tuples appear not "isolatedly" but appear always as a pair or a quadruple. This is an obvious "obstruction" if one want to proceed the analogous of classical proof of $K(\pi,1)$-ness for regular orbit spaces. This causes a doubt that the regular orbit space $(\widetilde{W}(R)\backslash\!\backslash\widetilde{\mathbb{E}})\backslash\widetilde{D}$ has non-trivial higher homotopy classes?

In order to answer (partially) to this doubt, we develop a new method to construct second homotopy classes associated to non-cancellative monoids [11].

[1]We call a defining relation of a discrete group *semi-positive* if it is given by an equality $P=Q$ between two semi-positive words P and Q in the letters of the generating set L. In particular, if the lengths of P and Q are equal, we call it homogeneous. See also Footnote 3.

[2]A monoid A is called canncellative if a relation $abd \sim acd$ holds for $a,b,c,d \in A$, then the relation $b \sim c$ holds. If $b \not\sim c$, we call (a,b,c,d) a non-canncellative tuple, and (b,c) its kernel.

Namely, we start with a connected CW-complex W equipped with a semi-positive presentation [3] $(L, \mathscr{R})$ of its fundamental group, and consider the associated monoid $A^+(L,\mathscr{R}) := \langle L, \mathscr{R}\rangle^+$. This algebraically defined monoid is geometrically realized as the equivalence classes of positive loops $L^* = \cup_{k=0}^{\infty} L^k$ divided by the relations generated by geometric liftings $\mathscr{R}_g$ of the relations $\mathscr{R}$ into the relative homotopy group $\pi_2(W, W_1, *)$. Let a twin $\tau := \{(a,b,c,d),(a',b,c,d')\}$, i.e. a pair over the same kernel (b,c) of non-cancellative tuples, be given. Using the above geometric description of the monoid $A^+(L,\mathscr{R})$, we construct the following two sets [11]:

- $\Pi(\tau) \subset \pi_2(W, *)$ a subset, called the Π-class for the twin τ,
- $G(\tau) \subset \pi_2(W, *)$ a subgroup, called the inertia group for the twin τ,

where $\Pi(\tau)$ form a single coset class of the group $G(\tau)$.

At this writing, we have neither a characterization nor a general criterion of the non-triviality of the Π-classes, but expect that the Π-classes are non-trivial for "good cases". Actually, we show that the second homotopy classes of the complement of 3 lines arrangement studied by A. Hattori (1974) [4] is reconstructed as the Π-classes w.r.t. Yoshinaga's presentation of the fundamental group [12].

Finaly, we conjecture

Conjecture. *The* Π*-classes associated with the twins of non-cancellatives of elliptic Artin monoids given in* [10] *are non-trivial, and the inertial groups are trivial.*

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[3] We mean that $(L, \mathscr{R})$ is a semi-positive presentation if L is a set of free generators of $\pi_1(W_1, *)$ (here, W_1 is the 1-skeleton and $*$ is the base point) and $\mathscr{R}$ is a system of fundamental relations whose elements has the form $P_i = Q_i$ $(i = 1, \cdots, m)$ s.t. P_i, Q_i are semi-positive words in L.

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Topology and combinatorics of decomposable arrangements

Alexander I. Suciu

1. Lie algebras of arrangements

The *holonomy Lie algebra* of a complex arrangement $\mathscr{A}$ is the quotient of the free Lie algebra on degree-1 generators indexed by the hyperplanes of $\mathscr{A}$, modulo the ideal generated by a certain set of quadratic relations depending only on the first two levels of the intersection lattice, $L(\mathscr{A})$,

$$\mathfrak{h}(\mathscr{A}) = \mathrm{Lie}(x_H : H \in \mathscr{A})\Big/\mathrm{ideal}\left\{\left[x_H, \sum\nolimits_{X\supset K} x_K\right] \;:\; \begin{matrix} X \in L_2(\mathscr{A}) \\ X \supset H \end{matrix}\right\}. \tag{1.1}$$

For any field $\daleth$, the universal enveloping algebra $U(\mathfrak{h}(\mathscr{A})\otimes\daleth)$ is isomorphic to $\overline{A}^{!}$, the quadratic dual of the quadratic closure of the Orlik–Solomon algebra over $\daleth$.

Now let $M = M(\mathscr{A})$ be the complement of $\mathscr{A}$, let $G = G(\mathscr{A})$ be its fundamental group, and let $\mathrm{gr}(G) = \bigoplus_{k\geqslant 1} \gamma_k(G)/\gamma_{k+1}(G)$ be the graded Lie algebra associated to the lower central series filtration of G, defined by $\gamma_1(G) = G$, $\gamma_2(G) = G'$, and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$. There is then a surjective morphism of graded Lie algebras, $\mathfrak{h}(\mathscr{A}) \twoheadrightarrow \mathrm{gr}(G)$, which is an isomorphism in degrees up to 3. The Chen Lie algebra of G is defined as $\mathrm{gr}(G/G'')$. There is also a surjection from $\mathrm{gr}_k(G) \twoheadrightarrow \mathrm{gr}_k(G/G'')$, which is an isomorphism for $k \leqslant 3$. Setting $\phi_k(G) = \mathrm{rank}\,\mathrm{gr}_k(G)$ and $\theta_k(G) = \mathrm{rank}\,\mathrm{gr}_k(G/G'')$, we have $\phi_k(G) \geqslant \theta_k(G)$, with equality for $k \leqslant 3$.

Since M is formal, the group G is 1-formal; therefore, the map $\mathfrak{h}(\mathscr{A})\otimes\mathbb{Q} \to \mathrm{gr}(G)\otimes\mathbb{Q}$ is an isomorphism, and the LCS ranks $\phi_k(G)$ are determined by $L_{\leqslant 2}(\mathscr{A})$. As noted in [9], though, the groups $\mathrm{gr}_k(G)$ may have non-zero torsion for k large, and this raised the question whether the torsion in $\mathrm{gr}(G)$ is combinatorially determined. This question was answered in the negative in [1]: there is a pair of arrangements with isomorphic lattices, $\mathscr{A}^{\pm}$, such that $\mathrm{gr}_k(G^+) \cong \mathrm{gr}_k(G^-)$ for $k \leqslant 3$, yet $\mathrm{tors}(\mathrm{gr}_4(G^+)) \ncong \mathrm{tors}(\mathrm{gr}_4(G^-))$.

2. Decomposable arrangements

For each flat $X \in L_2(\mathscr{A})$, consider the localized arrangement $\mathscr{A}_X = \{H \in \mathscr{A} : H \supset X\}$. This is a pencil of $|X| = \mu(X) + 1$ hyperplanes, where $\mu\colon L(\mathscr{A}) \to \mathbb{Z}$ is the Möbius function; thus, $M(\mathscr{A}_X)$ is a classifying space for the group $G(\mathscr{A}_X) \cong F_{\mu(X)} \times \mathbb{Z}$. The inclusion $\mathscr{A}_X \subset \mathscr{A}$ gives rise to an injective map, $j^X\colon M(\mathscr{A}) \hookrightarrow M(\mathscr{A}_X)$, which in turn induces a split surjection on fundamental groups, $j^X_\sharp\colon G(\mathscr{A}) \twoheadrightarrow G(\mathscr{A}_X)$. The maps j^X assemble into a map $j\colon M(\mathscr{A}) \to \prod_X M(\mathscr{A}_X)$; the induced homomorphism

$$j_\sharp\colon G(\mathscr{A}) \longrightarrow \prod\nolimits_{X\in L_2(\mathscr{A})} G(\mathscr{A}_X) =: G(\mathscr{A})^{\mathrm{loc}} \tag{2.1}$$

yields a morphism between the respective holonomy Lie algebras,

$$\mathfrak{h}(j_\sharp)\colon \mathfrak{h}(\mathscr{A}) \longrightarrow \prod_{X\in L_2(\mathscr{A})} \mathfrak{h}(\mathscr{A}_X) =: \mathfrak{h}(\mathscr{A})^{\mathrm{loc}}. \tag{2.2}$$

It was shown in [5] that the map $\mathfrak{h}_k(j_\sharp)$ is a surjection for $k \geqslant 3$ and an isomorphism for $k = 2$. Therefore, $\phi_k(G(\mathscr{A})) \geqslant \sum_X \phi_k(F_{\mu(X)})$ for $k \geqslant 2$, with equality for $k = 2$, thereby recovering a previous result of Falk.

The arrangement $\mathscr{A}$ is said to be *decomposable* if the map $\mathfrak{h}_3(j_\sharp)$ is an isomorphism; that is to say, $\mathfrak{h}_3(\mathscr{A})$ is free abelian of rank as small as possible, namely,

$$\operatorname{rank}\mathfrak{h}_3(\mathscr{A})^{\mathrm{loc}} = 2\sum_{X\in L_2(\mathscr{A})} \binom{\mu(X)+1}{3} \tag{2.3}$$

This purely combinatorial property is inherited by sub-arrangements and preserved under products of arrangements. When $\mathscr{A}$ is decomposable, the maps $\mathfrak{h}'(\mathscr{A}) \to \mathfrak{h}'(\mathscr{A})^{\mathrm{loc}}$ and $\mathfrak{h}(\mathscr{A}) \to \operatorname{gr}(G(\mathscr{A}))$ are isomorphisms, see [5]. It follows that $\mathfrak{h}_k(\mathscr{A}) \cong \operatorname{gr}_k(G(\mathscr{A}))$ for all $k \geqslant 1$, and all these groups are torsion-free, with ranks $\phi_k(G) = \sum_{X\in L_2(\mathscr{A})} \phi_k(F_{\mu(X)})$. Moreover, if $\mathscr{A}$ and $\mathscr{B}$ are decomposable and $L_{\leqslant 2}(\mathscr{A}) \cong L_{\leqslant 2}(\mathscr{B})$, then $\operatorname{gr}_{\geqslant 2}(G(\mathscr{A})) \cong \operatorname{gr}_{\geqslant 2}(G(\mathscr{B}))$. In fact, as shown in [7], the nilpotent quotients $G(\mathscr{A})/\gamma_k(G(\mathscr{A}))$ and $G(\mathscr{B})/\gamma_k(G(\mathscr{B}))$ are isomorphic, for all $k \geqslant 2$.

We say that $\mathscr{A}$ is decomposable over a field $\daleth$ if the map $\mathfrak{h}_3(j_\sharp) \otimes \daleth$ is an isomorphism (an equivalent definition for $\daleth = \mathbb{Q}$ appeared in [8]). Decomposability implies $\daleth$-decomposability, but the converse is not true, in general. On the other hand, for graphic arrangements all notions of decomposability are equivalent; moreover, $\mathscr{A} = \mathscr{A}(\Gamma)$ is decomposable if and only if Γ contains no K_4 subgraph, see [5].

3. Alexander invariants of arrangements

Once again, let $\mathscr{A}$ be an arrangement, with group $G = G(\mathscr{A})$. The *Alexander invariant of* $\mathscr{A}$ is defined as $B(\mathscr{A}) = G'/G''$, viewed as a module over the group ring $R = \mathbb{Z}[G/G'] = \mathbb{Z}[H_1(M;\mathbb{Z})]$, see [2, 11].

For each flat $X \in L_2(\mathscr{A})$, we also have a “local” Alexander invariant, $B(\mathscr{A}_X)$, viewed as a module over $R_X = \mathbb{Z}[H_1(M_X;\mathbb{Z})]$. The homomorphism $j^X_\sharp\colon G(\mathscr{A}) \to G(\mathscr{A}_X)$ induces a morphism $B(j^X_\sharp)\colon B(\mathscr{A}) \to B(\mathscr{A}_X)$, which covers the ring map $\tilde{j}^X_*\colon R \to R_X$ induced by $j^X_*\colon H_1(M;\mathbb{Z}) \twoheadrightarrow H_1(M_X;\mathbb{Z})$. Assembling these global-to-local morphisms, we obtain an R-morphism, $\Pi\colon B(\mathscr{A}) \to B(\mathscr{A})^{\mathrm{loc}}$, where $B(\mathscr{A})^{\mathrm{loc}} := \bigoplus_{X\in L_2(\mathscr{A})} B(\mathscr{A}_X)_{\tilde{j}^X_*}$ is the R-module obtained from $\bigoplus_X B(\mathscr{A}_X)$ by restriction of scalars.

Now let $I = \ker(\varepsilon\colon R \to \mathbb{Z})$ be the augmentation ideal, and let $\widehat{B}$ be the completion of $B = B(\mathscr{A})$ in the I-adic topology. The R-module B is said to be *separated* if $\bigcap_{k\geqslant 1} I^k B = \{0\}$, or, equivalently, the map $B \to \widehat{B}$ is injective (alternatively, the group G/G'' is residually nilpotent).

Next, we define the *infinitesimal Alexander invariant of* $\mathscr{A}$ as $\mathfrak{B}(\mathscr{A}) = \mathfrak{h}'(\mathscr{A})/\mathfrak{h}''(\mathscr{A})$, viewed as a graded module over the polynomial ring $S = \operatorname{Sym}[H_1(M(\mathscr{A});\mathbb{Z})] \cong \operatorname{gr}(R)$. This module is generated in degree 0, while $\mathfrak{B}_1(\mathscr{A}) \cong \operatorname{gr}_1(B(\mathscr{A})) \cong$

$\mathfrak{h}_3(\mathscr{A})$. Moreover, as shown in [4], $\sum_{n\geqslant 0}\theta_{k+2}(G(\mathscr{A}))t^k = \mathrm{Hill}(\mathfrak{B}(\mathscr{A})\otimes\mathbb{Q}, t)$. To each $X \in L_2(\mathscr{A})$ there corresponds $\mathfrak{B}(\mathscr{A}_X)$, a module over the ring $S_X = \mathrm{Sym}[H_1(M(\mathscr{A}_X);\mathbb{Z})] \cong \mathrm{gr}(R_X)$. As before, we obtain a morphism of graded S-modules, $\bar{\Pi}\colon \mathfrak{B}(\mathscr{A}) \to \mathfrak{B}(\mathscr{A})^{\mathrm{loc}}$.

Theorem 1 ([14])**.** *The morphisms $\mathfrak{B}(\mathscr{A}) \to \mathfrak{B}(\mathscr{A})^{\mathrm{loc}}$ and $B(\mathscr{A})\otimes\mathbb{Q} \to B(\mathscr{A})^{\mathrm{loc}}\otimes\mathbb{Q}$ are surjective.*

As an application of this theorem, we recover the following lower bound for the Chen ranks of arrangement groups, first established in [2] by other methods:

$$\theta_k(G(\mathscr{A})) \geqslant (k-1)\sum_{X\in L_2(\mathscr{A})}\binom{\mu(X)+k-2}{k}, \tag{3.1}$$

for all $k \geqslant 2$, with equality for $k = 2$.

4. Decomposable Alexander invariants

We say that the Alexander invariant of an arrangement $\mathscr{A}$ decomposes if the canonical map $B(\mathscr{A}) \to B(\mathscr{A})^{\mathrm{loc}}$ is an isomorphism. A similar definition was first made in [2] in regards to the I-adic completion of this map, $\widehat{B(\mathscr{A})} \to \widehat{B(\mathscr{A})}^{\mathrm{loc}}$. In the same spirit, we say that the infinitesimal Alexander invariant decomposes if the map $\mathfrak{B}(\mathscr{A}) \to \mathfrak{B}(\mathscr{A})^{\mathrm{loc}}$ is an isomorphism. In all three cases, analogous definitions work over a field $\daleth$. Furthermore, if $B(\mathscr{A})$ is decomposable (over $\daleth$), then $B(\mathscr{A})$ is separated (over $\daleth$).

A natural question arises: What is the relationship between the decomposability of an arrangement $\mathscr{A}$—a purely combinatorial notion, that depends only on $L_{\leqslant 2}(\mathscr{A})$—and that of $B(\mathscr{A})$—a notion that depends *a priori* on the topology of $M(\mathscr{A})$? The next result provides a fairly complete answer to this question.

Theorem 2 ([14])**.** *Let $\mathscr{A}$ be a hyperplane arrangement. Then,*

(1) $\mathfrak{B}(\mathscr{A})$ is decomposable (over $\mathbb{Q}$) if and only if $\mathscr{A}$ is decomposable (over $\mathbb{Q}$).
(2) $B(\mathscr{A})$ is decomposable (over $\mathbb{Q}$) if and only if $\mathscr{A}$ is decomposable and $B(\mathscr{A})$ is separated (over $\mathbb{Q}$).

As an application of this theorem, we show that equality holds in (3.1) for the Chen ranks of a $\mathbb{Q}$-decomposable arrangement. Similar formulas for $\theta_k(G(\mathscr{A}))$ were given in [2, 5] under the (possibly stronger) assumption that $\mathscr{A}$ is decomposable.

As another application of Theorem 2, we determine the degree-1 *resonance varieties* (the jump loci of the Koszul complex associated to the cohomology algebra) and the *characteristic varieties* (the jump loci for homology in rank 1 local systems) of the complement of a $\mathbb{Q}$-decomposable arrangement (see [3, 6, 9, 10] for background on the jump loci of arrangements.) Fix an ordering $\mathscr{A} = \{H_1,\dots,H_n\}$. For each rank-2 flat with $\mu(X) > 1$, consider the linear subspace $L_X = \big\{x \in \mathbb{C}^n : \sum_{H\in\mathscr{A}_X} x_H = 0 \text{ and } x_H = 0 \text{ if } H \notin \mathscr{A}_X\big\}$, and let $T_X = \exp(L_X) \subset (\mathbb{C}^*)^n$ be the corresponding algebraic subtorus.

Theorem 3 ([14]). *Let $\mathscr{A}$ be a $\mathbb{Q}$-decomposable arrangement. For each $s \geqslant 1$,*
(1) $\mathscr{R}_s(M) = \bigcup_{X \in L_2(\mathscr{A}): \mu(X) > s} L_X$.
(2) If $B(\mathscr{A}) \otimes \mathbb{Q}$ is separated, then $\mathscr{V}_s(M) = \bigcup_{X \in L_2(\mathscr{A}): \mu(X) > s} T_X$.

We do not know whether the separation hypothesis in part (2) may be dropped. Without it, all the components of $\mathscr{V}_s(M)$ passing through $\mathbf{1} \in (\mathbb{C}^*)^n$ are still of the form T_X for some $X \in L_2(\mathscr{A})$, but in principle there could also be isolated torsion points in $\mathscr{V}_s(M)$.

4.1. **Milnor fibrations.** Let $\mathscr{A}$ be an arrangement in $\mathbb{C}^{d+1}$. For each hyperplane $H \in \mathscr{A}$, let $f_H \colon \mathbb{C}^{d+1} \to \mathbb{C}$ be a linear form with kernel H. Assigning a multiplicity vector $\mathbf{m} = \{m_H\}_{H \in \mathscr{A}} \in \mathbb{N}^n$ to the hyperplanes, we obtain a polynomial map, $f_{\mathbf{m}} = \prod_{H \in \mathscr{A}} f_H^{m_H} \colon \mathbb{C}^{d+1} \to \mathbb{C}$, whose restriction to the complement, $f_{\mathbf{m}} \colon M(\mathscr{A}) \to \mathbb{C}^*$, is the projection map of a smooth, locally trivial bundle, known as the *Milnor fibration* of the multi-arrangement $(\mathscr{A}, \mathbf{m})$. Let $F_{\mathbf{m}}$ be the typical fiber and let $h \colon F_{\mathbf{m}} \to F_{\mathbf{m}}$ be the monodromy of the fibration. A much-studied problem is to compute the first Betti number of $F_{\mathbf{m}}$ and find the eigenvalues of the algebraic monodromy acting on $H_1(F_{\mathbf{m}}; \mathbb{C})$, see e.g. [3, 6]. As an application of Theorem 3, we prove the following result.

Theorem 4 ([14]). *Let $\mathscr{A}$ be an arrangement of rank at least 3. Suppose $\mathscr{A}$ is decomposable over $\mathbb{Q}$ and $B(\mathscr{A}) \otimes \mathbb{Q}$ is separated. Then, for any choice of multiplicities $\mathbf{m}$ on $\mathscr{A}$, the algebraic monodromy of the Milnor fibration, $h_* \colon H_1(F_{\mathbf{m}}; \mathbb{Q}) \to H_1(F_{\mathbf{m}}; \mathbb{Q})$, is trivial.*

Using results from [12, 13], it follows that $\phi_k(\pi_1(F_{\mathbf{m}}) = \phi_k(\pi_1(M))$ and $\theta_k(\pi_1(F_{\mathbf{m}}) = \theta_k(\pi_1(M))$, for all $k \geqslant 2$. The above theorem raises a two-part question.

Question. *Let $(\mathscr{A}, \mathbf{m})$ be a multi-arrangement, and let $h \colon F_{\mathbf{m}} \to F_{\mathbf{m}}$ be the monodromy of the corresponding Milnor fibration.*
(1) If $\mathscr{A}$ is decomposable, is the monodromy action on $H_1(F_{\mathbf{m}}; \mathbb{Z})$ trivial?
(2) If $\mathscr{A}$ is decomposable over $\mathbb{Q}$, is the monodromy action on $H_1(F_{\mathbf{m}}; \mathbb{Q})$ trivial?

In general, the group $H_1(F_{\mathbf{m}}; \mathbb{Z})$ may have torsion (see [3]), even for the usual Milnor fiber $F = F(\mathscr{A})$ when $m_H = 1$ for all $H \in \mathscr{A}$ (see [15]); thus, the monodromy h may act trivially on $H_1(F_{\mathbf{m}}; \mathbb{Q})$ but not on $H_1(F_{\mathbf{m}}; \mathbb{Z})$. Nevertheless, we do not know whether this can happen within the class of ($\mathbb{Q}$-)decomposable arrangements.

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Worpitzky-compatible sets and the freeness of arrangements between Shi and Catalan

Tan Nhat Tran
(joint work with Takuro Abe)

This presentation is based on our recent work [4]. This has a strong connection with my presentation at MFO 2021 [2].

1. Background and motivation

Let $V = \mathbb{R}^\ell$ with the standard inner product $(\cdot,\cdot)$. Let Φ be an irreducible (crystallographic) root system in V. Let $\Delta := \{\alpha_1, \dots, \alpha_\ell\}$ be a set of simple roots of Φ and Φ^+ the positive system associated to Δ. For $n \in \mathbb{Z}$ and $\alpha \in \Phi^+$, define an affine hyperplane $H_\alpha^n := \{x \in V \mid (\alpha, x) = n\}$ in V. For a hyperplane arrangement $\mathscr{A}$ in V, denote by $\mathbf{c}\mathscr{A}$ the *cone* of $\mathscr{A}$. An arrangement is called **free** if its *module of logarithmic derivations* is a free module (e.g. [7, Definition 4.5]).

Definition 1. For a nonnegative integer $k \in \mathbb{Z}_{\geqslant 0}$ and a subset $\Sigma \subseteq \Phi^+$, define the following hyperplane arrangement in V:

$$\mathscr{S}_\Sigma^k = \mathscr{S}_\Sigma^k(\Phi) := \{H_\alpha^n \mid \alpha \in \Phi^+, 1-k \leqslant n \leqslant k\} \cup \{H_\alpha^{-k} \mid \alpha \in \Sigma\}.$$

The subset Σ is called **Shi-free** (resp. **free**) if the cone $\mathbf{c}\mathscr{S}_\Sigma^k$ is a free arrangement for every $k > 0$ (resp. for $k = 0$).

The (Shi-)freeness of root systems has been a central topic in the study of free arrangements for decades. For simply-laced (type ADE) root systems, a characterization for the Shi-freeness is known due to Yoshinaga [11] (Theorem 2). Our goal is to complete this characterization for all root systems (Theorem 8). First let us give more information about the freeness of $\mathbf{c}\mathscr{S}_{\Sigma}^{k}$.

(1) Let $k = 0$. When $\Sigma = \Phi^+$, the arrangement $\mathscr{A}_{\Phi^+} := \mathscr{S}_{\Phi^+}^{0}$ is known as the **Weyl arrangement** of Φ. For arbitrary Σ, $\mathscr{A}_{\Sigma} := \mathscr{S}_{\Sigma}^{0}$ is a subarrangement of $\mathscr{A}_{\Phi^+}$. The Weyl arrangement is a well-known free arrangement, e.g. [8], [7, Theorem 6.60]. Apart from type A, the freeness of arbitrary Σ is unknown in general.

(2) Let $k > 0$. When $\Sigma = \varnothing$ and $\Sigma = \Phi^+$, the arrangements $\mathrm{Shi}_{\Phi}^{[1-k,k]} := \mathscr{S}_{\varnothing}^{k}$ and $\mathrm{Cat}_{\Phi}^{[-k,k]} := \mathscr{S}_{\Phi^+}^{k}$ are known as the **extended Shi arrangement** and **extended Catalan arrangement**, respectively. Thus the arrangement $\mathscr{S}_{\Sigma}^{k}$, when Σ varies, can be regarded as an interpolation between the extended Shi and Catalan arrangements. The freeness of $\mathbf{c}\mathrm{Shi}_{\Phi}^{[1-k,k]}$ and $\mathbf{c}\mathrm{Cat}_{\Phi}^{[-k,k]}$ had been conjectured by Edelman-Reiner [6] until they were affirmatively settled by Yoshinaga [10].

(3) The most significant class for which the (Shi-)freeness is known to be true for any root system is that of the *ideals*. The **root poset** $(\Phi^+, \geqslant)$ is the poset with partial order defined by $\beta_1 \geqslant \beta_2$ if $\beta_1 - \beta_2 \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geqslant 0}\alpha_i$. A subset $\Sigma \subseteq \Phi^+$ is called an **ideal** if for $\beta_1, \beta_2 \in \Phi^+$, $\beta_1 \geqslant \beta_2, \beta_1 \in \Sigma$ implies $\beta_2 \in \Sigma$. Then for any ideal Σ and $k \geqslant 0$, the cone $\mathbf{c}\mathscr{S}_{\Sigma}^{k}$ is always free. The case $k = 0$ was first partially proved by Sommers-Tymoczko [9] and later completely settled by Abe-Barakat-Cuntz-Hoge-Terao [1]. The case $k > 0$ was done in a follow-up paper of Abe-Terao [3].

(4) There is another arrangement closely related to $\mathscr{S}_{\Sigma}^{k}$. Define

$$\mathscr{S}_{-\Sigma}^{k} := \{H_{\alpha}^{n} \mid \alpha \in \Phi^+, 1 - k \leqslant n \leqslant k\} \backslash \{H_{\alpha}^{k} \mid \alpha \in \Sigma\}.$$

Abe-Terao [3] showed that if $k > 0$, then $\mathbf{c}\mathscr{S}_{\Sigma}^{k}$ and $\mathbf{c}\mathscr{S}_{-\Sigma}^{k}$ share the freeness, i.e. $\mathbf{c}\mathscr{S}_{\Sigma}^{k}$ is free if and only if $\mathbf{c}\mathscr{S}_{-\Sigma}^{k}$ is free. If this occurs for some $k > 0$, then $\mathscr{A}_{\Sigma} = \mathscr{S}_{\Sigma}^{0}$ is also free. Thus the freeness of Σ is a necessary (but not sufficient) condition for its Shi-freeness.

(5) Towards a search for a full characterization of the Shi-freeness, it is essential to extend the class of ideals. A subset $\Sigma \subseteq \Phi^+$ is called **coclosed** if for any $\alpha \in \Sigma$ and $\beta_1, \beta_2 \in \Phi^+$ such that $\alpha = d_1\beta_1 + d_2\beta_2$ with $d_1, d_2 \in \mathbb{Z}_{>0}$, either $\beta_1 \in \Sigma$ or $\beta_2 \in \Sigma$. It is easy to see that every ideal of a root system is coclosed. For simply-laced root systems, Yoshinaga showed that the coclosedness is the missing piece of a sufficient condition for the Shi-freeness.

Theorem 2. [11, Theorem 5.1] *Let Φ be an irreducible root system of type ADE and $\Sigma \subseteq \Phi^+$. Then Σ is Shi-free if and only if Σ is free and coclosed.*

However, the theorem above is not always true for doubly-laced root systems. We complete the characterization for every root system by replacing the coclosed sets by a more general concept, the so-called *Worpitzky-compatible* sets due to Ashraf-Tran-Yoshinaga [5]. The appearance of the Worpitzky-compatibility here is interesting and unexpected as this concept has original motivation from a geometric property of alcoves of root system and a lattice point counting problem seemingly unrelated to the freeness. The Worpitzky-compatibility was the main topic of my talk in [2].

2. The main results

Let us first recall the concept of compatibility. A connected component of $V \setminus \bigcup_{\alpha\in\Phi^+, n\in\mathbb{Z}} H_\alpha^n$ is called an **alcove**. Let A be an alcove. A **wall** of A is a hyperplane that supports a facet of A. The **ceilings** of A are the walls which do not pass through the origin and have the origin on the same side as A. The **upper closure** $A^\diamondsuit$ of A is the union of A and its facets supported by the ceilings of A. Let $P^\diamondsuit := \{x \in V \mid 0 < (\alpha_i, x) \leqslant 1 \ (1 \leqslant i \leqslant \ell)\}$ be the **fundamental parallelepiped** (of the coweight lattice) of Φ. Then $P^\diamondsuit$ has the following partition:

$$P^\diamondsuit = \bigsqcup_{A\colon \text{alcove},\, A\subseteq P^\diamondsuit} A^\diamondsuit,$$

which is known as the **Worpitzky partition** (e.g. [12, Proposition 2.5]).

Definition 3. [5, Definition 4.8] A subset $\Sigma \subseteq \Phi^+$ is called **Worpitzky-compatible** in Φ, or **compatible** for short, if for each alcove $A \subseteq P^\diamondsuit$, the intersection $A^\diamondsuit \cap H_\alpha^{n_\alpha}$ of its upper closure $A^\diamondsuit$ and any affine hyperplane $H_\alpha^{n_\alpha}$ for $\alpha \in \Sigma, n_\alpha \in \mathbb{Z}$ is either empty, or contained in a ceiling $H_\beta^{n_\beta}$ of A for some $\beta \in \Sigma, n_\beta \in \mathbb{Z}$. In short, every nonempty intersection can be lifted to a facet intersection.

The compatibility was originally defined in order to make a counting formula concerning the *characteristic* and *Ehrhart quasi-polynomials* valid [5, Theorem 4.11]. It is proved that every coclosed subset is compatible [5, Proof of Theorem 4.16].

We need a few more notations and definitions. For an arrangement $\mathscr{A}$ in V, denote by $L(\mathscr{A})$ the *intersection poset* of $\mathscr{A}$. Set $L_p(\mathscr{A}) := \{X \in L(\mathscr{A}) \mid \operatorname{codim}(X) = p\}$ for $0 \leqslant p \leqslant \ell$.

Definition & Notation 4. Let Φ be an irreducible root system and let $\mathscr{A} := \mathscr{A}_{\Phi^+}$ be the Weyl arrangement of Φ. If $X \in L_p(\mathscr{A})$, then $\Phi_X := \Phi \cap X^\perp$ is a rank p root subsystem (not necessarily irreducible) of Φ. A positive system of Φ_X is taken to be $\Phi_X^+ := \Phi^+ \cap \Phi_X$. Let Δ_X be the set of simple roots of Φ_X associated to Φ_X^+. For a subset $\Sigma \subseteq \Phi^+$, denote $\Sigma_X := \Sigma \cap \Phi_X^+$.

Definition 5. A subset $\Sigma \subseteq \Phi^+$ is called

(a) **negatively coclosed** if for any $\alpha \in \Sigma$ and $\beta_1, \beta_2 \in \Phi^+$ such that $\alpha = d_1\beta_1 + d_2\beta_2$ with $d_1, d_2 \in \mathbb{Z}_{>0}$ and $(\beta_1, \beta_2) < 0$, either $\beta_1 \in \Sigma$ or $\beta_2 \in \Sigma$,

(b) **2-locally compatible** if for any $X \in L_2(\mathscr{A})$ such that Φ_X is irreducible, the localization $\Sigma_X = \Sigma \cap \Phi_X^+$ is compatible in Φ_X,

(c) 2**-locally simple** if for any $X \in L_2(\mathscr{A})$ such that Φ_X is irreducible, either Σ_X contains a simple root of Φ_X (i.e. $\Sigma_X \cap \Delta_X \neq \varnothing$) or $\Sigma_X = \varnothing$.

In the subsequent characterizations, we must distinguish some particular subsets of positive roots in a root system of type G_2.

Definition 6. Given a root system $\Phi = G_2$ with $\Delta = \{\alpha_1, \alpha_2\}$ where α_2 is the unique long simple root, define the following subsets $\Sigma \subseteq \Phi^+$:
(a) $\Sigma = \{\alpha_2\} \cup S$ with $\varnothing \neq S \subseteq \{2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$.
(b) $\Sigma = \{\alpha_1, 3\alpha_1 + 2\alpha_2\} \cup S$ with $S \subseteq \{\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \}$.

We are ready to state our first main result connecting the compatibility, a geometric property of alcoves and the negative coclosedness, a combinatorial property of roots.

Theorem 7. *Let Φ be an irreducible root system and $\Sigma \subseteq \Phi^+$. The following are equivalent.*
(1) Σ is compatible.
(2) Σ is 2-locally compatible.
(3) One of the following occurs:
  (i) If $\Phi \neq G_2$, Σ is negatively coclosed.
  (ii) If $\Phi = G_2$, Σ is negatively coclosed, or one of the seven exceptions in Definition 6(a).

Our second main result is a generalization of Theorem 2 to any root system.

Theorem 8. *Let Φ be an irreducible root system and $\Sigma \subseteq \Phi^+$. The following are equivalent.*
(1) Σ is Shi-free.
(2) Σ is free and 2-locally simple.
(3) One of the following occurs:
  (i) If $\Phi \neq G_2$, Σ is compatible and free.
  (ii) If $\Phi = G_2$, Σ is compatible, or one of the four exceptions in Definition 6(b).

We emphasize that the proofs of Theorems 7 and 8 require only the classification of all rank 2 root systems ($A_1^2, A_2, B_2 = C_2, G_2$), and the fact that given a root system $\Phi \neq G_2$, any rank 2 irreducible root subsystem of Φ is of type A_2 or B_2.

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Non $K(\pi, 1)$ arrangements

Masahiko Yoshinaga

Let $\mathscr{A} = \{H_1, \dots, H_n\}$ be a central hyperplane arrangement in $\mathbb{R}^\ell$ with $\ell \geqslant 3$. Let us denote by $M(\mathscr{A})$ the complement of the complexified arrangement $\mathscr{A} \otimes \mathbb{C}$ in $\mathbb{C}^\ell$. One of the long-standing problems is to characterize $K(\pi, 1)$ property for such spaces. There are lots of results on $K(\pi, 1)$-ness. See [4, 2] for recent developments. There are also several results on non $K(\pi, 1)$-ness. See [1, 3, 5]. However, we are still far from understanding exactly when $M(\mathscr{A})$ is $K(\pi, 1)$.

In this talk, we discuss homotopical triviality/non-triviality of certain embedded spheres.

Our result is based on the description of the complex vector space $\mathbb{C}^\ell = \mathbb{R}^\ell \oplus \sqrt{-1}\cdot\mathbb{R}^\ell$. We can identify $\mathbb{C}^\ell$ with the total space of the tangent bundle $T\mathbb{R}^\ell$. More explicitly, we consider the tangent vector $v \in T_x\mathbb{R}^\ell$ is the point $x + \sqrt{-1}v \in \mathbb{C}^\ell$.

Now we consider the unit sphere $S^{\ell-1} \subset \mathbb{R}^\ell$. We shift the sphere $S^{\ell-1}$ into the imaginary direction so that we get an embedded sphere in $M(\mathscr{A})$. Such a shift naturally gives a system of half spaces $(H_1^{\varepsilon_1}, \dots, H_n^{\varepsilon_n})$, where $\varepsilon_i \in \{+, -\}$.

The main result asserts that such an embedded sphere is homotopically trivial if and only if $\bigcap_{i=1}^n H_i^{\varepsilon_i} \neq \varnothing$.

"If" part is easily proved just by filling the sphere by a Salvetti cell. To prove "only if", we construct an explicit Borel-Moore homology cycle of degree $(\ell + 1)$ and showing the twisted intersection number with the sphere is nonzero, which concludes homotopical nontriviality.

For details, see [6].

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Reporter: Tan Nhat Tran

Participants

Prof. Dr. Takuro Abe
Dept. of Mathematics
Rikkyo University
Nishi-Ikebukuro 3-34-1, Toshima-ku
Tokyo 1718501
JAPAN

Prof. Dr. Mohamed Barakat
Department Mathematik
Universität Siegen
Walter-Flex-Str. 3
57072 Siegen
GERMANY

Dr. Dan Bath
KU Leuven
Mathematics Department
Oude Mart 13
3000 Leuven
BELGIUM

Dr. Christin Bibby
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803-4918
UNITED STATES

Dr. Filippo Callegaro
Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo, 5
56127 Pisa
ITALY

Patty Commins
Department of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S. E.
Minneapolis, MN 55455
UNITED STATES

Dr. Basile Coron
School of Mathematical Sciences
Queen Mary
University of London
Mile End Road
London E1 4NS
UNITED KINGDOM

Prof. Dr. Michael Cuntz
Institut für Algebra, Zahlentheorie
und Diskrete Mathematik
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover
GERMANY

Prof. Dr. Emanuele Delucchi
University of Applied Arts and Sciences
of Southern Switzerland
Polo Universitario - Campus Est
Via la Santa 1
6962 Lugano
SWITZERLAND

Abraham del Valle Rodríguez
Departamento de Álgebra
Universidad de Sevilla
41012 Sevilla
SPAIN

Prof. Dr. Graham Denham
Department of Mathematics
Middlesex College
University of Western Ontario
London ON N6A 5B7
CANADA

Dr. Rodica Andreea Dinu
Fachbereich Mathematik u. Statistik
Universität Konstanz
Postfach 192
78464 Konstanz
GERMANY

Dr. Michael DiPasquale
Dept. of Mathematical Sciences
New Mexico State University
1290 Frenger Mall
Las Cruces, NM 88003-8001
UNITED STATES

Dr. Matthew Douglass
Division of Mathematical Sciences
National Science Foundation
2415 Eisenhower Ave.
Alexandria, VA 22314
UNITED STATES

Dr. Clément Dupont
Institut Montpelliérain Alexander
Grothendieck (IMAG)
Université de Montpellier
Place Eugène Bataillon
34090 Montpellier
FRANCE

Dr. Christopher Eur
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge, MA 02138-2901
UNITED STATES

Prof. Dr. Michael Falk
Department of Mathematics and
Statistics
Northern Arizona University
Flagstaff, AZ 86011-5717
UNITED STATES

Prof. Dr. Eva Maria Feichtner
Fachbereich 3
Mathematik und Informatik
Universität Bremen
Bibliothekstraße 1
28359 Bremen
GERMANY

Prof. Dr. Misha Feigin
School of Mathematics and Statistics
University of Glasgow
University Place
Glasgow G12 8QQ
UNITED KINGDOM

Dr. Luis Ferroni
Department of Mathematics
Royal Institute of Technology
100 44 Stockholm
SWEDEN

Mieke Fink
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Lorenzo Giordani
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum
GERMANY

Elena Hoster
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum
GERMANY

Leo Jiang
Department of Mathematics
University of Toronto
40 St. George Street
Toronto ON, M5S 2E4
CANADA

Jun.-Prof. Dr. Lukas Kühne
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Matt Larson
Apt 418A
Department of Mathematics
Stanford University
6 Comstock cl
Stanford, CA 94305-2125
UNITED STATES

Dr. Shiyue Li
School of Mathematics
Institute for Advanced Study
1 Einstein Drive
Princeton, NJ 08540
UNITED STATES

Dr. Jacob Matherne
Department of Mathematics
North Carolina State University
SAS Hall 3260
P.O. Box 8205
27695 Raleigh
UNITED STATES

Dr. Dmitry Mineev
Department of Computer Science
Bar-Ilan University
52900 Ramat-Gan
ISRAEL

Dr. Paul Mücksch
Institut für Algebra, Zahlentheorie
und Diskrete Mathematik
Leibniz Universität Hannover
30167 Hannover
GERMANY

Leonie Mühlherr
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Dr. Luis Narvaez Macarro
Departamento de Álgebra & Instituto de
Matemáticas (IMUS)
Universidad de Sevilla
41012 Sevilla
SPAIN

Anastasia Nathanson
School of Mathematics
University of Minnesota
Twin Cities
Minneapolis, MN 55455
UNITED STATES

Dr. Roberto Pagaria
Dipartimento di Matematica
Universita di Bologna
Piazza Porta S. Donato, 5
40127 Bologna
ITALY

Prof. Dr. Götz Pfeiffer
School of Mathematical and Statistical
Sciences
University of Galway
University Road
Galway H91 TK33
IRELAND

Prof. Dr. Gerhard Röhrle
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum
GERMANY

Prof. em. Dr. Kyoji Saito
Research Institute for Mathematical
Sciences
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Prof. Dr. Mario Salvetti
Dipartimento di Matematica "L.Tonelli"
Universita di Pisa
Largo Bruno Pontecorvo, 5
56127 Pisa
ITALY

Prof. Dr. Henry K. Schenck
Department of Mathematics
Auburn University
Auburn, AL 36849
UNITED STATES

Prof. Dr. Mathias Schulze
RPTU in Kaiserslautern
Fachbereich Mathematik
Postfach 3049
67653 Kaiserslautern
GERMANY

Dr. Avi Steiner
Fakultät für Mathematik
TU Chemnitz
09107 Chemnitz
GERMANY

Dr. Christian Stump
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum
GERMANY

Prof. Dr. Alexander I. Suciu
Department of Mathematics
Northeastern University
567 Lake Hall
Boston, MA 02115-5000
UNITED STATES

Prof. Dr. Tomasz Szemberg
Department of Mathematics
University of the National Education
Commission
ul. Podchorazych 2
30-084 Kraków
POLAND

Dr. Tan Nhat Tran
Institute of Algebra, Number Theory
and Discrete Mathematics,
Faculty of Mathematics and Physics
Leibniz University Hannover
Welfengarten 1
30167 Hannover
GERMANY

Prof. Dr. Shuhei Tsujie
Department of Mathematics
Hokkaido University of Education
Asahikawa Campus
9, Hokumon-Cho
0708621 Asahikawa
JAPAN

Lorenzo Vecchi
KTH Royal Institute of Technology
Stockholm 114 28
SWEDEN

Prof. Dr. Max D. Wakefield
Department of Mathematics
United States Naval Academy
572-C Holloway Rd
Annapolis, MD 21402
UNITED STATES

Prof. Dr. Uli Walther
Department of Mathematics
Purdue University
150 N University Street
West Lafayette, IN 47907-1395
UNITED STATES

Sven Wiesner
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum
GERMANY

Prof. Dr. Masahiko Yoshinaga
Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-1, Toyonaka
Osaka 560-0043
JAPAN