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Real and Logarithmic Enumerative Geometry

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Abstract. Key topics of the workshop included enumerative geometry, Gromov–Witten invariants, and their extensions to general ground fields and tropical counting. Significant contributions were made in refined invariants, topological recursion, and Givental reconstruction. Discussions also covered the log double ramification cycle, Brill-Noether theory, Hilbert schemes of points, moduli spaces, log smooth degenerations, mirror symmetry, and the topology of real algebraic varieties.

Mathematics Subject Classification (2020): 14-XX.

Introduction by the Organizers

The workshop *Real and Logarithmic Enumerative Geometry*, covered several key research topics in the field of algebraic geometry, focusing on recent advancements and methodologies.

One major area of discussion was Enumerative Geometry and Gromov-Witten invariants. Two talks reported about quadratic enrichments that extend inariants to general ground fields (Pauli, Solomon) and the correspondence with tropical counting (Pauli, Gräfnitz). New results about refined invariants in several directions were reported by Itenberg, Schuler, Bousseau and Kennedy-Hunt. Van Garrel covered refined BPS integrality and Parker emphasized the context of holomorphic Lagrangian correspondences coming from stable map markings constrainted in complex surfaces inside Calabi-Yau threefolds. New developments in topological recursion and Givental reconstruction in the real context were reported by Guidoni and Garcia-Failde.

Another significant topic is Moduli Spaces and Mirror Symmetry. The proof about the KSBA moduli space of stable log Calabi-Yau surfaces being a finite quotient of a toric variety was reported by Argüz. The enumerative geometry of intrinsic mirror families for maximal log smooth degenerations of Fano varieties was addressed in van Garrel's talk.

The topology and degenerations of algebraic varieties, both real and complex, was another theme of the workshop. Bounds for the individual Betti numbers of smooth real fibers using real logarithmic geometry were covered by Manzaroli. Degenerations of Hilbert schemes of points on surfaces were identified in the log context by Tschanz. Gräfnitz encoded the enumerative geometry of a maximal degeneration of the complex plane in scattering diagrams.

Further highlights include new results on Brill-Noether theory (Carocci), the construction of logarithmic tautological classes on moduli spaces of stable curves via the log double ramification cycle (Schmitt) and an elegant new formula for the genus one generating series of Gromov-Witten invariants of Hilbert scheme of points of $\mathbb{C}^2$ (Pandharipande).

Overall, the workshop report highlights the interplay between real and tropical geometry, mirror symmetry, and enumerative invariants, showcasing the latest research and methodologies in these interconnected fields.

We collect the personal summaries by the three organizers.

Penka Georgieva: I very much enjoyed to learn about the recent developments around the refined counts which we saw from 3 different points of view (real, log, and physics); another more unexpected common point was around the Givental formalisme that we saw in Rahul, Elba, and Thomas talks that I thought tight very well together and were hopefully a base for discussion (again very sorry I couldn't come in person !). Within topological recursion there are also refined invariants and I'll be interested to see if they are related to those discussed at the workshop. I also followed with interest the still mysterious $\mathbb{A}^1$ homotopy theory and the results around mirror symmetry. Besides the math, I also enjoyed the fact that we had I'd say more than usual number of women speakers and a very good distribution of early, mid, and advanced career speakers.

Dan Abramovich: I was particularly happy with the many fruitful discussions that took place informally. Strikingly, every evening Rahul Pandharipande held court, in group conversations eagerly attended by young participants, in a setting for which MFO is perfect. I myself discussed a joint project with Pandharipande. I also got to discuss an exciting idea of Jake Solomon regarding spin groups and their cousins "pin" groups. I learned from Sabrina Pauli about the subtle nuances of Chern classes and equivariant Chern classes when one attempts to enhance them quadratically. We were both frustrated by the lack of sufficiently introductory material, a frustration reinforced in discussion with Solomon, perhaps an impetus for a future endeavor. I discussed equivariant Chow groups in great detail with Barbara Fantechi, especially work of my students, as well as the work of her students. Argüz, Bousseau, and Carocci collaborated intensely on a lecture series we presented together in August.

Helge Ruddat: I spent four evenings working with Bernd Siebert and Michel van Garrel making good progress on explicit enumerative period integral computations for spaces constructed from wall structures. Missing out on the social parts of the evenings, I made up for that by various chats with different sets of participants over meals and in breaks, for example with Calla Tschanz and Patrick Kennedy-Hunt about log Hilbert schemes of points; with Argüz and Bousseau about virtual counts via tropical curves; with Andres Gomez and Bernd Siebert about K-stability and SYZ; with Sabrina Pauli, Jake Solomon and Rahul Pandharipande about the existence of an orientation on the space of stable maps to $\mathbb{P}^3$; while hiking, with Brett Parker about Hyperkähler structures in the boundary condition of enumerative geometry of log Calabi-Yau threefolds. I greatly benefited from presentations of speakers that I hadn't seen before, for example Thomas Guidoni, Johannes Schmitt, Matilde Manzaroli and Yannik Schuler.

We were pleased with the success of the three lightning talks by Anna-Maria Raukh, Andres Gomez and Xianyu Hu. These short presentations allowed these junior participants to introduce themselves and to share ideas about their PhD projects.

We also benefited from the possibility to attend lectures online because

(1) one of the organizers could not personally attend for private reasons and this new circumstance arose only two weeks before the conference,
(2) the local trains broke down on the day of scheduled arrival and were replaced by hard-to-find and irregularly departing coaches. One participant got stranded in a hotel elsewhere but he could watch the first day's lectures online before finally arriving,
(3) we had further online participants in the US for whom the travel didn't fit their busy schedule.

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Workshop: Real and Logarithmic Enumerative Geometry

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Abstracts

Refined invariants for real curves

Ilia Itenberg
(joint work with Eugenii Shustin)

Refined enumerative geometry, initiated in [1, 4], became one of the central topics in enumerative geometry with important links to closed and open Gromov-Witten invariants and to Donaldson-Thomas invariants. In a big part of known examples, refined invariants appear as one-parameter deformations of complex enumerative invariants (see, for example, [1, 2, 3, 4]). In his groundbreaking paper [5], G. Mikhalkin has proposed a refined invariant provided by enumeration of real rational curves and has related this invariant to the refined tropical invariants of F. Block and L. Göttsche [1]. Namely, he introduced an integer-valued *quantum index* for real algebraic curves in toric surfaces. To have a quantum index, a real curve should satisfy certain assumptions: it has to intersect toric divisors only at real points and to be irreducible and *separating*; the latter condition means that, in the complex point set of the normalization of the curve, the complement of the real part is disconnected, *i.e.*, formed by two halves exchanged by the complex conjugation (in fact, the quantum index is associated to a half of a separating real curve, while the other half has the opposite quantum index). Mikhalkin [5] showed that, for an appropriate kind of constraints, a Welschinger-type enumeration of real rational curves (*cf.* [7, 8]) in a given divisor class and with a given quantum index produces an invariant and can be directly related to the numerator of a Block-Göttsche refined tropical invariant (represented as a fraction with the standard denominator).

The main purpose of the talk is to introduce refined invariants of Mikhalkin's type in the case of curves of genus 1 and 2. We follow the ideas of [6] and choose constraints so that every counted real curve of genus $g = 1$ or 2 appears to be a *maximal* one (*i.e.*, it has $g + 1$ global real branches), and hence is separating. More precisely, given a toric surface with the tautological real structure and a very ample divisor class, we fix maximally many real points in a generic Menelaus position on the toric boundary of the positive quadrant, where genus g curves from the given linear system must be tangent to toric divisors with prescribed even intersection multiplicities, and we fix g more generic real points inside different non-positive quadrants as extra constraints (through which the curves under consideration should pass). There are finitely many real curves of genus g matching the constraints and all these curves are separating. Their halves have quantum index, and we equip each curve with a certain Welschinger-type sign.

For some toric surfaces (including the projective plane), we prove that the signed enumeration of (halves of) real curves of genus $g = 1$ or 2 that match given constraints, belong to a given linear system, and have a prescribed quantum index does not depend on the choice of a (generic) position of the constraints. The resulting invariants are said to be *refined*. In particular, we get new real enumerative invariants (without prescribing values for quantum index) in genus one and two.

We also discuss tropical counterparts of the above refined invariants and establish a tropical algorithm allowing one to compute them.

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Quadratically enriched Gromov-Witten invariants and how to compute them

Sabrina Pauli

We introduce quadratically enriched Gromov-Witten invariants, which specialize to both Gromov-Witten and Welschinger invariants and take their values in the Grothendieck-Witt ring. We then explain how to compute them using tropical methods based on joint work with Andrés Jaramillo Puentes [4] and work in progress with Andrés Jaramillo Puentes, Hannah Markwig, and Felix Röhrle [3].

Let's first recall the definition of a special case of a non-quadratically enriched Gromov-Witten invariants. Namely, let N_d be the number of rational degree d plane curves passing through $n = 3d-1$ points in general position. This number is an invariant, i.e. it is independent of the choice of the point configuration. There is a topological interpretation of this in terms of degree from algebraic topology. Let $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d)$ be the moduli space of n-marked stable maps of genus 0 to $\mathbb{P}^2$ of degree d. Then the degree of the evaluation map $\mathsf{ev}\colon \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2_{\mathbb{C}}, d) \to (\mathbb{P}^2_{\mathbb{C}})^n$ is equal to the number of n-marked stable maps whose image is a plane rational curve of degree d passing through n points in $\mathbb{P}^2$ in general position, i.e.

$$N_d = \deg \mathsf{ev}\,.$$

If you only consider curves defined over a non-algebraically closed field, you lose invariance. For example, the number of real degree 3 rational plane curves through $n = 3d-1 = 8$ real points can be 8, 10 or 12 depending on the point configuration. Welschinger restores invariance by introducing a signed real count

[1]. That is, he counts a real curve with a sign depending on the types of real nodes of the real curve. Namely, say a real node is of type $+1$ if it is split, i.e. locally given by $x^2 - y^2 = 0$ or in other words, the two branches live over $\mathbb{R}$, and say it is of type -1 if it is non-split, i.e. locally given by $x^2 + y^2 = 0$ and the two branches live over $\mathbb{C}$. For a real plane curve, let

$$\mathrm{Wel}_{\mathbb{R}}(C) := \prod_{\text{nodes } z} \mathrm{type}(z).$$

Consider a point configuration in $\mathbb{P}^2_{\mathbb{R}}$ consisting of n_1 real points and n_2 complex points in $\mathbb{P}^2_{\mathbb{R}}$ in general position with $n_1 + 2n_2 = n = 3d - 1$. Welschinger shows that

$$W_{d,n_2} := \sum_C \mathrm{Wel}_{\mathbb{R}}(C)$$

where the sum over all real rational degree d plane curves passing through such a point configuration is invariant, i.e. independent of the choice of point configuration. There is also a topological interpretation of W_{d,n_2} as the degree of an evaluation map.

In $\mathbb{A}^1$-homotopy theory one defines a homotopy category of smooth varieties over an arbitrary field k. Here one has most of the tools of classical algebraic topology at one's disposal, and analogous phenomena over $\mathbb{C}$ and $\mathbb{R}$ are usually specializations of a common result in $\mathbb{A}^1$-homotopy theory. Since both N_d and W_{d,n_2} can be interpreted as the degree of an evaluation map, the idea to define the quadratically enriched Gromov-Witten invariants is to define them as the $\mathbb{A}^1$-degree, i.e. the degree in $\mathbb{A}^1$-homotopy theory, of an evaluation map. Let k be a field of characteristic not equal to 2. Then the $\mathbb{A}^1$-degree takes values in the Grothendieck-Witt ring $\mathrm{GW}(k)$ of non-degenerate quadratic forms over k. Recall that $\mathrm{GW}(k)$ is generated by the isometry classes $\langle a \rangle$ of quadratic forms $x \mapsto ax^2$ with $a \in k^\times$ a unit and that for L/k a finite separable field extension there is a trace map $\mathrm{Tr}\colon \mathrm{GW}(L) \to \mathrm{GW}(k)$.

From now on let k be a perfect field of characteristic not equal to 2 or 3. Let $\sigma = (L_1, \dots, L_r)$ be a sequence of finite field extensions such that $\sum_{i=1}^r [L_i : k] = n = 3d - 1$. Then there exists a twisted evaluation map

$$\mathsf{ev}_\sigma \colon (\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2_k, d))_\sigma \to \prod_{i=1}^r \mathrm{Res}_{L_i/k} \mathbb{P}^2_k.$$

A point in the target corresponds to a point configuration of r points in $\mathbb{P}^2_k$ with residue fields $L_1, \dots, L_r$. Kass-Levine-Solomon-Wickelgren show that there is a well-defined $\mathbb{A}^1$-degree of this evaluation map valued in $\mathrm{GW}(k)$ [7, 6]. This $\mathbb{A}^1$-degree is equal to the sum of the local $\mathbb{A}^1$-degrees at the stable maps mapping to rational degree d plane curves through a point configuration of r points in $\mathbb{P}^2_k$ with residue fields $L_1, \dots, L_r$ in general position. Furthermore, Kass-Levine-Solomon-Wickelgren identify the local $\mathbb{A}^1$-degree at such a stable map with the following generalization of $\mathrm{Wel}_{\mathbb{R}}(C)$ [7]. Let C be a plane rational curve defined over $\kappa(C)$. Let z be a node of C with residue field $\kappa(z)$. Then the two branches of z are defined over $\kappa(z)(\sqrt{\alpha_z}) \in \kappa(z)^\times$. Set

$$\mathrm{Wel}^{\mathbb{A}^1}_{\kappa(C)}(C) := \left\langle \prod_{\text{nodes } z} N_{\kappa(z)/\kappa(C)}(\alpha_z) \right\rangle \in \mathrm{GW}(\kappa(C))$$

where $N_{\kappa(z)/\kappa(C)}$ is the field norm. Kass-Levine-Solomon-Wickelgren show that the local $\mathbb{A}^1$-degree at a stable map with image C equals $\mathrm{Tr}_{\kappa(C)/k}(\mathrm{Wel}^{\mathbb{A}^1}_{\kappa(C)}(C)) \in \mathrm{GW}(k)$ and thus

$$N^{\mathbb{A}^1}_{d,\sigma} := \deg^{\mathbb{A}^1}(\mathsf{ev}_\sigma) = \sum_C \mathrm{Tr}_{\kappa(C)/k}(\mathrm{Wel}^{\mathbb{A}^1}_{\kappa(C)}(C))$$

where the sum goes over all rational degree d plane curves through a point configuration through r points with residue fields $L_1, \dots, L_r$ in general position.

Now let's turn to the computation of $N^{\mathbb{A}^1}_{d,\sigma}$. Tropical geometry provides a powerful tool for solving problems in enumerative geometry, pioneered by Mikhalkin's correspondence theorem, which shows that the number of plane curves satisfying point conditions is equal to the number of its tropical counterpart counted with multiplicities. More specifically, Mikalkin proves that

$$N_d = N_d^{\mathrm{trop}} := \sum_\Gamma \mathrm{mult}_{\mathbb{C}}(\Gamma)$$

and

$$W_{d,0} = W_{d,0}^{\mathrm{trop}} := \sum_\Gamma \mathrm{mult}_{\mathbb{R}}(\Gamma)$$

where both sums are over all tropical rational degree d curves passing through a point configuration of $3d-1$ points in tropical general position.

Shustin proves a tropical correspondence theorem for computing W_{d,n_2}

$$W_{d,n_2} = W_{d,n_2}^{\mathrm{trop}} := \sum_\Gamma \mathrm{mult}_{\mathbb{R}}(\Gamma)$$

where the sum now goes over all so-called "Shustin (tropical) curves" through $n - 2n_2$ "thin" points (corresponding to real points) and n_2 "fat" points (corresponding complex points) in [2].

So one can ask if there is also a tropical correspondence theorem that identifies $N^{\mathbb{A}^1}_{d,\sigma}$ with some weighted count of tropical curves. The answer is partly yes. If $\sigma = (k, \dots, k)$ is just a sequence of n times k, then by joint work with Jaramillo Puentes [4] we have

$$N^{\mathbb{A}^1}_{d,\sigma} = N^{\mathrm{trop}}_{d,\sigma} := \sum_\Gamma \mathrm{mult}^{\mathbb{A}^1}(\Gamma)$$

where the sum goes over all tropical rational degree d curves passing through a point configuration of $n = 3d-1$ points in tropical general position. In particular, we sum over the same tropical curve as for N_d^{trop} and $W_{d,0}^{\mathrm{trop}}$. Also,

$$\mathrm{mult}^{\mathbb{A}^1}(\Gamma) = \begin{cases} \frac{\mathrm{mult}_{\mathbb{C}}(\Gamma)-1}{2}(\langle 1\rangle + \langle -1\rangle) + \langle \mathrm{mult}_{\mathbb{R}}(\Gamma)\rangle & \text{if } \mathrm{mult}_{\mathbb{C}}(\Gamma) \text{ odd} \\ \frac{\mathrm{mult}_{\mathbb{C}}(\Gamma)}{2}(\langle 1\rangle + \langle -1\rangle) & \text{if } \mathrm{mult}_{\mathbb{C}}(\Gamma) \text{ even} \end{cases}$$

A direct consequence is that

$$N_{d,\sigma}^{\mathbb{A}^1} = \frac{N_d - W_{d,0}}{2}(\langle 1\rangle + \langle -1\rangle) + W_{d,0}\langle 1\rangle$$

in $\mathrm{GW}(k)$. Furthermore, taking the rank respectively the signature of $N_{d,\sigma}^{\mathbb{A}^1}$ recovers N_d respectively $W_{d,0}$.

For more general σ there are first results in this direction the joint work in progress of Jaramillo Puentes, Markwig and Röhrle [3]. When $\sigma = (k(\sqrt{d_1}), \dots, k(\sqrt{d_{n_2}}), k, \dots, k)$ consists of quadratic and trivial field extensions, one can define $\mathrm{mult}^{\mathbb{A}^1}(\Gamma)$ such that

$$N_{d,\sigma}^{\mathbb{A}^1} = N_{d,\sigma}^{\mathbb{A}^1,\mathrm{trop}} := \sum_{\Gamma} \mathrm{mult}^{\mathbb{A}^1}(\Gamma)$$

where the sum now goes over all curves through $n - 2n_2$ "thin" points (corresponding to k-points) and n_2 "fat" points (corresponding to $k(\sqrt{d_i})$-points). Again taking the rank respectively the signature of $N_{d,\sigma}^{\mathbb{A}^1}$ recovers N_d respectively W_{d,n_2}.

The techniques in [3] in principle also work for general σ, but this still needs to be worked out.

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Expansions for Hilbert schemes of points

Calla Tschanz

The study of moduli spaces is a central topic in algebraic geometry; among moduli spaces, Hilbert schemes form an important class of examples. They have been widely studied in geometric representation theory, enumerative and combinatorial geometry and as two of the only four known deformation classes of hyperkähler manifolds, namely Hilbert schemes of points on K3 surfaces and generalised Kummer varieties. A prominent direction in this area is to understand the local moduli space of such objects and, in particular, methods for describing modular simple normal crossing degenerations of smooth Hilbert schemes. This talk focuses mainly on the results of our paper [17], in which we study how the technique of expanded degenerations applies to this problem for Hilbert schemes of points on surfaces.

Expanded degenerations are first introduced by Li [4] and then used by Li and Wu [5] to study Quot schemes on degenerations $\pi\colon X \to \mathbb{A}^1$ such that $(X, \pi^{-1}(0))$ is a simple normal crossing pair, where the singular locus of $\pi^{-1}(0)$ is smooth. We explore the connection between two ideas:

(1) The logarithmic geometry approach to this problem considered by Maulik and Ranganathan in [6], which builds upon previous work of Ranganathan [15] on logarithmic Gromov-Witten theory with expansions.
(2) The Geometric Invariant Theory (GIT) perspective of Gulbrandsen, Halle and Hulek [1].

This construction is the first instance of a logarithmic moduli space of coherent sheaves built using ideas from GIT. As, historically, GIT has been used to consider stability of objects, we hope that this work can provide insights into describing stability for logarithmic sheaves.

We construct two equivalent modular simple normal crossing degenerations of smooth Hilbert schemes of points on surfaces. These extend [5] and [1] to the case where the singular locus of $\pi^{-1}(0)$ is singular. The first is a stack $\mathfrak{M}^m_{\mathrm{LW}}$ which uses a generalisation of Li-Wu stability to this situation. The second is a stack $\mathfrak{M}^m_{\mathrm{SWS}}$ which uses a stability condition called SWS stability derived from GIT. This provides an explicit model of the degenerations theorised in [6] and we describe how these can be interpreted in the language of logarithmic geometry. The main results we present are the following.

Theorem 1. *The stacks $\mathfrak{M}^m_{LW}$ and $\mathfrak{M}^m_{SWS}$ are Deligne-Mumford and proper.*

Theorem 2. *There is an isomorphism of stacks*

$$\mathfrak{M}^m_{LW} \cong \mathfrak{M}^m_{SWS}.$$

Setup. Let k be an algebraically closed field of characteristic zero. Let $X \to C$ be a projective family of surfaces over a curve $C \cong \mathbb{A}^1$ such that the total space is smooth and the central fibre X_0 has simple normal crossing singularities. At a triple point of the singular fibre, X is étale locally given by $\mathrm{Spec}\, k[x, y, z, t]/(xyz - t)$. In this étale local model, the general fibres are smooth and the central fibre X_0 is given by three planes intersecting transversely in $\mathbb{A}^3$. We may rephrase the question of constructing simple normal crossing degenerations of Hilbert schemes of points as the following compactification problem. Let $X^\circ \coloneqq X \setminus X_0$, which lies over $C^\circ \coloneqq C \setminus \{0\}$. Given such a family $X \to C$, we explore how techniques of expanded degenerations may be used to construct good compactifications of the relative Hilbert scheme of m points $\mathrm{Hilb}^m(X^\circ/C^\circ)$.

The aim is to construct a modular compactification in which all limit subschemes can be chosen to satisfy some transversality condition in some modification of X_0. In the case which interests us here, namely Hilbert schemes of points, it will just mean that we would like the length m zero dimensional subschemes to have support in the smooth loci of the underlying two-dimensional fibres (we will refer to this condition as the subschemes being *smoothly supported*). The problem therefore is to construct *expansions* (birational modifications of the central fibre of X in a 1-parameter family) in which all limits of families of length m zero-dimensional subschemes needed to compactify $\mathrm{Hilb}^m(X^\circ/C^\circ)$ can be chosen to be smoothly supported. This allows us to break down the problem of studying Hilbert schemes of points on X_0 into smaller parts, by studying the products of Hilbert schemes of points on the irreducible components of the modifications of X. The work of Li and Wu only covers the case where the singular locus of X_0 is smooth. Understanding how these problems work in general for simple normal crossing surfaces is quite powerful, as we can always use semistable reduction to reduce to this case.

Application to hyperkähler varieties. So far, we have only required that X is a degeneration of surfaces with a simple normal crossing special fibre. A natural question is to study the more specific case where X is a type III good degeneration of K3 surfaces and try to construct a family of Hilbert schemes of points on X which will be minimal in the sense of the minimal model program, meaning a good or dlt minimal degeneration (see [11] and [7] for definitions of the minimality conditions). The singularities arising in such a degeneration X are of the type described here, i.e. we can restrict ourselves to the local problem where X_0 is thought of as given by $xyz = 0$ in $\mathbb{A}^3$. Among other reasons, Hilbert schemes of points on K3 surfaces are interesting to study because they form a class of examples of hyperkähler varieties. The question of minimality is addressed in [16].

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Tropical correspondence and mirror symmetry for log Calabi-Yau surfaces

Tim Gräfnitz

(joint work with Helge Ruddat, Eric Zaslow and Benjamin Zhou, and with Per Berglund and Michael Lathwood)

Log Calabi-Yau pairs (X, D) consisting of a smooth projective surface X and a smooth anticanonical divisor D correspond under mirror symmetry to Landau-Ginzburg models $W : \check{X} \to \mathbb{C}$, with a superpotential function W. For toric Fano varieties this is the Hori-Vafa potential given by a sum of toric monomials [1]. For more general cases, the potential is the primitive Gross-Siebert theta function ϑ_1, defined by a sum over broken lines in the consistent scattering diagram on the dual intersection complex of (X, D) [4]. This depends on the chamber. In the central chamber there is no scattering (in the Fano case), and ϑ_1 is equal to the Hori-Vafa potential. For $(X, D) = (\mathbb{P}^2, E)$, with E an elliptic curve, we have

$$\vartheta_1 = t \cdot \left(x + y + \frac{s}{xy} \right).$$

The corresponding broken lines are shown in Figure 1.

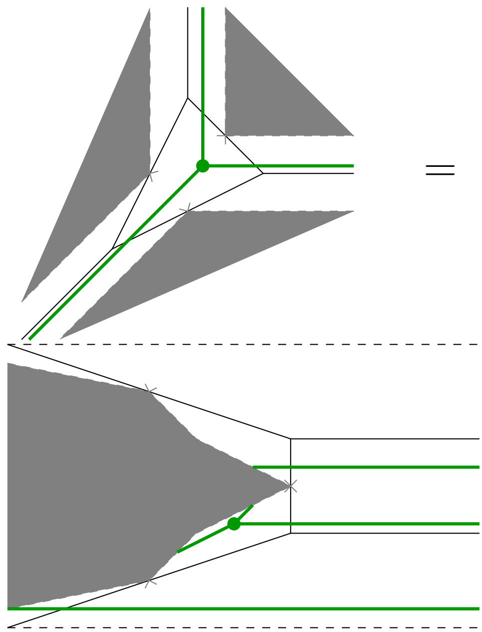

FIGURE 1. The potential ϑ_1 of $\mathbb{P}^2$ in the central chamber, in two different charts.

In an unbounded chamber (at "infinity"), by tropical correspondence [6], ϑ_1 is a generating function for 2-marked logarithmic Gromov-Witten invariants $R_{1,\beta\cdot D-1}(X,\beta)$, which can be seen as a logarithmic/relative analogue of Maslov index 2 disks:

$$\vartheta_1(y) = y + \sum_{\beta\in NE(X)} \tfrac{1}{\beta\cdot D-1} R_{1,\beta\cdot D-1}(X,\beta) s^\beta t^{\beta\cdot D} y^{-(\beta\cdot D-1)}.$$

It has infinitely many terms. For $(X,D)=(\mathbb{P}^2,E)$ the first terms are

$$\vartheta_1(y) = y + 2st^3y^{-2} + 5s^2t^6y^{-5} + 32s^3t^9y^{-8} + \dots$$

The corresponding broken lines are shown in Figure 2.

The open mirror map relates coordinates on X to coordinates on its mirror $\check{X}$. Its definition involves the classical period of the potential ϑ_1, which is a solution to a Picard-Fuchs type differential equation,

$$F_0(z) = \sum_{k>0} \frac{1}{k} \mathrm{coeff}_1(\vartheta_1^k) t^{-k}.$$

This is independent of the chamber by a result of [4]. We [7, 8] use a combinatorial identity of Bell polynomials to show that the open mirror map is equal to ϑ_1 at

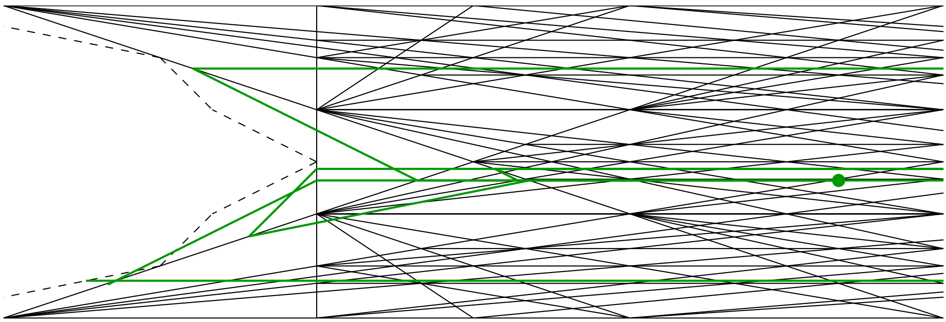

FIGURE 2. The potential ϑ_1 of $\mathbb{P}^2$ at infinity.

infinity: Consider the open mirror map $q_0 = z_0 e^{F_0(z)}$ and insert the inverses of the closed mirror maps $q_i = z_i e^{(\beta_i \cdot D) F_0(z)}$ to obtain $M(q) := e^{F_0(z(q))}$. Then, under the change of variables $q_i = z_i (t/y)^{\beta_i \cdot D}$, at infinity we have

$$\vartheta_1(y) = yM(q).$$

This shows that ϑ_1 is a coordinate on the mirror, as predicted by intrinsic mirror symmetry [5].

In the non-Fano case there is internal scattering, so ϑ_1 in the central chamber is different from the Hori-Vafa potential, with correction terms coming from scattering. For $\mathbb{F}_2$ and $\mathbb{F}_3$ this had been observed in [2] and [4]. With this corrected potential, the equality $\vartheta_1(y) = yM(q)$ is still valid [3]. Conjecturally, the Newton polytope of ϑ_1 gives a toric degeneration of X, which is related to other toric degenerations by mutations [9]. For $\mathbb{F}_4$, the potential is shown in Figure 3 and its mutation to the toric model $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ is shown in Figure 4.

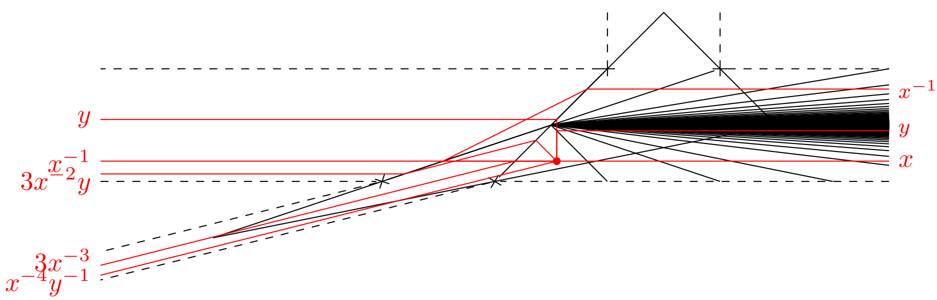

FIGURE 3. The potential ϑ_1 of $\mathbb{F}_4$ in the central chamber.

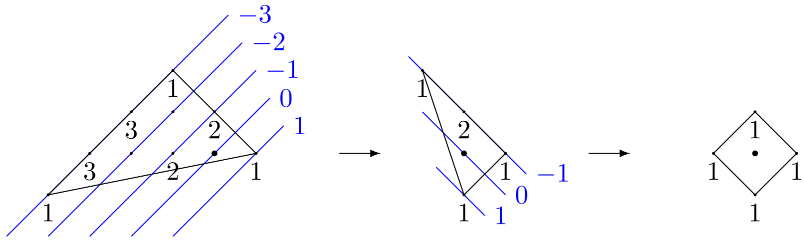

FIGURE 4. Mutations $\vartheta_1(\mathbb{F}_4) \to \vartheta_1(\mathbb{F}_2) \to \vartheta_1(\mathbb{F}_0)$.

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Hodge integrals, Abelian varieties, and the Hilbert scheme of points of the plane

RAHUL PANDHARIPANDE

The Deligne-Mumford moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ is by far the most studied moduli space of varieties in algebraic geometry. The Hilbert scheme $\mathsf{Hilb}(\mathbb{C}^2, d)$ of d points of the plane $\mathbb{C}^2$ is arguably the moduli of space of sheaves with the richest known structure. Gromov-Witten theory, via the virtual class of the moduli space of stable maps, provides a system of correspondences between these moduli spaces of varieties and sheaves:

$$\overline{\mathcal{M}}_{g,n} \leftarrow [\overline{\mathcal{M}}_{g,n}(\mathsf{Hilb}(\mathbb{C}^2, d), \beta)]^{\mathrm{vir}} \to \mathsf{Hilb}(\mathbb{C}^2, d)^n \,.$$

The data of all these correspondences as the genus g, the marking number n, and the curve class $\beta \in H_2(\mathsf{Hilb}(\mathbb{C}^2, d))$ vary constitutes the CohFT associated to $\mathsf{Hilb}(\mathbb{C}^2, d)$ called $\mathsf{GW}(\mathsf{Hilb}(\mathbb{C}^2, d))$.

The study of the genus 0 part of $\mathsf{GW}(\mathsf{Hilb}(\mathbb{C}^2,d))$ was undertaken 20 years ago by myself and A. Okounkov [OP1]. We found that the entire genus 0 part (in other words, the quantum cohomology of $\mathsf{Hilb}(\mathbb{C}^2,d)$) is controlled by the operator of quantum multiplication by the (unique up to scale) divisor class of $\mathsf{Hilb}(\mathbb{C}^2,d)$. The main result of [OP1] is the calculation of this operator in the Fock space description by Nakajima [N] and Grojnowski [G] of the cohomology of $\mathsf{Hilb}(\mathbb{C}^2,d)$. The genus 0 study played an important role in the investigation of the GW/DT correspondence of [MNOP] for local curves [BP,OP2]

The main goal of my lecture was to show the richness of the higher genus geometry of $\mathsf{GW}(\mathsf{Hilb}(\mathbb{C}^2,d))$. In the past year, the picture in genus 1 has become clearer. There are several approaches to $\mathsf{GW}(\mathsf{Hilb}(\mathbb{C}^2,d))$ in higher genus (see [PT]) including Hodge integrals for the families Gromov-Witten theory of the universal curve over $\overline{\mathcal{M}}_{g,n}$. In the case of the genus 1 series for $\mathsf{Hilb}(\mathbb{C}^2,d)$ corresponding to a single insertion of the divisor class (parallel to the fundamental genus 0 calculation discussed above), a complete solution is obtained via the Hodge integral study. Moreover, the result is connected in an essential way to the Noether-Lefschetz theory of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g as computed recently by A. Iribar López.

The formula of the basic genus 1 series (a result in 2024 of myself with A. Iribar López and H.-H. Tseng) and is:

$$-\big\langle(2)\big\rangle_1^{\mathsf{Hilb}(\mathbb{C}^2,d)} = -\frac{1}{24}\frac{(t_1+t_2)^2}{t_1t_2}\left(\mathsf{Tr}_d + \sum_{k=2}^{d-1}\frac{\sigma(d-k)}{d-k}\mathsf{Tr}_k\right).$$

Here, (2) denotes the divisor class of $\mathsf{Hilb}(\mathbb{C}^2,d)$ viewed as an element in Fock space. The variables t_1 and t_2 are the standard equivariant parameters for the scaling actions on the components of $\mathbb{C}^2$. The trace of the operator of quantum multiplication by the class $-(2)$ on the cohomology of $\mathsf{Hilb}(\mathbb{C}^2,d)$ is defined to be $(t_1+t_2)\cdot\mathsf{Tr}_d$. The function $\sigma(m)$ is the sum of the divisors of the integer m. We note that

$$\big\langle(2)\big\rangle_1^{\mathsf{Hilb}(\mathbb{C}^2,d)}, \mathsf{Tr}_2 \ldots, \mathsf{Tr}_d$$

are all q series, where q is the Novikov parameter associated to the curve classes of $\mathsf{Hilb}(\mathbb{C}^2,d)$. In fact, these are all rational functions in q.

The study of the above genus 1 formula relies also upon related results of S. Canning, F. Greer, C. Lian. D. Oprea, S. Molcho, and A. Pixton.

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Logarithmic Tautological Rings

Johannes Schmitt

The logarithmic Chow ring of a normal crossing pair (X, D) describes the intersection theory of all iterated boundary blow-ups of X simultaneously. We discuss past applications and explain how the language of Artin fans and cone stacks can be used to describe the blow-ups and to construct interesting cycle classes on them. We report some work in progress with R. Pandharipande, D. Ranganathan, and P. Spelier on logarithmic tautological classes on moduli spaces of stable curves.

1. Introduction

The study of smooth normal crossings pairs (X, D), where X is a smooth variety or algebraic stack and D is a normal crossings divisor, has significant implications in various areas of algebraic geometry, including moduli spaces and enumerative geometry. Motivated by the desire for a combinatorial framework for intersection theory on such pairs, we explore the notion of *logarithmic Chow rings* and their applications.

1.1. **Logarithmic Chow Rings.** Let (X, D) be a smooth normal crossings pair. A *log blow-up* of (X, D) is a birational map $\pi : \hat{X} \to X$ obtained by a sequence of blow-ups along smooth strata closures of D.

The *logarithmic Chow ring* $\mathrm{logCH}^*(X, D)$ is defined as the direct limit

$$\mathrm{logCH}^*(X, D) = \varinjlim_{\pi:\hat{X}\to X} \mathrm{CH}^*(\hat{X}),$$

where the limit is taken over all log blow-ups of (X, D) and the transition maps are pullbacks along the corresponding morphisms. This construction incorporates information from various compactifications of the interior $X \setminus D$ (which is unchanged by the log blow-ups) while maintaining a combinatorial perspective.

The logarithmic Chow ring is a graded $\mathbb{Q}$-algebra and admits a natural map from the usual Chow ring:

$$\mathrm{CH}^*(X) \to \mathrm{logCH}^*(X, D), \quad \alpha \mapsto [(X, \alpha)].$$

This map is injective, so the logarithmic Chow ring refines the information captured by the usual Chow ring.

1.2. **Artin Fans and Cone Stacks.** A crucial tool in understanding the structure of logarithmic Chow rings is the *Artin fan* $\mathcal{A}_{(X,D)}$. The Artin fan is an Artin stack of essentially combinatorial nature that encodes the stratification of X induced by D and provides a bridge between the geometry of (X, D) and the combinatorics of its boundary stratification.

- In the case of toric varieties, the Artin fan is simply the quotient stack $[X/T]$, where $T \subseteq X$ is the open torus. The Chow ring of the Artin fan is isomorphic to the ring of piecewise polynomials on the associated fan Σ_X.
- For general smooth normal crossings pairs, the Artin fan is constructed from a more general object called a *cone stack* $\Sigma_{(X,D)}$. This cone stack encodes the combinatorial data of the strata closures and their intersections, including information about monodromy actions (of the fundamental group of the strata S acting on the branches of D containing S).

There exists a smooth and surjective morphism

$$q_{(X,D)} : X \to \mathcal{A}_{(X,D)}$$

that relates the geometry of (X, D) to its Artin fan.

1.3. **Intersection Theory on Artin Fans.** The Chow ring of the Artin fan has an elegant description in terms of piecewise polynomials:

Theorem 1 ([1]). *The Chow ring of the Artin fan $\mathcal{A}_{(X,D)}$ admits an isomorphism*

$$\Phi : \mathrm{sPP}^*(\Sigma(X, D)) \to \mathrm{CH}^*(\mathcal{A}_{(X,D)})$$

from the ring of strict piecewise polynomials *on the cone stack $\Sigma(X, D)$.*

This theorem allows us to explicitly describe classes in $\mathrm{CH}^*(\mathcal{A}_{(X,D)})$ using convex geometric data. Furthermore, composing Φ with the pullback map

$$q^*_{X,D} : \mathrm{CH}^*(\mathcal{A}_{(X,D)}) \to \mathrm{CH}^*(X)$$

allows us to construct geometric classes on X (which turn out to be fundamental classes of strata closures decorated by Chern classes of their normal bundles).

The logarithmic Chow ring of $\mathcal{A}_{(X,D)}$ is similarly described in terms of *piecewise polynomials* on $\Sigma(X, D)$, but with the added flexibility of subdivisions of the cone stack, corresponding to log blow-ups of (X, D).

2. Applications to Moduli of Curves

The framework of logarithmic Chow rings proves particularly useful in studying the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ with its boundary divisor Δ.

2.1. **Logarithmic Double Ramification Cycle.** One important application is the construction of the *logarithmic double ramification cycle* $\mathrm{logDR}_g(A)$, which generalizes the classical double ramification cycle to the logarithmic setting. To introduce it, let $g \geq 0$ and $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $\sum a_i = 0$ (and $2g-2+n >$

0). Then we can first consider the locus $\mathcal{DR}_g^0(A) \subset \mathcal{M}_{g,n}$ of smooth curves $(C, p_1, \dots, p_n)$ satisfying the condition

$$\mathcal{O}_C\left(\sum a_i p_i\right) \cong \mathcal{O}_C.$$

In [2] Holmes constructs a compactification of (the fundamental class of) $\mathcal{DR}_g^0(A)$ by specifying a log blow-up $\pi : \widehat{\mathcal{M}}_g^A \to \overline{\mathcal{M}}_{g,n}$ and intersecting some Abel–Jacobi sections of the universal Jacobian over $\widehat{\mathcal{M}}_g^A$. Even though the construction involves some choices for the log blow-up $\widehat{\mathcal{M}}_g^A$, Holmes proves that the resulting cycle $\widehat{\mathrm{DR}}_g(A)$ on $\widehat{\mathcal{M}}_g^A$ gives a well-defined element

$$\mathrm{logDR}_g(A) = [(\widehat{\mathcal{M}}_g^A, \widehat{\mathrm{DR}}_g(A))] \in \mathrm{logCH}^g(\overline{\mathcal{M}}_{g,n}),$$

invariant under those choices. The logarithmic double ramification cycle exhibits remarkable properties that distinguish it from the classical double ramification cycle (obtained as $\mathrm{DR}_g(A) = \pi_* \widehat{\mathrm{DR}}_g(A)$):

Theorem 2 ([3], [4, 5], [6], [7])**.** *The log double ramification cycle satisfies:*

- $\mathrm{logDR}_g(A) \cdot \mathrm{logDR}_g(B) = \mathrm{logDR}_g(A) \cdot \mathrm{logDR}_g(A+B) \in \mathrm{logCH}^{2g}(\overline{\mathcal{M}}_{g,n})$.
- $\mathrm{logDR}_g(A)$ *belongs to the subring* $\mathrm{divlogCH}^*(\overline{\mathcal{M}}_{g,n})$ *of* $\mathrm{logCH}^*(\overline{\mathcal{M}}_{g,n})$ *generated by divisor classes.*
- *The double Hurwitz number*[1] $H_g(A)$ *can be expressed as an intersection number on* $\widehat{\mathcal{M}}_g^A$*:*

$$H_g(A) = \int_{\widehat{\mathcal{M}}_g^A} \mathrm{logDR}_g(A) \cdot \mathrm{br}_g(A),$$

  where $\mathrm{br}_g(A)$ *is a specific class encoding the fixed simple branch points.*
- *There are piecewise polynomials* $f_L, f_P \in \mathrm{PP}^*(\Sigma_{\overline{\mathcal{M}}_{g,n}})$ *such that*

$$\mathrm{logDR}_g(A) = \left[\exp\left(\eta + \Phi(f_L)\right) \cdot \Phi(f_P)\right]_g \in \mathrm{logCH}^g(\overline{\mathcal{M}}_{g,n}),$$

  where $\eta = \sum_i a_i^2/2 \cdot \psi_i$ *and* $[\ ]_g$ *denotes the codimension* g *part.*

These properties show that both the structural properties and the enumerative applications of logDR are more rich than those of the classical double-ramification cylces DR.

2.2. **Logarithmic tautological rings of moduli spaces of stable curves.** The talk concluded with a discussion of some work in progress:

Theorem 3 (joint with Pandharipande, Ranganathan, and Spelier)**.** *The ring homomorphism*

$$\mathrm{PP}^*(\Sigma_{\overline{\mathcal{M}}_{0,n}}) \to \mathrm{logCH}^*(\overline{\mathcal{M}}_{0,n})$$

is surjective, with kernel ideal generated by explicit piecewise-polynomial incarnations of the WDVV-relations.

[1]The number $H_g(A)$ counts branched covers of $\mathbb{P}^1$ with ramification profile A over 0 and ∞ and fixed simple branch points elsewhere.

The talk concluded with a discussion of more general tautological classes in $\log\mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$ which allow to combine boundary blow-ups with decorations via κ- and ψ-classes only defined on the blown-up strata.

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The negative counterpart of Witten's conjecture

Elba Garcia-Failde

In 1990, Witten [Wit90] conjectured that the generating series of intersection numbers of psi classes is a tau function of the KdV hierarchy. This was first proved by Kontsevich [Kon92]. In 2017, Norbury [Nor23] conjectured that the generating series of intersection numbers of psi classes times a negative square root of the canonical bundle is also a tau function of the KdV hierarchy, which corresponds to the Brézin–Gross–Witten (BGW) tau function. In joint work with N. Chidambaram and A. Giacchetto [CGG22], we prove Norbury's conjecture and obtain polynomial relations among kappa classes which were recently conjectured by Kazarian–Norbury [KN23].

Furthermore, we introduce a new collection of cohomology classes, which correspond to negative r-th roots ($r = 2$ in the previous paragraph) of the canonical bundle and form a cohomological field theory (CohFT), the negative analog of Witten's r-spin CohFT, which turns out to be geometrically much simpler. We prove that the corresponding intersection numbers can be computed recursively using a universal procedure called topological recursion and, equivalently, $\mathcal{W}$-constraints.

The strategy draws inspiration from our proof, together with S. Charbonnier [CCGG24], of Witten's r-spin conjecture from 1993 (Faber–Shadrin–Zvonkine's theorem from 2010) that claims that (positive) r-spin intersection numbers satisfy the r-KdV hierarchy. We also obtain new (tautological) relations on the moduli space of curves in a (negative) analogous way to Pandharipande–Pixton–Zvonkine [PPZ15]. The talk will be an overview of these four topics ($r = 2$ and $r > 2$; positive and negative) and their connections.

The Theta Θ^r classes. We are interested in the counterpart of Witten's r-spin class for "negative" spin. More precisely, for $r \in \mathbb{Z}_{\geq 2}$ and $(a_1, \dots, a_n) \in \{0, \dots, r-1\}$ (called the primary fields), consider the moduli space of twisted r-spin curves $\overline{\mathcal{M}}^{r,-1}_{g;a}$ which parametrizes the data of a stable curve $(C, x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g,n}$ together with a line bundle L on C such that

$$L^{\otimes r} \cong \omega_{\log}^{-1}\left(-\sum_{i=1}^{n} a_i x_i\right),$$

where $\omega_{\log}$ is the log canonical bundle of C. Let $\mathcal{C}$ be the universal curve to $\overline{\mathcal{M}}^{r,-1}_{g;a}$ and $\mathcal{L}$ be the universal line bundle on the universal curve: $\mathcal{L} \longrightarrow \mathcal{C} \xrightarrow{\pi} \overline{\mathcal{M}}^{r,-1}_{g;a}$. By forgetting the extra data of the line bundle L, we also have a forgetful map $f\colon \overline{\mathcal{M}}^{r,-1}_{g;a} \to \overline{\mathcal{M}}_{g,n}$ to the moduli space of stable curves.

Following Chiodo [Chi08], we take the derived pushforward $E \coloneqq R^{\bullet}\pi_*\mathcal{L}$. Given a point $p = (C, x_1, \dots, x_n, L) \in \overline{\mathcal{M}}^{r,-1}_{g;a}$, since $\deg L < 0$, we know that $H^0(C, L) = 0$, and hence E is an honest vector bundle over $\overline{\mathcal{M}}^{r,-1}_{g;a}$ (whose fiber over p is given by $H^1(C, L)$). One can compute the rank of this vector bundle using Riemann–Roch:

$$\mathrm{rk}_{\mathbb{C}} E = h^1(C, L) = -\deg L + g - 1 = \frac{(r+2)(g-1) + n + |a|}{r} =: D^r_{g;a} = D,$$

where $|a| \coloneqq \sum_{i=1}^{n} a_i$. Our main interest in this work is in the top Chern class of E, which is the top degree of the so-called Chiodo class. Let $W = \mathrm{span}_{\mathbb{Q}}(v_0, \dots, v_{r-1})$, and define the collection of maps $\Upsilon^r_{g,n}\colon W^{\otimes n} \longrightarrow H^{\bullet}(\overline{\mathcal{M}}_{g,n})$ as

$$\Upsilon^r_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) \coloneqq (-1)^n\, r^{\frac{2g-2+|a|+n}{r}}\, f_* c_{\mathrm{top}}(E) \in H^D(\overline{\mathcal{M}}_{g,n}).$$

Theorem 1. *Let $V \coloneqq \mathrm{span}_{\mathbb{Q}}(v_1, \dots, v_{r-1})$ and $\eta(v_a, v_b) = \delta_{a+b,r}$. If we restrict the arguments v_a of $\Upsilon^r_{g,n}$ to $1 \leq a \leq r-1$, then*

$$\Theta^r_{g,n} \coloneqq \Upsilon^r_{g,n}\big|_V$$

is a rank (r-1) (generically) non-semisimple CohFT on (V, η) with a modified unit v_{r-1}.

Having a non-vanishing cohomology class when setting some primary fields to zero (opposite to what happens for Witten's r-spin class (Ramond vanishing)), allows us to deform the Theta class along the direction v_0:

$$\Theta^{r,\epsilon}_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) \coloneqq \sum_{m \geq 0} \frac{\epsilon^m}{m!} p_{m,*}\Upsilon^r_{g,n+m}(v_{a_1} \otimes \cdots \otimes v_{a_n} \otimes v_0^{\otimes m}),$$

where $p_m\colon \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ is the map that forgets the last m marked points. As v_0 is not part of the vector space underlying the Theta class, the above deformation is not a shift along any direction of the associated Dubrovin–Frobenius manifold (differing from what occurs for shifted Witten classes).

Since $\deg p_{m,*}\Upsilon^r_{g,n+m}(v_{a_1} \otimes \cdots \otimes v_{a_n} \otimes v_0^{\otimes m}) = D - \frac{(r-1)m}{r}$ (which is negative for big m), the sum in the definition of the deformed Theta class is finite. This also implies that $\Theta^{r,\epsilon}_{g,n}$ is equal to the class $\Theta^r_{g,n}$ in top degree, with possibly some correction terms in strictly smaller degree (analogous to the situation of shifted Witten classes).

We prove that for any $\epsilon \neq 0$, the deformed Theta class is semisimple and homogeneous with respect to an Euler field. Then, we compute all the ingredients of the Teleman reconstruction theorem to find an expression for $\Theta^{r,\epsilon}$ in terms of tautological classes. Teleman's reconstruction theorem for CohFTs without a flat unit does not apply for $n = 0$, which results in the exception for $n = 0$.

Theorem 2. *For all stable (g,n), except for $(g,n) = (g,0)$ and degree $d = 3g-3$,*

$$\Theta^{r,\epsilon}_{g,n} = RTw_{g,n}\,,$$

where R, T and $w_{g,n}$ are an explicit R-transform, translation and topological field theory, respectively. The Theta class is the term of degree $d = D$:

$$\Theta^r_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) = [\deg_{\mathbb{C}} = D]\, RTw_{g,n}(v_{a_1} \otimes \cdots v_{a_n})\,.$$

All the terms of degree $d > D$ vanish and thus produce (new) tautological relations:

$$[d]\, RTw_{g,n}(v_{a_1} \otimes \cdots v_{a_n}) = 0\,, \ \text{for } d > D, \text{ except for } n = 0 \text{ and } d = 3g-3\,.$$

When $r = 2$ the statement takes a very simple form, involving κ-classes only, and proves a conjecture of Kazarian–Norbury [KN23, conjectures 1 and 4].

Topological recursion and integrability. A very useful tool to find connections between CohFTs and integrability is topological recursion, which is a universal formalism that takes as input an algebraic curve along with some extra data, called a spectral curve, and recursively constructs multidifferentials known as correlators on the underlying curve. From the spectral curve, one can build a semisimple CohFT such that the multidifferentials can be expressed in terms of descendant integrals of this CohFT [DOSS14]. We find a global spectral curve whose topological recursion correlators encode the descendant theory of the (deformed) Theta class.

Theorem 3 (Topological recursion)**.** *The CohFT associated to the* 1*-parameter family of spectral curves $\mathcal{S}_\epsilon$ on $\mathbb{P}^1$ given by*

$$x(z) = \frac{z^r}{r} - \epsilon z\,, \qquad y(z) = -\frac{1}{z}\,, \qquad B(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2}\,,$$

is $\Theta^{r,\epsilon}$. More precisely, the correlators corresponding to the spectral curve $\mathcal{S}_\epsilon$ are

$$\omega_{g,n}(z_1,\ldots,z_n) = \sum_{a_1,\ldots,a_n=1}^{r-1} \int_{\overline{\mathcal{M}}_{g,n}} \Theta^{r,\epsilon}_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) \prod_{i=1}^{n} \sum_{k_i \geq 0} \psi_i^{k_i}\, d\xi^{k_i,a_i}(z_i)\,,$$

where the $d\xi^{k,a}(z)$ are certain explicit differentials.

This is a powerful result, as it allows one to calculate any descendant integral by a recursion on $2g-2+n$. By taking the parameter $\epsilon \to 0$, we obtain as a corollary that the descendant integrals of the Theta class $\Theta^r_{g,n}$ are computed by the Bouchard–Eynard topological recursion on the *r-Bessel spectral curve* $\mathcal{S}_0$. The

Bouchard–Eynard topological recursion was analysed thoroughly in the context of higher Airy structures in [BBC+24] and proved to be equivalent to a set of $\mathcal{W}$-constraints in general. Putting together the identification of the correlators of the r-Bessel spectral curve with descendant integrals of the Theta class and the results of [BBC+24], we prove the negative spin analog of the Witten r-spin conjecture.

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Generalized Block-Göttsche polynomials and Welschinger invariants

Pierrick Bousseau

(joint work with Hülya Argüz)

Let S be a smooth projective rational surface over $\mathbb{C}$, obtained as a blow-up of the projective plane at n points in general position. For every homology class $\beta \in H_2(S, \mathbb{Z})$, there is a Gromov–Witten count $GW_{0,\beta}^S \in \mathbb{Z}$ of complex rational curves in S passing through $m := c_1(S) \cdot \beta - 1$ points in general position. The numbers $GW_{0,\beta}^S$ can be computed recursively using the WdVV relation in Gromov–Witten theory [4]. On the other hand, if all the n blown-up points are in the real projective plane, then S inherits a natural structure of real algebraic variety, and one can consider the counts with Welschinger signs $W_\beta^S \in \mathbb{Z}$ of real rational curves in S passing through m real points in general position [10]. The numbers $W_{0,\beta}^S$ can be computed recursively using the real WdVV equation [6].

When S is a toric del Pezzo surface, that is when $n \leq 3$, the counts $GW_{0,\beta}^S$ and W_β^S of complex and real rational curves can be determined tropically by counting tropical curves in $\mathbb{R}^2$ with appropriate multiplicities [7, 8] – see also [1]. According to Block-Göttsche, the tropical multiplicities admit a natural q-deformation. Moreover, counting tropical curves with these refined q-multiplicities produces Laurent polynomials $BG_\beta^S(q) \in \mathbb{Z}[q^\pm]$ which have the remarkable property to specialize to the complex Gromov–Witten counts $GW_{0,\beta}^S$ when $q = 1$, and to the real Welschinger counts W_β^S when $q = -1$ [3].

Our main goal is to provide a generalization of Block–Göttsche polynomials for general S. The tropical definition does not extend in an obvious way beyond the toric case. Instead, we will take as a starting point an algebro-geometric interpretation of Block-Göttsche polynomials in terms of higher genus Gromov–Witten invariants, as shown in [2]. For any $g \in \mathbb{Z}_{\geq 0}$, we define a higher genus generalization $GW_{g,\beta}^S$ of the invariants $GW_{0,\beta}^S$, using the insertion of a class λ_g to cut out the virtual dimension to zero. Our first main result is the following theorem.

Theorem [Argüz–Bousseau]**.** There exists a unique Laurent polynomial $BPS_\beta^S(q) \in \mathbb{Z}[q^\pm]$ such that, after the change of variables $q = e^{iu}$, we have the equality

$$BPS_\beta^S(q) = \frac{\sum_{g\geq 0} GW_{g,\beta}^S u^{2g-2+c_1(S)\cdot\beta}}{\left(2\sin\left(\frac{u}{2}\right)\right)^{c_1(S)\cdot\beta-2}} .$$

We propose to view the polynomials $BPS_\beta^S(q)$ as generalizations of Block-Göttsche polynomials. By the main result of [2], the polynomials $BPS_\beta^S(q)$ indeed recovers the Block-Göttsche polynomials when S is a toric del Pezzo surface. Moreover, for general S, the specialization at $q = 1$ of $BPS_\beta^S(q)$ recovers the complex Gromov–Witten count $GW_{0,\beta}^S$. We conjecture that we also recovers the real Welschinger counts at $q = -1$:

Conjecture [Argüz–Bousseau]**.** For every $n \geq 0$, we have the equality

$$BPS_\beta^S(-1) = W_\beta^S .$$

Using degeneration arguments, we prove this conjecture when S is a del Pezzo surface, that is, when $n \leq 8$.

Theorem [Argüz–Bousseau]**.** For every $n \leq 8$, we have the equality

$$BPS_\beta^S(-1) = W_\beta^S .$$

While no WdVV relation is known for the polynomials $BPS_\beta^S(q)$, we provide an effective way to compute the polynomials $BPS_\beta^S(q)$ using a q-generalization of the work of Parker computing the invariants $GW_{0,\beta}^S$ by degeneration [9].

Finally, we conjecture that the polynomials $BPS_\beta^S(q)$ have an alternative desciption in terms of refined Donaldson–Thomas theory of the Calabi–Yau 3-fold K_S given by the total space of the canonical line bundle of the surface S. This conjecture is a generalization of a conjectural interpretation proposed by Göttsche–Shende of Block-Göttsche polynomials using χ_y-genera of relative Hilbert schemes of points [5].

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Towards quadratically enriched Gromov-Witten theory

Jake Solomon

The talk discussed progress on a program to develop quadratically enriched Gromov–Witten theory. A relative orientation of a morphism $f : M \to N$ of smooth k-schemes is an invertible sheaf L on M together with an isomorphism $\rho : Hom(\det TM, \det TN) \to L^{\otimes 2}$. Let S be a del Pezzo surface over k, in the sense that S is a geometrically connected, smooth, projective k-scheme of dimension 2 with ample anticanonical bundle $-K_S$. Let $d_S = K_S \cdot K_S$ denote the degree of S.

Let $\bar{M}_{0,n}(S, D)$ denote the space of genus zero stable maps with n marked points in the class $D \in Pic(S)$ and consider the total evaluation map $ev : \bar{M}_{0,n}(S, D) \to S^n$. Let $\sigma = (L_1, \ldots, L_r)$ be an r-tuple of field extensions $k \subset L_i \subset \bar{k}$ such that $\sum_{i=1}^{k} [L_i : k] = n$. For an L-scheme X, let $\mathrm{Res}_{L/k} X$ denote the restriction of scalars to k. We construct a corresponding Galois twist

$$ev_\sigma : \bar{M}_{0,n}(S, D)_\sigma \to (S^n)_\sigma = \prod_{i=1}^{r} \mathrm{Res}_{L_i/k} S.$$

We fix $n = d - 1$ and work under the following hypothesis.

Hypothesis 1. *Assume that D is not an m-fold multiple of a -1-curve for $m > 1$. Moreover, assume that $d_S \geq 4$, or $d_S = 3$ and $d := -K_S \cdot D \neq 6$, or $d_S = 2$ and $d \geq 7$.*

1. Characteristic zero

Assume first that k has characteristic zero. In [KLSW23, Theorem 4.5], we identify a closed subset $A \subset (S^n)_\sigma$ such that $\bar{M}_{0,n}(S,D)_\sigma^{\text{good}} := \bar{M}_{0,n}(S,D)_\sigma \setminus ev^{-1}(A)$ has the following two properties.

(1) The restriction of the total evaluation map $ev_\sigma^{\text{good}} : \bar{M}_{0,n}(S,D)_\sigma^{\text{good}} \to (S^n)_\sigma$ is relatively oriented.
(2) The codimension of $A \subset (S^n)_\sigma$ is at least 2.

Properties 1 and 2 allow us to define the degree of $\mathsf{ev}_\sigma^{\text{good}}$. However, the degree is no longer valued in the integers $\mathbf{Z}$. Rather, we build on F. Morel's $\mathbb{A}^1$-degree [Mor04, KW19] to define a degree in the Grothendieck–Witt ring $\mathrm{GW}(k)$. We recall the definition and basic properties of $GW(k)$ in Section 3 below. One of our main results is the following.

Theorem 1. *Let S, D, σ satisfy Hypothesis 1 and assume that S is $\mathbb{A}^1$-connected. Then there exists an invariant $N_{S,D,\sigma}$ in the Grothendieck–Witt ring $\mathrm{GW}(k)$ given by the degree of ev_σ^{good}.*

2. Positive characteristic

We turn to the case when k has positive characteristic. Let $M_0^{\text{bir}}(S,D) \subset \overline{M}_0(S,D)$ be the open subscheme of maps $u : \mathbf{P} \to S$ from irreducible genus 0 curves such that $\mathbf{P} \to u(\mathbf{P})$ is birational. Such u is said to be *unramified* if $u^*T^*S \to T^*\mathbf{P}$ is surjective.

Hypothesis 2. *In addition to Hypothesis 1, assume k is perfect of characteristic not 2 or 3. If $d_S = 2$, assume additionally that for every effective $D' \in Pic(S)$, there is a geometric point f in each irreducible component of $M_0^{\text{bir}}(S,D')$ with f unramified.*

Let Λ be a complete discrete valuation ring with reside field k and quotient field K of characteristic 0. In [KLSW23, Section 9] we construct $\tilde{S} \to \operatorname{Spec}\Lambda$ a smooth del Pezzo surface equipped with an effective $\tilde{D} \in Pic(\tilde{S})$ with special fibers $\tilde{S}_k \cong S$ and $\tilde{D}_k \cong D$. We construct a Galois twist

$$\tilde{ev}_\sigma : \bar{M}_{0,n}(\tilde{S},\tilde{D})_\sigma \to (\tilde{S}^n)_\sigma$$

that agrees with ev_σ on the special fiber. Moreover, we identify a closed subset $\tilde{A} \subset (\tilde{S}^n)_\sigma$ such that $\bar{M}_{0,n}(\tilde{S},\tilde{D})_\sigma^{\text{good}} := \bar{M}_{0,n}(\tilde{S},\tilde{D})_\sigma \setminus ev^{-1}(\tilde{A})$ has the following two properties.

(1) The restriction of the total evaluation map $\tilde{ev}_\sigma^{\text{good}} : \bar{M}_{0,n}(\tilde{S},\tilde{D})_\sigma^{\text{good}} \to (\tilde{S}^n)_\sigma$ is relatively oriented.
(2) The codimension of $\tilde{A} \subset (\tilde{S}^n)_\sigma$ is at least 2.

Properties 1 and 2 again allow us to define the degree of $\tilde{ev}_\sigma^{\text{good}}$ in $\mathrm{GW}(k)$. Thus we obtain the following result.

Theorem 2. *Let S, D, σ satisfy Hypothesis 2 and assume that S is $\mathbb{A}^1$-connected. Then, there exists an invariant $N_{S,D,\sigma}$ in the Grothendieck–Witt ring $\mathrm{GW}(k)$ given by the degree of $\tilde{ev}_\sigma^{\text{good}}$. It is independent of the choice of $\tilde{S}, \tilde{D}$.*

3. The Grothendieck–Witt ring

In order to explain the enumerative meaning of the invariants $N_{S,D,\sigma}$, we recall the definition and basic properties of the Grothendieck-Witt ring $\mathrm{GW}(k)$. The Grothendieck–Witt ring is defined as the group completion of the semi-ring of non-degenerate symmetric bilinear forms over k. Since symmetric bilinear forms over a field are stably diagonalizable, an arbitrary element of this group can be expressed as a sum of rank 1 bilinear forms. Let $\langle a \rangle$ denote the element of $\mathrm{GW}(k)$ corresponding the rank 1 bilinear form $k \times k \to k$ given by $(x, y) \mapsto axy$ for a in k^*.

For finite rank field extensions $L \subseteq E$, there is an additive transfer map

$$\mathrm{Tr}_{E/L} : \mathrm{GW}(E) \to \mathrm{GW}(L),$$

which has the following simple description when $L \subseteq E$ is separable: for a symmetric, non-degenerate bilinear form $\beta : V \times V \to E$ over E, we can view V as a vector space over L and consider the composition

$$V \times V \xrightarrow{\beta} E \xrightarrow{\mathrm{Tr}_{E/L}} L$$

where $\mathrm{Tr}_{E/L}$ is the sum of the Galois conjugates. Since $L \subseteq E$ is separable, $Tr_{E/L} \circ \beta$ is a non-degenerate symmetric bilinear form over L. The value of the transfer map on the class $[\beta]$ of the form β is given $\mathrm{Tr}_{E/L}[\beta] = [Tr_{E/L} \circ \beta]$.

4. Enumerative meaning

To see the enumerative meaning of the degree $N_{S,D,\sigma}$, we generalize the sign associated to a node with two complex conjugate branches over $\mathbf{R}$. Suppose $u : \mathbf{P}_{k(u)} \to S$ is a rational curve on S defined over the field extension $k(u)$ of k. Let p be a node of $u(\mathbf{P}_{k(u)})$. The two tangent directions at p define a degree 2 field extension $k(p)[\sqrt{D(p)}]$ of $k(p)$, for a unique element $D(p)$ in $k(p)^*/(k(p)^*)^2$. By [SGA73, Exposé XV Théoréme 1.2.6], the extension $k(u) \subseteq k(p)$ is separable. Let $N_{k(p)/k(u)} : k(p)^* \to k(u)^*$ denote the norm of the field extension $k(u) \subseteq k(p)$ given by the product of the Galois conjugates.

Definition 4.1. *The* mass *of p is defined by*

$$\text{mass}(p) = \langle N_{k(p)/k(u)} D(p) \rangle \quad \textit{in } \mathrm{GW}(k(u)). \tag{1}$$

This makes sense because multiplying $D(p)$ by a square in $k(p)$ multiplies the norm by a square in $k(u)$.

The following is valid under the same hypotheses as Theorem 1 for k of characteristic zero and under the same hypotheses as Theorem 2 for k of positive characteristic.

Theorem 3. *If there exist $p_1, p_2, \ldots, p_r$ points of S with $k(p_i) \cong L_i$ in general position, we have the equality*

$$N_{S,D,\sigma} = \sum_{\substack{u \text{ degree } D \\ \text{rational curve} \\ \text{through the points} \\ p_1,\ldots,p_r}} \mathrm{Tr}_{k(u)/k} \prod_{p \text{ node of } u(\mathbf{P}^1)} \mathrm{mass}(p).$$

in $\mathrm{GW}(k)$*. So the weighted count of degree D rational plane curves through the points $p_1, p_2, \ldots, p_r$ given on the right hand side is independent of the general choice of points. When k is an infinite field and S is rational over k, such a general choice of points exists.*

Consequently, for $k = \mathbf{C}$ the rank of $N_{S,D,\sigma}$ coincides with the corresponding Gromov–Witten invariant. For $k = \mathbf{R}$, the signature of $N_{S,D,\sigma}$ recovers the signed counts of real rational curves of Degtyarev-Kharlamov and Welschinger. For $k = \mathbf{F}_p, \mathbb{Q}_p, \mathbb{Q}$ etc., one obtains a new Gromov–Witten invariant. Andrés Jaramillo Puentes and Sabrina Pauli have work in progress giving an enriched count of rational curves of a fixed degree through rational points on a toric surface via a tropical correspondence theorem, building on their previous work [JPP22].

General position of the points $p_1, p_2, \ldots, p_r$ of S with $k(p_i) \cong L_i$ means the following. There is a dense open subset U of $\prod_{i=1}^r \mathrm{Res}_{L_i/k} S$ such that for any rational point of U, the theorem holds for the corresponding r-tuple of points $p_1, p_2, \ldots, p_r$ of S with $k(p_i) \cong L_i$. The open subset U may not contain a rational point. Even for $S = \mathbb{P}^2$, this may happen over a finite field. Nonetheless, $N_{S,D,\sigma}$ is a meaningful invariant. It is the $\mathbb{A}^1$-degree of an evaluation map and an analogue of a Gromov–Witten invariant defined over perfect fields of characteristic not 2 or 3, including finite fields. Just as Gromov–Witten invariants make sense of curve counts when general position can not be achieved, these analogues give meaning to curve counts when rational points do not exist.

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Givental reconstruction for real Gromov-Witten invariants

Thomas Guidoni

The all-genus Gromov-Witten theory of targets with semi-simple small quantum cohomology can be expressed in terms of its genus 0 restriction through Givental reconstruction formula. It was discovered by Givental for the equivariant theory of projective spaces - and more generally of targets with hamiltonian toric actions with isolated fixed points - using localization formulas [1, 2, 3]. Teleman proved the general non-equivariant statement by classifying homogenous semi-simple cohomological field theories [6].

In this note, we describe a Givental-type reconstruction for real Gromov-Witten theory with conjugate constraints. We prove it for the equivariant theory of the targets $(\mathbb{P}_{\mathbb{C}}^{2N-1}, \theta_N)$, where

$$\theta_N : [x_{+1} : x_{-1} : \ldots : x_{+N} : x_{-N}] \mapsto [\overline{x_{-1}} : \overline{x_{+1}} : \ldots : \overline{x_{-N}} : \overline{x_{+N}}].$$

Equivariant refers to the action of the N-dimensional torus $\mathbb{T} = (\mathbb{C}^*)^N$, where the i-th factor $\mathbb{C}^*$ acts with weight $(0, 0, \ldots, -1, 1, \ldots, 0, 0)$ on $\mathbb{P}^{2N-1}$.

For $g, n, d \geq 0$, let

$$\overline{M}_{g,n}(\mathbb{P}^{2N-1}, \theta_N, d)$$

be the moduli space of stable real maps $f : (\Sigma, \sigma; p_1^+, p_1^-, \ldots, p_n^+, p_n^-) \to (\mathbb{P}^{2N-1}, \theta_N)$ of degree d, where Σ is a connected Riemann surface of genus g, σ is an anti-holomorphic involution on Σ, the $2n$-marked points $(p_i^{\pm})$ are pair-wise distinct and satisfy $\sigma(p_i^{\pm}) = p_i^{\mp}$, and $f \circ \sigma = \theta_N \circ f$. We also consider the moduli space $D\overline{M}_{g,n}(\mathbb{P}^{2N-1}, \theta_N, d)$ of genus g doublets, defined similarly with domains (Σ, σ) such that Σ is the disjoint union of two connected Riemann surfaces of genus g and σ exchanges the two connected components.

The target $(\mathbb{P}^{2N-1}, \theta_N)$ is real-orientable in the sense of Georgieva-Zinger and it carries a canonical real orientation [5]. It provides the moduli spaces of stable real maps with a virtual fundamental class [4] and allows to define the real Gromov-Witten invariants

$$\left\langle \psi^{k_1}\gamma_1, \ldots, \psi^{k_n}\gamma_n \right\rangle_g^{\mathbb{R}} = \sum_{d=0}^{\infty} q^d \int_{[\overline{M}_{g,n}(\mathbb{P}^{2N-1}, \theta_N, d)]_{\mathbb{T}}^{vir}} \prod_{i=1}^{n} \psi_i^{k_i} ev_i^*(\gamma_i) \in H_{\mathbb{T}}^{\bullet}(pt, \mathbb{C})[[q]]$$

for integers $k_1, \ldots, k_n \geq 0$ and classes $\gamma_1, \ldots, \gamma_n \in H_{\mathbb{T}}^{\bullet}(\mathbb{P}^{2N-1}, \mathbb{C})$. The ψ-class ψ_i and the evaluation map ev_i are taken at the positive marked point p_i^+. For $t \in H_{\mathbb{T}}^{\bullet}(\mathbb{P}^{2N-1}, \mathbb{C})$, let

$$\left\langle\left\langle \psi^{k_1}\gamma_1, \ldots, \psi^{k_n}\gamma_n \right\rangle\right\rangle_g^{\mathbb{R}} = \sum_{m=0}^{\infty} \frac{1}{m!} \left\langle \psi^{k_1}\gamma_1, \ldots, \psi^{k_n}\gamma_n, \underbrace{t/2, \ldots, t/2}_{m \text{ times}} \right\rangle_g^{\mathbb{R}}$$

We define similarly the doublet invariants $\left\langle\left\langle \psi^{k_1}\gamma_1, \ldots, \psi^{k_n}\gamma_n \right\rangle\right\rangle_g^{D}$ as integrals over the corresponding doublet moduli spaces. The real Givental-type reconstruction describes the invariants

$$\langle\langle\cdot,\dots,\cdot\rangle\rangle_g^{\mathbb{R}+D} = \langle\langle\cdot,\dots,\cdot\rangle\rangle_g^{\mathbb{R}} + \langle\langle\cdot,\dots,\cdot\rangle\rangle_{\frac{g+1}{2}}^{D}. \tag{1}$$

The doublet contribution vanishes for even g. We allow the value $g=-1$ for which there is no real contribution. The invariants (1) can be computed using localization [5]

The Givental-type reconstruction for the invariants (1) is expressed as a sum over real stable graphs. A real graph of type (g,n) is a connected graph Γ with some decorations. Each vertex $v\in V(\Gamma)$ has a genus $g_v\in\mathbb{Z}_+$, leaves labelled by a subset I_v of $\{1,\dots,n\}$ in such a way that (I_v) is a partition of $\{1,\dots,n\}$, and an arbitrary number of additional leaves that we refer to as real leaves. If n_v is the sum of the numbers of half-edges, marked and real leaves incident to v, we require

$$1-g-n=\sum_{v\in V(\Gamma)} 2-2g_v-n_v.$$

The real graph is stable if $2-2g_v-n_v<0$ for all $v\in V(\Gamma)$. Let $\Gamma_{g,n}^{\mathbb{R}}$ be the finite set of isomorphism classes of stable real graphs of type (g,n).

For each stable real graph $[\Gamma]\in\Gamma_{g,n}^{\mathbb{R}}$, we define

$$\Big\langle\Big\langle\psi^{k_1}\gamma_1,\dots,\psi^{k_n}\gamma_n\Big\rangle\Big\rangle^{\Gamma} = \prod_{v\in V(\Gamma)}\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k!}\int_{\overline{M}_{g_v,n_v+k}}\Omega^{t,q^2}_{g_v,n_v+k}(\cdots) \tag{2}$$

where $\Omega^{t,q}_{g_v,n_k+k}$ is the topological field theory

$$H^\bullet_{\mathbb{T}}(\mathbb{P}^{2N-1},\mathbb{C})^{\otimes(n_v+k)}\to H^0(\overline{M}_{g_v,n_v+k},\mathbb{C})\otimes H^\bullet_{\mathbb{T}}(pt,\mathbb{C})[[q]].$$

In order to describe the vectors inserted into these multilinear forms in (2), we need formal series involved in Givental reconstruction for the complex invariants :

- A formal series $\mathcal{S}^{t,q}(z)$ in $1/z$ with endomorphism coefficients.
- A formal series $\mathcal{R}^{t,q}(\psi)$ in ψ with endomorphism coefficients.
- A formal series $\mathcal{E}^{t,q}(\psi,\psi')$ in ψ,ψ' with bivector coefficients.
- A formal series $\mathcal{T}^{t,q}(\psi)$ in ψ with vector coefficients.

All these formal series are defined from the complex genus 0 theory of $\mathbb{P}^{2N-1}$, see [1, 2]. For the real Givental-type reconstruction, we also consider :

- The involution $\tau=-\theta_N^*$ on $H^\bullet_{\mathbb{T}}(\mathbb{P}^{2N-1},\mathbb{C})$. It induces the involution

$$\hat{\tau}\left(\sum_{k=0}^{\infty}\gamma_k\psi^k\right)=\sum_{k=0}^{\infty}\tau(\gamma_k)(-\psi)^k$$

  on $H^\bullet_{\mathbb{T}}(\mathbb{P}^{2N-1},\mathbb{C})[[\psi]]$.
- A formal series $\mathcal{U}^{t,q}(\psi')$ in ψ' with coefficients in linear forms. It is obtained from the asymptotic expansion at $\psi'=0$ of

$$\frac{\sqrt{-1}}{2}\Big\langle\Big\langle\frac{\cdot}{\psi'-\psi}\Big\rangle\Big\rangle_0^{\mathbb{R}}.$$

We can now describe the vectors inserted into the $(n_v + k)$-linear form associated to a vertex $v \in V(\Gamma)$ in (2). The insertions are labelled by the half-edges, marked and real leaves incident to v, and an arbitrary number k of additional insertions. Each insertion corresponds to one of the $n_v + k$ marked points in $\overline{M}_{g_v,n_v+k}$ - in particular each insertion corresponds to a ψ-class $\psi_j \in H^2(\overline{M}_{g_v,n_v+k}, \mathbb{C})$, at which we evaluate the formal variable ψ in what follows.

- In the insertion corresponding the i-th marked point, we plug the coefficient of $1/z^{k_i+1}$ in
$$(id + \hat{\tau}) \circ \frac{\mathcal{R}^{t,q^2}(-\psi)^T \circ \mathcal{S}^{t,q^2}(z)(\gamma_i)}{z - \psi}.$$
The transpose is taken with respect to the Poincaré pairing.
- An edge corresponds to two insertions - possibly at different vertices - one for each half-edge. We insert the bivector
$$\frac{(id + \hat{\tau}) \otimes (id + \hat{\tau}')}{2} \circ \mathcal{E}^{t,q^2}(\psi, \psi').$$
- The additional insertions are filled with
$$\frac{id + \hat{\tau}}{2} \circ \mathcal{T}^{t,q^2}(\psi).$$
- Into the real leaves, we plug
$$\mathcal{U}^{t,q}(-\psi)^T.$$

Theorem 1. *The real Gromov-Witten theory of* $(\mathbb{P}^{2N-1}, \theta_N)$ *satisfy*
$$\sqrt{-1}^{1-g} \left\langle\left\langle \psi^{k_1}\gamma_1, \dots, \psi^{k_n}\gamma_n \right\rangle\right\rangle_g^{\mathbb{R}+D} = \sum_{[\Gamma] \in \Gamma_{g,n}^{\mathbb{R}}} \frac{1}{|Aut(\Gamma)|} \left\langle\left\langle \psi^{k_1}\gamma_1, \dots, \psi^{k_n}\gamma_n \right\rangle\right\rangle^{\Gamma}$$
for $1 - g - n < 0$ *at any point* t *such that* $\tau(t) = t$.

The right-hand side involves only integrals of ψ-classes over the moduli spaces of curves and the genus 0 complex and real Gromov-Witten theory of the target.

Conjecture 1. *The real Gromov-Witten of every real-orientable target whose quantum coholomogy is semi-simple satisfy Theorem 1 for a suitable* $\mathcal{U}^{t,q}(\psi)$.

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Logarithmic linear series

Francesca Carocci
(joint work with Luca Battistella, Jonathan Wise)

1. Introduction: history and motivations

For C a smooth, projective curve a $\mathfrak{g}^r_d$, namely *rank r, degree d linear series* is the data of $L \in \mathrm{Pic}^d(C)$ together with $V \subseteq \mathrm{H}^1(C, L)$ of dimension $r + 1$. Linear series are the main object of study of classical *Brill–Noether theory.* The fundamental theorems in the theory were established in the 1980s, using *degenerations to singular curves* [5, 4]; among these classical results is the following:

Theorem 1. *Let C be a general smooth projectuve curve of genus g. Let $\rho(g, r, d) := g-(r+1)(g-d+r)$. Then the moduli space $\mathcal{G}_{r,d}(C)$ of $\mathfrak{g}^r_d$ on C is a smooth projective variety of the expected dimension $\rho(g, r, d) := g - (r + 1)(g - d + r)$.*

Although the original results of classical Brill-Noether theory can be proved without degenerations more recent progress depends on passing to nodal curves. It is then natural to ask:

Question 1.

(1) How does the data of a linear series degenerate when C degenerate to a nodal curve C_0, i.e.: How does the line bundle L degenerate and how does V degenerate ?
(2) What can we say about the moduli space of linear series on C_0 ?
(3) How can we detect when a linear series on C_0 is the limit of a linear series on a smooth curve?

In the talk I explain how in a joint work with Luca Battistella and Jonathan Wise we give an answer to this question using logarithmic geometry.

First, we will recall that for degenerations to nodal curves of *compact type* we have the satisfying theory of *limit linear series*, [3]. We review Eisenbud-Harris definition and at the end of the talk explain how to re-interpret it in terms of our logarithmic linear series.

Then we will introduce the necessary ingredients to define linear series on logarithmic curves.

2. Logarithmic ingredients

2.1. **Logarithmic curves and Artin Fans.** A *log curve* over S is a proper, integral and logarithmically smooth morphism $\pi \colon C \to S$ with connected one-dimensional (geometric) fibres. F. Kato [6] provided a classification of log smooth curves. To $C \xrightarrow{\pi} S$ a log curve we can associate $\mathcal{A}_C \xrightarrow{\bar{\pi}} \mathcal{A}_S$ a morphism of *Olsson Fans* [1]. We consider the Olsson fan relative to S $\Gamma = S \times_{\mathcal{A}_S} \mathcal{A}_C$ obtained by base change. By [2] the category of Olssons's fans is equivalent to the data of *generalized cone complexes.* Thus Γ (relative to S) should really be thought as a combinatorial object. We refer to τ as the tropicalization. Notice that Γ, thought as an algebraic stack, is naturally endowed with a logarithmic structure M_Γ such that the map τ is strict.

The following is the only example we need for the talk:

Example 2. *Let $C \to \mathrm{Spec}(\mathbb{N} \to k)$ be a log curve over the standard log point with $C \cong C_v \cup_{q_e} C_w$ a curve with one node q_e. Then $\mathcal{A}_S \cong [\mathbb{A}^1/\mathbb{G}_m]$ with associated cone complex the ray $\rho \cong \mathbb{R}_{\geq 0}$; $\mathcal{A}_C \cong [\mathbb{A}^2/\mathbb{G}_m^2]$ with associated cone complex the positive quadrant $\sigma \cong \mathbb{R}^2_{\geq 0}$. The morphism $\mathcal{A}_C \to \mathcal{A}_S$ is induced by the multiplication map $\mathbb{A}^1 \xrightarrow{t=xy} \mathbb{A}^1$. Then $\Gamma \cong [\mathrm{Spec}\, \frac{k[x,y]}{xy}/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts with opposite weights λ, λ^{-1} on x, y. The associated combinatorial onject is now a polyhedra cone complex, namely a edge e.*

2.2. Logarithmic Picard group.

Definition 3. *Let $C \xrightarrow{\pi} S$ be a log curve, then a* logarithmic line bundle *is an M_C^{gp}-torsor $\mathcal{L}$ such that on the fiber $\mathcal{L}|_{C_s}$ satisfy a technical condition called the* bounded monodromy condition*.*

Theorem 4. [8] *The logarithmic Picard group*

$$\mathrm{LogPic}(C/S) = \mathrm{R}^1\, \pi_* \mathbf{G}^\dagger_{\log}$$

is a stack over LogSch *in the strictly étale topology. It admits a logarithmic smooth cover by log smooth schemes and it is* proper *over S.*

In order to understand how to think of log line bundles for us will be important that *bounded monodromy* implies:

Fact. *For any $\mathcal{L} \in \mathrm{LogPic}(C/S)$ there exist a log modification $S' \to S$ and a subdivision $C' \xrightarrow{f} C \times_S S'$ such that the $\bar{M}_{C'}^{gp}$ torsor $\overline{f^*\mathcal{L}}$ induced by $f^*\mathcal{L}$ is trivial.*

For any trivialization $\gamma \in \overline{f^*\mathcal{L}}$ we have a representative line bundle $f^*\mathcal{L}(\gamma)$. These trivializations are a torsor under $\mathrm{CL}(\Gamma') \cong \pi'_* \overline{M}^{gp}_{C'}$. In particular, we have the following interpretation of LogPic.

$$\lim_{C' \to S'} \mathrm{Pic}(C'/S')/\pi'_* \overline{M}^{gp}_{C'} \cong \mathrm{LogPic}(C/S)$$

Since the map τ is strict, $\bar{M}_C^{gp} \cong \bar{M}_\Gamma^{gp}$. Thus we can identify $\bar{M}_C^{gp}$-torsors $\overline{L}$ with $\bar{M}_\Gamma^{gp}$-torsosr.

The theory of logarithmic line bundle tells us how to take *degenerations of line bundles* when the curve degenerate to a nodal one.

2.3. Vector bundle up to homothety.

Definition 5. *A $M_\Gamma^{gp} \otimes_{\mathcal{O}_\Gamma} \mathrm{GL}_n$-torsor $\mathcal{E}$ is called a* vector bundle up to homothety. *on the tropical curve Γ.*

To a vector bundle up to homothety we can associate a $\bar{M}_\Gamma^{gp}$-torsor $\overline{\mathcal{E}}$. Then, as for logarithmic line bundles, for each section $\gamma \in \overline{\mathcal{E}}$, there is a locally free sheaf $E(\gamma)$,representing $\mathcal{E}$. In the key example of the edge:

Example 6. *Using Klyachko's description of equivariant vector bundles on toric varieties (see for example* [7] *we will see that vector bundle on a edge e is the data of two filtered vector spaces* $E_1^\bullet, E_2^\bullet$ *together with an isomorphism of the associated graded* $\mathrm{gr}_\bullet(E_1^\bullet) \cong \mathrm{gr}_{-\bullet}(E_2^\bullet)$*. where the change of sign comes from the fact that the action of* $\mathbb{G}_m$ *on the two branches is with opposite weight. This will allow us to explain that a vector bundle up to homothety on e is the data of two vector bundles up to homothety on the rays with an isomorphism also only well defined up to some shifts* d $\mathrm{gr}_\bullet(E_1^\bullet) \cong \mathrm{gr}_{d-\bullet}(E_2^\bullet)$*.*

We explain that theory of vector bundles on families of tropical curve present some difficulties: the stack of vector bundles of rank r on Γ/S is *not algebraic*. We resolve this issue working with *realizable* bundles; the stack $\mathcal{B}_r$ of realizable bundles on $\Gamma \xrightarrow{p} S$ a tropical curve is algebraic and irreducible of dimension $-(r)^2$ over S.

Once we have all the ingredients, we define logarithmic linear series and state the main results.

3. Logarithmic linear series: definition and results

Definition 7. *A logarithmic linear series on* $C \to S$ *with tropicalization* $\tau : C \to \Gamma/S$ *;*

(1) a logarithmic line bundle L *on* C *of degree d with associated* $\bar{M}_C^{gp} = \bar{M}_\Gamma^{gp}$*-torsor* $\bar{L}$ *over* Γ;

(2) an homothety vector bundle $\mathcal{E}$ *of rank* $r+1$ *on* Γ *together with an isomorphism of* $\bar{M}_\Gamma^{gp}$*-torsors* $\bar{L} \cong \overline{\mathcal{E}}$*;*

(3) a morphism $\tau^*\mathcal{E} \xrightarrow{x} \mathcal{L}$ *of locally free sheaves up to homothety;*

such that

non-degeneracy: *For each (equivalently, any) local section* $\alpha \in \bar{L}$ *the morphism* $\mathcal{E}(\alpha) \xrightarrow{x} \tau_*\mathcal{L}(\alpha)$ *is universally injective.*

stability: *there is a stability condition.*

We explain how to interpret limit linear series of Eisenbud-Harris as logarithmic linear series.

Theorem 8. $G_r^d(C/S) \to \mathrm{LogSchs}^{fs}/S$ *is logarithmic stack in the strict étale topology, which is moreover algebraic of finite type, Deligne-Mumford and proper relative to* $\mathrm{LogPic}(C/S)$*. In particular* $G_r^d(C/S) \to S$ *is proper.*

Theorem 9. $G_r^d(C/S)$ *is endowed with a perfect obstruction theory* $Rp_*\mathcal{L} \otimes \mathcal{E}^\vee$ *relative to the irreducible stack* $\mathcal{B}_r$ *parametrizing log curves together with a realizable vector bundle. In particular* $G_r^d(C/S)$ *is endowed with a virtual class of expected dimension* $\rho(g,r,d)$ *relative to S.*

Corollary 10. *Every irreducible component of* $G_r^d(C/S)$ *has dimension at least* $\rho(g,r,d)$ *relative to S. A sufficient condition for a logarithmic linear series* $l\mathfrak{g}_r^d$ *to be smoothable is that the dimension of* $G_r^d(C/S)$ *at* $l\mathfrak{g}_r^d$ *over S is exactly* $\rho(g,r,d)$*.*

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Gromov–Witten invariants of log Calabi–Yau 3-folds are holomorphic Lagrangian correspondences

Brett Parker

Motivated by geometric quantisation, Alan Weinstein famously proposed using Lagrangian correspondences as morphisms in a symplectic category. Weinstein proposed this in unpublished work in the 1970s, but a modern survey can be found in [1]. Analytic difficulties plague this idea in the smooth setting, but a holomorphic version of Weinstein's symplectic category overcomes such difficulties. The evaluation space for the moduli stack of holomorphic curves in a log Calabi–Yau 3-fold has a natural holomorphic symplectic structure constructed from the 3-fold's holomorphic volume form. Moreover, the image of the moduli stack of holomorphic curves is a holomorphic Lagrangian, and the pushforward of the virtual fundamental is a holomorphic Lagrangian cycle.

A smooth symplectic manifold is a smooth manifold M with non-degenerate, closed 2-form ω, and a Lagrangian submanifold of M is a half–dimensional submanifold on which the symplectic form vanishes. Symplectic manifolds arise as phase spaces in classical mechanics. A primary example is the cotangent bundle T^*X of a smooth manifold X, and examples of Lagrangian submanifolds are cotangent fibers, and the graphs of exact 1–forms. To motivate the Weinstein category, consider a symplectomorphism $f : (M, \omega_M) \longrightarrow (N, \omega_N)$; that is is a diffeomorphism such that $f^*\omega_N = \omega_M$. A diffeomorphism f is a symplectomorphism if and only if the symplectic form $-\omega_M + \omega_N$ vanishes on graph of f within $M \times N$, so the graph of f is a Lagrangian submanifold of the symplectic manifold $M^- \times N$, where M^- indicates M with the symplectic form $-\omega_M$. Weinstein proposed expanding the definition of morphisms from M to N to include all Lagrangian submanifolds of $M^- \times N$, and composing these morphisms as correspondences between M and

N. This idea was partially motivated by pioneering work of Hörmander on the semiclassical limit of operators; [3]. In the case of cotangent bundles, Hörmander associates 'classical' Lagrangian correspondences in $(T^*X)^- \times T^*Y$ as wave-front sets of certain 'quantum' operators between the spaces of functions on X and Y. For example, Fourier integral operators in the form $f \mapsto \int e^{i\Phi(x,y)}a(x,y)f(x)dx$ are associated to the correspondence defined by the image of graph of $d\phi$ under the symplectic map $T^*X \times T^*Y \longrightarrow (T^*X)^- \times T^*Y$ defined by multiplying fibres of T^*X by -1. Moreover, when Lagrangian correspondences are sufficiently transverse, the classical composition of correspondences coincides with the quantum composition of operators. For a modern symplectic perspective, see [2].

The Weinstein symplectic category is a wonderfully enticing idea, but suffers difficulties: the composition of Lagrangian correspondences is only a Lagrangian correspondence when the correspondences intersect cleanly. In this talk, we consider various holomorphic versions of Weinstein's symplectic category. The pathologies of smooth intersection theory do not occur in the more rigid holomorphic or algebraic setting, so we can overcome the complications plaguing Weinstein's symplectic category in the smooth setting.

A correspondence between sets X and Y is a subset $R \subset X \times Y$. The set-theoretic composition of correspondences $R_1 \subset X \times Y$ and $R_2 \subset Y \times Z$ is the set $R_2 \circ R_1 \subset X \times Z$ consisting of points (x, z) such that there exists some point $y \in Y$ such that $(x, y) \in R_1$ and $(y, z) \in R_2$. So, $R_2 \circ R_1$ is constructed by taking the fibre product of R_1 and R_2 over Y, then projecting out the Y direction. This notion of relations as sets is not quite what we use for our holomorphic versions of the Weinstein category. Instead, our relations consist of a formal linear combination of Lagrangian cycles, and composition uses fibre products and pushforwards of cycles, so differs from the set theoretic composition in that it assigns canonical integer weights to the components of the set theoretic composition. With this version of relations, we can canonically encode Gromov–Witten invariants of log Calabi–Yau 3-folds as Lagrangian cycles, and composition of these Lagrangian cycles is compatible with gluing of Gromov–Witten invariants.

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The KSBA moduli space of stable log Calabi–Yau surfaces

Hülya Argüz
(joint work with Valery Alexeev, Pierrick Bousseau)

A higher dimensional analogue of a "stable curve" [5] is a stable pair, introduced by Kollár–Shepherd-Barron [9] and Alexeev [2], given by a tuple (Y, B), where Y is a projective variety, and B is a $\mathbb{Q}$-divisor satisfying the following conditions:

i) (Y, B) has semi-log-canonical (slc) singularities
ii) $K_Y + B$ is $\mathbb{Q}$-Cartier and ample.

The moduli space of stable pairs, often referred to as the KSBA moduli space, is defined as follows.

Fix $d \in \mathbb{Z}$, $\omega_1, \dots \omega_n \in \mathbb{Q}_{>0}$, and $v \in \mathbb{Q}_{>0}$. Then, the KSBA moduli space $\mathcal{M}^{KSBA}(d, \omega_1, \dots, \omega_N, v)$ is the moduli space parametrising all stable pairs (Y, B), where Y is a projective variety of dimension d, B is a $\mathbb{Q}$-divisor in Y which admits a decomposition $B = \sum_{i=1}^{n} \omega_i B_i$ where B_i's are effective divisors. Moreover, one requires the *volume* of (Y, B), which by definition is the intersection number $(K_Y + B)^d$, to be equal to $v \in \mathbb{Q}_{>0}$.

Note that choosing $d = 1$ and $\omega_i = 1$ for all $1 \leq i \leq n$, we get stable pairs $(C, \sum_{i=1}^{n} p_i)$ given by n-marked curves of arithmetic genus g such that the volume $K_C + \sum_{i=1}^{n} p_i = 3g - 3 + n = v$. In dimension one, the condition to have slc singularities amounts to requiring C to have at worst nodal singularities, and furthermore $K_C + \sum_{i=1}^{n} p_i$ being ample is equivalent to the automorphism group of $(C, \sum_{i=1}^{n} p_i)$ to be finite. Therefore, $\mathcal{M}^{KSBA}(1, 1, \dots, 1, v)$ is the classical Deligne–Mumford moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. The KSBA moduli space of higher dimensional stable pairs, similarly to $\overline{\mathcal{M}}_{g,n}$, is a Deligne–Mumford stack – in what follows we just consider the coarse moduli space and forget the stacky structure. However, the connected components of the KSBA moduli space are generally not irreducible, due to the fact that the deformation theory of slc singularities in high dimensions is significantly more challenging than the deformation theory of nodal curves. One generally may wish to describe the geometry of the irreducible components of the KSBA moduli space.

For instance, in the toric situation for a polarised toric variety Y with an ample line bundle L, denoting the toric boundary divisor by D, we define $\mathcal{M}_{(Y,D,L)}$ as the closure in $\mathcal{M}^{KSBA}$ of the locus of stable pairs $(Y, D + \epsilon C)$, where $C \in |L|$ and $0 < \epsilon << 1$. In this case, the geometry of the KSBA moduli space is described in [3], where Alexeev showed that the (normalization of the) moduli space $\mathcal{M}_{(Y,D,L)}$ is a toric variety with associated fan given by the secondary fan of the momentum polytope of (Y, D, L). Here the secondary fan for the momemtum polytope P of (Y, D, L), defined by Gelfand–Kapranov–Zelevinsky [6], also known as the GKZ fan is a toric fan whose maximal cones are in one-to-one correspondence with the regular triangulations of P. In [4], we generalize this result to the KSBA moduli space of log Calabi–Yau surfaces and show that (up to a finite cover) it is also a toric variety. In what follows we shortly explain how this is done, after defining the KSBA moduli space in the context of log Calabi–Yau surfaces.

Let (Y, D, L) be a polarized log Calabi–Yau surface given by a projective surface Y with an ample line bundle L on it, and $D \subset Y$ a reduced, anticanonical divisor. While mostly in the literature log Calabi–Yau surfaces are assumed to be given by smooth projective varieties with such divisors, we allow (Y, D) to be singular, and assume it has log canonical singularities [4]. Furthermore assume that (Y, D) is maximal, that is, $D \neq \emptyset$ and admits a 0-dimensional strata. In this case, the KSBA moduli space for (Y, D, L), denoted by $\mathcal{M}_{(Y,D,L)}$, is the closure in the moduli space of stable pairs of the locus of stable pairs deformation equivalent to $(Y, D+\epsilon C)$, where $C \in |L|$ and $0 < \epsilon \ll 1$. In this case, Hacking–Keel–Yu (HKY) conjectured that $\mathcal{M}_{(Y,D,L)}$ should be, up to a finite cover, a toric variety defined by a generalized version of the "secondary fan" coming from mirror symmetry [8] (in fact, the HKY conjecture concerns more generally log Calabi–Yau varieties of any dimension). Apart from the toric cases proven by Alexeev, this conjecture was previously proven for del Pezzo surfaces Y of degree n where $1 \leq n \leq 6$, with D a cycle of n many (-1)-curves, and moreover $L = -K_Y$. In [4] we prove this conjecture for all log Calabi–Yau surfaces. Our main result shows:

Theorem: The moduli space $\mathcal{M}_{(Y,D,L)}$ of polarized log Calabi–Yau surfaces, up to a finite cover, is a toric variety.

To prove this, we use mirror symmetry and study the "double mirror" to (Y, D, L). As a first step we construct semi-stable mirrors to maximal degenerations of (Y, D, L), which are given by projective crepant resolutions $\mathcal{X} \to \overline{\mathcal{X}} \to \text{Spec } \mathbf{C}[|t|]$ where $\overline{\mathcal{X}}$ is an affine threefold with canonical singularities. We then study the double mirror family, that is, the mirror to the semistable mirror $\mathcal{X} \to \overline{\mathcal{X}} \to \text{Spec } \mathbf{C}[|t|]$, using "intrinsic mirror symmetry" of Gross–Siebert [7], which uses as a main ingredient punctured log Gromov–Witten theory developed by Abramovich–Chen–Gross–Siebert [1]. We show that the double mirrors to all maximal degenerations of (Y, D, L), or the intrinsic mirrors to all crepant resolutions of $\overline{\mathcal{X}}$ glue to a family over the toric variety $\mathcal{M}_{sec}$ whose fan is the relative Mori fan of $\mathcal{X}/\overline{\mathcal{X}}$. One of our key results then shows that this family is a family of KSBA stable log Calabi–Yau surfaces, with general fiber deformation equivalent to (Y, D, L). By the universal property of KSBA moduli spaces this gives a map $\mathcal{M}_{sec} \to \mathcal{M}_{(Y,D,L)}$. As a final step we show that this map is finite and surjective.

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Topology of totally real degenerations

Matilde Manzaroli
(joint work with Emiliano Ambrosi)

We present the content of [AM22], where we study the topology of totally real semi-stable degenerations. The main result is a bound for the individual Betti numbers of a smooth real fiber in terms of the complex geometry of the degenerated fiber. The main ingredient is the use of real logarithmic geometry, which makes it possible to study degenerations that are not necessarily toric, and therefore to go beyond the case of smooth tropical degenerations, studied by Renaudineau-Shaw.

Let X be a real algebraic variety, let $X(\mathbb{C})$ be the set of its complex points and $X(\mathbb{R})$ the set of its real points. For a topological space Y, set $b_i(Y) := \dim_{\mathbb{F}_2}(H^i(Y, \mathbb{Z}/2\mathbb{Z}))$ for its i^{th} Betti number.

1. Complex semistable degeneration

A classical tool to study irreducible smooth projective varieties is to degenerate them to a union of irreducible simpler varieties. A celebrated theorem of Steenbrink [Stee76] shows that if C is a smooth complex curve and $X \to C$ is a semistable degeneration of projective varieties with singular fiber X_0, then the rational cohomology of a general smooth fiber X_t can be computed from the geometry of X_0. More precisely, Steenbrink shows that, for every $q \geq 0$, there exists a complex $E_2^{q,\bullet}$ of $\mathbb{Q}$-vector spaces, depending only on X_0, such that

$$\dim(H^i(X_t, \mathbb{Q})) = \sum_{p+q=i} \dim H^p(E_2^{q,\bullet}). \tag{1}$$

The goal of [AM22] is to try to extend this kind of results to real semistable degenerations and to understand to which extent the topology of the real special fiber control the topology of the real general fiber.

2. Real semistable degeneration

Assume from now on that $X \to C$ is a real semistable degeneration with singular fiber X_0. Of course, one cannot expect equalities similar to (1) to hold in the real setting, since the real topology of the special fiber can drastically change in different fibers near 0. The idea is then to try to compare the real part of the special fiber with its complex counterpart and, afterwards, to relate the latter with the real part of a fiber near 0. In order to do this, let us first recall the Smith-Thom inequality,

$$\sum_i b_i(X(\mathbb{R})) \leq \sum_i b_i(X(\mathbb{C})) \tag{2}$$

which bounds the total Betti number of the real topology of a real variety X with the one of its complexification. This is one of the few general results comparing the real and the complex topology of real algebraic varieties. A special role, in the study of the topology of real algebraic variety, is played by *maximal* varieties, i.e. the ones for which (2) is an equality.

Inspired by (2), one could hope to obtain a bound for the topology of a real fiber near 0 in terms of data that depends only on its complexification. Recently, using tropical geometry, Renaudineau-Shaw proved [RS23, Theorem 1.4], confirming a conjecture of Itenberg [Ite17], that for a fiber X_t near 0 of a real hypersurface inside a real *toric* degeneration constructed via *primitive combinatorial Viro patchworking* [Vi80], one has that

$$b_i(X_t(\mathbb{R})) \leq \sum_j h^{i,j}(X_t), \tag{3}$$

where $h^{i,j}(X) := \dim(H^i(X, \Omega^j_X))$.

While these conjectures and results were limited to combinatorial and toric situations, we go one step further giving a purely algebraic geometric setting in which an inequality close to (3) holds. Theorem 1 and Corollary 4) show that if X_0 can be stratified with components that are cohomologically simple, then X_t satisfies (3), up to the dimension of the 2-torsion in some cohomology group. The main novelty of our approach is the use of (real) logarithmic geometry ([Arg21]) [Kat89], which allows to extend previous results for toric degenerations to more general families.

3. Main result

Assume that C is a smooth real curve and $f : X \to C$ a real projective morphism which is smooth outside a real point $0 \in C(\mathbb{R})$ and strictly-semistable around 0, in the sense that the irreducible components of X_0 are smooth and, locally analytically around 0, the family $f : X(\mathbb{C}) \to C(\mathbb{C})$ is isomorphic to the standard semistable degeneration $\mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n, T]/(x_1 \ldots x_n - T)) \to \mathrm{Spec}(\mathbb{C}[T])$. Assume furthermore that $f : X \to C$ is totally real, i.e. that the irreducible components of $X_0(\mathbb{C})$ are real. Write

$$X_0 = \bigcup_{i \in I} X_i$$

for the decomposition of X_0 in irreducible components and for every subset $J \subseteq I$ set

$$X_J := \bigcap_{i \in J} X_i \quad \text{and} \quad X_J^0 := X_J \setminus \bigcup_{i \notin J} X_i.$$

Then

$$X_0 = \coprod_{J \subseteq I} X_J^0$$

is a stratification $\mathfrak{I} := \{X_J^0\}_{J \subseteq I}$ of X_0 by smooth real algebraic subvarieties. Fix a refinement $\mathfrak{Z} := \{X_\Delta^0\}$ of $\mathfrak{I}$, made of smooth real algebraic varieties.

In [AM22], we construct, for every ring A and every $q \geq 0$ a canonical cochain complex $C^\bullet_{q,\mathfrak{Z},A}$ of A-modules depending only on the complex geometry of the stratification $\mathfrak{Z}$. Inspired by the geometry of degenerations constructed via primitive patchworking (see Remark 1), we consider the following conditions on the members of $\mathfrak{Z}$.

(a) $H^i(X^0_\Delta(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) = 0$, for all $i \geq 1$ and $X^0_\Delta \in \mathfrak{Z}$;
(b) X^0_Δ is maximal, for all $X^0_\Delta \in \mathfrak{Z}$;
(c) the mixed Hodge structure on $H^i(X^0_\Delta(\mathbb{C}), \mathbb{Q})$ is pure of type (i, i) and $H^i(X^0_\Delta(\mathbb{C}), \mathbb{Z})$ is torsion free, for all $i \geq 1$ and $X^0_\Delta \in \mathfrak{Z}$.

Our main result is then the following.

Theorem 1.
(1) Assume that (a) and (b) hold. Then, for every $t \in C(\mathbb{R})$ close to 0 one has:

$$b_p(X_t(\mathbb{R})) \leq \sum_q dim(H^p(C^\bullet_{q,\mathfrak{Z},\mathbb{Z}/2\mathbb{Z}})).$$

(2) Assume that (a),(b) and (c) hold. Then for every $t \in C(\mathbb{R})$ close to 0, one has:
(i) $dim(H^p(C^\bullet_{q,\mathfrak{Z},\mathbb{Z}} \otimes \mathbb{Q})) = h^{p,q}(X_t)$
(ii) $C^\bullet_{q,\mathfrak{Z},\mathbb{Z}} \otimes \mathbb{Z}/2\mathbb{Z} \simeq C^\bullet_{q,\mathfrak{Z},\mathbb{Z}/2\mathbb{Z}}$.

Remark 1. If X_0 comes from a degeneration constructed via primitive combinatorial patchworking, then it admits a stratification made by complements of hyperplane arrangements, which satisfy the (a),(b),(c) above. This shows that Theorem 1 generalizes the main result of [RS23] to a more general setting. Recently, there have been other two generalizations of [RS23]: on the combinatorial side [BLMR23]; on the geometric side, [RaReSh23].

Theorem 1 implies the following (the main motivation for [AM22]).

Corollary 4. *Assume that (a), (b), (c) hold and that $H^p(C^\bullet_{q,\mathfrak{Z},A})$ is torsion free for every $p, q \in \mathbb{N}$. Then for every $t \in C(\mathbb{R})$ close to 0 and every $p \in \mathbb{N}$ one has*

$$b_p(X_t(\mathbb{R})) \leq \sum_q h^{p,q}(X_t).$$

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Gromov-Witten theory and the refined topological string

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(joint work with Andrea Brini)

Since the initiation of the field, developments in Gromov–Witten theory were often inspired by analogies with mathematical physics. Given a smooth Calabi–Yau threefold X and an effective curve class β, one such example is Gopakumar–Vafa or BPS integrality which predicts that there are a finite number of integer invariants underlying the generating series

$$\mathsf{GW}_\beta(X) \coloneqq \sum_{g\geq 0} u^{2g-2} \int_{[\overline{M}_g(X,\beta)]^{\mathrm{virt}}} 1 \in \mathbb{Q}(\!(u)\!)\,.$$

of Gromov–Witten invariants of all genera. For example when X is local $\mathbb{P}^2$ and $\beta = [L]$ the class of a line we find

$$\mathsf{GW}_{[L]}(K_{\mathbb{P}^2}) = 3 \cdot \left(2\sin\tfrac{u}{2}\right)^{-2}\,. \tag{1}$$

Here, the leading factor 3 should be viewed as the single integer invariant — called *BPS invariant* — governing the Gromov–Witten invariants in every genus. In my talk I will introduce a refinement of this BPS integrality conjecture in the context of equivariant Gromov–Witten theory of Calabi–Yau fivefolds of the form $X \times \mathbb{C}^2$ concentrating on the case where $X = K_S$ is a local surface.

The refinement. Let S be a smooth projective surface. There is a natural action of $(\mathbb{C}^\times)^3$ on the fibres of $K_S \times \mathbb{C}^2 \to S$ where the ith $\mathbb{C}^\times$-factor scales the ith fibre direction with weight one. Via the embedding

$$T \coloneqq (\mathbb{C}^\times)^2 \hookrightarrow (\mathbb{C}^\times)^3,\ (t_1,t_2) \mapsto \left((t_1t_2)^{-1}, t_1, t_2\right)$$

we obtain a T-action on $K_S \times \mathbb{C}^2$ which fixes the holomorphic five-form. This lifts to an action on the moduli space of stable maps with this target.

We define the generating series of T-equivariant Gromov–Witten invariants

$$\mathsf{GW}_\beta(K_S \times \mathbb{C}^2) \coloneqq \sum_{g\geq 0} {}^{T}\!\!\int_{[\overline{M}_g(K_S\times\mathbb{C}^2,\beta)]^{\mathrm{virt}}_T} 1$$

where the right-hand side integral is defined via virtual T-localisation. The sum is a well-defined element in the completed and localised T-equivariant Chow cohomology of a point. Since $H^T_*(\mathrm{pt}) \cong \mathbb{Q}[\epsilon_1,\epsilon_2]$ the expression can naturally be viewed as a power series in two parameters ϵ_1 and ϵ_2. In the one parameter limit $\epsilon_1 = -\epsilon_2 = \mathrm{i}u$, which we will refer to as the *unrefined limit*, we recover a quantity we have already encountered before:

Lemma 1. *We have*

$$\mathsf{GW}_\beta(K_S \times \mathbb{C}^2)\Big|_{\epsilon_1=-\epsilon_2=\mathrm{i}u} = \mathsf{GW}_\beta(K_S)\,.$$

Refined BPS invariants. Now let S be a del Pezzo surface. In this case we can recursively define power series

$$\mathsf{BPS}_\beta(S)(\epsilon_1,\epsilon_2) \in \mathbb{Q}[\![\epsilon_1,\epsilon_2]\!]$$

labelled by effective curve classes β in S by demanding that these series satisfy

$$\mathsf{GW}_\beta(K_S \times \mathbb{C}^2) =: \sum_{\substack{k\in\mathbb{Z}_{>0}\\ k\,\mid\,\beta}} \frac{1}{k}\frac{\mathsf{BPS}_{\beta/k}(S)(k\epsilon_1,k\epsilon_2)}{2\sinh\frac{k\epsilon_1}{2}\;2\sinh\frac{k\epsilon_2}{2}}\,.$$

Conjecture 1. $\mathsf{BPS}_\beta(S)$ *lifts to an integer valued Laurent polynomial in* $\mathrm{e}^{\frac{\epsilon_1}{2}}$, $\mathrm{e}^{\frac{\epsilon_2}{2}}$.

To come back to our earlier example, when $X = K_{\mathbb{P}^2}$ and $\beta = [L]$ a low-genus computer calculation yields

$$\mathsf{BPS}_{[L]}(\mathbb{P}^2) = \mathrm{e}^{\epsilon_1+\epsilon_2} + 1 + \mathrm{e}^{-\epsilon_1-\epsilon_2} + O(\epsilon_i^8)\,.$$

In the unrefined limit this expression specialises to

$$\mathsf{BPS}_{[L]}(\mathbb{P}^2)\Big|_{\epsilon_1=-\epsilon_2=\mathrm{i}u} = 3 + O(u^8)$$

which via Lemma 1 is consistent with equation (1). More generally, in the unrefined limit Conjecture 1 specialises to the original conjecture of Gopakumar and Vafa [3] which was proven by Ionel–Parker [5] and Doan–Ionel–Walpuski [2].

Geometric interpretation. We expect $\mathsf{BPS}_\beta(S)$ to have a geometric interpretation in terms of Gieseker stable sheaves on S with support β and fixed Euler characteristic one. If we denote by M_β the moduli space of such stable sheaves, then the Hilbert–Chow morphism induces a perverse grading on cohomology groups:

$$H^{i,j} := \mathrm{gr}_j^P H^i\big(M_\beta, \mathbb{Q}[\dim M_\beta]\big).$$

Conjecture 2. *Identifying* $q_\pm = \mathrm{e}^{\frac{\epsilon_1\pm\epsilon_2}{2}}$ *we have*

$$\mathsf{BPS}_\beta(S) = \sum_{i,j} q_+^i q_-^j\,(-1)^{i+j}\dim H^{i,j}\,. \tag{2}$$

In the unrefined limit the conjecture specialises to Maulik and Toda's proposal for Gopakumar–Vafa invariants [8].

Evidence. Orthogonal to the unrefined limit we have the following evidence for our conjectures.

Theorem 2. *Conjecture 1 and 2 hold for* $S = \mathbb{P}^2$ *in the limit* $\epsilon_2 = 0$.

Idea of the proof. The theorem is proven by passing through a correspondence with the Gromov–Witten theory of the surface relative a smooth anticanonical curve E. To be more precise, let us denote by $\overline{M}_g(S/E,\beta)$ the moduli stack of genus g, class β stable maps to S with a marking of maximal tangency $(E\cdot\beta)$ along E. Then via a degeneration to the normal cone argument one can show that

$$\epsilon_2\;\mathsf{GW}_\beta(K_S\times\mathbb{C}^2)\Big|_{\epsilon_2=0} = \frac{(-1)^{E\cdot\beta+1}}{E\cdot\beta}\sum_{g\geq 0}\epsilon_1^{2g-1}\int_{[\overline{M}_g(S/E,\beta)]^{\mathrm{virt}}}\lambda_g\,.$$

The theorem then follows from the work of Bousseau [1] (with the addition of [7]) who establishes the analogous statement of the theorem for the right-hand side of the last equation when $S = \mathbb{P}^2$. □

Motivation & context. Conjecture 1 and 2 are informed by analogies with mathematical physics and expected correspondences with other curve counting theories. In each case these relations are natural refinements of their already known unrefined versions. To give more detail, if X is a smooth Calabi–Yau threefold which admits a non-trivial $\mathbb{C}^\times$-action we expect the equivariant Gromov–Witten theory of $X \times \mathbb{C}^2$ to be the counterpart of

- **the refined topological string on** X as studied in physics literature. Especially, Conjecture 1 is very much informed by Huang–Klemm's study of the anticipated B-model interpretation of this quantum field theory [4].
- **K-theoretic Pandharipande–Thomas theory of** X as introduced by Nekrasov and Okounkov [9]. Indeed, Kononov–Pi–Shen observed that for $X = K_{\mathbb{P}^2}$ in low degree these invariants can be matched with the right-hand side polynomial in equation (2) [6].

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q-refined tropical curve counts with descendents

Patrick Kennedy-Hunt

Fix a multiset Δ° of vectors in $\mathbb{Z}^2 \setminus \{(0,0)\}$ with sum zero, together with non-negative integers n and $k_1, \ldots, k_n$ such that

$$n - 1 + |\Delta^\circ| = 2n + \sum_{i=1}^{n} k_i.$$

Associated to this discrete data are two enumerative invariants.

(1) The vectors Δ° determine a toric surface X equipped with curve class β. There is an associated moduli space $M_{g,\Delta}$ whose points correspond to (stable) parameterised curves. This moduli space comes equipped by a homology class $[\mathsf{M}_{g,\Delta}]^{\mathrm{vir}}$ and cohomology classes λ_g, ψ_i, see [15, Section 2.1.1]. For every genus $g \geq 0$, there is an associated *logarithmic Gromov–Witten* invariant with λ_g insertion defined as the following intersection product

$$N^{\mathbf{k}}_{g,\Delta} = \int_{[\mathsf{M}_{g,\Delta}]^{\mathrm{vir}}} (-1)^g \lambda_g \prod_{i=1}^{n} \mathsf{ev}_i^\star(\mathsf{pt})\, \psi_i^{k_i}.$$

The above logarithmic Gromov–Witten invariant captures information about algebraic curves passing through a generic collection of n points in X subject to stationary descendant constraints.

(2) Fixing a generic ordered tuple of n points $p = (p_1, \ldots, p_n)$ in $\mathbb{R}^2$, the data $(\Delta^\circ, \mathbf{k})$ defines a finite set of genus zero parameterised tropical curves $T^{\mathbf{k}}_{\Delta,p}$, see [15, Section 1.1] for the definition of parameterised tropical curves and the set $T^{\mathbf{k}}_{\Delta,p}$. The cardinality of $T^{\mathbf{k}}_{\Delta,p}$ is not constant for a dense choice of p. To obtain a quantity invariant of p, one assigns to each tropical curve $h \in T^{\mathbf{k}}_{\Delta,p}$ a rational function $m_h(q)$ of formal variable $q^{1/2}$. We define a count of tropical curves

$$N^{\Delta,\mathbf{k}}_{\mathsf{trop}}(q) = \sum_{h \in T^{\mathbf{k}}_{\Delta,p}} m_h(q).$$

The rational function $N^{\Delta,\mathbf{k}}_{\mathsf{trop}}(q)$ is independent of p, defined in terms of polyhedral geometry first defined by Blechman and Shustin [10], generalising work of Block and Göttsche [21].

Theorem A. *After the change of variables $q = e^{iu}$ we have the equality*

$$\sum_{g \geq 0} N^{\boldsymbol{k}}_{g,\Delta}\, u^{2g-2+|\Delta^\circ|-\sum_i k_i} = N^{\Delta,\boldsymbol{k}}_{\mathsf{trop}}(q).$$

Theorem A, established in joint work with Qaasim Shafi and Ajith Urundolil-Kumaran [15] is an example of a *tropical correspondence theorem* [23, 9, 16, 17, 18, 19, 20, 13, 6, 12]. The theorem generalises work of Bousseau to the descendant situation [1], and relates to work of Mandel and Ruddat who handled the case without λ_g descendents [13].

Under the (logarithmic) Gromov–Witten/ Donaldson–Thomas correspondence [2, 3, 22], the data of $N^{\mathbf{k}}_{g,\Delta}$ is encoded by a logarithmic *Donaldson–Thomas invariant.* Logarithmic Donaldson–Thomas invariants are defined in terms of the intersection theory of *logarithmic Hilbert schemes* which parameterise embedded curves in the setting of logarithmic geometry [8, 14, 7].

One can interpret Theorem A as asserting that the combinatorics of tropical curves is encoded in the intersection theory of $M_{g,\Delta}$. The logarithmic Gromov–Witten/ Donaldson–Thomas correspondence thus asserts that the combinatorics of tropical curves is reflected in the intersection theory of the logarithmic Hilbert scheme $\mathbb{P}$ of curves of class β in X. We will state Theorem B, see [14], which gives an alternative way to understand this fact.

Associated to polytope Δ is a fan $\mathbb{P}^{\mathsf{trop}}$ whose points parameterise tropical curves of degree Δ superimposed on the fan of X. The combinatorial complexity of $\mathbb{P}^{\mathsf{trop}}$ is entirely governed by the secondary fan of Δ, as defined by Gelfand, Kapranov, and Zelevinsky [4].

Theorem B. *The logarithmic Hilbert scheme of curves in X of degree Δ is a toric stack with fan $\mathbb{P}^{\mathsf{trop}}$.*

The intersection theory of the toric stack $\mathbb{P}$ is controlled by Minkowksi weights on the fan of $\mathbb{P}^{\mathsf{trop}}$. Thus the intersection theory of the logarithmic Hilbert scheme is controlled by the geometry of tropical curves and their moduli. Theorem B therefore gives a different perspective on how the structure of tropical curves arise in logarithmic enumerative geometry.

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Enumerative intrinsic mirror symmetry

Michael van Garrel

1. Intrinsic Mirror Construction

Definition LSD. *Let Y be a smooth projective variety of dimension N and let D be a smooth effective divisor. A maximal LSD (maximal log smooth degeneration) of (Y, D) is a log smooth morphism*

$$\mathfrak{f} : \mathcal{Y} \longrightarrow S$$

such that:

(1) S is (the germ of) an affine curve with distinguished point $0 \in S$ and generic point η, and the log structure of S is the divisorial log structure $(S, 0)$;
(2) $\mathcal{Y}_\eta = \mathfrak{f}^{-1}(\eta) = Y \otimes k(\eta)$;
(3) The log structure of $\mathcal{Y}$ is the divisorial log structure $(\mathcal{Y}, \mathcal{D})$ for $\mathcal{D}$ a maximal boundary normal crossings divisor such that $\mathcal{D} = \mathcal{D}_{\mathrm{hor}} + \mathcal{D}_{\mathrm{vert}}$ where:
  (a) $\mathcal{D}_{\mathrm{hor}}$ is irreducible and $\mathfrak{f}\big|_{\mathcal{D}_{\mathrm{hor}}}$ is surjective;
  (b) $\mathcal{D}_{\mathrm{vert}} = \mathfrak{f}^{-1}(0) = D_1 + \cdots + D_l$.
  In particular, $\log\dim(\mathcal{Y}, \mathcal{D}) = N + 1$ is maximal.

Definition LSD Fano. *A maximal LSD Fano variety is a triple $(Y, D, \mathfrak{f})$ consisting of:*

- *A smooth complex projective Fano variety Y.*
- *A* smooth *anticanonical divisor $D \in |-K_Y|$.*
- *A maximal LSD $\mathfrak{f} : \mathcal{Y} \to S$ of (Y, D).*

Remark 1. Given Y, it is an open question of how restrictive the existence of $(Y, D, \mathfrak{f})$ is. In what follows, we work with a maximal LSD Fano $(Y, D, \mathfrak{f})$.

Remark 2. Note that $\mathfrak{f}\big|_{\mathcal{D}_{\mathrm{hor}}} : \mathcal{D}_{\mathrm{hor}} \to S$ is a maximal log smooth degeneration of D. In particular, $\log\dim(\mathcal{D}_{\mathrm{hor}}, \mathcal{D}_{\mathrm{vert}} \cap \mathcal{D}_{\mathrm{hor}}) = N = \dim(D) + 1$ is maximal.

Given a maximal LSD Fano $(Y, D, \mathfrak{f})$, we apply the intrinsic mirror construction (relative case) [5] to $\mathfrak{f} : (\mathcal{Y}, \mathcal{D}) \to (S, 0)$. The outcome is the intrinsic mirror family over $\mathbb{C}[[\mathrm{NE(Y)}]]$, which we restrict to a family $\mathfrak{X}$ over $\mathbb{C}[[t]]$ via $z^\beta \mapsto t^{c_1\beta}$. Applying the analyticity result of [6], this is an analytic family over a disc in $\mathbb{C}$ centered around the origin. By abuse of notation, we also denote by $\mathfrak{X}_t$ the fiber over an element t of the disc.

Roughly speaking, $(\mathcal{Y}, \mathcal{D})$ determines a ring of theta functions, $\mathfrak{f}$ determines a grading of the theta functions and $\mathfrak{X}$ is given as the formal Proj construction of the ring of theta functions. Thus $\mathfrak{X}$ is projective over the ring of degree 0 theta functions $\mathrm{R}(Y, D)$, which is also obtained by applying the intrinsic mirror construction to (Y, D). As a vector space,

$$\mathrm{R}(Y, D) = \bigoplus_{i=0}^{\infty} \mathbb{C}[[t]]\vartheta_i,$$

with $\vartheta_0 = 1$ and the structural coefficients of the multiplication are given by punctured Gromov–Witten invariants [5, 1], in fact, in this setting, by 2-pointed log Gromov–Witten invariants using [7]. By composing with the natural isomorphism $\mathbb{C}[[t]][\vartheta_1] \to \mathrm{R}(Y, D)$, we obtain a commutative diagram

$$\begin{array}{ccc} & & \mathfrak{X} \\ & \swarrow & \downarrow{\scriptstyle \vartheta_1} \\ \mathrm{Spf}\,\mathrm{R}(Y, D) & \longrightarrow & \mathrm{Spf}\,\mathbb{C}[[t]][\vartheta_1]\,. \end{array}$$

The proper function ϑ_1 determines a Calabi–Yau $(N-1)$-fibration on $\mathfrak{X}$. We recall [2, Theorem 5.9]. There is an *asymptotic* open $\overline{\mathfrak{X}} \subset \mathfrak{X}$, a canonical proper function w on $\overline{\mathfrak{X}}$, and a commutative diagram

$$\begin{array}{ccccc} & & \mathfrak{X} & \hookleftarrow & \overline{\mathfrak{X}} \\ & \swarrow & \downarrow{\scriptstyle \vartheta_1} & & \downarrow{\scriptstyle w} \\ \mathrm{Spf}\,\mathrm{R}(Y, D) & \longrightarrow & \mathrm{Spf}\,\mathbb{C}[[t]][\vartheta_1] & \xleftarrow{\Phi} & \mathrm{Spf}\,\mathbb{C}[w^{\pm 1}][[t]] \end{array}$$

with

$$\Phi^{\#}(\vartheta_1) = w\left(1 + \sum_{p \geq 1} \sum_{\substack{\beta \in \mathrm{NE}(Y) \\ p = c_1\beta - 1}} p\, N^{\beta}_{p,1}(Y, D)\, \frac{t^{c_1\beta}}{w^{c_1\beta}}\right).$$

Here $N^{\beta}_{p,1}(Y, D)$ is the log Gromov–Witten invariant counting rational curves of class β that meet D in one *fixed* point of tangency $p = (\beta \cdot D) - 1$ and in one point of tangency 1. Following [2, Definition 5.8], we call Φ the *LG mirror map*, which was studied in [3].

Let $\overline{\mathfrak{X}}^1 := w^{-1}(1) \subset \overline{\mathfrak{X}}$. By [2, Corollary 5.13], the Calabi–Yau $(N-1)$-fibration

$$\overline{\mathfrak{X}}^1 \to \mathrm{Spf}\,\mathbb{C}[[t]]$$

is the *twisted* mirror family to D, twisted by a certain unidirectional wall structure. We denote by $\overline{\mathfrak{X}}^1_t \subset \overline{\mathfrak{X}}^1$ the ‘fiber’ over t with the same caveat as above, namely that by the analyticity result of [6], we alternately view t as a formal variable and as an element of a small analytic disc.

We recall the main result of [6] in this context. A tropical 1-cycle β_{trop} in the pseudo-manifold associated to $\mathfrak{f}\big|_{\mathcal{D}_{\mathrm{hor}}} : \mathcal{D}_{\mathrm{hor}} \to S$ determines a $(N-1)$-cycle $\beta_t \in \mathrm{H}_{N-1}(\overline{\mathfrak{X}}^1_t, \mathbb{Z})$. Then, by [6, Theorem 1.7],

$$\exp\left(\frac{1}{\left(2\pi\sqrt{-1}\right)^{N-2}} \int_{\beta_t} \Omega_{\overline{\mathfrak{X}}^1_t}\right) = \langle s(t), \beta_{\mathrm{trop}}\rangle\, t^{\langle c_1(\varphi), \beta_{\mathrm{trop}}\rangle},$$

where:

- The unidirectional wall structure paired with β_{trop} determines $\langle s(t), \beta_{\mathrm{trop}}\rangle \in \mathbb{C}[[t]]$, which is invertible under the multiplication of power series, i.e. has non-zero constant coefficient.
- The topological term $\langle c_1(\varphi), \beta_{\mathrm{trop}}\rangle \in \mathbb{Z}$ measures the monodromy of β_{trop}.

Theorem 1 (van Garrel–Ruddat–Siebert, [4])**.** *Assume there is a critical value* w_{crit} *of* ϑ_1 *with* $0 < w_{\mathrm{crit}} < 1$*. Then the cycle* $\beta_t \in \mathrm{H}_{N-1}(\overline{\mathfrak{X}}^1_t, \mathbb{Z})$ *extends to a relative cycle* $\beta_t(w_{\mathrm{crit}}, 1) \in \mathrm{H}_N(\mathfrak{X}_t, \overline{\mathfrak{X}}^1_t; \mathbb{Z})$*. Moreover, the period integral*

$$\frac{1}{\left(2\pi\sqrt{-1}\right)^{N-2}} \int_{\beta_t(w_{\mathrm{crit}},1)} \Omega_{\mathfrak{X}_t}$$

is the generating function of maximal tangency log Gromov–Witten invariants of (Y, D) *with insertion given by by a curve class* $\beta \in \mathrm{H}_2(D, \mathbb{Z})$ *whose tropicalization is* β_{trop}*.*

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