Iwasawa theory for branched $\mathbb{Z}_p$-towers of finite graphs

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Abstract. We initiate the study of Iwasawa theory for branched $\mathbb{Z}_p$-towers of finite connected graphs. These towers are more general than what have been studied so far, since the morphisms of graphs involved are branched covers, a particular kind of harmonic morphisms of graphs. We prove an analogue of Iwasawa's asymptotic class number formula for the p-part of the number of spanning trees in this setting. Moreover, we find an explicit generator for the characteristic ideal of the finitely generated torsion Iwasawa module governing the growth of the p-part of the number of spanning trees in such towers.

1. Introduction

Let p be a rational prime and let

$$K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots$$

be a tower of number fields for which K_n/K is Galois with Galois group isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. Iwasawa's famous asymptotic class number formula [9, 10] shows the existence of $\mu, \lambda, n_0 \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{Z}$ such that when $n \geq n_0$, one has

$$\operatorname{ord}_p\big(h(K_n)\big) = \mu p^n + \lambda n + \nu,$$

where $h(K_n)$ denotes the class number of K_n, and ord_p denotes the usual p-adic valuation on the field of rational numbers.

Motivated by the analogy between number theory and graph theory, one can study the variation of the p-part of the number of spanning trees as one goes up a $\mathbb{Z}_p$-tower of graphs

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots . \tag{1.1}$$

Such a $\mathbb{Z}_p$-tower consists of a sequence of covering maps $X_{n+1} \to X_n$ between finite connected graphs for which the composition $X_n \to X_{n-1} \to \cdots \to X_0 = X$ is Galois with Galois group isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. It turns out that in perfect analogy with Iwasawa's asymptotic class number formula, there exist $\mu, \lambda, n_0 \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{Z}$ such that

$$\operatorname{ord}_p\big(\kappa(X_n)\big) = \mu p^n + \lambda n + \nu, \tag{1.2}$$

provided $n \geq n_0$, where now $\kappa(X_n)$ is the number of spanning trees of X_n.

A proof of this result is given in increasing generality in [13, 14, 23]. The approach therein is based on Ihara zeta and L-functions and their special value at $u = 1$. The Ihara zeta function $Z_X(u)$ of a finite graph is the reciprocal of a polynomial. It is of the form $Z_X(u)^{-1} = (1 - u^2)^{-\chi(X)} h_X(u)$, where $\chi(X)$ denotes the Euler characteristic of X, and where $h_X(u)$ is some polynomial with integer coefficients given explicitly as the determinant of an operator via Ihara's determinant formula. A result of Hashimoto [8] gives

$$h_X'(1) = -2\chi(X)\kappa(X),$$

when X is a finite connected graph. Therefore, the special value at $u = 1$ of these functions encodes much information about the invariant $\kappa(X)$. These facts can be used to construct a power series $f(T) \in \mathbb{Z}_p[\![T]\!]$, and combined with the Artin formalism satisfied by the Ihara L-functions, one can get the Iwasawa invariants μ and λ from the power series $f(T)$ thereby obtaining (1.2).

On the other hand, another proof of (1.2) has been obtained by Gonet in [7]. Her approach is algebraic and makes use of the structure theorem for finitely generated Iwasawa modules. To every finite connected graph is associated a finite abelian group $\mathrm{Pic}^0(X)$ which is called the Picard group of degree zero of X (also known as the Jacobian, the sandpile, or still the critical group of X). It follows from a classical theorem of Kirchhoff in graph theory (see [21, Chapter VI.4] for instance) that its cardinality is precisely the invariant $\kappa(X)$. Any cover of finite connected graphs $f : Y \to X$ induces a surjective group morphism $f_* : \mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X)$. Thus from a $\mathbb{Z}_p$-tower of graphs such as in (1.1) above, one obtains a compatible system of maps $\mathrm{Pic}^0(X_{n+1})[p^\infty] \to \mathrm{Pic}^0(X_n)[p^\infty]$, where we write $A[p^\infty]$ for the Sylow p-subgroup of a finite abelian group A. Since each $\mathrm{Pic}^0(X_n)[p^\infty]$ is a $\mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}]$-module, the inverse limit

$$\mathrm{Pic}^0_\Lambda = \varprojlim_{n \geq 0} \mathrm{Pic}^0(X_n)[p^\infty] \tag{1.3}$$

becomes an Iwasawa module, i.e., a compact Λ-module, where Λ is the Iwasawa algebra of $\mathbb{Z}_p$. After showing that this Iwasawa module is finitely generated over Λ, one can use the structure theorem for finitely generated Λ-modules up to pseudo-isomorphisms to obtain a module theoretical proof of (1.2).

The two approaches are intimately related to one another. Using the usual non-canonical isomorphism $\Lambda \xrightarrow{\cong} \mathbb{Z}_p[\![T]\!]$ given by $\gamma \mapsto 1 + T$, every Λ-module can be viewed as a $\mathbb{Z}_p[\![T]\!]$-module, and Kleine and Müller proved [12, Theorem 5.2 and Remark 5.3] that

$$\mathrm{char}_{\mathbb{Z}_p[\![T]\!]}(\mathrm{Pic}^0_\Lambda) \cdot (T) = \left(f(-T)\right), \tag{1.4}$$

where $\mathrm{char}_{\mathbb{Z}_p[\![T]\!]}(M)$ denotes the characteristic ideal of a finitely generated $\mathbb{Z}_p[\![T]\!]$-module M and $f(T)$ is the power series constructed in [13, 14, 23]. For the module theoretical approach to both (1.2) and (1.4), in addition to [7, 12], see also [11].

In the analogy between graphs and compact Riemann surfaces, covering maps of graphs correspond to unramified covers of compact Riemann surfaces. On the other hand,

branched (or ramified) covers of compact Riemann surfaces correspond to harmonic morphisms, a broader class of morphisms of graphs than covering maps. Harmonic morphisms were introduced in [22] and studied further by many authors including for instance in [1, 4]. The goal of this paper is to study more general $\mathbb{Z}_p$-towers of graphs than the ones that have been considered so far in the literature. We will consider towers

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots$$

such as (1.1) above, but where the maps $X_{n+1} \to X_n$ will be *branched* (or *ramified*) covers of finite graphs rather than usual (unramified) covering maps. Branched covers of graphs form a class of graph morphisms that lies in between (unramified) covering maps and harmonic morphisms. We explain in Section 4.3 how to construct such branched $\mathbb{Z}_p$-towers of finite connected graphs by modifying the notion of voltage assignment accordingly. In this paper, we use the module theoretical approach. There should be a connection with Ihara zeta and L-functions as well in this situation, as studied for instance in [2, 25], and we would like to look at this point of view in a separate work. The reason why we adopt the module theoretical approach here is that given a branched cover of finite connected graphs $Y \to X$, it induces a natural surjective group morphism $\mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X)$ just as in the unramified situation. Therefore, starting with a branched $\mathbb{Z}_p$-tower of graphs, we obtain again a compatible system of morphisms $\mathrm{Pic}^0(X_{n+1})[p^\infty] \to \mathrm{Pic}^0(X_n)[p^\infty]$, and we can consider the Iwasawa module

$$\mathrm{Pic}^0_\Lambda = \varprojlim_{n \geq 0} \mathrm{Pic}^0(X_n)[p^\infty].$$

The study of this Iwasawa module leads us to our first main result.

Theorem A (Theorem 5.6). *Let*

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots$$

be a branched $\mathbb{Z}_p$-tower of finite connected graphs arising from a voltage assignment as explained in Section 4.3. Then Pic^0_Λ is a finitely generated torsion Λ-module. Moreover, letting $\mu = \mu(\mathrm{Pic}^0_\Lambda)$ and $\lambda = \lambda(\mathrm{Pic}^0_\Lambda)$, there exist $n_0 \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{Z}$ such that

$$\mathrm{ord}_p\big(\kappa(X_n)\big) = \mu p^n + \lambda n + \nu,$$

when $n \geq n_0$.

We then move on to study the characteristic ideal of Pic^0_Λ, and we prove the following result which we now state in an imprecise form. See Theorems 5.9 and 6.1 for the precise formulation.

Theorem B (Theorems 5.9 and 6.1). *Let*

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots$$

be a branched $\mathbb{Z}_p$-tower of finite connected graphs arising from a voltage assignment as explained in Section 4.3. We define an operator Δ on a free $\mathbb{Z}_p[\![T]\!]$-module M of finite rank for which

$$\mathrm{char}_{\mathbb{Z}_p[\![T]\!]}(\mathrm{Pic}_\Lambda^0) \cdot (T) = \big(\det(\Delta)\big).$$

Moreover, $\det(\Delta) \in \mathbb{Z}_p[\![T]\!]$ can be explicitly calculated in terms of the voltage assignment.

The paper is organized as follows. In Section 2, we gather together some basic facts that will be used throughout the paper. We remind the reader about Serre's formalism for graphs in Section 2.1, about the Picard group of degree zero in Section 2.2, and about groups acting on graphs in Section 2.3. When a group acts on a graph, the theory becomes an equivariant one, and we explain this for the Picard group in Section 2.4. Ultimately, we are interested in the p-part of the number of spanning trees, and thus we tensor everything with $\mathbb{Z}_p$ over $\mathbb{Z}$ in Section 2.5. We remind the reader about the Iwasawa algebra and Iwasawa modules in Section 2.6. In Section 3, we introduce the notion of branched cover of graphs $Y \to X$, and we explain a useful induced map on the Picard groups $\mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X)$. In Section 4, we introduce the branched $\mathbb{Z}_p$-towers of graphs that we will be studying in this paper. We construct branched covers explicitly using voltage assignments in Section 4.1 and we explain how this construction behaves functorially in Section 4.2. This allows us to construct branched $\mathbb{Z}_p$-towers of finite graphs in Section 4.3. To every branched $\mathbb{Z}_p$-tower of finite graphs is associated an unramified $\mathbb{Z}_p$-tower. This is explained in Sections 4.4, 4.5, and 4.6. In Section 5, we study the Iwasawa module Pic_Λ^0 associated to a branched $\mathbb{Z}_p$-tower of finite connected graphs. We prove the analogue of Iwasawa's asymptotic class number formula in Section 5.1, and we find a generator for the characteristic ideal of Pic_Λ^0 in Section 5.2. This allows us to end this paper with a few numerical examples in Section 6.

2. Preliminaries

2.1. Graph theory

Throughout this paper, we use Serre's formalism for graphs (see [18, 19]). Thus, a graph X consists of a vertex set V_X and a set of directed edges $\mathbf{E}_X$ equipped with an incidence function $\mathrm{inc} : \mathbf{E}_X \to V_X \times V_X$ given by $e \mapsto \mathrm{inc}(e) = (o(e), t(e))$ and an inversion function $\mathrm{inv} : \mathbf{E}_X \to \mathbf{E}_X$ also denoted by $e \mapsto \bar{e}$ satisfying the following conditions:

(1) $\bar{e} \neq e$,

(2) $\bar{\bar{e}} = e$,

(3) $o(\bar{e}) = t(e)$ and $t(\bar{e}) = o(e)$,

for all $e \in \mathbf{E}_X$. The vertex $o(e)$ is called the origin and the vertex $t(e)$ the terminus of the directed edge e. Given $v \in V_X$, we let

$$\mathbf{E}_{X,v} = \big\{ e \in \mathbf{E}_X : o(e) = v \big\}.$$

We will also write $\mathbf{E}^o_{X,v}$ instead of $\mathbf{E}_{X,v}$ at times. Similarly to $\mathbf{E}^o_{X,v} = \mathbf{E}_{X,v}$, one can define

$$\mathbf{E}^t_{X,v} = \{e \in \mathbf{E}_X : t(e) = v\},$$

and the inversion map induces bijections

$$\text{inv} : \mathbf{E}^o_{X,v} \to \mathbf{E}^t_{X,v}$$

for all $v \in V_X$. The set of undirected edges is obtained by identifying e with $\bar{e}$ and will be denoted by E_X. An orientation S for X consists of the image of a section for the natural map $\mathbf{E}_X \to E_X$. In other words, an orientation is obtained by choosing a direction for each undirected edge. A graph is finite if both V_X and $\mathbf{E}_X$ are finite sets, and locally finite if $\mathbf{E}_{X,v}$ is finite for all $v \in V_X$. In this case, we let

$$\text{val}_X(v) = |\mathbf{E}_{X,v}| = |\mathbf{E}^t_{X,v}|,$$

and this quantity is called the valency (or degree) of the vertex v. A path in X is a sequence of directed edges $c = e_1 \cdot \ldots \cdot e_n$ for which one has $t(e_i) = o(e_{i+1})$ for all $i = 1, \ldots, n-1$. The vertices $o(e_1)$ and $t(e_n)$ are called the origin and the terminus of the path c and will be denoted by $o(c)$ and $t(c)$, respectively. The graph X is called connected if given any two distinct vertices v_1 and v_2 of X, there exists a path c in X satisfying $o(c) = v_1$ and $t(c) = v_2$.

Definition 2.1. Let X and Y be graphs. A morphism $f : X \to Y$ of graphs consists of two functions $f_V : V_X \to V_Y$ and $f_E : \mathbf{E}_X \to \mathbf{E}_Y$ satisfying

(1) $f_V(o(e)) = o(f_E(e))$,

(2) $f_V(t(e)) = t(f_E(e))$,

(3) $\overline{f_E(e)} = f_E(\bar{e})$,

for all $e \in \mathbf{E}_X$.

We will usually denote both f_V and f_E simply by f. Note that given a morphism of graphs $f : X \to Y$ and any vertex $v \in V_X$, the restriction $f|_{\mathbf{E}_{X,v}}$ induces a function

$$f|_{\mathbf{E}_{X,v}} : \mathbf{E}_{X,v} \to \mathbf{E}_{Y,f(v)}.$$

Remark 2.2. There are more general notions of morphisms of graphs in the literature than Definition 2.1 for which one allows edges to be mapped to vertices. See for instance [4, Definition 2.1] in the case where the graphs are assumed to have no loops. In this paper, we restrict ourselves to Definition 2.1.

2.2. The Picard group of degree zero

Let $X = (V_X, \mathbf{E}_X)$ be a graph. We define $\text{Div}(X)$ to be the free abelian group on V_X, and we have a natural surjective group morphism $s : \text{Div}(X) \twoheadrightarrow \mathbb{Z}$ defined on vertices via $v \mapsto s(v) = 1$. We let $\text{Div}^0(X)$ be the kernel of s so that we have a short exact sequence

of abelian groups

$$0 \to \mathrm{Div}^0(X) \to \mathrm{Div}(X) \xrightarrow{s} \mathbb{Z} \to 0. \tag{2.1}$$

If X is locally finite, then we define a few operators on $\mathrm{Div}(X)$ as follows. The valency (or degree) operator $\mathcal{D}_X$ is defined on vertices via

$$v \mapsto \mathcal{D}_X(v) = \mathrm{val}_X(v)v,$$

whereas the adjacency operator $\mathcal{A}_X$ is defined on vertices via

$$v \mapsto \mathcal{A}_X(v) = \sum_{e \in \mathbf{E}_{X,v}} t(e).$$

The Laplacian operator $\mathcal{L}_X : \mathrm{Div}(X) \to \mathrm{Div}(X)$ is then defined to be

$$\mathcal{L}_X = \mathcal{D}_X - \mathcal{A}_X.$$

Moreover, we define $\mathrm{Pr}(X) = \mathrm{Im}(\mathcal{L}_X)$. It is simple to check from the definitions that $\mathrm{Pr}(X)$ is a subgroup of $\mathrm{Div}^0(X)$. In what follows, we let

$$\mathrm{Pic}(X) = \mathrm{Div}(X)/\mathrm{Pr}(X) \quad \text{and} \quad \mathrm{Pic}^0(X) = \mathrm{Div}^0(X)/\mathrm{Pr}(X).$$

We shall refer to $\mathrm{Pic}(X)$ as the Picard group of X, and to $\mathrm{Pic}^0(X)$ as the Picard group of degree zero of X.

Theorem 2.3. *If X is a finite connected graph, then one has a short exact sequence of abelian groups*

$$0 \to \mathbb{Z} \sum_{v \in V_X} v \to \mathrm{Div}(X) \xrightarrow{\mathcal{L}_X} \mathrm{Pr}(X) \to 0, \tag{2.2}$$

so that $\mathrm{rank}_{\mathbb{Z}}(\mathrm{Pr}(X)) = \mathrm{rank}_{\mathbb{Z}}(\mathrm{Div}^0(X)) = |V_X| - 1$ *from which it follows that* $\mathrm{Pic}^0(X)$ *is a finite abelian group. Moreover* $|\mathrm{Pic}^0(X)| = \kappa(X)$, *where* $\kappa(X)$ *is the number of spanning trees of* X.

Proof. On one hand, we have

$$\mathcal{L}_X \left(\sum_{v \in V_X} v \right) = \sum_{v \in V_X} \mathrm{val}_X(v)v - \sum_{v \in V_X} \sum_{e \in \mathbf{E}_{X,v}} t(e)$$

$$= \sum_{v \in V_X} \mathrm{val}_X(v)v - \sum_{e \in \mathbf{E}_X} t(e)$$

$$= \sum_{v \in V_X} \mathrm{val}_X(v)v - \sum_{v \in V_X} \sum_{e \in \mathbf{E}_{X,v}^t} v$$

$$= 0.$$

On the other hand, if $D = \sum_{v \in V_X} m_v v \in \ker(\mathcal{L}_X)$, then a simple calculation shows that

$$m_v = \frac{1}{\mathrm{val}_X(v)} \sum_{e \in \mathbf{E}_{X,v}^t} m_{o(e)}$$

for all $v \in V_X$. Choosing v so that m_v is maximal implies that $m_{o(e)} = m_v$ for all $e \in \mathbf{E}^t_{X,v}$, and the connectedness assumption allows us to deduce that

$$D \in \mathbb{Z} \sum_{v \in V_X} v.$$

This shows that (2.2) is a short exact sequence. The fact that $|\mathrm{Pic}^0(X)| = \kappa(X)$ follows from a classical theorem of Kirchhoff in graph theory. See any book on graph theory such as [5]. ∎

2.3. Groups acting on graphs

The group of automorphisms of a graph Y will be denoted as usual by $\mathrm{Aut}(Y)$. Throughout this paper, we will often have groups acting on graphs. The precise definition is as follows.

Definition 2.4. Let G be a group and Y a graph. We say that G acts on Y if we are given a group morphism $G \to \mathrm{Aut}(Y)$. The group G is said to act without inversion if for all $e \in \mathbf{E}_Y$ and all $\sigma \in G$, one has $\sigma \cdot e \neq \bar{e}$.

If G acts without inversion on a graph Y, then one obtains a graph $G \backslash Y$ by letting the vertices to be $G \backslash V_Y$ and the directed edges to be $G \backslash \mathbf{E}_Y$. The incidence map is given by $o(G \cdot e) = G \cdot o(e), t(G \cdot e) = G \cdot t(e)$, and the inversion map by $\overline{G \cdot e} = G \cdot \bar{e}$. We leave it to the reader to check that $G \backslash Y$ is a graph, and that the natural map $Y \to G \backslash Y$ given for $w \in V_Y$ and $e \in \mathbf{E}_Y$ by

$$w \mapsto G \cdot w \quad \text{and} \quad e \mapsto G \cdot e$$

is a morphism of graphs.

2.4. The Picard group of degree zero as a $\mathbb{Z}[G]$-module

Let Y be a graph and assume that a group G acts on Y without inversion. Then, $\mathrm{Div}(Y)$ is acted upon by G as well, and the short exact sequence

$$0 \to \mathrm{Div}^0(Y) \to \mathrm{Div}(Y) \xrightarrow{s} \mathbb{Z} \to 0$$

becomes one of $\mathbb{Z}[G]$-modules, where G acts trivially on $\mathbb{Z}$. Note that every $\sigma \in G$ induces a bijection

$$\mathbf{E}_{Y,w} \to \mathbf{E}_{Y,\sigma(w)} \tag{2.3}$$

for all $w \in V_Y$, so that if Y is locally finite, one has

$$\mathrm{val}_Y(\sigma \cdot w) = \mathrm{val}_Y(w), \tag{2.4}$$

for all $w \in V_Y$ and all $\sigma \in G$.

Proposition 2.5. *Assuming that Y is locally finite, the degree, the adjacency, and the Laplacian operators on $\mathrm{Div}(Y)$ are all morphisms of $\mathbb{Z}[G]$-modules.*

Proof. The degree operator $\mathcal{D}_Y$ is G-equivariant because of (2.4) above. The adjacency operator is G-equivariant, since

$$\sum_{e \in \mathbf{E}_{Y,\sigma \cdot w}} t(e) = \sigma \cdot \sum_{e \in \mathbf{E}_{Y,w}} t(e)$$

by (2.3) above. It then follows that the Laplacian operator $\mathcal{L}_Y$ is G-equivariant as well, since $\mathcal{L}_Y = \mathcal{D}_Y - \mathcal{A}_Y$. ∎

As a direct consequence, we obtain the following result.

Corollary 2.6. *Let Y be a locally finite graph, and let G be a group acting on Y without inversion. Then, both $\mathrm{Pic}(Y)$ and $\mathrm{Pic}^0(Y)$ are $\mathbb{Z}[G]$-modules.*

2.5. Tensoring with $\mathbb{Z}_p$ over $\mathbb{Z}$

Later on, we will be working over $\mathbb{Z}_p$, rather than over $\mathbb{Z}$, where p is a fixed rational prime. Since $\mathbb{Z}_p$ is flat over $\mathbb{Z}$, everything we did so far works equally well over $\mathbb{Z}_p$. If M is a $\mathbb{Z}$-module, then we let

$$M_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} M.$$

For instance, if X is a graph, then we have $\mathrm{Div}_p(X)$, $\mathrm{Div}_p^0(X)$, $\mathrm{Pr}_p(X)$, $\mathrm{Pic}_p(X)$, and $\mathrm{Pic}_p^0(X)$ whenever these groups are defined. After tensoring the short exact sequence (2.1) with $\mathbb{Z}_p$ over $\mathbb{Z}$, one gets

$$0 \to \mathrm{Div}_p^0(X) \to \mathrm{Div}_p(X) \xrightarrow{s_p} \mathbb{Z}_p \to 0. \tag{2.5}$$

If X is locally finite, then the two short exact sequences

$$0 \to \mathrm{Pr}(X) \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0,$$
$$0 \to \mathrm{Pr}(X) \to \mathrm{Div}^0(X) \to \mathrm{Pic}^0(X) \to 0$$

become after tensoring with $\mathbb{Z}_p$ over $\mathbb{Z}$ the short exact sequences

$$0 \to \mathrm{Pr}_p(X) \to \mathrm{Div}_p(X) \to \mathrm{Pic}_p(X) \to 0,$$
$$0 \to \mathrm{Pr}_p(X) \to \mathrm{Div}_p^0(X) \to \mathrm{Pic}_p^0(X) \to 0$$

of $\mathbb{Z}_p$-modules. If furthermore X is finite and connected, then the short exact sequence (2.2) becomes

$$0 \to \mathbb{Z}_p \sum_{v \in V_X} v \to \mathrm{Div}_p(X) \xrightarrow{\mathcal{L}_p} \mathrm{Pr}_p(X) \to 0.$$

In this situation, note that we have

$$\mathrm{Pic}_p^0(X) \simeq \mathrm{Div}_p^0(X)/\mathrm{Pr}_p(X) \simeq \mathrm{Pic}^0(X)[p^\infty],$$

where $\mathrm{Pic}^0(X)[p^\infty]$ denotes the Sylow p-subgroup of the finite abelian group $\mathrm{Pic}^0(X)$.

In particular, we have

$$\left|\mathrm{Pic}_p^0(X)\right| = \kappa_p(X),$$

where $\kappa_p(X) = p^{\mathrm{ord}_p(\kappa(X))}$. In other words, tensoring with $\mathbb{Z}_p$ over $\mathbb{Z}$ allows us to focus on the p-part of the invariant $\kappa(X)$. If moreover the graph X is acted upon by a group G, then all these $\mathbb{Z}_p$-modules become modules over the group ring $\mathbb{Z}_p[G]$.

2.6. The Iwasawa algebra and Iwasawa modules

In this section, we gather some well-known results on the Iwasawa algebra Λ of $\mathbb{Z}_p$ and finitely generated modules over Λ that we will use throughout. Our main references for this section are [3, 16, 24].

We prefer to work with a multiplicative notation, so we let Γ be a multiplicative group that is topologically isomorphic to $\mathbb{Z}_p$. We fix once and for all a topological generator γ for Γ. For each integer $n \geq 0$, we set $\Gamma_n = \Gamma/\Gamma^{p^n}$ and we let also $\gamma_n = \gamma^{p^n}$. The Iwasawa algebra Λ is the profinite completion of the group ring $R = \mathbb{Z}_p[\Gamma]$, in other words

$$\Lambda = \mathbb{Z}_p[\![\Gamma]\!] = \varprojlim_{n \geq 0} \mathbb{Z}_p[\Gamma_n],$$

where the compatible maps $\mathbb{Z}_p[\Gamma_{n+1}] \twoheadrightarrow \mathbb{Z}_p[\Gamma_n]$ are the natural projection maps. It is a unique factorization domain that is a Noetherian ring of Krull dimension two. Moreover it is a local ring with unique maximal ideal given by $\mathfrak{m} = (p, \gamma - 1)$. The ring Λ is a topological ring when endowed with the $\mathfrak{m}$-adic topology which is compact, and hence also complete.

The projection maps $\phi_n : \mathbb{Z}_p[\Gamma] \twoheadrightarrow \mathbb{Z}_p[\Gamma_n]$ induce a unital ring morphism $\mathbb{Z}_p[\Gamma] \to \Lambda$ which is injective, since $\bigcap_{n \geq 0} \ker(\phi_n) = 0$. Thus, we have a natural embedding of unital commutative rings $R \hookrightarrow \Lambda$. Throughout, we let

$$\omega_n = \gamma_n - 1 = \gamma^{p^n} - 1 \in \mathfrak{m}^n \subseteq R \subseteq \Lambda.$$

The natural projection maps $\Lambda \twoheadrightarrow \mathbb{Z}_p[\Gamma_n]$ and $R \twoheadrightarrow \mathbb{Z}_p[\Gamma_n]$ induce isomorphisms

$$\Lambda/\omega_n\Lambda \xrightarrow{\sim} \mathbb{Z}_p[\Gamma_n] \quad \text{and} \quad R/R \cap \omega_n\Lambda \xrightarrow{\sim} \mathbb{Z}_p[\Gamma_n] \tag{2.6}$$

of unital commutative rings. In addition to the elements ω_n, we will need the elements

$$\omega_{n,k} = \frac{\omega_n}{\omega_k} = 1 + \gamma_k + \gamma_k^2 + \cdots + \gamma_k^{p^{n-k}-1} \in R,$$

whenever $k, n \in \mathbb{Z}_{\geq 0}$ are such that $k \leq n$.

Any (Hausdorff) compact topological Λ-module is called an Iwasawa module. A finitely generated Λ-module M is in fact a topological Λ-module by taking for a basis of neighborhoods around $m \in M$, the sets $m + \mathfrak{m}^n \cdot M$, where $n \geq 0$, and any such module is an Iwasawa module, i.e., is compact. With this topology, any morphism of finitely generated Λ-modules is automatically continuous so when dealing with finitely generated

Λ-modules, the theory becomes purely algebraic. We can view $\mathbb{Z}_p$ as a Λ-module or an R-module with trivial action. As such, we have in particular

$$\mathbb{Z}_p \simeq \Lambda/\omega_0\Lambda \quad \text{and} \quad \mathbb{Z}_p \simeq R/R \cap \omega_0\Lambda. \tag{2.7}$$

A morphism of finitely generated Λ-modules $f : M \to N$ is called a pseudo-isomorphism if it has finite kernel and cokernel. This defines an equivalence relation on the collection of finitely generated torsion Λ-modules, and we will write $M \sim N$ if M and N are pseudo-isomorphic finitely generated torsion Λ-modules. To every finitely generated torsion Λ-module M is associated its Iwasawa invariants $\mu(M)$ and $\lambda(M)$ which are non-negative integers. Two pseudo-isomorphic finitely generated torsion Λ-modules have the same Iwasawa invariants.

Theorem 2.7. *Let M be a finitely generated Λ-module and assume that there exists $k \in \mathbb{Z}_{\geq 0}$ such that $M/\omega_{n,k}M$ is finite for all $n \geq k$. Then M is a finitely generated torsion Λ-module. Moreover, there exist $n_0 \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{Z}$ such that*

$$\operatorname{ord}_p\big(|M/\omega_{n,k}M|\big) = \mu(M)p^n + \lambda(M)n + \nu,$$

when $n \geq n_0$.

Proof. See [24, Theorem 13.19 and Lemma 13.21]. ∎

If M is a finitely generated torsion Λ-module, then we shall denote its characteristic ideal by $\operatorname{char}_\Lambda(M)$. The characteristic ideal is also an invariant that depends only on the pseudo-isomorphism class of the finitely generated torsion Λ-module. If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of finitely generated torsion Λ-modules, then one has

$$\operatorname{char}_\Lambda(M_2) = \operatorname{char}_\Lambda(M_1) \cdot \operatorname{char}_\Lambda(M_3). \tag{2.8}$$

The characteristic ideal can sometimes be calculated as follows.

Theorem 2.8. *Let F be a free Λ-module of finite rank and let M be a finitely generated torsion Iwasawa module. If we have a short exact sequence*

$$0 \to F \xrightarrow{f} F \to M \to 0$$

of Λ-modules, then

$$\operatorname{char}_\Lambda(M) = \big(\det(f)\big).$$

Proof. See [3, Chapitre VII, Section 4, Corollaire to Proposition 14]. ∎

There is a non-canonical isomorphism

$$\Lambda \xrightarrow{\simeq} \mathbb{Z}_p[\![T]\!] \tag{2.9}$$

given by $\gamma \mapsto 1 + T$. Therefore, any Λ-module can be viewed as a $\mathbb{Z}_p[\![T]\!]$-module, and finitely generated Λ-modules are in particular finitely generated $\mathbb{Z}_p[\![T]\!]$-modules. Every finitely generated $\mathbb{Z}_p[\![T]\!]$-module M is pseudo-isomorphic to a module of the form

$$\mathbb{Z}_p[\![T]\!]^r \oplus \bigoplus_{i=1}^{s} \mathbb{Z}_p[\![T]\!]/(p^{m_i}) \oplus \bigoplus_{j=1}^{t} \mathbb{Z}_p[\![T]\!]/(f_j(T)^{n_j}), \tag{2.10}$$

where $r, s, t, m_i, n_j \in \mathbb{Z}_{\geq 0}$ and the $f_j(T) \in \mathbb{Z}_p[T]$ are distinguished and irreducible polynomials. Recall that a polynomial $f(T) \in \mathbb{Z}_p[T]$ is called distinguished if it is monic and p divides every coefficient except the leading one. For instance, the elements

$$\omega_n(T) = (1 + T)^{p^n} - 1 \in \mathbb{Z}[T] \subseteq \mathbb{Z}_p[\![T]\!]$$

obtained from the ω_n via the isomorphism (2.9) are distinguished. A finitely generated $\mathbb{Z}_p[\![T]\!]$-module is torsion precisely when $r = 0$ in (2.10) above. If M is a finitely generated torsion $\mathbb{Z}_p[\![T]\!]$-module pseudo-isomorphic to a module of the form

$$\bigoplus_{i=1}^{s} \mathbb{Z}_p[\![T]\!]/(p^{m_i}) \oplus \bigoplus_{j=1}^{t} \mathbb{Z}_p[\![T]\!]/(f_j(T)^{n_j}),$$

then its characteristic ideal, as an ideal of $\mathbb{Z}_p[\![T]\!]$, is given by

$$\mathrm{char}_{\mathbb{Z}_p[\![T]\!]}(M) = (f(T)),$$

where

$$f(T) = \prod_{i=1}^{s} p^{m_i} \cdot \prod_{j=1}^{t} f_j(T)^{n_j} \in \mathbb{Z}_p[T]. \tag{2.11}$$

The Iwasawa invariants of M are given by

$$\mu(M) = \mu(f(T)) \quad \text{and} \quad \lambda(M) = \lambda(f(T)),$$

where we recall that given any

$$g(T) = a_0 + a_1 T + a_2 T^2 + \cdots \in \mathbb{Z}_p[\![T]\!],$$

one sets

$$\mu(g(T)) = \min\{\mathrm{ord}_p(a_i) : i \geq 0\},$$

and

$$\lambda(g(T)) = \min\{i \geq 0 : \mathrm{ord}_p(a_i) = \mu(g(T))\}.$$

For the polynomial $f(T)$ of (2.11) above, one has

$$\mu(f(T)) = \sum_{i=1}^{s} m_i,$$

$$\lambda(f(T)) = \deg(f(T)),$$

since the $f_j(T)$ are distinguished.

3. Branched covers of graphs

3.1. Branched covers

The following definition is taken from [19, p. 69].

Definition 3.1. Let X and Y be graphs and let $f : Y \to X$ be a morphism of graphs. Then f is called a branched (or ramified) cover if the following two conditions are satisfied:

(1) Both functions $f : V_Y \to V_X$ and $f : \mathbf{E}_Y \to \mathbf{E}_X$ are surjective,

(2) For all $w \in V_Y$, the cardinality m_w, also denoted by $m_f(w)$, of $(f|_{\mathbf{E}_{Y,w}})^{-1}(e)$ is independent of $e \in \mathbf{E}_{X,f(w)}$. The cardinality m_w is called the ramification index of the vertex w. If m_w is finite, then the function

$$f|_{\mathbf{E}_{Y,w}} : \mathbf{E}_{Y,w} \to \mathbf{E}_{X,f(w)}$$

is m_w-to-1.

If $f : Y \to X$ is a branched cover for which $m_w = 1$ for all $w \in V_Y$, then we will refer to f as an unramified cover or simply a cover. In this case, the functions $f|_{\mathbf{E}_{Y,w}}$ are bijections for all $w \in V_Y$. In the situation where both X and Y are locally finite and m_w is finite, we have

$$\mathrm{val}_Y(w) = m_w \cdot \mathrm{val}_X\big(f(w)\big). \tag{3.1}$$

If X, Y, Z are graphs and $f : Z \to Y$, $g : Y \to X$ are both branched covers of graphs, then so is the composition $g \circ f : Z \to X$. If w is a vertex of Z, then one has

$$m_{g \circ f}(w) = m_f(w) \cdot m_g\big(f(w)\big),$$

when all these quantities are finite. As pointed out in [19, p. 69], a branched cover in the sense of Definition 3.1 is an example of a harmonic morphism as defined for instance in [1, 22] under suitable assumptions on the graphs involved such as simplicity or looplessness.

Remark 3.2. If $f : Y \to X$ is a branched cover and Y is connected, then so is X. Indeed, if $v_1, v_2 \in V_X$, then let $w_1, w_2 \in V_Y$ be such that $f(w_i) = v_i$ for $i = 1, 2$. Since Y is connected, there exists a path c in Y going from w_1 to w_2. Then, the path $f(c)$ is a path in X going from v_1 to v_2 showing the claim.

Following [19, p. 69], given a branched cover $f : Y \to X$ and $v \in V_X$, let

$$d(v) = \sum_{w \in f^{-1}(v)} m_w,$$

provided this sum is finite. Since we have commutative diagrams

$$
\begin{array}{ccc}
\mathbf{E}^o_{Y,w} & \xrightarrow{f|_{\mathbf{E}^o_{Y,w}}} & \mathbf{E}^o_{X,f(w)} \\
{\scriptstyle \mathrm{inv}} \downarrow & & \downarrow {\scriptstyle \mathrm{inv}} \\
\mathbf{E}^t_{Y,w} & \xrightarrow{f|_{\mathbf{E}^t_{Y,w}}} & \mathbf{E}^t_{X,f(w)},
\end{array}
$$

it follows that the maps

$$f|_{\mathbf{E}_{Y,w}^t} : \mathbf{E}_{Y,w}^t \to \mathbf{E}_{X,f(w)}^t$$

are also m_w-to-1. Therefore, if $e \in \mathbf{E}_X$, then

$$d\big(o(e)\big) = \big|f^{-1}(e)\big| = d\big(t(e)\big),$$

and if X is connected, $d(v)$ is independent of the vertex $v \in V_X$. This nonnegative integer is called the degree of the branched cover f and is denoted by $[Y : X]$. One has

$$[Y : X] = \sum_{w \in f^{-1}(v)} m_w, \tag{3.2}$$

for all $v \in V_X$, when X is assumed to be connected.

Whenever a group G acts on a graph Y, one gets two G-sets, as both V_Y and $\mathbf{E}_Y$ are acted upon by G as well. Note that if G acts on a graph Y in a way that G acts freely on V_Y, then it necessarily acts freely on $\mathbf{E}_Y$ as well. Indeed, if $\sigma \cdot e = e$ for some $e \in \mathbf{E}_Y$, then $o(e) = o(\sigma \cdot e) = \sigma \cdot o(e)$ from which we deduce that σ is the neutral element of G provided G acts freely on V_Y.

Proposition 3.3. *Let Y be a graph and let G be a group acting without inversion on Y.*

(1) *If G acts freely on $\mathbf{E}_Y$, then the natural morphism of graphs $f : Y \to G \backslash Y$ is a branched cover. Moreover, for all $w \in V_Y$, one has $m_w = |S_w|$, where $S_w = \mathrm{Stab}_G(w)$.*

(2) *If G acts freely on V_Y, then the natural morphism of graphs $f : Y \to G \backslash Y$ is an unramified cover.*

Proof. It is clear that f is surjective both on vertices and directed edges. For simplicity let $X = G \backslash Y$ and let $G \cdot e \in \mathbf{E}_{X,f(w)}$, where $w \in V_Y$ is an arbitrary vertex of Y. Then $o(G \cdot e) = G \cdot w$, and there exists $\sigma \in G$ such that $o(\sigma \cdot e) = w$. A simple calculation shows that

$$(f|_{\mathbf{E}_{Y,w}})^{-1}(G \cdot e) = \{\tau \cdot \sigma \cdot e : \tau \in S_w\}.$$

Since G acts freely on $\mathbf{E}_Y$, all elements $\tau \cdot \sigma \cdot e$ are distinct as τ runs over S_w. This ends the proofs of both claims. ∎

3.2. The Picard group of degree zero and branched covers

Suppose now that we have a branched cover of graphs $f : Y \to X$, then we obtain a natural surjective group morphism

$$f_* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$$

given by $w \mapsto f_*(w) = f(w)$. If moreover m_w is finite for all $w \in V_Y$, then we have another natural group morphism

$$f_r : \mathrm{Div}(Y) \to \mathrm{Div}(X)$$

given by $w \mapsto f_r(w) = m_w \cdot f(w)$.

Proposition 3.4. *Assuming that X and Y are locally finite, and with the notation as above, we have a commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Div}(Y) & \xrightarrow{\;\mathscr{L}_Y\;} & \mathrm{Div}(Y) \\
{\scriptstyle f_r}\downarrow & & \downarrow{\scriptstyle f_*} \\
\mathrm{Div}(X) & \xrightarrow{\;\mathscr{L}_X\;} & \mathrm{Div}(X).
\end{array}
$$

Proof. The commutativity of the diagram follows from the commutativity of the two diagrams

$$
\begin{array}{ccc}
\mathrm{Div}(Y) & \xrightarrow{\;\mathscr{D}_Y\;} & \mathrm{Div}(Y) \\
{\scriptstyle f_r}\downarrow & & \downarrow{\scriptstyle f_*} \\
\mathrm{Div}(X) & \xrightarrow{\;\mathscr{D}_X\;} & \mathrm{Div}(X),
\end{array}
\quad\text{and}\quad
\begin{array}{ccc}
\mathrm{Div}(Y) & \xrightarrow{\;\mathscr{A}_Y\;} & \mathrm{Div}(Y) \\
{\scriptstyle f_r}\downarrow & & \downarrow{\scriptstyle f_*} \\
\mathrm{Div}(X) & \xrightarrow{\;\mathscr{A}_X\;} & \mathrm{Div}(X).
\end{array}
$$

The first diagram commutes because of (3.1), and the second diagram commutes since $f|_{\mathbf{E}_{Y,w}} : \mathbf{E}_{Y,w} \to \mathbf{E}_{X,f(w)}$ is m_w-to-1. ∎

Since $f_* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ is surjective, it follows from Proposition 3.4 that we have two surjective group morphisms

$$f_* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X) \quad\text{and}\quad f_* : \mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X). \tag{3.3}$$

The surjectivity of the second group morphism implies the divisibility $\kappa(X) \mid \kappa(Y)$ whenever $f : Y \to X$ is a branched cover of finite connected graphs.

Remark 3.5. The fact that $\kappa(X) \mid \kappa(Y)$ is known to hold true more generally for harmonic morphisms (see [1, Section 4]).

4. Constructions of branched covers via voltage assignments

4.1. The basic construction

Let X be a graph, G a group (for which we use the multiplicative notation), and $\alpha : \mathbf{E}_X \to G$ a function satisfying

$$\alpha(\bar{e}) = \alpha(e)^{-1}. \tag{4.1}$$

Such a function $\alpha : \mathbf{E}_X \to G$ satisfying (4.1) above is often called a voltage assignment on X with values in the group G. Note that if S is an orientation for X, then it suffices to specify α on S and set $\alpha(\bar{s}) = \alpha(s)^{-1}$ in order to get a function $\alpha : \mathbf{E}_X \to G$ satisfying (4.1) above. Let also

$$\mathcal{I} = \big\{(v, I_v) \mid v \in V_X \text{ and } I_v \leq G\big\},$$

be a collection of subgroups of G indexed by the vertices of X. We define a graph $X(G, \mathcal{I}, \alpha)$ as follows. The vertex set is the disjoint union

$$V = \bigsqcup_{v \in V_X} \{v\} \times G/I_v,$$

and the collection of directed edges is given by

$$\mathbf{E} = \mathbf{E}_X \times G.$$

The directed edge (e, σ) connects the vertex $(o(e), \sigma I_{o(e)})$ to the vertex $(t(e), \sigma\alpha(e)I_{t(e)})$. Furthermore, one lets

$$\overline{(e, \sigma)} = (\bar{e}, \sigma\alpha(e)).$$

We leave it to the reader to check that $X(G, \mathcal{I}, \alpha)$ is a graph.

The group G acts naturally on the graph $X(G, \mathcal{I}, \alpha)$ without inversion. Indeed, for simplicity, let $Y = X(G, \mathcal{I}, \alpha)$. If $\tau \in G$, let $\phi_\tau : Y \to Y$ be defined via

$$\phi_\tau(v, \sigma I_v) = (v, \tau\sigma I_v),$$

$$\phi_\tau(e, \sigma) = (e, \tau\sigma).$$

We leave it to the reader to check that ϕ_τ is an automorphism of graphs. It is then simple to check that the map $G \to \operatorname{Aut}(Y)$ defined via $\tau \mapsto \phi_\tau$ is a homomorphism of groups which gives an action of G on Y without inversion. It clearly acts freely on $\mathbf{E}_Y$, but not necessarily freely on V_Y. In fact, if $w = (v, \sigma I_v) \in V_Y$, then

$$S_w = \operatorname{Stab}_G(w) = \sigma I_v \sigma^{-1}.$$

We define the map

$$f : X(G, \mathcal{I}, \alpha) \to X$$

via $f(v, \sigma \cdot I_v) = v$ and $f(e, \sigma) = e$. It is simple to verify that f is a morphism of graphs. In fact, it is a branched cover, since for $w = (v, \sigma I_v) \in V_Y$, given $e \in \mathbf{E}_{X,v}$, one has

$$(f|_{\mathbf{E}_{Y,w}})^{-1}(e) = \{(e, \rho \cdot \sigma) : \rho \in S_w = \sigma I_v \sigma^{-1}\}.$$

Thus the cardinality of $(f|_{\mathbf{E}_{Y,w}})^{-1}(e)$ is $|I_v|$ and is independent of $e \in \mathbf{E}_{X,v}$. Summarizing the previous discussion, we have the following proposition.

Proposition 4.1. *Let X be a graph and $\alpha : \mathbf{E}_X \to G$ a function satisfying* (4.1). *Let also $\mathcal{I} = \{(v, I_v) : I_v \leq G\}$ be a collection of subgroups of G indexed by V_X and consider the graph $Y = X(G, \mathcal{I}, \alpha)$. Then, the natural morphism of graphs*

$$f : Y \to X$$

is a branched cover of graphs. The group G acts naturally on Y without inversion and freely on $\mathbf{E}_Y$. Moreover, for each vertex $w \in V_Y$, the ramification index $m_f(w)$ depends only on $v = f(w) \in V_X$ and is equal to $|I_v|$. If G is finite, then (3.2) *becomes*

$$[Y : X] = r_v \cdot m_v,$$

where $[Y : X] = |G|$, $m_v = |I_v|$ and $r_v = |f^{-1}(v)|$ for all $v \in V_X$.

4.2. Functoriality

More generally, assume that $\varphi : G_1 \to G_2$ is a surjective group morphism and that $\alpha : \mathbf{E}_X \to G_1$ is a voltage assignment with values in G_1. Composing with φ gives a voltage assignment $\varphi \circ \alpha : \mathbf{E}_X \to G_2$ with values in G_2. We let

$$\varphi(\mathcal{I}) = \{(v, \varphi(I_v)) : v \in V_X\},$$

so that $\varphi(\mathcal{I})$ is a collection of subgroups of G_2 indexed by the vertices of X. We obtain two graphs

$$X(G_1, \mathcal{I}, \alpha) \quad \text{and} \quad X(G_2, \varphi(\mathcal{I}), \varphi \circ \alpha),$$

and there is a natural map $\varphi_* : X(G_1, \mathcal{I}, \alpha) \to X(G_2, \varphi(\mathcal{I}), \varphi \circ \alpha)$ given by

$$\varphi_*(v, \sigma_1 I_v) = (v, \varphi(\sigma_1)\varphi(I_v)) \quad \text{and} \quad \varphi_*(e, \sigma_1) = (e, \varphi(\sigma_1)).$$

We leave it to the reader to check that $\varphi_* : X(G_1, \mathcal{I}, \alpha) \to X(G_2, \varphi(\mathcal{I}), \varphi \circ \alpha)$ is a morphism of graphs that is surjective on vertices and directed edges. In fact it is a branched cover of graphs. Indeed, for notational simplicity let $Z = X(G_1, \mathcal{I}, \alpha)$, and also $Y = X(G_2, \varphi(\mathcal{I}), \varphi \circ \alpha)$. Moreover, let $w = (v, \sigma_1 I_v)$ be a vertex of Z. If $(e, \sigma_2) \in \mathbf{E}_{Y, \varphi_*(w)}$, then $\sigma_2 = \varphi(\sigma_1 \tau)$ for some $\tau \in I_v$. A simple calculation shows that

$$(\varphi_*|_{\mathbf{E}_{Z,w}})^{-1}((e, \sigma_2)) = \{(e, \rho\sigma_1\tau) : \rho \in \sigma_1 I_v \sigma_1^{-1} \cap \ker(\varphi)\}, \tag{4.2}$$

so that the second condition of Definition 3.1 is satisfied with

$$m_w = |\sigma_1 I_v \sigma_1^{-1} \cap \ker(\varphi)|.$$

Note that if $\varphi : G_1 \to 1$ is the trivial group morphism, then we get back the situation of Proposition 4.1.

4.3. Branched $\mathbb{Z}_p$-towers of graphs

We can now explain the $\mathbb{Z}_p$-towers of graphs that we will be studying in this paper. As in Section 2.6, we let Γ be a multiplicative topological group isomorphic to $\mathbb{Z}_p$. Let $X = (V_X, \mathbf{E}_X)$ be a finite connected graph and let $\alpha : \mathbf{E}_X \to \Gamma$ be a function satisfying (4.1). Moreover, for each $v \in V_X$, choose a closed subgroup I_v of Γ, and as in Section 4.1, let $\mathcal{I} = \{(v, I_v) : v \in V_X\}$. We obtain a graph $X_\infty = X(\Gamma, \mathcal{I}, \alpha)$ and a branched cover $X_\infty \to X$. The graph X_∞ is infinite. In order to get finite graphs, consider for each integer $n \geq 1$ the natural surjective group morphism $\pi_n : \Gamma \twoheadrightarrow \Gamma_n$, where we recall from Section 2.6 that we set $\Gamma_n = \Gamma/\Gamma^{p^n} \simeq \mathbb{Z}/p^n\mathbb{Z}$. Let $\alpha_n : \mathbf{E}_X \to \Gamma_n$ be the composition $\pi_n \circ \alpha$. We let also

$$\mathcal{I}_n = \{(v, \pi_n(I_v)) : v \in V_X\}$$

which is a collection of subgroups of Γ_n indexed by the vertices of X. We obtain a family of finite graphs

$$X_n = X(\Gamma_n, \mathcal{I}_n, \alpha_n),$$

and the natural surjective group morphisms $\Gamma_{n+1} \twoheadrightarrow \Gamma_n$ induces by Section 4.2 branched covers $X_{n+1} \to X_n$ for each $n \geq 0$. We thus obtain a tower of graphs

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots , \tag{4.3}$$

where each map $X_{n+1} \to X_n$ is a branched cover satisfying $[X_{n+1} : X_n] = p$. We call such a tower a branched (or ramified) $\mathbb{Z}_p$-tower of finite graphs provided all X_n are connected. We explain a sufficient condition that guarantees the connectedness of all finite graphs X_n in Section 4.5 below. Each graph X_n in the tower (4.3) is acted upon by Γ_n by Corollary 2.6 and Proposition 4.1.

Proposition 4.2. *For the purpose of this proposition, let f denote the natural group morphism* $\mathrm{Pic}^0(X_{n+1}) \twoheadrightarrow \mathrm{Pic}^0(X_n)$ *from* (3.3) *and let φ denote the natural projection map* $\Gamma_{n+1} \twoheadrightarrow \Gamma_n$. *Then, for all $x \in \mathrm{Pic}^0(X_{n+1})$ and for all $\gamma \in \Gamma_{n+1}$, one has*

$$\varphi(\gamma) \cdot f(x) = f(\gamma \cdot x).$$

The same property holds true for the group morphism $\mathrm{Pic}(X_{n+1}) \twoheadrightarrow \mathrm{Pic}(X_n)$.

Proof. Let $w = (v, \sigma \pi_{n+1}(I_v))$ be a vertex of X_{n+1}. On one hand, we have

$$f\left(\gamma \cdot \left(v, \sigma \pi_{n+1}(I_v)\right)\right) = f\left(v, \gamma \sigma \pi_{n+1}(I_v)\right) = \left(v, \varphi(\gamma)\varphi(\sigma)\pi_n(I_v)\right),$$

and on the other hand

$$\varphi(\gamma) \cdot f\left(v, \sigma \pi_{n+1}(I_v)\right) = \varphi(\gamma)\left(v, \varphi(\sigma)\pi_n(I_v)\right) = \left(v, \varphi(\gamma)\varphi(\sigma)\pi_n(I_v)\right)$$

showing the desired claim. ∎

The group morphisms $\mathrm{Pic}^0(X_{n+1}) \to \mathrm{Pic}^0(X_n)$ induce maps

$$\mathrm{Pic}_p^0(X_{n+1}) \to \mathrm{Pic}_p^0(X_n)$$

that form a compatible system of $\mathbb{Z}_p$-module morphisms. By Proposition 4.2, the projective limit

$$\mathrm{Pic}_\Lambda^0 = \varprojlim_{n \geq 0} \mathrm{Pic}_p^0(X_n), \tag{4.4}$$

is an Iwasawa module. Similarly, the group morphisms $\mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$ induce maps after tensoring with $\mathbb{Z}_p$ over $\mathbb{Z}$ that also form a compatible system of $\mathbb{Z}_p$-module morphisms $\mathrm{Pic}_p(X_{n+1}) \to \mathrm{Pic}_p(X_n)$. Thus, we obtain another Iwasawa module

$$\mathrm{Pic}_\Lambda = \varprojlim_{n \geq 0} \mathrm{Pic}_p(X_n). \tag{4.5}$$

4.4. Immersions

In this paper, we will encounter another type of morphisms for which we now give the precise definition.

Definition 4.3. Let X and Y be graphs. A morphism of graphs $\iota : X \to Y$ is called an immersion if the induced function

$$\iota|_{\mathbf{E}_{X,v}} : \mathbf{E}_{X,v} \to \mathbf{E}_{Y,\iota(v)}$$

is injective for all $v \in V_X$.

Starting with a graph X, a voltage assignment $\alpha : \mathbf{E}_X \to G$, and a collection of subgroups $\mathcal{I}$ indexed by the vertices of X as in Section 4.1, one can forget about $\mathcal{I}$ (or take $I_v = 1$ for all vertices $v \in V_X$), and keep the same voltage assignment α. In this case, the morphism of graphs $X(G, \alpha) \to X$ is an unramified cover of graphs. Furthermore, we have a commutative diagram of the form

$$
\begin{array}{ccc}
X(G, \alpha) & \xrightarrow{\ \iota\ } & X(G, \mathcal{I}, \alpha) \\
\downarrow & \swarrow {\scriptstyle f} & \\
X & &
\end{array}
$$

where ι is the morphism of graphs given by

$$(v, \sigma) \mapsto \iota(v, \sigma) = (v, \sigma I_v) \quad \text{and} \quad (e, \sigma) \mapsto \iota(e, \sigma) = (e, \sigma).$$

Since ι is a bijection on directed edges, the map ι is an immersion of graphs. Note that the vertical map on the left is an unramified cover of graphs, whereas the map f is a branched cover as we explained above. The morphism of graphs ι is clearly G-equivariant.

4.5. Connectedness

We keep the same notation as in Section 4.4.

Lemma 4.4. *If $X(G, \alpha)$ is connected, then so is $X(G, \mathcal{I}, \alpha)$.*

Proof. Let $(v_1, \sigma_1 I_{v_1})$ and $(v_2, \sigma_2 I_{v_2})$ be two vertices of $X(G, \mathcal{I}, \alpha)$. Since $X(G, \alpha)$ is assumed to be connected, there exists a path c in $X(G, \alpha)$ going from (v_1, σ_1) to (v_2, σ_2). Then, the path $\iota(c)$ of $X(G, \mathcal{I}, \alpha)$ goes from $(v_1, \sigma_1 I_{v_1})$ to $(v_2, \sigma_2 I_{v_2})$ showing the claim. ∎

Let now $\alpha : \mathbf{E}_X \to \Gamma$ be a voltage assignment and let $\mathcal{I}$ be a collection of closed subgroups of Γ indexed by V_X as in Section 4.3. If we let $X_n^{\mathrm{unr}} = X(\Gamma_n, \alpha_n)$, then we have an unramified $\mathbb{Z}_p$-tower of finite graphs

$$X = X_0 \leftarrow X_1^{\mathrm{unr}} \leftarrow X_2^{\mathrm{unr}} \leftarrow \cdots \leftarrow X_n^{\mathrm{unr}} \leftarrow \cdots$$

that fits into a commutative diagram

$$
\begin{array}{ccccccccc}
X_1^{\mathrm{unr}} & \longleftarrow & X_2^{\mathrm{unr}} & \longleftarrow & \cdots & \longleftarrow & X_n^{\mathrm{unr}} & \longleftarrow & \cdots \\
\swarrow \ \ \downarrow {\scriptstyle \iota_1} & & \downarrow {\scriptstyle \iota_2} & & & & \downarrow {\scriptstyle \iota_n} & & \\
X \longleftarrow X_1 & \longleftarrow & X_2 & \longleftarrow & \cdots & \longleftarrow & X_n & \longleftarrow & \cdots,
\end{array}
$$

where the vertical maps are all immersions of graphs. Letting $\pi_1(X, v_0)$ be the fundamen-

tal group of X based at a vertex v_0, the function $\alpha : \mathbf{E}_X \to \Gamma$ induces a group morphism

$$\rho_\alpha : \pi_1(X, v_0) \to \Gamma.$$

Theorem 4.5. *With the notation as above, all the graphs X_n^{unr} are connected if and only if $\rho_\alpha(\pi_1(X, v_0))$ generates Γ topologically.*

Proof. This follows from [17, Theorem 2.11]. ∎

Combined with Lemma 4.4 and the discussion in [17, Section 2.3.1], one has a sufficient condition that can be checked explicitly to construct branched $\mathbb{Z}_p$-towers of graphs, and guarantee the connectedness of all the graphs X_n.

In Section 5.2 below, we will also consider the infinite graph $X_\infty^{\mathrm{unr}} = X(\Gamma, \alpha)$. Note that X_∞^{unr} is locally finite, whereas X_∞ is not necessarily.

4.6. A useful lemma

Let us keep again the same notation as in Section 4.4. The following lemma and corollary will be used in Section 5 below.

Lemma 4.6. *To simplify the notation, let $Y^{\mathrm{unr}} = X(G, \alpha)$, $Y = X(G, \mathcal{I}, \alpha)$, and assume that X, Y and Y^{unr} are all locally finite graphs. Since $\iota : Y^{\mathrm{unr}} \to Y$ is surjective on vertices, we obtain a natural surjective group morphism*

$$\iota_* : \mathrm{Div}(Y^{\mathrm{unr}}) \to \mathrm{Div}(Y).$$

Assuming that I_v is finite for all $v \in V_X$, then for all $v \in V_X$ and for all $\sigma \in G$, one has

$$\mathcal{L}_Y \circ \iota_* \big((v, \sigma) \big) = \sum_{i \in I_v} \iota_* \circ \mathcal{L}_{Y^{\mathrm{unr}}} \big((v, \sigma \cdot i) \big).$$

Proof. On one hand, we have

$$
\begin{aligned}
\mathcal{L}_Y \circ \iota_* \big((v, \sigma) \big) &= \mathcal{L}_Y \big((v, \sigma I_v) \big) \\
&= \mathcal{D}_Y \big((v, \sigma I_v) \big) - \mathcal{A}_Y \big((v, \sigma I_v) \big) \\
&= |I_v| \mathrm{val}_X(v)(v, \sigma I_v) - \sum_{\varepsilon \in \mathbf{E}_{Y,(v,\sigma I_v)}} t(\varepsilon) \\
&= |I_v| \mathrm{val}_X(v)(v, \sigma I_v) - \sum_{e \in \mathbf{E}_{X,v}} \sum_{\tau \in \sigma I_v} \big(t(e), \tau \alpha(e) I_{t(e)} \big) \\
&= |I_v| \mathrm{val}_X(v)(v, \sigma I_v) - \sum_{e \in \mathbf{E}_{X,v}} \sum_{i \in I_v} \big(t(e), \sigma i \alpha(e) I_{t(e)} \big).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\mathcal{L}_{Y^{\mathrm{unr}}} \big((v, \sigma i) \big) &= \mathrm{val}_X(v)(v, \sigma i) - \sum_{\varepsilon \in \mathbf{E}_{Y^{\mathrm{unr}},(v,\sigma i)}} t(\varepsilon) \\
&= \mathrm{val}_X(v)(v, \sigma i) - \sum_{e \in \mathbf{E}_{X,v}} \big(t(e), \sigma i \alpha(e) \big).
\end{aligned}
$$

Therefore, for each $i \in I_v$ we have

$$\iota_* \circ \mathcal{L}_{Y^{\mathrm{unr}}}\big((v, \sigma i)\big) = \mathrm{val}_X(v)(v, \sigma I_v) - \sum_{e \in \mathbf{E}_{X,v}} \big(t(e), \sigma i \alpha(e) I_{t(e)}\big)$$

from which it follows that

$$\sum_{i \in I_v} \iota_* \circ \mathcal{L}_{Y^{\mathrm{unr}}}\big((v, \sigma i)\big) = |I_v| \mathrm{val}_X(v)(v, \sigma I_v) - \sum_{e \in \mathbf{E}_{X,v}} \sum_{i \in I_v} \big(t(e), \sigma i \alpha(e) I_{t(e)}\big),$$

and this ends the proof. ∎

Corollary 4.7. *With the same notation as in Lemma 4.6, for each $v \in V_X$, let*

$$P_v = \iota_* \circ \mathcal{L}_{Y^{\mathrm{unr}}}\big((v, 1_G)\big) \in \mathrm{Div}(Y).$$

Then $\mathrm{Pr}(Y)$ is generated over $\mathbb{Z}[G]$ by

$$\sum_{i \in I_v} i P_v$$

as v runs over all vertices in X.

Proof. By definition, $\mathrm{Pr}(Y)$ is generated over $\mathbb{Z}[G]$ by the divisors $\mathcal{L}_Y \circ \iota_*((v, 1_G))$ as v runs through V_X. The result then follows directly from Lemma 4.6. ∎

5. Iwasawa theory for branched $\mathbb{Z}_p$-towers of graphs

We keep the same notation as before. So $X = (V_X, \mathbf{E}_X)$ is a finite connected graph, Γ is a multiplicative group that is topologically isomorphic to $\mathbb{Z}_p$ and $\alpha : \mathbf{E}_X \to \Gamma$ is a voltage assignment. For each $v \in V_X$, we pick a closed subgroup I_v of Γ, and this gives rise to a branched $\mathbb{Z}_p$-tower of graphs

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots$$

as explained Section 4.3. Recall that we always assume all of the graphs X_n to be connected. See Section 4.5 for a sufficient criterion that guarantees this condition is satisfied.

Our goal in this section is to study the Iwasawa module Pic^0_Λ that was defined above in (4.4). In order to do so, we will use the infinite graph $X_\infty = X(\Gamma, \mathcal{I}, \alpha)$. The short exact sequence (2.5) for the graph X_∞ gives a short exact sequence

$$0 \to \mathrm{Div}^0_p(X_\infty) \to \mathrm{Div}_p(X_\infty) \xrightarrow{s_p} \mathbb{Z}_p \to 0$$

of $\mathbb{Z}_p[\Gamma]$-modules, since Γ is acting on all of these modules, the action of Γ on $\mathbb{Z}_p$ being the trivial one. Recall from Section 2.6 that we set $R = \mathbb{Z}_p[\Gamma]$. Letting

$$\mathrm{Div}_\Lambda(X_\infty) = \Lambda \otimes_R \mathrm{Div}_p(X_\infty),$$

the surjective R-module morphism $s_p : \mathrm{Div}_p(X_\infty) \to \mathbb{Z}_p$ induces a surjective morphism

$$\mathrm{id} \otimes s_p : \mathrm{Div}_\Lambda(X_\infty) \to \Lambda \otimes_R \mathbb{Z}_p \tag{5.1}$$

of Λ-modules. We let

$$\mathrm{Div}_\Lambda^0(X_\infty) = \ker(\mathrm{id} \otimes s_p). \tag{5.2}$$

It follows from (2.7) that

$$\Lambda \otimes_R \mathbb{Z}_p \simeq \Lambda \otimes_R (R/R \cap \omega_0\Lambda) \simeq \Lambda/\omega_0\Lambda \simeq \mathbb{Z}_p$$

as Λ-modules, where the element $\omega_0 = \gamma_0 - 1 \in R \subseteq \Lambda$ was defined in Section 2.6. After making this identification, (5.1) and (5.2) gives us a short exact sequence

$$0 \to \mathrm{Div}_\Lambda^0(X_\infty) \to \mathrm{Div}_\Lambda(X_\infty) \xrightarrow{s_\Lambda} \mathbb{Z}_p \to 0 \tag{5.3}$$

of Λ-modules. At the finite level, for each $n \geq 0$ we also have a short exact sequence

$$0 \to \mathrm{Div}_p^0(X_n) \to \mathrm{Div}_p(X_n) \to \mathbb{Z}_p \to 0 \tag{5.4}$$

of $\mathbb{Z}_p[\Gamma_n]$-modules, since Γ_n acts on all of these modules, the action of Γ_n on $\mathbb{Z}_p$ being the trivial one. Via the natural projection map $\Lambda \twoheadrightarrow \Lambda/\omega_n\Lambda \simeq \mathbb{Z}_p[\Gamma_n]$, we can actually view the short exact sequence (5.4) as one of Λ-modules as well. The branched cover $\pi_n : X_\infty \to X_n$ induces a natural surjective morphism

$$\pi_n : \mathrm{Div}_p(X_\infty) \to \mathrm{Div}_p(X_n)$$

of R-modules. The group $\mathrm{Div}_p(X_\infty)$ is only a module over R, but since $\mathrm{Div}_p(X_n)$ is a module over Λ, we obtain a natural surjective morphism

$$\pi_n : \mathrm{Div}_\Lambda(X_\infty) \to \mathrm{Div}_p(X_n) \tag{5.5}$$

of Λ-modules which we denote by the same symbol. This map induces the commutative diagram of Λ-modules

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Div}_\Lambda^0(X_\infty) & \longrightarrow & \mathrm{Div}_\Lambda(X_\infty) & \xrightarrow{s_\Lambda} & \mathbb{Z}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi_n} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Div}_p^0(X_n) & \longrightarrow & \mathrm{Div}_p(X_n) & \xrightarrow{s_p} & \mathbb{Z}_p & \longrightarrow & 0,
\end{array}
$$

where the right vertical map is the identity, and both horizontal sequences are exact.

The structure of $\mathrm{Div}_\Lambda(X_\infty)$ can be understood as follows. For simplicity, we write V instead of V_X. We let V^{ram} be the set of ramified vertices, that is the collection of vertices $v \in V$ for which I_v is non-trivial. We also let $V^{\mathrm{unr}} = V \smallsetminus V^{\mathrm{ram}}$. From now on, we introduce a labeling of V, say $V = \{v_1, \ldots, v_s\}$, where we agree that

$$V^{\mathrm{unr}} = \{v_1, \ldots, v_r\} \quad \text{and} \quad V^{\mathrm{ram}} = \{v_{r+1}, \ldots, v_s\}.$$

For each $i = 1, \ldots, s$, we let $w_{i,\infty}$ be the vertex (v_i, I_i) of V_{X_∞}, where we write I_i instead of I_{v_i}. Then, as an R-module we have

$$\mathrm{Div}_p(X_\infty) = \bigoplus_{i=1}^r \mathbb{Z}_p[\Gamma] \cdot w_{i,\infty} \oplus \bigoplus_{i=r+1}^s \mathbb{Z}_p[\Gamma/I_i] \cdot w_{i,\infty}. \tag{5.6}$$

From now on, we let k_i be the non-negative integer such that $I_i = \Gamma^{p^{k_i}}$. Note that $\mathbb{Z}_p[\Gamma/I_i] \simeq \mathbb{Z}_p[\Gamma_{k_i}]$, and we have isomorphisms of Λ-modules

$$\Lambda \otimes_R \mathbb{Z}_p[\Gamma/I_i] \simeq \Lambda \otimes_R \mathbb{Z}_p[\Gamma_{k_i}] \simeq \Lambda \otimes_R (R/R \cap \omega_{k_i}\Lambda) \simeq \Lambda/\omega_{k_i}\Lambda \simeq \mathbb{Z}_p[\Gamma_{k_i}].$$

It follows that tensoring (5.6) with Λ over R gives

$$\mathrm{Div}_\Lambda(X_\infty) = \bigoplus_{i=1}^r \Lambda \cdot w_{i,\infty} \oplus \bigoplus_{i=r+1}^s \mathbb{Z}_p[\Gamma_{k_i}] \cdot w_{i,\infty}$$

$$\simeq \Lambda^r \oplus \bigoplus_{i=r+1}^s \mathbb{Z}_p[\Gamma_{k_i}]. \tag{5.7}$$

Similarly, at the finite level we set

$$I_{i,n} = I_i \cdot \Gamma^{p^n}/\Gamma^{p^n}$$

which is the image of I_i via the natural projection map $\pi_n : \Gamma \twoheadrightarrow \Gamma_n$, and we let $w_{i,n}$ be the vertex $(v_i, I_{i,n})$ of X_n. We then have

$$\mathrm{Div}_p(X_n) = \bigoplus_{i=1}^r \mathbb{Z}[\Gamma_n] \cdot w_{i,n} \oplus \bigoplus_{i=r+1}^s \mathbb{Z}_p[\Gamma_n/I_{i,n}] \cdot w_{i,n}$$

as $\mathbb{Z}_p[\Gamma_n]$-modules. Note that if $n \geq k_i$, then $I_{i,n} = \Gamma^{p^{k_i}}/\Gamma^{p^n}$.

Assumption 5.1. From now on, we assume that n is large enough so that

$$I_{i,n} = \Gamma^{p^{k_i}}/\Gamma^{p^n}$$

for all $i = r+1, \ldots, s$.

Under Assumption 5.1, we have

$$\mathrm{Div}_p(X_n) = \bigoplus_{i=1}^r \mathbb{Z}[\Gamma_n] \cdot w_{i,n} \oplus \bigoplus_{i=r+1}^s \mathbb{Z}_p[\Gamma_{k_i}] \cdot w_{i,n}$$

$$\simeq \mathbb{Z}[\Gamma_n]^r \oplus \bigoplus_{i=r+1}^s \mathbb{Z}_p[\Gamma_{k_i}], \tag{5.8}$$

since $\Gamma_n/I_{i,n} \simeq \Gamma/I_i \simeq \Gamma_{k_i}$.

Note that from (5.7) and (5.8), the Λ-module morphism (5.5) is given by the natural projection map $\Lambda \twoheadrightarrow \mathbb{Z}_p[\Gamma_n]$ on the unramified component and by the identity map on the ramified component.

Lemma 5.2. *The kernel of the Λ-module morphism π_n from (5.5) is given by*

$$\ker(\pi_n) = \omega_n \operatorname{Div}_\Lambda(X_\infty).$$

Proof. From (5.7) and (5.8), we get a short exact sequence

$$0 \to \bigoplus_{i=1}^{r} \omega_n \Lambda \cdot w_{i,\infty} \to \operatorname{Div}_\Lambda(X_\infty) \xrightarrow{\pi_n} \operatorname{Div}_p(X_n) \to 0$$

of Λ-modules. Because of Assumption 5.1, we have $\omega_{k_i} \mid \omega_n$. Therefore,

$$\omega_n \mathbb{Z}_p[\Gamma_{k_i}] \simeq \omega_n(\Lambda/\omega_{k_i}\Lambda) = 0.$$

It follows that

$$\omega_n \operatorname{Div}_\Lambda(X_\infty) = \bigoplus_{i=1}^{r} \omega_n \Lambda \cdot w_{i,\infty},$$

and the result follows. ∎

Note that it follows from (5.6) and (5.7) that the natural morphism of R-modules

$$\operatorname{Div}_p(X_\infty) \to \operatorname{Div}_\Lambda(X_\infty)$$

is injective, so that we can view divisors in $\operatorname{Div}_p(X_\infty)$ as lying in $\operatorname{Div}_\Lambda(X_\infty)$.

Definition 5.3. For each $i = 1, \ldots, s$, we let

$$P_{i,\infty} = \operatorname{val}_X(v_i) w_{i,\infty} - \sum_{e \in \mathbf{E}_{X,v_i}} \left(t(e), \alpha(e) I_{t(e)} \right) \in \operatorname{Div}_\Lambda(X_\infty). \qquad (5.9)$$

Moreover, we define $\operatorname{Pr}_\Lambda^{\mathrm{unr}}$ to be the Λ-submodule of $\operatorname{Div}_\Lambda(X_\infty)$ generated by

$$\{P_{i,\infty} : i = 1, \ldots, r\},$$

and we let $\operatorname{Pr}_{\Lambda,n}^{\mathrm{ram}}$ be the Λ-submodule of $\operatorname{Div}_\Lambda(X_\infty)$ generated by

$$\{\omega_{n,k_i} P_{i,\infty} : i = r + 1, \ldots, s\},$$

where the elements ω_{n,k_i} were defined in Section 2.6.

Theorem 5.4. *Under Assumption 5.1, the morphism*

$$\pi_n : \operatorname{Div}_\Lambda(X_\infty) \to \operatorname{Div}_p(X_n)$$

induces isomorphisms

$$\operatorname{Div}_\Lambda(X_\infty)/N_n \xrightarrow{\simeq} \operatorname{Pic}_p(X_n) \quad \text{and} \quad \operatorname{Div}_\Lambda^0(X_\infty)/N_n \xrightarrow{\simeq} \operatorname{Pic}_p^0(X_n),$$

of Λ-modules, where

$$N_n = \omega_n \operatorname{Div}_\Lambda(X_\infty) + \operatorname{Pr}_{\Lambda,n}^{\mathrm{ram}} + \operatorname{Pr}_\Lambda^{\mathrm{unr}}.$$

Proof. For each $n \geq 0$, let $P_{i,n} \in \mathrm{Div}_p(X_n)$ be the divisor P_{v_i} for the graph X_n defined in Corollary 4.7. Observe now that for $i = 1, \ldots, r$, one has

$$\pi_n(P_{i,\infty}) = P_{i,n},$$

whereas for $i = r + 1, \ldots, s$, one has

$$\pi_n(\omega_{n,k_i} P_{i,\infty}) = \sum_{j=0}^{p^{n-k_i}-1} \gamma^{jp^{k_i}} \cdot P_{i,n} = \sum_{\sigma \in I_{i,n}} \sigma P_{i,n}.$$

Corollary 4.7 and Lemma 5.2 imply the equality

$$\pi_n(N_n) = \mathrm{Pr}_p(X_n) \tag{5.10}$$

which implies also the inclusion $N_n \subseteq \pi_n^{-1}(\mathrm{Pr}_p(X_n))$. Conversely, if $D \in \pi_n^{-1}(\mathrm{Pr}_p(X_n))$, then $\pi_n(D) \in \mathrm{Pr}_p(X_n)$ so that by (5.10), one has $\pi_n(D) = \pi_n(D_0)$ for some $D_0 \in N_n$. Lemma 5.2 implies that $D - D_0 \in \omega_n \mathrm{Div}_\Lambda(X_\infty) \subseteq N_n$ and this shows the equality

$$N_n = \pi_n^{-1}(\mathrm{Pr}_p(X_n)).$$

The first isomorphism then follows.

For the second one, it suffices to notice that $\omega_n \mathrm{Div}_\Lambda(X_\infty) \subseteq \mathrm{Div}_\Lambda^0(X_\infty)$, and that from the definition of $P_{i,\infty}$ above in (5.9), we also clearly have $P_{i,\infty} \in \mathrm{Div}_\Lambda^0(X_\infty)$ for all $i = 1, \ldots, s$. Therefore, $N_n \subseteq \mathrm{Div}_\Lambda^0(X_\infty)$ for all $n \geq 0$, and this concludes the proof. ∎

As a consequence, we obtain the following concrete descriptions of Pic_Λ and Pic_Λ^0 that were defined above in (4.5) and (4.4).

Corollary 5.5. *With the same notation as above, one has*

$$\mathrm{Pic}_\Lambda \simeq \mathrm{Div}_\Lambda(X_\infty)/\mathrm{Pr}_\Lambda^{\mathrm{unr}} \quad and \quad \mathrm{Pic}_\Lambda^0 \simeq \mathrm{Div}_\Lambda^0(X_\infty)/\mathrm{Pr}_\Lambda^{\mathrm{unr}}.$$

Proof. As in Theorem 5.4, we let

$$N_n = \omega_n \mathrm{Div}_\Lambda(X_\infty) + \mathrm{Pr}_{\Lambda,n}^{\mathrm{ram}} + \mathrm{Pr}_\Lambda^{\mathrm{unr}}$$

as long as $n \geq k = \max\{k_i : i = r + 1, \ldots, s\}$. Observe first that the commutativity of the diagram

$$\mathrm{Div}_\Lambda(X_\infty) \xrightarrow{\pi_{n+1}} \mathrm{Div}_p(X_{n+1})$$
$$\searrow^{\pi_n} \qquad \downarrow$$
$$\mathrm{Div}_p(X_n)$$

induces yet another commutative diagram

$$\mathrm{Div}_\Lambda(X_\infty)/N_{n+1} \xrightarrow{\simeq} \mathrm{Pic}_p(X_{n+1})$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\mathrm{Div}_\Lambda(X_\infty)/N_n \xrightarrow{\simeq} \mathrm{Pic}_p(X_n),$$

where the left vertical arrow is the natural map induced from the inclusion $N_{n+1} \subseteq N_n$, and the two horizontal isomorphisms are given by Theorem 5.4. It follows that we have

$$\text{Pic}_\Lambda \simeq \varprojlim_{n \geq k} \text{Div}_\Lambda(X_\infty)/N_n.$$

Since $\text{Pr}_\Lambda^{\text{unr}} \subseteq N_n$, we have natural maps

$$\text{Div}_\Lambda(X_\infty)/\text{Pr}_\Lambda^{\text{unr}} \to \text{Div}_\Lambda(X_\infty)/N_n$$

that induce a Λ-module morphism

$$\text{Div}_\Lambda(X_\infty)/\text{Pr}_\Lambda^{\text{unr}} \to \varprojlim_{n \geq k} \text{Div}_\Lambda(X_\infty)/N_n, \tag{5.11}$$

and it remains to show that this last morphism is in fact an isomorphism. Observe that

$$\bigcap_{n \geq k} N_n = \text{Pr}_\Lambda^{\text{unr}}.$$

Indeed, the inclusion $\text{Pic}_\Lambda^{\text{unr}} \subseteq \bigcap_{n \geq k} N_n$ is clear by definition, and if $D \in \bigcap_{n \geq k} N_n$, then for each $n \geq k$, we have $D = D'_n + P_n$ for some $D'_n \in \omega_n \cdot \text{Div}_\Lambda(X_\infty) + \text{Pr}_{\Lambda,n}^{\text{ram}}$ and some $P_n \in \text{Pr}_\Lambda^{\text{unr}}$. Since $D'_n \to 0$ as $n \to \infty$, we have $P_n \to D$ as $n \to \infty$. But since $\text{Pr}_\Lambda^{\text{unr}}$ is finitely generated over Λ, it is compact. It follows that $D \in \text{Pr}_\Lambda^{\text{unr}}$, and this shows that (5.11) is injective. The surjectivity of (5.11) follows from a standard result about projective limits of compact modules (see for instance [15, Chapter IV, Proposition 2.7]). This ends the proof that

$$\text{Pic}_\Lambda \simeq \text{Div}_\Lambda(X_\infty)/\text{Pr}_\Lambda^{\text{unr}},$$

and the second isomorphism is proven similarly. ∎

Since $\text{Div}_\Lambda(X_\infty)$ is a finitely generated Λ-module, then so is Pic_Λ by Corollary 5.5. Moreover, since Λ is a Noetherian ring, $\text{Div}_\Lambda^0(X_\infty) = \ker(s_\Lambda)$ is a finitely generated Λ-module as well, and it follows from Corollary 5.5 that Pic_Λ^0 is also a finitely generated Λ-module.

5.1. The analogue of Iwasawa's asymptotic class number formula

We can now go ahead and prove our first main theorem (Theorem A from Section 1) of this paper.

Theorem 5.6. *Let $X = (V_X, \mathbf{E}_X)$ be a finite connected graph, and let $\alpha : \mathbf{E}_X \to \Gamma$ be a voltage assignment. Consider a family $\mathcal{I}$ of closed subgroups of Γ indexed by the vertices of V_X and consider the branched $\mathbb{Z}_p$-tower of graphs*

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots,$$

where $X_n = X(\Gamma_n, \mathcal{I}_n, \alpha_n)$ as explained in Section 4.3. Assume that all finite graphs X_n

are connected. Then $\operatorname{Pic}^0_\Lambda$ *is a finitely generated torsion Λ-module. Moreover, if we let*

$$\mu = \mu(\operatorname{Pic}^0_\Lambda) \quad and \quad \lambda = \lambda(\operatorname{Pic}^0_\Lambda),$$

then there exist $n_0 \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{Z}$ such that

$$\operatorname{ord}_p\big(\kappa(X_n)\big) = \mu p^n + \lambda n + \nu,$$

when $n \geq n_0$.

Proof. We keep the same notation as above. As in the proof of Corollary 5.5, set $k = \max\{k_i : i = r+1, \ldots, s\}$, and define the Λ-modules

$$M = \operatorname{Div}^0_\Lambda(X_\infty)/\operatorname{Pr}^{\mathrm{unr}}_\Lambda \quad and \quad A = N/\operatorname{Pr}^{\mathrm{unr}}_\Lambda,$$

where

$$N = \omega_k \operatorname{Div}_\Lambda(X_\infty) + \frac{\omega_k}{\omega_n}\operatorname{Pr}^{\mathrm{ram}}_{\Lambda,n} + \operatorname{Pr}^{\mathrm{unr}}_\Lambda.$$

Note that $M \simeq \operatorname{Pic}^0_\Lambda$ by Corollary 5.5, and that N does not depend on n. Theorem 5.4 implies that

$$\operatorname{Pic}^0_p(X_n) \simeq M/\omega_{n,k}A,$$

where the elements $\omega_{n,k} \in R$ were defined in Section 2.6. It follows that we have a short exact sequence

$$0 \to A/\omega_{n,k}A \to \operatorname{Pic}^0_p(X_n) \to M/A \to 0 \tag{5.12}$$

of Λ-modules. Since $\operatorname{Pic}^0_p(X_n) \simeq \operatorname{Pic}^0(X_n)[p^\infty]$ is finite, then so are $A/\omega_{n,k}A$ and M/A. By definition, A is a finitely generated Λ-module. It follows from Theorem 2.7 that A is torsion and that there exist $n_0 \in \mathbb{Z}_{\geq 0}$ and $\nu' \in \mathbb{Z}$ such that when $n \geq n_0$, one has

$$\operatorname{ord}_p\big(|A/\omega_{n,k}A|\big) = \mu(A)p^n + \lambda(A)n + \nu'.$$

Since $M/A \simeq \operatorname{Div}^0_\Lambda(X_\infty)/N$ does not depend on n, we have $|M/A| = p^{k'}$ for some $k' \geq 0$. Letting $\nu = \nu' + k'$, the short exact sequence (5.12) gives

$$\operatorname{ord}_p\big(\kappa(X_n)\big) = \mu(A)p^n + \lambda(A)n + \nu,$$

when n is large. The Λ-modules M and A are related to one another via the short exact sequence

$$0 \to A \to M \to M/A \to 0,$$

and since M/A is finite, A and M are pseudo-isomorphic. It follows that $M \simeq \operatorname{Pic}^0_\Lambda$ is also a finitely generated torsion Λ-module and

$$\mu(A) = \mu(M) \quad and \quad \lambda(A) = \lambda(M). \qquad \blacksquare$$

Remark 5.7. In the situation where $V^{\mathrm{ram}} = \varnothing$, note that we can take $k = 0$ and the short exact sequence (5.12) becomes

$$0 \to A/\omega_{n,0}A \to \operatorname{Pic}^0_p(X_n) \to \operatorname{Pic}^0_p(X) \to 0$$

for all $n \geq 0$. This corresponds to the situation originally studied from the module theoretical point of view in [7, 11, 12].

5.2. The analogue of Iwasawa's main conjecture

Let us start with the following proposition.

Proposition 5.8. *Let $X = (V_X, \mathbf{E}_X)$ be a finite connected graph, and let $\alpha : \mathbf{E}_X \to \Gamma$ be a voltage assignment. Consider a family $\mathcal{I}$ of closed subgroups of Γ indexed by the vertices of V_X and consider the branched $\mathbb{Z}_p$-tower of graphs*

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots ,$$

where $X_n = X(\Gamma_n, \mathcal{I}_n, \alpha_n)$ as explained in Section 4.3. Assume that all finite graphs X_n are connected. Then, one has a short exact sequence of finitely generated Λ-modules

$$0 \to \mathrm{Pic}_\Lambda^0 \to \mathrm{Pic}_\Lambda \to \mathbb{Z}_p \to 0,$$

where $\mathbb{Z}_p$ is viewed as a Λ-module with trivial action. Moreover, Pic_Λ is a finitely generated torsion Λ-module, and one has

$$\mathrm{char}_\Lambda(\mathrm{Pic}_\Lambda^0) \cdot \omega_0 \Lambda = \mathrm{char}_\Lambda(\mathrm{Pic}_\Lambda).$$

Proof. The short exact sequence (5.3) induces a short exact sequence

$$0 \to \mathrm{Div}_\Lambda^0(X_\infty)/\mathrm{Pr}_\Lambda^{\mathrm{unr}} \to \mathrm{Div}_\Lambda(X_\infty)/\mathrm{Pr}_\Lambda^{\mathrm{unr}} \xrightarrow{s_\Lambda} \mathbb{Z}_p \to 0,$$

which combined with Corollary 5.5 gives the desired short exact sequence. Since both $\mathbb{Z}_p$ and Pic_Λ^0 are finitely generated torsion Λ-modules, it follows that Pic_Λ is also a finitely generated torsion Λ-module. Moreover, since $\mathrm{char}_\Lambda(\mathbb{Z}_p) = \omega_0 \Lambda$, (2.8) implies that we have the equality of ideals

$$\mathrm{char}_\Lambda(\mathrm{Pic}_\Lambda^0) \cdot \omega_0 \Lambda = \mathrm{char}_\Lambda(\mathrm{Pic}_\Lambda). \qquad \blacksquare$$

It follows from Proposition 5.8 that in order to understand $\mathrm{char}_\Lambda(\mathrm{Pic}_\Lambda^0)$, it suffices to understand $\mathrm{char}_\Lambda(\mathrm{Pic}_\Lambda)$. In order to do so, consider $X_\infty^{\mathrm{unr}} = X(\Gamma, \alpha)$ as we did at the end of Section 4.5. Let

$$M_1 = \bigoplus_{i=1}^{r} \Lambda \cdot w_{i,\infty}^{\mathrm{unr}} \quad \text{and} \quad M_2 = \bigoplus_{i=r+1}^{s} \Lambda \cdot w_{i,\infty}^{\mathrm{unr}},$$

where $w_{i,\infty}^{\mathrm{unr}} = (v_i, 1_\Gamma)$ for $i = 1, \ldots, s$. Note that we have $\mathrm{Div}_\Lambda(X_\infty^{\mathrm{unr}}) = M_1 \oplus M_2$. We define an operator

$$\Delta : \mathrm{Div}_\Lambda(X_\infty^{\mathrm{unr}}) \to \mathrm{Div}_\Lambda(X_\infty^{\mathrm{unr}}) \tag{5.13}$$

as follows: If $i = 1, \ldots, r$, then

$$\Delta(w_{i,\infty}^{\mathrm{unr}}) = \mathcal{L}_\Lambda^{\mathrm{unr}}(w_{i,\infty}^{\mathrm{unr}}),$$

where $\mathcal{L}_\Lambda^{\mathrm{unr}}$ is the Laplacian operator on the locally finite graph X_∞^{unr}, and if $i = r+1, \ldots, s$, then

$$\Delta(w_{i,\infty}^{\mathrm{unr}}) = \omega_{k_i} \cdot w_{i,\infty}^{\mathrm{unr}}.$$

Since $\mathrm{Div}_\Lambda(X_\infty^{\mathrm{unr}})$ is a free Λ-module, it makes sense to talk about $\det(\Delta) \in \Lambda$. We are now ready to prove our second main theorem (Theorem B from Section 1) of this paper. (See also Theorem 6.1 below.)

Theorem 5.9. *Let $X = (V_X, \mathbf{E}_X)$ be a finite connected graph, and let $\alpha : \mathbf{E}_X \to \Gamma$ be a voltage assignment. Consider a family $\mathcal{I}$ of closed subgroups of Γ indexed by the vertices of V_X and consider the branched $\mathbb{Z}_p$-tower of graphs*

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots ,$$

where $X_n = X(\Gamma_n, \mathcal{I}_n, \alpha_n)$ as explained in Section 4.3. Assume that all finite graphs X_n are connected. With the notation as above, one has

$$\mathrm{char}_\Lambda(\mathrm{Pic}_\Lambda) = \big(\det(\Delta)\big),$$

where Δ is the operator defined above in (5.13). Moreover, one has

$$\mathrm{char}_\Lambda(\mathrm{Pic}_\Lambda^0) \cdot \omega_0 \Lambda = \big(\det(\Delta)\big).$$

Proof. Consider the natural surjective Λ-module morphism $\iota_* : \mathrm{Div}_\Lambda(X_\infty^{\mathrm{unr}}) \to \mathrm{Div}_\Lambda(X_\infty)$ induced by the immersion $\iota : X_\infty^{\mathrm{unr}} \to X_\infty$. It follows from the explicit description (5.7) of the Λ-module structure of $\mathrm{Div}_\Lambda(X_\infty)$ that

$$\ker(\iota_*) = \bigoplus_{i=r+1}^{s} \omega_{k_i} \Lambda \cdot w_{i,\infty}^{\mathrm{unr}}.$$

Therefore, we have an exact sequence of Λ-modules

$$\mathrm{Div}_\Lambda(X_\infty^{\mathrm{unr}}) \xrightarrow{\Delta} \mathrm{Div}_\Lambda(X_\infty^{\mathrm{unr}}) \to \mathrm{Div}_\Lambda(X_\infty)/\mathrm{Pr}_\Lambda^{\mathrm{unr}} \to 0, \tag{5.14}$$

where the map on the right is ι_* followed by the natural projection map $\mathrm{Div}_\Lambda(X_\infty) \to \mathrm{Div}_\Lambda(X_\infty)/\mathrm{Pr}_\Lambda^{\mathrm{unr}}$. By Corollary 5.5, we have $\mathrm{Div}_\Lambda(X_\infty)/\mathrm{Pr}_\Lambda^{\mathrm{unr}} \simeq \mathrm{Pic}_\Lambda$. Moreover, since Pic_Λ is a torsion Λ-module by Proposition 5.8 and $\mathrm{Div}_\Lambda(X_\infty^{\mathrm{unr}})$ is free of finite rank as a Λ-module, [11, Lemma A.3] implies that Δ is injective so that we have a short exact sequence

$$0 \to \mathrm{Div}_\Lambda(X_\infty^{\mathrm{unr}}) \xrightarrow{\Delta} \mathrm{Div}_\Lambda(X_\infty^{\mathrm{unr}}) \to \mathrm{Pic}_\Lambda \to 0.$$

By Theorem 2.8, we have

$$\mathrm{char}_\Lambda(\mathrm{Pic}_\Lambda) = \big(\det(\Delta)\big).$$

The second equality follows directly from Proposition 5.8. $\blacksquare$

Remark 5.10. When $V^{\mathrm{ram}} = \varnothing$, Theorem 5.9 reduces to [12, Theorem 5.2] in the situation where $l = 1$ in their notation. In this case, we have $X_n = X_n^{\mathrm{unr}}$ for all $n \geq 0$, and Δ is the Laplacian operator $\mathcal{L}_\Lambda^{\mathrm{unr}}$ on the locally finite graph $X_\infty = X_\infty^{\mathrm{unr}}$.

Remark 5.11. It follows from (5.14) that the projective dimension of Pic_Λ is smaller or equal than one. Therefore, by [16, Proposition 5.3.19], the Iwasawa module Pic_Λ has no finite nontrivial Λ-submodule.

6. Examples

In order to provide numerical examples, it is convenient to work with $\mathbb{Z}_p[\![T]\!]$ rather than Λ. So we now reexpress the result of Theorem 5.9 using the non-canonical isomorphism (2.9). Moreover, we prefer to work additively, so we work directly with $\mathbb{Z}_p$ rather than Γ. We will make use of the group morphism

$$\rho : \mathbb{Z}_p \to \mathbb{Z}_p[\![T]\!]^\times \tag{6.1}$$

given by

$$a \mapsto \rho(a) = (1 + T)^a = \sum_{i=0}^{\infty} \binom{a}{i} T^i \in \mathbb{Z}_p[\![T]\!]^\times,$$

where the $\binom{X}{i}$ are the usual integer-valued polynomials (which also satisfy $\binom{a}{i} \in \mathbb{Z}_p$ whenever $a \in \mathbb{Z}_p$) given by $\binom{X}{0} = 1$, $\binom{X}{1} = X$, and

$$\binom{X}{i} = \frac{X(X-1)\cdots(X-i+1)}{i!} \in \mathbb{Q}[X]$$

for $i \geq 2$. Note that the group morphism ρ corresponds to $\Gamma \hookrightarrow R^\times \hookrightarrow \Lambda^\times$. Starting with a voltage assignment $\alpha : \mathbf{E}_X \to \mathbb{Z}_p$, we define a few matrices as follows. First, we let $D = (d_{ij})$ be the diagonal matrix defined via

$$d_{ii} = \begin{cases} \mathrm{val}_X(v_i), & \text{if } i = 1, \dots, r; \\ 0, & \text{otherwise.} \end{cases}$$

Then, we define another matrix

$$B(T) = \big(b_{ij}(T)\big) \in M_{s \times s}\big(\mathbb{Z}_p[\![T]\!]\big)$$

as follows:

(1) If $i = 1, \dots, s$, and $j = 1, \dots, r$, then let

$$b_{ij}(T) = \sum_{\substack{e \in \mathbf{E}_X \\ \mathrm{inc}(e) = (v_j, v_i)}} \rho\big(\alpha(e)\big),$$

where ρ is the group morphism (6.1) above.

(2) If $i = j = r + 1, \dots, s$, then let

$$b_{ij}(T) = -\omega_{k_i}(T),$$

where $\omega_{k_i}(T) = (1 + T)^{p^{k_i}} - 1$ is the distinguished polynomial in $\mathbb{Z}_p[\![T]\!]$ corresponding to ω_{k_i} via the isomorphism (2.9).

(3) Set $b_{ij}(T) = 0$ otherwise.

A simple calculation based on the definition (5.13) of the operator Δ shows that

$$\det(\Delta) = \det\left(D - B(T)\right),$$

when Δ is viewed as a morphism of $\mathbb{Z}_p[\![T]\!]$-modules. We let

$$f_{X,\mathcal{I},\alpha}(T) = \det\left(D - B(T)\right) \in \mathbb{Z}_p[\![T]\!]. \tag{6.2}$$

Theorem 5.9 can now be phrased as follows.

Theorem 6.1. *Let $X = (V_X, \mathbf{E}_X)$ be a finite connected graph, and let $\alpha : \mathbf{E}_X \to \mathbb{Z}_p$ be a voltage assignment. Consider a family $\mathcal{I}$ of closed subgroups of $\mathbb{Z}_p$ indexed by the vertices of V_X and consider the branched $\mathbb{Z}_p$-tower of graphs*

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots ,$$

where $X_n = X(\mathbb{Z}/p^n\mathbb{Z}, \mathcal{I}_n, \alpha_n)$ as explained in Section 4.3. Assume that all finite graphs X_n are connected. Then, one has

$$\operatorname{char}_{\mathbb{Z}_p[\![T]\!]}(\operatorname{Pic}_\Lambda^0) \cdot (T) = \left(f_{X,\mathcal{I},\alpha}(T)\right),$$

where $f_{X,\mathcal{I},\alpha}(T)$ is the power series defined in (6.2) above.

It follows from Theorem 6.1 that we have

$$\mu(\operatorname{Pic}_\Lambda^0) = \mu\left(f_{X,\mathcal{I},\alpha}(T)\right) \quad \text{and} \quad \lambda(\operatorname{Pic}_\Lambda^0) = \lambda\left(f_{X,\mathcal{I},\alpha}(T)\right) - 1,$$

and this allows us to provide a few numerical examples below. The calculation of the number of spanning trees and of the power series $f_{X,\mathcal{I},\alpha}(T)$ below have been performed with the software [20]. For the second and third example, since we do not have an exact formula for the number of spanning trees of the graphs in the towers, we calculated the number of spanning trees only for the first seven layers. Both times, the p-adic valuation of the number of spanning trees seems to agree with a formula involving the Iwasawa invariants that we calculated independently from the corresponding power series.

(1) Let us revisit [23, Example 2, p. 451]. Let $p = 2$, and let X be the bouquet graph on two loops. Pick an arbitrary orientation S for X and consider the function $\alpha : S \to \mathbb{Z} \subseteq \mathbb{Z}_2$ given by

$$s_1 \mapsto \alpha(s_1) = 3 \quad \text{and} \quad s_2 \mapsto \alpha(s_2) = 5.$$

The corresponding (unramified) $\mathbb{Z}_2$-tower of graphs was studied in [23, p. 451]. One can use Theorem 4.5 to check that all the finite graphs in the $\mathbb{Z}_2$-tower of graphs are connected. Let us now introduce some ramification at the unique vertex of X. Say, we let $I_1 = 4\mathbb{Z}_2$. Then, we obtain the branched $\mathbb{Z}_2$-tower of graphs
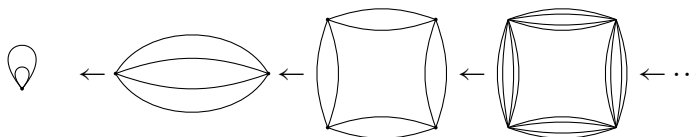
All the graphs X_n are connected by Lemma 4.4. The power series $f_{X,\mathcal{I},\alpha}(T)$ is given by

$$f_{X,\mathcal{I},\alpha}(T) = 4T + 6T^2 + 4T^3 + T^4 \in \mathbb{Z}[T] \subseteq \mathbb{Z}_2[\![T]\!],$$

so we should have $\mu(\mathrm{Pic}^0_\Lambda) = 0$ and $\lambda(\mathrm{Pic}^0_\Lambda) = 3$. We calculate

$$\kappa_0 = 1, \quad \kappa_1 = 2^2, \quad \kappa_2 = 2^5, \quad \kappa_3 = 2^8, \ldots,$$

where we write κ_n for $\kappa(X_n)$. For $n \geq 2$, the graph X_n has four vertices and there are 2^{n-1} undirected edges between two adjacent vertices. Thus $\kappa_n = 4 \cdot (2^{n-1})^3 = 2^{3n-1}$ when $n \geq 1$. We have

$$\mathrm{ord}_2\big(\kappa(X_n)\big) = 3n - 1,$$

for all $n \geq 1$.

(2) Let us revisit [6, Example 2, page 55]. Let $p = 3$ and let X be the dumbbell graph

Pick the orientation S to be arbitrary on the loops and going from the first vertex on the left to the vertex on the right for the non-loop edge. Label those directed edges from left to right so that s_2 corresponds to the non-loop directed edge going from left to right. Take the function $\alpha : S \to \mathbb{Z} \subseteq \mathbb{Z}_3$ given by

$$s_1 \mapsto \alpha(s_1) = 1, \quad s_2 \mapsto \alpha(s_2) = 0, \quad \text{and} \quad s_3 \mapsto \alpha(s_3) = 11.$$

One obtains the following (unramified) $\mathbb{Z}_3$-tower of finite graphs
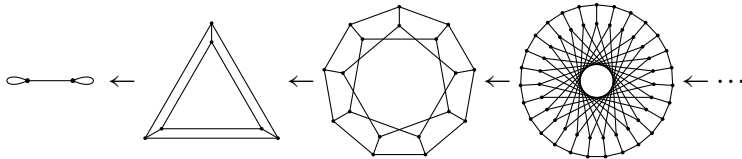
The graphs X_n are the generalized Petersen graphs $G(3^n, 11)$ and are connected by Theorem 4.5. The power series $f_{X,\alpha}(T)$ is given by

$$f_{X,\alpha}(T) = -122T^2 + 122T^3 - 1211T^4 + \cdots \in \mathbb{Z}[\![T]\!] \subseteq \mathbb{Z}_3[\![T]\!]$$

so we should have $\mu(\mathrm{Pic}^0_\Lambda) = 0$ and $\lambda(\mathrm{Pic}^0_\Lambda) = 1$. We calculate
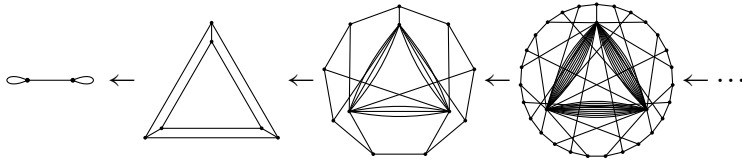
$$\kappa_0 = 1, \quad \kappa_1 = 3 \cdot 5^2, \quad \kappa_2 = 3^2 \cdot 5^2 \cdot 71^2, \quad \kappa_3 = 3^3 \cdot 5^2 \cdot 71^2 \cdot 109^2 \cdot 13931^2, \ldots,$$

where we write κ_n for $\kappa(X_n)$. We have

$$\mathrm{ord}_3\big(\kappa(X_n)\big) = n,$$

for all $n \geq 0$. Let us now introduce some ramification at the second vertex only,

so let $I_1 = 0$ and $I_2 = 3\mathbb{Z}_3$. Then, we obtain the branched $\mathbb{Z}_3$-tower of graphs

The graphs X_n are connected by Lemma 4.4. The power series $f_{X,\mathcal{I},\alpha}(T)$ is given by

$$f_{X,\mathcal{I},\alpha}(T) = 3T + 3T^2 - 2T^3 - \cdots \in \mathbb{Z}[\![T]\!] \subseteq \mathbb{Z}_3[\![T]\!],$$

so we should have $\mu(\mathrm{Pic}^0_\Lambda) = 0$ and $\lambda(\mathrm{Pic}^0_\Lambda) = 3 - 1 = 2$. We calculate

$$\kappa_0 = 1, \quad \kappa_1 = 3 \cdot 5^2, \quad \kappa_2 = 3^3 \cdot 5^2 \cdot 19^2, \quad \kappa_3 = 3^5 \cdot 5^2 \cdot 19^2 \cdot 5779^2,$$
$$\kappa_4 = 3^7 \cdot 5^2 \cdot 19^2 \cdot 3079^2 \cdot 5779^2 \cdot 62650261^2, \ldots,$$

where we let κ_n be $\kappa(X_n)$. We checked that we have

$$\mathrm{ord}_3\big(\kappa(X_n)\big) = 2n - 1,$$

for $1 \leq n \leq 7$ which seems to suggest that this formula holds true for all $n \geq 1$, although we do not have a proof of that claim.

(3) Let $p = 3$, and let X be the graph

Take the orientation S of X as follows: Direct the loops arbitrarily, and for the non-loop edges, direct them from left to right. Label the directed edges $S = \{s_1, s_2, \ldots s_7\}$ so that s_4, s_5, s_6 are the non-loop directed edges, s_1, s_2, s_3 the directed loops at the first vertex on the left, and s_7 the directed loop at the second vertex on the right. Take the function $\alpha : S \to \mathbb{Z} \subseteq \mathbb{Z}_3$ given by

$$s_i \mapsto \alpha(s_i) = 1, \quad s_j \mapsto \alpha(s_j) = 0, \quad \text{and} \quad s_7 \mapsto \alpha(s_7) = 11,$$

where $i = 1, 2, 3$, and $j = 4, 5, 6$. Let us introduce some ramification at the second vertex only, say $I_2 = 3\mathbb{Z}_3$. We obtain the branched $\mathbb{Z}_3$-tower of graphs
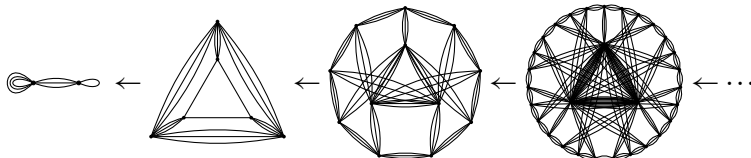
Using Theorem 4.5 and Lemma 4.4, one checks that all the finite graphs X_n are connected. The power series $f_{X,\mathcal{I},\alpha}(T)$ is given by

$$f_{X,\mathcal{I},\alpha}(T) = 3(3T + 3T^2 - 2T^3 + \cdots) \in \mathbb{Z}[\![T]\!] \subseteq \mathbb{Z}_3[\![T]\!],$$

so we should have $\mu(\text{Pic}_\Lambda^0) = 1$ and $\lambda(\text{Pic}_\Lambda^0) = 3 - 1 = 2$. We calculate

$$\kappa_0 = 3, \quad \kappa_1 = 3^4 \cdot 7^2, \quad \kappa_2 = 3^{12} \cdot 7^2 \cdot 19^2, \quad \kappa_3 = 3^{32} \cdot 7^2 \cdot 19^2 \cdot 5779^2,$$
$$\kappa_4 = 3^{88} \cdot 7^2 \cdot 19^2 \cdot 3079^2 \cdot 5779^2 \cdot 62650261^2, \dots,$$

where we let again $\kappa_n = \kappa(X_n)$. We checked that we have

$$\text{ord}_3\big(\kappa(X_n)\big) = 3^n + 2n - 1,$$

for all $1 \leq n \leq 7$. Again, this seems to suggest that this formula holds true for all $n \geq 1$.

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