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Connecting real and hyperarithmetical analysis

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Abstract. Going back to Kreisel in the sixties, *hyperarithmetical analysis* is a cluster of logical systems just beyond arithmetical comprehension. Only recently natural examples of theorems from the mathematical mainstream were identified that fit this category. In this paper, we provide *many* examples of theorems of real analysis that *sit within the range of hyperarithmetical analysis*, namely between the higher-order version of Σ^1_1-AC_0 and weak-Σ^1_1-AC_0, working in Kohlenbach's higher-order framework. Our example theorems are based on the *Jordan decomposition theorem*, *unordered sums*, *metric spaces*, and *semi-continuous functions*. Along the way, we identify a couple of new systems of hyperarithmetical analysis.

1. Introduction

1.1. Motivation and overview

The aim of this paper is to exhibit many natural examples of theorems from real analysis that exist *in the range of hyperarithmetical analysis*. The exact meaning of 'hyperarithmetical analysis' and the previous italicised text is discussed in Section 1.3, but intuitively speaking the latter amounts to being sandwiched between known systems of hyperarithmetical analysis or their higher-order extensions. We shall work in Kohlenbach's framework from [36], with which we assume basic familiarity.

We introduce some necessary definitions and axioms in Section 1.2. We shall establish that the following inhabit the range of hyperarithmetical analysis.

- Basic properties of (Lipschitz) continuous functions on compact metric spaces *without second-order representation/separability conditions*, including the generalised intermediate value theorem (Section 2).
- Properties of functions of *bounded variation*, including the *Jordan decomposition theorem*, where the total variation is given (Section 3).
- Properties of semi-continuous functions and closed sets (Section 4.1).
- Convergence properties of *unordered sums* (Section 4.2).

These results still go through if we restrict to arithmetically defined objects by Theorem 2.8. To pinpoint the exact location of the aforementioned principles, we introduce a new 'finite choice' principle based on finite-Σ^1_1-AC_0 from [26] (see Section 1.2), using Borel's notion of *height function* [9, 10].

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Finally, as to conceptual motivation, the historical examples of systems of hyperarithmetical analysis are rather logical in nature and natural examples from the mathematical mainstream are a relatively recent discovery, as discussed in Section 1.3. Our motivation is to show that third-order arithmetic exhibits *many* robust examples of theorems in the range of hyperarithmetical analysis, similar perhaps to how so-called splittings and disjunctions are much more plentiful in third-order arithmetic, as explored in [62]. In this paper, we merely develop certain examples and indicate the many possible variations.

1.2. Preliminaries

We introduce some basic definitions and axioms necessary for this paper. We note that subsets of $\mathbb{R}$ are given by their characteristic functions as in Definition 1.2, well-known from measure and probability theory. We shall generally work over ACA_0^ω – defined right below – as some definitions make little sense over the base theory RCA_0^ω. We refer to [36] for the latter.

First of all, full second-order arithmetic Z_2 is the 'upper limit' of second-order RM. The systems Z_2^ω and Z_2^Ω are conservative extensions of Z_2 by [30, Cor. 2.6]. The system Z_2^Ω is RCA_0^ω plus Kleene's quantifier $(\exists^3)$ (see e.g. [52] or [30]), while Z_2^ω is RCA_0^ω plus (S_k^2) for every $k \geq 1$; the latter axiom states the existence of a functional S_k^2 deciding Π_k^1-formulas in Kleene normal form. The system $\Pi_1^1\text{-}\mathsf{CA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\mathsf{S}_1^2)$ is a Π_3^1-conservative extension of $\Pi_1^1\text{-}\mathsf{CA}_0$ [60], where S_1^2 is also called the *Suslin functional*. We also write ACA_0^ω for $\mathsf{RCA}_0^\omega + (\exists^2)$ where the latter is as follows

$$(\exists E : \mathbb{N}^\mathbb{N} \to \{0,1\})(\forall f \in \mathbb{N}^\mathbb{N})\big[(\exists n \in \mathbb{N})(f(n) = 0) \leftrightarrow E(f) = 0\big]. \qquad (\exists^2)$$

The system ACA_0^ω is a conservative extension of ACA_0 by [30, Thm. 2.5]. Over RCA_0^ω, $(\exists^2)$ is equivalent to (μ^2), where the latter expresses the existence of Feferman's μ (see [36, Prop. 3.9]), defined as follows for all $f \in \mathbb{N}^\mathbb{N}$:

$$\mu(f) := \begin{cases} n & \text{if } n \text{ is the least natural number such that } f(n) = 0, \\ 0 & \text{if } f(n) > 0 \text{ for all } n \in \mathbb{N}. \end{cases}$$

The following schema is essential to our enterprise, as discussed in Section 1.3.

Principle 1.1 ($\mathsf{QF\text{-}AC}^{0,1}$). *For any $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$, if $(\forall n \in \mathbb{N})(\exists f \in \mathbb{N}^\mathbb{N})(Y(f,n) = 0)$, then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^\mathbb{N}$ with $(\forall n \in \mathbb{N})(Y(f_n, n) = 0)$.*

The local equivalence between sequential and 'epsilon-delta' continuity cannot be proved in ZF, but can be established in $\mathsf{RCA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ (see [36]). Thus, it should not be a surprise that the latter system is often used as a base theory too.

Secondly, we make use the following standard definitions concerning sets.

Definition 1.2 (Sets). Sets are defined via characteristic functions as follows.

- A subset $A \subset \mathbb{R}$ is given by its characteristic function $F_A : \mathbb{R} \to \{0,1\}$, i.e., we write $x \in A$ for $F_A(x) = 1$, for any $x \in \mathbb{R}$.

- A set $A \subset \mathbb{R}$ is *enumerable* if there is a sequence of reals that includes all elements of A.
- A set $A \subset \mathbb{R}$ is *countable* if there is $Y : \mathbb{R} \to \mathbb{N}$ that is injective on A, i.e.,

$$(\forall x, y \in A)\big(Y(x) =_0 Y(y) \to x =_{\mathbb{R}} y\big).$$

- A set $A \subset \mathbb{R}$ is *strongly countable* if there is $Y : \mathbb{R} \to \mathbb{N}$ that is injective and surjective on A; the latter means that $(\forall n \in \mathbb{N})(\exists x \in A)(Y(x) = n)$.
- A set $A \subset \mathbb{R}$ is *finite* in case there is $N \in \mathbb{N}$ such that for any finite sequence $(x_0, \dots, x_N)$, there is $i \leq N$ with $x_i \notin A$. We sometimes write '$|A| \leq N$'.

Thirdly, we list the following second-order system needed below.

Principle 1.3 (finite-Σ^1_1-AC_0, [26]). *The system* RCA_0 *plus for any arithmetical* φ*:*

$$(\forall n \in \mathbb{N})(\exists \textit{ nonzero finitely many } X \subset \mathbb{N})\varphi(n, X) \to (\exists (X_n)_{n\in\mathbb{N}})(\forall n \in \mathbb{N})\varphi(n, X_n),$$

where '$(\exists$ nonzero finitely many $X \subset \mathbb{N})\varphi(n, X)$' means that there is a non-empty sequence $(X_0, \dots, X_k)$ such that for any $X \subset \mathbb{N}$, $\varphi(n, X) \leftrightarrow (\exists i \leq k)(X_i = X)$.

We let height-Σ^1_1-AC_0 be finite-Σ^1_1-AC_0 where we additionally assume $g \in \mathbb{N}^{\mathbb{N}}$ to be given such that for all n, $g(n) \geq k + 1$ where $k + 1$ is the length of the sequence $(X_0, \dots, X_k)$ in the formula '$(\exists$ nonzero finitely many $X \subset \mathbb{N})\varphi(n, X)$'. We have the following straightforward connections:

$$\Sigma^1_1\text{-}\mathsf{AC}_0 \to \text{finite-}\Sigma^1_1\text{-}\mathsf{AC}_0 \to \text{height-}\Sigma^1_1\text{-}\mathsf{AC}_0 \to \text{weak-}\Sigma^1_1\text{-}\mathsf{AC}_0,$$

i.e., height-Σ^1_1-AC_0 is also a system of hyperarithmetical analysis by Section 1.3. In the grand scheme of things, g is a *height function*, a notion that goes back to Borel [8, 10] and is studied in RM in [61, 67].

1.3. On hyperarithmetical analysis

Going back to Kreisel [37], the notion of *hyperarithmetical set* (see e.g. [71, VIII.3]) gives rise to the second-order definition of *theory/theorem of hyperarithmetical analysis* (THA for brevity, see e.g. [4]). In this section, we recall known results regarding THAs, including the exact (rather technical) definition, for completeness.

First of all, well-known THAs are Σ^1_1-CA_0 and weak-Σ^1_1-CA_0 (see [71, VII.6.1 and VIII.4.12]), where the latter is the former with the antecedent restricted to unique existence. Any system *between* two THAs is *also* a THA, which is a convenient way of establishing that a given system is a THA.

Secondly, at the higher-order level, $\mathsf{ACA}^\omega_0 + \mathsf{QF}\text{-}\mathsf{AC}^{0,1}$ from Section 1.2 is a conservative extension of Σ^1_1-CA_0 by [30, Cor. 2.7]. This is established by extending any model $\mathcal{M}$ of Σ^1_1-AC_0 to a model $\mathcal{N}$ of $\mathsf{ACA}^\omega_0 + \mathsf{QF}\text{-}\mathsf{AC}^{0,1}$, where the second-order part of $\mathcal{N}$ is isomorphic to $\mathcal{M}$. In this paper, we study (higher-order) systems that imply weak-Σ^1_1-CA_0

and are implied by $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$. In light of the aforementioned conservation result, it is reasonable to refer to such intermediate third-order systems as *existing in the range of hyperarithmetical analysis*.

Thirdly, finding a *natural* THA, i.e., hailing from the mathematical mainstream, is surprisingly hard. Montalbán's INDEC from [46], a special case of Jullien's [34, IV.3.3], is generally considered to be the first such statement. The latter theorem by Jullien can be found in [25, 6.3.4 (3)] and [59, Lem. 10.3]. The monographs [25, 34, 59] are all 'rather logical' in nature and INDEC is the *restriction* of a higher-order statement to countable linear orders in the sense of RM [71, V.1.1], i.e., such orders are given by sequences. In [53, Rem. 2.8] and [63, Rem. 7 and §3.4], a number of third-order statements are identified, including the Bolzano–Weierstrass theorem and König's infinity lemma, that are in the range of hyperarithmetical analysis. Shore and others have studied a considerable number of THAs from graph theory [5, 26, 70]. A related concept is that of *almost theorem/theory of hyperarithmetical analysis* (ATHA for brevity, [4]), which is weaker than ACA_0 but becomes a THA when combined with the latter.

Finally, we consider the official definition of THA from [46] based on ω-models.

Definition 1.4. A system T of axioms of second-order arithmetic is a *theory/theorem of hyperarithmetical analysis* in case

- T holds in $\mathsf{HYP}(Y)$ for every $Y \subset \omega$, where $\mathsf{HYP}(Y)$ is the ω-model consisting of all sets hyperarithmetic in Y,
- all ω-models of T are hyperarithmetically closed.

Here, an ω-model is *hyperarithmetically closed* if it is closed under disjoint union and for every set $X, Y \subset \omega$, if X is hyperarithmetically reducible to Y and Y is in the model, then X is in the model too. In turn, this notion of reducibility means that $n \in X$ can be expressed by a Δ_1^1-formula with Y as a parameter; we refer to [46, Thm. 1.14] for equivalent formulations.

2. Metric spaces

We introduce the well-known definition of metric space (M, d) to be used in this paper (Section 2.1), where we always assume M to be a subset of $\mathbb{R}$, up to coding of finite sequences. We show that basic properties of (Lipschitz) continuous functions on such metric spaces exist in the range of hyperarithmetical analysis (Section 2.2), even if we restrict to arithmetically defined objects (Theorem 2.8). We have previously studied metric spaces in [66]; to our own surprise, some of these results have nice generalisations relevant to the study of hyperarithmetical analysis.

2.1. Basic definitions

We shall study metric spaces (M, d) as in Definition 2.1. We assume that M comes with its own equivalence relation '$=_M$' and that the metric d satisfies the axiom of extensionality relative to '$=_M$' as follows:

$$(\forall x, y, v, w \in M)\big([x =_M y \wedge v =_M w] \rightarrow d(x, v) =_{\mathbb{R}} d(y, w)\big).$$

Similarly to functions on the reals, '$F : M \rightarrow \mathbb{R}$' denotes a function from M to the reals that satisfies the following instance of the axiom of function extensionality:

$$(\forall x, y \in M)\big(x =_M y \rightarrow F(x) =_{\mathbb{R}} F(y)\big). \tag{E_M}$$

We recall that the study of metric space in second-order RM is at its core based on equivalence relations, as discussed explicitly in e.g. [71, I.4] or [24, §10.1].

Definition 2.1. A functional $d : M^2 \rightarrow \mathbb{R}$ is a *metric on M* if it satisfies the following properties for $x, y, z \in M$:

(a) $d(x, y) =_{\mathbb{R}} 0 \leftrightarrow x =_M y$,

(b) $0 \leq_{\mathbb{R}} d(x, y) =_{\mathbb{R}} d(y, x)$,

(c) $d(x, y) \leq_{\mathbb{R}} d(x, z) + d(z, y)$.

We shall only study metric spaces (M, d) with $M \subset \mathbb{N}^{\mathbb{N}}$ or $M \subset \mathbb{R}$. To be absolutely clear, quantifying over M amounts to quantifying over $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{R}$, perhaps modulo coding of finite sequences, i.e., the previous definition can be made in third-order arithmetic for the intents and purposes of this paper. Since we shall study compact metric spaces, this restriction is minimal in light of [17, Thm. 3.13].

Sub-sets of M are defined via characteristic functions, like for the reals in Definition 1.2, keeping in mind (E_M). In particular, we use standard notation like $B_d^M(x, r)$ to denote the open ball $\{y \in M : d(x, y) <_{\mathbb{R}} r\}$.

Secondly, the following definitions are now standard, where we note that a different nomenclature is sometimes used in second-order RM. A sequence $(w_n)_{n\in\mathbb{N}}$ in (M, d) is *Cauchy* if $(\forall k \in \mathbb{N})(\exists N \in \mathbb{N})(\forall m, n \geq N)(d(w_n, w_m) < \frac{1}{2^k})$.

Definition 2.2 (Compactness and around). For a metric space (M, d), we say that

- (M, d) is *weakly countably-compact* if for any $(a_n)_{n\in\mathbb{N}}$ in M and sequence of rationals $(r_n)_{n\in\mathbb{N}}$ such that we have $M \subset \cup_{n\in\mathbb{N}} B_d^M(a_n, r_n)$, there is $m \in \mathbb{N}$ such that $M \subset \cup_{n\leq m} B_d^M(a_n, r_n)$,
- (M, d) is *countably-compact* if for any sequence $(O_n)_{n\in\mathbb{N}}$ of open sets in M such that $M \subset \cup_{n\in\mathbb{N}} O_n$, there is $m \in \mathbb{N}$ such that $M \subset \cup_{n\leq m} O_n$,
- (M, d) is *compact* in case for any $\Psi : M \rightarrow \mathbb{R}^+$, there are $x_0, \dots, x_k \in M$ such that $\cup_{i\leq k} B_d^M(x_i, \Psi(x_i))$ covers M,
- (M, d) is *sequentially compact* if any sequence has a convergent sub-sequence,
- (M, d) is *limit point compact* if any infinite set in M has a limit point,
- (M, d) is *complete* in case every Cauchy sequence converges,
- (M, d) is *totally bounded* if for all $k \in \mathbb{N}$, there are $w_0, \dots, w_m \in M$ such that $\cup_{i\leq m} B_d^M(w_i, \frac{1}{2^k})$ covers M.

- a function $f : M \to \mathbb{R}$ is *topologically continuous* if for any open $V \subset \mathbb{R}$, the set $f^{-1}(V) = \{x \in M : f(x) \in V\}$ is also open.
- a function $f : M \to \mathbb{R}$ is *closed* if for any closed $C \subset M$, we have that $f(C)$ is also closed [35,41,49,69].

Regarding the final item, the set $f(C)$ does not necessarily exist in ACA_0^ω, but '$f(C)$ is closed' makes sense[1] as shorthand for the associated well-known definition. We could study other notions, e.g., the Lindelöf property or compactness based on nets, but have opted to stick to basic constructs already studied in second-order RM.

Finally, fragments of the induction axiom are sometimes used, even in an essential way, in second-order RM (see e.g. [4, 50]). The equivalence between induction and bounded comprehension is also well-known in second-order RM [71, X.4.4]. We shall need a little bit of the induction axiom as follows.

Principle 2.3 (IND_2). *Let* Y^2 *satisfy* $(\forall n \in \mathbb{N})(\exists f \in 2^{\mathbb{N}})[Y(n, f) = 0]$. *Then*

$$(\forall n \in \mathbb{N})(\exists w^{1^*})\big[|w| = n \wedge (\forall i < n)\big(Y(i, w(i)) = 0\big)\big].$$

We let IND_0 and IND_1 be IND_2 with '$(\exists f \in 2^{\mathbb{N}})$' restricted to respectively '($\exists$ at most one $f \in 2^{\mathbb{N}}$)' and '$(\exists! f \in 2^{\mathbb{N}})$'. We have previously used IND_i for $i = 0, 1, 2$ in the RM of the Jordan decomposition theorem [53]. By the proof of [53, Thm. 2.16], $\mathsf{Z}_2^\omega + \mathsf{IND}_2$ cannot prove the uncountability of the reals formulated as: *the unit interval is not strongly countable*.

2.2. Metric spaces and hyperarithmetical analysis

2.2.1. Introduction. In this section, we identify a number of the basic properties of metric spaces in the range of hyperarithmetical analysis, as listed on the next page. The Axiom of Choice for finite sets as in Principle 2.4 naturally comes to the fore. Clearly, the principle $\mathsf{Finite\ Choice}$ implies finite-Σ_1^1-AC_0 over ACA_0^ω.

Principle 2.4 ($\mathsf{Finite\ Choice}$). *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of non-empty finite sets in* $[0, 1]$. *Then there is* $(x_n)_{n\in\mathbb{N}}$ *such that* $x_n \in X_n$ *for all* $n \in \mathbb{N}$.

In more detail, we will establish that the following theorems are intermediate between $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ and $\mathsf{ACA}_0^\omega + \mathsf{Finite\ Choice}$.

- Basic properties of continuous functions on *sequentially* compact metric spaces (Section 2.2.2).
- Basic properties of *sequentially* continuous functions on (countably) compact metric spaces (Section 2.2.3).
- Restrictions of the previous results to arithmetically defined or *Lipschitz* continuous functions (Section 2.2.4).

[1] In particular, '$y \in f(C)$' means '$(\exists x \in C)(f(x) = y)$' and '$f(C)$ is closed' means '$(\forall y \notin f(C))(\exists N \in \mathbb{N})(\forall z \in B(z, \frac{1}{2^N}))(z \notin f(C))$', as expected.

- Basic properties of *connected* metric spaces, including the generalisation of the *intermediate value theorem* (Section 2.2.5).

We sometimes obtain elegant equivalences, like for the intermediate value theorem (Theorem 2.13). We believe there is no ‘universal’ approach to the previous results: each section is based on a very particular kind of metric space.

2.2.2. Sequentially compact spaces. In this section, we establish that basic properties of sequentially compact spaces inhabit the range of hyperarithmetical analysis. The following theorem is our first result, to be refined below.

Theorem 2.5 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_2$). *The principle* Finite Choice *follows from any of the items (a)–(j) where (M, d) is any metric space with $M \subset \mathbb{R}$; the principle* $\mathsf{QF\text{-}AC}^{0,1}$ *implies items (a)–(i).*

(a) *For sequentially compact (M, d), any continuous $f : M \to \mathbb{R}$ is bounded.*

(b) *The previous item with ‘is bounded’ replaced by ‘is uniformly continuous’.*

(c) *For sequentially compact (M, d) and continuous $f : M \to \mathbb{R}$ with $\inf_{x\in M} f(x) = y \in \mathbb{R}$ given, there is $x \in M$ with $f(x) = y$.*

(d) (Dini) *Let (M, d) be sequentially compact and let $f_n : (M \times \mathbb{N}) \to \mathbb{R}$ be a monotone sequence of continuous functions converging to continuous $f : M \to \mathbb{R}$. Then the convergence is uniform.*

(e) *For a sequentially compact metric space (M, d), equicontinuity implies uniform equicontinuity* [43, Prop. 4.25].

(f) *For a sequentially compact metric space (M, d) with $M \subset [0, 1]$ infinite, there is a discontinuous function $f : M \to \mathbb{R}$.*

(g) (Closed map lemma, [39, 41, 44, 49]) *For a sequentially compact metric space (M, d) any continuous function $f : M \to \mathbb{R}$ is closed.*

(h) *For sequentially compact (M, d) and disjoint closed $C, D \subset M$, $d(C, D) > 0$.*

(i) (Weak Cantor intersection theorem) *For a sequentially compact metric space (M, d) and a sequence of closed sets with $M \supseteq C_n \supseteq C_{n+1} \neq \emptyset$, such that* $\lim_{n\to\infty} \mathsf{diam}(C_n) = 0$, *there is a unique $w \in \bigcap_{n\in\mathbb{N}} C_n$.*

(j) (Ascoli–Arzelà) *For sequentially compact (M, d), a uniformly bounded equicontinuous sequence of functions on M has a uniformly convergent sub-sequence.*

The theorem still goes through if we require a modulus of continuity in item (a) or if we replace ‘continuity’ by ‘topological continuity’ in items (a)–(f).

Proof. We first derive Finite Choice from item (a) via a proof-by-contradiction. To this end, fix a sequence of non-empty finite sets of reals $(X_n)_{n\in\mathbb{N}}$. Suppose there is no sequence $(x_n)_{n\in\mathbb{N}}$ of reals such that $x_n \in X_n$ for all $n \in \mathbb{N}$. We now define

$$M_0 := \{w^{1^*} : (\forall i < |w|)(w(i) \in X_i)\}, \tag{2.1}$$

where w^{1^*} is a finite sequence of reals of length $|w|$, readily coded using $(\exists^2)$. We define the equivalence relation '$=_{M_0}$' as follows: the relation $w =_{M_0} v$ holds if $|w| =_0 |v|$, where $w, v \in M_0$. The metric $d_0 : M_0^2 \to \mathbb{R}$ is defined as $d_0(w, v) = |\frac{1}{2^{|v|}} - \frac{1}{2^{|w|}}|$ for any $w, v \in M_0$. We then have $d_0(v, w) =_{\mathbb{R}} 0 \leftrightarrow |v| =_0 |w| \leftrightarrow v =_{M_0} w$ as required. We also have $0 \leq d_0(v, w) =_{\mathbb{R}} d_0(w, v)$ for any $v, w \in M_0$, while for any $z \in M_0$ we observe:

$$\begin{aligned} d_0(v, w) &= \left|\tfrac{1}{2^{|v|}} - \tfrac{1}{2^{|w|}}\right| = \left|\tfrac{1}{2^{|v|}} - \tfrac{1}{2^{|z|}} + \tfrac{1}{2^{|z|}} - \tfrac{1}{2^{|w|}}\right| \leq \left|\tfrac{1}{2^{|v|}} - \tfrac{1}{2^{|z|}}\right| + \left|\tfrac{1}{2^{|z|}} - \tfrac{1}{2^{|w|}}\right| \\ &= d_0(v, z) + d_0(z, w) \end{aligned}$$

by the triangle equality of the absolute value on the reals. Hence, (M_0, d_0) is a metric space as in Definition 2.1.

To show that (M_0, d_0) is sequentially compact, let $(w_n)_{n\in\mathbb{N}}$ be a sequence in M_0 and consider the following case distinction. In case $(\forall n \in \mathbb{N})(|w_n| < m_0)$ for some fixed $m_0 \in \mathbb{N}$, then $(w_n)_{n\in\mathbb{N}}$ contains at most $(m_0 + 1)!$ different elements. The pigeon hole principle now implies that at least one w_{n_0} occurs infinitely often in $(w_n)_{n\in\mathbb{N}}$, i.e., $(w_{n_0})_{n\in\mathbb{N}}$ is a convergent sub-sequence. In case $(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})(|w_n| \geq m)$, the sequence $(w_n)_{n\in\mathbb{N}}$ yields a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in X_n$ for all $n \in \mathbb{N}$, which is impossible by assumption. Hence, (M_0, d_0) is a sequentially compact metric space.

Next, define $f : M_0 \to \mathbb{R}$ as $f(w) := |w|$, which is clearly not bounded on M_0, which one shows using IND_2. To show that f is continuous at $w_0 \in M_0$, consider the formula $|\frac{1}{2^{|w_0|}} - \frac{1}{2^{|v|}}| = d_0(v, w_0) < \frac{1}{2^N}$; the latter is false for $N \geq |w_0| + 2$ and any $v \neq_{M_0} w_0$. Hence, the following formula is vacuously true:

$$(\forall k \in \mathbb{N})(\exists N \in \mathbb{N})\big(\forall v \in B_{d_0}^{M_0}(w_0, \tfrac{1}{2^N})\big)\big(\big|f(w_0) - f(v)\big| <_{\mathbb{R}} \tfrac{1}{2^k}\big), \tag{2.2}$$

i.e., f is continuous at $w_0 \in M_0$, with a modulus of continuity given by $h(w, k) := \frac{1}{2^{|w|+k+2}}$. To see that f is also topologically continuous, fix an open set $V \subset \mathbb{R}$ and fix $w_0 \in f^{-1}(V)$. Then for $N_0 := |w_0| + 2$, one verifies that $B_d^{M_0}(w_0, \frac{1}{2^{N_0}}) \subset f^{-1}(V)$, i.e., $f^{-1}(V)$ is open. Thus, $f : M_0 \to \mathbb{R}$ is a continuous but unbounded function on a sequentially compact metric space (M_0, d_0), contradicting item (a). Item (b) also implies $\mathsf{Finite\ Choice}$ as f is not uniformly continuous. For item (c), $g : M_0 \to \mathbb{R}$ defined as $g(w) := \frac{1}{2^{|w|}}$ is continuous in the same way as for f. However, using IND_2, the infimum of g on M_0 is 0, but there is no $w \in M_0$ with $g(w) =_{\mathbb{R}} 0$, by definition. Hence, item (c) also implies $\mathsf{Finite\ Choice}$.

Now assume item (d) and suppose $\mathsf{Finite\ Choice}$ is again false; letting $(X_n)_{n\in\mathbb{N}}$ and (M_0, d_0) be as in the previous paragraph, we define $f_n : (\mathbb{N} \times M_0) \to \mathbb{R}$ as:

$$f_n(w) := \begin{cases} |w| & \text{if } |w| \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Clearly, $\lim_{n\to\infty} f_n(w) = f(w)$ and $f_n(w) \leq f_{n+1}(w)$ for $w \in M_0$; f_n is continuous in the same way as for f. Item (d) implies that the convergence is *uniform*, i.e.,

$$(\forall k \in \mathbb{N})(\exists N \in \mathbb{N})(\forall w \in M_0)(\forall n \geq N)\big(\big|f_n(w) - f(w)\big| < \tfrac{1}{2^k}\big), \tag{2.4}$$

which yields a contradiction by letting $N_1 \in \mathbb{N}$ be as in (2.4) for $k = 1$ and choosing $w_1 \in M$ of length $N_1 + 1$ using IND_2. One derives Finite Choice from item (e) in the same way.

Next, regarding item (g), suppose Finite Choice is false and consider again (M_0, d_0). Define the continuous function $f : M_0 \to \mathbb{R}$ by $f(w) = q_{|w|}$ where $(q_n)_{n\in\mathbb{N}}$ is an enumeration of the rationals without repetitions. Using IND_2, we have $f(M_0) = \mathbb{Q}$ and the latter is not closed while M_0 is, contradicting item (g), and Finite Choice must hold. To obtain the latter from item (f), note that (M_0, d_0) is infinite (using IND_2) while all functions $f : M_0 \to \mathbb{R}$ are continuous as (2.2) is vacuously true. Regarding item (j), assuming again that Finite Choice is false, the sequence $(f_n)_{n\in\mathbb{N}}$ as in (2.3) is equi-continuous:

$$(\forall k \in \mathbb{N},\ w \in M_0)(\exists N \in \mathbb{N})\big(\forall v \in B^{M_0}_{d_0}(w, \tfrac{1}{2^N})\big)(\forall n \in \mathbb{N})\big(\big|f_n(w) - f_n(v)\big| <_{\mathbb{R}} \tfrac{1}{2^k}\big),$$

which (vacuously) holds in the same way as for (2.2). However, as for item (d), uniform convergence (of a sub-sequence) is false, i.e., item (j) also implies Finite Choice. For item (i), suppose Finite Choice is false, define $C_n := \{w \in M_0 : |w| > n + 1\}$, and verify that this closed and non-empty set has diameter at most $\frac{1}{2^n}$, using IND_2. Since $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$, we obtain Finite Choice from item (i). For item (h), suppose Finite Choice is false, and define $C = \{w \in M_0 : |w| \text{ is odd}\}$ and $D = \{w \in M_0 : |w| \text{ is even}\}$. One readily verifies that $C \cap D = \emptyset$, C, D are closed, and $d(C, D) = 0$.

To establish the items in the theorem in $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$, the usual proof-by-contradiction goes through. A proof sketch of item (a) as follows: let (M, d) be as in the latter and suppose $f : M \to \mathbb{R}$ is continuous and unbounded, i.e.,

$$(\forall n \in \mathbb{N})(\exists x \in M)\big(\big|f(x)\big| > n\big).$$

Since $M \subset \mathbb{R}$ and real numbers are represented by elements of Baire space, we may apply $\mathsf{QF\text{-}AC}^{0,1}$ to obtain $(x_n)_{n\in\mathbb{N}}$ in M such that $|f(x_n)| > n$ for all $n \in \mathbb{N}$. Since M is sequentially compact, $(x_n)_{n\in\mathbb{N}}$ has a convergent sub-sequence, say with limit $y \in M$. Clearly, f is not continuous at $y \in M$, a contradiction. To obtain (f), apply $\mathsf{QF\text{-}AC}^{0,1}$ to the statement that $M \subset [0, 1]$ is infinite, yielding a sequence $(w_n)_{n\in\mathbb{N}}$ in M. Now define $f : M \to \mathbb{R}$ as $f(x_n) = n$ and $f(y) = 0$ for $y \neq x_m$ for all $m \in \mathbb{N}$. Since f is unbounded on M, it is discontinuous by item (a). Most other items are established using $\mathsf{QF\text{-}AC}^{0,1}$ in the same way.

We also sketch how $\mathsf{QF\text{-}AC}^{0,1}$ implies item (g). To this end, let f, M be as in the closed map lemma and suppose $f(C)$ is not closed for closed $C \subset M$. Hence, there is $y_0 \notin f(C)$ such that $(\forall k \in \mathbb{N})(\exists y \in B(y_0, \frac{1}{2^k}))(y \in f(C))$. By definition, the latter formula means

$$(\forall k \in \mathbb{N})(\exists x \in C)\big(\big|f(x) - y_0\big| < \tfrac{1}{2^k}\big).$$

Apply $\mathsf{QF\text{-}AC}^{0,1}$ to obtain a sequence $(x_n)_{n\in\mathbb{N}}$ in C with $(\forall k \in \mathbb{N})(|f(x_k) - y_0| < \frac{1}{2^k})$. By sequential compactness, there is a convergent sub-sequence $(z_n)_{n\in\mathbb{N}}$, say with limit z. Since C is closed, we have $z \in C$ and since f is continuous (and hence sequentially continuous) also $f(z) = y_0$. This contradicts $y_0 \notin f(C)$ and the closed map lemma therefore follows from $\mathsf{QF\text{-}AC}^{0,1}$. ∎

The final part of the proof also goes through if f is only usco (see Def. 4.1). As to other generalisations of Theorem 2.5, the latter still goes through for 'continuity' replaced by 'absolute differentiability' from [18] formulated[2] appropriately.

Finally, we observe that (M_0, d_0) from (2.1) is not (countably) compact, i.e., we need a slightly different approach for the latter, to be found in the next section.

2.2.3. Compact spaces. In this section, we establish that basic properties of (countably) compact spaces inhabit the range of hyperarithmetical analysis.

First of all, the following theorem is a version of Theorem 2.5 for (countably) compact spaces and sequential continuity. We seem to (only) need sequential compactness to guarantee that everything remains provable in $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$.

Theorem 2.6 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_2$). *The principle* Finite Choice *follows from any of the items (a)–(d) where (M, d) is any metric space with $M \subset \mathbb{R}$; the principle* $\mathsf{QF\text{-}AC}^{0,1}$ *implies all these items.*

(a) *For (weakly) countably-compact and sequentially compact (M, d), any sequentially continuous $f : M \to \mathbb{R}$ is bounded.*

(b) *The previous item with 'is bounded' replaced by 'is (uniformly) continuous'.*

(c) *For a (weakly) countably-compact (M, d) that is infinite, there is $f : M \to \mathbb{R}$ that is not sequentially continuous.*

(d) *The first item with '(weakly) countably-compact' replaced by 'compact' or 'complete and totally bounded'.*

Proof. We first derive Finite Choice from item (a) via a proof-by-contradiction. To this end, fix a sequence of non-empty finite sets of reals $(X_n)_{n\in\mathbb{N}}$. Suppose there is no sequence $(x_n)_{n\in\mathbb{N}}$ of reals such that $x_n \in X_n$ for all $n \in \mathbb{N}$ and recall M_0 from (2.1). Now define $M_1 = M_0 \cup \{0_{M_1}\}$ where 0_{M_1} is a new symbol such that $w \neq_{M_1} 0$ for $w \in M_0$ and '$=_{M_1}$' is '$=_{M_0}$' otherwise. Define $d_1 : M_1^2 \to \mathbb{R}$ as d_0 on M_0, as $d_1(w, 0_{M_1}) := d(0_{M_1}, w) = \frac{1}{2^{|w|}}$ for $w \in M_0$, and $d_1(0_{M_1}, 0_{M_1}) = 0$. Then (M_1, d_1) is a metric, which is shown in the same way as for (M_0, d_0).

To show that (M_1, d_1) is countably-compact, let $(O_n)_{n\in\mathbb{N}}$ be an open cover of M_1 and suppose $n_1 \in \mathbb{N}$ is such that $0_{M_1} \in O_{n_1}$. By definition, there is $N_1 \in \mathbb{N}$ such that $B_{d_1}^{M_1}(0_{M_1}, \frac{1}{2^{N_1}}) \subset O_{n_0}$, i.e., $d(0_{M_1}, w) = \frac{1}{2^{|w|}} < \frac{1}{2^{N_1}}$ implies $w \in O_{n_1}$ for $w \in M_0$. Now use IND_2 to enumerate the finitely many $v \in M_0$ such that $|v| \leq N_1$. This finite sequence is covered by some $\cup_{n\leq n_2} O_n$, i.e., we have obtained a finite sub-covering of M_1, namely $\cup_{n\leq \max(n_1,n_2)} O_n$. Moreover, (M_1, d_1) is sequentially compact, which can be proved via the same case distinction as for (M_0, d_0) in the proof of Theorem 2.5.

[2] The correct formulation based on [18] is that '$f : M \to \mathbb{R}$ is (absolutely) differentiable on the metric space (M, d)' in case we have

$$(\forall k \in \mathbb{N},\ p \in M)(\exists N \in \mathbb{N})(\forall x, y \in M)\big(0 < d(x,p), d(y,p) < \tfrac{1}{2^N} \to \big|\tfrac{|f(x)-f(p)|}{d(x,p)} - \tfrac{|f(y)-f(p)|}{d(y,p)}\big| < \tfrac{1}{2^k}\big),$$

which is the 'epsilon-delta' definition formulated to avoid the existence of the derivative.

Next, define $g : M_1 \to \mathbb{R}$ as $g(w) := |w|$ for $w \in M_0$ and $g(0_{M_1}) =_{\mathbb{R}} 0$, which is clearly not bounded on M_1; this follows again via IND_2. Then g is continuous at $w_0 \in M_0$ in the same way as f from the proof of Theorem 2.5, namely since (2.2) is vacuously true. To show that g is sequentially continuous at 0_{M_1}, let $(w_n)_{n\in\mathbb{N}}$ be a sequence converging to 0_{M_1}. In case this sequence is eventually constant 0_{M_1}, clearly $g(0_M) = \lim_{n\to\infty} g(w_n)$ as required. In case $(w_n)_{n\in\mathbb{N}}$ is not eventually constant 0_{M_1}, the convergence to 0_{M_1} in the d_1-metric implies that for any $n \in \mathbb{N}$, there is $m \geq n$ with $|w_m| > n$. Thus, $(w_n)_{n\in\mathbb{N}}$ yields a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in X_n$ for all $n \in \mathbb{N}$, which contradicts our assumptions, i.e., this case cannot occur. As a result, $g : M_1 \to \mathbb{R}$ is sequentially continuous. Since, it is also unbounded (thanks to IND_2), we obtain a contradiction with item (a). Thus, (a) implies Finite Choice, and the same for item (b). To obtain Finite Choice from item (c), note that (M_1, d_1) is infinite (using IND_2) while *all* functions $f : M_1 \to \mathbb{R}$ are *sequentially* continuous by the previous.

To show that (M_1, d_1) satisfies the properties in item (d), note that for $\Psi : M_1 \to \mathbb{R}^+$, the ball $B_{d_1}^{M_1}(0_{M_1}, \Psi(0_{M_1}))$ covers all but finitely many points of M_1 (in the same way as O_{n_0} from the second paragraph of the proof). Hence, (M_1, d_1) is compact, and totally boundedness follows in exactly the same way. For completeness, let $(w_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in M_1, i.e., we have

$$(\forall k \in \mathbb{N})(\exists N \in \mathbb{N})(\forall n, m \geq N)\big(d_1(w_n, w_m) < \tfrac{1}{2^k}\big).$$

As above, $(w_n)_{n\in\mathbb{N}}$ is either eventually constant or provides a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in X_n$ for all $n \in \mathbb{N}$. The latter case is impossible by assumption and the former case is trivial.

To establish the items in the theorem in $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$, the usual proof-by-contradiction goes through as in the proof of Theorem 2.5. ∎

We believe that we cannot use epsilon-delta or topological continuity in the previous theorem. Nonetheless, we have the following corollary that makes use of the sequential[3] definition of uniform continuity.

Corollary 2.7. *Over* $\mathsf{ACA}_0^\omega + \mathsf{IND}_2$, *items* (a)–(f) *in Theorem* 2.5 *and items* (a)–(d) *in Theorem* 2.6 *are intermediate between* $\mathsf{QF\text{-}AC}^{0,1}$ *and* Finite Choice *if we replace 'continuity' by 'sequential uniform continuity'.*

Proof. The usual proof-by-contradiction using $\mathsf{QF\text{-}AC}^{0,1}$ (and $(\exists^2)$) shows that sequential uniform continuity implies uniform continuity. For the remaining implications, consider $g : M_1 \to \mathbb{R}$ from the proof of Theorem 2.6. This function is sequentially continuous at 0_{M_1} since any sequence converging to 0_{M_1} must be eventually constant 0_{M_1}. Similarly, for sequences $(w_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ in M_1 $\lim_{n\to\infty} d_1(w_n, v_n) = 0$ implies that the sequences are eventually equal. Hence, g is also sequentially uniformly continuous. A similar proof goes through for (M_0, d_0) and f from Theorem 2.5. ∎

[3] A function $f : M \to \mathbb{R}$ is called *sequentially uniformly continuous* if for any sequences $(w_n)_{n\in\mathbb{N}}$, $(v_n)_{m\in\mathbb{N}}$ in M such that $\lim_{n\to\infty} d(w_n, v_n) = 0$, we have $\lim_{n\to\infty} |f(w_n) - f(v_n)| = 0$.

We have identified a number of basic properties of continuous functions on compact metric spaces that exist in the range of hyperarithmetical analysis. A number of restrictions and variations are possible, which is the topic of the next section.

2.2.4. Restrictions. We show that some the above principles still inhabit the range of hyperarithmetical analysis if we restrict to arithmetically defined objects or Lipschitz continuity.

First of all, the following theorem establishes that Theorem 2.5 holds if we restrict to arithmetically defined objects.

Theorem 2.8. *Item (a) from Theorem 2.5 still implies* weak-Σ^1_1-AC_0 *over* $\mathsf{ACA}_0^\omega + \mathsf{IND}_1$ *if all objects in the former item have an arithmetical definition.*

Proof. In a nutshell, we can modify the above proofs to obtain (only) weak-Σ^1_1-AC_0 while all relevant objects can be defined using $(\exists^2)$. To this end, let φ be arithmetical and such that $(\forall n \in \mathbb{N})(\exists! X \subset \mathbb{N})\varphi(n, X)$, but there is no sequence $(X_n)_{n\in\mathbb{N}}$ with $(\forall n \in \mathbb{N})$ $\varphi(n, X_n)$. Use $\exists^2$ to define $\eta : [0, 1] \to (2^{\mathbb{N}} \times 2^{\mathbb{N}})$ such that $\eta(x) = (f, g)$ outputs the binary expansions of x, taking $f = g$ if there is only one. Define the following set using $\exists^2$:

$$A := \big\{x \in [0, 1] : (\exists m \in \mathbb{N})\big(\varphi\big(m, \eta(x)(1)\big) \vee \varphi\big(m, \eta(x)(2)\big)\big)\big\}$$

and $Y(x) := (\mu m)[\varphi(m, \eta(x)(1) \vee \varphi(m, \eta(x)(2)))]$. Then Y is injective and surjective on A. In particular $A_m := \{x \in [0, 1] : \varphi(m, \eta(x)(1)) \vee \varphi(m, \eta(x)(2))\}$ contains exactly one element by definition. Using $X_n = A_n$, the metric space (M_0, d_0) as in (2.1) in the proof of Theorem 2.5 now has an arithmetical definition. The same holds for the function $F : M_0 \to \mathbb{R}$ where $F(w) := |w|$. The rest of the proof of item (a) of Theorem 2.5 now goes through, using IND_1 instead of IND_2 where relevant, yielding in particular a contradiction. Hence, there must be a sequence $(X_n)_{n\in\mathbb{N}}$ with $(\forall n \in \mathbb{N})\varphi(n, X_n)$, i.e., weak-$\Sigma^1_1$-$\mathsf{AC}_0$ follows as required. ∎

Secondly, we show that we may replace 'continuity' by 'Lipschitz continuity' in some of the above principles.

Definition 2.9. A function $f : M \to \mathbb{R}$ is *α-Hölder-continuous* in case there exist $M, \alpha > 0$ such that for any $v, w \in M$:

$$\big|f(v) - f(w)\big| \leq M d(v, w)^\alpha,$$

A function is *Lipschitz (continuous)* if it is 1-Hölder-continuous.

Theorem 2.10 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_2$). *The principle* Finite Choice *follows from any of the items (a)–(e) where (M, d) is any metric space with $M \subset \mathbb{R}$; the principle* QF-$\mathsf{AC}^{0,1}$ *implies items (a)–(e).*

(a) *For a metric space (M, d), any sequentially compact $C \subseteq M$ is bounded, i.e., there are $w \in M$, $m \in \mathbb{N}$ with $(\forall v \in C)(d(v, w) \leq m)$ (see* [6, p. 333]).

(b) *For sequentially compact* (M, d)*, any uniformly continuous* $f : M \to \mathbb{R}$ *is bounded.*

(c) *The previous item with 'uniformly' replaced by '*α*-Hölder' or 'Lipschitz'.*

(d) *For sequentially compact* (M, d) *that is infinite, there exists* $f : M \to \mathbb{R}$ *that is bounded but not Lipschitz continuous.*

(e) *For sequentially compact and bounded* (M, d) *and Lipschitz* $f : M \to \mathbb{R}$ *with* $\inf_{x\in M} f(x) = y \in \mathbb{R}$ *given, there is* $x \in M$ *with* $f(x) = y$*.*

Proof. First of all, to derive item (a) from $\textsf{QF-AC}^{0,1}$, fix a metric space (M, d) and let $C \subseteq M$ be sequentially compact. Suppose C is not bounded, i.e., for some fixed $w_0 \in M$, we have $(\forall m \in \mathbb{N})(\exists v \in C)(d(w_0, v) > m)$. Apply $\textsf{QF-AC}^{0,1}$ to obtain a sequence $(v_n)_{n\in\mathbb{N}}$ such that $d(w_0, v_n) > n$ for all $n \in \mathbb{N}$. Clearly, this sequence cannot have a convergent sub-sequence, a contradiction, and C must be bounded. To derive item (d) from $\textsf{QF-AC}^{0,1}$, apply $\textsf{QF-AC}^{0,1}$ to the statement that M is infinite. The resulting sequence $(w_n)_{n\in\mathbb{N}}$ has a convergent sub-sequence, say $(v_n)_{n\in\mathbb{N}}$ with limit v. Define $f(w) = 1$ (resp. $f(w) = -1$) if $v_n = w$ and n is even (resp. odd), and $f(w) = 0$ otherwise. Clearly, $f : M \to \mathbb{R}$ is bounded but not (Lipschitz) continuous. By, Theorem 2.5, the other items follow from $\textsf{QF-AC}^{0,1}$.

Secondly, to derive $\textsf{Finite Choice}$ from item (a), suppose $(X_n)_{n\in\mathbb{N}}$ is a sequence of finite sets such that there is no sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in X_n$ for all $n \in \mathbb{N}$. Recall the set M_0 from (2.1) and define $d_2 : M_0^2 \to \mathbb{R}$ as $d_2(v, w) = ||v| - |w||$ for $v, w \in M_0$. That d_2 is a metric is readily verified: the first and third item of Definition 2.1 hold by definition and the triangle equality of the absolute value; the second item in this definition holds since $d_2(v, w) = 0 \leftrightarrow |u| = |w| \leftrightarrow u =_{M_0} w$. Now, the set $C = \{w \in M_0 : |w| \text{ is even}\}$ is sequentially compact, as every sequence in C either has at most finitely many different members, or yields a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in X_n$ for all $n \in \mathbb{N}$. We have excluded the latter by assumption, while the former trivially yields a convergent sub-sequence. Using $\textsf{IND}_2$, C is however not bounded in (M_0, d_2), a contradiction, and item (a) implies $\textsf{Finite Choice}$.

Thirdly, to derive $\textsf{Finite Choice}$ from the remaining items, let (M_0, d_2) be as above and note that the latter is sequentially compact as in the previous paragraph. Now define $f : M_0 \to \mathbb{R}$ as $f(u) := \frac{|u|}{2}$ and observe that $|f(u) - f(v)| = \frac{1}{2}||u| - |v|| \leq \frac{1}{2}d_2(u, v)$, i.e., f is Lifschitz (and uniformly) continuous. However, $\textsf{IND}_2$ shows that f is not bounded, a contradiction, and items (b)–(c) imply $\textsf{Finite Choice}$. Similarly, item (d) implies $\textsf{Finite Choice}$ as (M_0, d_2) is such that every bounded function $f : M_0 \to \mathbb{R}$ is automatically Lipschitz. Indeed, if $|f(w)| \leq M_0$ for all $w \in M_0$, then the Lipschitz constant for f can be taken to be $2M_0$.

Finally, to derive $\textsf{Finite Choice}$ from item (e), suppose the former is false and consider again (M_0, d_0), which is trivially bounded due to the definition of d_0. Now define $g : M_0 \to \mathbb{R}$ as $g(u) := \frac{1}{2^{|u|+1}}$. This function is Lipschitz on (M_0, d_0) as

$$\big|g(u) - g(v)\big| = \big|\tfrac{1}{2^{|u|+1}} - \tfrac{1}{2^{|v|+1}}\big| = \tfrac{1}{2}\big|\tfrac{1}{2^{|u|}} - \tfrac{1}{2^{|v|}}\big| = \tfrac{1}{2}d(u, v).$$

However, g has infimum equal to zero (using IND_2) but is strictly positive on M_0, contradicting item (e), which establishes the theorem. ∎

In conclusion, many implications between the notions in Definition 2.2 exist in the range of hyperarithmetical analysis, as well as the associated Lebesgue number lemma for countable coverings of open sets. These are left to the reader.

2.2.5. Connectedness. We show that basic properties of connected metric spaces exist in the range of hyperarithmetical analysis, including the intermediate value theorem. We also obtain some elegant equivalences in Theorem 2.13.

First of all, Cantor and Jordan were the first to study connectedness [75], namely as in the first item in Definition 2.11. The connectedness notions from the latter are equivalent for compact metric spaces in light of [44, §4.39] or [56, p. 123].

Definition 2.11 (Connectedness). We define connectedness for metric spaces as follows.

- A metric space (M, d) is *chain connected* in case for any $w, v \in M$ and $\varepsilon > 0$, there is a sequence $w = x_0, x_1, \ldots, x_{n-1}, x_n = v \in M$ such that for all $i < n$ we have $d(x_i, x_{i+1}) < \varepsilon$.
- A metric space (M, d) is *connected* in case M is not the disjoint union of two non-empty open sets.

We shall study the following generalisation of the intermediate value theorem.

Principle 2.12 (Intermediate value theorem). *Let (M, d) be a sequentially compact and chain connected metric space and let $f : M \to \mathbb{R}$ be continuous. If $f(w) < c < f(v)$ for some $v, w \in M$ and $c \in \mathbb{R}$, then there is $u \in M$ with $f(u) = c$.*

The *approximate* intermediate value theorem is the previous principle with the conclusion weakened to 'then for any $\varepsilon > 0$ there is $u \in M$ with $|f(u) - c| < \varepsilon$'. The latter theorem is well-known from constructive mathematics (see e.g. [7, p. 40]).

Theorem 2.13 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_2$). *The principle* Finite Choice *follows from any of the items (a)–(g) where (M, d) is any metric space with $M \subset \mathbb{R}$; the principle* $\mathsf{QF\text{-}AC}^{0,1}$ *implies items (a)–(g).*

(a) *The intermediate value theorem as in Principle 2.12.*

(b) *Principle 2.12 for Lipschitz continuous functions.*

(c) *The approximate intermediate value theorem.*

(d) *Let (M, d) be a sequentially compact and chain connected metric space and let $f : M \to \{0, 1\}$ be continuous. Then f is constant on M.*

(e) *Let (M, d) be a sequentially compact and chain connected metric space and let $f : M \to \mathbb{R}$ be locally constant. Then f is constant on M.*

(f) *Let (M, d) be sequentially compact and chain connected and let $f : M \to \mathbb{R}$ be locally constant and continuous. Then f is constant on M.*

(g) *For a sequentially compact metric space* (M, d)*, chain connectedness implies connectedness.*

(h) *Let* (M, d) *be a sequentially compact and chain connected metric space and let* $f : M \to \mathbb{R}$ *be (Lipschitz) continuous. Then* f *is bounded on* M*.*

(i) *Item (h) with '*f *is bounded' replaced by '*$f(M)$ *is not dense in* $\mathbb{R}$*'.*

(j) *Item (h) with '*f *is bounded' replaced by '*$f(M)$ *is closed'.*

Moreover, items (a), (c), and (d)–(g) are equivalent.

Proof. First of all, we show that item (a) implies Finite Choice. To this end, suppose the latter is false and consider M_0 as in (2.1). Let $(q_n)_{n\in\mathbb{N}}$ be an enumeration of the rationals (without repetitions) and define $d_3 : M_0^2 \to \mathbb{R}$ as follows: $d_3(w, v) := |q_{|w|} - q_{|v|}|$ for $w, v \in M_0$. Then (M_0, d_3) is a sequentially compact metric space, which is proved in the same way as for the previous metrics d_0, d_1, d_2, namely that any sequence in M_0 can have at most finitely many different elements. That (M_0, d_3) is chain connected is proved using IND_2. Indeed, fix $u, w \in M_0, \varepsilon > 0$ and consider $d_3(w, v) = |q_{|w|} - q_{|v|}|$. Let $q_{|w|} = r_0, r_1, \ldots, r_{k-1}, r_k = q_{|v|} \in \mathbb{Q}$ be a finite sequence such that $|r_i - r_{i+1}| < \varepsilon$ for $i < k$. Using IND_2, there are $w_i \in M_0$ such that $q_{|w_i|} = r_i$ for $i < k$, and chain connectedness of M_0 follows.

Now define $f : M_0 \to \mathbb{R}$ by $f(w) = \frac{1}{2}q_{|w|}$, which is (Lipschitz) continuous, essentially by the definition of d_3, as we have:

$$\big|f(w) - f(v)\big| = \big|\tfrac{1}{2}q_{|w|} - \tfrac{1}{2}q_{|v|}\big| = \tfrac{1}{2}|q_{|w|} - q_{|v|}| \leq \tfrac{1}{2}d_3(w, v).$$

However, the range of f consists of rationals, i.e., it does not have the intermediate value property. This contradiction yields Finite Choice. The same proof goes through for items (h)–(j).

Secondly, assume $\mathsf{QF\text{-}AC}^{0,1}$ and let (M, d), $f : M \to \mathbb{R}$, $w, v \in M$, and $c \in \mathbb{R}$ be as in Principle 2.12. Since M is chain connected, we have

$$(\forall k \in \mathbb{N})(\exists z^{1^*})\big(z(0) = w \wedge z\big(|w| - 1\big) = v \wedge \big(\forall i < |z| - 1\big)d\big(z(i), z(i+1)\big) < \tfrac{1}{2^k}\big). \quad (2.5)$$

Apply $\mathsf{QF\text{-}AC}^{0,1}$ to obtain a sequence $(z_k)_{k\in\mathbb{N}}$ of finite sequences. Define a sequence $(t_k)_{k\in\mathbb{N}}$ in M where t_k is the first element t in z_k such that $f(t) \geq c$. By sequential completeness, there is a convergent sub-sequence $(s_k)_{k\in\mathbb{N}}$ with limit $s \in M$. Since f is continuous, we have $\lim_{k\to\infty} f(s_k) = f(s)$ and hence $f(s) = c$.

Thirdly, to show that item (g) implies Finite Choice, again suppose the latter is false and consider (M_0, d_3). By the above, the latter is sequentially compact and chain connected. To show that it is not connected, define

$$O_1 := \{w \in M_0 : q_{|w|} > \pi\} \quad \text{and} \quad O_2 := \{w \in M_0 : q_{|w|} < \pi\},$$

verify that they are open and disjoint, and observe that $M_0 = O_1 \cup O_2$, i.e., item (g) is false. Note also that $f : M_0 \to \mathbb{R}$ defined as $f(w) = 1$ if $w \in O_1$ and 0 otherwise, is continuous but not constant, i.e., item (d) also implies Finite Choice.

To derive item (g) from $\mathsf{QF\text{-}AC}^{0,1}$, let (M, d) be as in the former, i.e., sequentially compact and chain connected. Suppose M is not connected, i.e., $M = O_1 \cup O_2$ where the latter are open, disjoint, and non-empty. Now fix $v \in O_1$ and $w \in O_2$ and consider (2.5). Apply $\mathsf{QF\text{-}AC}^{0,1}$ to obtain a sequence $(z_k)_{k\in\mathbb{N}}$ of finite sequences. Define sequences $(s_k)_{k\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}}$ in M where t_k is the first element t in z_k such that $t \in O_2$ and s_k is the predecessor of t in z_k. By sequential completeness, $(s_k)_{k\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}}$ have convergent sub-sequences, with the same limit by construction. However, if this limit is in O_1, then so is $(t_k)_{k\in\mathbb{N}}$ eventually, a contradiction. Similarly, if this limit is in O_2, then so is $(s_k)_{k\in\mathbb{N}}$ eventually, a contradiction. In each case we obtain a contradiction, i.e., M must be connected, and item (g) follows. The same proof goes through for item (d).

Next, item (g) implies item (e) as in case the latter fails for $f : M \to \mathbb{R}$, say with $f(w) <_{\mathbb{R}} f(v)$, then $O_1 = \{z \in M : f(z) \leq f(w)\}$ and $O_2 = \{z \in M : f(z) > f(w)\}$ are open, disjoint, and non-empty sets such that $M = O_1 \cup O_2$, i.e., item (g) fails too. To show that item (e) implies $\mathsf{Finite\ Choice}$, suppose the latter is false and let (M_0, d_3) be as above. Define $g : M_0 \to \mathbb{R}$ as $g(w) = n$ in case $|q_{|w|}| \in [n\pi, (n+1)\pi]$. Clearly, f is locally constant but not constant, i.e., item (e) is false. To derive item (g) from item (e) (and item (f)), suppose the former is false, i.e., (M, d) is a sequentially compact and chain connected metric space that is not connected. Let $M = O_1 \cup O_2$ be the associated decomposition and note that $f : M \to \mathbb{R}$ defined by $f(w) = 1$ if $w \in O_1$ and 0 otherwise, is locally constant (and continuous) but not constant, i.e., item (e) (and (f)) also fails. The equivalence for item (d) follows in the same way.

To show that item (g) implies item (a), suppose the latter is false for $f : M \to \mathbb{R}$ and $c \in \mathbb{R}$, i.e., $f(w) \neq c$ for all $w \in M$. By assumption, $O_1 := \{w \in M : f(w) < c\}$ and $O_2 := \{w \in M : f(w) > c\}$ are open, disjoint, and non-empty, i.e., item (g) also fails. To show that item (a) (and item (c)) implies item (g), suppose the latter fails for $M = O_1 \cup O_2$, i.e., the latter are open, non-empty, and disjoint. Then $f : M \to \mathbb{R}$ defined by $f(w) = 1$ if $w \in O_1$ and 0 otherwise, is continuous but does not have the (approximate) intermediate value property. ∎

Regarding item (i), we could not find a way of replacing '$f(M)$ is not dense in $\mathbb{R}$' by '$f(M)$ has finite measure'. We could study *local connectedness* and obtain similar results, but feel this section is long enough as is.

In conclusion, we have identified many basic properties of metric spaces that exist in the range of hyperarithmetical analysis. We believe there to be many more such principles in e.g. topology.

3. Functions of bounded variation and around

We introduce functions of bounded variation (Section 3.1) and show that their basic properties exist in the range of hyperarithmetical analysis (Section 3.2). Similar to Theorem 2.8, we could restrict to arithmetically defined functions.

3.1. Bounded variation and variations

The notion of *bounded variation* (often abbreviated BV) was first explicitly[4] introduced by Jordan around 1881 [32] yielding a generalisation of Dirichlet's convergence theorems for Fourier series. Indeed, Dirichlet's convergence results are restricted to functions that are continuous except at a finite number of points, while BV-functions can have infinitely many points of discontinuity, as already studied by Jordan, namely in [32, p. 230]. In this context, the *total variation* $V_a^b(f)$ of $f : [a, b] \to \mathbb{R}$ is defined as:

$$\sup_{a \le x_0 < \cdots < x_n \le b} \sum_{i=0}^{n} \big| f(x_i) - f(x_{i+1}) \big|. \tag{3.1}$$

The following definition provides two ways of defining 'BV-function'. We have mostly studied the first one [53, 61, 67] but will use the second one in this paper.

Definition 3.1. Functions of bounded variation are defined as follows.

(a) The function $f : [a, b] \to \mathbb{R}$ *has bounded variation* on $[a, b]$ if there is $k_0 \in \mathbb{N}$ such that $k_0 \ge \sum_{i=0}^{n} |f(x_i) - f(x_{i+1})|$ for any partition $x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b$.

(b) The function $f : [a, b] \to \mathbb{R}$ *has total variation* $z \in \mathbb{R}$ on $[a, b]$ if $V_a^b(f) = z$.

We recall the 'virtual' or 'comparative' meaning of suprema in RM from e.g. [71, X.1]. In particular, a formula '$\sup A \le b$' is merely shorthand for (essentially) the well-known definition of the supremum.

Secondly, the fundamental theorem about BV-functions is formulated as follows.

Theorem 3.2 (Jordan decomposition theorem, [32, p. 229]). *A* BV*-function* $f : [0, 1] \to \mathbb{R}$ *is the difference of two non-decreasing functions* $g, h : [0, 1] \to \mathbb{R}$.

Theorem 3.2 has been studied via second-order representations in [27, 38, 51, 76]. The same holds for constructive analysis by [12, 13, 29, 57], involving different (but related) constructive enrichments. We have obtained many equivalences for the Jordan decomposition theorem, formulated using item (a) from Definition 3.1 in [53, 67], involving the following principle.

Principle 3.3 (cocode_0). *A countable set* $A \subset [0, 1]$ *can be enumerated.*

This principle is 'explosive' in that $\mathsf{ACA}_0^\omega + \mathsf{cocode}_0$ proves ATR_0 while $\Pi_1^1\text{-}\mathsf{CA}_0^\omega + \mathsf{cocode}_0$ proves $\Pi_2^1\text{-}\mathsf{CA}_0$ (see [54, §4]).

Thirdly, $f : \mathbb{R} \to \mathbb{R}$ is *regulated* if for every x_0 in the domain, the 'left' and 'right' limit $f(x_0-) = \lim_{x \to x_0-} f(x)$ and $f(x_0+) = \lim_{x \to x_0+} f(x)$ exist. Feferman's μ readily provides the limit of $(f(x + \frac{1}{2^n}))_{n \in \mathbb{N}}$ if it exists, i.e., the notation $\lambda x. f(x+)$ for regulated f makes sense in ACA_0^ω. On a historical note, Scheeffer and Darboux study discontinuous

[4] Lakatos in [40, p. 148] claims that Jordan did not invent or introduce the notion of bounded variation in [32], but rather discovered it in Dirichlet's 1829 paper [42].

regulated functions in [21,68] without using the term 'regulated', while Bourbaki develops Riemann integration based on regulated functions in [11]. Finally, BV-functions are regulated while Weierstrass' 'monster' function is a natural example of a regulated function not in BV.

3.2. Bounded variation and hyperarithmetical analysis

We identify a number of statements about BV-functions that exist within the range of hyperarithmetical analysis, assuming ACA_0^ω. We even obtain some elegant equivalences and discus the (plentiful) variations of these results in Section 4.3.

First of all, the following principle appears to be important, which is just cocode_0 from the previous section restricted to strongly countable sets.

Principle 3.4 (cocode_1). *A strongly countable set $A \subset [0,1]$ can be enumerated.*

Some RM-results for cocode_1 may be found in [53, §2.2.1]; many variations are possible and these systems all exist in the range of hyperarithmetical analysis. The cited results are not that satisfying as they mostly deal with properties of strongly countable sets, in contrast to the below.

Secondly, we have the following theorem, establishing that items (ii)–(v) exist in the range of hyperarithmetical analysis.

Theorem 3.5 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_1$). *The higher items imply the lower ones.*

(i) *The principle* $\mathsf{QF\text{-}AC}^{0,1}$.

(ii) (Jordan) *For $f \in \mathrm{BV}$ with $V_0^1(f) = 1$, there are non-decreasing $g, h : [0,1] \to \mathbb{R}$ such that $f = g - h$.*

(iii) *For $f \in \mathrm{BV}$ with $V_0^1(f) = 1$, there is a sequence that includes all points of discontinuity of f.*

(iv) *For $f \in \mathrm{BV}$ with $V_0^1(f) = 1$, the supremum*[5] $\sup_{x\in[p,q]} f(x)$ *exists for $p, q \in [0,1] \cap \mathbb{Q}$.*

(v) cocode_1.

(vi) $\mathsf{weak\text{-}\Sigma_1^1\text{-}AC_0}$.

Items (ii)–(iii) are equivalent; we only use IND_1 to derive cocode_1 from item (iv).

Proof. Assume $\mathsf{QF\text{-}AC}^{0,1}$ and let $f \in \mathrm{BV}$ be such that $V_0^1(f) = 1$. By [52, Thm. 2.16], ACA_0^ω suffices to enumerate all jump discontinuities of a regulated function, while f is regulated by [53, Thm. 3.33]. Then $V_0^1(f) = 1$ implies that

$$(\forall k \in \mathbb{N})(\exists x_0, \ldots, x_m \in I)\big[(\forall i < m)(x_i < x_{i+1}) \wedge 1 - \tfrac{1}{2^k} < \textstyle\sum_{j=0}^{m} \big|f(x_j) - f(x_{j+1})\big|\big].$$

[5] To be absolutely clear, we assume, for the existence of a functional $\Phi : \mathbb{Q}^2 \to \mathbb{R}$ such that $(\forall p, q \in \mathbb{Q} \cap [0,1])(\Phi(p,q) = \sup_{x\in[p,q]} f(x))$.

The formula in square brackets is arithmetical, i.e., since $(\exists^2)$ is available we may apply $\mathsf{QF\text{-}AC}^{0,1}$ to obtain a sequence of finite sequences $(w_n)_{n\in\mathbb{N}}$ witnessing the previous centred formula. This sequence includes all removable discontinuities of f. Indeed, suppose $y_0 \in [0,1]$ is such that $f(y_0-) = f(y_0+) \neq f(y_0)$ is not among the reals in $(w_n)_{n\in\mathbb{N}}$. Let $k_0 \in \mathbb{N}$ be such that $|f(y_0+) - f(y_0)| > \frac{1}{2^{k_0}}$ and note that

$$1 - \tfrac{1}{2^{k_0+1}} < \textstyle\sum_{j=0}^{m_{k_0+1}} \left|f(x_j) - f(x_{j+1})\right|$$

for $w_{k_0+1} = (x_0, \ldots, x_{m_{k_0+1}})$ by assumption. Extending w_{k_0+1} with y_0 and points $z_0 < y_0 < u_0$ close enough to y_0, we obtain a partition of $[0,1]$ that witnesses that $V_0^1(f) > 1$, contradicting our assumptions. Since f is regulated, it only has removable and jump discontinuities, i.e., item (iii) follows from $\mathsf{QF\text{-}AC}^{0,1}$ as required.

By [53, Thm. 3.33], ACA_0^ω suffices to enumerate the points of discontinuity of any monotone $g : [0,1] \to \mathbb{R}$, i.e., item (ii) implies item (iii). To obtain item (ii) from item (iii), note that the supremum over $\mathbb{R}$ in (3.1) can be replaced by a supremum over $\mathbb{Q}$ and any sequence that includes all points of discontinuity of f. Hence, we may use $(\exists^2)$ to define the weakly increasing function $g(x) := \lambda x.V_0^x(f)$. One readily verifies that $h(x) := g(x) - f(x)$ is also weakly increasing, i.e., $f = g - h$ as in item (ii) follows. To obtain item (iv) from item (iii), note that – similar to the previous – the supremum over $\mathbb{R}$ in $\sup_{x\in[p,q]} f(x)$ can be replaced by a supremum over $\mathbb{Q}$ and any sequence that includes all points of discontinuity of f.

To derive cocode_1 from item (iv), let $A \subset [0,1]$ and $Y : [0,1] \to \mathbb{R}$ such that the latter is injective and surjective on the former. Now define $f : [0,1] \to \mathbb{R}$ as follows: $f(x) := \frac{1}{2^{Y(x)}}$ if $x \in A$, and 0 otherwise. Using IND_1, f is in BV and $V_0^1(f) = 1$. Now use $(\exists^2)$ to decide whether $\sup_{x\in[0,\frac{1}{2}]} f(x) < 1$; if the latter holds, '1' is the first bit of the binary expansion of $x_0 \in [0,1]$ such that $Y(x_0) = 0$. Using the supremum functional and $(\exists^2)$, the usual interval-halving technique then allows us to enumerate A, as required for cocode_1. For the final part, let φ be arithmetical and such that $(\forall n \in \mathbb{N})(\exists! X \subset \mathbb{N})\varphi(n, X)$. Use $\exists^2$ to define $\eta : [0,1] \to (2^{\mathbb{N}} \times 2^{\mathbb{N}})$ such that $\eta(x) = (f, g)$ outputs the binary expansions of x, taking $f = g$ if there is only one. Then $E_n = \{x \in [0,1] : \varphi(n, \eta(x)(1)) \vee \varphi(n, \eta(x)(2))\}$ is a singleton and $Y(x) := (\mu n)(x \in E_n)$ is injective and surjective on $A = \cup_{n\in\mathbb{N}} E_n$. The enumeration of A provided by cocode_1 yields the consequent of weak-Σ_1^1-AC_0. ∎

As to the role of the Axiom of Choice in Theorem 3.5, we note that the items (ii)–(v) can also be proved *without* $\mathsf{QF\text{-}AC}^{0,1}$. Indeed, $\lambda x.V_0^x(f)$ as in (3.1) involves a supremum over $\mathbb{R}$, which can be defined in Z_2^Ω using the well-known interval-halving technique, i.e., the usual textbook proof (see e.g. [1]) goes through in Z_2^Ω.

Thirdly, we have the following corollary using slightly more induction.

Corollary 3.6. *Over $\mathsf{ACA}_0^\omega + \mathsf{IND}_2$, item (iii) from Theorem 3.5 is equivalent to:*

For $f \in$BV with $V_0^1(f) = 1$ and with Fourier coefficients given, there is a sequence $(x_n)_{n\in\mathbb{N}}$ outside of which the Fourier series converges to $f(x)$.

Proof. We note that IND_2 suffices to guarantee that BV-functions are regulated by [53, Thm. 3.33]. Now, the Fourier series of a BV-function always converges to $\frac{f(x+)+f(x-)}{2}$ *and* this fact is provable in ACA_0^ω if the Fourier coefficients are given, as discussed in (a lot of detail in) [61, §3.4.4]. Hence, item (iii) of Theorem 3.5 immediately implies the centred statement in item (a), while for the reversal, the centred statement provides a sequence that includes all removable discontinuities, i.e., where $f(x) \neq f(x+)$ but $f(x+) = f(x-)$. By [52, Thm. 2.16], ACA_0^ω suffices to enumerate all jump discontinuities of a regulated function. Since there are no other discontinuities for f, the corollary follows. ∎

We could obtain similar results for e.g. Bernstein or Hermite-Fejér polynomials as analogous results hold for BV-functions (see [67]). Other variations are discussed in Remark 4.3 below.

Fifth, as noted in Section 3.1, enumerating the points of discontinuity of a regulated function implies cocode_0; the latter yields ATR_0 when combined with ACA_0^ω. By contrast, item (ii) in the following theorem is much weaker.

Theorem 3.7 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_0$). *The higher items imply the lower ones.*

(i) *The principle* $\mathsf{QF\text{-}AC}^{0,1}$.

(ii) *For regulated* $f : [0,1] \to \mathbb{R}$ *with infinite* D_f, *there is a sequence of distinct points of discontinuity of* f.

(iii) *The principle* Finite Choice.

(iv) *The principle* finite-Σ_1^1-AC_0.

Proof. The first downward implication is immediate by applying $\mathsf{QF\text{-}AC}^{0,1}$ – modulo $(\exists^2)$ – to 'D_f is not finite'. The final implication is straightforward. For the second downward implication, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of non-empty finite sets and let $\eta : [0,1] \to \mathbb{R}$ be such that $\eta(x)$ is the binary expansion of x, choosing a tail of zeros if necessary. Define $h : [0,1] \to \mathbb{R}$ as:

$$h(x) := \begin{cases} \frac{1}{2^n} & \text{if } \eta(x) = \underbrace{11\dots11}_{k+1\text{-times}} * \langle 0\rangle * g_0 \oplus \dots \oplus g_k \text{ and } (\forall i \leq k)(g_i \in X_i), \\ 0 & \text{otherwise.} \end{cases}$$

Using IND_2, one readily shows that h is regulated (with left and right limits equal to zero) and that D_h is infinite if $\cup_{n\in\mathbb{N}} X_n$ is. Any sequence in D_h then yields a sequence as in the consequent of Finite Choice. ∎

An interesting variation is provided by the following corollary. We conjecture that Finite Choice cannot be obtained from the second item.

Corollary 3.8 (ACA_0^ω). *The higher items imply the lower ones.*

(i) *The principle* $\mathsf{QF\text{-}AC}^{0,1}$.

(ii) *For* $f : [0,1] \to \mathbb{R}$ *in* BV *with infinite* D_f, *there is a sequence of distinct points of discontinuity of* f.

(iii) (Finite Choice′) *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of non-empty finite sets in* $[0,1]$ *and let* $g \in \mathbb{N}^{\mathbb{N}}$ *be such that* $|X_n| \leq g(n)$. *Then there is a sequence* $(x_n)_{n\in\mathbb{N}}$ *such that* $x_n \in X_n$ *for all* $n \in \mathbb{N}$.

(iv) *The principle* height-Σ_1^1-AC_0.

Proof. The final implication is straightforward while the first one follows as in the proof of the theorem. For the second downward implication, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of non-empty finite sets with $|X_n| \leq g(n)$. Define $h : [0,1] \to \mathbb{R}$ as in the proof of the theorem but replacing '$\frac{1}{2^n}$' in the first case by $\frac{1}{2^n}\frac{1}{g(n)+1}$. By construction, h is in BV with $V_0^1(f) \leq 1$ and the set D_h is infinite if $\cup_{n\in\mathbb{N}} X_n$ is. Any sequence in D_h then yields the sequence as in the consequent of Finite Choice′. ∎

Finally, we discuss numerous possible variations of the above results in Section 4.3, including Riemann integration and rectifiability.

4. Other topics in hyperarithmetical analysis

4.1. Semi-continuity and closed sets

We show that basic properties of semi-continuous functions, like the extreme value theorem, exist in the range of hyperarithmetical analysis. Since upper semi-continuous functions are closely related to closed sets, the latter also feature prominently.

First of all, we need Baire's notion of semi-continuity first introduced in [3].

Definition 4.1. We have the following definitions.

- The function $f : [0,1] \to \mathbb{R}$ is *upper semi-continuous* at $x_0 \in [0,1]$ if for any $k \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that $(\forall y \in B(x_0, \frac{1}{2^N}))(f(y) < f(x_0) + \frac{1}{2^k})$.
- The function $f : [0,1] \to \mathbb{R}$ is *lower semi-continuous* at $x_0 \in [0,1]$ if for any $k \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that $(\forall y \in B(x_0, \frac{1}{2^N}))(f(y) > f(x_0) - \frac{1}{2^k})$.

We use the common abbreviations 'usco' and 'lsco' for the previous notions. We say that '$f : [0,1] \to \mathbb{R}$ is usco' if f is usco at every $x \in [0,1]$. Following [45], the extreme value theorem does not really generalise beyond semi-continuous functions.

Secondly, we have the following theorem, a weaker version of which is in [55]. We repeat that since the characteristic function of a closed set is usco, the connection between items (ii) and ClC is not that surprising.

Theorem 4.2 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_2$). *The higher items imply the lower ones.*

(i) *The principle* $\mathsf{QF\text{-}AC}^{0,1}$.

(ii) (Extreme value theorem) *For usco* $f : \mathbb{R} \to \mathbb{R}$ *with* $y = \sup_{x\in[n,n+1]} f(x)$ *for all* $n \in \mathbb{N}$, *there is* $(x_n)_{n\in\mathbb{N}}$ *such that* $(\forall n \in \mathbb{N})(x_n \in [n, n+1] \wedge f(x_n) = y)$.

(iii) (ClC, [55]) *Let* $(C_n)_{n\in\mathbb{N}}$ *be a sequence of non-empty closed sets in* $[0,1]$. *Then there is* $(x_n)_{n\in\mathbb{N}}$ *such that* $x_n \in C_n$ *for all* $n \in \mathbb{N}$.

(iv) *For usco and regulated* $f : \mathbb{R} \to \mathbb{R}$ *with* $y = \sup_{x\in[n,n+1]} f(x)$ *for all* $n \in \mathbb{N}$, *there is* $(x_n)_{n\in\mathbb{N}}$ *such that* $(\forall n \in \mathbb{N})(x_n \in [n, n+1] \wedge f(x_n) = y)$.

(v) (Finite Choice) *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of non-empty finite sets in* $[0, 1]$. *Then there is* $(x_n)_{n\in\mathbb{N}}$ *such that* $x_n \in X_n$ *for all* $n \in \mathbb{N}$.

(vi) *The principle* $\textsf{finite-}\Sigma^1_1\textsf{-AC}_0$.

Proof. For the first downward implication, if the supremum y is given, we have

$$(\forall n, k \in \mathbb{N})\big(\exists x \in [n, n+1]\big)\big(f(x) > y - \tfrac{1}{2^k}\big),$$

and applying $\textsf{QF-AC}^{0,1}$ yields a sequence $(x_{n,k})_{n,k\in\mathbb{N}}$. Since $(\exists^2) \to \textsf{ACA}_0$, the latter has a convergent sub-sequence (for fixed $n \in \mathbb{N}$), with limit say $y_n \in [n, n+1]$ by sequential completeness. One readily verifies that $f(y_n) = y$ for any $n \in \mathbb{N}$ as f is usco. For the second implication, fix a sequence $(C_n)_{n\in\mathbb{N}}$ of closed sets and define $h : [0, 1] \to \mathbb{R}$ as follows using Feferman's μ:

$$h(x) := \begin{cases} 1 & x - n \in C_n \wedge n > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Since h is essentially the characteristic function of closed sets, h is usco on $[n, n+1]$ by definition, for each $n \in \mathbb{N}$. The sequence provided by item (ii) then clearly satisfies $x_n \in C_n$. To show that $\textsf{ClC}$ implies item (ii), let $f : [0, 1] \to \mathbb{R}$ and $y \in \mathbb{R}$ be as in the latter and define $C_{n,k} = \{x \in [n, n+1] : f(x) \geq y - \frac{1}{2^k}\}$ which is non-empty by definition and closed as f is usco. The sequence provided by $\textsf{ClC}$ yields $x_n \in [n, n+1]$ such that $f(x_n) = y$. The function h from (4.1) is also regulated in case each C_n is finite, i.e., the fourth implication also follows. ∎

We note that item (ii) is equivalent to e.g. the sequential version of the Cantor intersection theorem [55].

Thirdly, $\textsf{ClC}$ from Theorem 4.2 is provable in $\textsf{WKL}_0$ if we assume that the closed sets are given by a sequence of RM-codes (see [71, IV.1.8]). We next study $\textsf{ClC}$ for an alternative representation of closed sets from [14–16] as follows.

Definition 4.3. A (code for a) *separably closed* set is a sequence $S = (x_n)_{n\in\mathbb{N}}$ of reals. We write '$x \in \overline{S}$' in case $(\forall k \in \mathbb{N})(\exists n \in \mathbb{N})(|x - x_n| < \frac{1}{2^k})$. A (code for a) separably open set is a code for the (separably closed) complement.

Next, item (i) in Theorem 4.4 is a weakening of [71, V.4.10], which in turn is a second-order version of the countable union theorem. In each case, the antecedent only expresses that for every n, there *exists* an enumeration of A_n; abusing notation[6] slightly, we still

[6]In particular, the formula '$X \in \overline{A_n}$' in Theorem 4.4 is short-hand for

$$(\exists (Y_m)_{m\in\mathbb{N}})\big[(\forall Y \subset \mathbb{N})(Y \in A_n \to (\exists m \in \mathbb{N})(Y_m = Y)) \wedge (\forall k \in \mathbb{N})(\exists l \in \mathbb{N})(\overline{X}k = \overline{Y_l}k \wedge Y_l \in A_n)\big],$$

which is slightly more unwieldy.

write '$X \in \overline{A_n}$' as in Definition 4.3, leaving the enumeration of A_n implicit. We sometimes identify subsets $X \subset \mathbb{N}$ and elements $f \in 2^{\mathbb{N}}$.

Theorem 4.4 (ACA_0). *The following items are intermediate between* Σ^1_1-AC_0 *and* weak-Σ^1_1-AC_0.

(i) *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of analytic codes such that each A_n is enumerable and non-empty. There is a sequence $(X_n)_{n\in\mathbb{N}}$ with $(\forall n \in \mathbb{N})(X_n \in \overline{A_n})$.*

(ii) *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of analytic codes such that A_n is enumerable and $\overline{A_n}$ has positive measure. There exists $(X_n)_{n\in\mathbb{N}}$ with $(\forall n \in \mathbb{N})(X_n \in \overline{A_n})$.*

(iii) *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of analytic codes such that A_n is enumerable and $\overline{A_n}$ is not enumerable. There exists $(X_n)_{n\in\mathbb{N}}$ with $(\forall n \in \mathbb{N})(X_n \in \overline{A_n})$.*

(iv) *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of analytic codes such that for all $n \in \mathbb{N}$, A_n is RM-open. There exists $(X_n)_{n\in\mathbb{N}}$ with $(\forall n \in \mathbb{N})(X_n \in A_n)$.*

Proof. To prove the items in Σ^1_1-AC_0, apply the latter to $(\forall n \in \mathbb{N})(\exists X \subset \mathbb{N})[X \in A_n]$, noting that the formula in square brackets is Σ^1_1 if A_n is an analytic code. To derive weak-Σ^1_1-AC_0 from item (i), let φ be arithmetical and such that $(\forall n \in \mathbb{N})(\exists! X \subset \mathbb{N})\varphi(X,n)$ and define '$X \in A_n$' as $\varphi(X,n)$ using [71, V.1.7′]. Clearly, $X \in \overline{A_n}$ then implies $\varphi(X,n)$ as A_n codes a singleton, i.e., item (i) implies weak-Σ^1_1-AC_0. To obtain weak-Σ^1_1-AC_0 from item (iv), let φ be as in the antecedent of the former and consider $\Psi(X,n,k)$ defined as

$$(\exists Y \subset \mathbb{N})\big[\varphi(Y,n) \wedge \overline{Y}k = \overline{X}k\big], \tag{4.2}$$

which yields a sequence of Σ^1_1-formulas, yielding in turn a sequence of analytic codes $(A_{n,k})_{n,k\in\mathbb{N}}$ by [71, V.1.7′]. In light of (4.2), $A_{n,k}$ is a basic open ball in $2^{\mathbb{N}}$. In case $X_{n,k} \in A_{n,k}$ for all $n,k \in \mathbb{N}$, define $Y_n := \lambda k.\overline{X_{n,k}}k$ and note that $\varphi(Y_n,n)$ for all $n \in \mathbb{N}$. To obtain weak-Σ^1_1-AC_0 from item (ii), let φ be as in the antecedent of the former and define $\Psi(X,n,k)$ as

$$(\exists Y \subset \mathbb{N})\big[\varphi(\overline{X}k * Y,n) \wedge (\exists \sigma \in 2^{<\mathbb{N}})(X = \sigma * 00\ldots)\big],$$

which yields a sequence of Σ^1_1-formulas, yielding in turn a sequence of analytic codes $(A_{n,k})_{n,k\in\mathbb{N}}$ by [71, V.1.7′]. For fixed $n_0 \in \mathbb{N}$, there is a unique $X_0 \subset \mathbb{N}$ such that $\varphi(X_0,n_0)$, immediately yielding an enumeration of $A_{n_0,k}$ for any $k \in \mathbb{N}$. Essentially by definition, $\overline{A_{n,k}}$ has measure $1/2^k$. In case $X_{n,k} \in \overline{A_{n,k}}$ for all $n,k \in \mathbb{N}$, define $Y_n := \lambda k.\overline{X_{n,k}}k$ and note that $\varphi(Y_n,n)$ for all $n \in \mathbb{N}$. Item (iii) also follows as enumerable sets have measure zero. ∎

We would like to formulate item (i) using Borel codes from [71, V.3], but the latter seem to need ATR_0 to express basic aspects. The items from the theorem also imply finite-Σ^1_1-AC_0, which is left as an exercise.

Finally, we formulate a higher-order result for comparison; we continue the abuse of notation involving $\overline{S_n}$ as in Theorem 4.2.

Theorem 4.5 (ACA_0^ω). *The higher items imply the lower ones.*

(i) *The principle* $\mathsf{QF\text{-}AC}^{0,1}$.

(ii) *Let* $(S_n)_{n\in\mathbb{N}}$ *be a sequence of sets in* $[0,1]$ *such that for all* $n \in \mathbb{N}$, S_n *is enumerable and non-empty. There is* $(x_n)_{n\in\mathbb{N}}$ *with* $(\forall n \in \mathbb{N})(x_n \in \overline{S_n})$.

(iii) *Let* $(S_n)_{n\in\mathbb{N}}$ *be a sequence of sets in* $[0,1]$ *such that for all* $n \in \mathbb{N}$, S_n *is enumerable and* $\overline{S_n}$ *has positive measure. There is* $(x_n)_{n\in\mathbb{N}}$ *with* $(\forall n \in \mathbb{N})(x_n \in \overline{S_n})$.

(iv) cocode_1.

Proof. The first downward implication follows by applying $\mathsf{QF\text{-}AC}^{0,1}$ to 'S_n is non-empty for all $n \in \mathbb{N}$'. For the third downward implication, let $Y : [0,1] \to \mathbb{R}$ and $A \subset [0,1]$ be such that $(\forall n \in \mathbb{N})(\exists! x \in A)(Y(x) = n)$. Define the set

$$E_{n,k} := \big\{x \in [n, n+1] : (\exists q \in \mathbb{Q})\big(Y(x-n+q) = n \wedge x-n+q \in A \wedge |q| \leq \tfrac{1}{2^{k+1}}\big)\big\}$$

and note that this sequence has a straightforward enumeration while the associated separably closed set has measure $\frac{1}{2^k}$. Let $(x_{n,k})_{n,k\in\mathbb{N}}$ be the sequence provided by item (iii). Using sequential compactness, $y_n = \lim_{k\to\infty} x_{n,k}$ is a real in $[n, n+1]$ satisfying $Y(y_n) = n$, for any $n \in \mathbb{N}$ as required. ■

Variations of the previous theorem are possible, e.g., replacing 'enumerable' by '(strongly) countable'. Nonetheless, we are not able to derive e.g. cocode_1 from ClC restricted to closed sets of positive measure, i.e., the previous two theorems may well be due to the coding of closed sets as in Definition 4.3.

4.2. Unordered sums

The notion of *unordered sum* is a device for bestowing meaning upon sums involving uncountable index sets. We introduce the relevant definitions and then prove that basic properties of unordered sums exist in the range of hyperarithmetical analysis.

First of all, unordered sums are essentially 'uncountable sums' $\sum_{x\in I} f(x)$ for *any* index set I and $f : I \to \mathbb{R}$. A central result is that if $\sum_{x\in I} f(x)$ somehow exists, it must be a 'normal' series of the form $\sum_{i\in\mathbb{N}} f(y_i)$, i.e., $f(x) = 0$ for all but countably many $x \in [0,1]$; Tao mentions this theorem in [73, p. xii].

By way of motivation, there is considerable historical and conceptual interest in this topic: Kelley notes in [35, p. 64] that E. H. Moore's study of unordered sums in [47] led to the concept of *net* with his student H. L. Smith [48]. Unordered sums can be found in (self-proclaimed) basic or applied textbooks [31, 72] and can be used to develop measure theory [35, p. 79]. Moreover, Tukey shows in [74] that topology can be developed using *phalanxes*, which are nets with the same index sets as unordered sums.

Secondly, as to notations, unordered sums are just a special kind of *net* and $a : [0,1] \to \mathbb{R}$ is therefore written $(a_x)_{x\in[0,1]}$ in this context to suggest the connection to nets. The associated notation $\sum_{x\in[0,1]} a_x$ is purely symbolic. We only need the following notions in the below. Let $\mathsf{fin}(\mathbb{R})$ be the set of all finite sequences of reals without repetitions.

Definition 4.6. Let $a : [0,1] \to \mathbb{R}$ be any mapping, also denoted $(a_x)_{x\in[0,1]}$.

- We say that $(a_x)_{x\in[0,1]}$ is *convergent to* $a \in \mathbb{R}$ if for all $k \in \mathbb{N}$, there is $I \in \mathsf{fin}(\mathbb{R})$ such that for $J \in \mathsf{fin}(\mathbb{R})$ with $I \subseteq J$, we have $|a - \sum_{x\in J} a_x| < \frac{1}{2^k}$.
- A *modulus of convergence* is any sequence $\Phi^{0\to 1^*}$ such that $\Phi(k) = I$ for all $k \in \mathbb{N}$ in the previous item.

For simplicity, we focus on *positive unordered sums*, i.e., $(a_x)_{x\in[0,1]}$ such that $a_x \geq 0$ for $x \in [0,1]$.

Thirdly, we establish that basic properties of unordered sums exist in the range of hyperarithmetical analysis.

Theorem 4.7 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_1$). *The higher items imply the lower ones.*

(i) $\mathsf{QF\text{-}AC}^{0,1}$.

(ii) *For a positive and convergent unordered sum* $\sum_{x\in[0,1]} a_x$*, there is a sequence* $(y_n)_{n\in\mathbb{N}}$ *of reals such that* $a_y = 0$ *for all* y *not in this sequence.*

(iii) *For a positive and convergent unordered sum* $\sum_{x\in[0,1]} a_x$*, there is a modulus of convergence.*

(iv) cocode_1.

Proof. Assume $\mathsf{QF\text{-}AC}^{0,1}$ and note that the convergence of an unordered sum to some $a \in \mathbb{R}$ implies

$$(\forall k \in \mathbb{N})\big(\exists I \in \mathsf{fin}(\mathbb{R})\big)\Big(\big|a - \textstyle\sum_{x\in I} a_x\big| < \frac{1}{2^k}\Big). \tag{4.3}$$

Apply $\mathsf{QF\text{-}AC}^{0,1}$ to (4.3) to obtain a sequence $(I_n)_{n\in\mathbb{N}}$ of finite sequences of reals. This sequence must contain all $y \in \mathbb{R}$ such that $a_y \neq 0$. Indeed, suppose $y_0 \in \mathbb{R}$ satisfies $a_{y_0} >_{\mathbb{R}} \frac{1}{2^{k_0}}$ for fixed $k_0 \in \mathbb{N}$ and y_0 is not included in $(I_n)_{n\in\mathbb{N}}$. By definition, I_{k_0+2} satisfies

$$\big|a - \textstyle\sum_{x\in I_{k_0+2}} a_x\big| < \frac{1}{2^{k_0+2}}.$$

However, for $J = I_{k_0+2} \cup \{y_0\}$, we have $a_J > a$, a contradiction. Hence, $\mathsf{QF\text{-}AC}^{0,1}$ implies item (ii). The second and third items are readily seen to be equivalent.

For the final downward application, let $A \subset [0,1]$ and $Y : [0,1] \to \mathbb{R}$ be such that the latter is injective and surjective on the former. Define $a_x := \frac{1}{2^{Y(x)+1}}$ if $x \in A$, and 0 otherwise. One readily proves that $\sum_{x\in[0,1]} a_x$ is convergent to 1, for which IND_1 is needed. The sequence from the second item now yields the enumeration of the set A required by cocode_1. ■

We note that height-Σ_1^1-AC_0 can be obtained from item (ii) in Theorem 4.7; we conjecture that finite-Σ_1^1-AC_0 cannot be obtained. Since unordered sums are just nets, one could study statements like

a convergent net has a convergent sub-sequence,

which for index sets defined over Baire space is equivalent to $\mathsf{QF\text{-}AC}^{0,1}$ [64].

4.3. Variations and generalisations

We discuss variations and generalisations of the above results.

First of all, many variations of the results in Section 3.2 exist for *rectifiable* functions. Now, Jordan proves in [33, §105] that BV-functions are exactly those for which the notion of 'length of the graph of the function' makes sense. In particular, $f \in \mathrm{BV}$ if and only if the 'length of the graph of f', defined as follows:

$$L\big(f, [0,1]\big) := \sup_{0=t_0<t_1<\cdots<t_m=1} \sum_{i=0}^{m-1} \sqrt{(t_i - t_{i+1})^2 + \big(f(t_i) - f(t_{i+1})\big)^2} \tag{4.4}$$

exists and is finite by [1, Thm. 3.28 (c)]. In case the supremum in (4.4) exists (and is finite), f is also called *rectifiable*. Rectifiable curves predate BV-functions: in [68, §1–2], it is claimed that (4.4) is essentially equivalent to Duhamel's 1866 approach from [23, Ch. VI]. Around 1833, Dirksen, the PhD supervisor of Jacobi and Heine, already provides a definition of arc length that is (very) similar to (4.4) (see [22, §2, p. 128]), but with some conceptual problems as discussed in [20, §3].

Secondly, regulated functions are not necessarily BV but have *bounded* Waterman variation $W_0^1(f)$ (see [1]), which is a generalisation of BV where the sum in (3.1) is weighted by a *Waterman sequence*, which is a sequence of positive reals that converges to zero and with a divergent series. Some of the above results generalise to regulated function for which the Waterman variation is known, say $W_0^1(f) = 1$.

Thirdly, one can replace the consequent of item (iii) in Theorem 3.5 by a number of similar conditions, like the existence of a Baire 1 representation (which can be defined in ACA_0^ω for monotone functions), the fundamental theorem of calculus at all reals but a given sequence, or the condition that if the Riemann integral of $f : [0,1] \to [0,1]$ in BV is zero, $f(x) = 0$ for all $x \in [0,1]$ but a given sequence. Many similar conditions may be found in [61, 65, 67].

Fourth, Theorem 3.7 is readily generalised to (almost) arbitrary functions on the reals. To make sure the resulting theorem is provable in $\mathsf{ACA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$, it seems we need *oscillation functions*[7]. Riemann, Ascoli, and Hankel already considered the notion of oscillation in the study of Riemann integration [2, 28, 58], i.e., there is ample historical precedent. In the same way as for Theorem 3.7, one proves that the higher items imply the lower ones over ACA_0^ω.

- The principle $\mathsf{QF\text{-}AC}^{0,1}$.
- Any infinite set $X \subset [0,1]$ has a limit point.
- For any $f : [0,1] \to \mathbb{R}$ with oscillation function $\mathsf{osc}_f : [0,1] \to \mathbb{R}$, the set

  $$D_f = \big\{x \in [0,1] : \mathsf{osc}_f(x) > 0\big\}$$

  is either finite or has a limit point.

[7] For any $f : \mathbb{R} \to \mathbb{R}$, the associated *oscillation functions* are defined as follows: $\mathsf{osc}_f([a,b]) := \sup_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x)$ and $\mathsf{osc}_f(x) := \lim_{k\to\infty} \mathsf{osc}_f(B(x, \frac{1}{2^k}))$.

- For a non-piecewise continuous $f:[0,1]\to\mathbb{R}$ with oscillation function $\mathtt{osc}_f:[0,1]\to\mathbb{R}$, the set $D_f = \{x \in [0,1] : \mathtt{osc}_f(x) > 0\}$ has a limit point.
- The arithmetical Bolzano–Weierstrass theorem ABW_0 [19].

We note that $\mathtt{osc}_f : [0,1] \to \mathbb{R}$ is necessary to make '$x \in D_f$' into an *arithmetical* formula while 'x is a limit point of D_f' is a meaningful (non-arithmetical) formula even if D_f does not exist as a set.

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