Doc. Math. 29 (2024), 1341–1366
DOI 10.4171/DM/975

Global Lipschitz geometry of conic singular sub-manifolds with applications to algebraic sets

André Costa, Vincent Grandjean, and Maria Michalska

Abstract. We prove that a connected globally conic singular sub-manifold of a Riemannian manifold, compact when the ambient manifold is non-Euclidean, is Lipschitz Normally Embedded: its outer and inner metric space structures are equivalent. Moreover, we show that generic $\mathbb{K}$-analytic germs as well as generic affine algebraic sets in $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, are globally conic singular sub-manifolds. Consequently, a generic $\mathbb{K}$-analytic germ or a generic algebraic subset of $\mathbb{K}^n$ is Lipschitz Normally Embedded.

1. Introduction

A subset S of a smooth Riemannian manifold M is called *quasi-convex, Whitney* 1*-regular* or *Lipschitz normally embedded* (abbreviated to LNE) if there exists a constant L such that

$$d_{\text{in}}^S \leq L \cdot d^S$$

where (S, d^S) is the outer metric structure, i.e., the distance in S is taken in the ambient space M, and (S, d_{in}^S) is the inner metric structure, i.e., the distance between two points of S is taken as the infimum of the length of the rectifiable curves in S connecting them. This notion was well-established first by Whitney in [40, 41], thereafter studied in [24, 37] for sub-analytic sets; and re-introduced under the name of quasi-convex sets in the investigation of length spaces, see [19]. The least ambivalent name of Lipschitz normally embedded sets, therefore the one we will use, was introduced by [1] in the semi-algebraic context.

The present paper continues the application of Riemannian geometric methods to global Lipschitz geometry of algebraic sets, which was initiated in the PhD thesis of the first author [6] and our articles [9, 11]. In this paper we deal with the classical type of singularities in the Riemannian setting: the conically singular ones.

The interest in a metrically conical point in Riemannian geometry dates back at least to the seminal works [2–5]. Although the notion of conic singular point of a subset in this paper is differential, whenever the subset inherits the inner metric from the ambient Riemannian manifold, it turns into a metrically conical point, compare with [17, 27] for

Mathematics Subject Classification 2020: 51K99 (primary); 14P99, 53B99, 58K99 (secondary).
Keywords: conic singularity, generic algebraic set, Lipschitz geometry, quasiconvex set, Lipschitz normally embedded set.

notable variation. In this article we study the global Lipschitz geometry of conic singular sub-manifolds. Interestingly, though investigation of Lipschitz properties in an unbounded metric space would usually involve studying the behaviour along geodesic rays [18, 19], methods of our paper [11] allow us to reduce equivalently the global LNE property of a conic singular sub-manifold of $\mathbb{R}^n$ to its one-point compactification in $\mathbf{S}^n$, see Corollaries 3.17 and 3.25. We obtain the following result.

Theorem 3.6. *A connected compact conic singular sub-manifold of a Riemannian manifold is Lipschitz normally embedded.*

Since even smooth sub-manifolds may fail to be LNE, if non-compact, we present the natural notion of conic at infinity and obtain the following result.

Theorem 3.20. *A connected globally conic singular sub-manifold of* $\mathbb{R}^n$ *is LNE.*

On the other hand, there is a growing body of work on LNE algebraic and analytic subset germs and one might consult [13, 29, 35] for an overview of the state of the art. The immediate natural question presents itself: *is the LNE property common among algebraic sets or analytic set germs?* The literature up to date fails to tackle this problem. The local LNE problem at infinity for complex affine algebraic sets was explicitly initiated in [15]. Besides the obvious examples of global LNE sets that are the compact connected sub-manifolds and $\mathbb{K}$-cones over these, prior to [6,9] completely characterizing LNE complex algebraic curves of $\mathbb{C}^n$, only the two concurrent papers [22, 23] presented non-trivial examples of globally LNE algebraic sets of $\mathbb{K}^n$.

We prove that indeed LNE sets are prevalent among analytic germs and affine algebraic sets, since we show in Theorems 4.1 and 4.5 that a generic germ of an analytic singularity in $(\mathbb{C}^n, \mathbf{0})$ of multiplicity m is conic at the origin and a generic algebraic subset of $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, is a globally conic singular sub-manifold. This implies the following.

Corollary 4.2. *A Generic germ of an analytic singularity in* $(\mathbb{C}^n, \mathbf{0})$ *of multiplicity* m *is LNE.*

Corollary 4.6. *Every connected component of a generic affine algebraic subset of* $\mathbb{K}^n$ *is LNE.*

We build up the proofs basically from scratch, as our methods are quite direct. The only previously known results on LNE sets used in this paper are Proposition 2.8 from [1] and properties of cones which we first encountered in [23]. Nevertheless, thanks to the literature on germs, one can combine Corollary 3.19 on affine trace of projective conic sub-manifolds with known results such as [31,33] to get various new examples of singular LNE affine algebraic sets, compare with Corollary 4.17 or Example 4.18. Additionally, we attempted to make the paper self-contained, due to it sitting at the intersection of Global Analysis and Algebraic Geometry.

2. Preliminaries

We will first establish some notation and thereafter present the notions of p-sub-manifold, spherical blow-up and Lipschitz normally embedded sets. Throughout the paper smooth means C^∞.

2.1. Notation

The Euclidean distance in $\mathbb{R}^n$ is denoted $|-|$. The open ball of radius r and centre $\mathbf{a}$ is $B^n(\mathbf{a}, r)$, its closure is $\mathbf{B}^n(\mathbf{a}, r)$ and $\mathbf{S}^{n-1}(\mathbf{a}, r)$ is its boundary. When $\mathbf{a}$ is the origin we only write B_r^n, $\mathbf{B}_r^n$ and $\mathbf{S}_r^{n-1}$ and among them the unit ball is B^n and the unit sphere is $\mathbf{S}^{n-1}$.

Denote $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{R}_{>0} := (0, \infty)$. The half-line in the oriented direction of the vector $\mathbf{u} \in \mathbb{R}^n \setminus \mathbf{0}$ is

$$\mathbb{R}_{\geq 0}\mathbf{u} := \{t\mathbf{u} : t \in \mathbb{R}_{\geq 0}\}.$$

The non-negative cone over the subset X of $\mathbb{R}^n$ with vertex $\mathbf{a}$ is defined as

$$\widehat{X}^+ := \mathbf{a} + \bigcup_{\mathbf{x} \in X \setminus \mathbf{a}} \mathbb{R}_{\geq 0}(\mathbf{x} - \mathbf{a}).$$

Let X be a subset of $\mathbb{R}^n$ containing $\mathbf{a}$. Let $\triangleleft$ be any element of $\{<, \leq, >, \geq\}$. We denote

$$\begin{aligned} X(\mathbf{a})_r &:= X \cap \mathbf{S}^{n-1}(\mathbf{a}, r), & X(\mathbf{a})_{\triangleleft r} &:= X \cap \big\{|\mathbf{x} - \mathbf{a}| \triangleleft r\big\}, \\ X_r &:= X(\mathbf{0})_r, & X_{\triangleleft r} &:= X(\mathbf{0})_{\triangleleft r}. \end{aligned} \tag{2.1}$$

By $\operatorname{clos}_M(X)$ denote the closure of a subset X in the topological space M.

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We identify $\mathbb{R}^{2n} = (\mathbb{R}^2)^n$ with $\mathbb{C}^n$ via the mapping

$$\mathbf{x} = (x_j, y_j)_{j=1,\ldots,n} \to \mathbf{x}^{\mathbb{C}} := (x_j + i y_j)_{j=1,\ldots,n}$$

with inverse $\mathbf{z} = (z_j)_{j=1,\ldots,n} \to \mathbf{z}^{\mathbb{R}} := (\operatorname{Re}(z_j), \operatorname{Im}(z_j))_{j=1,\ldots,n}$.

We embed the affine space $\mathbb{K}^n$ into the projective space $\mathbb{KP}^n$ as $\mathbf{x} \mapsto [\mathbf{x} : 1]$. Let $\mathbf{H}_\infty = \mathbb{KP}^n \setminus \mathbb{K}^n$ be the hyperplane at infinity of $\mathbb{KP}^n$, that is

$$\mathbf{H}_\infty = \big\{[\mathbf{x} : 0] : \mathbf{x} \in \mathbb{K}^n \setminus \mathbf{0}\big\}.$$

Given a subset X of $\mathbb{KP}^n$, we call the set $X \setminus \mathbf{H}_\infty$ its affine part.

2.2. p-sub-manifolds and spherical blowing-up

We present here notions from [26] adapted to our elementary embedded context.

Let $(M, \partial M)$ be a smooth manifold with boundary ∂M (possibly empty). Throughout the paper we consider embedded smooth sub-manifolds: a subset N of $(M, \partial M)$ is a sub-manifold with boundary ∂N (possibly empty) if (i) it is a smooth manifold with boundary $(N, \partial N)$; (ii) its manifold topology coincides with the induced topology from M; (iii) the inclusion mapping $N \hookrightarrow M$ is a smooth injective immersion.

Definition 2.1. Let $(M, \partial M)$ be a smooth manifold with boundary. A subset N of M is a *p-sub-manifold* if: (i) it is a smooth sub-manifold with boundary ∂N of M; (ii) ∂N is contained in ∂M; (iii) N is transverse to ∂M:

$$\mathbf{x} \in \partial N \implies T_{\mathbf{x}}N + T_{\mathbf{x}}\partial M = T_{\mathbf{x}}M.$$

The spherical blowing-up of $\mathbb{R}^n$ with centre the point $\mathbf{a}$ of $\mathbb{R}^n$ is the mapping

$$\mathbf{bl_a} : [\mathbb{R}^n, \mathbf{a}] = \mathbb{R}_{\geq 0} \times \mathbf{S}^{n-1} \to \mathbb{R}^n, \quad (r, \mathbf{u}) \to r\mathbf{u} + \mathbf{a}.$$

Its domain $[\mathbb{R}^n, \mathbf{a}]$ is a smooth manifold with smooth compact boundary $\partial[\mathbb{R}^n, \mathbf{a}] = 0 \times \mathbf{S}^{n-1}$ equipped with the metric tensor

$$\mathbf{h} := \mathrm{eucl}|_{\mathbb{R}_{\geq 0}} \otimes \mathrm{eucl}|_{\mathbf{S}^{n-1}}. \tag{2.2}$$

Restriction of blowing-up to the cylinder $(0, \infty) \times \mathbf{S}^{n-1}$ is a diffeomorphism onto $\mathbb{R}^n \setminus \mathbf{a}$.

Definition 2.2. Let X be a subset of $\mathbb{R}^n$. The *strict transform* of X by $\mathbf{bl_a}$ is the subset of $[\mathbb{R}^n, \mathbf{a}]$ defined as

$$[X, \mathbf{a}] := \mathrm{clos}_{[\mathbb{R}^n,\mathbf{a}]}\big(\mathbf{bl_a}^{-1}(X \setminus \mathbf{a})\big).$$

The *front face* of the strict transform of X is

$$\mathrm{ff}\big([X, \mathbf{a}]\big) := [X, \mathbf{a}] \cap \partial\big([\mathbb{R}^n, \mathbf{a}]\big) = 0 \times S_{\mathbf{a}}X,$$

where

$$S_{\mathbf{a}}X := \left\{\mathbf{u} \in \mathbf{S}^{n-1} : \exists (\mathbf{x}_k)_k \subset X \setminus \mathbf{a} \text{ with } \mathbf{x}_k \to \mathbf{a} \text{ and } \frac{\mathbf{x}_k - \mathbf{a}}{|\mathbf{x}_k - \mathbf{a}|} \to \mathbf{u}\right\}.$$

The non-negative cone $\widehat{S_{\mathbf{a}}X}^+$ over $S_{\mathbf{a}}X$ is the *tangent cone of X at* $\mathbf{a}$.

If X is a sub-manifold of $\mathbb{R}^n$, then the strict transform $[X, \mathbf{a}]$ is a p-sub-manifold of $[\mathbb{R}^n, \mathbf{a}]$. Moreover, if X is a sub-manifold without boundary containing the point $\mathbf{a}$, its strict transform $[X, \mathbf{a}]$ is a smooth sub-manifold with smooth compact boundary $\partial[X, \mathbf{a}] = \mathrm{ff}([X, \mathbf{a}])$. In particular, the front face of $[\mathbb{R}^n, \mathbf{a}]$ is $\partial[\mathbb{R}^n, \mathbf{a}]$.

The following result is well-known, so we omit its straightforward proof.

Proposition 2.3. *Let $\phi : N \to N'$ be a smooth diffeomorphism between the smooth sub-manifolds N of $\mathbb{R}^n$ and N' of $\mathbb{R}^{n'}$. If $\mathbf{a}$ is a point of N, then the smooth diffeomorphism*

$$\mathbf{bl}_{\phi(\mathbf{a})}^{-1} \circ \phi \circ \mathbf{bl_a} : [N, \mathbf{a}] \setminus \mathrm{ff}\big([N, \mathbf{a}]\big) \to \big[N', \phi(\mathbf{a})\big] \setminus \mathrm{ff}\big(\big[N', \phi(\mathbf{a})\big]\big)$$

extends to a smooth diffeomorphism $[N, \mathbf{a}] \to [N', \phi(\mathbf{a})]$ between two smooth sub-manifolds with smooth compact non-empty boundaries.

Let us finish this section with spherical blowing-up in manifolds adapted to the embedded context, as in some consecutive sections we will rely on Nash embeddings.

Let N be a sub-manifold of $\mathbb{R}^n$, and let $\mathbf{a}$ be a point of N. The blowing-up of $\mathbf{a}$ in N is the mapping

$$\mathbf{bl}_{\mathbf{a}}^{N} := \mathbf{bl_a}|_{[N,\mathbf{a}]} : [N, \mathbf{a}] \to N.$$

Definition 2.4. If X is a subset of N, its *strict transform by* $\mathbf{bl}_{\mathbf{a}}^N$ is

$$[X, \mathbf{a}]^N := \mathrm{clos}_{[N,\mathbf{a}]}\big((\mathbf{bl}_{\mathbf{a}}^N)^{-1}(X \setminus \mathbf{a})\big)$$

and *front face* of the strict transform of X is defined as

$$\mathrm{ff}^N\left([X, \mathbf{a}]^N\right) := [X, \mathbf{a}]^N \cap \mathrm{ff}\left([N, \mathbf{a}]\right).$$

The following identities hold true

$$[X, \mathbf{a}]^N = [X, \mathbf{a}] \quad \text{and} \quad \mathrm{ff}^N\left([X, \mathbf{a}]^N\right) = \mathrm{ff}\left([X, \mathbf{a}]\right).$$

In particular it follows naturally that if $(X, \mathbf{a})$ is a germ of a sub-manifold of $\mathbb{R}^n$ at the point $\mathbf{a}$, then the strict transform $[X_{<r}, \mathbf{a}]$ of a representative $X_{<r}$ of this germ is a p-sub-manifold of $[\mathbb{R}^n, \mathbf{a}]$.

2.3. Lipschitz normally embedded sets

As remarked in the introduction alternative names for the notion of Lipschitz normally embedded (LNE) sets are quasi-convex and Whitney 1-regular, see [19, 37, 40].

Let (M, g_M) be a smooth Riemannian manifold (with boundary or without). The metric tensor g_M induces the natural distance d^M on M which is the infimum of the lengths of rectifiable curves connecting any given pair of points. Any subset X of M admits two natural metric space structures inherited from (M, g_M).

Definition 2.5.

(i) The *outer metric space structure* (X, d^X), where the *outer distance* function d^X is the restriction of d^M to $X \times X$.

(ii) The *inner metric space structure* (X, d_{in}^X), where the *inner distance* function d_{in}^X is defined as follows: given $\mathbf{x}, \mathbf{x}' \in X$, the number $d_{\mathrm{in}}^X(\mathbf{x}, \mathbf{x}')$ is the infimum of the lengths of rectifiable paths lying in X joining $\mathbf{x}$ and $\mathbf{x}'$.

Observe that $d^X \leq d_{\mathrm{in}}^X$ and $d^M = d_{\mathrm{in}}^M$. When X is a sub-manifold of M, then d_{in}^X is equal to the natural distance function induced by the tensor g_M restricted to X.

Definition 2.6.

(i) A subset X of (M, g_M) is *Lipschitz normally embedded* (shortened to LNE) in M if there exists a positive constant L such that

$$\mathbf{x}, \mathbf{x}' \in X \implies d_{\mathrm{in}}^X(\mathbf{x}, \mathbf{x}') \leq L \cdot d^X(\mathbf{x}, \mathbf{x}').$$

(ii) The subset X of (M, g_M) is *locally LNE at* $\mathbf{x}$ if there exists a neighbourhood U of $\mathbf{x}$ in M such that $X \cap U$ is LNE in M.

(iii) The subset X of (M, g_M) is *locally LNE* if it is locally LNE at each point of its closure in M.

We will simply say a set is LNE, but it is always understood that it is with respect to the metric space structure of the given ambient space.

Remark 2.7. If the manifold M is compact, the property of being LNE depends only on the C^1 structure of M, compare [9, Remark 2.4].

Note that any embedded sub-manifold is locally LNE. Thus the next result describes global Lipschitz geometry of compact sub-manifolds (see [1, Proposition 2.1] without proof, [23, Proposition 2.4] and [9, Lemma 2.6]).

Proposition 2.8. *A connected compact subset of the smooth Riemannian manifold (M, g_M) is LNE if and only if it is locally LNE. In particular, any connected compact C^1 embedded sub-manifold, possibly with boundary, is LNE.*

The claim of Proposition 2.8 above fails to be true when the subset is just closed, even in a complete Riemannian manifold, see Example 3.12.

A useful trait is that the LNE property is hereditary in the sense of Proposition 2.9 below. In fact, equivalence of induced metrics on an LNE subset in the proof below shows that LNE subsets preserve ambient Lipschitz properties, for example Hölder exponents.

Proposition 2.9. *Let X be a subset of a smooth Riemannian manifold (M, g_M) which is locally LNE at a point $\mathbf{x}$. A subset Y of X is locally LNE at $\mathbf{x}$ in (X, d_{in}^X) if and only if it is locally LNE at $\mathbf{x}$ in (M, d^M).*

Proof. The length of any curve in Y gives the same value regardless if it is considered in M or in X. Thus the inner metric on Y induced by the metric d_{in}^X of X is the same as the inner metric on Y induced by the metric structure of M, therefore there is a unique d_{in}^Y. Moreover, the outer metric structure of Y inherited from X is $d^Y(X) := d^X_{\text{in}|Y\times Y}$, while the outer metric structure inherited from M is $d^Y(M) := (d^M)_{|Y\times Y}$. Since the distance $d^X = (d^M)_{|X\times X}$ is equivalent near the point $\mathbf{x}$ to d_{in}^X by assumption, we get the claim. ∎

A special case of interest is the following result (see [23, Proposition 2.8] or [8, Lemma 2.5] for a detailed proof).

Proposition 2.10. *A non-negative cone over an LNE subset of the unit sphere $\mathbf{S}^{n-1}$ is LNE in $\mathbb{R}^n$.*

From Proposition 2.10 and the law of cosines, under notation (2.1), we conclude what follows.

Corollary 2.11. *A non-negative cone X is LNE if and only if every connected component of its link $X \cap \mathbf{S}^{n-1}$ is LNE.*

Corollary 2.12. *Let X be a non-negative cone over a connected set and $\triangleleft$ an element of $\{<, \leq, >, \geq, =\}$. The following conditions are equivalent:* (i) *X is LNE,* (ii) *$X_{\triangleleft r}$ is LNE for every $r > 0$,* (iii) *$X_{\triangleleft r}$ is LNE for a radius $r > 0$.*

3. Conic singular sub-manifolds are Lipschitz normally embedded

In this section we prove the main theorems. First, Theorem 3.6 states that every compact connected conic singular sub-manifold is Lipschitz normally embedded. Then we indicate that the result is no longer true without assumption of compactness. Thus we present the conic condition at infinity and prove in Theorem 3.20 that every connected globally conic singular sub-manifold of the Euclidean space is Lipschitz normally embedded.

3.1. Conic singular points

The notion of conic singular point is classical and we follow the seminal works [2–5]. Yet our embedded context dictates presentation of definitions using charts, while Proposition 2.3 guarantees that the notion is still intrinsic and independent of any $\mathscr{C}^2$ Riemannian structure. Note that although a conic singular point is a differential notion, whenever a subset inherits the inner metric from the ambient Riemannian manifold, its conic singular point turns into a metrically conical point.

Definition 3.1. A *singular point* $\mathbf{a}$ of a subset X of a smooth manifold is a point at which the subset germ $(X, \mathbf{a})$ is not that of a smooth sub-manifold. Otherwise we say that the point $\mathbf{a}$ *is a smooth point* of X. The *singular locus* X_{sing} *of the set* X is the set of its singular points.

Definition 3.2.

(i) Let X be a subset of $\mathbb{R}^n$ and $\mathbf{a}$ be a point of X. The point $\mathbf{a}$ is *a conic point* of X, if there exists a positive radius r such that the strict transform $[X(\mathbf{a})_{<r}, \mathbf{a}]$ is a closed subset and a p-sub-manifold of $[\mathbb{R}^n(\mathbf{a})_{<r}, \mathbf{a}]$.

(ii) Let X be a subset of a smooth manifold M and $\mathbf{a}$ be a point of X. The point $\mathbf{a}$ is *a conic point* of X, if there exists a smooth chart $\psi : \mathcal{U} \to B^n$ of M centred at $\mathbf{a}$ such that $\psi(X \cap \mathcal{U})$ is conic at $\psi(\mathbf{a})$.

The second part of Definition 3.2 is well-posed: let $\psi_i : \mathcal{U}_i \to B^n$, $i = 1, 2$, be two smooth charts centred at $\mathbf{a}$. Denote $\mathbf{a}_i := \psi_i(\mathbf{a})$ and $X_i := \psi_i(X \cap \mathcal{U}_i)$. Proposition 2.3 implies that X_1 is conic at $\mathbf{a}_1$ if and only if X_2 is conic at $\mathbf{a}_2$, and similarly X_1 is singularly conic at $\mathbf{a}_1$ if and only if X_2 is singularly conic at $\mathbf{a}_2$.

When X is a subset of a sub-manifold M of $\mathbb{R}^n$, then the two parts of definition of conic points agree, see Proposition 3.3 below. In this elementary context it is a variation of the commutativity of the blowing-up with the base change, compare [26, Chapter V].

Proposition 3.3. *The subset X of the sub-manifold N of $\mathbb{R}^n$ is conic in N at* $\mathbf{a}$*, if and only if the subset X of $\mathbb{R}^n$ is conic at* $\mathbf{a}$*.*

Proof. We have $\mathbf{a} \in X \subset N \subset \mathbb{R}^n$, each set being locally closed at $\mathbf{a}$, and thus $[X, \mathbf{a}]^N = [X, \mathbf{a}]$ and $\text{ff}^N[X, \mathbf{a}] = \text{ff}[X, \mathbf{a}] \subset \text{ff}[N, \mathbf{a}]$. ■

The following Collar Neighbourhood Lemma is a key ingredient of this paper.

Proposition 3.4 (Collar Neighbourhood Lemma). *Let X be a subset of $\mathbb{R}^n$. If $\mathbf{a}$ is a conic point of X, there exists a radius r_0 and a smooth diffeomorphism*

$$\tau_{\mathbf{a}} : [0, r_0] \times S_{\mathbf{a}}X \to [X, \mathbf{a}] \cap \big([0, r_0] \times \mathbf{S}^{n-1}\big)$$

which is: (i) *link-preserving, i.e., it is of the form* $(r, \mathbf{u}) \mapsto (r, \mu(r, \mathbf{u}))$ *for all* $0 \le r \le r_0$, (ii) *its restriction to* $\mathrm{ff}[X, \mathbf{a}]$ *is the identity mapping.*

Since the Collar Neighbourhood Lemma is a classical result (for instance in [27, Theorem 1.2] one can find a much more general context and precise statement), let us provide simply an outline of the proof.

Sketch of proof. Recall that $[\mathbb{R}^n, \mathbf{a}] = \mathbb{R}_{\ge 0} \times \mathbf{S}^{n-1}$ is equipped with the Riemannian product metric $\mathbf{h} = \mathrm{d}r^2 \otimes \mathrm{d}\mathbf{u}^2$ as in (2.2), where $\mathrm{d}\mathbf{u}^2$ is the restriction $\mathrm{eucl}|_{\mathbf{S}^{n-1}}$ of the Euclidean metric tensor to $\mathbf{S}^{n-1}$. Without loss of generality, we can assume that $\mathcal{X} := [X, \mathbf{a}]$ is a closed p-sub-manifold with boundary $\mathrm{ff}([X, \mathbf{a}]) = 0 \times S_{\mathbf{a}}X$. Define the smooth function

$$r_{\mathcal{X}} : \mathcal{X} \to \mathbb{R}, \quad (r, \mathbf{u}) \mapsto r.$$

By hypotheses, the gradient $\nabla_{\mathbf{g}} r_{\mathcal{X}}$, for $\mathbf{g} = \mathbf{h}|_{\mathcal{X}}$, does not vanish on $\mathcal{X} \cap [0, r_0] \times \mathbf{S}^{n-1}$ for some $r_0 > 0$. The flow Ψ of the renormalized gradient $\frac{\nabla_{\mathbf{g}} r_{\mathcal{X}}}{|\nabla_{\mathbf{g}} r_{\mathcal{X}}|^2}$, satisfying the Cauchy problem $\Psi(0, \mathbf{x}) = \mathbf{x}$, will give the link-preserving diffeomorphism. ■

Note that every smooth point is a conic point. If $\mathbf{a}$ is a conic point of the subset X of a manifold M, then $S_{\mathbf{a}}\psi(X)$ is a smooth compact sub-manifold of $\mathbf{S}^{n-1}$ for a chart ψ of M. Moreover, the germ $(X \setminus \mathbf{a}, \mathbf{a})$ is that of a smooth sub-manifold. Thus any conic singular point of X is an isolated singular point of X.

Definition 3.5. A subset X of a smooth manifold M is a *conic singular sub-manifold* if (i) it is a closed subset of M, (ii) the singular locus X_{sing} of X is finite or empty, and (iii) every singular point is conic singular.

In particular, a non-negative cone $\widehat{X}^+$ over a closed smooth sub-manifold X of $\mathbf{S}^{n-1}$ is a conic singular sub-manifold of $\mathbb{R}^n$ as it has a conic point at $\mathbf{0}$ which is singular conic whenever it is not a linear sub-space of $\mathbb{R}^n$. Note that if X is a conic singular sub-manifold, the connected components of $X \setminus X_{\mathrm{sing}}$ may have different dimensions.

3.2. Compact conic singular sub-manifolds are LNE

Theorem 3.6. *Any connected component of a compact conic singular sub-manifold of a smooth Riemannian manifold is Lipschitz normally embedded.*

Proof. Let (M, g) be a smooth Riemannian manifold. Nash Embedding Theorem [32] states that any smooth Riemannian manifold (M, g) embeds isometrically in $\mathbb{R}^n$ as a sub-manifold equipped with the inner metric, i.e., there exists a smooth embedding

$$u : (M, g) \to \big(u(M), \mathrm{eucl}|_{u(M)}\big) \subset \mathbb{R}^n \quad \text{such that} \quad u^*(\mathrm{eucl}|_{u(M)}) = g.$$

In particular, $u(M)$ equipped with its inner distance $d_{\text{in}}^{u(M)} = (u^{-1})^* d^M$ is locally LNE in $\mathbb{R}^n$. Since u is an isometry, by Propositions 2.8 and 2.9 we get the lemma below.

Lemma 3.7. *Any compact connected subset of M is LNE in M if and only if its image by u is LNE in $\mathbb{R}^n$.*

Moreover, by Propositions 3.3 and 2.3 the following is immediate.

Lemma 3.8. *A subset X of M is conic at $\mathbf{a}$ (respectively singularly conic at $\mathbf{a}$) if and only if $u(X)$ is conic in $u(M)$ at $u(\mathbf{a})$ (respectively singularly conic at $u(\mathbf{a})$).*

By Nash Embedding Theorem and Lemmas 3.7 and 3.8, in order to prove Theorem 3.6 it suffices to consider a compact connected conic singular sub-manifold X of $\mathbb{R}^n$.

Take any point of a connected compact conic singular sub-manifold X of $\mathbb{R}^n$ and without loss of generality assume it is the origin $\mathbf{0}$. The proof below works for any point of X, but since X is locally LNE at each of its smooth points, it is the singular points that are of interest.

Given a positive radius r, recall notation (2.1) below

$$X_r := X \cap \mathbf{S}_r^{n-1} \quad \text{and} \quad X_{\leq r} := X \cap \mathbf{B}_r^n.$$

Denote the tangent cone of X at $\mathbf{0}$ as

$$\mathbf{C} := \widehat{S_\mathbf{0} X}^+.$$

Lemma 3.9. *The smooth diffeomorphism $\mathbf{bl_0} \circ \tau_\mathbf{0} \circ \mathbf{bl_0}^{-1} : \mathbf{C}_{\leq r_0} \setminus \mathbf{0} \to X_{\leq r_0} \setminus \mathbf{0}$, where $r_0 > 0$ and $\tau_\mathbf{0}$ are as in the Collar Neighbourhood Lemma (Lemma 3.4), extends to a radius-preserving homeomorphism*

$$\phi_\mathbf{0} : \mathbf{C}_{\leq r_0} \to X_{\leq r_0}.$$

The mapping $\phi_\mathbf{0}$ is outer bi-Lipschitz, i.e. bi-Lipschitz when the source and target spaces are equipped with their respective outer metric space structures.

Proof. Consider $\mathbf{M} := [\mathbb{R}^n, \mathbf{0}]$ with the Riemannian metric $\mathbf{h} := \text{eucl}|_{\mathbb{R}_{\geq 0}} \otimes \text{eucl}|_{\mathbf{S}^{n-1}}$. Recall that the diffeomorphism $\tau_\mathbf{0} =: \tau$ of Proposition 3.4 is of the form

$$\tau(r, \mathbf{u}) = \big(r, \mu(r, \mathbf{u})\big) \in \mathbf{M},$$

where $\mu : \mathbf{bl_0}^{-1}(\mathbf{C}_{\leq r_0}) = [0, r_0] \times S_\mathbf{0} X \to \mathbf{S}^{n-1}$ is a smooth submersion. Therefore $D\mu$ takes values in $T\mathbf{S}^{n-1}$ and its norm is uniformly bounded over the compact cylinder $\mathbf{bl_0}^{-1}(\mathbf{C}_{\leq r_0})$. Let L be a positive constant such that

$$\|D_\mathbf{y}\mu\|_\mathbf{h} \leq L \quad \text{for all } \mathbf{y} \in \mathbf{bl_0}^{-1}(\mathbf{C}_{\leq r_0}).$$

Let $r > 0$. At every point $\mathbf{y} = (r, \mathbf{u}) \in \mathbf{M}$ the tangent space is

$$T_\mathbf{y}\mathbf{M} = \mathbb{R} \times T_\mathbf{u}\mathbf{S}^{n-1}.$$

Let $\mathbf{x} := r\mathbf{u} = \mathbf{bl_0}(\mathbf{y})$, then $T_\mathbf{x}\mathbb{R}^n$ decomposes as the orthogonal sum

$$T_\mathbf{x}\mathbb{R}^n = \mathbb{R}\partial_r \oplus T_{r\mathbf{u}}\mathbf{S}_r^{n-1}, \quad \text{where } \partial_r(\mathbf{x}) := \mathbf{u}. \tag{3.1}$$

As vector sub-spaces of $\mathbb{R}^n$, note that $T_\mathbf{u}\mathbf{S}^{n-1} = T_{t\mathbf{u}}\mathbf{S}_t^{n-1}$. Thus

$$D_\mathbf{y}\mathbf{bl_0} = \mathbf{1}_\mathbb{R} \oplus r \cdot \mathrm{Id}_{T_\mathbf{u}\mathbf{S}^{n-1}} \quad \text{and} \quad D_\mathbf{x}\mathbf{bl_0}^{-1} = \left(\mathbf{1}_\mathbb{R}^{-1}, \frac{1}{|\mathbf{x}|} \cdot \mathrm{Id}_{T_{\partial_r(\mathbf{x})}\mathbf{S}^{n-1}}\right),$$

where $\mathbf{1}_\mathbb{R} : \mathbb{R} \ni t \mapsto t \cdot \partial_r \in \mathbb{R}\mathbf{x}$.

For any

$$\xi = (t, \xi_\mathbf{S}) \in T_\mathbf{y}\left(\mathbf{bl_0}^{-1}(\mathbf{C}_{\le r_0})\right) = \mathbb{R} \times T_\mathbf{u}(S_\mathbf{0}X),$$

with $\xi_\mathbf{S} \in T_\mathbf{u}\mathbf{S}^{n-1}$, we have

$$D_\mathbf{y}\tau \cdot \xi = (t, \zeta_\mathbf{S}) \in \mathbb{R} \times T_{\mu(r,\mathbf{u})}\mathbf{S}^{n-1},$$

where $\zeta_\mathbf{S} := D_\mathbf{y}\mu \cdot \xi$. Note that

$$|\zeta_\mathbf{S}|^2 \le L^2 \cdot \left(t^2 + |\xi_\mathbf{S}^2|\right).$$

If $\vec{v} := t\partial_r \oplus \upsilon \in T_\mathbf{x}\mathbf{C} = \mathbb{R} \times T_\mathbf{u}(S_\mathbf{0}X)$ with $\upsilon \in T_\mathbf{u}(S_\mathbf{0}X)$ we find

$$D_\mathbf{x}\phi_\mathbf{0} \cdot \vec{v} = t\partial_r \oplus D_\mathbf{y}\mu \cdot \left(|\mathbf{x}|t, \upsilon\right).$$

Therefore the norm of $D\phi_\mathbf{0}$ is uniformly bounded over $\mathbf{C}_{\le r_0} \setminus \mathbf{0}$ by $1 + r_0L$. Since the inverse of τ is a link-preserving diffeomorphism $(r, \mathbf{u}) \to (r, \widetilde{\mu}(r, \mathbf{u}))$, analogous reasoning shows that the norm of $D(\phi_\mathbf{0})^{-1}$ is also uniformly bounded over $X_{\le r_0} \setminus \mathbf{0}$. Thus $\phi_\mathbf{0}$ is bi-Lipschitz over the closure of each connected component of $\mathbf{C}_{\le r_0} \setminus \mathbf{0}$. By properties of blow-up and Proposition 3.4 it also preserves radius.

Let $S_1, \ldots, S_k$ be connected components of the link $S_\mathbf{0}X$. Since the S_i are pairwise disjoint compact sets, we have

$$2\delta := \min\left\{\mathrm{dist}(S_i, S_j) : 1 \le i < j \le k\right\} > 0.$$

where dist is taken in $\mathbb{R}^n$. Consider the truncated non-negative cones

$$C_i := (\widehat{S_i}^+)_{\le r_0}.$$

Of course, $\mathbf{C}_{\le r_0} = \bigcup_{i=1}^k C_i$. Let $\mathbf{s}_i \in S_i$ and $0 \le r_i \le r_0$. For $i < j$ the law of cosines yields

$$\delta(r_i + r_j) \le |r_i\mathbf{s}_i - r_j\mathbf{s}_j| \le r_i + r_j.$$

Therefore

$$\left|\phi_\mathbf{0}(r_i\mathbf{s}_i) - \phi_\mathbf{0}(r_j\mathbf{s}_j)\right| \le \left|\phi_\mathbf{0}(r_i\mathbf{s}_i)\right| + \left|\phi_\mathbf{0}(r_j\mathbf{s}_j)\right| = r_i + r_j \le \frac{1}{\delta}|r_i\mathbf{s}_i - r_j\mathbf{s}_j|$$

and $\phi_\mathbf{0}$ is Lipschitz.

To show that the inverse of $\phi_{\mathbf{0}}$ is Lipschitz, let

$$X_i := \phi_{\mathbf{0}}(C_i).$$

Since $\phi_{\mathbf{0}}$ is a homeomorphism at $\mathbf{0}$ and $S_{\mathbf{0}}X_i = S_i$, we can assume that r_0 is small enough so that

$$\min\left\{\operatorname{dist}\left(\frac{\mathbf{x}_i}{|\mathbf{x}_i|}, \frac{\mathbf{x}_j}{|\mathbf{x}_j|}\right), \ \mathbf{x}_i \in X_i,\ \mathbf{x}_j \in X_j,\ 1 \leq i < j \leq k\right\} \geq \frac{\delta}{2}.$$

Similarly to the proof that $\phi_{\mathbf{0}}$ is Lipschitz, this estimate implies that the inverse mapping $(\phi_{\mathbf{0}})^{-1}$ is also Lipschitz. ■

By Lemma 3.9 the mapping $\phi_{\mathbf{0}}$ is outer bi-Lipschitz, thus it is also bi-Lipschitz with respect to the inner metrics on source and target. Note that $\mathbf{C}$ is a non-negative cone over a smooth set, hence by Corollary 2.12 the set $\mathbf{C}_{<r_0}$ is LNE in $\mathbb{R}^n$. Thus the set $X_{<r_0} = \phi_{\mathbf{0}}(\mathbf{C}_{<r_0})$ is LNE.

To end the proof apply Proposition 2.8 to the compact and locally LNE set X. ■

An alternative proof of Theorem 3.6 follows by showing that τ_0 is a Lipschitz trivialization of μ and similarly to [7, Lemma 3.9] one gets that μ does not have any critical points on $[0, r_0) \times S_{\mathbf{a}}X$, thus its derivative is bounded away from zero.

From Lemma 3.9 one immediately obtains the following two properties.

Corollary 3.10. *Let X be a subset of $\mathbb{R}^n$ which is conic at the point $\mathbf{a}$. There exists a small positive radius $r_{\mathbf{a}}$ such that each representative $X(\mathbf{a})_{\leq r}$, $X(\mathbf{a})_{<r}$, $X(\mathbf{a})_r$ is LNE for all $r \leq r_{\mathbf{a}}$. Moreover, the link $X(\mathbf{a})_r = X \cap \mathbf{S}^{n-1}(\mathbf{a}, r)$ is Lipschitz homotopic in X to a point, i.e., there exists a Lipschitz continuous map $H : X(\mathbf{a})_r \times [0,1] \to X$ with $H_{|X(\mathbf{a})_r \times \{0\}} = \mathrm{id}_{X(\mathbf{a})_r}$ and $H_{X(\mathbf{a})_r \times \{1\}} = \mathrm{id}_{\{\mathbf{a}\}}$.*

Corollary 3.11. *Let X be a compact conic singular sub-manifold of a smooth manifold M. Assume X has local dimension at least 2 and let Σ be a finite subset. If $X \setminus \Sigma$ is connected, then it is LNE.*

3.3. Conic singularities at infinity

We will show that a connected unbounded conic singular sub-manifold in $\mathbb{R}^n$ is LNE once it is also conic at infinity. A condition at infinity is necessary, since even an arbitrary smooth sub-manifold of $\mathbb{R}^n$ may fail to be LNE.

Example 3.12. The plane parabola $y = x^2$ (real and complex) is not LNE (see for instance [6, 9, 15]).

Thus in Definition 3.13 we introduce the conical condition at infinity for unbounded sets. It is a natural counterpart to conical property at a point: the precise relationship will be established in Propositions 3.15 and 3.16.

Let $[\mathbb{R}^n,\infty] := \mathbb{R}_{\geq 0}\times \mathbf{S}^{n-1}$ with the boundary $\partial[\mathbb{R}^n,\infty] = 0\times\mathbf{S}^{n-1}$. *The spherical blowing-up at infinity* is the smooth rational diffeomorphism

$$\mathbf{bl}_\infty : [\mathbb{R}^n,\infty]\setminus\partial[\mathbb{R}^n,\infty] = \mathbb{R}_{>0}\times\mathbf{S}^{n-1}\to\mathbb{R}^n\setminus\mathbf{0}, \quad (r,\mathbf{u})\mapsto\frac{\mathbf{u}}{r}.$$

It introduces a natural compactification of $\mathbb{R}^n$ as the smooth manifold with boundary

$$\big(\overline{\mathbb{R}^n},\mathbf{S}^{n-1}_\infty\big) := \big([\mathbb{R}^n,\infty]\sqcup\mathbf{0}, 0\times\mathbf{S}^{n-1}\big)$$

using the spherical blowing-up mapping at infinity to identify $\mathbb{R}^n\setminus\mathbf{0}$ with $[\mathbb{R}^n,\infty]\setminus(0\times\mathbf{S}^{n-1})$. The *sphere at infinity* $\mathbf{S}^{n-1}_\infty$ is the boundary $0\times\mathbf{S}^{n-1}$ of $\overline{\mathbb{R}^n}$.

For any subset X of $\mathbb{R}^n$ denote by X^∞ its set of accumulation points at infinity

$$0\times X^\infty := \operatorname{clos}_{\overline{\mathbb{R}^n}}\big(\mathbf{bl}_\infty^{-1}(X)\big)\cap(0\times\mathbf{S}^{n-1}_\infty).$$

Analogously to Definition 2.2 we can equivalently write

$$X^\infty := \Big\{\mathbf{u}\in\mathbf{S}^{n-1} : \exists(\mathbf{x}_k)_k\subset X\setminus\mathbf{0}\text{ with }\mathbf{x}_k\to\infty\text{ and }\frac{\mathbf{x}_k}{|\mathbf{x}_k|}\to\mathbf{u}\Big\}.$$

The strict transform of X by $\mathbf{bl}_\infty$ in $\overline{\mathbb{R}^n}$ is defined as

$$\bar{X} := \mathbf{bl}_\infty^{-1}(X)\cup(0\times X^\infty).$$

In particular, we have $\overline{\mathbb{R}^n_{>r}} = \mathbf{bl}_\infty^{-1}(\mathbb{R}^n_{>r})\cup\mathbf{S}^{n-1}_\infty = [0,\frac{1}{r})\times\mathbf{S}^{n-1}$.

Definition 3.13. A subset X of $\mathbb{R}^n$ is *conic at* ∞, if either it is bounded or there exists a radius r such that $(\overline{X_{>r}}, 0\times X^\infty)$ is a closed subset and a smooth p-sub-manifold of $(\overline{\mathbb{R}^n_{>r}},\mathbf{S}^{n-1}_\infty)$.

Definition 3.14. A subset of $\mathbb{R}^n$ is a *globally conic singular sub-manifold* if it is a conic singular sub-manifold of $\mathbb{R}^n$ which is conic at ∞.

3.4. Localization of conic singularities at infinity

In this section we show how a conical singularity at infinity can be viewed simply as a conical singularity at a point.

The inversion of $\mathbb{R}^n$ is the following rational isomorphism

$$\iota:\mathbb{R}^n\setminus\mathbf{0}\to\mathbb{R}^n\setminus\mathbf{0}, \quad \mathbf{x}\mapsto\frac{\mathbf{x}}{|\mathbf{x}|^2}.$$

We have

$$\iota = \mathbf{bl}_\infty\circ\mathbf{bl}_\mathbf{0}^{-1} = \mathbf{bl}_\mathbf{0}\circ\mathbf{bl}_\infty^{-1} = \iota^{-1} \tag{3.2}$$

since by definition the mappings $\mathbf{bl}_\infty$ and $\mathbf{bl}_\mathbf{0}$ admit an inverse only over $\mathbb{R}^n\setminus\mathbf{0}$. Moreover, by Definition 2.2 of $S_\mathbf{0}X$, the definition of X^∞ and equation (3.2), the following holds true

$$\big(\iota(X\setminus\mathbf{0})\big)^\infty = S_\mathbf{0}X \quad\text{and}\quad S_\mathbf{0}\big(\iota(X\setminus\mathbf{0})\big) = X^\infty \tag{3.3}$$

for subsets X of $\mathbb{R}^n$. Thus the accumulation set at infinity can be viewed as the tangent link at infinity. In particular for a non-negative cone X with vertex $\mathbf{0}$ we have

$$\iota(X \setminus \mathbf{0}) = X \setminus \mathbf{0}.$$

For a subset X of $\mathbb{R}^n$ define $\widetilde{X}$ to be the Euclidean closure of $\iota(X \setminus \mathbf{0})$, thus if X is an unbounded closed set, we get

$$\widetilde{X} = \iota(X \setminus \mathbf{0}) \cup \mathbf{0}.$$

Proposition 3.15. *An unbounded subset X of $\mathbb{R}^n$ is conic at ∞ if and only if its image by the inversion $\widetilde{X}$ in $\mathbb{R}^n$ is conic at $\mathbf{0}$.*

Proof. The closure of the image by inversion of a closed unbounded set $X_{>r}$ in $\mathbb{R}^n_{>r}$ is the closed set $\widetilde{X}_{<\frac{1}{r}}$ in $\mathbb{R}^n_{<\frac{1}{r}}$ containing $\mathbf{0}$. Recall that

$$[\mathbb{R}^n_{>r}, \infty] = \left[0, \frac{1}{r}\right) \times \mathbf{S}^{n-1} = \left[\mathbb{R}^n_{<\frac{1}{r}}, \mathbf{0}\right].$$

Hence by identity (3.2) the set $\overline{X_{>r}}$ is equal to the strict transform by $\mathbf{bl_0}$ of $\widetilde{X}_{<\frac{1}{r}}$ with the same boundary $0 \times X^\infty$. Therefore, the claim follows directly from Definitions 3.2 and 3.13 of conic property at a point and at ∞. ∎

Let ω be the north-pole $(0, \ldots, 0, 1)$ of the sphere $\mathbf{S}^n$. The stereographic projection centred at ω is a smooth semi-algebraic diffeomorphism with the inverse

$$\sigma : \mathbb{R}^n \to \mathbf{S}^n \setminus \omega, \quad \mathbf{x} \to \left(\frac{2\mathbf{x}}{|\mathbf{x}|^2 + 1}, \frac{|\mathbf{x}|^2 - 1}{|\mathbf{x}|^2 + 1}\right).$$

Consider the smooth semi-algebraic diffeomorphism

$$\varphi_r : \mathbf{B}^n_r \to \mathbf{S}^n, \quad \mathbf{x} \to \left(\frac{2\mathbf{x}}{1 + |\mathbf{x}|^2}, \frac{1 - |\mathbf{x}|^2}{1 + |\mathbf{x}|^2}\right)$$

which is a bi-Lipschitz mapping onto its image between manifolds with boundary. Its inverse is a restriction of the stereographic projection from the south pole, in particular it is a chart on $\mathbf{S}^n$ sending the north pole ω to $\mathbf{0}$. For $\mathbf{x} \in \mathbb{R}^n_{\geq r}$ we get

$$\left(\varphi^{-1}_{\frac{1}{r}} \circ \sigma\right)(\mathbf{x}) = \iota(\mathbf{x}). \tag{3.4}$$

Proposition 3.16. *An unbounded subset X of $\mathbb{R}^n$ is conic at ∞ if and only if $\sigma(X) \cup \omega$ is conic at the north pole ω of $\mathbf{S}^n$.*

Proof. By Proposition 3.15 the set X is conic at ∞ if and only if $\widetilde{X}$ is conic at $\mathbf{0}$. Now, use the chart φ_r restricted to the open ball and apply the second part of Definition 3.2 of conical points on manifolds: the point ω is a conical point of $\sigma(X) \cup \omega$ if and only if $\widetilde{X}$ is conic at 0. Thus by identity (3.4) we get the claim. ∎

Since the inversion ι is a smooth diffeomorphism, the image of a conic singular sub-manifold of $\mathbb{R}^n \setminus \mathbf{0}$ is a conic singular sub-manifold in $\mathbb{R}^n \setminus \mathbf{0}$ and vice versa by Proposition 2.3 applied at every singular point. Similarly, stereographic projection of a conic singular sub-manifold of $\mathbf{S}^n \setminus \omega$ is a conic singular sub-manifold of $\mathbb{R}^n$. Hence from Propositions 3.15 and 3.16 we get immediately the following.

Corollary 3.17. *For a subset of* $\mathbb{R}^n$ *the following conditions are equivalent:* (i) *it is a globally conic singular sub-manifold,* (ii) *the closure of its image under inversion is a globally conic singular sub-manifold,* (iii) *it is a stereographic projection of a conic singular sub-manifold of* $\mathbf{S}^n$.

Last but not least, let us consider the compactification of the Euclidean space as the affine part of a (real or complex) projective space. Projectivization is especially of interest from the point of view of algebraic geometry, even though it generally does not preserve Lipschitz geometry, see Example 4.20.

Let $n_{\mathbb{K}}$ be equal to n if $\mathbb{K} = \mathbb{R}$ and $2n$ if $\mathbb{K} = \mathbb{C}$. Define

$$\pi_n^{\mathbb{K}} : \overline{\mathbb{R}^{n_{\mathbb{K}}}} \to \mathbb{KP}^n.$$

When $\mathbb{K} = \mathbb{R}$, put

$$\pi_n^{\mathbb{R}}(\mathbf{x}) := \begin{cases} [\mathbf{0} : 1] & \text{if } \mathbf{x} = \mathbf{0}, \\ [\mathbf{u} : r] & \text{if } \mathbf{x} = (r, \mathbf{u}) \in [\mathbb{R}^n, \infty] \end{cases}$$

and when $\mathbb{K} = \mathbb{C}$, put

$$\pi_n^{\mathbb{C}}(\mathbf{x}) := \begin{cases} [\mathbf{0} : 1] & \text{if } \mathbf{x} = \mathbf{0}, \\ [\mathbf{u}^{\mathbb{C}} : r] & \text{if } \mathbf{x}^{\mathbb{R}} = (r, \mathbf{u}) \in [\mathbb{R}^{2n}, \infty]. \end{cases}$$

The mapping $\pi_n^{\mathbb{K}}$ is a smooth semi-algebraic submersion which maps $\mathbb{K}^n$ onto $\mathbb{K}^n = \mathbb{KP}^n \setminus \mathbf{H}_\infty$ as the standard embedding $\mathbf{x} \to [\mathbf{x} : 1]$.

Proposition 3.18. *Let* X *be a subset of* $\mathbb{KP}^n$ *such that* $(X, \mathbf{H}_\infty)$ *is a germ of a smooth sub-manifold along the hyperplane at infinity. If* X *is transverse to* $\mathbf{H}_\infty$*, then the affine part* $X \setminus \mathbf{H}_\infty$ *is conic at* ∞ *as a sub-manifold of* $\mathbb{R}^{n_{\mathbb{K}}}$.

Proof. In the real case at any boundary point of $[\mathbb{R}^n, \infty]$, the mapping $\pi_n^{\mathbb{R}}$ is a local diffeomorphism, thus by Proposition 2.3 we get the claim.

In the complex case, let $\mathbf{x} = (0, \mathbf{u}) \in \mathbf{S}_\infty^{2n-1}$ be a boundary point of $\overline{\mathbb{R}^{2n}}$ and $\mathbb{C}_{\mathbf{u}}$ be complex line $\mathbb{C}\mathbf{u}^{\mathbb{C}} \subset \mathbb{C}^n$. The link $\mathbb{C}_{\mathbf{u}} \cap \mathbf{S}^{2n-1}$ is a one-dimensional manifold with the tangent space $L_{\mathbf{u}}$ at the point $\mathbf{u}$. Let $N_{\mathbf{u}}$ be the orthogonal complement of $L_{\mathbf{u}}$ in the real plane $\mathbb{C}_{\mathbf{u}}$. Using notation (3.1), the linear mapping $D_{\mathbf{x}}\pi_n^{\mathbb{C}}$ maps isomorphically $\mathbb{R}\partial_r \oplus T_{\mathbf{u}}\mathbf{S}^{2n-1}$ onto $T_{[\mathbf{u}^{\mathbb{C}}:0]}\mathbb{KP}^n = \mathbb{C}_{\mathbf{u}} \oplus T_{\mathbf{u}^{\mathbb{C}}}\mathbf{H}_\infty$ and maps isomorphically $T_{\mathbf{u}}\mathbf{S}^{2n-1}$ onto $N_{\mathbf{u}} \times T_{\mathbf{u}^{\mathbb{C}}}\mathbf{H}_\infty$. Since X is transverse to $\mathbf{H}_\infty$, the surjectivity of $D_{\mathbf{x}}\pi_n^{\mathbb{C}}$ at any boundary point $\mathbf{x} \in \mathbf{S}_\infty^{2n-1}$ yields that $\overline{X}$ is a germ of a p-sub-manifold along the sphere at infinity. ∎

Corollary 3.19. *Affine part of a conic singular sub-manifold of* $\mathbb{KP}^n$ *that meets the hyper-plane at infinity transversally is a globally conic singular sub-manifold of* $\mathbb{K}^n$.

3.5. Globally conic singular sub-manifolds are LNE

Theorem 3.20. *Every connected component of a globally conic singular sub-manifold of* $\mathbb{R}^n$ *is Lipschitz normally embedded.*

Proof. Without loss of generality, assume the globally conic singular sub-manifold X is connected and unbounded. From Proposition 3.15 follows immediately that similarly to the local case, if X is conic at ∞, then its link at infinity X^∞ by identity (3.3) is equal to $\iota(S_{\mathbf{0}}\widetilde{X})$ and is a smooth compact sub-manifold of $\mathbb{R}^n$. There exists a positive radius r_0 such that for every $r > r_0$ the finite singular set X_{sing} is contained in the open ball B^n_r and

$$X_{\geq r} = X \setminus B^n_r$$

is a smooth p-sub-manifold of $\mathbb{R}^n_{\geq r}$. Moreover, possibly taking r_0 larger, for any $r > r_0$ the set

$$X_{\leq r} = X \cap \mathbf{B}_r$$

is connected with a smooth compact boundary $\partial X_{\leq r} = X \cap \mathbf{S}^{n-1}_r = \partial X_{\geq r}$.

Lemma 3.21. *For all* $r > r_0$ *the set* $X_{\leq r}$ *is LNE.*

Proof. The set $X_{\leq r}$ is a germ of a sub-manifold (possibly with smooth boundary) at its every non-singular point. A germ of a sub-manifold with boundary is locally LNE by [9, Corollary 2.7] and a subset is locally LNE at each of its conic points by Theorem 3.6 and Proposition 2.8. Thus by Proposition 2.8 we get the claim. ■

The set $\widetilde{X}_{\leq\frac{1}{r}}$ is conic at $\mathbf{0}$ by Proposition 3.15, thus by Lemma 3.9 it is outer bi-Lipschitz to the truncated cone $\widehat{S_{\mathbf{0}}\widetilde{X}}^+_{\leq\frac{1}{r}}$, which by identity (3.3) is equal to $\widehat{X^\infty}^+_{\leq\frac{1}{r}}$. Hence there is an outer bi-Lipschitz homeomorphism

$$\phi_{\mathbf{0}} : \widetilde{X}_{\leq\frac{1}{r}} \to \widehat{X^\infty}^+_{\leq\frac{1}{r}}$$

which satisfies $|\phi_{\mathbf{0}}(\mathbf{x})| = |\mathbf{x}|$.

Lemma 3.22. *The homeomorphism*

$$\phi_\infty : X_{\geq r} \to \widehat{X^\infty}^+_{\geq r}, \quad \phi_\infty := \iota \circ \phi_{\mathbf{0}} \circ \iota$$

is outer bi-Lipschitz.

Proof. Use the main result of [16]. ■

If two sets are outer bi-Lipschitz equivalent, then they are inner bi-Lipschitz equivalent, which follows easily from expressing the length of an arc as an appropriate limit of sums of outer distances, compare for instance [13]. Thus every connected component of the set $X_{\geq r}$ is LNE as an image by an outer bi-Lipschitz mapping of an LNE set, since every connected component of $\widehat{X^{\infty}}^{+}_{\geq r}$ is LNE in $\mathbb{R}^n$ by Corollary 2.12.

Lemma 3.23. *If X is a connected closed set which is locally LNE and it is conic at ∞, then X is LNE.*

Proof. Suppose X is not LNE, i.e. there exist sequences $(\mathbf{x}_k)_k$ and $(\mathbf{y}_k)_k$ in X such that

$$d_k := \frac{d_{\text{in}}^X(\mathbf{x}_k, \mathbf{y}_k)}{|\mathbf{x}_k - \mathbf{y}_k|} \to \infty \quad \text{as } k \to \infty.$$

Since $X_{\leq r}$ is LNE by Proposition 2.8 and $X_{\geq r}$ is an outer bi-Lipschitz image of an LNE cone without its origin, thus we may suppose that $(\mathbf{x}_k)_k \subset X_{\leq r}$ and $(\mathbf{y}_k)_k$ lies in a given connected component Y of $X_{\geq r}$, and furthermore that both the sequences converge (possibly to infinity).

Since $X_{\leq r}$ and Y are LNE, there exists a constant L such that for any $\mathbf{z}$ in the compact boundary $Y \cap \mathbf{S}_r^{n-1} \subset X \cap \mathbf{S}_r^{n-1}$ we have

$$d_{\text{in}}^X(\mathbf{x}_k, \mathbf{y}_k) \leq L|\mathbf{x}_k - \mathbf{z}| + L|\mathbf{y}_k - \mathbf{z}|.$$

This estimate and the assumption that $d_k \to \infty$ imply that both $(\mathbf{x}_k)_k$ and $(\mathbf{y}_k)_k$ converge to the same point $\mathbf{z}_0 \in X \cap \mathbf{S}_r^{n-1}$. But this means that both sequences lie in $X_{\leq R}$ with $R > \max |\mathbf{y}_k|$, which is a LNE set by Lemma 3.21. Thus d_k is finite, which gives a contradiction. ∎

Lemma 3.23 concludes the proof of Theorem 3.20, since X as a conic sub-manifold is locally LNE by Theorem 3.6 and Proposition 2.8. ∎

One can prove Theorem 3.20 by an alternative argument. Namely, since $[\mathbb{R}^n, \infty] = [\mathbb{R}^n, \mathbf{0}]$ and X is conic at ∞, Proposition 3.4 holds true at infinity: *there exists a smooth link-preserving diffeomorphism $\tau_\infty : \overline{X} \cap [0, r_0] \times \mathbf{S}^{n-1}$ onto $[0, r_0] \times X^\infty$*. Thus one can obtain a version at infinity of Lemma 3.9, proved exactly in the same way, that $\mathbf{bl}_\infty \circ \tau_\infty \circ \mathbf{bl}_\infty^{-1} : X \setminus B_R^n \to \widehat{X^{\infty}}^{+} \setminus B_R^n$ *is bi-Lipschitz and preserves radii.*

Remark 3.24. Theorems 3.6 and 3.20 still hold true if we were to work with C^2 conic singular sub-manifolds instead of C^∞ smooth ones.

As a direct consequence of Theorems 3.6 and 3.20, using gluing as in Lemma 3.23, one obtains.

Corollary 3.25. *Let X be a closed connected subset of $\mathbb{R}^n$ which is conic at infinity. It is LNE in $\mathbb{R}^n$ if and only if $\sigma(X) \cup \omega$ is LNE in $\mathbf{S}^n$.*

Interestingly, the claim of Corollary 3.25 holds under an alternative assumption that X is definable in an o-minimal structure, see the result of [11]. The methods of proof vastly differ though. It is an open question on what is the common underlying structure of the set that would allow for a uniform statement and proof, both for singular manifolds and definable sets.

Remark 3.26. Provided a compact Riemannian manifold with boundary (M, g) is endowed with a scattering metric near its boundary, as for instance in [20, 28, 36], one can define which subsets are conic at infinity and the following should hold: *Every connected and globally conic singular sub-manifold of the manifold* (M, g) *is Lipschitz normally embedded in* M. Since preparing this paper we have confirmed this to be true in the upcoming preprint [10].

4. Conic structure and Lipschitz geometry of generic algebraic sets

In this section we investigate the conic nature and global Lipschitz geometry of generic algebraic sets. To avoid ambiguity, we will be explicit in what we mean by generic in each context. Notation and notions, though standard, can be found in Appendix A. Throughout this section we assume that the number of equations k is smaller than the dimension of the space n, since by Bezout's theorem a general intersection of n or more algebraic hypersurfaces is at most finite thus satisfies our claims trivially.

We will show that generic algebraic sets in the real or complex affine space are globally conic singular sub-manifolds. We do not undertake here the interesting open question of which algebraic sets are conic sub-manifolds. Note that there exist singular algebraic sets which are analytic manifolds, see for instance [30], thus even the two most elementary categories: of non-singular algebraic subsets of the affine space conic at infinity and that of algebraic sets which are smooth globally conic singular sub-manifolds of the Euclidean space, are not the same.

4.1. Complex analytic set germs

Most work on the LNE nature of analytic sets up to date concerns the local case of analytic singularity germs. Minimal surface singularities [33], general determinantal singularities [23], explicit super-isolated singularities [31] are LNE (for an overview see [13]). Yet we found no result addressing the basic question – *are LNE singularities common?* – to which we give a positive answer in Corollary 4.2.

Theorem 4.1. A generic analytic singularity germ of $(\mathbb{C}^n, \mathbf{0})$ of multiplicity m is conic at the origin*: for every* $\mathbf{m} \in \mathbb{N}^k$ *with* $m = \Pi_{j=1}^k m_j$*, there exists a Zariski open dense subset* $U(\mathbf{m})$ *of* $\mathbb{K}_{\mathbf{m},k}^{\mathrm{hom}}[\mathbf{x}]$ *such that the analytic set germ* $(\mathcal{Z}(\mathbf{f}), \mathbf{0})$ *is conic singular at the origin for any analytic map germ* $\mathbf{f} \in \mathcal{O}_{n,k}$ *with* $\mathrm{in}_{\mathbf{0}}(\mathbf{f}) \in U(\mathbf{m})$.

Proof. Consider the non-empty Zariski open set $S(\mathbf{m}) = U(\mathbf{m}) \cap P(\mathbf{m})$ of Propositions A.1 and A.3. Take $\mathbf{f} \in \mathcal{O}_{n,k}$ with $\mathrm{in}_{\mathbf{0}}(\mathbf{f}) \in S(\mathbf{m})$. Denote $X := \mathcal{Z}(\mathbf{f})$. The projective variety $\mathcal{Z}(\mathrm{in}_{\mathbf{0}}(\mathbf{f}))$ of $\mathbb{CP}^{n-1}$ is non-singular by assumption, hence $\mathrm{crit}(\mathbf{f}) \cap X$ is contained in $\{\mathbf{0}\}$. The Collar Neighbourhood Lemma (Lemma 3.4) implies that the germ

$$\big([X, \mathbf{0}], \mathrm{ff}\,\big([\mathbb{R}^{2n}, \mathbf{0}]\big)\big)$$

is that of a p-sub-manifold of $([\mathbb{R}^{2n}, \mathbf{0}], \mathrm{ff}([\mathbb{R}^{2n}, \mathbf{0}]))$. Thus the origin is a conic singular point of X. Moreover, due to Proposition A.3 the set X is irreducible such that its multiplicity is the same as the multiplicity of its tangent cone at $\mathbf{0}$ which is

$$\Pi_{j=1}^{k} \deg \mathrm{in}_{\mathbf{0}}(f_j) = m. \qquad \blacksquare$$

We say that a *germ of a set in* $(\mathbb{K}^n, \mathbf{0})$ *is LNE* if it has an LNE representative of arbitrarily small diameter.

Corollary 4.2. *A generic analytic singularity germ in* $(\mathbb{C}^n, \mathbf{0})$ *of given multiplicity is LNE.*

Proof. From Theorem 4.1 and Theorem 3.6 follows that the set $\mathcal{Z}(\mathbf{f})$ is locally LNE at $\mathbf{0}$. Since the germ $\mathcal{Z}(\mathbf{f})$ is definable as a sub-analytic set, by [34] or [11] there exists a positive radius $r_{\mathbf{f}}$ such that for all positive $r \leq r_{\mathbf{f}}$ the subset $\mathcal{Z}(\mathbf{f})_{\leq r}$ is LNE. This ends the proof. $\blacksquare$

Recall that an analytic singularity germ of $(\mathbb{C}^n, \mathbf{0})$ is an *isolated complete intersection singularity (ICIS)* if the singular point is isolated and the number of generators of its vanishing ideal is equal to its codimension, see for instance [25, p. 329]. Thus by Corollary 4.2 and Propositions A.1 and A.3 we obtain the following.

Corollary 4.3. *A generic ICIS germ is LNE.*

Example 4.4. We illustrate the results of this section as follows.

(1) A generic hypersurface singularity of multiplicity m in $(\mathbb{C}^n, \mathbf{0})$ is LNE.

(2) A generic curve germ of multiplicity m in $(\mathbb{C}^n, \mathbf{0})$ is LNE. One can infer it also using the classical result of [39] which states that a curve germ is outer bi-Lipschitz homeomorphic to its generic planar projection, a hypersurface singularity.

Note that there are many interesting open questions on genericity of LNE singularity germs in families: for instance when a versal unfolding of a LNE singularity germ admits LNE fibers (for global behaviour compare with Example 2).

Using the same methods, one can obtain similar results for real analytic germs, though one needs to take into consideration the real phenomena, such as positive polynomials.

4.2. Affine algebraic sets

This section presents global results. We show that almost all algebraic sets (real and complex) are globally conic sub-manifolds. We always take the algebraic set as zero locus of a polynomial ideal in $\mathbb{K}[\mathbf{x}]$, thus it is a-priori naturally embedded in $\mathbb{K}^n$. There were only a few examples of non-trivial LNE algebraic sets [7, 12, 23], until our recent works on curves [6, 9].

Theorem 4.5. A generic affine algebraic subset of $\mathbb{K}^n$ is a globally conic singular sub-manifold*: for every* $\mathbf{d} \in \mathbb{N}^k$ *there exists a Zariski open dense subset* $V(\mathbf{d})$ *of* $\mathbb{K}_{\mathbf{d},k}[\mathbf{x}]$ *such that the algebraic set* $\mathcal{Z}(\mathbf{f})$ *is a globally conic singular sub-manifold of* $\mathbb{K}^n$.

Proof. Apply Corollary 3.19 to Theorem 4.13, since a generic algebraic set meets the hyperplane at infinity transversely, see Proposition A.2. ∎

Corollary 4.6. *Every connected component of a generic affine algebraic subset of* $\mathbb{K}^n$ *is LNE.*

Example 4.7. In particular, the following algebraic sets satisfy the results of this section.

(1) A generic complex affine hypersurface is LNE.

(2) A generic complex affine curve is LNE. In fact by [9] a complex affine curve is LNE if and only if it has ordinary singularities and meets the hyperplane at infinity transversely, compare Section 4.3.

4.3. Curves

A special case worth mentioning are curves: we show in this section that connected and globally conic is equivalent to the LNE property. A paper of note on the subject of global Lipschitz geometry of complex curves is [38], which presents a complete bi-Lipschitz invariant of the outer metric space structure of complex affine plane curves, yet does not describe which ones are LNE.

Rephrasing [9, Proposition 4.6] in the terms of Theorem 3.6 yields.

Corollary 4.8. *A closed* $\mathbb{K}$*-analytic curve of a compact* $\mathbb{K}$*-analytic manifold is LNE if and only if it is a connected conic singular sub-manifold.*

Whereas, the result [9, Theorem 8.1] combined with Corollary 4.8 implies.

Corollary 4.9. *An algebraic curve is LNE in* $\mathbb{K}^n$ *if and only if it is a connected globally conic singular sub-manifold.*

The curve case explicitly shows how the LNE property is dependent on the embedding. In case of algebraic curves in $\mathbb{K}^n$ the main characteristic is the degree of the embedding, for instance: the embeddings of the line $t \to (t, t)$ and $t \to (t, t^2)$ into $\mathbb{K}^2$ induce very different Lipschitz properties of its image, see Example 3.12.

4.4. Fibers of polynomial mappings

Similarly to Section 4.2, one can prove conical structure and thus LNE property of typical fiber of generic polynomial mappings. The LNE property of fibers already appeared in [7] for levels of polynomial mappings over Lipschitz trivial values, even though such mappings are sparse, see also [14]. Therefore, our results imply prevalence of polynomial mappings with LNE general fiber and without Lipschitz trivial values. Example 2 shows that LNE property may appear non-generically in a family of fibers.

Recall that a fiber of a mapping $\mathbf{f} : \mathbb{K}^n \to \mathbb{K}^k$ over the value $\mathbf{c}$ is *typical* if $\mathbf{f}$ is a locally smooth trivial fibration over some open neighbourhood of $\mathbf{c}$. By a Bertini–Sard Theorem, non-typical values of polynomial mappings are contained in a nowhere dense Zariski closed subset of $\mathbb{K}^k$, see for instance [21].

Theorem 4.10. Every typical fiber of a generic polynomial mapping is a globally conic singular sub-manifold*: for every* $\mathbf{d} \in \mathbb{N}^k$ *there exists a Zariski open dense subset* $W(\mathbf{d})$ *of* $\mathbb{K}_{\mathbf{d},k}[\mathbf{x}]$ *such that the fiber* $\mathcal{Z}(\mathbf{f} - \mathbf{c})$ *is a globally conic singular sub-manifold of* $\mathbb{K}^n$ *for* $\mathbf{f} \in W(\mathbf{d})$ *and a typical value* $\mathbf{c}$ *of* $\mathbf{f}$.

Proof. If the typical level of a generic polynomial map meets the hyperplane at infinity transversely, then all typical levels either meet the hyperplane transversely or are compact (the latter only in the real case). The zero level is a typical level of generic polynomial mapping: for any $\mathbf{d} \in \mathbb{N}^k$ there exists a Zariski open dense set U of $\mathbb{K}_{\mathbf{d},k}[\mathbf{x}]$ such that $\mathbf{0}$ is a typical level of $\mathbf{f} \in U$. Now, since meeting transversely the hyperplane at infinity is also generic, see Proposition A.2, we get the claim. ∎

Corollary 4.11. *Every connected component of a typical level of a generic polynomial mapping*

$$\mathbf{f} : \mathbb{K}^n \to \mathbb{K}^k$$

is LNE.

Example 4.12. We can for instance single out the following cases.

(1) All typical levels of a complex polynomial f are LNE, provided the zero level of its highest degree form f_d is reduced and intersects the hyperplane $\mathbf{H}_\infty$ transversely. If the polynomial f is real, the same holds true for every connected component of its typical fibers.

(2) Let $f : \mathbb{C}^2 \to \mathbb{C}$, $(x, y) \to x^2 y$. No fiber of f is LNE, except for the critical reducible zero level fiber (by LNE curve characterization of [9]).

(3) All fibers of the polynomial map $(x, y, z) \to (x, xy + z)$ are affine lines, but the mapping is not a Lipschitz trivial fibration over any value, see [7, Example 5.6].

4.5. General affine trace of projective algebraic sets

In this section, we show that a generic algebraic subset of the projective space and its general affine part are LNE, even though the change from the Fubini–Study (or any Riemannian) metric of the projective space to the Euclidean metric is quite drastic, thus the relation seems non-intuitive. This behaviour was already observed for complex algebraic curves in [9] and we feel that it is of similar flavor as the result of [12]. We end this section by examples showing that unfortunately complex projective closure is not the right compactification to study LNE properties of affine subsets.

Although Theorem 4.13 below restates a well-known property, we give it for reference here.

Theorem 4.13. A generic algebraic subset of $\mathbb{KP}^n$ is a conic singular sub-manifold*: for every* $\mathbf{d} \in \mathbb{N}^k$ *there exists a Zariski open dense subset* $U(\mathbf{d})$ *of* $\mathbb{K}^{\mathrm{hom}}_{\mathbf{d},k}[\mathbf{x}]$ *such that the algebraic set* $\mathcal{Z}(\mathbf{f})$ *is a globally conic singular sub-manifold of* $\mathbb{KP}^n$ *for* $\mathbf{f} \in U(\mathbf{d})$.

Proof. A generic projective algebraic subset is non-singular, see Proposition A.1, and non-singular varieties are smooth. ∎

Corollary 4.14. *Every connected component of a generic projective algebraic set in* $\mathbb{KP}^n$ *is LNE.*

Given a hyperplane H of $\mathbb{KP}^n$ consider the affine space

$$\mathbb{K}^n_H := \mathbb{KP}^n \setminus H$$

naturally equipped with its Euclidean structure eucl_H. A $PGL_{n+1}(\mathbb{K})$ automorphism geometrically maps the hyperplane at infinity to another hyperplane H, thus it induces a unitary $\mathbb{K}$-linear mapping between $(\mathbb{K}^n, \text{eucl})$ and $(\mathbb{K}^n_H, \text{eucl}_H)$. If X is a subset of $\mathbb{KP}^n$, then the set

$$X \setminus H \subset \mathbb{K}^n_H$$

is called the *affine trace of X with respect to the hyperplane H* and we treat it with respect to the Euclidean metric of the affine space. A *general affine trace* of a projective algebraic set is its affine trace with respect to a generic hyperplane H.

Theorem 4.15. A general affine trace of a generic algebraic set in the projective space $\mathbb{KP}^n$ is a globally conic singular sub-manifold in $\mathbb{K}^n$ with respect to the standard Euclidean metric*: if X is a non-singular algebraic subset of* $\mathbb{KP}^n$*, there exists a Zariski open dense subset $H(X)$ of the dual projective space of hyperplanes $\check{\mathbb{KP}}^n$ such that for every $H \in H(X)$ the affine trace $X \setminus H$ is a globally conic singular sub-manifold of $\mathbb{K}^n = \mathbb{KP}^n \setminus H$.*

Proof. Apply Corollary 3.19 to Theorem 4.13, since a generic hyperplane meets the algebraic set transversely, see Proposition A.2. ■

Quite non-intuitively Corollary 4.16 below indicates a relation between Lipschitz properties in the projective space and the affine space, even though the metrics vastly differ.

Corollary 4.16. *Every connected component of a general affine trace of a generic projective algebraic set is LNE in* $\mathbb{K}^n$.

Corollary 4.17. *A general affine trace of a projective algebraic subset of* $\mathbb{KP}^n$ *with isolated singularities is LNE in* $\mathbb{K}^n$ *if and only if it is connected and all singularities are LNE.*

Proof. Immediate by Proposition 2.8, Corollary 3.19 and genericity of transversality at infinity, see Proposition A.2. ■

Example 4.18. The following situations, although simple, are worth mentioning.

(1) All conic curves are LNE in $\mathbb{C}^2$ with the exception of parabola: indeed, they are general affine traces of a LNE projective curve, compare with [9].

(2) A General affine trace in $\mathbb{K}^n$ of a complex curve or normal surface with LNE singularities is LNE (for instance the singularities can be taken as in [23, 31, 33] or Section 4.1).

Remark 4.19. The Zariski open conditions of Propositions A.1 and A.2 yield non-empty Zariski open subsets in the moduli space (under the projective linear action) of projective varieties in $\mathbb{KP}^n$ of given degree, transverse to a given hyperplane. Thus the notion of genericity we work with is essentially the same as that we would obtain working with such moduli spaces instead. So one can interpret the results of this section as: *almost every element of the orbit under unitary action of an algebraic set with isolated singularities has LNE affine trace with respect to the hyperplane H_∞ if and only if all singularities of the set are LNE.*

Example 4.20 below shows that even though projective closure behaves well under genericity conditions, in general it breaks Lipschitz properties, contrary to one-point compactification or localization via inversion, see Section 3.4.

Example 4.20.

(1) The parabola $x = y^2$ is *not LNE in* $\mathbb{K}^2$ (Example 3.12), but it is an affine trace of a projective non-singular conic curve which is *LNE in* $\mathbb{KP}^2$.

(2) The real cubic $x = y^3$ *is LNE in* $\mathbb{R}^2$, because it is smooth and its image under inversion is LNE at $\mathbf{0}$ (see [11]). But its projective closure has a cuspidal singularity at $(1 : 0 : 0)$, thus it is *not LNE in* $\mathbb{RP}^2$.

A. On genericity

First we recall some standard notations, then we will present genericity Bertini-like theorems that are used in the proofs of Section 4. Throughout the Appendix we consider polynomials in at least two variables. Although the results are part of folklore, demonstrations are available in [8, Appendix].

Denote $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Let $\mathbb{K}[\mathbf{x}] := \mathbb{K}[x_1, \dots, x_n]$ be the $\mathbb{K}$-algebra of polynomial functions over $\mathbb{K}^n$. Let $\mathbb{K}_d^{\text{hom}}[\mathbf{x}]$ be the set of all homogeneous polynomials of degree d in n variables and $\mathbb{K}_d[\mathbf{x}]$ be the set of polynomials of degree at most d. For $\mathbf{d} = (d_1, \dots, d_k) \in \mathbb{N}^k$ denote by

$$\mathbb{K}_{\mathbf{d},k}^{\text{hom}}[\mathbf{x}] := \mathbb{K}_{d_1}^{\text{hom}}[\mathbf{x}] \times \cdots \times \mathbb{K}_{d_k}^{\text{hom}}[\mathbf{x}]$$

the set of polynomial mappings $\mathbb{K}^n \to \mathbb{K}^k$ with homogeneous coordinates of degree vector $\mathbf{d}$ and by

$$\mathbb{K}_{\mathbf{d},k}[\mathbf{x}] := \mathbb{K}_{d_1}[\mathbf{x}] \times \cdots \times \mathbb{K}_{d_k}[\mathbf{x}]$$

the set of polynomial mappings $\mathbb{K}^n \to \mathbb{K}^k$ with coordinates of degree at most $\mathbf{d}$.

Let $\mathcal{O}_n$ be the local $\mathbb{C}$-algebra of $\mathbb{C}$-analytic function germs and $\mathfrak{m}_n$ be its maximal ideal. Denote by $\mathcal{O}_{n,k}$ the $\mathcal{O}_n$-module of analytic map germs $(\mathbb{C}^n, \mathbf{0}) \to \mathbb{C}^k$. The multiplicity m_f of $f \in \mathcal{O}_n$ is the non-negative integer d such that $f \in \mathfrak{m}_n^d \setminus \mathfrak{m}_n^{d+1}$. The initial form of f at the origin is

$$\text{in}_{\mathbf{0}}(f) := f_{m_f},$$

where $f_{m_f} \in \mathbb{K}^{\hom}_{m_f}[\mathbf{x}]$ is such that $m_{f-f_{m_f}} > m_f$. For an analytic map germ

$$\mathbf{f} = (f_1, \dots, f_k) \in \mathcal{O}_{n,k}$$

denote $\mathrm{in}_{\mathbf{0}}(\mathbf{f}) := (\mathrm{in}_{\mathbf{0}}(f_1), \dots, \mathrm{in}_{\mathbf{0}}(f_k))$. The multiplicity of a pure dimensional analytic set germ $(X, \mathbf{0})$ in $(\mathbb{C}^n, \mathbf{0})$ is its intersection multiplicity with a generic $(n - \dim X)$-dimensional linear space.

Proposition A.1. Generic algebraic set in the projective space $\mathbb{KP}^n$ is non-singular*: for any given* $\mathbf{d} \in \mathbb{N}^k$ *there exists a Zariski open dense subset* $U(\mathbf{d})$ *of* $\mathbb{K}^{\hom}_{\mathbf{d},k}[\mathbf{x}]$ *such that the projective set* $\mathcal{Z}(\mathbf{f})$ *is non-singular for every* $\mathbf{f} \in U(\mathbf{d})$.

There is a $\mathbb{K}$-linear isomorphism

$$f = f_0 + f_1 + \cdots + f_d \to f^{\hom} := x_{n+1}^d f_0 + x_{n+1}^{d-1} f_1 + \cdots + f_d$$

between polynomials $\mathbb{K}_d[\mathbf{x}]$ and homogeneous polynomials $\mathbb{K}^{\hom}_d[\mathbf{x}, x_{n+1}]$. In particular, it maps Zariski open sets onto Zariski open sets.

This correspondence yields the hyperplane at infinity $\mathbf{H}_\infty$ as the vanishing locus of x_{n+1} as well as the natural embedding of $\mathbb{K}^n$ into $\mathbb{KP}^n$ as $\mathbf{x} \to [\mathbf{x} : 1]$. Therefore, Proposition A.1 is equivalent to an analogous statement for affine algebraic sets. In view of Theorem 4.1, it is worth recalling that the zero set of $f^{\hom}$ is a $\mathbb{K}$-cone over the zero set of f.

Proposition A.2. Intersection of a non-singular algebraic set and a hyperplane is generically transverse*: Given a non-singular algebraic subset* X *of* $\mathbb{KP}^n$*, there exists a Zariski open dense subset* $H(X)$ *of the projective space of hyperplanes* $\check{\mathbb{KP}}^n$ *such that every* $H \in H(X)$ *intersects* X *transversely. Conversely, if* H *is a hyperplane in* $\mathbb{KP}^n$ *and* $\mathbf{d} \in \mathbb{N}^k$*, then there exists a Zariski open dense set* $V(H, \mathbf{d})$ *of* $\mathbb{K}^{\hom}_{\mathbf{d},k}[\mathbf{x}]$ *such that* $\mathcal{Z}(\mathbf{f})$ *intersects* H *transversely for every* $\mathbf{f} \in V(H, \mathbf{d})$.

Proposition A.3. A generic ideal of complex polynomials is prime*: for any given* $\mathbf{d} \in \mathbb{N}^k$ *with* $k \leq n-2$ *there exists a Zariski open dense subset* $P(\mathbf{d})$ *of* $\mathbb{C}^{\hom}_{\mathbf{d},k}[\mathbf{x}]$ *such that the ideal generated by coordinates of* $\mathbf{f}$ *is prime for every* $\mathbf{f} \in P(\mathbf{d})$.

Proof of the latter can be found in [42]. Note that in particular, the degree of $\mathcal{Z}(\mathbf{f})$ for $\mathbf{f} \in P(\mathbf{d})$ is equal to $\Pi_{j=1,\dots,k} \deg f_j$, where the degree of a pure dimensional algebraic subset X of $\mathbb{CP}^n$ is its intersection number with the general linear space of dimension $n - \dim X$.

Acknowledgments. The second named author is very grateful to Daniel Cibotaru for motivating discussions and comments. The last author would like to thank Helge Pedersen for sharing examples of bad behaviour of LNE property under blowing-up, inspiring Example 4.20.

Funding. The last author was partially supported by grant FAPESP 2019/21181-02.

References

[1] L. Birbrair and T. Mostowski, Normal embeddings of semialgebraic sets. *Michigan Math. J.* **47** (2000), no. 1, 125–132 Zbl 0983.32005 MR 1755260

[2] J. Cheeger, On the spectral geometry of spaces with cone-like singularities. *Proc. Nat. Acad. Sci. U.S.A.* **76** (1979), no. 5, 2103–2106 Zbl 0411.58003 MR 0530173

[3] J. Cheeger, Spectral geometry of singular Riemannian spaces. *J. Differential Geom.* **18** (1983), no. 4, 575–657 Zbl 0529.58034 MR 0730920

[4] J. Cheeger and M. Taylor, On the diffraction of waves by conical singularities. I. *Comm. Pure Appl. Math.* **35** (1982), no. 3, 275–331 Zbl 0526.58049 MR 0649347

[5] J. Cheeger and M. Taylor, On the diffraction of waves by conical singularities. II. *Comm. Pure Appl. Math.* **35** (1982), no. 4, 487–529 Zbl 0536.58032 MR 0657825

[6] A. Costa, *Characterization of Lipschitz normally embedded complex curves and Lipschitz trivial values of polynomial mappings*. Ph.D. thesis, Unversidade Federal do Ceará, 2023, available at https://repositorio.ufc.br/handle/riufc/70235 visited on 25 September 2024

[7] A. Costa, V. Grandjean, and M. Michalska, Lipschitz trivial values of polynomial mappings. *J. Geom. Anal.* **32** (2022), no. 11, article no. 269 Zbl 1493.14100 MR 4470306

[8] A. Costa, V. Grandjean, and M. Michalska, Global Lipschitz geometry of conic singular submanifolds with applications to algebraic sets. 2023, arXiv:2306.14854v1

[9] A. Costa, V. Grandjean, and M. Michalska, One point compactification and Lipschitz normally embedded definable subsets. 2023, arXiv:2304.08555v4

[10] A. Costa, V. Grandjean, and M. Michalska, Characterization of Lipschitz normally embedded complex curves. *Bull. Sci. Math.* **190** (2024), article no. 103369 Zbl 1529.14016 MR 4673264

[11] A. Costa, V. Grandjean, and M. Michalska, Remarks on Lipschitz geometry on globally conic singular manifolds. 2024, in preparation

[12] L. R. G. Dias and N. R. Ribeiro, Lipschitz normally embedded set and tangent cones at infinity. *J. Geom. Anal.* **32** (2022), no. 2, article no. 51 Zbl 1479.58028 MR 4358703

[13] L. Fantini and A. Pichon, On Lipschitz normally embedded singularities. In *Handbook of geometry and topology of singularities IV*, pp. 497–519, Springer, Cham, 2023 MR 4676328

[14] A. Fernandes, V. Grandjean, and H. Soares, A note on the local Lipschitz triviality of values of complex polynomial functions. *Math. Z.* **296** (2020), no. 1-2, 861–874 Zbl 1441.14012 MR 4140767

[15] A. Fernandes and J. E. Sampaio, On Lipschitz rigidity of complex analytic sets. *J. Geom. Anal.* **30** (2020), no. 1, 706–718 Zbl 1448.14002 MR 4058534

[16] V. Grandjean and R. Oliveira, Stereographic compactification and affine bi-Lipschitz homeomorphisms. *Glasg. Math. J.* (2024), DOI 10.1017/S001708952400017X

[17] D. Grieser, A natural differential operator on conic spaces. *Discrete Contin. Dyn. Syst.* **2011** (2011), 568–577 Zbl 1306.58010 MR 2987439

[18] M. Gromov, Hyperbolic manifolds, groups and actions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, pp. 183–213, Ann. of Math. Stud 97, Princeton University Press, Princeton, NJ, 1981 Zbl 0467.53035 MR 0624814

[19] M. Gromov, *Structures métriques pour les variétés riemanniennes*. Text. Math. 1, CEDIC, Paris, 1981 Zbl 0509.53034 MR 0682063

[20] A. Hassell, T. Tao, and J. Wunsch, Sharp Strichartz estimates on nontrapping asymptotically conic manifolds. *Amer. J. Math.* **128** (2006), no. 4, 963–1024 Zbl 1177.58019 MR 2251591

[21] Z. Jelonek and K. Kurdyka, Quantitative generalized Bertini-Sard theorem for smooth affine varieties. *Discrete Comput. Geom.* **34** (2005), no. 4, 659–678 Zbl 1083.14069 MR 2173932

[22] K. Katz, M. Katz, D. Kerner, and Y. Liokumovich, Determinantal variety and normal embedding. *J. Topol. Anal.* **10** (2018), no. 1, 27–34 Zbl 1406.53048 MR 3737507

[23] D. Kerner, H. M. Pedersen, and M. A. S. Ruas, Lipschitz normal embeddings in the space of matrices. *Math. Z.* **290** (2018), no. 1-2, 485–507 Zbl 1419.14003 MR 3848442

[24] K. Kurdyka, On a subanalytic stratification satisfying a Whitney property with exponent 1. In *Real algebraic geometry (Rennes, 1991)*, pp. 316–322, Lecture Notes in Math. 1524, Springer, Berlin, 1992 Zbl 0779.32006 MR 1226263

[25] S. Łojasiewicz, *Introduction to complex analytic geometry*. Birkhäuser, Basel, 1991 Zbl 0747.32001 MR 1131081

[26] R. Melrose, Differential analysis on manifolds with corners. 1996, unfinished book, http://math.mit.edu/~rbm visited on 25 September 2024

[27] R. Melrose and J. Wunsch, Propagation of singularities for the wave equation on conic manifolds. *Invent. Math.* **156** (2004), no. 2, 235–299 Zbl 1088.58011 MR 2052609

[28] R. Melrose and M. Zworski, Scattering metrics and geodesic flow at infinity. *Invent. Math.* **124** (1996), no. 1-3, 389–436 Zbl 0855.58058 MR 1369423

[29] R. Mendes and J. E. Sampaio, On the link of Lipschitz normally embedded sets. *Int. Math. Res. Not. IMRN* **2024** (2024), no. 9, 7488–7501 MR 4742831

[30] J. Milnor, *Singular points of complex hypersurfaces*. Ann. of Math. Stud. 61, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1968 Zbl 0184.48405 MR 0239612

[31] F. Misev and A. Pichon, Lipschitz normal embedding among superisolated singularities. *Int. Math. Res. Not. IMRN* **2021** (2021), no. 17, 13546–13569 Zbl 1490.14006 MR 4307695

[32] J. Nash, The imbedding problem for Riemannian manifolds. *Ann. of Math. (2)* **63** (1956), 20–63 Zbl 0070.38603 MR 0075639

[33] W. D. Neumann, H. M. Pedersen, and A. Pichon, Minimal surface singularities are Lipschitz normally embedded. *J. Lond. Math. Soc. (2)* **101** (2020), no. 2, 641–658 Zbl 1441.14016 MR 4093969

[34] N. X. V. Nhan, On a link criterion for Lipschitz normal embeddings among definable sets. *Math. Nachr.* **296** (2023), no. 7, 2958–2974 Zbl 1532.14022 MR 4626868

[35] A. Pichon, An introduction to Lipschitz geometry of complex singularities. In *Introduction to Lipschitz geometry of singularities*, pp. 167–216, Lecture Notes in Math. 2280, Springer, Cham, 2020 Zbl 1457.32073 MR 4200099

[36] W. Schlag, A. Soffer, and W. Staubach, Decay for the wave and Schrödinger evolutions on manifolds with conical ends. I. *Trans. Amer. Math. Soc.* **362** (2010), no. 1, 19–52 Zbl 1185.35046 MR 2550144

[37] J. Stasica, The Whitney condition for subanalytic sets. *Zeszyty Nauk. Uniw. Jagielloń. Prace Mat.* **23** (1982), 211–221 Zbl 0508.32001 MR 0670588

[38] R. Targino, Outer Lipschitz geometry of complex algebraic plane curves. *Int. Math. Res. Not. IMRN* **2023** (2023), no. 15, 13255–13289 Zbl 1531.30044 MR 4621864

[39] B. Teissier, Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney. In *Algebraic geometry (La Rábida, 1981)*, pp. 314–491, Lecture Notes in Math. 961, Springer, Berlin, 1982 MR 0708342

[40] H. Whitney, Analytic extensions of differentiable functions defined in closed sets. *Trans. Amer. Math. Soc.* **36** (1934), no. 1, 63–89 Zbl 0008.24902 MR 1501735

[41] H. Whitney, Functions differentiable on the boundaries of regions. *Ann. of Math. (2)* **35** (1934), no. 3, 482–485 Zbl 0009.30901 MR 1503174

[42] J. Yu, Do most polynomials generate a prime ideal? *J. Algebra* **459** (2016), 468–474 Zbl 1370.13018 MR 3503982

Communicated by Christian Bär

Received 24 October 2023; revised 27 June 2024.

André Costa
Centro de Ciências e Tecnologias, Universidade Estadual do Ceará, Campus do Itaperi, 60714-903 Fortaleza - CE, Brazil; andrecosta.math@gmail.com

Vincent Grandjean
Departamento de Matemática, Universidade Federal de Santa Catarina, Campus de Trindade, 80040-900 Florianópolis - SC, Brazil; vincent.grandjean@ufsc.br

Maria Michalska
Wydział Matematyki i Informatyki, Uniwersytet Łódzki, Banacha 22, 90-238 Łódź, Poland; maria.michalska@wmii.uni.lodz.pl