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Epsilon-regularity for almost-minimizers of anisotropic free interface problem with Hölder dependence on the position

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Abstract. We establish an ε-regularity result for almost-minimizers of a class of variational problems involving both bulk and interface energies. The bulk energy is of Dirichlet type. The surface energy exhibits anisotropic behavior and is defined by means of an ellipsoidal density that is Hölder continuous with respect to the position variable.

1. Introduction and statements

The existence and regularity of solutions to variational problems, encompassing both bulk and interface energies, have been extensively studied across various disciplines and remain a focal point of much mathematical research. These problems serve to describe a broad spectrum of phenomena in applied sciences, including nonlinear elasticity, materials science and image segmentation in computer vision (see, for instance, [2,4,12,27–30,32,35]). A model integral functional initially introduced to study minimal energy configurations of two conducting materials by R. V. Kohn, G. Strang in [33] and F. Murat, L. Tartar in [41], and later recovered by L. Ambrosio, G. Buttazzo in [3] and F. H. Lin in [37] is the following:

$$\int_\Omega \sigma_E(x)|\nabla u|^2\,dx + P(E;\Omega), \tag{1.1}$$

where $\sigma_E := \alpha \mathbb{1}_E + \beta \mathbb{1}_{\Omega\setminus E}$, $0 < \alpha < \beta$, with $E \subset \Omega \subset \mathbb{R}^n$ and $u \in H^1(\Omega)$. Here, $\mathbb{1}_E$ stands for the characteristic function of E and $P(E;\Omega)$ denotes the perimeter of the set E in Ω. In [3,37], the authors proved the existence and regularity for minimal configurations (E,u) of (1.1). In a broader context, F. H. Lin and R. V. Kohn addressed more generalized Dirichlet energies as outlined in [38],

$$\int_\Omega (F(x,u,\nabla u) + \mathbb{1}_E G(x,u,\nabla u))dx + \int_{\Omega\cap\partial^* E} \Psi(x,\nu_E(x))d\mathcal{H}^{n-1}(x), \tag{1.2}$$

with the constraints

$$u = u_0 \quad \text{on } \partial\Omega \quad \text{and} \quad |E| = d.$$

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Here, ν_E is the measure-theoretic outward unit normal to the reduced boundary $\partial^* E$ of E. The regularity for the minimizing pair (E, u) is quite intricate to establish, especially concerning the free boundary ∂E due to the interaction between the bulk term and the perimeter term. In order to illustrate the regularity results of the free boundary, we introduce the following notations. We define the set of regular points of ∂E as follows:

$$\begin{aligned}&\mathrm{Reg}(E)\\ &:= \big\{x \in \partial E \mid \partial E \text{ is a } C^{1,\gamma} \text{ hypersurface in } B_\varepsilon(x) \subset \Omega \text{ for some } \varepsilon > 0 \text{ and } \gamma \in (0,1)\big\},\end{aligned}$$

and accordingly, we define the set of singular points of ∂E as

$$\Sigma(E) := (\partial E \cap \Omega) \setminus \mathrm{Reg}(E).$$

The most notable advancements in regularity results regarding the free boundary ∂E of minimizers of the functional (1.1) have been accomplished by G. De Philippis, A. Figalli in [16] and N. Fusco, V. Julin in [29]. They proved that for minimal configurations of the functional (1.1) it turns out that

$$\dim_{\mathcal{H}}(\Sigma(E)) \leq n - 1 - \eta \tag{1.3}$$

for some $\eta > 0$ depending only on α, β. In the more general case of integral functionals of the type (1.2), the theory of regularity is much less developed. The first regularity result established in the broader context of integral energies of the type (1.2) was accomplished by F. H. Lin and R. V. Kohn in 1999; indeed, in [38], they proved, for minimal configurations (E, u) of (1.2), that $\mathcal{H}^{n-1}(\Sigma(E)) = 0$. The assumptions made by Lin and Kohn to achieve such regularity results require C^k differentiability for F, G, and Ψ as they appear in (1.2), where $k \geq 2$ and F, G grow quadratically with respect to the gradient variable. On the other hand, nothing is proved concerning the Hausdorff dimension of $\Sigma(E)$ like in (1.3).

We also point out that the problem can be set in a non-quadratic framework as well. This instance is less studied and only few regularity results are available (see [4,9–11,21,27,34]).

In some recent papers, such as [23,24], the Hausdorff dimension estimate of $\Sigma(E)$ has been attained, significantly relaxing the differentiability assumption required on F and G. Indeed, in [23], only Hölder continuous dependence of F and G with respect to x and u is necessary. However, it is worth noting that the aforementioned result is demonstrated under the assumption that $\Psi(\nu) = |\nu|$, representing the conventional perimeter.

In this paper, we will deal with the anisotropic case. Anisotropic surface energies manifest in various physical phenomena, such as crystal formation (refer to [5,6]), liquid droplets (see [12,18,26,36,40,44]), and capillary surfaces (see [17,19]). F. J. Almgren was a pioneer in investigating the regularity of surfaces that minimize anisotropic variational problems in his seminal paper [1]. Early studies in this field were primarily conducted within the framework of varifolds and currents. While these results can be applied to

surfaces of any codimension, they necessitate relatively stringent regularity assumptions on the integrands of the anisotropic energies, as outlined in [7, 42]. More recently, the regularity assumptions on the integrands Ψ of the anisotropic energies have been relaxed, as highlighted in [20, 25], where it is assumed that $\Psi(x,\cdot)$ is of class C^1 and $\Psi(\cdot,\xi)$ is Hölder continuous.

In this context, it is worth mentioning a very recent paper [43], which establishes the regularity result for quasi-minimizers of anisotropic surface energies within the class of sets of finite perimeter, under the assumption of Hölder continuous dependence of Ψ on x. This outcome is derived within the scope of ellipsoidal variational energies, as detailed in (1.4). Notably, surface energies of this specific form were initially introduced in a paper [45] by J. Taylor. In more detail, in the case that the elliptic integrand is given by $\Psi(x,\nu) = \langle A(x)\nu,\nu\rangle^{1/2}$, where $A(x) = (a_{ij}(x))_{i,j=1}^n$ is an elliptic and bounded matrix, the surface energy takes the form

$$\mathbf{\Phi}_A(E;G) := \int_{G\cap\partial^* E} \langle A(x)\nu_E,\nu_E\rangle^{1/2}\, d\mathcal{H}^{n-1}(x). \tag{1.4}$$

We assume that A is uniformly elliptic; that is, there exist two constants $0<\lambda\le\Lambda<+\infty$ such that

$$\lambda|\xi|^2 \le \langle A(x)\xi,\xi\rangle \le \Lambda|\xi|^2 \quad \forall x\in\Omega,\ \forall\xi\in\mathbb{R}^n.$$

We require that A is Hölder continuous with exponent $\mu\in(0,1]$, that is,

$$[A]_{C^\mu(\Omega)} = \sup_{\substack{x\neq y\\ x,y\in\Omega}} \frac{|A(x)-A(y)|}{|x-y|^\mu} < +\infty.$$

To avoid excessive technicalities, we assume that the bulk energy follows a Dirichlet-type distribution, although the outcome could be generalized to functionals of type (1.2). Given a bounded open set $\Omega\subset\mathbb{R}^n$, we consider the following functional:

$$\mathcal{F}_A(E,u;\Omega) = \int_\Omega \sigma_E|\nabla u|^2\,dx + \mathbf{\Phi}_A(E;\Omega), \tag{1.5}$$

where $\sigma_E = \alpha\mathbb{1}_E + \beta\mathbb{1}_{\Omega\setminus E}$, $0<\alpha<\beta$, and $E\subset\Omega$. The achieved ε-regularity result is presented within the scope of local almost-minimizers. This makes it applicable in a variety of concrete applications, as we will demonstrate, for example, in the case of constrained problems. The following definition naturally arises in several problems from material sciences (see, for example, [3,22,37–39], compare also with [43, Definition 2.2]).

Definition 1.1 ((κ,μ)-minimizers). Let $U\Subset\Omega$. The energy pair (E,u) is a (κ,μ)-minimizer in U of the functional $\mathcal{F}_A$, defined in (1.5), if for every $B_r(x_0)\subset U$

$$\mathcal{F}_A(E,u;B_r(x_0)) \le \mathcal{F}_A(F,v;B_r(x_0)) + \kappa|E\,\triangle\,F|^{\frac{n-1}{n}+\frac{\mu}{n}},$$

whenever (F,v) is an admissible test pair, namely, F is a set of finite perimeter with $F\,\triangle\,E\Subset B_r(x_0)$ and $v-u\in H^1_0(B_r(x_0))$.

The main theorem proved in this paper is the following theorem.

Theorem 1.2. *Let (E, u) be a (κ, μ)-minimizer of $\mathcal{F}_A$. Then, the following statements hold:*

(i) *there exists a relatively open set $\Gamma \subset \partial E$ such that Γ is a $C^{1,\sigma}$-hypersurface for all $0 < \sigma < \frac{\mu}{2}$;*

(ii) *there exists $\eta > 0$ depending on n, α, β such that*

$$\mathcal{H}^{n-1-\eta}((\partial E \setminus \Gamma) \cap \Omega) = 0.$$

We outline the strategy adopted to prove this result. In the regularity theory for Λ-minimizers of the perimeter, the regular part Γ of the boundary of E is detected by the points that have a uniformly small excess in some ball (see Definition 2.3). A decay relation for the excess plays a crucial role, as it triggers an iteration argument that shows that the unitary normal vector varies continuously along Γ, thus ensuring its smoothness.

For our problem, it is not possible to prove a decay relation for the excess without considering the interaction between the surface and the bulk energy. Indeed, as outlined in Section 7, if the excess of a point x_0 in ∂E in some ball $B_r(x_0)$ is small, we are only able to prove an improvement relation for the excess, which involves the rescaled Dirichlet integral of u in $B_r(x_0)$ as well.

In this context, Γ is defined as the collection of the points of the boundary of E that are centers of balls $B_r(x_0)$ where the excess $\mathbf{e}(E, x_0, r)$ and the rescaled Dirichlet integral $\mathcal{D}_u(x_0, r)$ (see (4.1)) of u are sufficiently small. In order to prove the smoothness of Γ, a decay relation for the sum of these quantities is required.

A decay relation for the rescaled Dirichlet integral of u around points of small excess is proved separately in Proposition 4.7.

A much finer argument is needed to establish an improvement relation for the excess (see Theorem 7.3). One of the key concepts enabling us to adapt the standard excess-decay arguments, commonly used in the context of perimeter minimizers, to the anisotropic setting is a specific change of variable T_{x_0}. This affine transformation, already used in [13,31, 43], maps Wulff shapes of Φ_A, which are ellipsoids, into balls $B_r(x_0)$ (see Section 2). We first prove a version of the excess improvement theorem for transformed couples $(\widetilde{E}, \tilde{u}) = (T_{x_0}(E), u \circ T_{x_0}^{-1})$, which are $(\kappa\lambda^{-\frac{n}{2}}, \mu)$-minimizers of $\mathcal{F}_{x_0, A_{x_0}}$ (see (3.1)). The proof of the latter is carried out by contradiction and is based on a blow-up argument. In this step, we can benefit of using the classical perimeter instead of the anisotropic one around x_0, being $A_{x_0}(x_0) = I$ (see Proposition 6.1). The main ideas can be summarized as follows.

(1) Density lower and upper bounds on the perimeter (see Theorems 4.2 and 4.4) guarantee that around points x_0 of $\partial\widetilde{E}$ with small excess, the boundary of $\widetilde{E}$ almost coincides with the graph of a Lipschitz function f (see Theorem 4.9). Therefore, it is possible to apply the area formula directly along $\partial\widetilde{E}$ up to a small error. The portion of the boundary that does not match is controlled by the excess at that scale.

(2) The function f is quasi-harmonic. We need a quantitative estimate of its quasi-harmonicity by a power of the excess with an exponent greater than $\frac{1}{2}$. In this step, the minimality of the optimal couple $(\widetilde{E}, \tilde{u})$ and the first variation formulae play a crucial role.

(3) The direction of improvement of the excess is detected by the unitary normal vector to the graph of f. By means of a reverse Poincaré inequality (see Theorem 6.3), the excess at a smaller scale at x_0 is controlled by the flatness of $\partial\widetilde{E}$ around x_0, which is in turn estimated by the excess via the good properties of f.

The paper is divided into sections, which reflect the proof strategy. Section 2 collects notation and preliminary definitions. In Section 3, some invariance properties of the excess and minimality under the transformation T_{x_0} and rescaling are proved. In Section 4, we establish density lower and upper bounds for the perimeter of E and their consequences, which are the decay of the rescaled Dirichlet energy and the Lipschitz approximation theorem. Section 5 is devoted to prove a compactness result for sequences of (κ, α)-minimizers, which serves as a crucial tool for estimating the size of the singular set of E, as stated in Theorem 1.2. Section 6 includes the reverse Poincaré inequality, which is the counterpart of the well-known Caccioppoli's inequality for weak solutions of elliptic equations. Section 7 contains the proof of Theorem 1.2. The main ingredients to achieve such a result are a first variation formula for the bulk energy of the functional $\mathcal{F}_{x_0,A_{x_0}}$ and two versions of the excess improvement theorem. Finally, Section 8 deals with the application of the regularity result to a volume constrained problem via a penalization argument.

2. Notation and preliminaries

In the rest of the paper, we will write $\langle \xi, \eta \rangle$ for the inner product of vectors $\xi, \eta \in \mathbb{R}^n$, and consequently, $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ will be the corresponding Euclidean norm. As usual, ω_n stands for the Lebesgue measure of the unit ball in $\mathbb{R}^n$. We denote by $\mathbb{S}^{n-1}$ the unit sphere of $\mathbb{R}^n$.

We will write $x = (x', x_n)$ for all $x \in \mathbb{R}^n$, where $x' \in \mathbb{R}^{n-1}$ collects the first $n-1$ components of x and $x_n \in \mathbb{R}$ is its nth component. Accordingly, we denote by $\nabla' = (\partial_{x_1}, \dots, \partial_{x_{n-1}})$ the gradient with respect to the first $n-1$ components.

The n-dimensional ball in $\mathbb{R}^n$ with center x_0 and radius $r > 0$ is denoted as

$$B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}.$$

If $x_0 = 0$, we simply write B_r in place of $B_r(x_0)$. The $(n-1)$-dimensional ball in $\mathbb{R}^{n-1}$ with center x_0' and radius $r > 0$ is denoted by

$$\mathbf{D}_r(x_0') = \{x' \in \mathbb{R}^{n-1} : |x' - x_0'| < R\}.$$

If u is integrable in $B_R(x_0)$, we set

$$u_{x_0,r} = \frac{1}{\omega_n r^n} \int_{B_r(x_0)} u \, dx = \fint_{B_r(x_0)} u \, dx.$$

If $E \subset \mathbb{R}^n$ and $t \in [0,1]$, the set of points of E of density t is defined as

$$E^{(t)} = \left\{x \in \mathbb{R}^n : |E \cap B_r(x)| = t|B_r(x)| + o(r^n) \text{ as } r \to 0^+\right\}.$$

Given a Lebesgue measurable set $E \subset \mathbb{R}^n$, we say that E is of locally finite perimeter if there exists a $\mathbb{R}^n$-valued Radon measure μ_E (called the Gauss–Green measure of E) such that

$$\int_E \nabla\phi \, dx = \int_{\mathbb{R}^n} \phi \, d\mu_E \quad \forall \phi \in C_c^1(\mathbb{R}^n).$$

Moreover, we denote the perimeter of E relative to $G \subset \mathbb{R}^n$ by $P(E;G) = |\mu_E|(G)$.

The support of μ_E can be characterized by

$$\text{spt}\mu_E = \left\{x \in \mathbb{R}^n : 0 < |E \cap B_r(x)| < \omega_n r^n, \ \forall r > 0\right\}$$

(see [39, Proposition 12.19]). It holds that $\text{spt}\mu_E \subset \partial E$. If E is of locally finite perimeter, then the *reduced boundary* $\partial^* E$ of E is the set of those $x \in \mathbb{R}^n$ such that

$$\nu_E(x) := \lim_{r \to 0^+} \frac{\mu_E(B_r(x))}{|\mu_E|(B_r(x))}$$

exists and belongs to $\mathbb{S}^{n-1}$. In the following, the topological boundary ∂E must be understood by considering the representative $E^{(1)}$ of E, for which it holds that $\overline{\partial^* E} = \partial E$.

The properties of the matrix A in the definition of the anisotropic perimeter (1.4) guarantee that $\mathbf{\Phi}_A(E;F)$ is comparable to the classical perimeter, as observed in [43].

Remark 2.1 (Comparability to perimeter). $\mathbf{\Phi}_A(E;\cdot)$ is comparable to $P(E;\cdot)$, since for Borel sets $F \subset \mathbb{R}^n$, by the uniform ellipticity of A, it follows that

$$\lambda^{1/2} P(E;F) \le \mathbf{\Phi}_A(E;F) \le \Lambda^{1/2} P(E;F). \tag{2.1}$$

If A equals the identity matrix I, we have the isotropic case $\mathbf{\Phi}_A(E;\cdot) = P(E;\cdot)$.

It will be useful in the sequel to build comparison sets by replacing regions within an open set. The anisotropic perimeter can be split as in the isotropic case.

Proposition 2.2 (Comparison by replacement). *If E and F are sets of locally finite perimeter in $\mathbb{R}^n$ and G is an open set of finite perimeter in $\mathbb{R}^n$ such that*

$$\mathcal{H}^{n-1}(\partial^* G \cap \partial^* E) = \mathcal{H}^{n-1}(\partial^* G \cap \partial^* F) = 0,$$

then the set defined by

$$F_0 = (F \cap G) \cup (E \setminus G)$$

is a set of locally finite perimeter in $\mathbb{R}^n$. Moreover, if $G \Subset U$ and U is an open subset of $\mathbb{R}^n$, then

$$\mathbf{\Phi}_A(F_0;U) = \mathbf{\Phi}_A(F;G) + \mathbf{\Phi}_A(E;U \setminus \bar{G}) + \mathbf{\Phi}_A(G;E^{(1)} \triangle F^{(1)}).$$

Proof. The proof can be easily obtained from [39, Theorem 16.16]. Its details can be found in [43, Proposition 4.3]. ∎

In the following, for $R > 0$ and $\nu \in \mathbb{S}^{n-1}$, we will denote the cylinder centered in x_0 with radius R oriented in the direction ν by

$$\mathbf{C}_R(x_0, \nu) := x_0 + \big\{y \in \mathbb{R}^n : |\langle y, \nu\rangle| < R, |y - \langle y, \nu\rangle \nu| < R\big\},$$

and the cylinder of radius R oriented in the direction e_n with height 2 by

$$\mathbf{K}_R(x_0) := \mathbf{D}_R(x_0') \times (-1, 1).$$

In the following, for simplicity of notation, we will write

$$\mathbf{C}_R = \mathbf{C}_R(0, e_n) \quad \text{and} \quad \mathbf{K}_R = \mathbf{K}_R(0).$$

In addition, we introduce some usual quantities involved in regularity theory.

Definition 2.3 (Excess). Let E be a set of locally finite perimeter, $x \in \partial E$, $r > 0$ and $\nu \in \mathbb{S}^{n-1}$. We give the following definitions.

(1) The *cylindrical excess* of E at the point x, at the scale r and with respect to the direction ν is defined as

$$\begin{aligned}\mathbf{e}^C(E, x, r, \nu) &:= \frac{1}{r^{n-1}} \int_{\mathbf{C}_r(x,\nu)\cap\partial^* E} \frac{|\nu_E - \nu|^2}{2} d\,\mathcal{H}^{n-1} \\ &= \frac{1}{r^{n-1}} \int_{\mathbf{C}_r(x,\nu)\cap\partial^* E} [1 - \langle \nu_E, \nu\rangle] d\,\mathcal{H}^{n-1}.\end{aligned}$$

(2) The *spherical excess* of E at the point x, at the scale r and with respect to the direction ν is defined as

$$\mathbf{e}(E, x, r, \nu) := \frac{1}{r^{n-1}} \int_{\partial^* E\cap B_r(x)} \frac{|\nu_E - \nu|^2}{2} d\,\mathcal{H}^{n-1}.$$

(3) The *spherical excess* of E at the point x and at the scale r is defined as

$$\mathbf{e}(E, x, r) := \min_{\nu\in\mathbb{S}^{n-1}} \mathbf{e}(E, x, r, \nu).$$

We omit the dependence on the set when it is clear from the context.

3. Scaling and change of variables

Given a symmetric positive matrix A, the diagonal matrix D of his eigenvalues and the matrix V of orthonormal eigenvectors, we have $A = VDV^{-1}$. Accordingly, we define $A^{1/2} = VD^{1/2}V^{-1}$. Being $A^{-1/2}A^{1/2} = I$, the anisotropic perimeter of E coincides with

the standard perimeter of the image of E under the affine change of variable $y = A^{-1/2}x$ up to the scaling factor $\det(A^{-1/2})$. Localizing this argument freezing the matrix $A^{-1/2}(x)$ in a point $x_0 \in \partial E$, we define the affine change of variables

$$T_{x_0}(x) = A^{-1/2}(x_0)(x - x_0) + x_0, \quad T_{x_0}^{-1}(y) = A^{1/2}(x_0)(y - x_0) + x_0 \quad \forall x, y \in \mathbb{R}^n,$$

and the matrix-valued function

$$A_{x_0}(y) := A^{-\frac{1}{2}}(x_0)A(T_{x_0}^{-1}(y))A^{-\frac{1}{2}}(x_0) \quad \forall y \in \mathbb{R}^n,$$

which satisfies $A_{x_0}(x_0) = I$. It can be easily verified that the set $T_{x_0}^{-1}(B_r(x_0))$ is the Wulff shape of $\mathbf{\Phi}_{A(x_0)}$. Moreover, the following inclusions hold:

$$B_{\lambda^{\frac{1}{2}}r}(x_0) \subset T_{x_0}^{-1}(B_r(x_0)) \subset B_{\Lambda^{\frac{1}{2}}r}(x_0)$$

for any $r > 0$ and $x_0 \in \mathbb{R}^n$. Under the affine change of variable T_{x_0}, the minimality with respect to the functional $\mathcal{F}_A$ will be rephrased through the following functional:

$$\mathcal{F}_{x_0,D}(E, u; \Omega) = \int_\Omega \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2 \, dx + \mathbf{\Phi}_D(E; \Omega) \tag{3.1}$$

(see Proposition 3.1).

In the sequel, we collect two invariance properties of (κ, μ)-minimizers under the transformation T_{x_0} and rescaling.

Proposition 3.1 (Invariance of almost-minimizers under T_{x_0}). *Let (E, u) be a (κ, μ)-minimizer of $\mathcal{F}_A$ in Ω and $x_0 \in \Omega$. Then, $(T_{x_0}(E), u \circ T_{x_0}^{-1})$ is a $(\kappa\lambda^{-\frac{n}{2}}, \mu)$-minimizer of $\mathcal{F}_{x_0,A_{x_0}}$ in the balls $B_r(z)$ such that $B_{(\Lambda/\lambda)^{\frac{1}{2}}r}(z) \subset T_{x_0}(\Omega)$.*

Proof. We use the notation $E_0 := T_{x_0}(E)$ and $u_0 := u \circ T_{x_0}^{-1}$. Let $B_r(z) \subset T_{x_0}(\Omega)$ be such that $B_{(\Lambda/\lambda)^{\frac{1}{2}}r}(z) \subset T_{x_0}(\Omega)$ and (F_0, v_0) be an admissible test pair, i.e., F_0 is a set of finite perimeter with $F_0 \,\triangle\, E_0 \Subset B_r(z)$ and $v_0 - u_0 \in H_0^1(B_r(z))$. First, we notice that, setting $F = T_{x_0}^{-1}(F_0)$, as in [43, Proposition 4.1], we have

$$E \,\triangle\, F \Subset T_{x_0}^{-1}(B_r(z)) \subset B_{\Lambda^{\frac{1}{2}}r}(T_{x_0}^{-1}(z)) \subset \Omega,$$

where the last condition is satisfied because $r < \Lambda^{-\frac{1}{2}}\operatorname{dist}(T_{x_0}^{-1}(z), \partial\Omega)$. Moreover, for $v = v_0 \circ T_{x_0}$, we have $v - u \in H_0^1(T_{x_0}^{-1}(B_r(z)))$ whose extension to zero in $B_{\Lambda^{\frac{1}{2}}r}(T_{x_0}^{-1}(z))$, denoted again by $v - u$, belongs to $H_0^1(B_{\Lambda^{\frac{1}{2}}r}(T_{x_0}^{-1}(z)))$. It follows, by the hypothesis of (κ, μ)-minimality of the pair (E, u), that

$$\mathcal{F}_A\big(E, u; B_{\Lambda^{\frac{1}{2}}r}(T_{x_0}^{-1}(z))\big) \le \mathcal{F}_A\big(F, v; B_{\Lambda^{\frac{1}{2}}r}(T_{x_0}^{-1}(z))\big) + \kappa|E \,\triangle\, F|^{\frac{n-1+\mu}{n}}.$$

This simplifies to

$$\mathcal{F}_A\big(E, u; T_{x_0}^{-1}(B_r(z))\big) \le \mathcal{F}_A\big(F, v; T_{x_0}^{-1}(B_r(z))\big) + \kappa|E \,\triangle\, F|^{\frac{n-1+\mu}{n}}.$$

We now calculate, using [43, formula (4.9)] and the change of variables $y = T_{x_0}(x)$,

$$\begin{aligned}
&\mathcal{F}_{x_0,A_{x_0}}(E_0,u_0;B_r(z))\\
&= \int_{B_r(z)} \sigma_{E_0}|\nabla u_0 A^{-\frac{1}{2}}(x_0)|^2\,dy + \mathbf{\Phi}_{A_{x_0}}(E_0;B_r(z))\\
&= \det(A^{-\frac{1}{2}}(x_0))\Big(\int_{T_{x_0}^{-1}(B_r(z))} \sigma_E|\nabla u|^2\,dx + \mathbf{\Phi}_A\big(E;T_{x_0}^{-1}(B_r(z))\big)\Big)\\
&= \det(A^{-\frac{1}{2}}(x_0))\mathcal{F}_A\big(E,u;T_{x_0}^{-1}(B_r(z)))\big)\\
&\le \det(A^{-\frac{1}{2}}(x_0))\big(\mathcal{F}_A\big(F,v;T_{x_0}^{-1}(B_r(z))\big) + \kappa|E\,\triangle\,F|^{\frac{n-1+\mu}{n}}\big),
\end{aligned}$$

where with a slight abuse of notation we have denoted $\sigma_{E_0} = \alpha\mathbb{1}_{E_0} + \beta\mathbb{1}_{T_{x_0}(\Omega)\setminus E_0}$. The result follows by observing that $\det(A^{-\frac{1}{2}}(x_0)) \le \lambda^{-\frac{n}{2}}$ and

$$\det(A^{-\frac{1}{2}}(x_0))\mathcal{F}_A\big(F,v;T_{x_0}^{-1}(B_r(z))\big) = \mathcal{F}_{x_0,A_{x_0}}(F_0,v_0;B_r(z)).$$ ∎

Proposition 3.2 (Scaling of (κ,μ)-minimizers). *For $x_0 \in \Omega$ and $r > 0$, let (E,u) be a (κ,μ)-minimizer of $\mathcal{F}_{x_0,A_{x_0}}$ in Ω (or a (κ,μ)-minimizer of $\mathcal{F}_A$). Then, $\big(\Psi_{x_0,r}(E), r^{-\frac{1}{2}}u\circ\Psi_{x_0,r}^{-1}\big)$ is a $(\kappa r^\mu,\mu)$-minimizer of $\mathcal{F}_{x_0,A_{x_0}\circ\Psi_{x_0,r}^{-1}}$ in $\Psi_{x_0,r}(\Omega)$, (or respectively, a $(\kappa r^\mu,\mu)$-minimizer of $\mathcal{F}_A$ in $\Psi_{x_0,r}(\Omega)$), where*

$$\Psi_{x_0,r}(x) := \frac{x-x_0}{r} \quad \forall x\in\mathbb{R}^n.$$

Proof. Let $B_s(z) \subset \Psi_{x_0,r}(\Omega)$. Applying the change of variables $y = \Psi_{x_0,r}(x)$, we deduce that

$$\begin{aligned}
&\mathcal{F}_{x_0,A_{x_0}\circ\Psi_{x_0,r}^{-1}}\big(\Psi_{x_0,r}(E), u\circ\Psi_{x_0,r}^{-1};B_s(z)\big)\\
&= \int_{B_s(z)} \sigma_{\Psi_{x_0,r}(E)}|\nabla(u\circ\Psi_{x_0,r}^{-1})A^{-\frac{1}{2}}(x_0)|^2\,dy + \mathbf{\Phi}_{A_{x_0}\circ\Psi_{x_0,r}^{-1}}(\Psi_{x_0,r}(E);B_s(z))\\
&= \frac{1}{r^{n-1}}\Big(\int_{B_{rs}(x_0+rz)} \sigma_E|\nabla u A^{-\frac{1}{2}}(x_0)|^2\,dx + \mathbf{\Phi}_{A_{x_0}}(E;B_{rs}(x_0+rz))\Big)\\
&= \frac{1}{r^{n-1}}\mathcal{F}_{x_0,A_{x_0}}(E,u;B_{rs}(x_0+rz)).
\end{aligned}$$

Let (F,v) be such that F is a set of finite perimeter, $F\,\triangle\,\Psi_{x_0,r}(E) \Subset B_s(z)$ and $v \in (u\circ\Psi_{x_0,r}^{-1}) + H_0^1(B_s(z))$. It holds that $\Psi_{x_0,r}^{-1}(F)\,\triangle\,E \Subset B_{rs}(x_0+rz) \subset \Omega$ and $v\circ\Psi_{x_0,r} - u \in H_0^1(B_{rs}(x_0+rz))$. Using the (κ,μ)-minimality of (E,u), we get

$$\begin{aligned}
&\mathcal{F}_{x_0,A_{x_0}\circ\Psi_{x_0,r}^{-1}}\big(\Psi_{x_0,r}(E), u\circ\Psi_{x_0,r}^{-1};B_s(z)\big) = \frac{1}{r^{n-1}}\mathcal{F}_{x_0,A_{x_0}}(E,u;B_{rs}(x_0+rz))\\
&\le \frac{1}{r^{n-1}}\big[\mathcal{F}_{x_0,A_{x_0}}\big(\Psi_{x_0,r}^{-1}(F), v\circ\Psi_{x_0,r};B_{rs}(x_0+rz)\big) + \kappa|\Psi_{x_0,r}(E)\,\triangle\,\Psi_{x_0,r}(F)|^{\frac{n-1+\mu}{n}}\big]\\
&= \mathcal{F}_{x_0,A_{x_0}\circ\Psi_{x_0,r}^{-1}}(F,v;B_s(z)) + \kappa r^\mu|E\,\triangle\,F|^{\frac{n-1+\mu}{n}},
\end{aligned}$$

which means that $\big(\Psi_{x_0,r}(E), r^{-\frac{1}{2}} u \circ \Psi^{-1}_{x_0,r}\big)$ is a $(\kappa r^{\mu}, \mu)$-minimizer of $\mathcal{F}_{x_0, A_{x_0} \circ \Psi^{-1}_{x_0,r}}$ in $\Psi_{x_0,r}(\Omega)$. ∎

The following proposition will be useful in the proof of Theorem 7.3, where we need to compare the excess of the transformed couple and of the original couple.

Proposition 3.3 (Comparability of the excess under change of variable T_{x_0} and radius). *There exists a positive constant $c_1 = c_1(n, \lambda, \Lambda)$ such that if E is a set of locally finite perimeter and $x_0 \in \partial E$, then, for any $r > 0$,*

$$c_1^{-1} \mathbf{e}\big(T_{x_0}(E), x_0, \Lambda^{-\frac{1}{2}} r\big) \le \mathbf{e}(E, x_0, r) \le c_1 \mathbf{e}\big(T_{x_0}(E), x_0, \lambda^{-\frac{1}{2}} r\big).$$

Proof. For $r > 0$ and $\nu \in \mathbb{S}^{n-1}$, we define the ellipsoidal excess at the point x_0, at the scale r and with respect to the direction ν the following quantity:

$$\mathbf{e}_W(E, x_0, r, \nu) := \min_{\nu \in \mathbb{S}^{n-1}} \frac{1}{r^{n-1}} \int_{T_{x_0}^{-1}(B_r(x_0)) \cap \partial^* E} \frac{|\nu_E - \nu|^2}{2} d\mathcal{H}^{n-1}.$$

Leveraging the inclusions

$$T_{x_0}^{-1}\big(B_{\Lambda^{-\frac{1}{2}} r}(x_0)\big) \subset B_r(x_0) \subset T_{x_0}^{-1}\big(B_{\lambda^{-\frac{1}{2}} r}(x_0)\big),$$

we infer that

$$c^{-1} \mathbf{e}_W\big(E, x_0, \Lambda^{-\frac{1}{2}} r, \nu\big) \le \mathbf{e}(E, x_0, r, \nu) \le c\, \mathbf{e}_W\big(E, x_0, \lambda^{-\frac{1}{2}} r, \nu\big) \tag{3.2}$$

for some positive constant $c = c(n, \lambda, \Lambda)$. In [43, Proposition 5.1], it is shown that

$$c^{-1} \mathbf{e}\left(T_{x_0}(E), x_0, s, \frac{A^{\frac{1}{2}}(x_0)\nu}{|A^{\frac{1}{2}}(x_0)\nu|}\right) \le \mathbf{e}_W(E, x_0, s, \nu) \le c\, \mathbf{e}\left(T_{x_0}(E), x_0, s, \frac{A^{\frac{1}{2}}(x_0)\nu}{|A^{\frac{1}{2}}(x_0)\nu|}\right)$$

for any $s > 0$. Using the previous inequalities from below for $s = \Lambda^{-\frac{1}{2}} r$ and from above for $s = \lambda^{-\frac{1}{2}} r$, and inserting them in (3.2), we get

$$c^{-1} \mathbf{e}\left(T_{x_0}(E), x_0, \Lambda^{-\frac{1}{2}} r, \frac{A^{\frac{1}{2}}(x_0)\nu}{|A^{\frac{1}{2}}(x_0)\nu|}\right) \le \mathbf{e}(E, x_0, r, \nu) \le c\, \mathbf{e}\left(T_{x_0}(E), x_0, \lambda^{-\frac{1}{2}} r, \frac{A^{\frac{1}{2}}(x_0)\nu}{|A^{\frac{1}{2}}(x_0)\nu|}\right).$$

Minimizing over $\nu \in \mathbb{S}^{n-1}$, we obtain the thesis. ∎

4. Energy density estimates

The main goal of this section is to prove density lower and upper bounds for the perimeter of a (κ, μ)-minimizer. As consequences, the decay of the associate rescaled Dirichlet energy, defined for $u \in H^1(B_r(x_0))$ as

$$\mathcal{D}_u(x_0, r) := \frac{1}{r^{n-1}} \int_{B_r(x_0)} |\nabla u|^2 \, dx, \tag{4.1}$$

and the Lipshitz approximation theorem will follow. In view of this aim, we mention a result stating a decay estimate for elastic minima around points where either the density of E is close to 0 or 1, or the set E is asymptotically close to a hyperplane. We address the reader to [29, Proposition 2.4] for the proof.

Lemma 4.1. *Let (E,u) be a (κ,μ)-minimizer of the functional $\mathcal{F}_A$. There exists $\tau_0 \in (0,1)$ such that the following statement is true: for all $\tau \in (0,\tau_0)$ there exists $\varepsilon_0 = \varepsilon_0(\tau) > 0$ such that if $B_r(x_0) \Subset \Omega$ and one of the following conditions holds:*

(i) $|E \cap B_r(x_0)| < \varepsilon_0 |B_r(x_0)|$,

(ii) $|B_r(x_0) \setminus E| < \varepsilon_0 |B_r(x_0)|$,

(iii) *there exists a halfspace H such that* $\frac{|(E \triangle H) \cap B_r(x_0)|}{|B_r(x_0)|} < \varepsilon_0$,

then

$$\mathcal{D}_u(x_0, \tau r) \le c_2 \tau \mathcal{D}_u(x_0, r)$$

for some positive constant $c_2 = (c_2 n, \alpha, \beta)$.

The second result we want to mention, which will be used later, provides an upper bound for the whole energy $\mathcal{F}_A$ on balls. The proof is rather standard and we address the reader to [23, Theorem 3] for the details. Here, we just give a sketch of the proof, underlining the only points where the presence of the anisotropy entails different computations.

Theorem 4.2 (Energy upper bound). *Let (E,u) be a (κ,μ)-minimizer of $\mathcal{F}_A$ in Ω. Then, for every open set $U \Subset \Omega$ there exists a positive constant $c_3 = c_3(n,\alpha,\beta,\Lambda,\kappa,\mu,U, \|\nabla u\|_{L^2(\Omega)})$ such that for every $B_r(x_0) \subset U$ it holds*

$$\mathcal{F}_A(E,u;B_r(x_0)) \le c_3 r^{n-1}.$$

Proof. Let $B_r(x_0) \subset U \Subset \Omega$. Testing the minimality of (E,u) with $(E \cup B_r(x_0), u)$, we deduce that

$$\mathcal{F}_A(E,u;\Omega) \le \mathcal{F}_A(E \cup B_r(x_0), u; \Omega) + \kappa |B_r(x_0) \setminus E|^{\frac{\mu+n-1}{n}}. \tag{4.2}$$

The only difference in our proof, compared to the isotropic case, is the use of the following formula concerning anisotropic perimeter and set operations. The latter follows from Proposition 2.2 applied with $F = U$, $G = B_r(x_0)$, that is, for almost all $r > 0$,

$$\mathbf{\Phi}_A(E \cup B_r(x_0); U) = \mathbf{\Phi}_A(E; U \setminus \overline{B_r(x_0)}) + \mathbf{\Phi}_A(B_r(x_0); U \setminus E).$$

Making $\mathcal{F}_A$ explicit and getting rid of the common terms in (4.2) we obtain the following energy estimate on $B_r(x_0) \setminus E$:

$$\begin{aligned}\int_{B_r(x_0)\setminus E} (\beta-\alpha)|\nabla u|^2\,dx + \mathbf{\Phi}_A(E;B_r(x_0)) &\le \mathbf{\Phi}_A(B_r(x_0);U\setminus E) + \kappa |B_r(x_0)\setminus E|^{\frac{\mu+n-1}{n}} \\ &\le \Lambda^{1/2}\mathcal{H}^{n-1}(\partial B_r(x_0)) + c(n,\kappa) r^{n-1} \\ &\le c(n,\Lambda,\kappa) r^{n-1}.\end{aligned}$$

Starting from this estimate the proof follows verbatim the argument used in [23, Theorem 3], because henceforth only variations of the function u are used and the perimeter is not involved anymore. Indeed, using a blow-up argument, it can be proved that there exist $M = M(n, \alpha, \beta) > 0$ and $\tau \in (0, \frac{1}{2})$, depending on Λ/λ, such that for every $\delta \in (0, 1)$ there exists $h_0 \in \mathbb{N}$ such that for any $B_r(x_0) \subset U$, we have

$$\int_{B_r(x_0)} |\nabla u|^2 \le h_0 r^{n-1} \quad \text{or} \quad \int_{B_{\tau r}(x_0)} |\nabla u|^2 \, dx \le M\tau^{n-\delta} \int_{B_r(x_0)} |\nabla u|^2 \, dx,$$

from which the thesis follows. ∎

In the following lemma, we show that the energy $\mathcal{F}_A$ decays "fast" in the balls where the perimeter of E is "small". Lemma 4.1 is utilized in its proof, specifically in instances (i) and (ii).

Lemma 4.3. *Let (E, u) be a (κ, μ)-minimizer in Ω of the functional $\mathcal{F}_A$. For every $\tau \in (0, 1)$ there exists $\varepsilon_1 = \varepsilon_1(\tau) > 0$ such that if $B_r(x_0) \subset \Omega$ and $P(E; B_r(x_0)) < \varepsilon_1 r^{n-1}$, then*

$$\mathcal{F}_A(E, u; B_{\tau r}(x_0)) \le c_4\big(\tau^n \mathcal{F}_A(E, u; B_r(x_0)) + (\tau r)^{\mu+n-1}\big)$$

for some positive constant $c_4 = c_4(n, \alpha, \beta, \lambda, \Lambda, \kappa, \mu, \|\nabla u\|_{L^2(\Omega)}) > 0$ independent of τ and r.

Proof. Let $\tau \in (0, 1)$ and $B_r(x_0) \subset \Omega$. Without loss of generality, we may assume that $\tau < \frac{1}{2}$. We rescale (E, u) in B_1 by setting $E_r = \frac{E - x_0}{r}$ and $u_r(y) = r^{-\frac{1}{2}} u(x_0 + ry)$, for $y \in B_1$. Applying Proposition 3.2, we have that (E_r, u_r) is a $(\kappa r^\mu, \mu)$-minimizer of $\mathcal{F}_{\widetilde{A}}$ in B_1, where $\widetilde{A} = A \circ \Psi^{-1}_{x_0, r}$. Observing that

$$r^{n-1} \mathcal{F}_{\widetilde{A}}(E_r, u_r; B_\tau) = \mathcal{F}_A(E, u; B_{\tau r}(x_0)),$$

we have to prove that there exists $\varepsilon_1 = \varepsilon_1(\tau)$ such that, if $P(E; B_1) < \varepsilon_1$, then

$$\mathcal{F}_{\widetilde{A}}(E, u; B_\tau) \le c_4\big(\tau^n \mathcal{F}_{\widetilde{A}}(E, u; B_1) + \tau^{\mu+n-1} r^\mu\big).$$

For simplicity of notation, we will still denote E_r by E, u_r by u and $\widetilde{A}$ by A. We note that, since $P(E; B_1) < \varepsilon_1$, by the relative isoperimetric inequality, either $|B_1 \cap E|$ or $|B_1 \setminus E|$ is small, and thus, Lemma 4.1 can be applied. We assume that $|B_1 \setminus E| \le |B_1 \cap E|$, the other case being similar. By the coarea formula and the relative isoperimetric inequality, we get

$$\int_\tau^{2\tau} \mathcal{H}^{n-1}(\partial B_\rho \setminus E) \, d\rho \le |B_1 \setminus E| \le c(n) P(E; B_1)^{\frac{n}{n-1}}.$$

Therefore, we may choose $\rho \in (\tau, 2\tau)$, independent of n, such that it holds $\mathcal{H}^{n-1}(\partial^* E \cap \partial B_\rho) = 0$ and

$$\mathcal{H}^{n-1}(\partial B_\rho \setminus E) \le \frac{c(n)}{\tau} P(E; B_1)^{\frac{n}{n-1}} \le \frac{c(n) \varepsilon_1^{\frac{1}{n-1}}}{\tau} P(E; B_1). \tag{4.3}$$

Now, we test the minimality of (E,u) with (F_0,u), where $F_0 := E \cup B_\rho$. We remark that, being $\mathcal{H}^{n-1}(\partial^* E \cap \partial B_\rho) = 0$, we can apply Proposition 2.2 with $U = F = B_1$ and $G = B_\rho$, thus obtaining

$$\mathbf{\Phi}_A(F_0; B_1) = \mathbf{\Phi}_A(E \cup B_\rho; B_1) = \mathbf{\Phi}_A(E; B_1 \setminus \overline{B_\rho}) + \mathbf{\Phi}_A(B_\rho; B_1 \setminus E^{(1)}). \tag{4.4}$$

The $(\kappa r^\mu, \mu)$-minimality of (E,u) supplies

$$\int_{B_1} \sigma_E |\nabla u|^2 \, dx + \mathbf{\Phi}_A(E; B_1) \le \int_{B_1} \sigma_{F_0} |\nabla u|^2 \, dx + \mathbf{\Phi}_A(F_0; B_1) + \kappa r^\mu |E \,\triangle\, F_0|^{\frac{\mu+n-1}{n}}.$$

Using (4.4) to get rid of the common perimeter terms and recalling that $E = E^{(1)}$, we deduce that

$$\int_{B_1} \sigma_E |\nabla u|^2 \, dx + \mathbf{\Phi}_A(E; B_\rho) \le \int_{B_1} \sigma_{F_0} |\nabla u|^2 \, dx + \mathbf{\Phi}_A(B_\rho; B_1 \setminus E) + \kappa r^\mu |E \,\triangle\, F_0|^{\frac{\mu+n-1}{n}}.$$

Taking into account the comparability to the perimeter (2.1) and perimeter estimate (4.3), recalling that $\rho \in (\tau, 2\tau)$ and getting rid of the common Dirichlet terms, we deduce that

$$\begin{aligned}
\int_{B_\tau} \sigma_E |\nabla u|^2 \, dx + \lambda^{1/2} P(E; B_\tau) &\le \beta \int_{B_{2\tau}} |\nabla u|^2 \, dx \\
&\quad + \Lambda^{1/2} \mathcal{H}^{n-1}(\partial B_\rho \setminus E) + c(n,\kappa) r^\mu \tau^{\mu+n-1} \\
&\le \beta \int_{B_{2\tau}} |\nabla u|^2 \, dx \\
&\quad + \frac{c(n)\Lambda^{1/2}}{\tau} \varepsilon_1^{\frac{1}{n-1}} P(E; B_1) + c(n,\kappa) r^\mu \tau^{\mu+n-1}.
\end{aligned}$$

Finally, we choose ε_1 such that

$$c(n)\Lambda^{1/2} \varepsilon_1^{\frac{1}{n-1}} \le \tau^{n+1} \quad \text{and} \quad c(n) \varepsilon_1^{\frac{n}{n-1}} \le \varepsilon_0(2\tau)|B_1|,$$

where ε_0 is from Lemma 4.1, thus getting

$$\int_{B_{2\tau}} |\nabla u|^2 \, dx \le 2^n c_2 \tau^n \int_{B_1} |\nabla u|^2 \, dx.$$

From these estimates, the result easily follows, applying again the comparability to the perimeter. ∎

Taking advantage of the established results, we are able to deduce a density lower bound estimate for the perimeter of a (κ,μ)-minimizer of $\mathcal{F}_A$.

Theorem 4.4 (Density lower bound). *Let (E,u) be a (κ,μ)-minimizer of $\mathcal{F}_A$ in Ω and $U \Subset \Omega$ be an open set. Then, there exists a constant $c_5 = c_5(n,\alpha,\beta,\lambda,\Lambda,\kappa,\mu,U, \|\nabla u\|_{L^2(\Omega)}) > 0$ such that for every $x_0 \in \partial E$ and $B_r(x_0) \subset U$, it holds that*

$$P(E; B_r(x_0)) \ge c_5 r^{n-1}. \tag{4.5}$$

Moreover, $\mathcal{H}^{n-1}((\partial E \setminus \partial^* E) \cap \Omega) = 0$.

Proof. The proof matches that of [23, Theorem 4] exactly, given the comparability to the perimeter. We start by assuming that $x_0 \in \partial^* E$. Without loss of generality, we may also assume that $x_0 = 0$. Arguing by contradiction on (4.5), by using Theorem 4.2 and Lemma 4.3, we can easily prove by induction (see [23, Theorem 4] for the details) that

$$\mathcal{F}(E, u; B_{\sigma\tau^h r}) \leq \varepsilon_1(\tau)\tau^{\mu h}(\sigma\tau^h r)^{n-1},$$

where τ and σ are sufficiently small and ε_1 is from Lemma 4.3. Starting from this, we deduce that

$$\lim_{\rho\to 0^+} \frac{P(E; B_\rho)}{\rho^{n-1}} = \lim_{h\to+\infty} \frac{P(E; B_{\sigma\tau^h r})}{(\sigma\tau^h r)^{n-1}} \leq \lim_{h\to+\infty} 2\varepsilon_1(\tau)\tau^{\mu h} = 0,$$

which implies that $x_0 \notin \partial^* E$, that is a contradiction. We recall that we chose the representative of ∂E such that $\partial E = \overline{\partial^* E}$. Thus, if $x_0 \in \partial E$, there exists $(x_h)_{h\in\mathbb{N}} \subset \partial^* E$ such that $x_h \to x_0$ as $h \to +\infty$,

$$P(E; B_r(x_h)) \geq C r^{n-1}$$

and $B_r(x_h) \subset U$, for h large enough. Passing to the limit as $h \to +\infty$, we get the result. ∎

Definition 4.5 (Ahlfors regularity). A Borel measure μ on $\mathbb{R}^n$ is said to be d-Ahlfors regular if there exist two positive constants c_A and r_0 such that

$$c_A^{-1} r^d \leq \mu(B_r(x)) \leq c_A r^d$$

for all $x \in \mathrm{spt}\mu$ and $0 < r < r_0$. According to the notation used in [8], we denote

$$\begin{aligned}\mathcal{A}(c_A, r_0) := \Big\{ & E \subset \mathbb{R}^n : E \text{ is a set of locally finite perimeter satisfying} \\ & \partial E = \mathrm{spt}\mu_E \text{ and its perimeter measure } |\mu_E| \text{ is} \\ & (n-1)\text{-Ahlfors regular with constants } r_0 \text{ and } c_A \Big\}.\end{aligned}$$

Remark 4.6. It is evident that Theorems 4.2 and 4.4 ensure the belonging of the (κ, μ)-minimizers of $\mathcal{F}_A$ to the class $\mathcal{A}(c_A, r_0)$, for some constant c_A identified in such theorems. Naturally, for $x_0 \in \Omega$ and $r > 0$, the $(\kappa\lambda^{-\frac{n}{2}} r^\mu, \mu)$-minimizers of $\mathcal{F}_{x_0, A_{x_0}\circ\Psi^{-1}_{x_0,r}}$ obtained through the affine transformation T_{x_0} and the scaling $\Psi^{-1}_{x_0,r}$ (see Proposition 3.1 and Proposition 3.2) belong to the class $\mathcal{A}\big(c_A(\frac{\Lambda}{\lambda})^{\frac{n}{2}}, \frac{r_0}{r\Lambda^{\frac{1}{2}}}\big)$.

The next result of this section establishes that around the points of the boundary of the set where the excess is "small", the Dirichlet integral decays "fast". In its proof, Lemma 4.1 plays a crucial role in instance (iii).

Proposition 4.7 (Decay of the rescaled Dirichlet integral). *For every $\tau \in (0, 1)$ there exists $\varepsilon_2 = \varepsilon_2(\tau) > 0$ such that if (E, u) is a (κ, μ)-minimizer of $\mathcal{F}_A$ in $B_r(x_0)$, with $x_0 \in \partial E$, and $\mathbf{e}(x_0, r) \leq \varepsilon_2$, then*

$$\mathcal{D}_u(x_0, \tau r) \leq c_6 \tau \mathcal{D}_u(x_0, r)$$

for some positive constant $c_6 = c_6(n, \alpha, \beta, \|\nabla u\|_{L^2(\Omega)})$.

Proof. Applying a usual scaling argument, by Proposition 3.2, we assume by contradiction that for some $\tau \in (0,1)$ there exist two positive sequences $(\varepsilon_h)_{h\in\mathbb{N}}$ and $(r_h)_{h\in\mathbb{N}}$ and a sequence $((E_h,u_h))_{h\in\mathbb{N}}$ of $(\kappa r_h^{\mu},\mu)$-minimizers of $\mathcal{F}_{A\circ\Psi^{-1}_{x_0,r_h}}$ in B_1 with equibounded energies such that $0\in\partial E_h$,

$$\mathbf{e}(E_h,0,1)=\varepsilon_h\to 0 \quad \text{and} \quad \mathcal{D}_{u_h}(0,\tau)>\bar{C}\,\tau\,\mathcal{D}_{u_h}(0,1) \tag{4.6}$$

for some positive constant $\bar{C}$ to be chosen. Thanks to the energy upper bound (Theorem 4.2) and the compactness of $(E_h)_{h\in\mathbb{N}}$, we may assume that $E_h\to E$ in $L^1(B_1)$ and $0\in\partial E$. Since, by lower semicontinuity, the excess of E at 0 is null, E is a half-space in B_1, say H. In particular, for h large, it holds that

$$|(E_h\,\triangle\,H)\cap B_1|<\varepsilon_0(\tau)|B_1|,$$

where ε_0 is from Lemma 4.1, which gives a contradiction with the inequality (4.6), provided we choose $\bar{C}>c_2$, where c_2 is also from Lemma 4.1. ∎

The last results also come as consequences of the density lower and upper bounds proved above. The height bound lemma is a standard step in the proof of regularity because it is one of the main ingredients to prove the Lipschitz approximation theorem. We remark that this is stated for $(\kappa r^{\mu},\mu)$-minimizers of $\mathcal{F}_{A_{x_0}\circ\Psi_{x_0,r}}$, which are still Ahlfors regular (see Remark 4.6). The proof of this result can be found in [8, Theorem A.2].

Lemma 4.8 (Height bound). *For $x_0\in\Omega$ and $r>0$, let (E,u) be a $(\kappa r^{\mu},\mu)$-minimizer of $\mathcal{F}_{A_{x_0}\circ\Psi^{-1}_{x_0,r}}$ in B_1. There exist two positive constants ε_3 and c_7, depending on n, α, β, λ, Λ, κ, μ, $\|\nabla u\|_{L^2(B_1)}$, such that if $0\in\partial E$ and*

$$\mathbf{e}(0,1,e_n)<\varepsilon_3,$$

then

$$\sup_{y\in\partial E\cap B_{1/2}}|y_n-(x_0)_n|\le c_7\mathbf{e}(0,1,e_n)^{\frac{1}{2(n-1)}}.$$

Proceeding as in [39], we state the following Lipschitz approximation lemma, which is a consequence of the height bound lemma. Its proof follows exactly as in [8, Theorem A.3]. It is a fundamental step in the long journey to the regularity because it provides a connection between the regularity theories for parametric and non-parametric variational problems. Indeed, we are able to prove for $(\kappa r^{\mu},\mu)$-minimizers that the smallness of the excess guaranties that ∂E can be locally almost entirely covered by the graph of a Lipschitz function.

Theorem 4.9 (Lipschitz approximation). *For $x_0\in\Omega$ and $r>0$, let (E,u) be a $(\kappa r^{\mu},\mu)$-minimizer of $\mathcal{F}_{A_{x_0}\circ\Psi^{-1}_{x_0,r}}$ in B_1. There exist two positive constants ε_4 and c_8, depending on $n,\alpha,\beta,\lambda,\Lambda,\|\nabla u\|_{L^2(B_1)}$, such that if $0\in\partial E$ and*

$$\mathbf{e}(0,1,e_n)<\varepsilon_4,$$

then there exists a Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ *such that*

$$\sup_{x' \in \mathbb{R}^{n-1}} |f(x')| \leq c_8 \mathbf{e}(0, 1, e_n)^{\frac{1}{2(n-1)}}, \quad \|\nabla' f\|_{L^\infty} \leq 1$$

and

$$\mathcal{H}^{n-1}((\partial E \,\triangle\, \Gamma_f) \cap B_{1/2}) \leq c_8 \mathbf{e}(0, 1, e_n),$$

where Γ_f *is the graph of* f*. Moreover,*

$$\int_{\mathbf{D}_{\frac{1}{2}}} |\nabla' f|^2 \, dx' \leq c_8 \mathbf{e}(0, 1, e_n).$$

5. Compactness for sequences of minimizers

In this section, we prove a standard compactness result for sequences of (κ, μ)-minimizers. Given two positive constants M_1 and M_2, we set

$$B_{M_1,M_2} := \big\{A \in C^\gamma(\mathbb{R}^n; \mathbb{R}^n \otimes \mathbb{R}^n) : A \text{ is symmetric}, [A]_{C^\gamma} < M_1, \ \|A\|_\infty < M_2\big\}.$$

We define

$$\mathcal{A} = \big\{A \in C^\gamma(\mathbb{R}^n; \mathbb{R}^n \otimes \mathbb{R}^n) : \lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2, \ \forall x, \xi \in \mathbb{R}^n\big\} \cap B_{M_1,M_2}. \tag{5.1}$$

Lemma 5.1 (Compactness). *Let* (E_h, u_h) *be a sequence of* (κ_h, μ)*-minimizers of* $\mathcal{F}_{A_h}$ *in* Ω *such that* $\sup_h \mathcal{F}_{A_h}(E_h, u_h; \Omega) < +\infty$*,* $A_h \to A_\infty$ *uniformly on compact sets, where the matrices* A_∞, A_h *are in the class* $\mathcal{A}$ *defined in* (5.1)*,* $\kappa_h \to \kappa \in \mathbb{R}^+$*. There exist a (not relabeled) subsequence and a* (κ, μ)*-minimizer* (E, u) *of* $\mathcal{F}_{A_\infty}$ *in* Ω *such that, for every open set* $U \Subset \Omega$*, it holds that*

$$E_h \to E \text{ in } L^1(U), \quad u_h \to u \text{ in } H^1(U), \quad \mathbf{\Phi}_{A_h}(E_h; U) \to \mathbf{\Phi}_{A_\infty}(E; U).$$

In addition,

$$\text{if } x_h \in \partial E_h \cap U \text{ and } x_h \to x \in U, \text{ then } x \in \partial E \cap U, \tag{5.2}$$

$$\text{if } x \in \partial E \cap U, \text{ there exists } x_h \in \partial E_h \cap U \text{ such that } x_h \to x. \tag{5.3}$$

Finally, if we assume also that $\nabla u_h \rightharpoonup 0$ *weakly in* $L^2_{loc}(\Omega; \mathbb{R}^n)$ *and* $\kappa_h \to 0$*, as* $h \to +\infty$*, then* E *is a local minimizer of* $\mathbf{\Phi}_{A_\infty}$*, that is*

$$\mathbf{\Phi}_{A_\infty}(E; B_r(x_0)) \leq \mathbf{\Phi}_{A_\infty}(F; B_r(x_0)), \tag{5.4}$$

for every set F *of locally finite perimeter such that* $F \,\triangle\, E \Subset B_r(x_0) \subset \Omega$*.*

Proof. Using the boundedness condition on $\sup_h \mathcal{F}_{A_h}(E_h, u_h; \Omega)$, we may assume that u_h weakly converges to u in $H^1(U)$ and strongly in $L^2(U)$, and $\mathbb{1}_{E_h}$ converges to $\mathbb{1}_E$ in $L^1(U)$, as $h \to +\infty$. By a lower semicontinuity argument, we start proving the (κ, μ)-minimality of (E, u). Let us fix $B_r(x_0) \Subset U$ and assume for simplicity of notation that $x_0 = 0$. Let (F, v) be a test pair such that F is a set of locally finite perimeter, $F \,\triangle\, E \Subset B_r$ and $\mathrm{supp}(u - v) \Subset B_r$. Possibly passing to a subsequence and using Fubini's theorem, we may choose $0 < r_0 < \rho < r$ such that $F \,\triangle\, E \Subset B_\rho$, $E \setminus B_{r_0} = F \setminus B_{r_0}$, $\mathrm{supp}(u - v) \Subset B_\rho$, and in addition,

$$\mathcal{H}^{n-1}(\partial B_\rho \cap \partial^* E) = \mathcal{H}^{n-1}(\partial B_\rho \cap \partial^* E_h) = 0$$

and

$$\lim_{h\to 0} \mathcal{H}^{n-1}(\partial B_\rho \cap (F^{(1)} \,\triangle\, E_h^{(1)})) = 0. \tag{5.5}$$

Now, we choose a cut-off function $\psi \in C_0^1(B_r)$ such that $\psi \equiv 1$ in B_ρ and define

$$v_h = \psi v + (1 - \psi)u_h, \quad F_h := (F \cap B_\rho) \cup (E_h \setminus B_\rho)$$

to test the minimality of (E_h, u_h). Thanks to the (κ_h, μ)-minimality of (E_h, u_h) and using also Proposition 2.2, we deduce that

$$\begin{aligned}
&\int_{B_r} \sigma_{E_h} |\nabla u_h|^2 dx + \mathbf{\Phi}_{A_h}(E_h; B_r) \\
&\quad \le \int_{B_r} \sigma_{F_h} |\nabla v_h|^2 dx + \mathbf{\Phi}_{A_h}(F_h; B_r) + \kappa_h |F_h \,\triangle\, E_h|^{\frac{\mu+n-1}{n}} \\
&\quad \le \int_{B_r} \sigma_{E_h}(1 - \psi)|\nabla u_h|^2 dx + \int_{B_r} \sigma_{F_h} \psi |\nabla v|^2 dx + \int_{B_r \setminus B_\rho} C|\nabla \psi|^2 |u - u_h|^2 dx \\
&\qquad + \mathbf{\Phi}_{A_h}(F; B_\rho) + \mathbf{\Phi}_{A_h}(E_h; B_r \setminus \bar{B}_\rho) + \mathbf{\Phi}_{A_h}(B_\rho; F \,\triangle\, E_h) + \kappa_h |F_h \,\triangle\, E_h|^{\frac{\mu+n-1}{n}}.
\end{aligned} \tag{5.6}$$

Using the uniform convergence $A_h \to A_\infty$, the strong convergence $u_h \to u$ in L^2, condition (5.5), and getting rid of common terms, from the latter estimate we can write

$$\begin{aligned}
&\int_{B_r} \sigma_{E_h} \psi |\nabla u_h|^2 dx + \mathbf{\Phi}_{A_\infty}(E_h; B_\rho) \\
&\qquad \le \int_{B_r} \sigma_{F_h} \psi |\nabla v|^2 dx + \mathbf{\Phi}_{A_\infty}(F; B_\rho) + \kappa_h |F_h \,\triangle\, E_h|^{\frac{\mu+n-1}{n}} + \varepsilon_h
\end{aligned}$$

for some $\varepsilon_h \to 0$. By the lower semicontinuity of the anisotropic perimeter (see [43, Proposition 3.1]), the equi-integrability of $(\nabla u_h)_{h\in\mathbb{N}}$ and the lower semicontinuity of Dirichlet integral, we infer that

$$\int_{B_r} \sigma_E \psi |\nabla u|^2 dx + \mathbf{\Phi}_{A_\infty}(E; B_\rho) \le \int_{B_r} \sigma_F \psi |\nabla v|^2 dx + \mathbf{\Phi}_{A_\infty}(F; B_\rho) + \kappa |F \,\triangle\, E|^{\frac{\mu+n-1}{n}}.$$

Letting $\psi \downarrow \chi_{B_\rho}$, we get

$$\int_{B_\rho} \sigma_E |\nabla u|^2 dx + \mathbf{\Phi}_{A_\infty}(E; B_\rho) \leq \int_{B_\rho} \sigma_F |\nabla v|^2 dx + \mathbf{\Phi}_{A_\infty}(F; B_\rho) + \kappa |F \,\triangle\, E|^{\frac{\mu+n-1}{n}}. \tag{5.7}$$

Similarly, choosing $E = F$ and $u = v$ in (5.6), and arguing as before, we get

$$\limsup_{h\to+\infty} \left(\int_{B_\rho} \sigma_{E_h} \psi |\nabla u_h|^2 dx + \mathbf{\Phi}_{A_\infty}(E_h; B_\rho) \right) \leq \int_{B_\rho} \sigma_E \psi |\nabla u|^2 dx + \mathbf{\Phi}_{A_h}(E; B_\rho).$$

Letting $\psi \downarrow \chi_{B_\rho}$, we conclude

$$\lim_{h\to+\infty} \mathbf{\Phi}_{A_h}(E_h; B_\rho) = \mathbf{\Phi}_{A_\infty}(E; B_\rho), \quad \lim_{h\to+\infty} \int_{B_\rho} \sigma_{E_h} |\nabla u_h|^2 dx = \int_{B_\rho} \sigma_E |\nabla u|^2 dx.$$

With a usual argument we can deduce $u_h \to u$ in $W^{1,2}(U)$, $\mathbf{\Phi}_{A_h}(E_h; U) \to \mathbf{\Phi}_{A_\infty}(E; U)$ for every open set $U \Subset \Omega$. The topological information stated in (5.2) and (5.3) follows as in [39, Theorem 21.14], indeed they are a consequence of the lower and upper density estimates given above. Finally, if $\nabla u_h \rightharpoonup 0$ weakly in $L^2_{loc}(\Omega; \mathbb{R}^n)$ and $\kappa_h \to 0$, we can choose $v = u$ in (5.7), deriving (5.4). ∎

6. Reverse Poincaré inequality

In this section, we derive a reverse Poincaré inequality which lets us estimate the excess around a point of the boundary of the transformed set with its flatness. The first step in the proof is to establish a weak form of this inequality.

In the following proposition, it is proved that if the anisotropy matrix valued in a point x_0 is the identity, then around x_0 the anisotropic perimeter is comparable to the perimeter.

Proposition 6.1. *Let $x_0 \in \Omega$ and $r > 0$. There exists a positive constant $c_9 = c_9(n, \alpha, \beta, \lambda, \Lambda, \kappa, \mu, \|\nabla u\|_{L^2(\Omega)})$ such that if (E, u) is a $(\kappa r^\mu, \mu)$-minimizer of $\mathcal{F}_{x_0, A_{x_0} \circ \Psi^{-1}_{x_0,r}}$ in B_1, with $0 \in \partial E \cap B_1$, then*

$$\begin{aligned} &\int_{B_\rho} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2 \, dx + P(E; B_\rho) \\ &\qquad \leq \int_{B_\rho} \sigma_F |\nabla v A^{-\frac{1}{2}}(x_0)|^2 \, dx + P(F; B_\rho) + c_9 \big(\kappa + [A]_{C^\gamma}\big) r^\mu \rho^{n-1+\mu} \end{aligned}$$

for every (F, v) such that $F \,\triangle\, E \Subset B_\rho \subset B_1$ and $v \in u + H^1_0(B_\rho)$.

Proof. Let (F, v) be such that $F \,\triangle\, E \Subset B_\rho$ and $v \in u + H^1_0(B_\rho)$. We can assume that

$$\int_{B_\rho} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2 \, dx + P(E; B_\rho) \geq \int_{B_\rho} \sigma_F |\nabla v A^{-\frac{1}{2}}(x_0)|^2 \, dx + P(F; B_\rho).$$

We remark that $A_{x_0} \circ \Psi_{x_0,r}^{-1}$ is Hölder continuous and

$$\left[A_{x_0} \circ \Psi_{x_0,r}^{-1}\right]_{C^\mu} \leq \frac{\Lambda^{\frac{\mu}{2}}}{\lambda}[A]_{C^\mu} r^\mu.$$

Since $(A_{x_0} \circ \Psi_{x_0,r}^{-1})(0) = I$, by the Hölder continuity of $A_{x_0} \circ \Psi_{x_0,r}^{-1}$, we infer that

$$\begin{aligned}
|\nu_E| &= \left\langle \left(A_{x_0} \circ \Psi_{x_0,r}^{-1}\right)(0)\nu_E, \nu_E\right\rangle^{\frac{1}{2}} \\
&\leq \left\langle \left(A_{x_0} \circ \Psi_{x_0,r}^{-1}\right)(x)\,\nu_E, \nu_E\right\rangle^{\frac{1}{2}} + \frac{1}{2\lambda}\left[A_{x_0} \circ \Psi_{x_0,r}^{-1}\right]_{C^\mu}\rho^\mu \\
&\leq \left\langle \left(A_{x_0} \circ \Psi_{x_0,r}^{-1}\right)(x)\,\nu_E, \nu_E\right\rangle^{\frac{1}{2}} + \frac{\Lambda^{\frac{\mu}{2}}}{2\lambda^2}[A]_{C^\mu}(r\rho)^\mu
\end{aligned}$$

for any $x \in B_\rho$. Integrating over B_ρ with respect to the measure $\mathcal{H}^{n-1} \llcorner \partial^* E$ and adding to both sides the term $\int_{B_\rho} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2$, we obtain

$$\begin{aligned}
&\int_{B_\rho} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2\,dx + P(E; B_\rho) \\
&\quad\leq \int_{B_\rho} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2\,dx + \mathbf{\Phi}_{A_{x_0} \circ \Psi_{x_0,r}^{-1}}(E; B_\rho) + \frac{\Lambda^{\frac{\mu}{2}}}{2\lambda^2}[A]_{C^\mu}(r\rho)^\mu P(E; B_\rho).
\end{aligned}$$

Arguing in a similar way, we get

$$\begin{aligned}
&\int_{B_\rho} \sigma_F |\nabla v A^{-\frac{1}{2}}(x_0)|^2\,dx + \mathbf{\Phi}_{A_{x_0} \circ \Psi_{x_0,r}^{-1}}(F; B_\rho) \\
&\quad\leq \int_{B_\rho} \sigma_F |\nabla v A^{-\frac{1}{2}}(x_0)|^2\,dx + P(F; B_\rho) + \frac{\Lambda^{\frac{\mu}{2}}}{2\lambda^2}[A]_{C^\mu}(r\rho)^\mu P(F; B_\rho).
\end{aligned}$$

Applying the definition of $(\kappa r^\mu, \mu)$-minimality of (E, u) and using the previous two inequalities, we write

$$\begin{aligned}
&\int_{B_\rho} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2\,dx + P(E; B_\rho) \\
&\leq \int_{B_\rho} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2\,dx + \mathbf{\Phi}_{A_{x_0} \circ \Psi_{x_0,r}^{-1}}(E; B_\rho) + \frac{\Lambda^{\frac{\mu}{2}}}{2\lambda^2}[A]_{C^\mu}(r\rho)^\mu P(E; B_\rho) \\
&\leq \int_{B_\rho} \sigma_F |\nabla v A^{-\frac{1}{2}}(x_0)|^2\,dx + \mathbf{\Phi}_{A_{x_0} \circ \Psi_{x_0,r}^{-1}}(F; B_\rho) \\
&\quad + \kappa r^\mu |E \,\triangle\, F|^{\frac{n-1+\mu}{n}} + \frac{\Lambda^{\frac{\mu}{2}}}{2\lambda^2}[A]_{C^\mu}(r\rho)^\mu P(E; B_\rho) \\
&\leq \int_{B_\rho} \sigma_F |\nabla v A^{-\frac{1}{2}}(x_0)|^2\,dx + P(F; B_\rho) + c(n)\kappa r^\mu \rho^{n-1+\mu} \\
&\quad + \frac{\Lambda^{\frac{\mu}{2}}}{2\lambda^2}[A]_{C^\mu}(r\rho)^\mu [P(E; B_\rho) + P(F; B_\rho)]
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{B_\rho} \sigma_F |\nabla v A^{-\frac{1}{2}}(x_0)|^2 \, dx + P(F; B_\rho) + c(n)\kappa r^\mu \rho^{n-1+\mu} \\
&\quad + \frac{\Lambda^{\frac{\mu}{2}}}{2\lambda^2} [A]_{C^\mu} (r\rho)^\mu \left[2P(E; B_\rho) + \int_{B_\rho} \sigma_E |\nabla u|^2 \, dx \right] \\
&\leq \int_{B_\rho} \sigma_F |\nabla v A^{-\frac{1}{2}}(x_0)|^2 \, dx + P(F; B_\rho) + c(n, \lambda, \Lambda, c_3)(\kappa + [A]_{C^\gamma}) r^\mu \rho^{n-1+\mu},
\end{aligned}
$$

where c_3 is the constant appearing in Theorem 4.2, which leads to the thesis. ∎

At this point, we are able to establish a weak form of the reverse Poincaré inequality. The strategy for its proof is the same outlined in [39, Lemma 24.9] (see also [43, Lemma 7.3] or [23, Lemma 10]).

Lemma 6.2 (Weak reverse Poincaré inequality). *Let $x_0 \in \Omega$ and $r > 0$. If (E, u) is a $(\kappa r^\mu, \mu)$-minimizer of $\mathcal{F}_{x_0, A_{x_0} \circ \Psi_{x_0, r}^{-1}}$ in $\mathbf{C}_4$ such that*

$$
\begin{gathered}
|x_n| < \frac{1}{8} \quad \forall x \in \mathbf{C}_2 \cap \partial E, \\
\left| \left\{ x \in \mathbf{C}_2 \setminus E : x_n < -\frac{1}{8} \right\} \right| = \left| \left\{ x \in \mathbf{C}_2 \cap E : x_n > \frac{1}{8} \right\} \right| = 0,
\end{gathered}
$$

and if $z \in \mathbb{R}^{n-1}$ and $s > 0$ are such that

$$
\mathbf{K}_s(z) \subset \mathbf{C}_2, \quad \mathcal{H}^{n-1}(\partial E \cap \partial \mathbf{K}_s(z)) = 0, \tag{6.1}
$$

then, for every $|c| < \frac{1}{4}$,

$$
\begin{aligned}
&P(E; \mathbf{K}_{\frac{s}{2}}(z)) - \mathcal{H}^{n-1}(\mathbf{D}_{\frac{s}{2}}(z)) \\
&\quad \leq c_{10} \left\{ \left[\left(P(E; \mathbf{K}_s(z)) - \mathcal{H}^{n-1}(\mathbf{D}_s(z)) \right) \right. \right. \\
&\qquad \left. \left. \times \int_{\mathbf{K}_s(z) \cap \partial^* E} \frac{(x_n - c)^2}{s^2} d\, \mathcal{H}^{n-1} \right]^{\frac{1}{2}} + \int_{\mathbf{K}_s(z)} |\nabla u|^2 \, dx + \left(\kappa + [A]_{C^\mu} \right) r^\mu \right\}
\end{aligned}
$$

for some positive constant $c_{10} = c_{10}(n, \alpha, \beta, \lambda, \Lambda, \kappa, \mu, \|\nabla u\|_{L^2(\Omega)})$.

Proof. We may assume that $z = 0$. The set function

$$
m(G) = P(E; \mathbf{C}_2 \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G) \quad \text{for } G \subset \mathbf{D}_2,
$$

where p denotes the projection on the plane $\{x_n = 0\}$, defines a Radon measure on $\mathbb{R}^{n-1}$, supported in $\mathbf{D}_2$. Since E is a set of locally finite perimeter, by [39, Theorem 13.8] there exist a sequence $(E_h)_{h \in \mathbb{N}}$ of open subsets of $\mathbb{R}^n$ with smooth boundary and a vanishing sequence $(\varepsilon_h)_{h \in \mathbb{N}} \subset \mathbb{R}^+$ such that

$$
E_h \xrightarrow{\text{loc}} E, \quad \mathcal{H}^{n-1} \llcorner \partial E_h \to \mathcal{H}^{n-1} \llcorner \partial E, \quad \partial E_h \subset I_{\varepsilon_h}(\partial E),
$$

as $h \to +\infty$, where $I_{\varepsilon_h}(\partial E)$ is a tubular neighborhood of ∂E with half-length ε_h. By the coarea formula, we get

$$\mathcal{H}^{n-1}(\partial \mathbf{K}_{\rho s} \cap (E^{(1)} \triangle E_h)) \to 0 \quad \text{for a.e. } \rho \in \Big(\frac{2}{3}, \frac{3}{4}\Big).$$

Moreover, provided h is large enough, by $\partial E_h \subset I_{\varepsilon_h}(\partial E)$, we get

$$|x_n| < \frac{1}{4} \quad \forall x \in \mathbf{C}_2 \cap \partial E_h,$$
$$\Big\{x \in \mathbf{C}_2 : x_n < -\frac{1}{4}\Big\} \subset \mathbf{C}_2 \cap E_h \subset \Big\{x \in \mathbf{C}_2 : x_n < \frac{1}{4}\Big\}.$$

Therefore, given $\lambda \in (0, \frac{1}{4})$ and $|c| < \frac{1}{4}$, we are in position to apply [39, Lemma 24.8] to every E_h to deduce that there exists $I_h \subset (\frac{2}{3}, \frac{3}{4})$, with $|I_h| \geq \frac{1}{24}$, and for any $\rho \in I_h$, there exists an open subset F_h of $\mathbb{R}^n$ of locally finite perimeter such that

$$F_h \cap \partial \mathbf{K}_{\rho s} = E_h \cap \partial \mathbf{K}_{\rho s}, \tag{6.2}$$
$$\mathbf{K}_{\frac{s}{2}} \cap \partial F_h = \mathbf{D}_{\frac{s}{2}} \times \{c\},$$
$$P(F_h; \mathbf{K}_{\rho s}) - \mathcal{H}^{n-1}(\mathbf{D}_{\rho s}) \leq c(n)\Big\{\lambda(P(E_h; \mathbf{K}_s) - \mathcal{H}^{n-1}(\mathbf{D}_s)) + \frac{1}{\lambda}\int_{\mathbf{K}_s \cap \partial E_h} \frac{|x_n - c|^2}{s^2}\, d\mathcal{H}^{n-1}\Big\}. \tag{6.3}$$

Clearly, $\bigcap_{h\in\mathbb{N}} \bigcup_{k\geq h} |I_k| \geq \frac{1}{24} > 0$, and thus, there exist a divergent subsequence $(h_k)_{k\in\mathbb{N}}$ and $\rho \in (\frac{2}{3}, \frac{3}{4})$ such that

$$\rho \in \bigcap_{k\in\mathbb{N}} I_{h_k} \quad \text{and} \quad \lim_{k\to+\infty} \mathcal{H}^{n-1}(\partial \mathbf{K}_{\rho s} \cap (E^{(1)} \triangle E_{h_k})) = 0.$$

We will write F_k in place of F_{h_k}. We consider the comparison set $G_k = (F_k \cap \mathbf{K}_{\rho s}) \cup (E \setminus \mathbf{K}_{\rho s})$. By applying [39, formula (16.33)], we infer that

$$P(G_k; \mathbf{K}_s) = P(F_k; \mathbf{K}_{\rho s}) + P(E; \mathbf{K}_s \setminus \mathbf{K}_{\rho s}) + \sigma_k,$$

where, thanks to (6.2),

$$\sigma_k = \mathcal{H}^{n-1}(\partial \mathbf{K}_{\rho s} \cap (E^{(1)} \triangle F_k)) = \mathcal{H}^{n-1}(\partial \mathbf{K}_{\rho s} \cap (E^{(1)} \triangle E_{h_k})) \to 0,$$

as $k \to +\infty$. We apply Proposition 6.1, deducing the following relation:

$$\int_{\mathrm{spt}(u-v)} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2\, dx + P(E; B_{\tilde\rho}) \leq \int_{\mathrm{spt}(u-v)} \sigma_G |\nabla v A^{-\frac{1}{2}}(x_0)|^2\, dx + P(G; B_{\tilde\rho}) + c(n, \alpha, \beta, \lambda, \Lambda, \kappa, \mu, \|\nabla u\|_{L^2(\Omega)})(\kappa r^\mu + [A]_{C^\mu} r^\mu)\tilde\rho^{n-1+\mu}$$

for every (G, v) such that $G \triangle E \Subset B_{\tilde\rho} \subset \mathbf{C}_4$ and $v \in u + H^1_0(B_{\tilde\rho})$.

Now, we test the previous relation of minimality with (G_k, u), as $E \,\triangle\, G_k \Subset \mathbf{K}_s \subset B_4 \subset \mathbf{C}_4$, and get rid of the common terms obtaining

$$P(E;\mathbf{K}_{\rho s}) \le P(F_k;\mathbf{K}_{\rho s}) + \sigma_k + c\big(n,\alpha,\beta,\lambda,\Lambda,\kappa,\mu,\|\nabla u\|_{L^2(\Omega)}\big)\bigg[\int_{\mathbf{K}_{\rho s}} |\nabla u|^2\,dx + (\kappa r^\mu + [A]_{C^\mu} r^\mu)\bigg]. \tag{6.4}$$

Thus, since m is nondecreasing and $\rho \in \big(\frac{2}{3}, \frac{3}{4}\big)$, by (6.4) and (6.3), we deduce that

$$\begin{aligned}
&P(E;\mathbf{K}_{\frac{s}{2}}) - \mathcal{H}^{n-1}(\mathbf{D}_{\frac{s}{2}}) \\
&\quad = m(\mathbf{D}_{\frac{s}{2}}) \le m(\mathbf{D}_{\rho s}) = P(E;\mathbf{K}_{\rho s}) - \mathcal{H}^{n-1}(\mathbf{D}_{\rho s}) \\
&\quad \le P(F_k;\mathbf{K}_{\rho s}) - \mathcal{H}^{n-1}(\mathbf{D}_{\rho s}) + \sigma_k + c\bigg[\int_{\mathbf{K}_{\rho s}} |\nabla u|^2\,dx + \big(\kappa + [A]_{C^\mu}\big) r^\mu\bigg] \\
&\quad \le c(n)\bigg\{\lambda\big(P(E_{h_k};\mathbf{K}_s) - \mathcal{H}^{n-1}(\mathbf{D}_s)\big) + \frac{1}{\lambda}\int_{\mathbf{K}_s \cap \partial E_{h_k}} \frac{|x_n - c|^2}{s^2}\,d\,\mathcal{H}^{n-1}\bigg\} \\
&\qquad + c\bigg[\int_{\mathbf{K}_s} |\nabla u|^2\,dx + (\kappa + [A]_{C^\mu}) r^\mu\bigg],
\end{aligned}$$

where $c = c(n,\alpha,\beta,\lambda,\Lambda,\kappa,\mu,\|\nabla u\|_{L^2(\Omega)})$. Letting $k \to +\infty$, assumption (6.1) implies that $P(E_{h(k)};\mathbf{K}_s) \to P(E;\mathbf{K}_s)$, and therefore,

$$P(E;\mathbf{K}_{\frac{s}{2}}) - \mathcal{H}^{n-1}(\mathbf{D}_{\frac{s}{2}}) \le c\bigg\{\lambda\big(P(E;\mathbf{K}_s) - \mathcal{H}^{n-1}(\mathbf{D}_s)\big) + \frac{1}{\lambda}\int_{\mathbf{K}_s \cap \partial E} \frac{|x_n - c|^2}{s^2}\,d\,\mathcal{H}^{n-1} + \int_{\mathbf{K}_{rs}} |\nabla u|^2\,dx + (\kappa + [A]_{C^\mu}) r^\mu\bigg\} \tag{6.5}$$

for any $\lambda \in (0, \frac{1}{4})$. If $\lambda > \frac{1}{4}$, then

$$\begin{aligned}
&P(E;\mathbf{K}_{\frac{s}{2}}) - \mathcal{H}^{n-1}(\mathbf{D}_{\frac{s}{2}}) = m(\mathbf{D}_{\frac{s}{2}}) \le m(\mathbf{D}_{\rho s}) \\
&\qquad \le 4\lambda P(E;\mathbf{K}_{\rho s}) - \mathcal{H}^{n-1}(\mathbf{D}_{\rho s}) \le c(n)\lambda\big(P(E;\mathbf{K}_s) - \mathcal{H}^{n-1}(\mathbf{D}_s)\big),
\end{aligned}$$

and thus (6.5) holds true for $\lambda > 0$, provided we choose $c(n) \ge 4$. Minimizing over λ, we get the thesis. ∎

Finally, we are able to prove the main result of this section.

Theorem 6.3 (Reverse Poincaré inequality). *Let $x_0 \in \Omega$ and $r > 0$. There exist two positive constants $c_{11} = c_{11}(n,\alpha,\beta,\lambda,\Lambda,\kappa,\mu,\|\nabla u\|_{L^2(\Omega)})$ and $\varepsilon_5 = \varepsilon_5(n)$ such that if (E,u) is a $(\kappa r^\mu,\mu)$-minimizer of $\mathcal{F}_{x_0,A_{x_0}\circ\Psi^{-1}_{x_0,r}}$ in $\mathbf{C}_{4\tau}(0,\nu)$, with $0 \in \partial E$, $\tau > 0$ and*

$$\mathbf{e}^C(0,4\tau,\nu) < \varepsilon_5,$$

then

$$\mathbf{e}^C(0,\tau,\nu) \le c_{11}\Bigg(\frac{1}{\tau^{n+1}}\int_{\partial E\cap \mathbf{C}_{2\tau}(0,\nu)} |\langle \nu, x\rangle - c|^2 d\,\mathcal{H}^{n-1} + \frac{1}{\tau^{n-1}}\int_{\mathbf{C}_{2\tau}(0,\nu)} |\nabla u|^2\,dx + (\kappa + [A]_{C^\mu})(\tau r)^\mu\Bigg) \tag{6.6}$$

for every $c \in \mathbb{R}$.

Proof. The proof of this result follows the same strategy employed in [23, Theorem 6]. We emphasize only small differences between the two proofs. Up to a rotation and employing a usual scaling argument, by Proposition 3.2, with a small abuse of notation, we may assume that (E,u) is a $(\kappa(\tau r)^\mu,\mu)$-minimizer of $\mathcal{F}_{x_0,A_{x_0}\circ\Psi^{-1}_{x_0,\tau r}}$ in $\mathbf{C}_4$, with $0\in\partial\widetilde{E}$. Leveraging the compactness of the perimeter and Theorem 4.9, it is possible to show that

$$|x_n| < \frac{1}{4} \quad \forall x \in \mathbf{C}_2 \cap \partial E,$$
$$\left|\left\{x \in \mathbf{C}_2 \setminus E : x_n < -\frac{1}{8}\right\}\right| = \left|\left\{x \in \mathbf{C}_2 \cap E : x_n > \frac{1}{8}\right\}\right| = 0.$$

Thus, for any $z \in \mathbb{R}^{n-1}$ and $s>0$ such that

$$\mathbf{K}_s(z) \subset \mathbf{C}_2, \quad \mathcal{H}^{n-1}(\partial E \cap \partial \mathbf{K}_s(z)) = 0,$$

we apply Lemma 6.2, deducing that, for every $|c| < \frac{1}{4}$,

$$\begin{aligned} &P(E;\mathbf{K}_s(z)) - \mathcal{H}^{n-1}(\mathbf{D}_s(z)) \\ &\le c\Bigg\{\Bigg[\left[P(E;\mathbf{K}_{2s}(z)) - \mathcal{H}^{n-1}(\mathbf{D}_{2s}(z))\right]\inf_{|c|<\frac{1}{4}}\int_{\mathbf{C}_2\cap\partial E}|x_n - c|^2\,d\,\mathcal{H}^{n-1}\Bigg]^{\frac{1}{2}} \\ &\quad + \int_{\mathbf{K}_s}|\nabla u|^2\,dx + \kappa\tau r^\mu + [A]_{C^\mu}(\tau r)^\mu\Bigg\} \end{aligned} \tag{6.7}$$

for some positive constant $c = c(n,\alpha,\beta,\lambda,\Lambda,\kappa,\mu,\|\nabla u\|_{L^2(\Omega)})$. Hence, proceeding as in [23, Theorem 6], by a covering argument, it is possible to show that (6.7) implies (6.6). ∎

7. Proof of the main theorem

The strategy adopted to establish the main result involves two key steps: first proving a first variation formula for the bulk energy of $\mathcal{F}_{x_0,A_{x_0}\circ\Psi^{-1}_{x_0,r}}$, then establishing an excess improvement theorem for transformed couples, which in turn implies an analogous theorem for the original ones.

Proposition 7.1 (First variation formula for the bulk term). *$x_0 \in \Omega$, $u \in H^1(B_1)$ and $X \in C^1_c(B_1;\mathbb{R}^n)$. We define $\Phi_t(x) = x + tX(x)$ for any $x \in \mathbb{R}^n$ and $t > 0$. Accordingly, we define*

$$E_t := \Phi_t(E), \quad u_t := u \circ \Phi_t^{-1}.$$

There exist two constants $c_{12} = c_{12}(\beta, \lambda, \nabla X) > 0$ and $t_0 > 0$ such that it holds that

$$\int_{B_1} \sigma_{E_t} |\nabla u_t A^{-\frac{1}{2}}(x_0)|^2 \, dx - \int_{B_1} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2 \, dx \le c_{12}(t + o(t)) \int_{B_1} |\nabla u|^2 \, dx$$

for any $0 < t < t_0$.

Proof. Taking into account that

$$\nabla \Phi_t^{-1}(\Phi_t(x)) = I - t\nabla X(x) + o(t), \quad \mathrm{J}\Phi_t(x) = 1 + t\mathrm{div}X(x) + o(t)$$

for any $x \in \mathbb{R}^n$ and $t > 0$, by the change of variable $y = \Phi_t(x)$, we obtain

$$\begin{aligned}
&\int_{B_1} \sigma_{E_t} |\nabla u_t A^{-\frac{1}{2}}(x_0)|^2 \, dy - \int_{B_1} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2 \, dx \\
&\quad = \int_{B_1} \sigma_E \left|[\nabla u - t\nabla u \nabla X + \nabla u \, o(t)] A^{-\frac{1}{2}}(x_0)\right|^2 (1 + t\mathrm{div}X + o(t)) \, dx \\
&\qquad - \int_{B_1} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2 \, dx \\
&\quad = \int_{B_1} \sigma_E \big[|\nabla u A^{-\frac{1}{2}}(x_0)|^2 + |\nabla u A^{-\frac{1}{2}}(x_0)|^2 (t\mathrm{div}X + o(t))\big] dx + H(t, \nabla u, \nabla X) \\
&\qquad - \int_{B_1} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2 \, dx \\
&\quad = \int_{B_1} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2 (t\mathrm{div}X + o(t)) dx + H(t, \nabla u, \nabla X), \qquad (7.1)
\end{aligned}$$

where

$$\begin{aligned}
&H(t, \nabla u, \nabla X) \\
&\quad = \int_{B_1} \sigma_E \left|[-t\nabla u \nabla X + \nabla u \, o(t)] A^{-\frac{1}{2}}(x_0)\right|^2 (1 + t\mathrm{div}X + o(t)) dx \\
&\qquad + \int_{B_1} 2\big\langle \nabla u A^{-\frac{1}{2}}(x_0), (-t\nabla u \nabla X + \nabla u \, o(t)) A^{-\frac{1}{2}}(x_0)\big\rangle (1 + t\mathrm{div}X + o(t)) dx.
\end{aligned}$$

We estimate

$$\int_{B_1} \sigma_E |\nabla u A^{-\frac{1}{2}}(x_0)|^2 (t\mathrm{div}X + o(t)) \, dx \le c(\beta, \lambda, \nabla X)(t + o(t)) \int_{B_1} |\nabla u|^2 \, dx \qquad (7.2)$$

and

$$H(t,\nabla u,\nabla X) \le c(\beta,\lambda,\nabla X)\int_{B_1}(t+o(t))^2|\nabla u|^2(1+t\operatorname{div}X+o(t))\,dx$$
$$+c(\beta,\lambda,\nabla X)\int_{B_1}(t+o(t))|\nabla u|^2(1+t\operatorname{div}X+o(t))\,dx$$
$$\le c(\beta,\lambda,\nabla X)(t+o(t))\int_{B_1}|\nabla u|^2\,dx. \tag{7.3}$$

Inserting (7.3) and (7.2) in (7.1), we get the desired inequality. ∎

Here, we present the proof of the excess improvement theorem for transformed couples.

Theorem 7.2 (Excess improvement for the transformed couple). *For any $\omega\in(0,1)$, $\tilde\sigma\in(0,1)$, $\widetilde M>0$, $\tilde\tau\in(0,\frac{1}{16})$ there exists a constant $\tilde\varepsilon=\tilde\varepsilon(\tilde\sigma,\widetilde M,\tilde\tau)>0$ such that if $(\widetilde E,\tilde u)$ is a $(\tilde\kappa,\mu)$-minimizer of $\mathcal{F}_{x_0,A_{x_0}}$ in $B_{\tilde r}(x_0)$, with $x_0\in\partial\widetilde E$ such that*

$$\mathbf{e}(\widetilde E,x_0,\tilde r)\le\tilde\varepsilon,\quad \mathcal{D}_{\tilde u}(x_0,\tilde r)+\tilde r^{(1-\omega)\mu}\le\widetilde M\mathbf{e}(\widetilde E,x_0,\tilde\sigma\tilde r),$$

then there exists a constant $c_{13}=c_{13}(n,\alpha,\beta,\lambda,\Lambda,\kappa,\mu,\|\nabla u\|_{L^2(\Omega)})>0$ such that

$$\mathbf{e}(\widetilde E,x_0,\tilde\tau\tilde r)\le c_{13}\big(\tilde\tau^2\mathbf{e}(\widetilde E,x_0,\tilde r)+\mathcal{D}_{\tilde u}(x_0,4\tilde\tau\tilde r)+(\tilde\tau\tilde r)^\mu\big).$$

Proof. Let us assume by contradiction that there exist a vanishing sequence $(\tilde r_h)_{h\in\mathbb{N}}\subset\mathbb{R}^+$ and a sequence $((\widetilde E_h,\tilde u_h))_{h\in\mathbb{N}}$ of $(\tilde\kappa,\mu)$-minimizers of $\mathcal{F}_{x_0,A}$ in $B_{\tilde r_h}(x_0)$, with $x_0\in\partial\widetilde E_h$, such that

$$\mathbf{e}(\widetilde E_h,x_0,\tilde r_h)=:\varepsilon_h\to0,\quad \mathcal{D}_{\tilde u_h}(x_0,\tilde r_h)+\tilde r_h^{(1-\omega)\mu}\le\widetilde M\mathbf{e}(\widetilde E_h,x_0,\tilde\sigma\tilde r_h)$$

and

$$\mathbf{e}(\widetilde E_h,x_0,\tilde\tau\tilde r_h)>\bar C\big(\tilde\tau^2\mathbf{e}(\widetilde E_h,x_0,\tilde r_h)+\mathcal{D}_{\tilde u_h}(x_0,4\tilde\tau\tilde r_h)+(\tilde\tau\tilde r_h)^\mu\big)$$

for some constant $\bar C>0$ to be chosen. Employing the usual scaling argument and applying Proposition 3.2, with a small abuse of notation, we may assume that $((\widetilde E_h,\tilde u_h))_{h\in\mathbb{N}}$ is a sequence of $(\tilde\kappa\tilde r_h^\mu,\mu)$-minimizers of $\mathcal{F}_{x_0,A_{x_0}\circ\Psi^{-1}_{x_0,\tilde r_h}}$ in B_1, with $0\in\partial\widetilde E_h$ such that

$$\mathbf{e}(\widetilde E_h,0,1)=\varepsilon_h\to0,\quad \mathcal{D}_{\tilde u_h}(0,1)+\tilde r_h^{(1-\omega)\mu}\le\widetilde M\mathbf{e}(\widetilde E_h,0,\tilde\sigma) \tag{7.4}$$

and

$$\mathbf{e}(\widetilde E_h,0,\tilde\tau)>\bar C\big(\tilde\tau^2\mathbf{e}(\widetilde E_h,0,1)+\mathcal{D}_{\tilde u_h}(0,4\tilde\tau)+(\tilde\tau\tilde r_h)^\mu\big).$$

Up to rotating each $\widetilde E_h$, we may also assume that, for all $h\in\mathbb{N}$,

$$\mathbf{e}(\widetilde E_h,0,1)=\frac12\int_{\partial\widetilde E_h\cap B_1}|\nu_{E_h}-e_n|^2\,d\mathcal{H}^{n-1}.$$

Step 1. Thanks to the Lipschitz approximation theorem, for h sufficiently large, there exists a 1-Lipschitz function $f_h\colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\sup_{\mathbb{R}^{n-1}} |f_h| \le c_8 \varepsilon_h^{\frac{1}{2(n-1)}}, \quad \mathcal{H}^{n-1}((\partial \widetilde{E}_h \,\triangle\, \Gamma_{f_h}) \cap B_{\frac{1}{2}}) \le c_8 \varepsilon_h, \quad \int_{\mathbf{D}_{\frac{1}{2}}} |\nabla' f_h|^2 \, dx' \le c_8 \varepsilon_h. \tag{7.5}$$

We define

$$g_h(x') := \frac{f_h(x') - a_h}{\sqrt{\varepsilon_h}}, \quad \text{where } a_h = \fint_{\mathbf{D}_{\frac{1}{2}}} f_h \, dx',$$

and we assume, up to a subsequence, that $\{g_h\}_{h\in\mathbb{N}}$ converges weakly in $H^1(\mathbf{D}_{\frac{1}{2}})$ and strongly in $L^2(\mathbf{D}_{\frac{1}{2}})$ to a function g. We prove that g is harmonic in $\mathbf{D}_{\frac{1}{2}}$. It is enough to show that

$$\lim_{h\to+\infty} \frac{1}{\sqrt{\varepsilon_h}} \int_{\mathbf{D}_{\frac{1}{2}}} \frac{\langle \nabla' f_h, \nabla' \phi \rangle}{\sqrt{1 + |\nabla' f_h|^2}} \, dx' = 0$$

for all $\phi \in C_0^1(\mathbf{D}_{\frac{1}{2}})$. We fix $\delta > 0$ so that $\operatorname{supp}\phi \times [-2\delta, 2\delta] \subset B_{\frac{1}{2}}$ and choose a cut-off function $\psi\colon \mathbb{R} \to [0,1]$ with $\operatorname{supp}\psi \subset (-2\delta, 2\delta)$, $\psi = 1$ in $(-\delta, \delta)$. Let us define

$$\Phi_h(x) := x + \tilde{r}_h^{\omega\mu} X(x), \quad \text{where } X(x) = \phi(x')\psi(x_n) e_n$$

for $x \in \mathbb{R}^n$. We apply Proposition 6.1 to deduce that

$$\begin{aligned} &P(\widetilde{E}_h; B_{\frac{1}{2}}) - P(\Phi_h(\widetilde{E}_h); B_{\frac{1}{2}}) \\ &\quad \le \int_{B_{\frac{1}{2}}} \sigma_{\Phi_h(\widetilde{E}_h)} |\nabla(\tilde{u}_h \circ \Phi_h^{-1}) A^{-\frac{1}{2}}(x_0)|^2 \, dx - \int_{B_{\frac{1}{2}}} \sigma_{\widetilde{E}_h} |\nabla \tilde{u}_h A^{-\frac{1}{2}}(x_0)|^2 \, dx \\ &\qquad + c\big(n, \alpha, \beta, \lambda, \Lambda, \kappa, \mu, \|\nabla u\|_{L^2(\Omega)}\big)\big(\tilde{\kappa} \tilde{r}_h^{\mu} + [A]_{C^\mu} \tilde{r}_h^{\mu}\big) \frac{1}{2^{n-1+\mu}}. \end{aligned} \tag{7.6}$$

Using the first variation formula for the perimeter and Proposition 7.1, for h sufficiently large, we get

$$P(\widetilde{E}_h; B_{\frac{1}{2}}) - P(\Phi_h(\widetilde{E}_h); B_{\frac{1}{2}}) = (\tilde{r}_h^{\omega\mu} + O(\tilde{r}_h^{2\omega\mu})) \int_{\partial \widetilde{E}_h \cap B_{\frac{1}{2}}} \langle \nu_{\widetilde{E}_h}, e_n \rangle \langle \nabla' \phi, \nu'_{\widetilde{E}_h} \rangle \, d\mathcal{H}^{n-1} \tag{7.7}$$

and

$$\begin{aligned} &\int_{B_{\frac{1}{2}}} \sigma_{\Phi_h(\widetilde{E}_h)} |\nabla(\tilde{u}_h \circ \Phi_h^{-1}) A^{-\frac{1}{2}}(x_0)|^2 \, dx - \int_{B_{\frac{1}{2}}} \sigma_{\widetilde{E}_h} |\nabla \tilde{u}_h A^{-\frac{1}{2}}(x_0)|^2 \, dx \\ &\qquad \le c\big(\tilde{r}_h^{\omega\mu} + o(\tilde{r}_h^{\omega\mu})\big) \int_{B_{\frac{1}{2}}} |\nabla \tilde{u}_h|^2 \, dx \end{aligned} \tag{7.8}$$

for some $c = c(\beta, \lambda, \nabla\phi, \nabla\psi) > 0$. Inserting inequalities (7.8) and (7.7) in (7.6), dividing by $\sqrt{\varepsilon_h}(\tilde{r}_h^{\omega\mu} + O(\tilde{r}_h^{2\omega\mu}))$ and taking (7.4) into account, we get

$$\begin{aligned}
&\frac{1}{\sqrt{\varepsilon_h}} \int_{\partial \widetilde{E}_h \cap B_{\frac{1}{2}}} \langle \nu_{\widetilde{E}_h}, e_n \rangle \langle \nabla' \phi, \nu'_{\widetilde{E}_h} \rangle \, d\mathcal{H}^{n-1} \\
&\quad \leq \frac{c}{\sqrt{\varepsilon_h}\big(\tilde{r}_h^{\omega\mu} + O\big(\tilde{r}_h^{\omega\mu}\big)\big)} \left(\big(\tilde{r}_h^{\omega\mu} + o\big(\tilde{r}_h^{\omega\mu}\big)\big) \int_{B_{\frac{1}{2}}} |\nabla \tilde{u}_h|^2 \, dx + \tilde{r}_h^{\mu} \right) \\
&\quad \leq \frac{c}{\sqrt{\varepsilon_h}} \big(\mathcal{D}_{\tilde{u}_h}(0,1) + \tilde{r}_h^{(1-\omega)\mu} \big) \\
&\quad \leq \frac{c}{\sqrt{\varepsilon_h}} \mathbf{e}(\widetilde{E}_h, 0, \tilde{\sigma}) \leq c\sqrt{\varepsilon_h}
\end{aligned}$$

for some $c = c(n, \alpha, \beta, \lambda, \Lambda, \tilde{\kappa}, \mu, [A]_{C^\mu}, \tilde{\sigma}, \widetilde{M}, \nabla\phi, \nabla\psi) > 0$. Replacing ϕ with $-\phi$, we infer that

$$\lim_{h \to +\infty} \frac{1}{\sqrt{\varepsilon_h}} \left| \int_{\partial \widetilde{E}_h \cap B_{\frac{1}{2}}} \langle \nu_{\widetilde{E}_h}, e_n \rangle \langle \nabla' \phi, \nu'_{\widetilde{E}_h} \rangle \, d\mathcal{H}^{n-1} \right| = 0. \tag{7.9}$$

Decomposing $\partial \widetilde{E}_h \cap B_{\frac{1}{2}} = \big([\Gamma_{f_h} \cup (\partial \widetilde{E}_h \setminus \Gamma_{f_h})] \setminus (\Gamma_{f_h} \setminus \partial \widetilde{E}_h) \big) \cap B_{\frac{1}{2}}$, we deduce

$$\begin{aligned}
&- \int_{\partial \widetilde{E}_h \cap B_{\frac{1}{2}}} \langle \nu_{\widetilde{E}_h}, e_n \rangle \langle \nabla' \phi, \nu'_{\widetilde{E}_h} \rangle \, d\mathcal{H}^{n-1} \\
&\quad = - \int_{\Gamma_{f_h} \cap B_{\frac{1}{2}}} \langle \nu_{\widetilde{E}_h}, e_n \rangle \langle \nabla' \phi, \nu'_{\widetilde{E}_h} \rangle \, d\mathcal{H}^{n-1} \\
&\qquad - \int_{(\partial \widetilde{E}_h \setminus \Gamma_{f_h}) \cap B_{\frac{1}{2}}} \langle \nu_{\widetilde{E}_h}, e_n \rangle \langle \nabla' \phi, \nu'_{\widetilde{E}_h} \rangle \, d\mathcal{H}^{n-1} \\
&\qquad + \int_{(\Gamma_{f_h} \setminus \partial \widetilde{E}_h) \cap B_{\frac{1}{2}}} \langle \nu_{\widetilde{E}_h}, e_n \rangle \langle \nabla' \phi, \nu'_{\widetilde{E}_h} \rangle \, d\mathcal{H}^{n-1}.
\end{aligned}$$

Since by the second inequality in (7.5), we have

$$\left| \frac{1}{\sqrt{\varepsilon_h}} \int_{(\partial \widetilde{E}_h \setminus \Gamma_{f_h}) \cap B_{\frac{1}{2}}} \langle \nu_{\widetilde{E}_h}, e_n \rangle \langle \nabla' \phi, \nu'_{\widetilde{E}_h} \rangle \, d\mathcal{H}^{n-1} \right| \leq c_8 \sqrt{\varepsilon_h} \sup_{\mathbb{R}^{n-1}} |\nabla' \phi|,$$

$$\left| \frac{1}{\sqrt{\varepsilon_h}} \int_{(\Gamma_{f_h} \setminus \partial \widetilde{E}_h) \cap B_{\frac{1}{2}}} \langle \nu_{\widetilde{E}_h}, e_n \rangle \langle \nabla' \phi, \nu'_{\widetilde{E}_h} \rangle \, d\mathcal{H}^{n-1} \right| \leq c_8 \sqrt{\varepsilon_h} \sup_{\mathbb{R}^{n-1}} |\nabla' \phi|,$$

then, by (7.9) and the area formula, we infer that

$$0 = \lim_{h \to +\infty} \frac{-1}{\sqrt{\varepsilon_h}} \int_{\Gamma_{f_h} \cap B_{\frac{1}{2}}} \langle \nu_{\widetilde{E}_h}, e_n \rangle \langle \nabla' \phi, \nu'_{\widetilde{E}_h} \rangle \, d\mathcal{H}^{n-1} = \lim_{h \to +\infty} \frac{1}{\sqrt{\varepsilon_h}} \int_{\mathbf{D}_{\frac{1}{2}}} \frac{\langle \nabla' f_h, \nabla' \phi \rangle}{\sqrt{1 + |\nabla' f_h|^2}} \, dx'.$$

This proves that g is harmonic in $\mathbf{D}_{\frac{1}{2}}$.

Step 2. The proof of this step now follows exactly as in [29] using the height bound lemma and the reverse Poincaré inequality. We give here the proof for the sake of completeness. Setting

$$b_h := \frac{(f_h)_{4\tilde\tau}}{\sqrt{1+|(\nabla' f_h)_{4\tilde\tau}|^2}},$$

$$\nu_h := \frac{(-(\nabla' f_h)_{4\tilde\tau}, 1)}{\sqrt{1+|(\nabla' f_h)_{4\tilde\tau}|^2}}.$$

We want to estimate from above the flatness of $\partial\widetilde{E}_h$ towards the hyperplane $\{y \in \mathbb{R}^n : \langle y, \nu_h\rangle = b_h\}$ in $B_{4\tilde\tau}$ with the excess. More precisely, we show that

$$\limsup_{h\to+\infty} \frac{1}{\varepsilon_h \tilde\tau^{n+1}} \int_{\partial\widetilde{E}_h \cap B_{4\tilde\tau}} |\langle \nu_h, x\rangle - b_h|^2 \, d\mathcal{H}^{n-1} \le c(n, c_8)\tilde\tau^2. \tag{7.10}$$

On one hand, by the mean value property of harmonic functions (see [39, Lemma 25.1]), Jensen's inequality, semicontinuity, and the third inequality in (7.5), we deduce that

$$\begin{aligned}
&\limsup_{h\to+\infty} \frac{1}{\varepsilon_h \tilde\tau^{n+1}} \int_{\partial\widetilde{E}_h \cap \Gamma_{f_h} \cap B_{4\tilde\tau}} |\langle \nu_h, x\rangle - b_h|^2 \, d\mathcal{H}^{n-1} \\
&= \limsup_{h\to+\infty} \frac{1}{\varepsilon_h \tilde\tau^{n+1}} \int_{\partial\widetilde{E}_h \cap \Gamma_{f_h} \cap B_{4\tilde\tau}} \frac{|\langle -(\nabla' f_h)_{4\tilde\tau}, x'\rangle + f_h(x') - (f_h)_{4\tilde\tau}|^2}{1+|(\nabla' f_h)_{4\tilde\tau}|^2} \sqrt{1+|\nabla' f_h(x')|^2}\, dx' \\
&\le \limsup_{h\to+\infty} \frac{1}{\varepsilon_h \tilde\tau^{n+1}} \int_{\mathbf{D}_{4\tilde\tau}} |f_h(x') - (f_h)_{4\tilde\tau} - \langle (\nabla' f_h)_{4\tilde\tau}, x'\rangle|^2 \, dx' \\
&= \frac{1}{\tilde\tau^{n+1}} \int_{\mathbf{D}_{4\tilde\tau}} |g(x') - (g)_{4\tilde\tau} - \langle (\nabla' g)_{4\tilde\tau}, x'\rangle|^2 \, dx' \\
&= \frac{1}{\tilde\tau^{n+1}} \int_{\mathbf{D}_{4\tilde\tau}} |g(x') - g(0) - \langle \nabla' g(0), x'\rangle|^2 \, dx' \\
&\le c(n)\tilde\tau^2 \sup_{x'\in\mathbf{D}_{4\tilde\tau}} |g(x') - g(0) - \langle \nabla' g(0), x'\rangle|^2 \\
&\le c(n)\tilde\tau^2 \int_{\mathbf{D}_{\frac12}} |\nabla' g|^2 \, dx' \le c(n)\tilde\tau^2 \liminf_{h\to+\infty} \int_{\mathbf{D}_{\frac12}} |\nabla' g_h|^2 \, dx' \le c(n, c_8)\tilde\tau^2,
\end{aligned} \tag{7.11}$$

where we used that $\mathbf{D}_{4\tilde\tau} \subset \mathbf{D}_{\frac14}$, since $\tilde\tau < \frac{1}{16}$. On the other hand, from the height bound lemma (see Lemma 4.8) and (7.5), we immediately get that

$$\lim_{h\to+\infty} \frac{1}{\varepsilon_h} \int_{(\partial\widetilde{E}_h \setminus \Gamma_{f_h}) \cap B_{2\tilde\tau}} |\langle \nu_h, x\rangle - b_h|^2 \, d\mathcal{H}^{n-1} = 0. \tag{7.12}$$

Hence, combining inequalities (7.11) and (7.12), we conclude that inequality (7.10) is satisfied. In order to apply the reverse Poincaré inequality, we show that the sequence

$\{\mathbf{e}_h(\widetilde{E}_h, 4\tilde{\tau}, \nu_h)\}_{h\in\mathbb{N}}$ is infinitesimal; indeed, by the definition of excess, Jensen's inequality and the third inequality in (7.5), we have

$$
\begin{aligned}
&2(4\tilde{\tau})^{n-1} \limsup_{h\to+\infty} \mathbf{e}_h(\widetilde{E}_h, 0, 4\tilde{\tau}, \nu_h)\\
&\quad= \limsup_{h\to+\infty} \int_{\partial\widetilde{E}_h\cap B_{4\tilde{\tau}}} |\nu_{\widetilde{E}_h} - \nu_h|^2 \, d\mathcal{H}^{n-1}\\
&\quad\le \limsup_{h\to+\infty} \left[2\int_{\partial\widetilde{E}_h\cap B_{4\tilde{\tau}}} |\nu_{\widetilde{E}_h} - e_n|^2 \, d\mathcal{H}^{n-1} + 2|e_n - \nu_h|^2 \mathcal{H}^{n-1}(\partial\widetilde{E}_h \cap B_{4\tilde{\tau}}) \right]\\
&\quad\le \limsup_{h\to+\infty} \left[4\varepsilon_h + 2\mathcal{H}^{n-1}(\partial\widetilde{E}_h \cap B_{4\tilde{\tau}}) \frac{|((\nabla' f_h)_{4\tilde{\tau}}, \sqrt{1+|(\nabla' f_h)_{4\tilde{\tau}}|^2} - 1)|^2}{1+|(\nabla' f_h)_{4\tilde{\tau}}|^2} \right]\\
&\quad\le \limsup_{h\to+\infty} \left[4\varepsilon_h + 4\mathcal{H}^{n-1}(\partial E_h \cap B_{4\tilde{\tau}}) |(\nabla' f_h)_{4\tilde{\tau}}|^2 \right]\\
&\quad\le \limsup_{h\to+\infty} \left[4\varepsilon_h + 4\int_{\mathbf{D}_{\frac{1}{2}}} |\nabla' f_h|^2 \, dx' \right] \le \lim_{h\to+\infty} [4\varepsilon_h + 4c_8\varepsilon_h] = 0.
\end{aligned}
$$

Therefore, applying the reverse Poincaré inequality, (7.10) and observing that $\mathbf{C}_{2\tilde{\tau}} \subset B_{4\tilde{\tau}}$, we have for h large that

$$
\begin{aligned}
\mathbf{e}(\widetilde{E}_h, 0, \tau) &\le \mathbf{e}(\widetilde{E}_h, 0, \tau, \nu_h)\\
&\le c_{11}\Bigg(\frac{1}{(2\tilde{\tau})^{n+1}} \int_{\partial\widetilde{E}_h\cap \mathbf{C}_{2\tilde{\tau}}(0,\tilde{\tau})} |\langle \nu_h, x\rangle - b_h|^2 \, d\mathcal{H}^{n-1}\\
&\qquad + \mathcal{D}_{\tilde{u}_h}(0, 4\tilde{\tau}) + \big(\tilde{\kappa} + [A]_{C^\mu}\big)(2\tilde{\tau}\tilde{r}_h)^\mu \Bigg)\\
&\le \tilde{C}\big(n, \alpha, \beta, \lambda, \Lambda, \kappa, \mu, \|\nabla u\|_{L^2(\Omega)}\big)\big(\tilde{\tau}^2 \mathbf{e}(\widetilde{E}_h, 0, 1) + \mathcal{D}_{\tilde{u}_h}(0, 4\tilde{\tau}) + (\tilde{\tau}\tilde{r}_h)^\mu\big),
\end{aligned}
$$

which is a contradiction if we choose $\bar{C} > \tilde{C}$. ∎

We use the previous theorem in the proof of the next result.

Theorem 7.3 (Excess improvement). *For any $\omega \in (0,1)$, $\sigma \in \big(0, \frac{\lambda^{\frac{1}{2}}}{\Lambda^{\frac{1}{2}}}\big)$, $M > 0$, $\tau \in \big(0, \frac{\lambda^{\frac{1}{2}}}{16\Lambda^{\frac{1}{2}}}\big)$ there exists a constant $\varepsilon_6 = \varepsilon_6(\sigma, M, \tau) > 0$ such that if (E, u) is a (κ, μ)-minimizer of $\mathcal{F}_A$ in $B_r(x_0)$, with $x_0 \in \partial E$ such that*

$$
\mathbf{e}(E, x_0, r) \le \varepsilon_6, \quad \mathcal{D}_u(x_0, r) + r^{(1-\omega)\mu} \le M\,\mathbf{e}(E, x_0, \sigma r),
$$

then there exists a constant $c_{14} = c_{14}(n, \alpha, \beta, \lambda, \Lambda, \kappa, \mu, \|\nabla u\|_{L^2(\Omega)}) > 0$ such that

$$
\mathbf{e}(E, x_0, \tau r) \le c_{14}\tau^\mu\big(\mathbf{e}(E, x_0, r) + \mathcal{D}_u(x_0, r) + r^\mu\big).
$$

Proof. Let $\sigma \in \big(0, \frac{\lambda^{\frac{1}{2}}}{\Lambda^{\frac{1}{2}}}\big)$, $M > 0$, $\tau \in \big(0, \frac{\lambda^{\frac{1}{2}}}{16\Lambda^{\frac{1}{2}}}\big)$, and let (E, u) be a (κ, μ)-minimizer of $\mathcal{F}_A$ in $B_r(x_0)$, with $x_0 \in \partial E$ such that

$$
\mathbf{e}(E, x_0, r) \le \varepsilon_6, \quad \mathcal{D}_u(x_0, r) + r^{(1-\omega)\mu} \le M\,\mathbf{e}(E, x_0, \sigma r).
$$

Setting

$$(\widetilde{E},\tilde{u}) := \big(T_{x_0}(E), u\circ T_{x_0}^{-1}\big), \quad \tilde{\kappa} := \lambda^{-\frac{n}{2}}\kappa, \quad \tilde{r} := \theta r,$$

where $\theta \in (0, \min\{\Lambda^{-\frac{1}{2}}, 1\})$, by Proposition 3.1 we have that $(\widetilde{E},\tilde{u})$ is a $(\tilde{\kappa},\mu)$-minimizer of $\mathcal{F}_{x_0,A_{x_0}}$ in $B_{\tilde{r}}(x_0)$. By Proposition 3.3, it holds that

$$\mathbf{e}(\widetilde{E},x_0,\tilde{r}) \le \overline{C}_1\mathbf{e}(E,x_0,r)$$

for some positive constant $\overline{C}_1 = \overline{C}_1(n,\lambda,\Lambda)$. Furthermore, estimating

$$\begin{aligned}\mathcal{D}_{\tilde{u}}(x_0,\tilde{r}) &= \frac{1}{(\theta r)^{n-1}}\int_{B_{\tilde{r}}(x_0)}|\nabla\tilde{u}|^2\,dy = \frac{\det(A^{-\frac{1}{2}}(x_0))}{(\theta r)^{n-1}}\int_{T_{x_0}^{-1}(B_r(x_0))}|\nabla u A^{\frac{1}{2}}(x_0)|^2\,dy\\ &\le \frac{\lambda^{-\frac{n}{2}}\Lambda}{(\theta r)^{n-1}}\int_{B_r(x_0)}|\nabla u|^2\,dy = \frac{\lambda^{-\frac{n}{2}}\Lambda}{\theta^{n-1}}\mathcal{D}_u(x_0,r)\end{aligned}$$

and applying again Proposition 3.3, we get

$$\begin{aligned}\mathcal{D}_{\tilde{u}}(x_0,\tilde{r}) + \tilde{r}^{(1-\omega)\mu} &\le c(n,\lambda,\Lambda)\big(\mathcal{D}_u(x_0,r) + r^{(1-\omega)\mu}\big) \le c(n,\lambda,\Lambda)M\mathbf{e}(E,x_0,\sigma r)\\ &\le c(n,\lambda,\Lambda)M\mathbf{e}\big(\widetilde{E},x_0,\sigma\lambda^{-\frac{1}{2}}r\big) \le \overline{C}_2M\mathbf{e}(\widetilde{E},x_0,\tilde{\sigma}\tilde{r})\end{aligned}$$

for some positive constant $\overline{C}_2 = \overline{C}_2(n,\lambda,\Lambda)$, where $\tilde{\sigma} := \frac{\lambda^{-\frac{1}{2}}}{\theta}\sigma < 1$, since $\sigma < \theta\lambda^{\frac{1}{2}}$. Choosing $\varepsilon_6 > 0$ such that $\overline{C}_1\varepsilon_6 < \tilde{\varepsilon}$ and setting $\widetilde{M} := \overline{C}_2M$, we apply Theorem 7.2 to obtain

$$\mathbf{e}(\widetilde{E},x_0,\tilde{\tau}\tilde{r}) \le C\big(\tilde{\tau}^2\mathbf{e}(\widetilde{E},x_0,\tilde{r}) + \mathcal{D}_{\tilde{u}}(x_0,4\tilde{\tau}\tilde{r}) + (\tilde{\tau}\tilde{r})^{\mu}\big)$$

for some positive constant $C = C(n,\alpha,\beta,\lambda,\Lambda,\kappa,\mu,\|\nabla u\|_{L^2(\Omega)})$, where $\tilde{\tau} := \frac{\tau}{\lambda^{\frac{1}{2}}\theta} < \frac{1}{16}$, since $\tau < \frac{\lambda^{\frac{1}{2}}\theta}{16}$. Leveraging Proposition 3.3, we get

$$\begin{aligned}\mathbf{e}(E,x_0,\tau r) = \mathbf{e}\big(E,x_0,\tilde{\tau}\lambda^{\frac{1}{2}}\tilde{r}\big) &\le c(n,\lambda,\Lambda)\mathbf{e}(\widetilde{E},x_0,\tilde{\tau}\tilde{r})\\ &\le C\big(\tilde{\tau}^2\mathbf{e}(\widetilde{E},x_0,\tilde{r}) + \mathcal{D}_{\tilde{u}}(x_0,4\tilde{\tau}\tilde{r}) + (\tilde{\tau}\tilde{r})^{\mu}\big).\end{aligned} \tag{7.13}$$

On one hand, by Proposition 3.3, we observe that

$$\mathbf{e}(\widetilde{E},x_0,\tilde{r}) \le c(n,\lambda,\Lambda)\mathbf{e}\big(E,x_0,\Lambda^{\frac{1}{2}}\tilde{r}\big) \le c(n,\lambda,\Lambda)\mathbf{e}(E,x_0,r), \tag{7.14}$$

being $\Lambda^{\frac{1}{2}}\tilde{r} \le r$. One the other hand, choosing $\varepsilon_6 < \varepsilon_2$, by Proposition 4.7 it follows that

$$\mathcal{D}_{\tilde{u}}(x_0,4\tilde{\tau}\tilde{r}) \le c(n,\lambda,\Lambda)\mathcal{D}_u\big(x_0,4\tilde{\tau}\Lambda^{\frac{1}{2}}\tilde{r}\big) \le c(n,\lambda,\Lambda)\tilde{\tau}\mathcal{D}_u(x_0,r), \tag{7.15}$$

since $4\tilde{\tau}\Lambda^{\frac{1}{2}}\tilde{r} \le r$, being $\tau < \frac{\lambda^{\frac{1}{2}}}{4\Lambda^{\frac{1}{2}}}$. Inserting (7.14) and (7.15) in (7.13), we obtain

$$\begin{aligned}\mathbf{e}(E,x_0,\tau r) &\le C\big(\tilde{\tau}^2\mathbf{e}(E,x_0,r) + \tilde{\tau}\mathcal{D}_u(x_0,r) + \tilde{\tau}^{\mu}r^{\mu}\big)\\ &\le C\tilde{\tau}^{\mu}\big(\mathbf{e}(E,x_0,r) + \mathcal{D}_u(x_0,r) + r^{\mu}\big)\\ &\le C\tau^{\mu}\big(\mathbf{e}(E,x_0,r) + \mathcal{D}_u(x_0,r) + r^{\mu}\big),\end{aligned}$$

which is the thesis. ∎

Leveraging the results proved in the previous sections, we are able to prove Theorem 1.2.

Proof of Theorem 1.2. Let $U \Subset \Omega$ be an open set. We prove that for every $\omega \in (0,1)$ and $\tau \in (0,1)$ there exist two positive constants $\overline{\varepsilon} = \overline{\varepsilon}(\tau, U)$ and $\overline{C}$ such that if $x_0 \in \partial E$, $B_r(x_0) \subset U$ and $\mathbf{e}(x_0, r) + \mathcal{D}(x_0, r) + r^{(1-\omega)\mu} < \overline{\varepsilon}$, then

$$\mathbf{e}(x_0, \tau r) + \mathcal{D}(x_0, \tau r) + (\tau r)^{(1-\omega)\mu} \leq \overline{C}\tau^{(1-\omega)\mu}\big(\mathbf{e}(x_0, r) + \mathcal{D}(x_0, r) + r^{(1-\omega)\mu}\big). \tag{7.16}$$

We fix $\tau \in (0,1)$. Setting

$$\overline{\tau} := \frac{\lambda^{\frac{1}{2}}}{16\Lambda^{\frac{1}{2}}}, \quad \overline{\sigma} := \frac{\lambda^{\frac{1}{2}}}{\Lambda^{\frac{1}{2}}},$$

we may assume without loss of generality that

$$\tau < \min\left\{\overline{\tau}, \frac{\overline{\sigma}}{2}\right\} = \overline{\tau}.$$

Furthermore, we fix $\sigma := 2\tau < \overline{\sigma}$. We distinguish two cases.

Case 1. $\mathcal{D}_u(x_0, r) + r^{(1-\omega)\mu} \leq \tau^{-1}\mathbf{e}(x_0, \sigma r)$. Choosing $\overline{\varepsilon} < \varepsilon_6(\sigma, \tau, \tau)$ it follows from Theorem 7.3 that

$$\mathbf{e}(x_0, \tau r) \leq c_{14}\tau^{\mu}\big(\mathbf{e}(x_0, r) + \mathcal{D}_u(x_0, r) + r^{\mu}\big).$$

Furthermore, choosing $\overline{\varepsilon} < \varepsilon_2(\tau)$, applying Proposition 4.7, we get (7.16).

Case 2. $\mathbf{e}(x_0, \sigma r) \leq \tau(\mathcal{D}_u(x_0, r) + r^{(1-\omega)\mu})$. By the property of the excess at different scales, we infer that

$$\mathbf{e}(x_0, \tau r) \leq 2^{n-1}\mathbf{e}(x_0, \sigma r) \leq 2^{n-1}\tau\big(\mathcal{D}_u(x_0, r) + r^{(1-\omega)\mu}\big),$$

obtaining (7.16).

Thus, choosing $\overline{\varepsilon} = \min\{\varepsilon_2(\tau), \varepsilon_6(2\tau, \tau, \tau)\}$, we conclude that the inequality (7.16) is verified.

We fix $\sigma \in (0, \frac{(1-\omega)\mu}{2})$ and choose $\tau_0 \in (0,1)$ such that $\overline{C}\tau_0^{(1-\omega)\mu} \leq \tau_0^{2\sigma}$ and we define

$$\Gamma \cap U := \Big\{x \in \partial E \cap U : \mathbf{e}(x, r) + \mathcal{D}(x, r) + r^{(1-\omega)\mu} < \overline{\varepsilon}(\tau_0, U) \text{ for some } r > 0 \text{ such that } B_r(x_0) \subset U\Big\}.$$

We note that $\Gamma \cap U$ is relatively open in ∂E. We show $\Gamma \cap U$ is a $C^{1,\sigma}$-hypersurface. Indeed, inequality (7.16) implies via standard iteration argument that if $x_0 \in \Gamma \cap U$ there exist $r_0 > 0$ and a neighborhood V of x_0 such that for every $x \in \partial E \cap V$ it holds

$$\mathbf{e}(x, \tau_0^k r_0) + \mathcal{D}(x, \tau_0^k r_0) + (\tau_0^k r_0)^{(1-\omega)\mu} \leq \tau_0^{2\sigma k} \quad \text{for } k \in \mathbb{N}_0.$$

In particular, $\mathbf{e}(x, \tau_0^k r_0) \leq \tau_0^{2\sigma k}$ and, arguing as in [29], we obtain that for every $x \in \partial E \cap V$ and $0 < s < t < r_0$ it holds

$$|(\nu_E)_s(x) - (\nu_E)_t(x)| \leq c t^{\sigma}$$

for some constant $c = c(n, \tau_0, r_0)$, where

$$(\nu_E)_t(x) = \fint_{\partial E \cap B_t(x)} \nu_E \, d\mathcal{H}^{n-1}.$$

The previous estimate first implies that $\Gamma \cap U$ is C^1. By a standard argument, we then deduce again from the same estimate that $\Gamma \cap U$ is a $C^{1,\sigma}$-hypersurface. Since ω is arbitrary, we gain that Γ is a $C^{1,\sigma}$-hypersurface, for any $\sigma \in (0, \frac{\mu}{2})$. We define $\Gamma := \bigcup_i (\Gamma \cap U_i)$, where $(U_i)_i$ is an increasing sequence of open sets such that $U_i \Subset \Omega$ and $\Omega = \bigcup_i U_i$. We are left to prove that there exists $\eta > 0$ such that

$$\mathcal{H}^{n-1-\eta}(\partial E \setminus \Gamma) = 0.$$

Setting

$$\Sigma = \left\{x \in \partial E \setminus \Gamma : \lim_{r \to 0} \mathcal{D}(x, r) = 0\right\},$$

by [29, Lemma 2.1], we have that $\nabla u \in L^{2(1+\eta)}_{loc}(\Omega)$ for some $\eta = \eta(n, \alpha, \beta) > 1$ and we have that

$$\dim_{\mathcal{H}} \left(\left\{x \in \Omega : \limsup_{r \to 0} \mathcal{D}(x, r) > 0\right\}\right) \leq n - 1 - \eta.$$

The conclusion follows in a standard way as in [29] (see also [14] and [15]) showing that $\Sigma = \emptyset$ if $n \leq 7$ and $\dim_{\mathcal{H}}(\Sigma) \leq n - 8$ if $n \geq 8$. In both cases, Lemma 5.1 will be employed. ∎

8. An application to a constrained problem

In this section, we show an application of Theorem 1.2 to the following constrained problem:

$$\min_{\substack{E \in \mathcal{A}(\Omega) \\ v \in u_0 + H^1_0(\Omega)}} \left\{\mathcal{F}_A(E, v; \Omega) : |E| = d\right\}, \tag{P_c}$$

where $u_0 \in H^1(\Omega)$, $0 < d < |\Omega|$ are given and $\mathcal{A}(\Omega)$ is the class of all subsets of Ω with finite perimeter. We assume that Ω is connected.

In this perspective, we need to distinguish the Hölder exponent of the matrix A, which we denote here by γ, from the exponent μ appearing in the Definition 1.1. In Theorem 8.2, we show that, for sufficiently large values of $\kappa > 0$, minimizing couples of (P_c) are solutions of the following penalized problem:

$$\min_{\substack{E \in \mathcal{A}(\Omega) \\ v \in u_0 + H^1_0(\Omega)}} \mathcal{F}_\kappa(E, v; \Omega), \tag{P}$$

where the functional $\mathcal{F}_\kappa$ is defined by

$$\mathcal{F}_\kappa(E, v; \Omega) := \mathcal{F}_A(E, v; \Omega) + \kappa \big||E| - d\big|^\gamma.$$

Now, we prove the penalization theorem. For simplicity of notation, we denote

$$a(x, \nu) = \big\langle A(x)\nu, \nu\big\rangle^{\frac{1}{2}} \quad \forall x, \nu \in \mathbb{R}^n.$$

It will be advantageous to have some estimates about the dependence of the integrand a on x and ν.

Remark 8.1 (Continuity of a with respect to x and ν). It is straightforward to check that the following inequalities hold:

$$|a(x, \nu) - a(y, \nu)| \le \frac{1}{2\sqrt{\lambda}}[A]_{C^\mu}|x - y|^\mu \quad \forall x, y \in \mathbb{R}^n, \quad |\nu| = 1, \tag{8.1}$$

$$|a(x, \xi) - a(x, \eta)| \le \frac{\Lambda}{\sqrt{\lambda}}|\xi - \eta| \quad \forall x \in \mathbb{R}^n, \ \forall \xi, \eta \in \mathbb{R}^n. \tag{8.2}$$

As explained before, the proof of the equivalence between the solution of the constrained problem (P_c) and the penalized problem (P) follows. We adapt a result proved in [22] to our setting.

Theorem 8.2. *There exists $\kappa_0 > 0$ such that if (E, u) is a minimizer of the functional*

$$\mathcal{F}_\kappa(F, w) = \int_\Omega \sigma_F |\nabla w|^2 \, dx + \mathbf{\Phi}_A(F; \Omega) + \kappa \big||F| - d\big|^\gamma \tag{8.3}$$

for some $\kappa \ge \kappa_0 > 0$, among all configurations (F, w) such that $w = u_0$ on $\partial\Omega$, then $|E| = d$ and (E, u) is a minimizer of problem (P_c). Conversely, if (E, u) is a minimizer of problem (P_c), then it is a minimizer of (8.3), for all $\kappa \ge \kappa_0$.

Proof. The argument is very similar to the one in [22, Theorem 1] (see also [24] and [23]). For the reader's convenience, we give here the sketch of the proof, emphasizing main ideas and some differences with respect to the case treated in [22].

The first part of the theorem can be proved by contradiction. We assume that there exist a positive sequence $(\kappa_h)_{h\in\mathbb{N}}$ such that $\kappa_h \to +\infty$, as $h \to +\infty$, and a sequence of configurations (E_h, u_h) minimizing $\mathcal{F}_{\kappa_h}$ such that $u_h = u_0$ on $\partial\Omega$ and $|E_h| \neq d$, for all $h \in \mathbb{N}$. We choose an arbitrary fixed $E_0 \subset \Omega$ with finite perimeter such that $|E_0| = d$. We point out that

$$\mathcal{F}_{\kappa_h}(E_h, u_h) \le \mathcal{F}(E_0, u_0) := \Theta. \tag{8.4}$$

Without loss of generality, we can assume that $|E_h| < d$, the case $|E_h| > d$ being similar. Our aim is to show that for h sufficiently large, there exists a configuration $(\widetilde{E}_h, \tilde{u}_h)$ such that $\mathcal{F}_{\kappa_h}(\widetilde{E}_h, \tilde{u}_h) < \mathcal{F}_{\kappa_h}(E_h, u_h)$, thus proving the result by contradiction.

By condition (8.4), it follows that the sequence $(u_h)_{h\in\mathbb{N}}$ is bounded in $H^1(\Omega)$, the perimeters of the sets E_h in Ω are uniformly bounded and $|E_h| \to d$. Therefore, possibly

extracting a not relabeled subsequence, we may assume that there exists a configuration (E, u) such that $u_h \to u$ weakly in $H^1(\Omega)$, $\mathbb{1}_{E_h} \to \mathbb{1}_E$ a.e. in Ω, where the set E is of finite perimeter in Ω and $|E| = d$. The couple (E, u) will be used as a reference configuration for the definition of $(\widetilde{E}_h, \tilde{u}_h)$.

Step 1. Construction of $(\widetilde{E}_h, \tilde{u}_h)$. Proceeding exactly as in [22], since Ω is connected, we can take a point $x \in \partial^* E \cap \Omega$. We observe that, given $\varepsilon > 0$ sufficiently small, we can find around x a point x' and $r > 0$ such that

$$|E \cap B_{r/2}(x')| < \varepsilon r^n, \quad |E \cap B_r(x')| > \frac{\omega_n r^n}{2^{n+2}}.$$

We assume without loss of generality that $x' = 0$, and from now on, we denote by B_r the balls centered at the origin. From the convergence of E_h to E we have that, for h sufficiently large,

$$|E_h \cap B_{r/2}| < \varepsilon r^n, \quad |E_h \cap B_r| > \frac{\omega_n r^n}{2^{n+2}}. \tag{8.5}$$

Now, we define the following bi-Lipschitz map used in [22] which maps B_r into itself:

$$f(x) := \begin{cases} (1 - \sigma_h(2^n - 1))x & \text{if } |x| < \dfrac{r}{2}, \\ x + \sigma_h\left(1 - \dfrac{r^n}{|x|^n}\right)x & \text{if } \dfrac{r}{2} \le |x| < r, \\ x & \text{if } |x| \ge r, \end{cases} \tag{8.6}$$

for some $0 < \sigma_h < 1/2^n$ such that, setting

$$\widetilde{E}_h = f(E_h), \quad \tilde{u}_h = u_h \circ f^{-1},$$

we have $|\widetilde{E}_h| < d$. It holds that

$$\begin{aligned} \mathcal{F}_{\kappa_h}(u_h, E_h) - \mathcal{F}_{\kappa_h}(\tilde{u}_h, \widetilde{E}_h) &= \left[\int_{B_r} \sigma_{E_h} |\nabla u_h|^2 \, dx - \int_{B_r} \sigma_{\widetilde{E}_h} |\nabla \tilde{u}_h|^2 \, dx\right] \\ &\quad + \left[\boldsymbol{\Phi}_A(E_h, \bar{B}_r) - \boldsymbol{\Phi}_A(\widetilde{E}_h, \bar{B}_r)\right] \\ &\quad + \kappa_h\left[(d - |E_h|)^\gamma - (d - |\widetilde{E}_h|)^\gamma\right] \\ &= I_{1,h} + I_{2,h} + I_{3,h}. \end{aligned} \tag{8.7}$$

For simplicity of notation, we will denote in the following:

$$g(y) = f^{-1}(y) \quad \forall y \in \mathbb{R}^n.$$

We will use in the sequel some estimates for the map f that can be easily obtained by direct computations (see [22] for the explicit calculations). These estimates are trivial for $|x| < r/2$, whereas they can be deduced by the explicit expression of ∇f for $r/2 < |x| < r$, that is,

$$\frac{\partial f_i}{\partial x_j}(x) = \delta_{ij} + \sigma_h\left[\left(1 - \frac{r^n}{|x|^n}\right)\delta_{ij} + n r^n \frac{x_i x_j}{|x|^{n+2}}\right] \quad \forall\, i, j \in \{1, \dots, n\}.$$

There exists a constant $C = C(n)$ depending only on n such that

$$\big\|\nabla g(y) - I\big\| \le C(n)\sigma_h \quad \forall y \in B_r, \tag{8.8}$$

$$1 + C(n)\sigma_h \le Jf(x) \le 1 + 2^n n\sigma_h \quad \forall x \in B_r. \tag{8.9}$$

Step 2. Estimate of $I_{1,h}$. Performing the change of variables $y = f(x)$, and observing that $\mathbb{1}_{\widetilde{E}_h} \circ f = \mathbb{1}_{E_h}$, we get

$$I_{1,h} = \int_{B_r} \sigma_{E_h}(x)\big[|\nabla u_h(x)|^2 - \big|\nabla u_h(x) \circ \nabla f^{-1}(f(x))\big|^2 Jf(x)\big]dx.$$

By means of the same computation as in [22], using (8.8) and (8.9), we deduce that

$$I_{1,h} \ge -\overline{C}_1 \Theta \sigma_h \tag{8.10}$$

for some positive constant $\overline{C}_1 = \overline{C}_1(n)$.

Step 3. Estimate of $I_{2,h}$. . In order to estimate $I_{2,h}$ we can use a generalized area formula for maps between rectifiable sets involving anistropies. We recall that (see [39, Proposition 17.1]), if E is a set of locally finite perimeter in $\mathbb{R}^n$, then $f(E)$ is a set of locally finite perimeter in $\mathbb{R}^n$ and

$$\partial^* f(E) = f(\partial^* E), \quad \nu_{f(E)}(y) = \frac{[\nabla g(y)]^t(\nu_E(y))}{|[\nabla g(y)]^t(\nu_E(y))|} \quad \forall y \in \partial^* f(E).$$

Using [39, formula (17.6)], we can easily deduce that

$$\int_{\partial^* f(E)} \phi(f^{-1}(y))\, d\mathcal{H}^{n-1}_y = \int_{\partial^* E} \phi(x) Jf(x) |(\nabla g \circ f)^t \nu_E(x)|\, d\mathcal{H}^{n-1}_x \tag{8.11}$$

for any Borel function ϕ defined on $\partial^* E$. If we choose $\phi(x) = a(f(x), \nu_{f(E)}(f(x)))$ in (8.11), we deduce that

$$\mathbf{\Phi}_A(f(E)) = \int_{\partial^* E} a\big(f(x), [\nabla g(f(x))]^t \nu_E(x)\big) Jf(x)\, d\mathcal{H}^{n-1}_x.$$

For the proof of the aforementioned formula in a more general framework, the reader is addressed to [43, Proposition A.1]. Now, we are ready to estimate the following quantity:

$$\begin{aligned} I_{2,h} &= \big[\mathbf{\Phi}_A(E_h, \overline{B}_r) - \mathbf{\Phi}_A(\widetilde{E}_h, \overline{B}_r)\big] \\ &= \int_{\partial^* E_h \cap \overline{B}_r} \big[a\big(f(x), [\nabla g(f(x))]^t \nu_{E_h}(x)\big) - a\big(f(x), \nu_{E_h}(x)\big)\big] Jf(x)\, d\mathcal{H}^{n-1}_x \\ &\quad + \int_{\partial^* E_h \cap \overline{B}_r} \big[a\big(f(x), \nu_{E_h}(x)\big) - a\big(x, \nu_{E_h}(x)\big)\big] Jf(x)\, d\mathcal{H}^{n-1}_x \\ &\quad + \int_{\partial^* E_h \cap \overline{B}_r} [Jf(x) - 1] a(x, \nu_{E_h}(x))\, d\mathcal{H}^{n-1}_x = J_{1,h} + J_{2,h} + J_{3,h}. \end{aligned}$$

Using (8.8) and (8.2), we deduce that

$$|J_{1,h}| \le \frac{\Lambda}{\sqrt{\lambda}} \int_{\partial^* E_h \cap \bar{B}_r} Jf(x) \left| \left[(\nabla g)^t - I \right] \nu_{E_h}(x) \right| d\mathcal{H}^{n-1}_x \le C(n)\Theta \frac{\Lambda}{\sqrt{\lambda}} \sigma_h.$$

Applying (8.1), we obtain

$$|J_{2,h}| \le \frac{[A]_{C^\gamma}}{2\sqrt{\lambda}} \int_{\partial^* E_h \cap \bar{B}_r} Jf(x) |f(x) - x|^\gamma \, d\mathcal{H}^{n-1}_x \le \frac{[A]_{C^\gamma}}{2\sqrt{\lambda}} C(n)\Theta \sigma_h^\gamma.$$

Finally, from (8.9), we have

$$|J_{3,h}| \le \int_{\partial^* E_h \cap \bar{B}_r} n 2^n a(x, \nu_{E_h}(x)) \sigma_h \, d\mathcal{H}^{n-1}_x \le n 2^n \Theta \sqrt{\Lambda} \sigma_h.$$

Summarizing, we conclude that

$$I_{2,h} \ge -\bar{C}_2 \Theta \sigma_h^\gamma \tag{8.12}$$

for some positive constant $\bar{C}_2 = \bar{C}_2(n, \lambda, \Lambda, [A]_{C^\gamma})$.

Step 4. Estimate of $I_{3,h}$. The following estimate is contained in [23, Theorem 2] and we detail it for reader's convenience.

First, we recall (8.5), (8.6), and (8.9), thus getting

$$\begin{aligned} |\widetilde{E}_h| - |E_h| &= \int_{E_h \cap B_r \setminus B_{r/2}} (Jf(x) - 1) dx + \int_{E_h \cap B_{r/2}} (Jf(x) - 1) dx \\ &\ge \left(\frac{\omega_n}{2^{n+2}} - \varepsilon \right) \sigma_h r^n - \left[1 - \left(1 - (2^n - 1)\sigma_h \right)^n \right] \varepsilon r^n \\ &\ge \sigma_h r^n \left[\frac{\omega_n}{2^{n+2}} - \varepsilon - (2^n - 1) n \varepsilon \right]. \end{aligned}$$

Therefore, if we choose $0 < \varepsilon < \bar{\varepsilon}(n)$ for some $\bar{\varepsilon}$ sufficiently small, we have that

$$\kappa_h(|\widetilde{E}_h| - |E_h|) \ge \kappa_h C(n) \sigma_h r^n. \tag{8.13}$$

Moreover, if denoting $\delta_h := d - |E_h|$, we choose σ_h in such a way that $|\widetilde{E}_h| - |E_h| \le \delta_h/2$ thus respecting the condition $|\widetilde{E}_h| < d$. Taking this into account, proceeding as before and using (8.9), we have

$$|\widetilde{E}_h| - |E_h| = \int_{E_h \cap B_r} (Jf(x) - 1) dx \le n 2^n \sigma_h r^n.$$

Then, we choose σ_h such that

$$\delta_h \le \sigma_h \le \frac{\delta_h}{n 2^{n+1} r^n}.$$

We remark that in the last condition we imposed also that σ_h is comparable with δ_h, which is crucial in the following estimate. Resuming (8.13) we can conclude

$$\begin{aligned} I_{3,h} &= \kappa_h\big[(d-|E_h|)^\gamma - (d-|\widetilde{E}_h|)^\gamma\big] \geq \kappa_h \frac{\gamma}{(d-|E_h|)^{1-\gamma}}(|\widetilde{E}_h|-|E_h|) \\ &= \kappa_h\gamma(d-|E_h|)^\gamma \frac{|\widetilde{E}_h|-|E_h|}{d-|E_h|} \geq \kappa_h\gamma\delta_h^\gamma \frac{c_2(n)\sigma_h r^n}{\delta_h} \\ &\geq \kappa_h \bar{C}_3 \sigma_h^\gamma r^n \end{aligned}$$

for some positive constant $\bar{C}_3 = \bar{C}_3(n,\gamma)$.

From the previous inequality, recalling (8.7), (8.10), and (8.12), we obtain

$$\mathcal{F}_{\kappa_h}(u_h, E_h) - \mathcal{F}_{\kappa_h}(\tilde{u}_h, \widetilde{E}_h) \geq \sigma_h^\gamma\big(\kappa_h\bar{C}_3 r^n - \Theta(\bar{C}_1+\bar{C}_2)\big) > 0,$$

if κ_h is sufficiently large. This contradicts the minimality of (E_h, u_h), thus concluding the proof. ∎

Remark 8.3. Theorem 8.2 allows us to prove the regularity of solutions of the free boundary problem under the constraint $|E| = d$. Under the assumption

$$\gamma \in \Big(\frac{n-1}{n}, 1\Big),$$

the parameter $\mu := \gamma n - n + 1$ is positive and, by Theorem 8.2, any minimizing couple (E,u) of (P_c) is a (κ,μ)-minimizer of $\mathcal{F}_\kappa$, for $\kappa \geq \kappa_0$, where κ_0 is the constant appearing in Theorem 8.2. Thus, we are in position to implement the regularity theory of the previous sections to (E,u) by applying Theorem 1.2.

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