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Chapple porism: a visual spell

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1 Introduction

Any triangle has a unique inscribed circle and circumcircle. The metric relation between their radii and the distance between their centers, proved by Leonard Euler (1707–1783) and first published in [5] (see also [2, pp. 74–75], [6, pp. 186–187], or [1, p. 85]), leaves no doubt that a random pair of nested circles is not generally the incircle and circumcircle of some triangle. Euler did not mention the poristic nature of the inscribed and circumscribed circles; according to [4], this was first spotted by William Chapple (1718–1781) in [3]: any point of the circumcircle $\mathcal{D}$ is a vertex of some triangle simultaneously inscribed in $\mathcal{D}$ and

Der Inkreis und der Umkreis eines Dreiecks bilden ein sogenanntes Chapple-Paar. Aufgrund des Schliessungssatzes von Poncelet enthält jedes Chapple-Paar nicht nur das ursprüngliche Dreieck, sondern jeder Punkt des Umkreises ist die Ecke eines Dreiecks mit demselben Inkreis. Um ein Chapple-Paar zu bilden, müssen zwei Kreise eine metrische Bedingung erfüllen, die den Abstand zwischen ihren Mittelpunkten mit ihren Radien verknüpft: die Euler-Relation. Der Satz von Poncelet gilt allgemein sogar für zwei Kegelschnitte anstelle der beiden Kreise. In der vorliegenden Arbeit wird gezeigt, dass eine Ellipse und deren Leitkreis, und gleichzeitig auch der Leitkreis und der Fokalkreis der Ellipse, die Schliessungseigenschaft von Poncelet für Dreiecke besitzen. Interessanterweise führen diese beiden neuen Paare zu einem natürlicheren Verständnis der Eulerschen Beziehung, und ein spezielles Triplett von Kreisen bietet einen visuellen Test für die Schliessungseigenschaft.

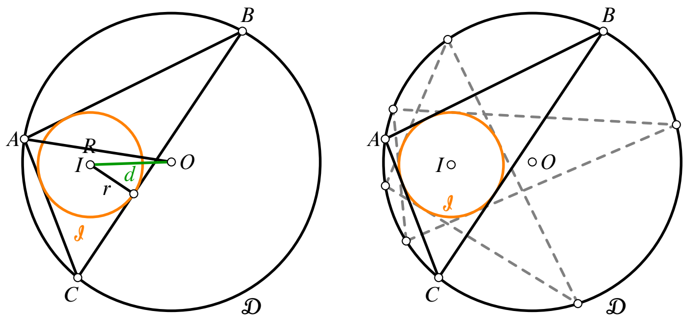

Figure 1. **Left:** the Euler condition $R^2 - 2Rr = d^2$. **Right:** a one-dimensional family of triangular orbits sharing the incircle and circumcircle.

circumscribed to $\mathcal{I}$ (see Figure 1). As a tribute to his pioneering contribution, the poristic pair circumcircle-inscribed circle bear his name. Chapple's arguments did not properly prove the porism. Soon after, in 1755, Landen (1719–1790) provided a full, constructive proof for both Chapple's porism, and the metric relations between the radii and the distance between the centers. His proofs cover both the nested and the secant case; see [4, 9]. Another proof for the Chapple porism, and its equivalence with the Euler formula, is due to M. Lhuilier (1750–1840); see [8]. The porism between one circumconic and a pencil of inscribed conics was formulated and proved by Poncelet already in 1813, and first published in [11].

Main results

Our search for a more natural approach to the Chapple porism was based on the idea that conics, rather than circles, offer a more suitable framework for Poncelet porisms. Firstly, we prove that any central conic generates two porisms. The first one is a porism between the conic and its director circle; see Theorem 1. A second porism is that between the focal circle and the director circle. As a mater of fact, any Chapple poristic pair consists of the director and the focal circle of some conic; we prove this in Theorem 2. These facts lead us to a concise Euclidean proof for the Chapple porism.

Visual proofs for the Euler inequality and Euler formula, using a special triplet of circles, are given in Corollary 1 and Corollary 2. A metric test for porisms for triangles is in Corollary 3.

On methods

Apart from polar reciprocity and a brief calculation using complex numbers, the proofs are Euclidean, concise, and notably more readable than the established ones. The references provided in the extensive introduction reinforce this claim.

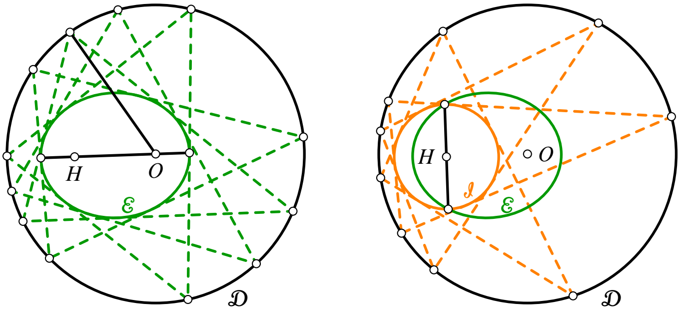

Figure 2. **Left:** the pair $\{\mathcal{D}, \mathcal{E}\}$ consisting of an ellipse and its director circle is poristic. **Right:** the pair $\{\mathcal{D}, \mathcal{I}\}$ consisting of the focal circle and the director circle of the same ellipse is a Chapple porism.

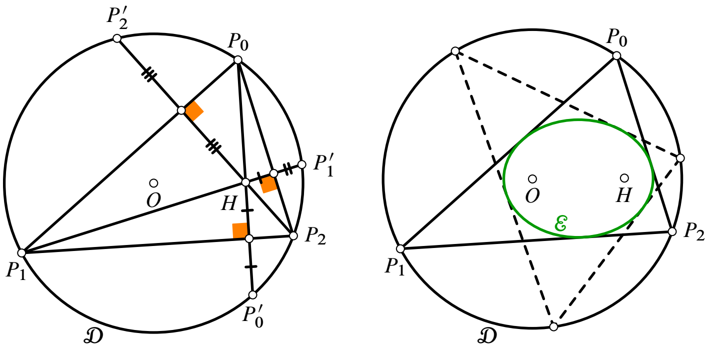

Figure 3. **Left:** the reflections of the orthocenter about the sides are on the circumcircle. **Right:** an ellipse and its director circle form a poristic pair, a director porism. In a director porism, the triangular orbits share the orthocenter.

2 One conic, two porisms

There are two circles intrinsically linked to any central conic that act as players in poristic pairs: its director and its focal circles.

2.1 A conic and its director circle

The director circle of a central conic (either ellipse or hyperbola) is the circle centered at one focus and of radius equal to the major axis; there are two of them, one for each focus.

The poristic nature of an ellipse and of its director circle, shown in Figure 2, has already been spotted by E. Lemoine in [7]; nevertheless, the proof uses the Poncelet porism. A proof that is self-contained is as follows.

Theorem 1. *The pair $\{\mathcal{D}, \mathcal{E}\}$ that consists of an ellipse $\mathcal{E}$ and its director circle $\mathcal{D}$ is poristic.*

Proof. Fix a point H inside $\mathcal{D}$; starting with an arbitrary point P_0 on $\mathcal{D}$, let the line P_0H meet the circle again at P_0', as in Figure 3. Let the perpendicular bisector of HP_0' meet $\mathcal{D}$ at P_1 and P_2; then the line P_0H is an altitude in $\triangle P_0P_1P_2$. Since H is the point on the altitude whose reflection about P_1P_2 is on the circumcircle, from the mirror property, H is the orthocenter. It then has the mirror property for all three sides and, by Apollonius [10, Book 3, Propositions 48 and 52], the ellipse with foci O, H and major axis R is inscribed in $\triangle P_0P_1P_2$. Since the point P_0 is arbitrary, this produces a one-dimensional family of triangles inscribed in $\mathcal{D}$ and circumscribed to $\mathcal{E}$, ending the proof. ■

Remark 1. Minor changes prove a similar result when the point H is outside $\mathcal{D}$; in this case, the inscribed conic is a hyperbola; see Figure 5.

2.2 The focal circle and the director circle

Let an ellipse $\mathcal{E}$ be focused in O and I, and let $\mathcal{D}$, the director circle of $\mathcal{E}$, be centered in O, as in Figure 4. Consider the circle $\mathcal{I}$, centered at the focus I, and whose diameter is the latus rectum[1] of $\mathcal{E}$; we shall call it the focal circle.

The following poristic pair, also intrinsic to any conic, seems new.

Theorem 2. *Let $\{\mathcal{D}, \mathcal{I}\}$ be a pair of nested circles. Let O and I be their centers, and let R and r be their radii. Let $\mathcal{E}$ be the ellipse focused in O and I and of major axis R. Then the pair $\{\mathcal{D}, \mathcal{I}\}$ is a Chapple poristic pair if and only if $\mathcal{D}$ and $\mathcal{I}$ are the director circle and the focal circle of $\mathcal{E}$, respectively.*

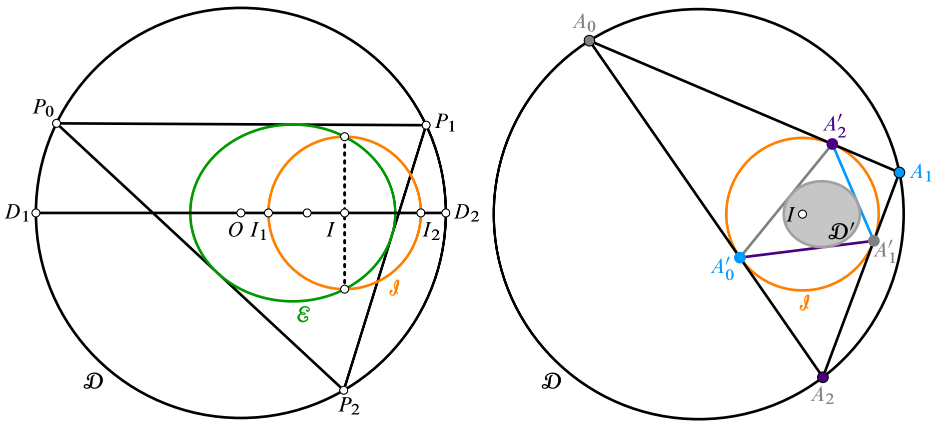

Figure 4. **Left:** the director porism $\{\mathcal{D}, \mathcal{E}\}$. **Right:** $\mathcal{I}$, the focal circle of $\mathcal{E}$, is the caustic of a Chapple porism $\{\mathcal{D}, \mathcal{I}\}$. As A_0 sweeps the circle $\mathcal{D}$, $A_1'A_2'$, its polar with respect to $\mathcal{I}$ envelops the ellipse $\mathcal{D}'$, the polar dual of $\mathcal{D}$ with respect to $\mathcal{I}$. The pair $\{\mathcal{I}, \mathcal{D}'\}$ is (a new) director porism.

[1]The chord of minimum length through the focus, e.g., the chord passing through the focus and perpendicular to the main axis.

Proof. To show that the pair $\{\mathcal{D}, \mathcal{I}\}$ is poristic, we use polar reciprocity; see [11, pp. 95–121] or [12, Chapter XV, &308–309, pp. 253–260]. We choose the focal circle $\mathcal{I}$ as the inversion circle, and prove that the pair $\{\mathcal{D}', \mathcal{I}'\}$ that consists of the polar duals of $\mathcal{D}$ and $\mathcal{I}$, with respect to $\mathcal{I}$, forms a poristic pair; see Figure 4. Since I, the center of the inversion circle, is inside $\mathcal{D}$, the polar of $\mathcal{D}$ with respect to I is an ellipse, $\mathcal{D}'$, with focus at I. If D_1D_2 is the diameter of $\mathcal{D}$ that passes through I, then by [12, &309, pp. 259–260], the main axis of the dual conic $\mathcal{D}'$ is the segment $D_1'D_2'$, delimited by the inverses of D_1 and D_2 with respect to $\mathcal{I}$.

To finish the proof, it suffices to show that $\mathcal{I}$ is the director circle of $\mathcal{D}'$ or, equivalently, that $D_1'D_2' = \frac{1}{2}I_1I_2$. We shall use complex numbers. Choose the origin at I. Then let $2r_1, 2r_2$ be the distances from I to the points D_1, D_2. It is easy to see that A_1, A_2 are precisely the midpoints of ID_1, ID_2, respectively; finally, let $I_1I_2 = 2r$. With these notations,

$$D_1 = -2r_1, \quad D_2 = 2r_2 \quad A_1 = -r_1, \quad A_2 = r_2 \quad I_1 = ir, \quad I_2 = -ir.$$

The inversion with respect to the circle $\mathcal{I}$ is given by

$$\mathcal{I}(z) = \frac{r^2}{\overline{z}}; \quad \text{therefore,} \quad D_1' = -\frac{r^2}{2r_1}, \quad D_2' = \frac{r^2}{2r_2};$$

hence

$$D_1'D_2' = |D_1' - D_2'| = \frac{r^2}{2}\Big(\frac{1}{r_1} + \frac{1}{r_2}\Big).$$

If a and c are, respectively, the main axis and the focal axis of $\mathcal{E}$, then

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{a-c} + \frac{1}{a+c} = \frac{2a}{a^2-c^2} = \frac{2a}{b^2} = \frac{2}{r},$$

where, at the last step, we used the length of the latus rectum

$$I_1I_2 = 2r = \frac{2b^2}{a}.$$

Thus, the polar dual of $\mathcal{D}$ with respect to $\mathcal{I}$ is the ellipse $\mathcal{D}'$, which has a focus in I and whose main axis is r. The polar dual of $\mathcal{I}$ is, of course, $\mathcal{I}$ itself. Theorem 1 guarantees that the pair consisting of an ellipse and its director circle is poristic, ending the proof.

In order to prove the converse, let $\mathcal{I}'$ be the focal circle of $\mathcal{E}$, centered at the focus I; by the first part, $(\mathcal{D}, \mathcal{I}')$ is a Chapple pair. The circles $\mathcal{I}$ and $\mathcal{I}'$ are both centered in I and both are caustics in a Chapple porism with $\mathcal{D}$; hence they necessarily coincide. ∎

Remark 2. If the circles $\mathcal{D}$ and $\mathcal{I}$ are secant, instead of an ellipse, we obtain a hyperbola, as shown in Figure 5; the proofs are similar and we shall omit.

3 Euler formula and Chapple porism: visual proofs

Theorem 2 guarantees that a pair of circles $\{\mathcal{D}, \mathcal{I}\}$ is poristic if and only if $\mathcal{I}$ and $\mathcal{D}$ are respectively the focal and director circle of a conic, with the foci in I and O, the centers of the two. This leads to the following visual test for a Chapple porism, via a triplet of circles.

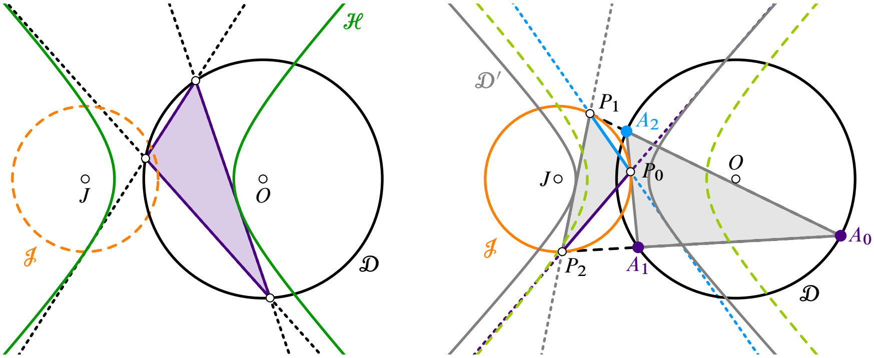

Figure 5. **Left:** a hyperbola, its director circle, and a secant director porism $\{\mathcal{D}, \mathcal{H}\}$. **Right:** the focal circle and the director circle form a secant Chapple porism $\{\mathcal{D}, \mathcal{J}\}$. The polars of the vertices A_0, A_1, A_2 with respect to $\mathcal{J}$ are the sides of the poristic orbit $P_0 P_1 P_2$, which envelops the hyperbola $\mathcal{D}'$, the polar dual of $\mathcal{D}$, with respect to $\mathcal{J}$.

Corollary 1. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be a pair of tangent, congruent circles. Let a third circle $\mathcal{I}$, congruent with the former two, be centered at the tangency point of $\mathcal{I}_1$ and $\mathcal{I}_2$, and let $\mathcal{D}$ be any circle. Then the pair $\{\mathcal{D}, \mathcal{I}\}$ forms a nested (respectively, secant) Chapple pair if and only if $\mathcal{I}_1$ and $\mathcal{I}_2$ are internally (respectively, externally) tangent to $\mathcal{D}$.*

Proof. First, assume that $\{\mathcal{D}, \mathcal{I}\}$ is a nested Chapple pair. Then, by Theorem 2, $\mathcal{D}$ is the director circle and $\mathcal{I}$ is the focal circle of the ellipse focused in O and I and director circle $\mathcal{D}$, shown in Figure 6. Let

$$d_1 = OI_1, \quad d = IO = 2c, \quad r = I_1 I = \frac{b^2}{a}, \quad R = 2a.$$

By the Pythagoras theorem,

$$\begin{aligned} d_1^2 = r^2 + d^2 = 4c^2 + \frac{b^4}{a^2} &= 4(a^2 - b^2) + \frac{b^4}{a^2} \\ &= \frac{(2a^2 - b^2)^2}{a^2} = \frac{(a^2 + c^2)^2}{a^2}; \end{aligned}$$

hence

$$d_1 + r = \frac{a^2 + c^2}{a} + \frac{b^2}{a} = \frac{a^2 + (a^2 - b^2) + b^2}{a} = 2a = R,$$

which proves that the circle $\mathcal{I}_1$, centered in I_1 and with radius r, is internally tangent to $\mathcal{D}$. The converse, as well as the externally tangent case, can be proved similarly and we omit the proof. ∎

Corollary 1 leads to a visual proof for Euler's inequality.

Corollary 2. *In any triangle, the diameter of the inscribed circle is smaller than the radius of the circumcircle.*

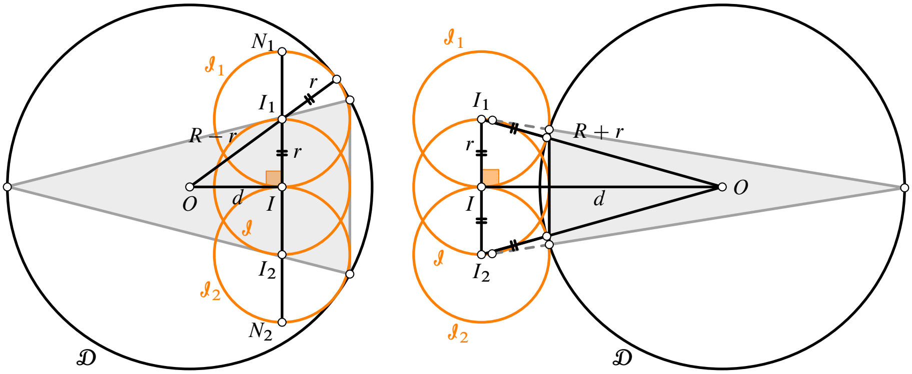

Figure 6. **Left:** $\mathcal{I}$ is the incircle in a Chapple porism if and only if the circles $\mathcal{I}_1, \mathcal{I}_2$, congruent with $\mathcal{I}$ and mutually tangent at $\mathcal{I}$, are internally tangent to $\mathcal{D}$. A visual proof for the Euler inequality $4r = N_1N_2 \leq 2R$. **Right:** a circle $\mathcal{I}$ is the excircle in a Chapple porism if and only if the circles $\mathcal{I}_1, \mathcal{I}_2$, congruent with $\mathcal{I}$ and mutually tangent at $\mathcal{I}$, are externally tangent to $\mathcal{D}$.

Proof. Refer to Figure 6. If $\mathcal{I}$ is the inscribed circle of a triangle, then by Corollary 1, the circles $\mathcal{I}_1$ and $\mathcal{I}_2$, which are congruent to $\mathcal{I}$ and externally tangent at its center I, are internally tangent to the circumcircle $\mathcal{D}$. Thus, any point on the circles $\mathcal{I}_1$ and $\mathcal{I}_2$ is in $\mathcal{D}$. In particular, the points N_1 and N_2, defined as the second intersection points of the line I_1I_2 with these circles, lie inside D; hence the distance between them is less than the diameter of $\mathcal{D}$. Since $N_1N_2 = 4r$, this establishes the Euler inequality $2r \leq R$. Equality holds if and only if N_1N_2 is the diameter of $\mathcal{D}$, which means that the points I_1, O, and I_2 are collinear. Since I_1I_2 is perpendicular to OI at I, this can only occur if $O = I$, which means the triangle is equilateral. ■

Corollary 1 also leads to a new, visual proof for the Euler relation.

Corollary 3. *Let $\{\mathcal{D}, \mathcal{I}\}$ be two circles centered at O, I and with radii R, r; let $d = OI$ be the distance between their centers. Then*

(i) *if $d < R$, the pair $\{\mathcal{D}, \mathcal{I}\}$ is a Chapple poristic pair if and only if*

$$(R-r)^2 = d^2 + r^2; \tag{1}$$

(ii) *if $d > R$, then the pair $\{\mathcal{D}, \mathcal{I}\}$ is a secant Chapple pair if and only if*

$$(R+r)^2 = d^2 + r^2. \tag{2}$$

Proof. (i) First assume that $\{\mathcal{D}, \mathcal{I}\}$ is a nested Chapple pair. Let $\mathcal{I}_1$ be a circle centered in I_1, congruent with $\mathcal{I}$ and such that $II_1 \perp OI$, as in Figure 6. Then, by Corollary 1, equation (1) is verified.

Conversely, if $\{\mathcal{D}, \mathcal{I}\}$ verify equation (1), then proceeding as in the proof of Corollary 1, it is easy to see that $I_1O = R - r$; this means that $\mathcal{I}_1$ is internally tangent to $\mathcal{D}$; hence the pair $\{\mathcal{D}, \mathcal{I}\}$ is poristic.

(ii) The secant case, shown on the right side of Figure 6, can be proved similarly and we omit the proof. ■

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