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Dimensions of a class of nonautonomous carpets and measures on $\mathbb{R}^2$

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Abstract. For each integer $k > 0$, let n_k and m_k be integers such that $n_k \geq 2$, $m_k \geq 2$, and let $\mathcal{D}_k$ be a subset of $\{0, \ldots, n_k - 1\} \times \{0, \ldots, m_k - 1\}$. For each $w = (i, j) \in \mathcal{D}_k$, we define an affine transformation on $\mathbb{R}^2$ by

$$\Phi_w(x) = T_k(x + w), \qquad w \in \mathcal{D}_k,$$

where $T_k = \operatorname{diag}(n_k^{-1}, m_k^{-1})$. The non-empty compact set

$$E = \bigcap_{k=1}^{\infty} \bigcup_{(w_1 w_2 \ldots w_k) \in \prod_{i=1}^{k} \mathcal{D}_i} \Phi_{w_1} \circ \Phi_{w_2} \circ \cdots \circ \Phi_{w_k}$$

is called a *nonautonomous carpet*.

In the paper, we provide the lower, packing, box-counting and Assouad dimensions of the nonautonomous carpets E. We also explore the dimension properties of nonautonomous measures μ supported on E, and we provide Hausdorff, packing and entropy dimension formulas of μ.

1. Introduction

1.1. Dimensions of measures

In the dimension theory of fractal geometry and dynamical systems, the dimensions of invariant measures are important objects to investigate, and the most frequently used dimensions are Hausdorff, packing and entropy dimensions.

Let μ be a finite Borel measure on $\mathbb{R}^d$. The *Hausdorff and packing dimensions* of μ, respectively, are defined as

$$\begin{aligned} \dim_{\mathrm{H}} \mu &= \inf\{\dim_{\mathrm{H}} A : \mu(A^c) = 0\}, \\ \dim_{\mathrm{P}} \mu &= \inf\{\dim_{\mathrm{P}} A : \mu(A^c) = 0\}. \end{aligned}$$

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The *lower and upper local dimensions* of μ are given by

$$\underline{\dim}_{\mathrm{loc}}\,\mu(x) = \liminf_{r\to 0} \frac{\log \mu\big(B(x,r)\big)}{\log r}, \qquad \overline{\dim}_{\mathrm{loc}}\,\mu(x) = \limsup_{r\to 0} \frac{\log \mu\big(B(x,r)\big)}{\log r},$$

and we say that the *local dimension* exists at x if these are equal, writing $\dim_{\mathrm{loc}}\mu(x)$ for the common value. Let $\mathcal{M}_n$ be the partition of $\mathbb{R}^d$ into grid boxes

$$\prod_{i=1}^{d} [2^{-n} j_i, 2^{-n}(j_i+1)]$$

with integers j_i. The *lower and upper entropy dimensions* of μ, respectively, are defined as

$$\underline{\dim}_e\,\mu = \liminf_{n\to\infty} \frac{H_n(\mu)}{\log 2^n}, \qquad \overline{\dim}_e\,\mu = \limsup_{n\to\infty} \frac{H_n(\mu)}{\log 2^n},$$

where

$$H_n(\mu) = -\sum_{Q\in\mathcal{M}_n} \mu(Q)\log\mu(Q).$$

If these are equal, we refer to the common value as the *entropy dimension* of μ. We refer the readers to [10, 13] for the background reading.

The following well-known theorem of Young [40] shows the connection of these dimensions.

Theorem 1.1. *Let μ be a probability measure on $\mathbb{R}^d$. Suppose that the local dimension*

$$\dim_{\mathrm{loc}}\mu(x) = \alpha, \qquad \mu\text{-a.e. } x\in\mathbb{R}^d.$$

Then $\dim_e \mu = \dim_{\mathrm{H}} \mu = \alpha$.

In 2002, Fan, Lau and Rao improved the conclusion of the theorem to $\dim_e\mu = \dim_{\mathrm{P}}\mu = \dim_{\mathrm{H}}\mu = \alpha$, see [13]. Determination of the dimensions of fractal sets is a challenging problem, see [2, 8, 11, 12, 14, 19, 21, 25, 36, 38] for various studies on the dimension theory of fractal sets. In particular, for self-affine sets with grid structure, which are often called non-typical self-affine sets such as Bedford–McMullen carpets, Gatzouras–Lalley sets, Barański sets, see [2,3,5,26,29,33], one strategy is to compute the Hausdorff dimensions of measures supported on the fractal set via local dimensions, and the supreme dimension of measures often gives the Hausdorff dimension of the set, that is

$$\dim_{\mathrm{H}} E = \sup\{\dim_{\mathrm{H}}\nu : \nu \text{ is a Borel probability measure supported on } E\}.$$

Since the lower and upper local dimensions often give the Hausdorff and packing dimensions of measures (see Lemma 4.4 in Section 4), it is important to investigate the local dimensions and the dimension of measures in the dimension theory of fractal sets. In many cases, people found that local dimensions exist and equal a constant almost surely, that is to say, in these studies the Hausdorff, packing and entropy dimensions of measures are identical, see [13]. However, there are fractal measures whose local dimensions do not necessarily exist, and it is an interesting question to investigate the dimension theory of such measures.

1.2. Self-affine sets

First, we review a class of non-typical self-affine sets, and we refer the readers to [10–12] for the study of typical self-affine sets.

Given integers m and n such that $n > m \geq 2$, let $\mathcal{D}$ be a subset of $\{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$. For each $w \in \mathcal{D}$, we define an affine transformation Φ_w on $\mathbb{R}^2$ by

$$\Phi_w(x) = T(x + w), \tag{1.1}$$

where $T = \mathrm{diag}(n^{-1}, m^{-1})$. Then $\{\Phi_w\}_{w\in\mathcal{D}}$ forms a *self-affine iterated function system* (IFS). By the well-known theorem of Hutchinson, see [10,24], this self-affine IFS has a unique self-affine attractor, that is a unique non-empty compact set $E \subset \mathbb{R}^2$ such that

$$E = \bigcup_{i=1}^{m} \Phi_i(E).$$

The self-affine set E is also called a *Bedford–McMullen set* or a *Bedford–McMullen carpet* [5,33].

Various dimensions of Bedford–McMullen carpets have been investigated, see [5, 15,32,33], and these sets are often used as good examples for the following dimension inequalities

$$\dim_{\mathrm{L}} E \leq \dim_{\mathrm{H}} E \leq \dim_{\mathrm{B}} E \leq \dim_{\mathrm{A}} E, \tag{1.2}$$

where $E \subset \mathbb{R}^d$ is compact, and where $\dim_{\mathrm{L}}$ and $\dim_{\mathrm{A}}$ denote lower dimension and Assouad dimension, respectively, see Section 5 for the definitions. Note that the lower dimension is only a lower bound to the Hausdorff dimension with additional assumptions such as the set E is closed, and we refer readers to [16] for details of Assouad type dimensions. Since Bedford–McMullen carpets are a class of simplest self-affine sets, they are frequently used as a testing ground on questions and conjectures of fractals, and we refer readers to [1, 17, 22, 23, 27, 30, 31, 34] for various studies on Bedford–McMullen carpets.

There are many different generalisations for Bedford–McMullen carpets, see [3,4, 14,15,17,26,29]. In [26], Kenyon and Peres studied the self-affine sponge E, which is a generalization of Bedford–McMullen carpets in $\mathbb{R}^d$, and they found the Hausdorff dimensions of self-affine measures by using ergodic property to show that the local dimension exists. Moreover, they proved that there exists a unique ergodic self-affine measure of full Hausdorff dimension. The research on the generalisations of Bedford–McMullen carpets is also an active area, and we refer readers to [8, 17, 18, 28, 37] for the related studies on different generalizations and the references therein.

In this paper, we study a class of fractals, named nonautonomous carpets (see Subsection 1.3), which may also be regarded as a generalisation of Bedford–McMullen carpets. Since we apply different affine IFSs at the different levels in the iterating process, such sets do not have dynamical properties any more. Therefore, the tools of ergodic theory cannot be invoked, which causes that the local dimensions of measures supported on these sets do not necessarily exist, and this leads to the difficulties to determine their dimensions of the sets and measures.

1.3. Nonautonomous carpets

Given a sequence $\{(n_k, m_k)\}_{k=1}^{\infty}$, where m_k and n_k are integers such that $n_k \geq 2$ and $m_k \geq 2$. For each integer k, let $\mathcal{D}_k$ be a subset of $\{0, \dots, n_k - 1\} \times \{0, \dots, m_k - 1\}$. We write $r_k = \operatorname{card}(\mathcal{D}_k)$ and always assume that $r_k \geq 2$. The set of all finite sequences with length k and the set of infinite sequences are denoted by

$$\Sigma^k = \prod_{j=1}^{k} \mathcal{D}_j, \qquad \Sigma_l^k = \prod_{j=l+1}^{k} \mathcal{D}_j, \qquad \Sigma^\infty = \prod_{j=1}^{\infty} \mathcal{D}_j.$$

For $\mathbf{w} = w_1 \dots w_k \in \Sigma^k$, $\mathbf{v} = v_1 \dots v_l \in \Sigma_k^{k+l}$, write

$$\mathbf{w} * \mathbf{v} = w_1 \dots w_k v_1 \dots v_l \in \Sigma^{k+l}.$$

We write $\mathbf{w}|k = (w_1 \dots w_k)$ for the *curtailment* after k terms of the infinite sequence $\mathbf{w} = (w_1 w_2 \dots) \in \Sigma^\infty$. We write $\mathbf{w} \preceq \mathbf{v}$ if $\mathbf{w}$ is a curtailment of $\mathbf{v}$. We call the set $[\mathbf{w}] = \{\mathbf{v} \in \Sigma^\infty : \mathbf{w} \preceq \mathbf{v}\}$ the *cylinder* of $\mathbf{w}$, where $\mathbf{w} \in \Sigma^*$. If $\mathbf{w} = \emptyset$, its cylinder is $[\mathbf{w}] = \Sigma^\infty$.

Given an integer $k > 0$ for each $w = (i, j) \in \mathcal{D}_k$, we define an affine transformation on $\mathbb{R}^2$ by

$$\Phi_w(x) = T_k(x + w), \quad w \in \mathcal{D}_k, \tag{1.3}$$

where $T_k = \operatorname{diag}(n_k^{-1}, m_k^{-1})$. For each $\mathbf{w} = (w_1 w_2 \dots w_k) \in \Sigma^k$, we write

$$\Phi_{\mathbf{w}} = \Phi_{w_1} \circ \Phi_{w_2} \circ \dots \circ \Phi_{w_k}.$$

Suppose that $J = [0, 1]^2 \subset \mathbb{R}^2$. For each integer $k > 0$, let $\{\Phi_w\}_{w \in \mathcal{D}_k}$ be the self-affine IFS as in (1.3). For each $\mathbf{w} \in \Sigma^k$, the set $J_{\mathbf{w}}$ is a geometrical affine copy to J, i.e., there exists an affine mapping $\Phi_{\mathbf{w}} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $J_{\mathbf{w}} = \Phi_{\mathbf{w}}(J)$. The non-empty compact set

$$E = \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{w}\in\Sigma^k} J_{\mathbf{w}} \tag{1.4}$$

is called a *nonautonomous carpet or self-affine Moran set* $\{(n_k, m_k, \mathcal{D}_k)\}_{k=1}^{\infty}$. For all $\mathbf{w} \in \Sigma^k$, the elements $J_{\mathbf{w}}$ are called *k-th-level basic sets* of E, see Figure 1 for the first three levels.

Note that this may also be regarded as a generalization of Moran fractals where only similarity contractions are used in the construction, see [35, 39]. In [20], the authors studied a special case of these sets where they require that $n_k \geq m_k$ for all $k > 0$, and they provided the Assouad, packing and box-counting dimensions of the sets. They also obtained the Hausdorff dimension formula under some strong technique assumptions. In this paper, we are interested in investigating the dimension properties of measures supported on nonautonomous sets, and we also provide the Assouad, packing and box-counting dimension formulas of sets which extend the conclusions in [20]. Furthermore, we study the lower dimension of the nonautonomous set which has not been studied, and this conclusion completes the dimension formulas in inequality (1.2). All these studies strongly rely on the fixed translations at each level, which gives the fine grid structure, and this is different to the classic Moran sets where the translations are very flexible in the Moran structure. Recently, Gu and Miao in [21] have studied a class of sets, called nonautonomous iterated functions systems and nonautonomous fractals, where they replaced the similarities by affine contractions and removed the separation assumption in the Moran construction. Like typical self-affine fractals, they obtained various almost sure results on dimensions.

Let $\Pi : \Sigma^{\infty} \to \mathbb{R}^2$ be the projection given by

$$\Pi(\mathbf{w}) = \sum_{k=1}^{\infty} \operatorname{diag}\Big(\prod_{h=1}^{k} n_h^{-1}, \prod_{h=1}^{k} m_h^{-1}\Big) w_k.$$

Then the nonautonomous carpet E is the image of Π, i.e., $E = \Pi(\Sigma^{\infty})$. Note that the range restriction of Π to E is surjective, i.e., $\Pi : \Sigma^{\infty} \to E$ is surjective.

Let $\mathcal{P}_k$ denote the collection of all probability vectors on $\mathcal{D}_k$, and $\mathcal{P} = \prod_{k=1}^{\infty} \mathcal{P}_k$. Given $\mathbf{p} = (\mathbf{p}_k)_{k=1}^{\infty} \in \mathcal{P}$, where $\mathbf{p}_k = (p_k(ij))_{(i,j)\in\mathcal{D}_k} \in \mathcal{P}_k$ is a probability vector. For each $\mathbf{w} = w_1 w_2 \ldots w_k \in \Sigma^k$, we write

$$\nu_{\mathbf{p}}\big([\mathbf{w}]\big) = p_{\mathbf{w}} = p_1(w_1)p_2(w_2)\cdots p_k(w_k). \tag{1.5}$$

Note that equation (1.5) uniquely determines a Borel probability measure on Σ^{∞} by Kolmogorov's existence theorem, see [7], and $\nu_{\mathbf{p}}$ is the distribution of a sequence of

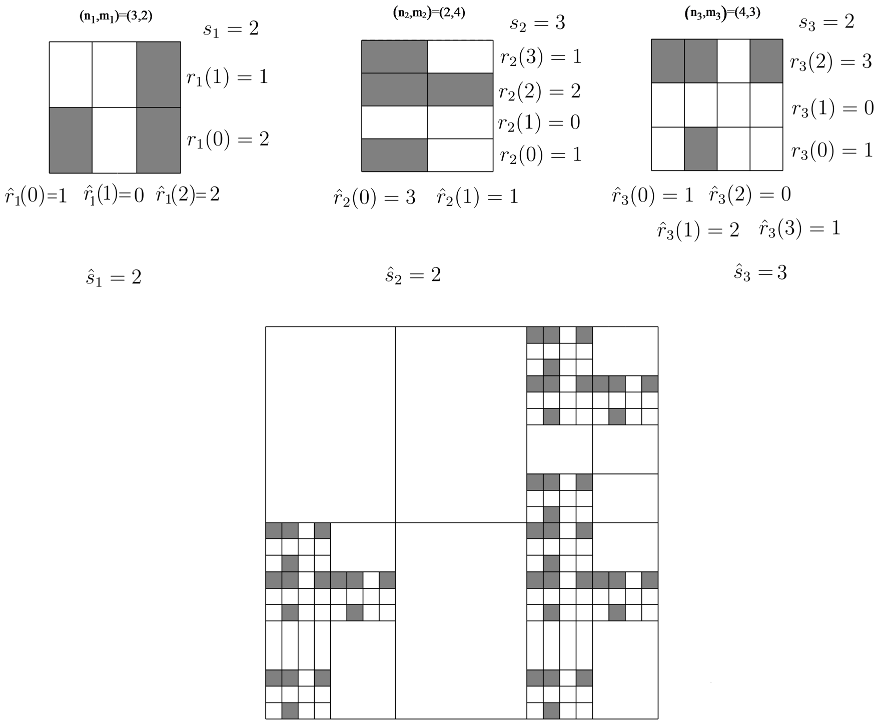

Figure 1. Nonautonomous carpet constructed to Level 3, where $\mathcal{D}_1 = \{(0,0),(2,0),(2,1)\}$, $\mathcal{D}_2 = \{(0,0),(0,2),(0,3),(1,2)\}$ and $\mathcal{D}_3 = \{(0,2),(1,0),(1,2),(3,2)\}$.

independent $\mathcal{D}_k$-valued random vectors $\mathbf{X}_k$ which have distributions $\mathbf{p}_k$. It is clear that

$$\mu_{\mathbf{p}}(A) = \nu_{\mathbf{p}}(\Pi^{-1}A) \tag{1.6}$$

is a Borel probability measure on E, and we call it a *nonautonomous measure* on E.

For each $k > 0$, we write that, for $w = (i,j) \in \mathcal{D}_k$,

$$q_k(w) = q_k(j) = \sum_{(i,j)\in\mathcal{D}_k} p_k(i,j), \qquad \widehat{q}_k(w) = \widehat{q}_k(i) = \sum_{(i,j)\in\mathcal{D}_k} p_k(i,j).$$

Note that $(q_k(j))_{j=0}^{m_k-1}$ and $(\widehat{q}_k(i))_{i=0}^{n_k-1}$ are also probability vectors, where $q_k(j)$ is the measure distributed on j-th row, and $\widehat{q}_k(i)$ is the measure distributed on i-th column.

For each $\delta > 0$, let $k = k(\delta)$ be the unique integer satisfying

$$\frac{1}{m_1}\frac{1}{m_2}\cdots\frac{1}{m_k} \le \delta < \frac{1}{m_1}\frac{1}{m_2}\cdots\frac{1}{m_{k-1}}. \tag{1.7}$$

Note that if there is no integer satisfying the above equation, we always set $k = 1$. For each given integer k, let $l = l(k)$ be the unique integer satisfying

$$\frac{1}{n_1}\frac{1}{n_2}\cdots\frac{1}{n_l} \le \frac{1}{m_1}\frac{1}{m_2}\cdots\frac{1}{m_k} < \frac{1}{n_1}\frac{1}{n_2}\cdots\frac{1}{n_{l-1}}. \tag{1.8}$$

We sometimes write $l(\delta)$ for $l(k)$ if $k = k(\delta)$ is given by (1.7). If there is no ambiguity in the context, we just write l instead of $l(k)$ for simplicity.

There are certain sets essential to the arguments of the paper which are called approximate squares. Such analogous sets were defined in [3, 29, 33]. In this paper, we use such sets repeatedly in calculations involving dimensions. For each $\delta > 0$ and every $\mathbf{w} = w_1 w_2 \ldots w_n \ldots \in \Sigma^\infty$, where $w_n = (i_n, j_n)$, we define the δ-approximate square containing $\mathbf{w}$ by

$$U(\delta, \mathbf{w}) = \big\{\mathbf{v} = v_1 v_2 \ldots v_n \ldots \in \Sigma^\infty : i_n = i'_n, n = 1, \ldots, l(\delta), \\ j_n = j'_n, n = 1, \ldots, k(\delta), v_n = (i'_n, j'_n)\big\},$$

and we write $\mathcal{U}_\delta$ for the collection of all such sets, i.e.,

$$\mathcal{U}_\delta = \big\{U(\delta, \mathbf{w}) : \mathbf{w} \in \Sigma^\infty\big\}.$$

We write

$$\mathcal{S}_\delta = \big\{\Pi(U) : U \in \mathcal{U}_\delta\big\}. \tag{1.9}$$

For simplicity, we also call the elements S of $\mathcal{S}_\delta$ the *δ-approximate squares* if there is no ambiguity. The measure distributed on approximate squares is essential in finding the dimensions of sets and measure.

Let $\nu_{\mathbf{p}}$ and $\mu_{\mathbf{p}}$ be the measures given by (1.5) and (1.6). Given $\delta > 0$, for each $U(\delta, \mathbf{w}) \in \mathcal{U}_\delta$, we have that

$$\nu_{\mathbf{p}}\big(U(\delta, \mathbf{w})\big) = \begin{cases} p_1(w_1)\cdots p_l(w_l) q_{l+1}(w_{l+1})\cdots q_k(w_k), & l \le k, \\ p_1(w_1)\cdots p_k(w_k)\widehat{q}_{k+1}(w_{k+1})\cdots \widehat{q}_l(w_l), & l > k. \end{cases} \tag{1.10}$$

where $k = k(\delta)$ and $l = l(\delta)$ are given by (1.7) and (1.8). For each $S(\delta, x) \in \mathcal{S}_\delta$ where $x \in S(\delta, x) \cap E$, there exists $\mathbf{w} \in \Sigma^\infty$ such that $\Pi(\mathbf{w}) = x$ and $\Pi(U(\delta, x)) = S(\delta, x)$. Then

$$\mu_{\mathbf{p}}\big(S(\delta, x)\big) = \nu_{\mathbf{p}}\big(U(\delta, \mathbf{w})\big). \tag{1.11}$$

Approximate squares are an essential tool in studying non-typical self-affine fractals, see [3, 5, 17, 29, 33], and we may also apply this tool to study the dimensions of the nonautonomous carpets and nonautonomous measures.

2. Main Results

In this section, we state our main conclusions. Let

$$N^+ = \sup\{n_k, m_k : k = 1, 2, \dots\}. \tag{2.1}$$

We always assume that N^+ is finite in the paper. Given $\mathbf{p} \in \mathcal{P}$, for each integer $k > 0$, the k*-th entropy* is defined as

$$H_k(\mathbf{p}) = \begin{cases} -\sum_{i=1}^{l} \sum_{w\in\mathcal{D}_i} p_i(w)\log p_i(w) - \sum_{i=l+1}^{k} \sum_{w\in\mathcal{D}_i} p_i(w)\log q_i(w), & l \le k; \\ -\sum_{i=1}^{k} \sum_{w\in\mathcal{D}_i} p_i(w)\log p_i(w) - \sum_{i=k+1}^{l} \sum_{w\in\mathcal{D}_i} p_i(w)\log \widehat{q}_i(w), & l > k, \end{cases} \tag{2.2}$$

where $l = l(k)$ is given by (1.8).

First, we give formulas of the upper and lower entropy dimensions by using k-th entropy.

Theorem 2.1. *Let E be the nonautonomous carpet defined by (1.4) with $N^+ < \infty$. Given $\mathbf{p} \in \mathcal{P}$, let $\mu_{\mathbf{p}}$ be the nonautonomous measure defined by (1.6). Then*

$$\begin{aligned} \overline{\dim}_e \mu_{\mathbf{p}} &= \limsup_{k\to\infty} \frac{H_k(\mathbf{p})}{\sum_{i=1}^{k} \log m_i}; \\ \underline{\dim}_e \mu_{\mathbf{p}} &= \liminf_{k\to\infty} \frac{H_k(\mathbf{p})}{\sum_{i=1}^{k} \log m_i}. \end{aligned}$$

To study the Hausdorff dimension of the nonautonomous measures, it often requires certain separation conditions. We introduce two such conditions from geometric and measure aspects.

Given $k > 0$, the set $\mathcal{D}_k$ is *centred* if $i \notin \{0, N-1\}$ and $j \notin \{0, M-1\}$ for every $(i, j) \in \mathcal{D}_k$. That $\mathcal{D}_k$ is centred implies that the rectangles corresponding to the affine mappings at k-th level do not connect to the boundary of square $[0, 1]^2$. We say nonautonomous carpet E satisfies the *frequency separation condition (FSC)* if there exists $c > 0$ such that

$$\lim_{n\to\infty} \frac{\mathrm{card}\{k : \mathcal{D}_k \text{ is centred for } k = 1, \dots, n\}}{n} = c.$$

Given $\mathbf{p} \in \mathcal{P}$, we say the nonautonomous measure $\mu_{\mathbf{p}}$ satisfies the *measure separation condition (MSC)* if there exists a constant $0 < C < 1$ such that for each $k > 0$,

$$\max\big\{q_k(0), q_k(m_k - 1), \widehat{q}_k(0), \widehat{q}_k(n_k - 1)\big\} < C.$$

This condition guarantees that the measure is not supported only on one of the four sides of the square $[0, 1]^2$, and it implies that the measure on the sides of approximate squares is zero. If we do not assume the strong separation condition on the nonautonomous carpets, this condition is important for the proof of Hausdorff and packing dimensions of measures supported on the sets.

Next, we state that the dimension formulas hold under either of FSC and MCS.

Theorem 2.2. *Let E be the nonautonomous carpet defined by (1.4) with $N^+ < \infty$. Given $\mathbf{p} \in \mathcal{P}$, let $\mu_{\mathbf{p}}$ be the nonautonomous measure defined by (1.6). Suppose that either E satisfies FSC or $\mu_{\mathbf{p}}$ satisfies MSC. Then*

$$\begin{aligned} \dim_{\mathrm{H}} \mu_{\mathbf{p}} &= \liminf_{k\to\infty} \frac{H_k(\mathbf{p})}{\sum_{i=1}^{k} \log m_i}; \\ \dim_{\mathrm{P}} \mu_{\mathbf{p}} &= \limsup_{k\to\infty} \frac{H_k(\mathbf{p})}{\sum_{i=1}^{k} \log m_i}. \end{aligned}$$

In Section 4, the FSC is replaced by a weaker condition, called boundary separation condition, see Theorem 4.1, and the Hausdorff and packing dimensions of measures are studies under the weak condition. Such geometric separation conditions are also useful to study the dimensions of sets.

It would be ideal that the supreme dimension of nonautonomous measures equals the dimension of the sets, but we only obtain the equality under geometric separation conditions in the following special case, see Corollary 4.2 as well.

Corollary 2.3. *Let E be the nonautonomous carpet defined by (1.4) with $N^+ < \infty$. Suppose that E satisfies FSC, and for all $k > 0$, $n_k \geq m_k$, and $r_k(j) = c_k$ for all j such that $r_k(j) \neq 0$. Then there exists $\mathbf{p} \in \mathcal{P}$ such that*

$$\begin{aligned} \dim_{\mathrm{H}} \mu_{\mathbf{p}} &= \max\{\dim_{\mathrm{H}} \mu_{\mathbf{p}'} : \mathbf{p}' \in \mathcal{P}\} = \dim_{\mathrm{H}} E; \\ \dim_{\mathrm{P}} \mu_{\mathbf{p}} &= \max\{\dim_{\mathrm{P}} \mu_{\mathbf{p}'} : \mathbf{p}' \in \mathcal{P}\} = \dim_{\mathrm{P}} E. \end{aligned}$$

Next, we state our conclusions on the dimension of nonautonomous carpets. For each integer $k > 0$, we write

$$\begin{aligned} r_k(j) &= \operatorname{card}\{i : (i, j) \in \mathcal{D}_k \text{ for each } j\}, \\ r_k^+ &= \max\{r_k(j) : j = 0, 1, \ldots, m_k - 1\}, \\ r_k^- &= \min\{r_k(j) : r_k(j) \neq 0, j = 0, 1, \ldots, m_k - 1\}, \\ s_k &= \operatorname{card}\{j : (i, j) \in \mathcal{D}_k \text{ for some } i\}. \end{aligned}$$

These count the rectangles from horizontal direction at k-th level, simply to say, $r_k(j)$ is the number of rectangles in j-th row, r_k^+ is the largest number of rectangles in these rows, r_k^- is the smallest number of rectangles in these rows, and s_k is the number of non-empty rows, see Figure 1.

Since it may happen that $m_k \geq n_k$ in our setting, we have to count the rectangles from vertical direction at each level as well. Similarly, for each integer $k > 0$, we write

$$\begin{aligned}
\widehat{r}_k(i) &= \operatorname{card}\{j : (i,j) \in \mathcal{D}_k \text{ for each } i\}, \\
\widehat{r}_k^+ &= \max\{\widehat{r}_k(i) : i = 0, 1, \ldots, n_k - 1\}, \\
\widehat{r}_k^- &= \min\{\widehat{r}_k(i) : \widehat{r}_k(i) \neq 0, i = 0, 1, \ldots, n_k - 1\}, \\
\widehat{s}_k &= \operatorname{card}\{i : (i,j) \in \mathcal{D}_k \text{ for some } j\},
\end{aligned}$$

where $\widehat{r}_k(i)$ is the number of rectangles in i-th column, $\widehat{r}_k^+$ is the largest number of rectangles in these columns, $\widehat{r}_k^-$ is the smallest number of rectangles in these columns, and $\widehat{s}_k$ is the number of non-empty columns, see Figure 1.

For each integer $k > 0$, let $l = l(k)$ be given by (1.8), and we write

$$N_{l,k}(E) = \begin{cases} r_1 \cdots r_l s_{l+1} \cdots s_k, & l \leq k, \\ r_1 \cdots r_k s_{k+1} \cdots s_l, & l > k. \end{cases} \tag{2.3}$$

We write

$$d^* = \limsup_{k\to\infty} \frac{N_{l,k}(E)}{\log m_1 \cdots m_k}, \qquad d_* = \liminf_{k\to\infty} \frac{N_{l,k}(E)}{\log m_1 \cdots m_k}. \tag{2.4}$$

The box dimension and packing dimension of E are given by d^* and d_*, respectively.

Theorem 2.4. *Let E be the nonautonomous carpet defined by (1.4) with $N^+ < \infty$. The packing dimension, upper box dimension and lower box dimension of E are given by*

$$\dim_{\mathrm{P}} E = \overline{\dim}_{\mathrm{B}} E = d^*, \qquad \underline{\dim}_{\mathrm{B}} E = d_*.$$

The proof of the theorem is similar to the one of [20, Theorem 2.1], and we omit it.

Finally, we state the conclusions on lower and Assouad dimensions for nonautonomous carpets, see Section 5 for the definitions and [17] for details. For integers k and k' such that $k' > k > 1$, let $l = l(k)$ and $l' = l'(k')$ be given by (1.8), and we have that $l' > l$. Hence, there are 6 different permutations for k', k, l' and l. To obtain the lower dimension formula, we have to find the smallest number of approximate squares with side length $\frac{1}{m_1}\frac{1}{m_2}\cdots\frac{1}{m_{k'}}$ covering the approximate square with side length $\frac{1}{m_1}\frac{1}{m_2}\cdots\frac{1}{m_k}$, and this number is given by the following formula according to the permutations of k', k, l' and l,

$$N^{-}_{k,k'}(E) = \begin{cases} r^{-}_{l+1}\cdots r^{-}_{l'} s_{k+1}\cdots s_{k'}, & l < l' \le k < k', \\ r^{-}_{l+1}\cdots r^{-}_{k} r_{k+1}\cdots r_{l'} s_{l'+1}\cdots s_{k'}, & l \le k < l' \le k', \\ r^{-}_{l+1}\cdots r^{-}_{k} r_{k+1}\cdots r_{k'} \widehat{s}_{k'+1}\cdots \widehat{s}_{l'}, & l \le k < k' \le l', \\ \widehat{r}^{-}_{k+1}\cdots \widehat{r}^{-}_{l} r_{l+1}\cdots r_{l'} s_{l'+1}\ldots s_{k'}, & k \le l < l' \le k', \\ \widehat{r}^{-}_{k+1}\cdots \widehat{r}^{-}_{l} r_{l+1}\cdots r_{k'} \widehat{s}_{k'+1}\cdots \widehat{s}_{l'}, & k \le l < k' \le l', \\ \widehat{r}^{-}_{k+1}\cdots \widehat{r}^{-}_{k'} \widehat{s}_{l+1}\cdots \widehat{s}_{l'}, & k < k' \le l < l'. \end{cases} \tag{2.5}$$

Similarly, to obtain the Assouad dimension, we must find the greatest number of approximate squares with side length $\frac{1}{m_1}\frac{1}{m_2}\cdots\frac{1}{m_{k'}}$ covering the approximate square with side length $\frac{1}{m_1}\frac{1}{m_2}\cdots\frac{1}{m_k}$, and this number is given by the following formula,

$$N^{+}_{k,k'}(E) = \begin{cases} r^{+}_{l+1}\cdots r^{+}_{l'} s_{k+1}\cdots s_{k'}, & l < l' \le k < k', \\ r^{+}_{l+1}\cdots r^{+}_{k} r_{k+1}\cdots r_{l'} s_{l'+1}\cdots s_{k'}, & l \le k < l' \le k', \\ r^{+}_{l+1}\cdots r^{+}_{k} r_{k+1}\cdots r_{k'} \widehat{s}_{k'+1}\cdots \widehat{s}_{l'}, & l \le k < k' \le l', \\ \widehat{r}^{+}_{k+1}\cdots \widehat{r}^{+}_{l} r_{l+1}\cdots r_{l'} s_{l'+1}\cdots s_{k'}, & k \le l < l' \le k', \\ \widehat{r}^{+}_{k+1}\cdots \widehat{r}^{+}_{l} r_{l+1}\cdots r_{k'} \widehat{s}_{k'+1}\cdots \widehat{s}_{l'}, & k \le l < k' \le l', \\ \widehat{r}^{+}_{k+1}\cdots \widehat{r}^{+}_{k'} \widehat{s}_{l+1}\cdots \widehat{s}_{l'}, & k < k' \le l < l'. \end{cases} \tag{2.6}$$

The following theorem shows that the lower and Assouad dimension of E are given by different limits involving $N^{-}_{k,k'}(E)$ and $N^{+}_{k,k'}(E)$.

Theorem 2.5. *Let E be the nonautonomous carpet defined by (1.4) with $N^+ < \infty$. The lower dimension and the Assouad dimension of E are given by*

$$\dim_{\mathrm{L}} E = \lim_{m\to\infty} \inf_{k} \left\{ \frac{\log N^{-}_{k,k+m}(E)}{\log m_{k+1}\cdots m_{k+m}} \right\};$$

$$\dim_{\mathrm{A}} E = \lim_{m\to\infty} \sup_{k} \left\{ \frac{\log N^{+}_{k,k+m}(E)}{\log m_{k+1}\cdots m_{k+m}} \right\}.$$

3. Entropy dimensions of nonautonomous measures

To prove the entropy dimension, we need the following well-known inequality.

Lemma 3.1. *The function $f : [0,\infty) \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ -x\log x & \text{if } x \ne 0 \end{cases}$$

is strictly concave, and for all $x_1, \ldots, x_N \ge 0$,

$$f\left(\sum_{i=1}^{N} x_i\right) \le \sum_{i=1}^{N} f(x_i) \le f\left(\sum_{i=1}^{N} x_i\right) + \left(\sum_{i=1}^{N} x_i\right) \log N.$$

Let f be the function defined in Lemma 3.1. The lower and upper entropy dimension may be rewritten as

$$\underline{\dim}_e \mu_{\mathbf{p}} = \liminf_{n\to\infty} \frac{\sum_{Q\in\mathcal{M}_n} f\big(\mu_{\mathbf{p}}(Q)\big)}{\log 2^{-n}};$$
$$\overline{\dim}_e \mu_{\mathbf{p}} = \limsup_{n\to\infty} \frac{\sum_{Q\in\mathcal{M}_n} f\big(\mu_{\mathbf{p}}(Q)\big)}{\log 2^{-n}}.$$

Proof of Theorem 2.1. For each $\delta > 0$, let n be the integer such that $2^{-n} \le \delta < 2^{-n+1}$. Then for each $Q \in \mathcal{M}_n$, it intersects at most $C_1 = 4(N^+)^3$ approximate squares of $\mathcal{S}_\delta$, and for each $S \in \mathcal{S}_\delta$, it intersects at most 3^2 cubes in $\mathcal{M}_n$. Therefore, by Lemma 3.1, it follows that for each $Q \in \mathcal{M}_n$,

$$f\big(\mu_{\mathbf{p}}(Q)\big) \le \sum_{S\in\mathcal{S}_\delta} f\big(\mu_{\mathbf{p}}(S\cap Q)\big) \le f\big(\mu_{\mathbf{p}}(Q)\big) + (\log C_1)\mu_{\mathbf{p}}(Q) \tag{3.1}$$

and for each $S \in \mathcal{S}_\delta$,

$$f\big(\mu_{\mathbf{p}}(S)\big) \le \sum_{Q\in\mathcal{M}_n} f\big(\mu_{\mathbf{p}}(S\cap Q)\big) \le f\big(\mu_{\mathbf{p}}(S)\big) + (2\log 3)\mu_{\mathbf{p}}(S). \tag{3.2}$$

Summing up (3.2) and (3.1) respectively, we obtain that

$$\sum_{Q\in\mathcal{M}_n} f\big(\mu_{\mathbf{p}}(Q)\big) \le \sum_{Q\in\mathcal{M}_n}\sum_{S\in\mathcal{S}_\delta} f\big(\mu_{\mathbf{p}}(S\cap Q)\big) \le \sum_{Q\in\mathcal{M}_n} f\big(\mu_{\mathbf{p}}(Q)\big) + \log C_1$$

and

$$\sum_{S\in\mathcal{S}_\delta} f\big(\mu_{\mathbf{p}}(S)\big) \le \sum_{S\in\mathcal{S}_\delta}\sum_{Q\in\mathcal{M}_n} f\big(\mu_{\mathbf{p}}(Q\cap S)\big) \le \sum_{S\in\mathcal{S}_\delta} f\big(\mu_{\mathbf{p}}(S)\big) + 2\log 3.$$

It follows that

$$\Big|\sum_{S\in\mathcal{S}_\delta} f\big(\mu_{\mathbf{p}}(S)\big) - \sum_{Q\in\mathcal{M}_n} f\big(\mu_{\mathbf{p}}(Q)\big)\Big| \le 2\log 3 + \log C_1. \tag{3.3}$$

Let $l = l(\delta)$ and $k = k(\delta)$ be given by (1.8) and (1.7). For $l \le k$, by induction, it follows that

$$\begin{aligned}
&\sum_{S\in\mathcal{S}_\delta} f\big(\mu_{\mathbf{p}}(S)\big)\\
&\quad=\sum_{U(\delta,\mathbf{w})\in\mathcal{U}_\delta} p_1(w_1)\cdots p_l(w_l)q_{l+1}(w_{l+1})\cdots q_k(w_k)\\
&\qquad\quad\cdot\log p_1(w_1)\cdots p_l(w_l)q_{l+1}(w_{l+1})\cdots q_k(w_k)\\
&\quad=\sum_{\mathbf{w}\in\Sigma^k} p_1(w_1)\cdots p_l(w_l)q_{l+1}(w_{l+1})\cdots q_k(w_k)\log p_1(w_1)\cdots p_l(w_l)\\
&\qquad+\sum_{\mathbf{w}\in\Sigma^k} p_1(w_1)\cdots p_l(w_l)q_{l+1}(w_{l+1})\cdots q_k(w_k)\log q_{l+1}(w_{l+1})\cdots q_k(w_k)\\
&\quad=\sum_{i=1}^{l}\sum_{w\in\mathcal{D}_i} p_i(w)\log p_i(w)+\sum_{i=l+1}^{k}\sum_{w\in\mathcal{D}_i} p_i(w)\log q_i(w).
\end{aligned}$$

For $l > k$, similarly, we have that

$$\begin{aligned}
\sum_{S\in\mathcal{S}_\delta} f\big(\mu_{\mathbf{p}}(S)\big) &= \sum_{U(\delta,\mathbf{w})\in\mathcal{U}_\delta} p_1(w_1)\cdots p_k(w_k)\widehat{q}_{k+1}(w_{k+1})\cdots\widehat{q}_l(w_l)\\
&\qquad\qquad\cdot\log p_1(w_1)\cdots p_k(w_k)\widehat{q}_{k+1}(w_{k+1})\cdots\widehat{q}_l(w_l)\\
&=\sum_{i=1}^{k}\sum_{w\in\mathcal{D}_i} p_i(w)\log p_i(w)+\sum_{i=k+1}^{l}\sum_{w\in\mathcal{D}_i} p_i(w)\log\widehat{q}_i(w).
\end{aligned}$$

Hence, by (2.2), we obtain that

$$\sum_{S\in\mathcal{S}_\delta} f\big(\mu_{\mathbf{p}}(S)\big) = -H_k(\mathbf{p}).$$

Combining this with (3.3), we have that

$$\begin{aligned}
\overline{\dim}_e\mu_{\mathbf{p}} &= \limsup_{n\to\infty}\frac{\sum_{Q\in\mathcal{M}_n} f\big(\mu_{\mathbf{p}}(Q)\big)}{\log 2^{-n}}\\
&= \limsup_{\delta\to 0}\frac{\sum_{S\in\mathcal{S}_\delta} f\big(\mu_{\mathbf{p}}(S)\big)}{\log\delta}\\
&= \limsup_{k\to\infty}\frac{H_k(\mathbf{p})}{\sum_{i=1}^k\log m_i}.
\end{aligned}$$

By the same argument, we have that

$$\underline{\dim}_e\mu_{\mathbf{p}} = \liminf_{n\to\infty}\frac{\sum_{Q\in\mathcal{M}_n} f\big(\mu_{\mathbf{p}}(Q)\big)}{\log 2^{-n}} = \liminf_{k\to\infty}\frac{H_k(\mathbf{p})}{\sum_{i=1}^k\log m_i},$$

which completes the proof. ■

4. Hausdorff and packing dimensions of nonautonomous measures

In this section, we study the Hausdorff and packing dimensions of nonautonomous measures under a weak condition.

Given $k > 0$, the set $\mathcal{D}_k$ is *left (right, bottom, top) empty* if $i \neq 0$ ($i \neq n_k - 1$, $j \neq 0$, $j \neq m_k - 1$). We say E is *left separated* if

$$\lim_{n\to\infty} \frac{\mathrm{card}\{k : \mathcal{D}_k \text{ is left empty for } k = 1, \dots, n\}}{n} = c_L > 0.$$

Similarly, we may define E is *right separated, top separated and bottom separated* where the limits are denoted by c_R, c_T, c_B, respectively. If E is left, right, bottom and top separated, we say E satisfies the *boundary separation condition (BSC)*.

Since BSC is weaker than FSC, we prove the dimension formulas of nonautonomous measures under the assumption of the BSC.

Theorem 4.1. *Let E be the nonautonomous carpet defined by (1.4) with $N^+ < \infty$. Given $\mathbf{p} \in \mathcal{P}$, let $\mu_{\mathbf{p}}$ be the nonautonomous measure defined by (1.6). Suppose that either E satisfies BSC or $\mu_{\mathbf{p}}$ satisfies MSC. Then*

$$\begin{aligned}
\dim_{\mathrm{H}} \mu_{\mathbf{p}} &= \liminf_{k\to\infty} \frac{H_k(\mathbf{p})}{\sum_{i=1}^{k} \log m_i};\\
\dim_{\mathrm{P}} \mu_{\mathbf{p}} &= \limsup_{k\to\infty} \frac{H_k(\mathbf{p})}{\sum_{i=1}^{k} \log m_i}.
\end{aligned}$$

Corollary 4.2. *Let E be an arbitrary nonautonomous carpet defined by (1.4). Suppose that E satisfies BSC, and for all $k > 0$, $n_k \geq m_k$, and $r_k(j) = c_k$ for all j such that $r_k(j) \neq 0$. Then there exists $\mathbf{p} \in \mathcal{P}$ such that*

$$\begin{aligned}
\dim_{\mathrm{H}} \mu_{\mathbf{p}} &= \max\{\dim_{\mathrm{H}} \mu_{\mathbf{p}'} : \mathbf{p}' \in \mathcal{P}\} = \dim_{\mathrm{H}} E;\\
\dim_{\mathrm{P}} \mu_{\mathbf{p}} &= \max\{\dim_{\mathrm{P}} \mu_{\mathbf{p}'} : \mathbf{p}' \in \mathcal{P}\} = \dim_{\mathrm{P}} E.
\end{aligned}$$

To study the dimensions of nonautonomous measures, we need a version of the law of large numbers. For the readers' convenience, we cite it here, see, for example, [6, Corollary A.8] for details.

Theorem 4.3. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables which are bounded in L^2 and such that*

$$\mathbf{E}(X_n | X_1, \dots, X_{n-1}) = 0,$$

for all $n \geq 1$. Then the sequence $\frac{1}{n}\sum_{i=1}^{n} X_i$ converges to 0 almost surely and in L^2.

To estimate the Hausdorff dimension, we need the following well-known fact, which is often called Frostman's Lemma, see [9].

Lemma 4.4. *Let μ be a finite Borel measure on $\mathbb{R}^d$.*

(1) *If* $\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s$ *for μ-almost every x, then* $\dim_{\mathrm{H}} \mu \geq s$.

(2) *If* $\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \leq s$ *for μ-almost every x, then* $\dim_{\mathrm{H}} \mu \leq s$.

(3) *If* $\limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s$ *for μ-almost every x, then* $\dim_{\mathrm{P}} \mu \geq s$.

(4) *If* $\limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \leq s$ *for μ-almost every x, then* $\dim_{\mathrm{P}} \mu \leq s$.

Given an integer $k > 0$, let $\delta = (m_1 \cdots m_k)^{-1}$, and we write $U(k, \mathbf{w}) = U(\delta, \mathbf{w})$ and $S_k(\mathbf{w}) = \Pi(U(k, \mathbf{w}))$ for simplicity in the rest of this section.

Proof of Theorem 4.1. First, we show that the conclusion holds under the assumption of the BSC. Given $\mathbf{w} = w_1 w_2 \ldots w_k \ldots \in \Sigma^\infty$, since $(\log p_k(w_k))_{k\in\mathbb{N}}$ is a sequence of independent random variables, their variances are uniformly bounded by

$$\mathbf{Var}\big(\log p_k(w_k)\big) \leq (N^+)^2 \max_{x\in[0.1]} x \log^2 x.$$

By Theorem 4.3, we have

$$-\sum_{k=1}^{N} \log p_k(w_k) = \sum_{k=1}^{N} \sum_{w\in\mathcal{D}_k} p_k(w) \log p_k(w) + o(N),$$

almost surely.

Similarly, the following equalities hold almost surely:

$$\begin{aligned} -\sum_{k=1}^{N} \log q_k(w_k) &= \sum_{k=1}^{N} \sum_{w\in\mathcal{D}_k} p_k(w) \log q_k(w) + o(N); \\ -\sum_{k=1}^{N} \log \widehat{q}_k(w_k) &= \sum_{k=1}^{N} \sum_{w\in\mathcal{D}_k} p_k(w) \log \widehat{q}_k(w) + o(N). \end{aligned}$$

For each integer $k > 0$, recall that $U(k, \mathbf{w}) = U(\delta, \mathbf{w})$ and $S_k(\mathbf{w}) = \Pi(U(k, \mathbf{w}))$ where $\delta = (m_1 \cdots m_k)^{-1}$. By (1.10), we have that for $k \geq l$,

$$\begin{aligned} \log \nu_{\mathbf{p}}\big(U(k, \mathbf{w})\big) &= \sum_{i=1}^{l} p_i(w_i) + \sum_{i=l+1}^{k} q_i(w_i) \\ &= \sum_{i=1}^{l} \sum_{w\in\mathcal{D}_k} p_k(w) \log p_k(w) \\ &\qquad + \sum_{i=l+1}^{k} \sum_{w\in\mathcal{D}_k} p_k(w) \log q_k(w) + o(k), \end{aligned}$$

and for $k < l$,

$$\begin{aligned}\log \nu_{\mathbf{p}}\big(U(k,\mathbf{w})\big) &= \sum_{i=1}^{k} p_i(w_i) + \sum_{i=k+1}^{l} q_i(w_i)\\ &= \sum_{i=1}^{k} \sum_{w\in\mathcal{D}_k} p_k(w)\log p_k(w)\\ &\qquad + \sum_{i=k+1}^{l} \sum_{w\in\mathcal{D}_k} p_k(w)\log \widehat{q}_k(w) + o(k),\end{aligned}$$

almost surely. Hence, by (1.11) and (2.2), it follows that

$$\log \mu_{\mathbf{p}}\big(S_k(\mathbf{w})\big) = \log \nu_{\mathbf{p}}\big(U(k,\mathbf{w})\big) = -H_k(\mathbf{p}) + o(k), \tag{4.1}$$

almost surely.

Fix $\varepsilon > 0$, and let $\xi = \frac{2\varepsilon}{1-\varepsilon}$. It is clear that $\xi \to 0$ as ε tends to 0. Since E satisfies the separation condition, there exists $K_0 > 0$ such that for $k > K_0$,

$$\begin{aligned}&\text{card}\{h : \mathcal{D}_h \text{ is left empty for } (1-\xi)k < h < k\} \geq 1, \qquad (4.2)\\ &\text{card}\{h : \mathcal{D}_h \text{ is right empty for } (1-\xi)k < h < k\} \geq 1,\\ &\text{card}\{h : \mathcal{D}_h \text{ is top empty for } (1-\xi)k < h < k\} \geq 1,\\ &\text{card}\{h : \mathcal{D}_h \text{ is bottom empty for } (1-\xi)k < h < k\} \geq 1.\end{aligned}$$

For sufficiently small $\rho > 0$, let k be the integer such that

$$\prod_{i=1}^{k} m_i \leq \rho < \prod_{i=1}^{k-1} m_i \leq (N^+)^{-1} \prod_{i=1}^{k} m_i.$$

Let $l = l(k)$ be given by (1.8). Setting

$$k' = k+1, \quad k'' = \min\big\{(1-\xi)k, k\big((1-\xi)^2 l\big)\big\},$$

where $k((1-\xi)^2 l)$ denotes the largest integer β such that $l(\beta) \leq (1-\xi)^2 l$. Then, by (4.2), we have that

$$\begin{aligned}&\text{card}\{h : \mathcal{D}_h \text{ is left for } l''(k'') < h < (1-\xi)l\} \geq 1,\\ &\text{card}\{h : \mathcal{D}_h \text{ is right for } l''(k'') < h < (1-\xi)l\} \geq 1,\\ &\text{card}\{h : \mathcal{D}_h \text{ is top empty for } k'' < h < k\} \geq 1,\\ &\text{card}\{h : \mathcal{D}_h \text{ is bottom empty for } k'' < h < k\} \geq 1.\end{aligned}$$

Next, we show that the distance from $\Pi(\mathbf{w})$ to the each side of $S_{k''}(\mathbf{w})$ is greater than ρ. We first consider the distance from $\Pi(\mathbf{w})$ to the left side of $S_{k''}(\mathbf{w})$. Let l_0 an integer satisfy $l''(k'') < l_0 < (1-\xi)l$. Then the distance from $\Pi(\mathbf{w})$ to the left side of $S_{k''}(\mathbf{w})$ is no less than $(n_1 \cdots n_{l_0})^{-1}$. Since l is sufficiently large, $\xi l \geq \frac{\log(N^+)^2}{\log 2}$, it is clear that

$$(n_1 \cdots n_{l_0})^{-1} \geq 2^{\xi l}(n_1 \cdots n_l)^{-1} \geq (N^+)^2 (n_1 \cdots n_l)^{-1} \geq \rho.$$

Hence, the distance from $\Pi(\mathbf{w})$ to the left side of $S_{k''}(\mathbf{w})$ is greater than ρ. For the distance from $\Pi(\mathbf{w})$ to the bottom side of $S_{k''}(\mathbf{w})$. Similarly, we may find an integer k_0 $k'' < k_0 < k$ such that $\mathcal{D}_{k_0}$ is bottom empty. Then the distance from $\Pi(\mathbf{w})$ to the bottom side of $S_{k''}(\mathbf{w})$ is no less than $(m_1 \cdots m_{k_0})^{-1}$, which is greater than ρ.

Similarly, the distances from $\Pi(\mathbf{w})$ to the top and right sides of $S_{k''}(\mathbf{w})$ are greater than ρ as well. This implies that $B(\Pi(\mathbf{w}), \rho) \subset S_{k''}(\mathbf{w})$. From the inequalities $(m_1 \cdots m_{k'})^{-1} < (m_1 \cdots m_k)^{-1} \leq \rho$, we have $S_{k'}(\mathbf{w}) \subset B(\Pi(\mathbf{w}), \rho)$.

Therefore, we obtain that

$$S_{k'}(\mathbf{w}) \subset B\big(\Pi(\mathbf{w}), \rho\big) \subset S_{k''}(\mathbf{w}). \tag{4.3}$$

By (4.1), immediately, we have that

$$\liminf_{k\to\infty} \frac{H_{k'}(\mathbf{p}) + o(k)}{\sum_{i=1}^k \log m_i} \geq \liminf_{\rho\to 0} \frac{\log \mu_{\mathbf{p}}\big(B\big(\Pi(\mathbf{w}), \rho\big)\big)}{\log \rho} \geq \liminf_{k\to\infty} \frac{H_{k''}(\mathbf{p}) + o(k)}{\sum_{i=1}^k \log m_i},$$

almost surely. Note that

$$H_{k'}(\mathbf{p}) \to H_k(\mathbf{p}), \qquad H_{k''}(\mathbf{p}) \to H_k(\mathbf{p})$$

as ε tends to 0, they imply that

$$\liminf_{\rho\to 0} \frac{\log \mu_{\mathbf{p}}\big(B\big(\Pi(\mathbf{w}), \rho\big)\big)}{\log \rho} = \liminf_{k\to\infty} \frac{H_k(\mathbf{p})}{\sum_{i=1}^k \log m_i},$$

almost surely. By Lemma 4.4, it follows that

$$\dim_{\mathrm{H}} \mu_{\mathbf{p}} = \liminf_{k\to\infty} \frac{H_k(\mathbf{p})}{\sum_{i=1}^k \log m_i}.$$

Similarly, by (4.3) and (4.1), we have that

$$\limsup_{k\to\infty} \frac{H_{k'}(\mathbf{p}) + o(k)}{\sum_{i=1}^k \log m_i} \geq \limsup_{\rho\to 0} \frac{\log \mu_{\mathbf{p}}\big(B\big(\Pi(\mathbf{w}), \rho\big)\big)}{\log \rho} \geq \limsup_{k\to\infty} \frac{H_{k''}(\mathbf{p}) + o(k)}{\sum_{i=1}^k \log m_i}.$$

By Lemma 4.4, we have that

$$\dim_{\mathrm{P}} \mu_{\mathbf{p}} = \limsup_{k\to\infty} \frac{H_k(\mathbf{p})}{\sum_{i=1}^k \log m_i}.$$

Next, we prove that the conclusion holds for MSC. For each integer $k > 0$, we write

$$A_k = \{x = \Pi(\mathbf{w}) \in E : B\big(x, (m_1 \cdots m_k)^{-1} e^{-\sqrt{k}}\big) \cap E \subset S_k(\mathbf{w})\}.$$

Let L_k be the collection of $x \in E$ such that the distance from x to the bottom side of $S_k(\mathbf{w})$ is less than $(m_1 \cdots m_k)^{-1} e^{-\sqrt{k}}$, where $x = \Pi(\mathbf{w})$ and $\mathbf{w} = (i_1, j_1) \dots (i_k, j_k) \dots \in \Sigma^\infty$. It is clear that $j_{k+1} = \cdots = j_{k+[\sqrt{k}/\log N^+]} = 0$. Hence, the measure of L_k is bounded by

$$\mu_{\mathbf{p}}(L_k) \le q_{k+1}(0) \cdots q_{k+[\sqrt{k}/\log N^+]}(0) \le C_1^{\sqrt{k}},$$

where $C_1 = C^{1/(\log N^+ + 1)} < 1$. We apply the similar argument to other three sides and obtain that

$$\sum_{k=1}^{\infty} \mu_{\mathbf{p}}(A_k^c) < 4 \sum_{k=1}^{\infty} C_1^{\sqrt{k}} < \infty.$$

By Borel–Cantelli Lemma, it follows that

$$\mu_{\mathbf{p}}(A_k^c \ i.o.) = 0.$$

Therefore, for $\mu_{\mathbf{p}}$-almost all x, we have that

$$\mu_{\mathbf{p}}\big(B\big(x, (m_1 \cdots m_k)^{-1} e^{-\sqrt{k}}\big)\big) \le \mu_{\mathbf{p}}\big(S_k(\mathbf{w})\big),$$

for sufficiently large k. For each $\rho > 0$, there exists a unique integer k such that

$$(m_1 \cdots m_{k+1})^{-1} e^{-\sqrt{k+1}} \le \rho < (m_1 \cdots m_k)^{-1} e^{-\sqrt{k}},$$

which implies that $\mu_{\mathbf{p}}(B(x,\rho)) \le \mu_{\mathbf{p}}(S_k(x))$. Therefore, we have that

$$\liminf_{\rho \to 0} \frac{\log \mu_{\mathbf{p}}\big(B(x,\rho)\big)}{\log \rho} \ge \liminf_{k \to \infty} \frac{\log \mu_{\mathbf{p}}\big(S_k(\mathbf{w})\big)}{-\sum_{i=1}^k \log m_i} = \liminf_{k \to \infty} \frac{H_k(\mathbf{p})}{\sum_{i=1}^k \log m_i}.$$

On the other hand, for each $\rho > 0$, let k be the integer such that

$$\prod_{i=1}^{k} m_i < \rho \le \prod_{i=1}^{k-1} m_i.$$

Then for all $x \in E$, choose $\mathbf{w} \in \Pi^{-1}(x)$, and we have $S_k(\mathbf{w}) \subset B(x,\rho)$, which implies $\mu_{\mathbf{p}}(B(x,\rho)) \ge \mu_{\mathbf{p}}(S_k(x))$. Therefore, by (4.1), we have that

$$\liminf_{\rho \to 0} \frac{\log \mu_{\mathbf{p}}\big(B(x,\rho)\big)}{\log \rho} \le \liminf_{k \to \infty} \frac{\log \mu_{\mathbf{p}}\big(S_k(\mathbf{w})\big)}{-\sum_{i=1}^k \log m_i} = \liminf_{k \to \infty} \frac{H_k(\mathbf{p})}{\sum_{i=1}^k \log m_i}.$$

It follows that

$$\liminf_{\rho\to 0}\frac{\log\mu_{\mathbf{p}}\big(B(x,\rho)\big)}{\log\rho}=\liminf_{k\to\infty}\frac{H_k(\mathbf{p})}{\sum_{i=1}^{k}\log m_i}.$$

almost surely. By Lemma 4.4, the Hausdorff dimension of $\mu_{\mathbf{p}}$ is given by

$$\dim_{\mathrm{H}}\mu_{\mathbf{p}}=\liminf_{k\to\infty}\frac{H_k(\mathbf{p})}{\sum_{i=1}^{k}\log m_i}.$$

Similarly, for $\mu_{\mathbf{p}}$-almost all x, we have that

$$\limsup_{r\to 0}\frac{\log\mu_{\mathbf{p}}\big(B(x,\rho)\big)}{\log\rho}=\limsup_{k\to\infty}\frac{H_k(\mathbf{p})}{\sum_{i=1}^{k}\log m_i},$$

and by Lemma 4.4, the packing dimension of $\mu_{\mathbf{p}}$ is given by

$$\dim_{\mathrm{P}}\mu_{\mathbf{p}}=\limsup_{k\to\infty}\frac{H_k(\mathbf{p})}{\sum_{i=1}^{k}\log m_i}.\qquad\blacksquare$$

Proof of Corollary 4.2. For each $k>0$, let $p_k(w)=\frac{1}{r_k}$ for all $w\in\mathcal{D}_k$. Since $r_k(j)=c_k$ for all j such that $r_k(j)\neq 0$, we have $r_k=c_ks_k$, and it implies $q_k(w)=\frac{1}{s_k}$. Recall that $U(k,\mathbf{w})=U(\delta,\mathbf{w})$ and $S_k(\mathbf{w})=\Pi(U(k,\mathbf{w}))$ where $\delta=(m_1\cdots m_k)^{-1}$, by (1.10), we have that

$$\begin{aligned}\log\mu_{\mathbf{p}}\big(S_k(\mathbf{w})\big)&=\log\nu_{\mathbf{p}}\big(U(k,\mathbf{w})\big)\\&=\sum_{i=1}^{l}\log p_k(w)+\sum_{k=l+1}^{k}\log q_k(w)\\&=-\sum_{i=1}^{l}\log r_k-\sum_{k=l+1}^{k}\log s_k,\end{aligned}$$

for all $\mathbf{w}\in\Sigma^\infty$ and $k>0$. By the same argument in Theorem 2.2, we have that

$$\liminf_{\rho\to 0}\frac{\log\mu_{\mathbf{p}}\big(B(x,\rho)\big)}{\log\rho}=\liminf_{k\to\infty}\frac{\sum_{i=1}^{l}\log r_k+\sum_{k=l+1}^{k}\log s_k}{\sum_{i=1}^{k}\log m_i}.$$

for all $x\in E$. Then by [9, Proposition 2.3],

$$\dim_{\mathrm{H}}E=\dim_{\mathrm{H}}\mu_{\mathbf{p}}=\liminf_{k\to\infty}\frac{\sum_{i=1}^{l}\log r_k+\sum_{k=l+1}^{k}\log s_k}{\sum_{i=1}^{k}\log m_i}.$$

Since $\dim_{\mathrm{H}}E=\sup\{\dim_{\mathrm{H}}\mu;\ \text{for all Borel }\mu\text{ on }E\text{ such that }0<\mu(E)<\infty\}$, we have that

$$\dim_{\mathrm{H}}\mu_{\mathbf{p}}=\max\{\dim_{\mathrm{H}}\mu_{\mathbf{p}'}:\mathbf{p}'\in\mathcal{P}\}=\dim_{\mathrm{H}}E.$$

Similarly, we have that

$$\dim_{\mathrm{P}} E = \dim_{\mathrm{P}} \mu_{\mathbf{p}} = \limsup_{k\to\infty} \frac{\sum_{i=1}^{l} \log r_k + \sum_{k=l+1}^{k} \log s_k}{\sum_{i=1}^{k} \log m_i},$$

and this implies that

$$\dim_{\mathrm{P}} \mu_{\mathbf{p}} = \max\{\dim_{\mathrm{P}} \mu_{\mathbf{p}'} : \mathbf{p}' \in \mathcal{P}\} = \dim_{\mathrm{P}} E. \qquad \blacksquare$$

Proof of Theorem 2.2. Since E satisfies the frequency separation condition, there exists $c > 0$ such that

$$\lim_{n\to\infty} \frac{\mathrm{card}\{k : \mathcal{D}_k \text{ is centred for } k = 1, \dots, n\}}{n} = c.$$

It is clear that if $\mathcal{D}_k$ is centred, then $\mathcal{D}_k$ is left, right, top and bottom separated and $c_L = c_R = c_T = c_B = c$, that is, FSC implies BSC. Hence, E satisfies boundary separation condition, and by Theorem 4.1, the conclusion holds. ■

Proof of Corollary 2.3. Since FSC implies BSC, by Corollary 4.2, the conclusion holds. ■

5. Lower and Assouad dimensions of nonautonomous carpets

In this section, we give the proofs for the lower and the Assouad dimension of nonautonomous carpets.

First, we show the connection between approximate squares and balls which is fundamental to our proofs. For simplicity, let

$$R_k = (m_1 \cdots m_k)^{-1},$$

and we write $\mathcal{S}_k$ for the collection of R_k-approximate squares,

$$\mathcal{S}_k = \mathcal{S}_{R_k},$$

where $\mathcal{S}_{R_k}$ is given by (1.9).

Lemma 5.1. *Given an integer* $k > 0$, *for every approximate square* $S \in \mathcal{S}_k$, *there exists* $x \in S$ *such that* $B(x, (N^+)^{-3} R_k) \cap E \subset S$.

Proof. Given $k > 0$, let $l = l(k)$ be given by (1.8). Without loss of generality, we assume that $l \le k$ (if $l > k$, the conclusion follows by exchanging the x and y axes).

Given $S \in \mathcal{S}_k$, there exist $y \in S$ and an approximate square $U(\mathbf{w})$ such that $S = S(y) = \Pi(U(\mathbf{w}))$ for some $\mathbf{w} = w_1 w_2 \ldots = (i_1, j_1)(i_2, j_2)\ldots$, where $y = \Pi(\mathbf{w}) \in E$ and

$$U(\mathbf{w}) = \big\{\mathbf{v} = v_1 v_2 \ldots v_n \ldots : i'_n = i_n, n = 1, \ldots, l, \\ j'_n = j_n, n = 1, \ldots, k, v_n = (i'_n, j'_n)\big\}.$$

We write

$$\partial U(\mathbf{w}) = \big\{\mathbf{v} \in U(\mathbf{w}) : i'_{l+1} = i'_{l+2} = 0 \text{ or } i'_{l+1} = n_{l+1} - 1, i'_{l+2} = n_{l+2} - 1, \\ j'_{k+1} = j'_{k+2} = 0 \text{ or } j'_{k+1} = m'_{k+1} - 1, j'_{k+2} = m_{k+2} - 1, \\ v_n = (i'_n, j'_n)\big\},$$

geometrically, it means that $\partial U(\mathbf{w})$ containing all the rectangles going down 2 levels from $U(\mathbf{w})$ and connecting to the side of $U(\mathbf{w})$, and all the elements in $U(\mathbf{w})$ whose distances to boundary of $U(\mathbf{w})$ are less than $(N^+)^{-3} R_k$ are contained in $\partial U(\mathbf{w})$.

The conclusion immediately follows if there exists an element $\mathbf{v} \in U(\mathbf{w})$ such that the distance from $\Pi(\mathbf{v})$ to the side of S greater than $(N^+)^{-3} R_k$. Precisely, if there exists $\mathbf{v} \in U(\mathbf{w})$ such that $\mathbf{v} \notin \partial U(\mathbf{w})$, then by taking $x = \Pi(\mathbf{v})$, we have that $B(x, (N^+)^{-3} R_k) \cap E \subset S$. Therefore, the key is to consider the approximate squares S empty in the middle, that is, the distance from $\Pi(\mathbf{v})$ to the side of S is less than or equal to $(N^+)^{-3} R_k$ for all $\mathbf{v} \in U(\mathbf{w})$.

Since $N^+ = \sup_k \{n_k, m_k\}$, it is sufficient to assume that $U \backslash \partial U = \emptyset$ and $\partial U \neq \emptyset$. Hence, it is clear that $\mathbf{v} = v_1 v_2 \ldots \in \partial U$ for all $\mathbf{v} \in U$. Recall that $i'_n = i_n, n = 1, \ldots, l$ and $j'_n = j_n, n = 1, \ldots, k$, $v_n = (i'_n, j'_n)$ for all $\mathbf{v} = v_1 v_2 \ldots \in \partial U$.

First, we show the conclusion holds if there exists $\mathbf{v} \in \partial U$ that satisfies one of the following six cases:

(1) $i'_{l+1} = i'_{l+2} = 0$, and $j'_{k+1} \notin \{0, m_{k+1} - 1\}$.

(2) $i'_{l+1} = i'_{l+2} = 0$, $j'_{k+1} = 0$ and $j'_{k+2} \neq 0$.

(3) $i'_{l+1} = i'_{l+2} = 0$, $j'_{k+1} = m_{k+1} - 1$ and $j'_{k+2} \neq m_{k+2} - 1$.

(4) $i'_{l+1} = n_{l+1} - 1, i'_{l+2} = n_{l+2} - 1$ and $j'_{k+1} \notin \{0, m_{k+1} - 1\}$.

(5) $i'_{l+1} = n_{l+1} - 1, i'_{l+2} = n_{l+2} - 1$, $j'_{k+1} = 0$ and $j'_{k+2} \neq 0$.

(6) $i'_{l+1} = n_{l+1} - 1, i'_{l+2} = n_{l+2} - 1$, $j'_{k+1} = m_{k+1} - 1$ and $j'_{k+2} \neq m_{k+2} - 1$.

Suppose that $\mathbf{v}$ satisfies one of (1)–(3). Let $x = \Pi(\mathbf{v})$. The distance from x to the top and bottom sides are more than $(N^+)^{-3} R_k$. Since $U \backslash \partial U = \emptyset$, if $k \geq l + 1$, we have that

$$\big(n_{l+1} - 1, j_{l+1}(S)\big) \notin \mathcal{D}_{l+1},$$

and if $k = l$, we have that

$$\widehat{r}_l(n_{l+1} - 1) = 0.$$

This implies that the intersection of $B(x,(N^+)^{-3}R_k)$ and the approximate square on the left of S is empty, and it immediately follows that

$$B\big(x,(N^+)^{-3}R_k\big)\cap E\subset S.$$

Hence, the conclusion holds. The proofs for (4)–(6) are identical.

Otherwise, if there is no element $\mathbf{v}\in\partial U$ satisfying (1)–(6), then every $\mathbf{v}=v_1v_2\dots v_n\dots\in\partial U$ satisfies either $j'_{k+1}=j'_{k+2}=0$ or $j'_{k+1}=m_{k+1}-1$, $j'_{k+2}=m_{k+2}-1$. Since $U\backslash\partial U=\emptyset$, the identity $j'_{k+1}=j'_{k+2}=0$ implies that $r_{k+1}(m_{k+1}-1)=0$, and $j'_{k+1}=m_{k+1}-1$, $j'_{k+2}=m_{k+2}-1$ implies that $r_{k+1}(0)=0$. Next, we show the conclusion holds if there exists $\mathbf{v}\in\partial U$ satisfying one of the following six cases:

(7) $j'_{k+1}=j'_{k+2}=0, i'_{l+1}\notin\{0,n_{l+1}-1\}$.

(8) $j'_{k+1}=j'_{k+2}=0, i'_{l+1}=0, i'_{l+2}\neq 0$.

(9) $j'_{k+1}=j'_{k+2}=0, i'_{l+1}=n_{l+1}, i'_{l+2}\neq n_{l+2}$.

(10) $j'_{k+1}=m_{k+1}-1, j'_{k+2}=m_{k+2}-1, i'_{l+1}\notin\{0,n_{l+1}-1\}$.

(11) $j'_{k+1}=m_{k+1}-1, j'_{k+2}=m_{k+2}-1, i'_{l+1}=0, i'_{l+2}\neq 0$.

(12) $j'_{k+1}=m_{k+1}-1, j'_{k+2}=m_{k+2}-1, i'_{l+1}=n_{l+1}, i'_{l+2}\neq n_{l+2}$.

Suppose that $\mathbf{v}$ satisfies one of (7)–(9). Let $x=\Pi(\mathbf{v})$. The distance from x to the left and right sides are more than $(N^+)^{-3}R_k$. Since $r_{k+1}(m_{k+1}-1)=0$, the intersection of $B(x,(N^+)^{-3}R_k)$ and approximate squares above S is empty. It implies that

$$B\big(x,(N^+)^{-3}R_k\big)\cap E\subset S,$$

and the conclusion holds. The proofs for (10)–(12) are similar.

Finally, suppose that there is no $\mathbf{v}\in\partial U$ satisfying any of (1)–(12), which implies that for each $\mathbf{v}=v_1v_2\dots v_n\dots\in\partial U$, $v_n=(i'_n,j'_n)$. Then there exists $\mathbf{v}\in\partial U$ satisfying one of the following case:

(13) $i'_{l+1}=i'_{l+2}=0, j'_{k+1}=j'_{k+2}=0$.

(14) $i'_{l+1}=i'_{l+2}=0, j'_{k+1}=m_{k+1}-1, j'_{k+2}=m_{k+2}-1$.

(15) $i'_{l+1}=n_{l+1}-1, i'_{l+2}=n_{l+2}-1, j'_{k+1}=j'_{k+2}=0$.

(16) $i'_{l+1}=n_{l+1}-1, i'_{l+2}=n_{l+2}-1, j'_{k+1}=m_{k+1}-1, j'_{k+2}=m_{k+2}-1$.

Suppose that $\mathbf{v}$ satisfies (13). Let $x=\Pi(\mathbf{v})$. Since $r_{k+1}(m_{k+1}-1)=0$, the intersection of $B(x,(N^+)^{-3}R_k)$ and any approximate square above S is empty. Since $\mathbf{v}$ does not satisfy (7), we have that $(n_{l+1}-1, j_{l+1}(S))\notin\mathcal{D}_{l+1}$ if $k\geq l+1$, and $\widehat{r}_l(n_{l+1}-1)=0$ if $k=l$. Hence, the intersection of $B(x,(N^+)^{-3}R_k)$ and the approximate square on the left of S is empty. It follows that

$$B\big(x,(N^+)^{-3}R_k\big)\cap E\subset S,$$

and the conclusion holds. The proofs for (14)–(16) are similar. ■

The following three lemmas are the key ingredients for the proof of lower dimensions. For each integer $k > 0$, we write $\mathcal{S}_k = \mathcal{S}_\delta$ and $\mathcal{U}_k = \mathcal{U}_\delta$ for $\delta = (m_1 \cdots m_k)^{-1}$. For all integers $k' > k > 0$, we write

$$\Gamma^-_{k,k'}(E) = \min_{S \in \mathcal{S}_k} \Gamma_{k,k'}(S), \qquad \Gamma^+_{k,k'}(E) = \max_{S \in \mathcal{S}_k} \operatorname{card}\{S' \in \mathcal{S}_{k'} : S' \subset S\},$$
$$\Gamma_{k,k'}(S) = \operatorname{card}\{S' \in \mathcal{S}_{k'} : S' \subset S\}.$$

The next lemma shows that $\Gamma^-_{k,k'}(E)$ is bounded by the number $N^-_{k,k'}(E)$.

Lemma 5.2. *For all integers $k' > k \geq 1$, let $N^-_{k,k'}(E)$ be given by (2.5). Then*

$$\Gamma^-_{k,k'}(E) = N^-_{k,k'}(E).$$

Proof. Fix k and k', let $l = l(k)$ and $l' = l'(k')$ be given by (1.8). For each $S(x) \in \mathcal{S}_k$ where $x \in S(x) \cap E$, there exists a unique $U(\mathbf{w}) \in \mathcal{U}_k$, such that $S(x) = \Pi(U(\mathbf{w}))$ and $x = \Pi(\mathbf{w})$. For each $S'(x') \in \mathcal{S}_{k'}$ such that $S'(x') \subset S(x)$, let $\mathbf{w} = w_1 w_2 \ldots w_n \ldots$ and $\mathbf{w}' = w'_1 w'_2 \ldots w'_n \ldots$ be infinite sequences such that $\Pi(w) = x$ and $\Pi(w') = x'$, where $w_n = (i_n, j_n)$, $w'_n = (i'_n, j'_n) \in \mathcal{D}_n$, and we have that

$$\begin{aligned} i_n &= i'_n, \quad n = 1, \ldots, l, \\ j_n &= j'_n, \quad n = 1, \ldots, k. \end{aligned}$$

Computing $\Gamma_{k,k'}(S(x))$ is equivalent to counting the number of sequences $\mathbf{w}'$ such that $\Pi(\mathbf{w}') \in \mathcal{S}_{k'}$ satisfying the above property. Therefore, it is divided into six cases: $l < l' \leq k < k'$, $l \leq k < l' \leq k'$, $l \leq k < k' \leq l'$, $k \leq l < l' \leq k'$, $k \leq l < k' \leq l'$ and $k < k' \leq l < l'$. We only prove the first three cases, and the other three cases are the same by interchanging the directions.

(1) For $l < l' \leq k < k'$. We have that

$$\operatorname{card}\left\{i'_n : (i'_n, j_n) \in \mathcal{D}_n\right\} = r_n(j_n) \geq r^-_n, \qquad \text{for } n = l+1, \ldots, l',$$

and

$$\operatorname{card}\left\{j'_n : (i'_n, j'_n) \in \mathcal{D}_n, \text{ for some } i'_n\right\} = s_n, \qquad \text{for } n = k+1, \ldots, k'.$$

Therefore,

$$\begin{aligned} \Gamma_{k,k'}\big(S(x)\big) &= r_{l+1}(j_{l+1}) \cdots r_{l'}(j_{l'}) s_{k+1} \cdots s_{k'} \\ &\geq r^-_{l+1} \cdots r^-_{l'} s_{k+1} \cdots s_{k'} = N^-_{k,k'}(E). \end{aligned}$$

Since it holds for all $x \in E$, we have that

$$\Gamma^-_{k,k'}(E) \geq N^-_{k,k'}(E).$$

On the other hand, we choose $\mathbf{w} = w_1 w_2 \dots w_n \dots \in \Sigma^\infty$, where $w_n = (i_n, j_n) \in \mathcal{D}_n$ such that $r_n(j_n) = r_n^-$ for $n = 1, 2, 3, \dots$ Let $x = \Pi(\mathbf{w})$ and $S(x) = \Pi(U(\delta, \mathbf{w}))$. Then, we have that

$$\Gamma^-_{k,k'}(E) \le \Gamma_{k,k'}\big(S(x)\big) = r^-_{l+1} \cdots r^-_{l'} s_{k+1} \cdots s_{k'} = N^-_{k,k'}(E).$$

Hence, for $l < l' \le k < k'$, it is true that $\Gamma^-_{k,k'}(E) = N^-_{k,k'}(E)$.
(2) For $l \le k < l' \le k'$, we have that

$$\mathrm{card}\big\{i_n : (i_n, j_n) \in \mathcal{D}_n\big\} = r_n(j_n) \ge r_n^-, \qquad \text{for } n = l+1, \dots, k,$$

$$\mathrm{card}\big\{(i_n, j_n) : (i_n, j_n) \in \mathcal{D}_n\big\} = r_n, \qquad \text{for } n = k+1, \dots, l',$$

and

$$\mathrm{card}\big\{j_n : (\widetilde{i_n}, j_n) \in \mathcal{D}_n \text{ for some } \widetilde{i_n}\big\} = s_n, \qquad \text{for } n = l'+1, \dots, k'.$$

Therefore, we have that

$$\begin{aligned} \Gamma_{k,k'}\big(S(x)\big) &= r_{l+1}(j_{l+1}) \cdots r_k(j_k) r_{k+1} \cdots r_{l'} s_{l'+1} \cdots s_{k'} \\ &\ge r^-_{l+1} \cdots r^-_k r_{k+1} \cdots r_{l'} s_{l'+1} \cdots s_{k'} = N^-_{k,k'}(E) \end{aligned}$$

Since it holds for all $x \in E$, we have that

$$\Gamma^-_{k,k'}(E) \ge N^-_{k,k'}(E).$$

On the other hand, we choose $\mathbf{w} = w_1 w_2 \dots w_n \dots \in \Sigma^\infty$, where $w_n = (i_n, j_n) \in \mathcal{D}_n$ such that $r_n(j_n) = r_n^-$ for $n = 1, 2, 3, \dots$ Let $x = \Pi(\mathbf{w})$ and $S(x) = \Pi(U(\delta, \mathbf{w}))$. Then, we have that

$$\Gamma_{k,k'}\big(S(x)\big) = r^-_{l+1} \cdots r^-_k r_{k+1} \cdots r_{l'} s_{l'+1} \cdots s_{k'} = N^-_{k,k'}(E).$$

It follows that

$$\Gamma^-_{k,k'}(E) \le \Gamma_{k,k'}\big(S(x)\big) = N^-_{k,k'}(E).$$

Hence, for $l \le k < l' \le k'$, it is true that $\Gamma^-_{k,k'}(E) = N^-_{k,k'}(E)$.
(3) For $l \le k < k' \le l'$, we have that

$$\mathrm{card}\big\{i_n : (i_n, j_n) \in \mathcal{D}_n\big\} = r_n(j_n) \ge r_n^-, \qquad \text{for } n = l+1, \dots, k,$$

$$\mathrm{card}\big\{(i_n, j_n) : (i_n, j_n) \in \mathcal{D}_n\big\} = r_n, \qquad \text{for } n = k+1, \dots, k',$$

and

$$\mathrm{card}\big\{i_n : (i_n, \widetilde{j_n}) \in \mathcal{D}_n \text{ for some } \widetilde{j_n}\big\} = \widehat{s}_n, \qquad \text{for } n = k'+1, \dots, l'.$$

Therefore, we have that

$$\begin{aligned}\Gamma_{k,k'}(S(x)) &= r_{l+1}(j_{l+1})\cdots r_k(j_k)r_{k+1}\cdots r_{k'}\widehat{s}_{k'+1}\cdots\widehat{s}_{l'}\\ &\geq r^-_{l+1}\cdots r^-_k r_{k+1}\cdots r_{k'}\widehat{s}_{k'+1}\cdots\widehat{s}_{l'} = N^-_{k,k'}(E).\end{aligned}$$

Since it holds for all $x \in E$, we have that

$$\Gamma^-_{k,k'}(E) \geq N^-_{k,k'}(E).$$

On the other hand, we choose an infinite sequence $\mathbf{w} = w_1 w_2 \ldots w_n \ldots \in \Sigma^\infty$, where $w_n = (i_n, j_n) \in \mathcal{D}_n$ such that $r_n(j_n) = r^-_n$ for $n = 1, 2, 3, \ldots$ Let $x = \Pi(\mathbf{w})$ and $S(x) = \Pi(U(\delta, \mathbf{w}))$. Then, we have that

$$\Gamma^-_{k,k'}(E) \leq \Gamma_{k,k'}\big(S(x)\big) = r^-_{l+1}\cdots r^-_k r_{k+1}\cdots r_{k'}\widehat{s}_{k'+1}\cdots\widehat{s}_{l'} = N^-_{k,k'}(E).$$

Hence, for $l \leq k < k' \leq l'$, it is true that $\Gamma^-_{k,k'}(E) = N^-_{k,k'}(E)$. Therefore, the conclusion holds. ∎

Lemma 5.3. *Given $\beta > 0$, there exists a constant C such that*

$$N^-_{k,k'}(E) > C\left(\frac{R_k}{R_{k'}}\right)^\beta, \qquad \textit{for all } 1 \leq k \leq k',$$

if and only if there exists a constant C' such that $\inf_{x\in E} N_r(B(x,R)\cap E) > C'\big(\frac{R}{r}\big)^\beta$ *for all* $0 < r < R < \frac{1}{N^+}$.

Proof. For all reals r, R satisfying $0 < r < R < \frac{1}{N^+}$, there exist integers k, k' such that

$$R_{k'} \leq r < R_{k'-1}, \qquad R_k \leq R < R_{k-1}.$$

Immediately, we have that

$$(N^+)^{-\beta}\left(\frac{R}{r}\right)^\beta \leq \left(\frac{R_k}{R_{k'}}\right)^\beta \leq (N^+)^\beta\left(\frac{R}{r}\right)^\beta. \tag{5.1}$$

First, assume that $N^-_{k,k'}(E) > C\big(\frac{R_k}{R_{k'}}\big)^\beta$ for every $1 \leq k \leq k'$. Arbitrarily, choose $x \in E$. The ball $B(x, 2R)$ contains at least one approximate square in $\mathcal{S}_k$, and any set with diameter no more than r intersects at most $(N^+ + 1)^3$ approximate squares in $\mathcal{S}_{k'}$. Hence, for all $0 < r < R < \frac{1}{N^+}$, by Lemma 5.2 and (5.1), we have that

$$\begin{aligned}N_r\big(B(x,2R)\cap E\big) &\geq (N^+ + 1)^{-3} N^-_{k,k'}(E)\\ &> (N^+ + 1)^{-3} C\left(\frac{R_k}{R_{k'}}\right)^\beta\\ &\geq (N^+ + 1)^{-3} C (C_1 N^+)^{-\beta}\left(\frac{2R}{r}\right)^\beta.\end{aligned}$$

By taking $C' = (N^+ + 1)^{-3} C (2N^+)^{-\beta}$, we have that

$$\inf_{x \in E} N_r\big(B(x,R) \cap E\big) > C' \bigg(\frac{R}{r}\bigg)^{\beta},$$

for all $0 < r < R < \frac{1}{N^+}$. Next, assume that $\inf_{x \in E} N_r(B(x,R) \cap E) > C'(\frac{R}{r})^{\beta}$ holds for any $0 < r < R < \frac{1}{N^+}$. Therefore, by Lemma 5.2 and (5.1), for all $1 \leq k \leq k'$ and $S \in \mathcal{S}_k$, if $\frac{R_k}{R_{k'}} > 2(N^+)^3$, we have that

$$\begin{aligned} \Gamma_{k,k'}(S) &\geq N_{2R_{k'}}\big(B\big(x,(N^+)^{-3}R_k\big) \cap E\big) \\ &> C' 2^{-\beta} (N^+)^{-3\beta} \bigg(\frac{R_k}{R_{k'}}\bigg)^{\beta}, \end{aligned}$$

and if $\frac{R_k}{R_{k'}} \leq 2(N^+)^3$, we have that

$$\Gamma_{k,k'}(S) \geq 1 > \frac{2^{-\beta}(N^+)^{-3\beta}}{2} \bigg(\frac{R_k}{R_{k'}}\bigg)^{\beta}.$$

By taking $C = \min\big\{C' 2^{-\beta}(N^+)^{-3\beta}, \frac{2^{-\beta}(N^+)^{-3\beta}}{2}\big\}$, we have that

$$N^-_{k,k'}(E) > C \left(\frac{R_k}{R_{k'}}\right)^{\beta}$$

for all $1 \leq k \leq k'$. The desired conclusion then follows. ∎

We write that

$$\Psi_{k,k'}(\xi) = N^-_{k,k'}(E)(m_{k+1} \cdots m_{k'})^{-\xi}.$$

Clearly, the function $\Psi_{k,k'}(\xi)$ is decreasing in ξ. For all $k < k'$, we write $\xi_{k,k'}$ for the unique solution $\Psi_{k,k'}(\xi) = 1$, and it is clear that

$$\xi_{k,k'} = \frac{\log N^-_{k,k'}(E)}{\log m_{k+1} \cdots m_{k'}}. \tag{5.2}$$

For all integers $k'' > k' > k > 1$, by Lemma 5.2, we have that

$$N^-_{k,k''}(E) \geq N^-_{k,k'}(E) N^-_{k',k''}(E).$$

Immediately, it follows that

$$\Psi_{k,k''}(\xi) \geq \Psi_{k,k'}(\xi)\Psi_{k',k''}(\xi). \tag{5.3}$$

Lemma 5.4. *The sequence $\{\inf_k \xi_{k,k+m}\}_{m=1}^{\infty}$ is convergent.*

Proof. For each integer $m > 0$, we write $\zeta_m = \inf_k \xi_{k,k+m}$. Since $\zeta_m \le \xi_{k,k+m}$, it is clear that for all integers $i > 0$ and $k > 0$,

$$\Psi_{k+im,k+(i+1)m}(\zeta_m) \ge 1.$$

Fix an integer $m > 0$. For each $\xi < \zeta_m$, for all integers $k > 0$, $p > 0$ and $n > 0$ such that $0 \le n \le m-1$, by (5.3), we obtain that

$$\begin{aligned}
\Psi_{k,k+pm+n}(\xi) &\ge \left(\prod_{i=0}^{p-1} \Psi_{k+im,k+(i+1)m}(\xi)\right) \cdot \Psi_{k+pm,k+pm+n}(\xi) \\
&= \left(\prod_{i=0}^{p-1} \Psi_{k+im,k+(i+1)m}(\zeta_m)\big(m_{k+im+1}\cdots m_{k+(i+1)m}\big)^{\zeta_m-\xi}\right) \\
&\qquad \cdot\Psi_{k+pm,k+pm+n}(\zeta_n)\big(m_{k+pm+1}\cdots m_{k+pm+n}\big)^{\zeta_n-\xi} \\
&\ge 2^{p(\zeta_m-\xi)} \min\big\{2^{n(\zeta_n-\xi)}, (N^+)^{n(\zeta_n-\xi)}\big\}.
\end{aligned}$$

Since $\xi < \zeta_m$, there exists an integer K_0 such that for all $p \ge K_0$,

$$\Psi_{k,k+pm+n}(\xi) \ge 1.$$

Hence, for all integers $p \ge K_0$, $k \ge 0$ and n such that $0 \le n \le m-1$, we have that $\xi_{k,k+pm+n} \ge \xi$, and this implies that $\zeta_{pm+n} \ge \xi$. Therefore, for all integers n such that $0 \le n \le m-1$, we have that $\liminf_{p\to\infty} \zeta_{pm+n} \ge \xi$. Since it holds for all $\xi < \zeta_m$, we obtain that

$$\liminf_{m\to\infty} \zeta_m \ge \limsup_{m\to\infty} \zeta_m.$$

Therefore, $\{\zeta_m\}$ is convergent, and the conclusion holds. ■

To prove the formula for the Assouad dimension, we need the following three lemmas. Since the Assouad dimension is the dual of the lower dimension, the proofs of these lemmas are similar to the lemmas used for the lower dimensions, and we skip them.

Lemma 5.5. *For all integers $k' > k \ge 1$, we have that*

$$\Gamma^+_{k,k'}(E) = N^+_{k,k'}(E).$$

Lemma 5.6. *Given $\beta > 0$, there exists a constant C such that*

$$N^+_{k,k'}(E) < C\left(\frac{R_k}{R_{k'}}\right)^\beta \text{ for all } 1 \le k \le k',$$

if and only if there exists a constant C' such that

$$\sup_{x\in E} N_r\big(B(x,R)\cap E\big) < C'\left(\frac{R}{r}\right)^\beta \text{ for all } 0 < r < R < \frac{1}{N^+}.$$

We write that

$$\Delta_{k,k'}(\beta) = N^{+}_{k,k'}(E)(m_{k+1}\cdots m_{k'})^{-\beta},$$

and we write $\beta_{k,k'}$ for the unique solution $\Delta_{k,k'}(\beta) = 1$.

Lemma 5.7. *The sequence* $\{\sup_k \beta_{k,k+m}\}_{m=1}^{\infty}$ *is convergent.*

Now, we are ready to prove the lower and Assouad dimension for nonautonomous carpets.

Proof of Theorem 2.5. By (5.2) and Lemma 5.4, we write that

$$\xi_* = \lim_{m\to\infty}\inf_k \xi_{k,k+m} = \lim_{m\to\infty}\inf_k \frac{\log N^{-}_{k,k+m}(E)}{\log m_{k+1}\cdots m_{k+m}}. \tag{5.4}$$

To prove that ξ_* is the upper bound for the lower dimension of E, we choose a subsequence $\{(k_n,k'_n)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} k'_n - k_n = \infty$ and $\lim_{n\to\infty}\xi_{k_n,k'_n} = \xi_*$. For all $\xi > \xi_*$, there exists an integer $K' > 0$ such that, for all $n > K'$,

$$\xi > \xi_{k_n,k'_n}.$$

Combining with (5.2), we obtain that

$$N^{-}_{k_n,k'_n}(E) = \left(\frac{R_{k_n}}{R_{k'_n}}\right)^{\xi_{k_n,k'_n}} \le \left(\frac{R_{k_n}}{R_{k'_n}}\right)^{\xi}.$$

For all $\varepsilon > 0$, by the definition of lower dimension, there exists C_ε such that

$$\inf_{x\in E} N_r\big(B(x,R)\cap E\big) \ge C_\varepsilon\left(\frac{R}{r}\right)^{\dim_{\mathrm{L}} E-\varepsilon}.$$

By Lemma 5.3, this is equivalent to

$$N^{-}_{k_n,k'_n}(E) \ge C_\varepsilon\left(\frac{R_{k_n}}{R_{k'_n}}\right)^{\dim_{\mathrm{L}} E-\varepsilon},$$

for all $n > 0$. Immediately, we obtain that

$$\left(\frac{R_{k_n}}{R_{k'_n}}\right)^{\xi} \ge C_\varepsilon\left(\frac{R_{k_n}}{R_{k'_n}}\right)^{\dim_{\mathrm{L}} E-\varepsilon},$$

and it implies that

$$\xi \ge \dim_{\mathrm{L}} E - \varepsilon + \frac{\log C_\varepsilon}{\log R_{k_n} - \log R_{k'_n}}.$$

by taking n tend to ∞, we have that $\xi \geq \dim_L E - \varepsilon$. Since ε is arbitrarily chosen, we obtain that

$$\dim_L E \leq \xi,$$

for all $\xi > \xi_*$. Thus, the inequality $\dim_L E \leq \xi_*$ holds.

Next, we prove that ξ_* is the lower bound. The conclusion holds for $\xi_* = 0$, and we only consider that $\xi_* > 0$.

Arbitrarily choose $0 < \xi < \xi_*$, by (5.4), there exists an integer $K'' > 0$ such that, for all $m > K''$, we have that $\xi < \xi_{k,k+m}$, for all integers $k > 0$. Combining with (5.2), it follows that for all $k' - k > K''$,

$$N^-_{k,k'}(E) = (m_{k+1} \cdots m_{k'})^{\xi_{k,k'}} = \left(\frac{R_k}{R_{k'}}\right)^{\xi_{k,k'}} > \left(\frac{R_k}{R_{k'}}\right)^{\xi}.$$

For all $k' - k \leq K''$, since $\xi > 0$, we obtain that

$$N^-_{k,k'}(E) \quad \geq \quad \left(\frac{R_k}{R_{k'}}\right)^{\xi-\xi} \geq \left(\frac{R_k}{R_{k'}}\right)^{\xi} (N^+)^{-K''\xi}.$$

Let $C_\xi = (N^+)^{-K''\xi}$, we have

$$N^-_{k,k'}(E) \geq C_\xi \left(\frac{R_k}{R_{k'}}\right)^{\xi},$$

for all $\xi > \xi_*$. By Lemma 5.3, $\dim_L E \geq \xi_*$. Hence, the lower dimension formula holds.

The proof for Assouad dimension is similar to the lower dimensions, where Lemma 5.2, Lemma 5.3 and Lemma 5.4 are replaced by Lemma 5.5, Lemma 5.6 and Lemma 5.7 and we omit it. ∎

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